Proceedings of the 27th Annual Conference on Research in Undergraduate Mathematics Education

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Preface

As part of its on-going activities to foster research in undergraduate mathematics education and the dissemination of such research, the Special Interest Group of the Mathematics Association of America on Research in Undergraduate Mathematics Education (SIGMAA on RUME) held its twenty-seventh annual Conference on Research in Undergraduate Mathematics Education in Alexandria, Virginia from February 27 – Mach 1, 2025.

The 27th RUME Conference enabled presenters and attendees the option to participate fully online such that travel was not a requirement, approximately 22% of participants were online.

The program included plenary addresses by Dr. Sandra Laursen and Dr. Megan Wawro and the presentation of 174 contributed, preliminary, and theoretical research reports and 120 posters. The conference was organized around the following themes: results of current research, contemporary theoretical perspectives and research paradigms, and innovative methodologies and analytic approaches as they pertain to the study of undergraduate mathematics education.

The proceedings include several types of papers that represent current work in undergraduate mathematics education, each underwent a rigorous review by three or more reviewers:

- Contributed Research Reports describe completed research studies
- Preliminary Research Reports describe ongoing research in early stages of analysis
- Theoretical Research Reports describe new theoretical perspectives for research
- Posters may fall into any of Contributed, Preliminary, or Theoretical and were presented in poster format. Authors contributed the poster itself, a summary of the work, or both.

The conference was hosted by Virginia Tech's Department of Mathematics. Many members of the RUME community volunteered for the Program Committee where they reviewed many submissions such that every submission was reviewed by at least one member of the Program Committee. The Program Committee aided us in putting together this program and their hard work is greatly appreciated. The Local Organizing Committee were responsible for the smooth running of the presentations and on-site activities, which would not have been possible without the help of many volunteers, and we thank them for their tireless efforts to host a conference that runs smoothly.

Thank you to all of the researchers who submitted such strong proposals and ultimately made the conference a fun and joyous event.

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Norms and Values Upheld by Different Mathematical Activities: Proving, Defining, and Algorithmatizing

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Algorithms, proofs, and definitions all play important roles in mathematics as products of research and objects worthy of study in the classroom. Nevertheless, limited research has examined mathematicians' views of algorithms or compared their views across these objects and related activities. Based on interviews with nine discrete or computational mathematicians, we highlight similarities and differences between the values associated with these mathematical activities and relative importance placed on the varied activities.

Keywords: norms, values, proofs, definitions, algorithms

The mathematical community holds many views about what it means to do mathematics. While research has previously examined individual mathematicians' beliefs about what it means to do and teach mathematics (e.g., Rupnow, 2023; Woods & Weber, 2020), new attention has been focused on the values and norms of the mathematical community (Dawkins & Weber, 2017; Rupnow & Randazzo, 2023; Vroom et al., 2024). These works have focused on mathematicians' perceptions of how particular tools like proof (Dawkins & Weber, 2017) and definitions (Rupnow & Randazzo, 2023; Vroom et al., 2024) permit particular values to be upheld via norms of practice in the mathematical community. To build on this work, we examine a third tool, algorithms, and relate it to proof and definitions as we address the research question: How do norms and values around algorithms harmonize or suggest tension with norms and values around proving and defining?

Background Literature

Mathematicians engage in a variety of disciplinary practices that occur across subdisciplines of mathematics, including symbolizing, algorithmatizing, defining, justifying (Rasmussen et al., 2005), and computing (Lockwood et al., 2019). However, most examinations of these practices or of understandings of the corresponding objects (symbols, algorithms, definitions, proofs) have occurred in isolation. Algorithms and computing have received more attention recently because of their importance to computational thinking in mathematics and science (e.g., Lockwood et al, 2019; Weintrop et al., 2016), though algorithms have sometimes been viewed negatively as reinforcing procedural (as opposed to conceptual) understanding (Kamil & Dominick, 1997).

In contrast, extensive literature has examined the roles of definitions and proofs in mathematics education. Works focused on definitions have highlighted students' frequent use of extracted rather than stipulated definitions (e.g., Alcock & Simpson, 2011; Edwards & Ward, 2004; 2008), that is, self-created definitions extracted from known examples rather than stated definitions that pre-specify (stipulate) meaning. Research has also characterized the expectations of the criteria of a definition, such as existence and specification of necessary and sufficient conditions (Harel et al., 2006; van Dormolen & Zaslavsky, 2003) and pedagogical advantages of some definitions over others (Zazkis & Leikin, 2008). In addition to this extensive work on definitions, based on interviews with algebraists, Rupnow & Randazzo (2023) highlighted values and norms of the mathematical community related to definitions and the practice of defining.

Research attending to proof and proving is also myriad. Examples include examining

different purposes that a proof can have (e.g., Weber, 2002), how different formats of instruction can align to different proof-focused pedagogical goals (e.g., Weber, 2004), different approaches to creating proofs in different mathematical subdisciplines (Dawkins & Karunakaran, 2016), how students understand proofs (e.g., Inglis et al., 2007; Stylianides & Stylianides, 2009), and differences in students' and mathematicians' approaches to creating proofs (e.g., Weber & Alcock, 2004). Additionally, Dawkins and Weber (2017) highlighted mathematicians' values and norms around proof production based on an examination of the published mathematics and mathematics education literature. Here we seek to compare and contrast mathematicians' views of algorithms, definitions, and proofs and the corresponding activity around them.

Theoretical Perspective

The Triadic Network of Justification (Laudan, 1984) is a theoretical framework that characterizes the conceptual connections among axiology, methodology, and factual claims in a community. Axiology relates to the ethics and values of a community, methodology relates to commonly accepted norms of practice in a community, and factual claims relate to what is accepted as settled fact in a community. While this framework was designed to conceptualize the nature of paradigmatic changes in the scientific community, this framework has been applied in mathematics education to characterize the values (axiology), norms (methodology), and theorems (factual claims) of the mathematical community with respect to proof (Dawkins & Weber, 2017) and definitions (Rupnow & Randazzo, 2023; Vroom et al., 2024).

Based on the existing literature (Dawkins & Weber, 2017) or interviews with mathematicians (Rupnow & Randazzo, 2023), these studies generated values upheld by specific norms related to proving or defining to characterize rationales behind practices of mathematicians. Vroom et al. (2024) used the defining norms and values to assist students in adhering to these expectations. Dawkins and Weber (2017) listed four proof-related values, each upheld by various norms: "mathematical knowledge is justified by a priori arguments", "mathematical knowledge and justifications should be a-contextual and specifically be independent of time and author", "mathematicians desire to increase their understanding of mathematics", and "mathematicians desire a set of consistent proof standards" (p. 128). Rupnow and Randazzo (2023) highlighted two definition-related values, each upheld by various norms: "clarity in and for communication" and "freedom of choice for the use and creation of definitions" (p. 303). In this study, we build on the previously observed values and norms related to proofs and definitions while also using Laudan (1984) to conceptualize values and norms related to algorithms.

Methods

The first author emailed mathematicians at R1 institutions in the United States who were listed on their departmental website as researching applied/computational mathematics or combinatorics (or a similar description). That author then conducted 60-minute remote, screen-recorded interviews with nine such mathematicians. Five participants' research aligned more with applied/computational mathematics and four aligned more with combinatorics. All names are gender-neutral pseudonyms. Interview questions focused on participants' views of algorithms and their use, proofs and proving, definitions and defining, and how these areas interrelate.

Three researchers coded the interviews in MAXQDA 2020 (VERBI Software, 2019), with at least two researchers coding each interview. We entered coding with existing norms (codes) and values (categories) for proving (Dawkins & Weber, 2017) and defining (Rupnow & Randazzo, 2023) but were open to adding new codes and categories if needed. All three researchers initially open-coded two transcripts to create norm codes and value categories for algorithms in

alignment with Laudan (1984). After discussing our open codes, we created a codebook for algorithm norms and added some new norms for proof. We then re-coded and came to consensus on the first two interviews. Finally, two researchers individually coded each of the remaining seven interviews and then discussed and came to consensus on the norms discussed. This process aligns with codebook thematic analysis (Braun et al., 2019) in that we entered analysis with some idea of the codes and categories (norms and values) we would use but were open to adding new codes as needed.

Results

We coalesced the themes (categories) we observed associated with algorithms in mathematics into three values:

1. Algorithms are useful for solving problems
2. Correctness of algorithms needs to be verified
3. Algorithms need to be clear to people and communicable to others.

Here we focus on the interplay between these values and the values for proving and defining in mathematics observed in the literature and our data. The first four of the following values for proving were introduced by Dawkins & Weber (2017), to which we added a fifth:

1. Mathematical truth is a priori
2. Mathematical knowledge and justifications should be a-contextual and independent of time and author
3. Mathematicians desire to increase their understanding
4. Mathematicians desire a consistent set of proof standards
5. Proving is socially mediated.

The following two values for definitions were introduced by Rupnow & Randazzo (2023):

1. Clarity in and for communication
2. Freedom of choice for the use and creation of definitions.

Below we highlight how these values arose in our data by discussing some of the norms (italicized) that upheld these values and then compare the values related to each mathematical practice to those in other practices.

Algorithm Values

Algorithms are useful for solving problems. The interviewees were unanimous, (n=9), in saying *algorithms are a tool*, although the interpretations of "tool" were myriad. For example, Becker took a high-level view on the definition of an algorithm and stated that "algorithms are a tool for us to systematically describe our way of thinking for a certain solution process." Similarly, Emery stated, "I'm thinking of an algorithm as meaning a procedure for accomplishing something," and explained that certain mathematical proofs can be viewed as algorithms by stating "there are constructive proofs which, in some sense, are inherently algorithms." On the other hand, Foster viewed algorithms more through the lens of a computational tool and stated, "Well, the purpose of an algorithm is always to compute something. Or to end up with something."

There was further consensus that *problems/contexts drive the use/creation of algorithms*. This norm conveys that there is a real-world problem that requires a solution, and this is the main motivating factor in algorithmic development. For example, as a researcher in graph theory and combinatorics, Emery stated that, "I mean, algorithmic research is devoted to finding procedures that will quickly answer certain questions." As a researcher in combinatorics, Harley stated that, "So I think that a problem is not really answered until you have some kind of algorithm that

gives the answer. An object is not understood until you have an algorithm that constructs it." On the other hand, as a mathematical biologist, Ira stated that, "The goal is always to help me do a computation," with the implication that there is always some motivation from a biological problem for the algorithms required in Ira's work.

Correctness of algorithms needs to be verified. All interviewees observed that *deductive logic is important for algorithms,* as algorithms involve a sequence of steps and deductive logic is required for the algorithm to flow from one step to the next. For instance, Becker stated, "So algorithms are a way of describing patterns. Generally, what happens is you notice that certain things happen when you do things in a certain order", which highlights the importance of noticing and describing the steps within the algorithm. Devin made a similar comment but focused more on the big-picture sequence of subtasks within an algorithm: "What the algorithm does, the algorithm is going to allow you to compartmentalize what is happening, what is it going to, how is it getting there, and where are you going."

All but one of the interviewees suggested that *algorithms should be presented/tested with examples*. Emery viewed examples as a check on solutions: "Considering examples can be effective in understanding what an algorithm does." Ashton also highlighted the utility of examples for showing the purpose of steps in the algorithm: "You start with the simplest example you could possibly come up with and then just slowly ratchet up the level of complexity until you hit a point where—what should happen as you do that is that each step of the algorithm becomes clear why it's needed." As someone who has worked in a variety of contexts involving applied mathematics, Foster's experience indicated the importance of testing algorithms relative to examples with known solutions: "I never write an algorithm without testing it on things. Usually, you want to test it on things where you know the answer, and you want to see how well it's doing."

Three participants also highlighted that *using/creating algorithms drives research on convergence/stability properties*. Foster stated that:

> Sometimes we worry about [the fact that the algorithm does not satisfy all hypotheses of existing theorems for convergence] a little bit, and sometimes we just compute. And it depends on how confident we are that the algorithm is working. If we're pretty sure it works, we may not stop off and prove a theorem. If we have a little bit of a doubt, something is happening we're not expecting, then we might get into seeing what we could do about showing the algorithm really does converge to the answer that we're looking for.

Note that Foster sometimes will use an algorithm without proof of convergence, but this lack of rigor can drive future work for finding such a proof.

Algorithms need to be clear to people and communicable to others. All interviewees agreed that *algorithms need to be understood (enough, depending on context)*. Harley expected deep understanding, saying, "Everyone needs to have a conceptual understanding, a big picture, and a qualitative understanding." Others also conveyed expectations for understanding, such as Cooper, who highlighted the importance of using context:

> I've had these conversations also with students, and I ask them to show it in class where it's like, "Look, this fit is fantastic. If time was going the other way, no problem at all. But if you've got a fit and you want to predict the future and that fit is fantastic except for the last three weeks, then you look for something else, right?"

On the other hand, Ashton stated: "you can develop, use, and understand algorithms without even knowing that they actually truthfully do what you think they do… and just to be clear,

sometimes it's separate because it's literally too hard" indicating that there can be situations, such as using algorithms in machine learning, where using algorithms as a black box is acceptable.

Algorithms and Proving

There are many similarities in the values common to proving and algorithms. Here we draw parallels between norms and values related to these practices. We first highlight the algorithm norm *deductive logic is important for algorithms* that upholds the value 'correctness of algorithms needs to be verified'. The argument can be made that every proof is an algorithm if one considers a sequence of logically justifiable steps starting from some assumptions and eventually reaching the desired outcome to describe both a mathematically rigorous proof and an algorithm. As noted above, all nine interviewees touched upon the importance of deductive logic for algorithms, and seven were coded for the proof norm *justification should be deductive and not admit rebuttals* that upholds the value 'mathematical truth is a priori'. Ashton stated, "a proof is an attempt to justify or argue that something is correct within a system," and Cooper stated that "If I've got a proof that is 99% correct in Mathematics, I've got nothing."

The development of both proofs and algorithms typically involves a period of inductive reasoning where various strategies are employed until either the proof is worked out in the former case, or the algorithm is constructed in the latter. The proof norm *proofs arise from inductive thinking but are written deductively* that upholds the value 'mathematical knowledge and justifications should be a-contextual and independent of time and author' was supported by seven interviewees. Foster said:

> I let them incubate for a while, and then I go to work on them more seriously and hope something drops. And I have techniques and machinery that I understand and can use, but you have to figure out which machine or which technique might be useful.

Harley stated that "…you try and see the pattern. And then you make a conjecture that this construction will produce the general object or will always produce whatever object you're looking for. And then you try to prove it. That's the hard part." Both quotes suggest the importance of looking for patterns (engaging in inductive thinking) as a precursor to writing a normative deductive proof.

While correct proofs and algorithms stand by themselves, a modification of the hypotheses of a theorem or a modification of the class of inputs to an algorithm can provide impetus for future work. The algorithm norm *problems/contexts drive the use/creation of algorithms*, upholding the value 'algorithms are useful for solving problems' is similar to proof norms of *proofs are tools for future problem solving* and *proofs lead to questions which lead to theorems (cyclic nature)* that uphold the proof value 'mathematicians desire to increase their understanding'. These proof norms were mentioned by seven and eight participants, respectively. For instance, Ashton stated, "Every proof is a possible technique you can apply in the future," Emery said, "You really try to get in and understand what the previous proof does in order to tailor it to new needs," and Harley remarked, "Generally, there will be lemmas and theorems that you can use, and you see how the tools have been applied and whether they can apply in your case." Notice these quotes all relate to the utility of knowing existing proofs for future proving.

Across the Three Practices

No matter the venue, anything mathematicians do needs to be clearly communicable to others, whether to other researchers or to students in the classroom. The algorithm value 'algorithms need to be clear to people and communicable to others', the proving values

‘mathematicians desire to increase their understanding’ and ‘proving is socially mediated’, and the defining value ‘clarity in and for communication’ all speak to the utmost importance that is placed upon the communication of mathematics to others. As Cooper states, “You got to know exactly what you’re talking about before you talk about it,” illustrating that there must be clear communication between mathematicians regarding agreed upon definitions. Harley stated, “there are proofs that are enlightening that really explain why something is true,” illustrating the point that mathematicians value proofs which clearly communicate the big ideas, over obfuscation of the important points.

Additionally, the utility of definitions, proofs, and algorithms are what imbues them with meaning. The algorithm value ‘algorithms are useful for solving problems’, the proving value ‘mathematicians desire to increase their understanding’, and the defining value ‘freedom of choice for the use and creation of definitions’ all support the idea that mathematics is more powerful when there is a larger aim in mind. Ashton illustrated the myriad utility of algorithms:

> Algorithms, to me, have-- okay, they have several purposes. One is to construct mathematical objects or examples. Another is as a proof technique. So often what you want to show is that a certain condition holds. And to do that, you construct an algorithm, prove that the algorithm terminates, and the existence of such an algorithm then demonstrates that the claimed result holds. And then a third would be just for computational purposes… I use them in all three of those ways.

On the topic of proofs, Harley discussed the utility of proofs by stating:

> So what are the purposes of proof? As I say, proofs are the binding cement or ligaments of mathematics. They’re what cement it into place, what make it solid or maybe the glue that holds it in place. Proofs are not in themselves the aim of mathematics, but they’re what hold it together. They’re what make it reliable.

Regarding defining, Ashton pointed out that definitions are in service of the mathematics, and the correct definition is important in order to be able to say something useful: “it depends on what definitions you need in order to be able to say something meaningful. So a lot of times, a theorem is only true if you restrict the definition in the correct ways.” Across these quotes, we see that algorithms, proofs, and definitions are not generally goals, of themselves, but rather tools to accomplish some broader purpose, such as solving problems, increasing understanding, or clearly specifying the limits of a claim so it can be proven.

Relative Importance of the Three Practices

Six of the participants intimated that algorithms, proofs and definitions are all equally important in mathematics. Ashton argued in this direction, stating:

> They’re all mutually supportive. So certainly, you can talk about each of those in isolation, and you should, and I do. But for me, I-- I really do want students to see that those are interrelated ideas. I mean, again, sometimes in order to define something, you have to define it by constructing it with an algorithm. And sometimes in order to prove something, you prove it with an algorithm. And sometimes what you’re proving is that an algorithm works. And sometimes in order to do that, you have to define a new thing. So these don’t have fixed roles in how they interact, right? They all interact in complicated ways and can show up in different spots.

Devin, an applied mathematician, similarly argued in this direction, stating:

> We have the clinical information, we have the model verification, and we have the theoretical proof. That’s the Holy Trinity, baby. That’s it. You are good to go. Man, you have it. And it works altogether. You need all of them working in tandem. Because a lot

> of people, they get the theoretical component, and they get the actual mathematical algorithms, and that's good. But what are these things helping you answer? They're helping you answer the real-world problems.

However, different participants expressed different views on the relative importance of the three practices in mathematics, with some interviewees pointing out that for many users of algorithms, the fact they are useful is the most important thing, not that they can be proved correct. Cooper drew on their own experience and stated, "To be a competently functioning professional, you don't need to be able to do a convergence proof like my numerical analysis colleagues. Because I've never done that for an algorithm, and yet my algorithms seem to be doing fine." Relatedly, some interviewees observed a disconnect between pure mathematicians, for whom proof is paramount, and their applied mathematics colleagues, for whom the utility of mathematical tools is more important. In response to the question of the purpose of proof in mathematics, Ira stated, "That's a loaded question. Proof is important in some branches of mathematics. It is unimportant in other branches of mathematics." In contrast, others viewed proof as more important than algorithms, such as Emery:

> I don't care so much about the algorithms except in the sense of finding such a proof, a constructive proof, which, again, is equivalent to an algorithm, but my focus is not usually on the algorithms, but rather on, again, on using them to prove something.

Still others highlighted the relative importance of definitions, such as Ira: "I tell students that clear, precise definitions are the glory of mathematics. They're the foundation. It cannot be overemphasized that a clear definition is extremely powerful."

Discussion

This study aimed to examine mathematicians' norms and values associated with the mathematical practices of algorithmatizing, proving, and defining. We observed three values related to algorithms, five related to proving, and two related to defining in our data and the literature. These included the new algorithm values of 'algorithms are useful for solving problems', 'correctness of algorithms needs to be verified', and 'algorithms need to be clear to people and communicable to others' as well as the new proof value of 'proving is socially mediated'.

Looking across the values related to the three practices, we observe similarities such as the importance of clear communication and the utility of the practice for consistency or useful applications. However, while many of our participants viewed all three practices as interrelated and mutually important, others strongly viewed some practice or practices as more important than others, potentially based on the subdiscipline of mathematics they were engaged in. Moreover, within each practice we also observe some tensions between different values, potentially related to whether the emphasis is on the process or resulting object. Proofs establish mathematical truth (given specified assumptions) beyond doubt, but proving is socially mediated and can change over time as conventions and expectations change. When defining a new concept, mathematicians have freedom to create a definition in the way that suits them, but definitions, once established, should remain consistent. Algorithms can be useful tools to solve problems, including problems that cannot reasonably be solved manually, but the process being followed in the algorithm needs to be understood, or at least verifiable to some degree, to be sure the tool is aiding in providing a true solution. Future research is needed to better understand which practices are more central to specific subdisciplines of mathematics and how to help students navigate these distinctions across different areas of mathematics and personal views of instructors.

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To Whom Are Instructors Obligated?: An investigation of Individual Obligations in Quotient Group Instruction

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Teaching is a complex process in which mathematicians are expected to make many decisions before, during, and after the instructional time. These decisions can be shaped by various factors such as pedagogical or content knowledge, professional obligations, or the personal beliefs and values of the mathematician. In this study, we draw on the theory of Practical Rationality as a means to examine the role that professional obligations play in the pedagogical decisions that mathematicians make. Specifically, we examine the role that individual obligations play in instruction noting that the way in which the obligation is viewed can lead to different decisions.

Keywords: Teacher Decision Making, Undergraduate Instruction, Professional Obligations

Teaching is a complex process in which instructors are expected to make a myriad of decisions before, during, and after instruction. Researchers at the K-12 literature have suggested that teacher decision making is a crucial area to better understand instruction and its impact on student learning (Fennema, et al., 1989). The decisions instructors make are affected by various factors such as pedagogical and content knowledge and belief systems (e.g., Ball, et al., 2008). In addition to individual factors (knowledge and beliefs), societal and institutional factors play a substantial role in instructor decision making. That is, instructional decisions are also informed by professional obligations that inform, justify, and constrain said decisions (Chazan et al., 2016).

In contrast to the K-12 setting, less research has explored instructor decision making at the undergraduate level. The work that does exist has primarily attended to the decision to lecture or using inquiry materials (e.g., Fukawa-Connelly et al., 2016) or inquiry practices (Shultz, 2022), and decisions made during a mandated inquiry-based linear algebra course (Mesa et al., 2020). Lande and Mesa (2016) have also examined decisions by community college faculty shaped by their status as full-time or part-time status. In general, little is known about the day-to-day decision making, especially in advanced undergraduate courses. In this report, we begin to unpack this level of decision making through a study with 13 abstract algebra instructors. We provided the instructors with five distinct images of quotient group instruction and analyzed their reflections on instructional decisions using the construct of professional obligation. In particular, we ask: In what ways do individual obligations influence the content coverage decisions made in Abstract Algebra courses?

Theory

Practical Rationality holds that instructors exist in complex environment with four types of stakeholders (e.g., Chazan et al., 2016) leading to a set of professional obligations that influence instruction. The four professional obligations are disciplinary obligations, individual obligations, institutional obligations, and interpersonal obligations. Disciplinary obligations stem from the perspective of knowledge being a stakeholder within the environment. This poses on the instructor that they are "obligated to steward a valid representation of the discipline of mathematics" (Herbst & Chazan, 2011, p. 450). This obligation holds that instructors hold their students to certain standards of proficiency, that the authentic mathematical knowledge is taught,

and that they steward authentic mathematical values and practices. Individual obligations are derived from a client perspective in which students are a stakeholder in their own education. This lends itself to the idea that the instructor is obligated to do what is best for the student (Herbst & Chazan, 2011). The notion of "best" has the potential to vary based on factors such as cultural or historical contexts (Chazan, et al., 2016). Institutional obligations stem from the institution in which the instructor is situated. This places obligations on the instructor that they attend to various policies from the institution, course schedules, and so on. Lastly, interpersonal obligations stem from the perspective that society has a stake in education. The instructor is then "obligated to share and steward their medium of interaction with other human beings in the classroom" (Herbst & Chazan, 2011, p. 450). Additionally, as is consistent with past studies on teacher-decision making (e.g., Lande & Mesa, 2016), we hold that in addition to professional obligations shaping the decisions made, the beliefs, values, and knowledge of the instructor also plays a role.

Instructional decisions are shaped by the professional obligations at play. At times, professional obligations may be in conflict. At this point, an instructor may attend more substantially to one or more obligation to make their choice. For example, a mathematician's obligation to the mathematics discipline may conflict with the required tasks in a given curriculum, an institutional obligation (Chazan et al., 2015). Similarly, a desire to follow a students' reasoning (obligation to the individual) and to enculturate students into proof norms (obligation to the discipline) may be in tension leading to what Herbst (2002) calls the double bind.

Literature Review

The literature base on instructional decision-making in undergraduate mathematics has focused on a few studies and reflections. Schoenfeld et al. (2016) and Woods and Weber (2020) examined university mathematicians using resources, orientations, and goals framework. Schoenfeld et al.'s piece argued that the synergy or relationship between pedagogical content knowledge and mathematical content knowledge influenced the decision making of the mathematician. In using this framing, they found that mathematical correctness or precision was a driving factor in decision making for mathematicians. Woods and Weber (2020) interviewed eight mathematicians about findings from five different studies related to instructional practices in undergraduate mathematics. They concluded that goals of content coverage contributed to the use of lecture practices. Mathematicians' orientation towards precision similarly contributed to the traditional "chalk-talk" practices as it was viewed as allowing appropriate formal content coverage.

A well-studied decision is the decision to lecture. Woods and Weber found that while mathematicians recognized that lecture had shortcomings, they felt it allowed them to accomplish the goals that they had for the course such as content coverage goals, modeling authentic mathematical practices, or preparing future mathematicians. In a large-scale survey of abstract algebra instructors, Fukawa-Connelly et al. (2016) and Johnson et al. (2018) found that 85% of mathematicians chose to use lecture over non-lecture curricula. Further questioning showed that mathematicians often cited time constraints or lack of resources as common reasons why. Johnson et al. suggested institutional obligations accounted for mathematicians' decision to lecture such as time constraints or lack of support from the department. In spite of reporting these perceived constraints, the participants additionally explained that they did not feel pressure to cover specific content and that their departments would possibly support curricular redesigns

suggesting the link between institutional obligations and the decision to lecture was not clear cut. Several other studies have considered some of the decisions instructors make in the context of inquiry learning. For example, Mesa et al. (2020) identified three tensions that mathematicians experienced during the implementation of a mandatory IBL Linear Algebra curriculum. These tensions were between perceptions of IBL and institutional expectations; instructor knowledge and the textbook; and a final tension with pacing of the course. These tensions were underpinned by obligations to the discipline and institutional obligations conflicting with one another. Mesa et al. reported that disciplinary obligations guided a large portion of the decisions made. While the mathematicians were constrained by the curriculum imposed by their department, the mathematicians were in turn selective about definitions, examples, and application-based problems that they used. Mesa et al. did not identify any individual and interpersonal obligations in their data, and there were no claims that could be made regarding how those obligations contributed to double-bind situation. Johnson et al. (2013) studied some of the instructional decisions made by three instructors voluntarily teaching an inquiry-oriented abstract algebra curriculum. They found that the instructors differed in the weight they gave to content coverage aims and the engaging students in authentic mathematical practice. This could reflect a tension related to disciplinary aims (both practice and content) and institutional aims (expected course coverage). The role of course coverage was even more pronounced in Mesa's (2014) study of community college trigonometry instructors whose agency to make such decisions was limited.

Across these studies, disciplinary and institutional obligations are most salient. Our knowledge of these obligations is largely in the context of major instructional choices such as the decision to lecture. The purpose of this study was to examine instructional decisions as a more micro-level by focusing on the specifics of instruction related to quotient groups with attention to obligations to the individual.

Methods

This data comes from a larger dissertation study that examined the pedagogical decisions and approaches related to quotient group instruction. Data collection for this portion of the study involved 13 mathematicians from 6 universities across the United States. These mathematicians were selected via a combination of convenience and snowball sampling (mathematicians suggested other instructors who would be good to interview either because they were thoughtful about their practice or they approached instruction differently). Mathematicians additionally met the criterion of having taught an abstract algebra course since 2019. This criterion was added for the sake of recall. These mathematicians all took part in an initial semi-structured interview where they described their approaches to teaching quotient group. From these interviews five distinct profiles of instruction were developed and shared during the second interview. It is this interview that is the focus of the present paper.

During the second interview, the five profiles or vignettes (each of roughly one page in length) captured an array of instructional decisions including both decisions related to teaching style and those related to content presentation. Mathematicians read through these during the interview (one at a time) and responded to the vignettes as they went through the interview. By examining the decisions of others, we hoped to perturb the mathematicians to be more reflective of the decisions they make and articulate the obligations behind their choices. The five vignettes will be referred to as IOAA (Inquiry Oriented Abstract Algebra), IBL (Inquiry Based Learning), Worksheet, Equivalence, and DTP (Definition, Theorem, Proof). The inquiry vignettes were

heavily student-centered with the IOAA vignette having students reinvent quotient groups from informal ideas of partitioning, and the second having students work in the formal system to develop proofs of major ideas. The equivalence vignette contained a lecture where ideas of quotient groups were rooted in equivalence relations. The DTP vignette was also a lecture-based image of teaching, but it took a traditional approach of providing major definitions including a process for constructing cosets from a normal subgroup. Finally, the worksheet vignette included a similar approach but provided students time to work on worksheets after introducing the content.

Each vignette-response interview lasted approximately 75 minutes and was video recorded and transcribed. As the mathematicians read the vignettes, they were prompted to discuss anything they noticed (positive or negative) or that stuck out to them. For consistency across interviews, there were several talking points across the vignettes that I would bring up if it did not come up naturally. Those talking points included questions regarding training mathematicians and its relation to the participants job and comparing and contrasting the various informal understandings across the vignettes, examples, or questioning practices during instruction.

The data was analyzed for professional obligations using methods described in Shultz et al., (2019). This involved identifying references to each of the four obligations using codebook descriptions developed by Shultz et al. The first author coded each interview completely. Once coded, 10% of the transcript (for each interview) was sent to a second coder. The portion sent to the second coder was randomly selected post removal of discussion not relevant to the study (e.g., talking about technical difficulties was removed before selection). At this stage, there was mostly consistency (76%) on interpretations of obligations. Any disagreements were discussed and resolved.

Results

As results are discussed, we will present some of the findings of the study related to the individual obligations that mathematicians expressed with regards to content coverage. We begin by looking at the role of equivalence within content coverage followed by an examination of mathematicians' decisions in choosing curriculum. Please note that participants in the study are referred to as Dr. A through Dr. N. The vignette instructor does not have a title.

Obligation to the Individual in Content Choices: Equivalence and Intuition

Mathematicians were presented with various formal and informal understandings of quotient groups. In the IOAA vignette, the instructor emphasized the ideas of partitioning in which quotient groups were created via identifying subsets that build a new group structure. This hypothetical vignette instructor put forth that they made the decision to not emphasize formal equivalence relations as it was unintuitive for students leaving that exploration to homework. This is juxtaposed with the Equivalence vignette in which the instructor emphasized defining quotient groups using equivalence relations and classes first then determining that the equivalence relation being explored leads to classes that form a quotient group.

The decision to focus heavily or lightly on equivalence relations were both motivated by obligation to the individual (the students). For example, Dr. D (lecturer – equivalence)[1] stated "I'm very persuaded by the idea of equivalence relations being far more intuitive than cosets. I a

[1] The tag on mathematicians refers to the vignette that I suspected they most likely identified with based on initial interview.

million percent agree with them [the equivalence vignette instructor]." In this quote, Dr. D expressed that the decision to emphasize equivalence due to obligation to students to teach in an intuitive way. Dr. A (lecturer – worksheet) also expressed that she thinks equivalence relations are somewhat intuitive for students and should be covered stating "I think it's a lot more intuitive for students than this person thinks. How intuitive and how well it fits depends on what they've seen before and what they mean about it." Thus, Dr. A is making the decision to build off what students have covered previously which was equivalence relations. In these quotes from both participants, curricular decisions are being justified by their obligation to the individual to teach in a way that is intuitive for students.

In contrast, Dr. B (student centered - IOAA) rationalized the decision to focus on partitioning rather than equivalence relations because:

> You know, I think equivalence relations are very formal and borderline impenetrable for students in terms of them having like a conceptual footing in it and it ends with quotient group structure versus I have students begin with quotient group structures, abstract that idea and then we move to cosets.

This motivation is similarly made via the lens of obligation to the individual and the desire to teach in a way that is intuitive to students. However, the instructors appeared to have different conceptions and evaluations of what is intuitive to students. We hypothesize this may connect to what is valued in the discipline (reasoning from formal definitions) and differing pedagogical knowledge for teaching.

Obligation to the Individual: Teaching Practice and the Choice to Lecture/Not Lecture

As mathematicians engaged with the vignettes, there were often claims made about various teaching styles that extended more broadly than the context of quotient groups. For example, Dr. B (student centered – IOAA) shared their reason for why they adopted student-centered tasks saying, "I don't take curriculum tasks on lightly, without really evaluating whether I think that this is something that makes sense and that this is a way that's likely to be meaningful for my students." Dr. B makes the decision to adopt tasks (in particular, IOAA tasks) based on the perception of students which is an explicit acknowledgement that there is an obligation to individuals. Other instructors commented on the IOAA vignette illustrating something "accessible" for students or providing them opportunity to deeply learn particular content (D4 in this case).

However, unlike what is established in prior research, it was often obligation to the individual that also supported the decision to lecture and not use inquiry approaches. Dr. D (lecturer - equivalence) shared concern that students would not understand the point of an activity without the lecture. Other instructors (Dr. G and Dr. H) explained they thought students would "lose interest" or become bored. Thus, there is an obligation to keep a student engaged and to lead them to a satisfying conclusion, which instructors may see inquiry activities as not meeting.

Finally, we share that for many instructors, the obligation to the individual was not treated as a monolith, but rather the obligation was to support the subset of students continuing to graduate school. Dr I (lecturer - DTP) explained that he felt "inquiry-based learning, especially in something like abstract algebra and deprive[s] anybody, that somebody that wants to go on to grad school." He explained this approach would be good for a class of high school teachers.

In this case, the decision to train mathematicians stems from the academic need of the students going on to graduate school rather than it strictly being a disciplinary obligation to

steward mathematical practice. Dr. G (lecturer – DTP) expressed a similar sentiment with regards to graduate school by saying:

> And of course, if you want your students to have a chance to get into a good graduate program, you have to make sure they are not too much behind compared to students who apply from a school at [University]. You see, Yeah, well, it's difficult because we do get these good students in our classes too. To get them, they are just not as many as at [neighboring school] And so it's hard to cater to them without neglecting the whole rest. At the same time, it's hard to cater to everyone without neglecting the good students. So, it's really difficult.

Unlike Dr. I, Dr. G (lecturer) expressed a tension between meeting the needs of the "good" students without neglecting the other students in this class. As he talked through the amount of material he should cover in class, he seemed to express two separate obligations to two separate groups of individuals – those going to grad school in mathematics and those who are not. Those two sets of obligations created a tension in how he covers and presents material.

In contrast to concerns that the inquiry is not challenging enough for students, some instructors voiced that the inquiry-based learning approach would only work in situations where students came into the course with a strong grasp on proving. Dr. K explained:

> The one thing with our [university] seniors when I'm teaching Modern algebra is, is I also feel to some extent I'm teaching an introduction to proof course. Even though they've already had an introduction to proof it, it tends to be not evident. In terms of what they come with, and one thing that one reason I stick to the equivalence classes is the definition. The definitions of reflexive, symmetric and transitive are things with. Very structured definitions that inform the structure of the proof, so I like them to work with that definition and writing proofs so this may be so there […] I think it would allow me to do a little bit more of just what are the ideas and they can handle the proofs and the definitions later. And it might also it might make this approaching a bit more feasible that I'm saying. Because if they're strong in writing formal proofs, that may be actually decent of formulating definitions for their concepts too.

We point out that such responses reflect an obligation to the student (and that this obligation is shaped by the particular population of students at a given institution) and the decision to lecture is supported by the desire to provide students with material they can succeed at.

Discussion

Due to space constraints, we shared only some of the data related specifically to individual obligations. Our guiding research question informing this report was: In what ways do individual obligations influence the content coverage decisions made in Abstract Algebra courses? We begin by discussing variation in pedagogical decisions that stem from individual obligations followed by a brief discussion about the composition of individual obligations and their role in decision-making.

Individual Obligations on a Spectrum

As a result of the data analysis, we noted that individual obligations seem to exist on a spectrum which allows for the possibility of opposing decisions to be made with the same justification. While mathematically equivalent, a major distinction between the mathematicians was how they defined quotient groups: either as a collection of cosets along with a binary operation that partition a group G (building from informal) or imposing an equivalence relation ~

onto a group G (building from formal). These alternative approaches caused a divide between what participants emphasized. Justifications regarding this divide stemmed largely from obligations to the individual. Recall that mathematicians such as Dr. A or Dr. D felt that equivalence was an important topic that required more attention than just being a portion of the homework. However, on the other end of that spectrum, Dr. B felt as if it was a better decision to not emphasize formal equivalence relations outside of the homework. All of these instructors' justifications were rooted in ideas of what is intuitive and accessible for students. That is, the same perceived obligations can lead to different outcomes, and this influences not just general approaches to instruction, but also smaller content decisions.

Tensions within Individual Obligations

Previous research such as Woods and Weber (2020) noted that mathematicians viewed training mathematicians as an important part of their job, and this goal led mathematicians to engage in lecture practices. While the data supports this conclusion, it also provided more insight into the role that individuals have in the decision-making process. As participants discussed their curricular decisions and what approaches to take, it became apparent that participants viewed their classes as having two competing groups of individuals that had a stake in what was taught in the course. As Dr. G expressed, there were students in the class who were going on to pursue more mathematics and then students who were not. In a sense there was an obligation to train or prepare the students going on to graduate school. However, the students not going to graduate school in mathematics presented a different set of needs. Dr. I framed these individuals as teachers/ those not pursuing more mathematics and those going on to graduate school. Regardless of the title given to the groups of individuals, the two groups imposed competing choices on the mathematician with respect to the content that they covered. In some cases, the instructor prioritized the students going to graduate school and in other cases, the tension remained unresolved as to what was the best course of action to take. We emphasize that the decision to lecture or not lecture was very much shaped by obligations to the individual not just towards obligations to the discipline or to the institution.

Conclusion

In this report responses from a study including 13 mathematicians regarding instructional decisions were discussed. By and large, the mathematicians' choices were influenced by obligations to the individual (the students). However, this obligation appeared to play out differently related to several dimensions including: what the instructor perceives as intuitive to students (which we conjecture may be related to both pedagogical content knowledge and distinctions between intuitive, accessible, and its meaning in deductive mathematics), what group of students is prioritized (those going to graduate school or the broader class), and instructor beliefs about what students are capable of. (such as elite students needed to engage in IBL).

As we consider future directions, we intend to further examine the role that professional obligations play in the teaching of quotient groups and abstract algebra. We note a limitation of this analysis and future analysis is that the data set is limited in terms of reflection rather than in the moment teaching decisions. Even so, we intend to expand from individual obligations to also examining the role of disciplinary, interpersonal, and institutional obligations. Furthermore, we feel that it is worth examining how these decisions impact student learning. We also intend to examine the role of intuition further. This includes investigation into what mathematicians consider to be intuitive and how that is related to the decisions that they make.

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Using Schemas to Model in APOS: The Case of the Differential Calculus of Two-Variable Functions

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In this report, we use the Schema portion of Action-Process-Object-Schema (APOS) theory to frame a study of student understanding of the differential calculus of two-variable functions. The use of Schemas allows us to obtain a more holistic perspective of students' understanding. Data was obtained from semi-structured interviews with eleven students. We argue that the APO portion of the theory can give detailed information about specific Schema components but that it does not inform about the interrelations between the different components. However, the notion of Schema and its instrumentalization via the notions of genetic decomposition, types of relations between components, and the triad stages of Schema development can provide a more comprehensive picture encompassing the different components of the differential calculus Schema. The notions of the scope of an instrument and the graph of a Schema are introduced to afford a visual representation of a student's Schema.

Keywords: APOS, schema, calculus, two-variable functions, differential multivariable calculus

APOS is a well-known cognitive theory that proposes that knowledge of a specific mathematical notion is constructed in stages of development, starting with the Action stage and going through the Process and Object stages of conceptualization. The different mental structures constructed in these stages can be organized in Schemas that evolve with time. When learning mathematics, students frequently meet complex situations in which different notions are interrelated to build a more developed structure. In these cases, the focus on single structures of the APO part of the theory is insufficient to capture the situation's complexity. In these cases, it is convenient to use Schemas to model understanding. Schemas provide a more holistic view of knowledge construction and development. In this article, we use the context of the differential calculus of two-variable functions (DC) to exemplify the use of Schemas and propose a way to represent graphically a student's DC Schema at a moment in time.

Literature review

Several early articles introduced APOS theory and its foundational ideas (e.g., Asiala et al., 1996; Dubinsky, 1991, 1994; Dubinsky & McDonald, 2001) with a more recent book (Arnon et al., 2014) summarizing the theory's development. The notion of Schema in APOS is developed from Piaget's ideas (1971, 1972). Since Piaget's ideas were developed in a context different from that of advanced mathematical thinking, they must be adapted to be useful in this new context. Different researchers have made their own adaptations, even for the context of mathematics education (e.g., Vergnaud, 1990), emphasizing different aspects. As with any theory, APOS grows with the contributions of the community of researchers that apply the theory to understand didactic phenomena. The need to further the application of Schemas to understand the didactics of the chain rule led a group of researchers to develop the notion of Schemas in APOS by introducing the idea of the triad of Schema development (Clark et al., 1997). Further development was spurred by attempting to discuss the complexity of a linear algebra course, leading to the notion of types of relations between Schema components (Trigueros, 2019).

Throughout the years, there has been research applying the APOS notion of Schema that deal with different notions, for example, calculus graphing Schema (Baker et al., 2000), implicit differentiation (Borji & Martínez-Planell, 2020), chain rule (Clark et al., 1997), limits (Cottrill et al., 1996), sequences (McDonald et al., 2000), two-variable function optimization (Martínez-Planell et al., 2023), differential equations (Trigueros, 2000), linear algebra (Trigueros, 2019), and modeling (Trigueros & Martínez-Planell, 2024).

The idea of a genetic decomposition for Schemas as a list of Schema components, relations between the components, and a description of the triad has been gaining traction, as may be seen in different publications (e.g., Martínez-Planell et al., 2023). This study's context is the differential calculus of two variable functions; for a literature review of the didactics of this context, we refer the reader to the survey in Martínez-Planell and Trigueros (2021). The article by Borji et al. (2023) is also helpful in this regard. It discusses different research studies on the differential calculus of two-variable functions underscoring their geometrical interpretation.

Theoretical Framework

Since APOS is a well-known theory, we only define what we need from the Schema portion of the theory. We refer the reader to Arnon et al. (2014) for definitions of Action, Process, and Object. A Schema is a coherent collection of Actions, Processes, Objects, and other previously constructed Schemas, dealing with a specific mathematical notion or topic. A Schema is coherent in the sense that the individual can determine when a problem situation falls within the scope of the Schema. This definition of Schema suggests that they may be modeled with a list of components, a list of relations between Schema components, and a description of how the Schema may develop. This model is called a *genetic decomposition* (GD) of a Schema. A GD is neither unique nor meant to describe the best way a student may learn a particular notion. The notion of types of relations between Schema components, introduced by Trigueros (2019), has proved to be useful. A *correspondence* relation is established out of habit or repetition, but the student cannot justify it. It can result from repeatedly seeing two components used in tandem or from superficial comparisons of similarities between components. A *transformation* relation groups different components in a normative way, with perhaps one of them being used to justify or explain the relation between the others. The relation is used normatively in a given problem situation, although not necessarily so across different problem situations. A *conservation* relation is used normatively and fluently across different problem situations. Properties in one structure are conserved or translated to properties in other structures so that the structures seem to be interchangeably used whenever it is convenient. Types of relations are a tool to use when deciding the stage of Schema development of a student with respect to a particular notion N. A Schema is at the *Intra*-N stage of development when the components are isolated or mostly related by correspondence relations; at the *Inter*-N stage of development when there are almost no isolated components and transformation relations start to appear, grouping different components; at the *Trans*-N stage of development when almost all the components are interrelated by conservation relations. The use of the adjectives “mostly” and “almost” in the previous definition is an acceptance of the fact that a student might be in transition from one stage of Schema development to another.

Our research questions are:

How do we study students’ Schemas for the differential calculus of two-variable functions?

What didactical suggestions can we deduce from the study of students’ Schemas?

Methodology

Eleven engineering students spanning the range from above-average (3), average (5), to below-average (3), as chosen by the professor, agreed to participate in two semi-structured interviews two weeks after completing an introductory multivariable calculus course at a mid-tier Iranian university. The two interviews were held on different days, each lasting approximately 1 hour. Interviews were video and audio recorded, transcribed, individually analyzed by the researchers, and discussed among them until a consensus was reached. There were 17 problems in the interviews[1]. The students used activities[2] designed to help them do the constructions proposed in the revised GD described in Martínez-Planell et al. (2015, 2017) and Trigueros et al. (2018). The activities used the ACE (Activities in class, Classroom discussion, Exercises as homework) pedagogical strategy (Arnon et al., 2014). In particular, students worked collaboratively in groups of three or four; each group turned in their collective work, which was promptly graded later; there were group presentations in which their answers were discussed; activities not finished in class were completed as homework; and common difficulties deduced were discussed in class. We do not detail the interview instrument or the activities since this report focuses on how Schemas were used to analyze the data.

While typically one starts research focusing on the Schema portion of the theory by proposing a GD for a Schema, this is not what happened in this study. A GD for the differential calculus of two-variable function (DC) using the APO portion of the theory had been previously proposed, tested with data from students, and revised by Martínez-Planell et al. (2015, 2017) and Trigueros et al. (2018). The revised GD was also expressed in terms of the APO portion of the theory. This was followed by a second research cycle to test the revised GD. The data obtained in the second research cycle, when analyzed using the APO portion of the theory, allowed us to obtain information about some specific components of the DC Schema: partial derivatives (Borji et al., 2024), directional derivatives (Borji et al., 2023), and slope (Martínez-Planell et al., 2022). However, focusing on the APO portion of the theory did not allow us to discuss students' construction of relations between these components and other components like tangent plane, total differential, gradient, and two-variable function, which is what we intend to do in the present study. In this report, we look at the same data but from the perspective of Schemas.

To examine the existing data from the perspective of Schemas, we proposed a list of components and relations. We start by considering the main notions that were tested with the interview instrument as components: two-variable function (F), slope (S), tangent plane (TP), partial derivative (PD), directional derivative (DD), gradient (G), and total differential (TD). To propose the relations, we ask: What can the interview instrument reveal about the relations students establish between different notions of the differential calculus of two-variable functions? The idea of the scope of an instrument proposes one answer to this question. The scope of an instrument is the collection of relations that are directly explored by the instrument or that may be introduced by the student in answering the questions in the instrument. The scope of an instrument may be represented graphically as follows. Each interview question explores at least one such relation. For example, in question 3a, we gave students the graph of a two-variable function and asked about the sign of the partial derivative at a point. A function was given (graphically), and information about a partial derivative (its sign) was requested. So, the question directly explores a relation between the Schema components of two-variable function and partial derivative. We represent a relation that is directly explored by the interview instrument with a

[1] The instrument is at https://drive.google.com/file/d/1ebXyptqwq2BAYYgkq-s02ijmpVfYBtx3/view?usp=sharing.
[2] See activities at https://drive.google.com/file/d/1cef_UjVXt9KxQemVK7hDFBioNND4CfzZ/view?usp=sharing.

solid line, as in Figure 1 (left). Further, the same question allows us to uncover other relations that students may establish. For example, in the same question, a student may decide the sign of the partial derivative by arguing about the slope of the tangent line at the point and thus use the component of slope to mediate the answer. We represent relations that students may indirectly address when answering an interview question with thinner lines, as in Figure 1 (right).

The relations that are directly or may be indirectly explored by the interview instrument are represented in Figure 2. This figure allows us to visually assess how complete an exploration of relations between components an interview instrument can afford. In this case, we see that the interview instrument allows us to inquire into the construction of many of the possible relations in the Schema.

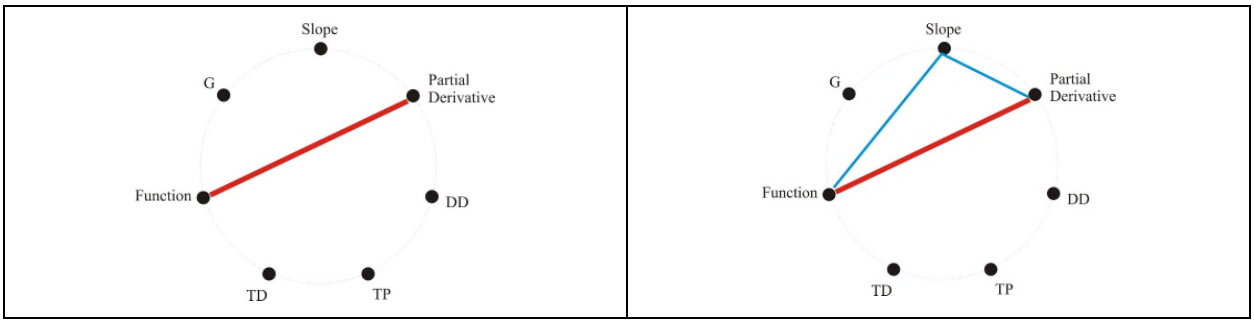


Fig. 1. Relation directly explored by interview question 3a (left) and relations that may be exhibited by a student when answering the same interview question (right).

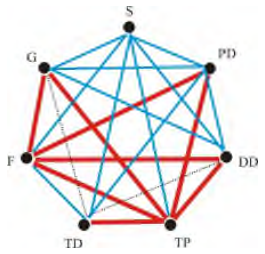


Fig. 2. Scope of the interview instrument. Relations that are directly (thicker lines) or indirectly (thinner lines) explored by the interview instrument. Dotted lines represent relations unlikely to be constructed by students.

After having analyzed the data using the APO portion of the theory, the data was analyzed again using codes that referred to the components and relations obtained from the interview instrument. For Schemas, we needed three distinct rounds of analysis. In the first round, we considered the relation established by the student in each question, if any, and classified its type. For example, if the student was given the graph of a two-variable function and asked for the sign of a specific partial derivative, we could use the codes F-S-PDc (the "c" is for correspondence), F-S-PDt (the "t" is for transformation), or F-S-PDe (the "e" is for equivalence or conservation) if the student talked about slope when answering the question, or perhaps F-PDc, F-PDt, or F-PDe if the student decided the sign based on properties of the function (increasing, decreasing) without mentioning slope. While the first round looked at the data question by question, the second round looked at it longitudinally. That is, we looked at all questions for a given student and verified that if an "e" had been used, the relation had been used in a normative way throughout the interview. We could also upgrade a relation initially classified as "t" to "e" if it had been used fluently and in a normative way throughout the entire interview. In the third round, we classified the students' Schema development as Intra-DC, Inter-DC, or Trans-DC. To

do so, we used the students' "graph of Schema development" to obtain a visual representation of the Schema development. To construct such a graph, we draw line segments between nodes equally distributed in a circle (like in the graph for the scope of an instrument), but this time, the thickness of the line segment depends on the type of transformation (conservation-thick, transformation-thin, correspondence-dotted).

Results

We first analyzed students' construction of relations and the types of relations. We only include an example of one problem situation. For a more complete discussion of the difference between conservation, transformation, and correspondence relations, see Trigueros et al. (2023). Given the graph of $z = f(x, y)$ (Figure 3), students were asked for the sign of $f_{yx}(3,0)$.

Hamid: This second partial derivative is the derivative respect to *y* and then respect to *x*. So I need to compare the slope of tangent lines in the *y* direction while we move in the *x* direction… Okay. Before the point (3,0) I consider a point like this, umm the tangent line to the curve of the surface in the *y* direction is a line like this and its slope seems to be negative because it's a decreasing line.

Interviewer: Why did you consider the tangent line in the y direction?

Hamid: Because the first derivative is respect to *y*.

Interviewer: Okay, continue.

Hamid: I go in the *x* direction because the second derivative is respect to *x*. If I draw the tangent line at the point (3,0) in the *y* direction it will be umm a line like this and it's a horizontal line. Comparing the value of the slopes between these two lines we see that their changes is positive so the answer is positive.

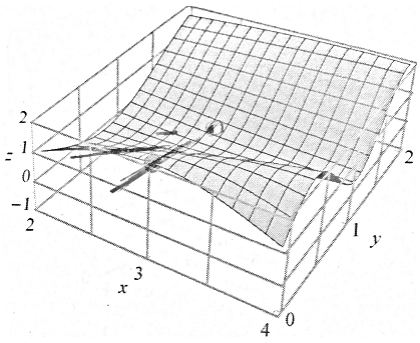


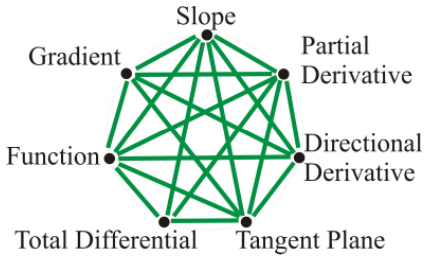
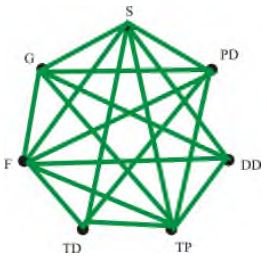
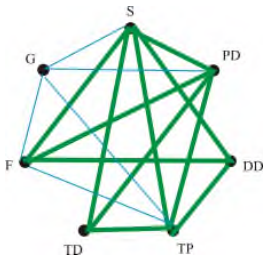
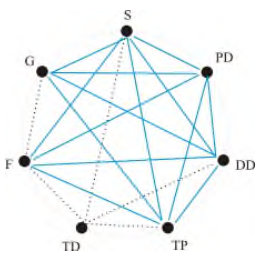
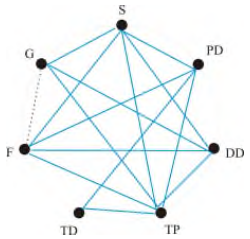
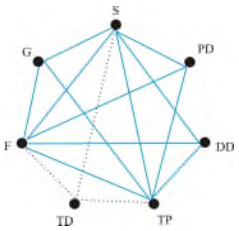
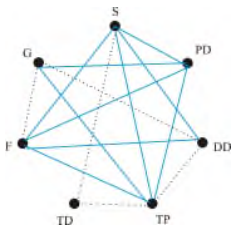
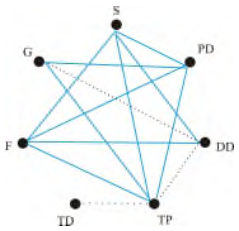
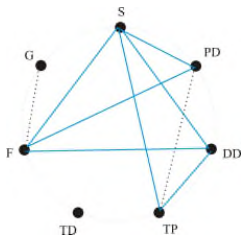
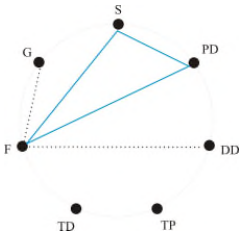
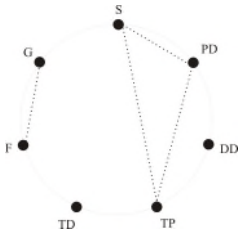
Fig. 3. Hamid's work on the question.

Hamid was given a function (F) and was asked about a (second) partial derivative (PD). He mediated his answer using slope (S) ("I need to compare the slope of tangent lines…"). So, this relation is labeled F-S-PD. His answer is normatively correct and justified, so that the relation is at least a transformation relation (F-S-PDt). Further (not shown here), his answers to questions involving F, S, and PD showed that he could move fluently between these components interpreting properties of one in terms of the other. This was done consistently and in a normative way throughout the interview. F-S-PD was labeled a conservation relation (F-S-PDe).

We then considered students' Schema development using the Schema development graphs in Table 1. Note that these graphs allow us to easily represent students' Schema development using the Intra-DC, Inter-DC, and Trans-DC triad. Results about students' Schema development can inform of their overall understanding and how to better support their learning. We don't show it

in this paper but results about students' stages of Schema development may be used to compare different didactical approaches.

Table 1. Schema development graphs of the participating students.

A1 Trans	A3 Trans	A2 transition from Inter to Trans	A5 Inter
			
A6 Inter	**A7 Inter**	**A4 Inter**	**A8 Inter**
			
A10 transition from Intra to Inter	**A11 Intra**	**A9 Intra**	
			

From Table 1, we can list the different relations between Schema components and the type of relation established by the students. This is shown in Table 2. This table will enable us to uncover the relations that seem to require more student support or increased classroom time or emphasis. This is important since it suggests possible problem situations to help students establish the relations. Thus, the investigation of students' stages of Schema development informs us about the overall performance of students and how to improve it.

Discussion and Conclusions

Our study contributes several observations pertinent to the didactics of the two-variable function differential calculus. Knowing what relations cause difficulty is useful. To see this, we consider the eight relations with the lowest total of conservation and transformation relations and consider the didactic implications of each one (Table 2). Each of F-G and DD-G have only four conservation or transformation relations. This suggests that more attention (class time and/or activities) has to be devoted to studying the properties of the gradient vector, relating it to the contour lines of a two-variable function or the level surfaces of a three-variable function, and to the direction of the largest instantaneous rate of change (largest directional derivative) of the

function. There are only four TP-TD and two TD-F conservation or transformation relations. This suggests that the relation between a tangent plane at a point, the translation to the origin of such a plane and the total differential at the point has to be further explored, together with graphical comparisons of tangent plane and function. Students should be able to think of the total differential as approximating locally the change in functional values geometrically, symbolically, and numerically. There were only two PD-DD conservation or transformation relations. We conjecture that this results from how students constructed directional derivatives (see Borji et al., 2023). They did so by exploiting local linearity and using the tangent plane at a point to talk about slopes in the *x* and *y* direction rather than referring to partial derivatives at the point. This result suggests that it is important that students not only relate tangent plane and slopes for developing the directional derivative but also relate this to the usual formula for directional derivative as the dot product of gradient vector and unit direction vector. Students did not construct the relations TD-G and DD-TD. While they would be unusual constructions and thus unexpected for students, they provide ideas for activities to help students further the coherence of their Schemas.

Table 2. Relations and types of relations established by students.

Relation/type	Conservation	Transformation	Correspondence	No rel.
Slope-Partial Derivative	3	7	1	0
Slope-Directional Derivative	3	6	0	2
Slope-Tangent Plane	3	6	1	1
Slope-Total Differential	3	0	3	5
Slope-Function	3	7	0	1
Slope-Gradient	2	4	0	5
Partial Der.-Directional Der.	1	1	0	9
Partial Der.-Tangent Plane	3	5	2	1
Partial Der.-Total Differential	3	1	0	7
Partial Derivative-Function	3	7	0	1
Partial Derivative-Gradient	2	4	0	5
Direct. Der.-Tangent Plane	3	4	2	2
Direct. Der.-Total differential	0	0	1	10
Direct. Der.-Function	3	6	1	1
Direct. Der.-Gradient	2	2	2	5
Tan. Plane-Total Differential	3	1	4	3
Tangent Plane-Function	2	6	0	3
Tangent Plane-Gradient	2	6	0	3
Total Differential-Function	2	0	2	7
Total Differential-Gradient	0	0	0	11
Function-Gradient	2	2	6	1

Another contribution of the study is introducing the notion of the scope of an instrument. This allows examining the suitability of an instrument to explore Schema development. The study also contributes the notion of a student's Schema development graph. It gives a visual way to represent the Schema development of an individual at a moment in time. This simplifies the analysis of students' DC-Schema development using the Intra-DC, Inter-DC, and Trans-DC triad. It can also inform possible didactical interventions, as the above discussion shows.

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Partitioning Physical Attributes for Riemann Sum Approximations: Underexamined Competency

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Researchers have investigated several interpretations of definite integrals and their notation, and they have recognized that one particular interpretation, Adding Up Pieces (e.g., Jones, 2013), is especially important for applying definite integrals to physics and other domains. This same research has paid less attention to how students partition physical attributes, and thus construct pieces, in ways consistent with constructing Riemann sum approximations and definite integrals. We analyzed how 8 students taking calculus-based physics courses constructed Riemann sum approximations and integrals for one task about water pressure on a tank wall. All of the students recognized that integration was relevant, but half did not partition the wall in ways compatible with constructing Riemann sum approximations or definite integrals that solved the task. We conjecture that seeing integrands as rates of change and structuring situations in terms of level curves for rates of change is a significant, but underexamined competency.

Keywords: Calculus, Integration, Coordination Class Theory

Introduction

A growing body of research has reported on students' interpretation and construction of integrals, especially when using integrals to solve problems in physics and other applications outside of pure mathematics (e.g., Cui et al., 2006; Jones & Ely, 2023; Nguyen & Rebello, 2011; Oehrtman & Simmons, 2023; Thompson, 1994a). One key issue is whether and, if so, how students connect definite integrals with adding up small amounts of a target quantity (Ely, 2017; Jones, 2013, 2015). We present interview data in which college students constructed small amounts when approximating a target quantity in one physical situation. We found that even when students interpreted integrals in terms of Adding Up Pieces (e.g., Jones, 2013, 2015), ways that they partitioned physical attributes were not always compatible with forming normatively correct Riemann sums. This implies significant competencies for applying definite integrals to problems outside of pure mathematics that to date have been left underexamined.

In the sections that follow, we review key strands of research on students' reasoning when applying definite integrals to solve problems, primarily from physics. We then summarize our methods, which were based on interviews with eight students who had credit for at least first-semester calculus and who were enrolled in calculus-based physics courses. We then present four cases in which students partitioned a physical attribute, but not in ways compatible with constructing Riemann sum approximations for definite integrals that solved the task. We use these cases to argue that current research has not paid enough attention to competencies for partitioning in anticipation of forming Riemann sums for approximating definite integrals.

Literature Review and Research Questions

Applying definite integrals to solve problems about physical situations requires reasoning in terms of quantities, such as lengths, time, work, electric charge, etc. Past research has provided a variety of perspectives on such reasoning, and much of it has been grounded in Thompson's

account of quantitative reasoning (e.g., Thompson, 1994b; 2011). One line of work has investigated different interpretations of definite integrals and their notation. Thompson and colleagues (e.g., Thompson, 1994a; Thompson & Silverman, 2008) have argued for interpretating integrals as accumulations from rates. Jones (e.g., 2013, 2015) has found that college students can interpret the $\int_a^b f(x)\,dx$ notation as indicating addition of small pieces (Adding Up Pieces), denotating boundaries of a region in the plane, and signifying a function whose derivative is the integrand. The Adding up Pieces interpretation is particularly relevant to the present study and consists of (a) partitioning a physical situation into small pieces, (b) approximating the target quantity within each partition piece, and (c) summing the small amounts of the target quantity across all of the partition pieces to arrive at an approximation of the total amount. Further work has considered whether, when using the Adding Up Pieces interpretation, students should understand those pieces as products composed through multiplication (e.g., Ely, 2017; Jones et al., 2017; Thompson & Harel, 2021). Exactly how students partition physical attributes in service of constructing Riemann sums and definite integrals, however, has not been highlighted in these studies.

A second set of perspectives has decomposed reasoning about definite integrals into layers that include ones focused on local relationships (e.g., Oehrtman & Chhetri, 2015; Oehrtman & Simmons, 2023; Sealey, 2014; Stevens & Jones, 2023; Von Korff & Rebello, 2012). As one example, Oehrtman and Simmons (2023) proposed *Emergent Quantitative Models* as a tool for tracing how students' attention shifts across three scales when constructing definite integrals. They named these scales the Basic, Local, and Global models. Anticipating the main example from our results section, a Basic model relates quantities with constant values (e.g., pressure • area = force, $P \bullet A = F$). A Local model restricts a basic model to small regions where varying quantities are approximately constant (e.g., pressure • "small amount" of area = "small amount" of force, $P \bullet \Delta A = \Delta F$). A Global model is an accumulation based on local models (e.g., sum of the small amounts of force, $\sum P \bullet \Delta A = F$). Oehrtman and Simmons made four points. First, students can derive different local models from the same basic model. To illustrate, there are at least two possible local models from the basic $P \bullet A = F$ model, $P \bullet \Delta A = \Delta F$ and $\Delta P \bullet A = \Delta F$. Second, students must make decisions about whether or not to partition. This involves recognizing that variation in values prevents constructing a close approximation using a single, global application of the Basic model. Third, students must decide *how* to partition. Fourth, students must decide whether and how to sum small pieces.

The present study makes closest contact with the third point above. In contrast to Oehrtman and Simmons (2023)—who presented a single example in which students partitioned the plane into concentric circles smoothly and appropriately for the task on which they were working—we found that for eight college students enrolled in calculus-based physics courses partitioning in service of constructing appropriate Riemann sums was not always straightforward. For the present report, we focused on one task and asked: *How did the students partition physical attributes and to what extent was that partitioning compatible with setting up a Riemann Sum and concomitant definite integral?* Although all the students recognized that definite integrals were relevant, we present examples in which they partitioned physical attributes in ways that did not lead to Riemann sum approximations and definite integrals that would solve the task.

Theoretical Framework

Our theoretical frame combines expert interpretation of definite integrals with cognitive components evidenced by students. Vergnaud (1983, 1988) has analyzed mathematical structures

for multiplying with quantities and distinguished two subtypes of situations. *Isomorphism-of-measure-spaces* (I-o-M) situations are ones where the multiplicative structure consists of a simple direct proportion between two measure spaces M_1 and M_2 (1983, p. 129). Here the direct proportion can be viewed as a rate of change. One common example is speed • time = distance, where the units might be miles/hour • hours = miles. *Product-of-measure-spaces* situations (P-o-M) are the Cartesian composition of two measure spaces, M_1 and M_2, into a third, M_3 (1983, p. 134). One common example is Newtons • distance = force, where the units might be Newtons • meters = Joules. Riemann sums use rectangular areas to model both I-o-M situations (e.g., integrating velocity to get distance) and P-o-M situations (e.g., integrating force to get work). For the present study, we focus on pressure as a rate (e.g., Newtons per square meter). More generally, if in the $f(x)dx$ notation one interprets $f(x)$ as a rate, then partitioning in anticipation of a Riemann sum involves locating subsets along which that rate is approximately constant. We that researchers have taken different positions on whether one can always interpret $f(x)$ as a rate when modeling situations with definite integral, $\int_a^b f(x)\,dx$.

Our cognitive perspective is informed by Coordination Class theory (e.g., diSessa, 2002; diSessa & Sherin, 1998; diSessa & Wagner, 2005), a strand of theory within the knowledge-in-pieces epistemological perspective (e.g., diSessa, 1993, 2006). diSessa (2002) motivated the notion of coordination classes with the following description of expert accomplishment: "The fundamental assumption behind the idea of coordination classes is that information is not transparently available in the world. Instead, we have to learn how to access different kinds. Indeed, in different circumstances, we may need to use very different means to determine the same kind of information" (p. 43). This description implies that a collection of knowledge resources allows people to see the same concept-specific information across a range of situations, a phenomenon referred to as *span*, and that different subsets of concept-related resources might be used in different situations, a phenomenon referred to as *concept projections* (diSessa & Wagner, 2005; Wagner, 2010). Foreshadowing our concluding discussion, we conjecture that students may need various subsets of resources (an issue of concept projections) to "see" rates and their "level curves" across an expanding range of situations (an issue of span). Moreover, it seems plausible that information about rates of change and level curves are not "transparently available" across both I-o-M situations and P-o-M situations.

Methods

We interviewed five students enrolled in first semester, calculus-based physics (Spring, 2023) and three students enrolled in second semester, calculus-based physics (Fall, 2023). All eight students attended the same selective university in the northeast United Sates. Each student participated in a series of three, one-on-one, semi-structured interviews (e.g., Bernard, 1994; Ginsburg, 1997). The interviews were conducted by the second author and were spaced a few weeks apart. Each interview lasted approximately 1 hour. The students worked on tasks that afforded opportunities to construct approximations through addition of pieces, consistent with constructing Riemann sums. In addition to text, several of the tasks included diagrams or graphs.

We recorded the interviews using two cameras, one to capture the student and the interviewer and one to capture close ups of written work. At the end of each interview, we collected all written work for later analysis. We used a computer to synchronize and combine the two video recordings into a single file, and we transcribed the interviews verbatim.

We analyzed videos for all eight students, attending to their moment-to-moment utterances, inscriptions, and hand gestures. In so doing, it became clear that one task in particular, the Water

Pressure task provided particularly good access to students' construction of approximations in a situation that, for them, was novel. This was in contrast to tasks where students accessed more rehearsed approaches, such as when using sampled velocity to approximate distance traveled. Furthermore, we observed a variety of approaches to partitioning in the Water Pressure task that were not well aligned with constructing normatively correct Riemann sum approximations.

Figure 1 shows the Water Pressure task used in Spring 2023, a task we adapted from Sealey (2014). In contrast to Sealey, we deliberately omitted formulae that might scaffold students' reasoning about multiplication and possible ways to interpret the $f(x)dx$ notation. Because this first group of students appeared to rely on rote recall when relating length and area units, in Fall 2023 we used a width of 4 meters and a depth of 3 feet to see how readily students would measure the tank wall area in meter-feet. One normatively correct approach to the Water Pressure task is to view pressure as a rate: so many units of force per unit of area. A definite integral for computing the total force can then be set up by attending to level curves where the pressure is constant. In this case, those curves are horizontal lines that extend from left to right across the tank wall. As a consequence, the small regions on which to approximate pressure with constant values are thin rectangles that also extend from left to right across the tank wall. Such rectangles were only one of four different approaches we observed when students connected areas and definite integrals. We discuss the three alternative approaches in the next section.

Pressure, *P*, applied across a surface area, *A*, creates a total force, *F*. Consider a vertical side wall of a tank with a width of 4 feet and a maximum depth of 3 feet (see picture below). Assume that the tank is full of water. The pressure on the tank wall increases with the depth of the water according to the following law: $P = 15x$, where x is the depth of the water (the deeper the water, the greater the pressure). Given this information, describe a method for approximating the total force from the water pressure on the wall.

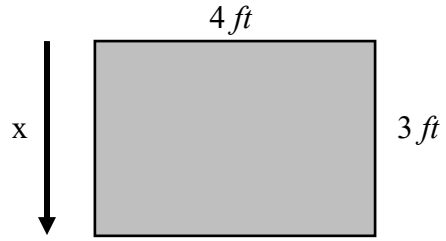


Figure 1. The Water Pressure task

Results

None of the eight students was confident about the relationship between pressure and force, but they all recognized that integration was likely relevant and discussed the $15x$ expression at some point. Thus, they all evidenced knowledge resources spontaneously that we recognized as useful for making progress on the task.

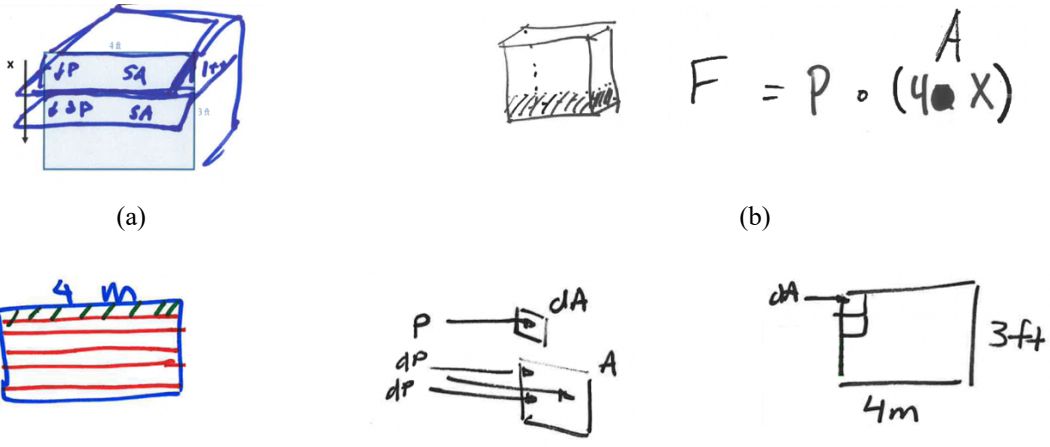
Approach 1: Areas Are Horizonal Cross Sections Parallel to the Bottom

Iliana (SP 23) discussed adding up pieces and located areas as horizontal cross sections of the tank. She read the Water Pressure task and wrote: "To calculate the total force you would take the integral of $P = 15x$ to find the sum of all the surface areas in the tank and evaluate on the bounds (0, x)." She continued by writing "$\int 15x$" and explaining:

Iliana: I just said that to calculate the total force you would have to take the integral of the pressure, and then that is like, integral is really just like finding the sum of all the surface

areas in the tank. The like, because like the pressure is just applied across each little area for like *x* [gestured with open palms to indicate horizontal slices].

Iliana commented "I don't feel like the 4 feet of the width has anything to do with that, but it might. I am not completely sure." When the interviewer asked where Iliana saw surface area, she drew horizontal slices (Figure 2a) and confirmed that her arrows pointed "downward." Thus, her verbal explanation, hand gestures, and drawing gave consistent evidence that she imagined pressure applied to cross sections parallel to the bottom of the tank. Because Iliana may have misread the description of the situation, the interviewer asked whether she considered pressure on the tank wall. Iliana reported that she had not, quickly recognized that "the pressure is like pushing outward onto the side with the force," and stated "But my picture still, this still goes like that [retracing horizontal slices in Figure 2a]." Thus, Iliana's orientation to areas did not shift hand-in-hand with her orientation to pressure.



(a) (b) (c) (d) (e)

Figure 2. Pieces of area. (a) Iliana, (b) Alex, (c) Melinda, (d) Matthew, (e) Matthew

Approach 2: Areas Are Vertical Regions of the Tank Wall (4*x*)

Alex (SP 23) and Melinda (FL 23) made comments that referenced adding up pieces, and they discussed pressure and area as functions of *x*. As soon as he finished reading the Water Pressure task, Alex used rate language to describe pressure: "We can say that this up here is pressure, which is in some units of force per square unit area." He considered downward pressure momentarily, similar to Iliana, but then turned his attention to pressure on the vertical wall, as we intended. He explained:

> *Alex*: If we're talking about the pressure out from the water, then what we have to talk about is *P* times the area of the tank wall. So *P*, which is 15*x*, and then the area of my tank wall is going to be 4 times *x* [wrote $(4 \bullet x)$, Figure 2b]….So, I want to figure out how much pressure is being applied to that little rectangle [shaded region in Figure 2b]. So, then this would be the pressure applied across the surface area. So, this is my *A* here [wrote A above $(4 \bullet x)$] creating my total force *F* on the sidewall of the tank [completed writing $F = P \bullet (4 \bullet x)$].

An exchange later, Alex explained that "my intuition there is if I'm doing something across an area, I usually use multiplication." He then used multiplication to describe the normatively correct relationship between pressure, area, and force:

> *Alex*: It's not that I have a total number *P*, and I'm splitting it up into smaller chunks. It's that I need to make sure that *P* is being applied equally to each one of these parts of *A*. So I

would need to be multiplying *P* by *A* in order to get that force over the area, or that pressure over the area to find the total force.

Alex's comments left unclear precisely where he located "each one of these parts of *A*." Although his discussion of parts was disjoint from using a Basic model, he continued using $(4 \bullet x)$ for area as he worked on the task.

Melinda initially wrote $A = 4 \bullet (3 - x)$ and $P = 15x$. She then graphed $P = 15x$, wrote $\int_0^3 15x\ dx$, and explained approximation in terms of adding pressures at different depths: "If we, say, measure at like more points, right? At all these, all these like *x* values and then we add them all up, then we can get a better approximation of the total pressure that's going against the wall." A few exchanges later, she shaded a rectangular strip at the top of the wall (Figure 2c) and changed her area expression to $4 \bullet (x)$. At the same time, when the interviewer asked Melinda about her $\int_0^3 15x\ dx$ integral, she explained:

> *Melinda*: You would want to find like the area of like a little piece, or like one of these, right. And that would be *dx*, which is the change in *x*, times the height, which is 15 times *x* or like it follows this function. And so, to find the area of that, it's like a rectangle. So, you just multiply the base times the height and then you find the area of that.

Melinda recognized that rectangles on the wall and on her graph of $P = 15x$ referred to different combinations of quantities, but she did not connect local rectangular regions on her graph to local rectangular regions on the tank wall.

Approach 3: Areas Are Regions That Tile But Do Not Span the Tank Wall

Matthew (FL 23) considered two ways to construct a small piece. His first was to consider pressure applied to little bits of area. He wrote $F = \int P\ dA$, a first step toward a normatively correct solution. His second was to consider little bits of pressure applied over the entire tank wall. He then provided a reason for choosing the latter over the former:

> *Matthew*: I don't know if it's going to be area, *dP*, like a little bit of pressure over the entire area, or it's this total pressure over the entire space of the area. But given that the area is constant, and I guess the pressure's changing, I think I would actually not go with this and do, maybe *dP* times *A*.

Next, Matthew wrote $F = \int dP\ A$, plugged in information provided in the task statement, and arrived at $F = 4m \bullet 3\ ft \int_0^3 15\ dx$. A few exchanges later, he reiterated:

> *Matthew*: The reason I said *F* equals *dP* times *A* instead of pressure *dA*, which is the area, is because the, the area isn't changing, like the size of the, the wall isn't changing, but the pressure is with the depth of the water.

Matthew then produced drawings to indicate how he visualized his two approaches (Figure 2d). For the drawing with several arrows labeled "*dP,*" he explained, "And when you, when you sum up all these *dP*s, you would get the pressure." Figure 2e shows how Matthew responded when asked where on the wall he saw *dA*. Notice that he apparently intended to tile the wall, but not with rectangles that spanned the width of the tank wall. A few exchanges later he explained that the pressure was constant on horizontal lines that spanned the width of the tank wall, but he did not revise his approach for setting up an integral.

Approach 4: Areas Are Normatively Correct Thin Slices That Span the Tank Wall (4d*x*)

Three of the remaining four students located a small piece of area as a thin rectangle that spanned the tank wall, even if they encountered other challenges constructing and interpreting

integrals to solve the task. Jack (SP 23) quickly drew a horizontal line across the tank wall to indicate a generic small area and wrote the expressions 4 *dx*. He then used the phrase "pressure over area" and tried to reason about *P*/*A* as the integrand, a misdirection that he eventually recognized and corrected. Izsák and Sus (2024) reported on Larisa (SP 23), who generated 15 * .04 = .6*N*, a normatively correct approximation for a small piece of force at $x = 1$. She then dismissed such a small contribution as inconsequential, in essence questioning the logic of Adding Up Pieces. Torin (FL 23) generated $F = \int_0^3 15x \bullet 4\ dx$, drew lines across the tank wall, but stated repeatedly that *dx* was simply a notation that did not carry meaning in the situation. In particular, he did not connect *dx* with a small change in height. Finally, Kathrine (SP 23) generated $\int 15x\ dx$, interpreted this integral as the area under the graph of 15*x*, and discussed trapezoids for determining that area, but she made no reference to rectangular areas on the tank.

Conclusion

We examined how students partitioned a physical attribute in service of constructing an approximation in a situation that, for them, was novel. Our results demonstrate that invoking the Adding Up Pieces interpretation of definite integrals (e.g., Jones, 2013, 2015), though often necessary, is not always sufficient for constructing appropriate Riemann Sum approximations for definite integrals. All eight students we interviewed considered using a definite integral for solving the Water Pressure task, and all eight discussed Adding Up Pieces art some point. At the same time, we observed four different approaches to partitioning the tank wall. One approach was normatively correct, but the remaining three were incompatible with constructing approximations for integrals that would solve the task. This result leads us to ask what knowledge would support students' approaches to partitioning physical attributes and generating Riemann Sum approximations, especially for situations with which they are less familiar.

In the Background section, we organized past research on definite integrals into two large strands, one that has identified different interpretations of integrals and one that has delineated multiple layers of attention when generating integrals. Compared to past research, our results suggest that more precise theory is necessary if we are to account for students' reasoning about definite integrals in ways that can ultimately inform instruction. We conjecture that tools from Coordination Class theory could help. Wagner (2016) took a first step in this direction when arguing that students' various interpretations of definite integrals could be understood as different concept projections. Future research could investigate in greater detail how students connect the $f(x)dx$ notation to problem situations. As an example, Matthew's decisions were driven by the fact that the tank wall area was known and constant, which led him away from normatively correct reasoning. Future work should examine how students might look for rates of change and their level curves across diverse situations. As noted by diSessa (2002), relevant information is not always transparently available, and students may need different means to determine how to partition physical attributes when applying definite integrals to solve problems outside of pure mathematics.

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Coping With Coercion in Logic: A Case study

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The ability to make a logical inference is at the heart of mathematical experience. However, a personal reasoning does not always follow the rules of formal logic. In this case study we focus on responses of one participant, a secondary school mathematics teacher, to scenarios involving coercive logic (the construct is defined and exemplified in the paper). This allows us to take a detailed look into what guides and what influences the participant's approaches to various scenarios. We argue for the relevance of coercive logic to mathematics education, in particular, in exploring mathematical reasoning of students.

Keywords: logic, reasoning, coercive logic, mental models, illusory inference

Logical reasoning is inseparable from mathematics. However, research has shown that the study of formal logic does not always result in improved logical derivations (O'Brien, 1973; Wason & Johnson-Laird, 1972; Johnson-Laird & Byrne, 1991; Inglis & Attridge, 2016). Moreover, while there are disagreements within psychological reasoning theories, many theoretical perspectives that explain human reasoning are strongly linked to mathematical theories, such as probability, formal logic, or model theory, to mention a few.

In what follows we situate our investigation within ample research on mathematical reasoning and logical inferences. We then introduce the notion of coercive logic and analyze responses of one participant to scenarios involving coercive logic.

On Logic and Reasoning in Mathematics Education

Numerous studies have explored diverse aspects of formal logic in mathematics education. There are studies that concentrate on negation (Antonini, 2001; Barnard, 1995), logic connectives (Dawkins, 2017; Dawkins & Cook, 2017), axioms (Dawkins, 2018), contrapositive reasoning (Stylianides, et al., 2004), and quantifiers (Dawkins & Roh, 2020), among others. A general aspect of this research is the advocacy for the understanding of formal logic as a tool that supports mathematics learning.

One of the fundamental aspects of logic and reasoning is implication (Dawkins & Norton, 2022; Dawkins, et al., 2021; Hub & Dawkins, 2018), that is, conditional statements. As mathematical theorems, or propositions in general, can be presented in a form of conditional statements, students' understanding of implication attracted interests of researchers interested in learning proof.

Conditional reasoning has an extensive literature in the cognitive sciences (Nickerson, 2015). In particular, research surrounding the Wason's selection task (Wason, 1968) showed that not all reasoning behind deduction or inference belongs to the formal logic realm; there is a lot of deduction that is context dependent (e.g., Cheng & Holyoak, 1985). Further, mathematical background of participants influences their decisions, as noted by Stylianides et al. (2004) in their study of on interpretation of contrapositive statements, in comparing everyday scenarios and the related symbolic form.

Introducing Coercive Logic

The concept of coercive logic was developed and popularized by Raymond Smullyan (1997) in his book "*The Riddle of Scheherazade and Other Amazing Puzzles, Ancient & Modern*". Smullyan (1997) defines coercive logic as "...an almost magical quality in that it forces [someone] to do something [they] wouldn't otherwise have done..." (p. 256)." (He credits the name to his son-in-law Dr. Jack Kotik.)

Example of Coercive Logic

Smullyan (1997) introduces coercive logic with a story of Scheherazade (a character from the Arabic collection of stories known as "One thousand and one nights"), who every night escapes the execution by entertaining the Sultan with stories. Smullyan changes the stories to puzzles, where Scheherazade gets the Sultan's promise to answer truthfully to one question, indicating only YES or NO. What would be the question that can spare her life? One of the suggested solutions for this logical puzzle is for Scheherazade to ask: "Will you either answer NO to this question or spare my life?". Note that the NO answer implies that neither of the two alternatives holds, but this would be the false answer, which negates the Sultan's promise to answer truthfully. As such, the only truthful answer is YES, which means that at least one of the two alternatives holds, and it must be the latter. With such a question Scheherazade can coerce the Sultan to spare her life.

Analyzing the case with the formalism of mathematical logic, we use:

N = you answer is NO ; L = spare Life

So, the given condition is "N or L", which is true if at least one of the alternatives is true.

Let us examine the outcome of each possible answer. The NO answer negates both N or L, so we get "¬ (N or L)", which is equivalent to "¬ N and ¬ L", which is a contradiction.

As such, the YES answer is the only option to avoid contradiction, because in "N or L", N is false, thus L must be true for "N or L" to obtain a valid true value.

To the best of our knowledge, there is no research in mathematics education that explicitly attends to coercive logic.

Coercive Logic, Brainteasers and Unfamiliar Environments

Paradoxes, puzzles, riddles, brain teasers, conundrums, etc. (basically any instrument that could create uncertainty or cognitive conflict) can enrich teaching and learning of mathematics (Kleiner & Movshovitz-Hadar, 1994). They have been part of mathematics and mathematics education throughout history. Scenarios that involved coercive logic have features that are similar to other kinds of recreational mathematics.

Coercive logic has characteristics related to cognitive conflict, inference, and uncertainty. These moments of doubt are opportunities to learn, teach, and learn as a teacher. Zaslavsky (2005) argued in favour of awakening certain dynamics that occur when dealing with cognitive conflict. She further mentioned that uncertainty is relevant to develop explanations and in general contribute to learning mathematics. We observe that coercive logic has the potential to become a motivational tool for the study of mathematical logic.

Scenarios of coercive logic, often presented as puzzles, are related to logic paradoxes, but there is an important difference. A paradox presents a self-contradictory statement or a situation that leads to contradictory conclusions. For example, in the famous Russell's paradox (The barber is the "one who shaves all those, and those only, who do not shave themselves". The question is, does the barber shave himself?), a barber who does not shave himself, actually shaves himself. However, in the coercive logic scenarios there is a "way out", that is, a

possibility of introducing an appropriate statement that will avoid a presumed paradox and lead to a desired outcome.

Research in mathematics education has demonstrated that asking learners to operate in unfamiliar environments can reveal their understanding of the elements of more familiar situations. For example, asking students about fractional numbers written in different bases reveals their understanding of place value (Khoury & Zazkis, 1994, Zazkis 1999); asking students to work with non-Cartesian coordinates highlights their knowledge of the Cartesian coordinate system (Zaslavsky et al., 2002, Zazkis, 2008). In a similar way, we expect to learn about participants' reasoning in the unfamiliar environment of coercive logic. In particular, we are interested in aspects of participants' reasoning and logical inference that are revealed when they respond to scenarios involving coercive logic.

The Study

Participants in our study were secondary mathematics teachers enrolled in a master's program in mathematics education. Our data sources included two written response questionnaires, which included coercive logic scenarios, and a semi-structured interview. As an introductory case study, which is a part of larger research, we focus here on one participant, Sal, whose responses were detailed and featured his approaches and hesitations.

Coercive Logic Scenarios

For the purpose of our study, we define coercive logic scenario as a fictional story culminating in a set of two conditional statements with different consequents, and a viable sought-after-statement that nullifies one or both conditionals to reach a unique valid and desirable outcome. This statement introduces a coercion. Consider the following example, used in one of our tasks:

> *You were trapped by cannibals, and they want to eat you. Their chief, who is known as a perfect logician, tells you that you can speak your last words, which should indicate your choice* (this is the story).
> *If what you say is true, then they will cook you in water.*
> *If what you say is false, then they will cook you in oil.*
> (this is the set of two conditionals)

The sought-after-statement is: *You will cook me in oil.* This statement nullifies both consequents: you cannot be cooked in water because the statements is not true. You cannot be cooked in oil, because "cook you in oil" is an implication of the false statement, and your statement is not false. In our design of coercive logic scenarios, we have constructed a variety of stories with sets of conditionals, in which the sought-after-statement leads to coercion and desirable outcomes. The participants were asked either to evaluate the given statements, or to formulate a statement that would lead to a desired outcome, such as avoiding a penalty or obtaining a reward.

Data Collection

The data collection was based on two questionnaires, administrated two weeks apart, and a semi structured interview scheduled after the completion of questionnaires.

Questionnaire 1 aimed at establishing a basic familiarity with formal rules of inference, and then introduced coercive logic scenarios. It included 6 multiple choice items. All items in Questionnaire 1 asked participants to explain their choice briefly for each case.

The first 4 items focused on conditionals with modus ponens structure, as well as 'and' and 'or' operators. In particular, acknowledging the differences in interpreting these operators in natural language vs. mathematical logic (exclusive vs. inclusive 'or'; consecutive vs. symmetrical 'and'), we were interested in participants' choices as the basis for their familiarity with formal rules of inference.

For example, Item #3 presented the following statements: (1) All flowers are red or beautiful, and (2) All Tulips are flowers, followed by questions: (i) Are all tulips red? (ii) Are all tulips beautiful? (iii) Are all tulips red or beautiful? (iv) Are all tulips red and beautiful? Item #4 presented the following: Your dinner includes an alcoholic drink or a mock cocktail with your meal. Are you having only one drink with your meal? For each question the participants had to indicate (a) Yes, (b) No, or (c) Not enough information.

Items #5 and #6 introduced coercive logic scenarios. They presented a story followed up by two conditional statements with different consequents, and a sought-after-statement to achieve the desired outcome. The participants were asked to examine the outcome of the statement. In particular, Item #5 included the Cannibals scenario presented above, with the sought-after statement "You will cook me in oil". The participants were asked to indicate (a) Yes, (b) No, or (c) Not enough information, to the two questions – Are they cooking you on oil? Are they cooking you in water" – and explain their reasoning. Item #6 included a scenario isomorphic to the Cannibals, but in a different (non-violent) context.

Questionnaire 2 included four coercive logic scenarios. Unlike items #5 and #6 in Questionnaire 1, in which the sought-after-statement was presented for participants' evaluation, the first two scenarios presented multiple choice options, only one of which included the sought-after-statement that results in coercion. In the next two scenarios the participants were asked to formulate by themselves the sought-after-statement that would reach the desired outcome. As an example, Fig 1 presents the second item of Questionnaire 2.

Scenario 2: Hogwarts

You are walking through the halls of Hogwarts School of Witchcraft and Wizardry when you suddenly bump into Professor Severus Snape making him drop all of his books and potions. He then tells you that as a punishment, he will put a spell on you. He will give you the chance to make a statement to choose what spell he'll curse you with.

You will make a statement such that

- if what you say is true, then the spell will be Petrificus Totalus (A body-binding curse that causes a person to experience temporary full-body paralysis.), and
- if what you say is false then the spell will be Obliviate (a spell that will make you forget all the content of your Witchcraft courses).

Then you have to choose one of the following statements to tell Professor Snape:

- Statement 1: You will give me the Petrificus Totalus.
- Statement 2: You will give me the Obliviate.
- Statement 3: You will either put a spell of Petrificus Totalus or of Obliviate.

Please chose one of the statements and explain the outcome of your choice.

Figure 1: Example of coercive logic scenario from Questionnaire 2

In an individual **semi-structured interview** setting selected participants were asked to examine and elaborate further on their written responses. The interviews lasted about 50 min; they were audiotaped and later transcribed. Participants were provided with an opportunity to change their responses and to reflect on their experience of addressing the questionnaire.

Theoretical Framing: Mental Models

The construct of "mental model" was initially introduced by Craik (1943) as mental representations of situations, which can be real, hypothetical or imaginary. Such models may contain perceptual qualities and may be abstract in nature. Johnson-Laird (1983) extended this idea to create a *mental model theory* of reasoning and deduction (Johnson-Laird & Byrne, 2002) Johnson-Laird (2012) summarised the three principles of the theory developed in the previous studies: (1) The principle of *iconicity*: A mental model has a structure that corresponds to the known structure of what it represents, (2) The principle of *possibilities*: Each mental model represents a distinct possibility, by capturing common features in different ways of occurrence, and (3) The principle of *truth*: Mental models focus on what is possible or true in the premisses. That is, what is impossible or deemed false is not commonly represented.

Mental model theory led to debates in the reasoning and cognitive sciences, as it advocates strongly for an effective prototype of inference non-dependent of formal rules. This prototype was able to predict systematic fallacies that cannot be explained by theories based solely on valid rules of inference (such as mental logic). In particular, Johnson-Laird & Savary (1999) introduced the notion of *illusory inferences*, that is, inferences that are compelling but invalid.

Example: Suppose that the following claims apply to a specific hand of cards, where only one of the two is true.

(a) If there is a king in the hand then there is an ace in the hand, or else,

(b) If there is a queen in the hand then there is an ace in the hand.

There is a king in the hand. What, if anything, follows?

Note that the conclusion: "There is an ace in the hand", is an invalid inference that is predicted by the mental model theory, based on the principles of possibility and truth. Given an exclusive disjunction of the two conditionals, (a) or (b), but not both, must be the case, however the further assertion – that there is a king in the hand – is definite. However, the disjunction is compatible with the falsity of one of the two conditionals. Since which one is false is unknown, there is no guarantee that there is an ace, even though there is a king. In particular, if (a) is false, the possibility of "ace in the hand" cannot be confirmed.

These illusions arise from the principle of truth. That is, the mental model for individuals who erroneously concluded "There is an ace in the hand", did not attend to the possibility that (a) in the premises can be false. It is shown that inferences that attend to truth only alleviate the illusions (Goldvarg & Johnson-Laird, 2000, 2001; Tabossi, et al., 1999).

Furthermore, according to mental model theory, people have a tendency to reason their way to consistency, that is, if they detect an inconsistency, they try to determine which premise to abandon to create consistency within a situation (for model accounts of these processes see e.g., Johnson-Laird, et al., 2000).

Attending to the case of Sal and his logical inference in considering coercive logic scenarios, we focus our research question on the particular case study: *How can mental models account for Sal's logical inference when responding to scenarios involving coercive logic*?

Results and Analysis

As mentioned above, our analysis is a case study of Sal. We chose Sal as his arguments appeared detailed but inconsistent, and, as described below, he provided unexpected ideas related to the coercive logic scenarios.

Item #4 on Questionnaire 1 is a specific question to evaluate the distinction between inclusive and exclusive "or". Sal's answer (Yes) to this question (having only one drink) points to Sal's understanding of disjunction as an exclusive 'or' in natural language. However, Sal's response to Item #3-iv (are tulips red and beautiful) was "Not enough information", which is inconsistent with exclusive disjunction. Further, for Items #5 and #6 of Questionnaire 1, which introduce scenarios of coercive logic, Sal indicated "not enough information" for each of the two questions in both items. Sal was not able to resolve the scenarios, and this can be further appreciated in his comment "we can't test whether the statement is true or not?". It appears that Sal was seeking an a-priori indication of the truth value of the claim in order to be able to examine the implication, rather than proceeding with hypothetical assignment of truth. Without such knowledge he was not able to proceed to examining the conditionals.

In Questionnaire 2 Sal chose the option that included the 'or' clause when it was one of the available choices. That is, for Item #2 he chose the statement "You will either put a spell of Petrificus Totalus or of Obliviate." Furthermore, he formulated the disjunctive statement when on the two items that invited participants to formulate a claim.

In the interview, Sal explicitly acknowledged being very confused on items #5 and #6 of Questionnaire 1. However, when asked to consider his responses to the items of Questionnaire 2, he noted, with respect to the Hogwarts scenario (see Fig. 1):

> *Sal*: But by saying this entire thing, it becomes true though, like it cannot be false [points out statement 3], this cannot be false is what I would like to say, cause whatever I say I am either gonna get either PT [refering to Petrificus Totalus] or Obliviate, right? But then the entire statement is true. So, I thought I would be getting the PT spell.
>
> *Interviewer*: Ok, how was that you come to the idea that this one is true then the second one [pointing to the second conditional] is not happening, right?
>
> *Sal*: Right, yes, that's right the second one, so I won't be getting the Obliviate if I say statement 3, it's what I was thinking.

The selection of disjunctive statement, which is tautological but true, invalidates one of the conditionals in the presented scenario. That is, the second conditional becomes irrelevant, yet the exclusive disjunction between the conditionals holds valid.

A disjunction in which components hold opposite truth values produces a tautology. Nevertheless, a choice of tautology resolved the situation for Sal, as he considers the consequent associated with truth as a preferred choice. That is, PT is preferable as it is temporary.

> *Sal*: Oh yeah, so, this PT spell says temporary full body paralysis something that's temporary. My paralysis will be gone, that's what I was thinking, versus this obliviate in that one you just forget all of the contents of the stuff that you learn from the course, right? So, I've rather had a temporary one rather than just forgetting everything that I've learnt. So, that's why I choose it, I'll go with the truth. Choose the PT one and say the truth.

This could have been an oversight in our design as we considered both choices to be undesirable and avoidable via appropriate logical coercion. However, for Sal the "truth" was associated with a desirable outcome, which is the less hurtful of the provided unfavorable choices.

During the interview Sal emphasized the fact that he wanted to choose the statement that contained a truth antecedent, and thus simplify the scenario to a unique conditional to deal with.

> *Interviewer:* I can see that there's some logic over all in the general aspect of your answers, for example, you were congruent in the choices that you were making [...] in this scenario 4 [item #4 in questionnaire 2] that you were asked to give a statement, you actually selected something similar to the previous scenario. So, the question now is: how come? Why did you decide on a statement of a similar form to the previous ones?
>
> *Sal*: If I had not read these first two questions, because it actually gave me the [pause] by saying "either or" I am making a true statement and can choose whatever I want to […] So, I wanted to get something off of the statement, so let's make a truth statement, so I think the first two question gave me a way of [pause], in my mind getting me definitive truth statements. So, then I can use that "either or" logic in here so that I am choosing the truth. So, I am making a truth statement […].

We note in this excerpt that the principle of truth manifested itself to increase the possibilities of a mental model that addresses the given logic scenario without the need to consider a possibility of a false statement. We consider Sal's responses as an example of *illusory inferences,* guided by the principle of truth.

But it is not enough to deal with one conditional to find a way out of coercive scenarios. This is clearer when there is no preference in the consequents of the conditional, such as in Cannibals scenario. It is the independent evaluation of both conditionals that allows participants to formulate or choose an appropriate sought-out-statement. Sal was not able to reason on items in which he did not identify a desirable outcome. However, when a choice was possible, he corroborated a mental model principle of truth that did not result in coercion, but allowed for a desirable (that is, less hurtful) choice.

Considering Sal's written responses, and his elaborations during the interview, we conclude that the principle of truth guided the construction of the mental models that Sal used to resolve the coercive logic scenarios. This guidance was effective in the scenarios in which there was a preference for an outcome within the coercive logic scenarios, but failed when it was necessary to construct dual mental models to deal with two conditionals presented in the scenarios.

Discussion and Future Research

Our research aimed at studying participants' reasoning and logical inference when they respond to scenarios involving coercive logic. In particular, we focused on the case of Sal and identified the prevalence of the truth principle in his mental models. Our contribution can be seen in introducing coercive logic scenarios as a method for studying participants' reasoning that may not be revealed by other means.

A limitation of the study is in our initial design of tasks that requires a minor adjustment. Our design of scenarios progressed from (a) asking participants to evaluate a sought-after-statement in Questionnaire 1, to (b) choose an appropriate claim from the presented possibilities, and (c) to formulate a claim in Questionnaire 2. However, the presented tautology and the disjunctive claim in (b) has influenced Sal's reasoning and directed his subsequent choice to an option that is "always true" and in such avoided coercion. We will eliminate such an option as we continue our research.

Our future research will explore additional theoretical constructs that may explain Sal's (and other participants') reasoning, either intertwining with mental models theory or presenting an alternative. For example, how can the notion of reducing abstraction (Hazzan, 1999; Hazzan & Zazkis, 2005) explain Sal's approach, especially the prevalence of the truth principle?

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Student Understanding of Solids of Revolution: Refining a Previous Model

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In this study, we consider students' performance on tasks involving solids of revolution to refine a recently introduced, research-based model. An activity was designed with tasks to help students do the constructions proposed in the model. After implementing the activity in the classroom, we suggest additions to the model using data from the written responses of 22 university calculus students' quiz and exam tasks. The results provide information about the challenges that students face when dealing with solids of revolution and allow us to refine the model to add more detail that might be used to help students overcome observed challenges. We discuss some limitations of the study and share teaching implications.

Keywords: Solids of revolution, Calculus, APOS theory, Riemann sums

Introduction and Literature Review

There is considerable research on student understanding of one-variable function integration (e.g., Jones, 2015; Orton, 1983; Sealey, 2014; Thompson, 1994; Thompson & Silverman, 2008). In one of the first studies dedicated to integration, Orton (1983) found that volumes of revolution were one of the three most difficult integration topics for students; the others were Riemann sums and limits of Riemann sums. Jones (2013) observed three major meanings for symbolic forms of integrals that emerged from student data: adding up pieces, perimeter and area, and function matching that roughly correspond respectively to a definite integral's relation to Riemann sums, the area-under-a-curve conception, and the integral as an anti-derivative. Jones (2013) observed the importance of the "adding up pieces" interpretation for applications and, like Orton (1983), suggested it needed more emphasis. He later reported (Jones, 2015) that the area-under-a-curve and anti-derivative conceptions were highly prevalent, with the Riemann sum-based conception being underutilized. This underutilization was also observed by Akrouti (2020), even in tasks where using Riemann sums was the only reasonable approach. Speer and Kung (2016) observed that there is little research dealing with volumes of solids of revolution (SoR). Some studies (e.g., Paul et al., 2024) consider the use of manipulatives when teaching this topic. However, few studies unpack students' difficulties with the topic. In one such article, Mofolo-Mbokane et al. (2013) found that students had struggled with graphing skills, going from continuous to discrete (drawing approximating rectangles and visualizing rotation to decide whether to use dx or dy), translating from 2D to 3D diagrams (producing drawings), manipulation skills (computing the definite integral), and general cognitive skills (putting everything together). In another study dedicated to SoR, Andrade and Montecinos (2013) argued that students need help to interpret $f(x)$ as something more than just "the function to be integrated," coinciding with Sealey's (2014) need for an orienting pre-layer. They suggested, as did Blanco et al. (2019), that students need support to generate an image of a 3D solid starting from a 2D drawing. Given students' difficulties in visualizing SoR, Juárez and colleagues (2020) designed a questionnaire to evaluate students' skills in visualizing such solids. Later, Juárez and different colleagues (2022) designed a GeoGebra intervention to improve students' visuospatial abilities when dealing with SoR. Bernard and Jones (2016) studied the steps students chose when computing several volumes of SoR, the order of the steps, and the reasoning for this order. Their findings suggest that visualizing the rotated solid may be a factor that influences students' ability

to solve SoR problems that should be explicitly addressed. They also found that some students might not relate the differentials in the integral expression with the thickness of disks or washers and that students frequently rely on memorized procedures in setting up integral expressions. These observations underscore the need to help students visualize SoR and relate Riemann sums to definite integrals, as is done in the present study.

The research goal for this study was to use data collected from students' performance on SoR tasks to refine and suggest revisions for the theoretical model proposed by Borji and Martínez-Planell (2023) to explain how students come to understand volumes of revolution and what obstacles they might face in this process. We designed and implemented an activity to help students make the constructions proposed in the model. The results are discussed, and a revision of the model is proposed.

Theoretical Framework

We use Action-Process-Object-Schema (APOS) theory (Arnon et al., 2014), focusing only on the A, P, and O conceptions. An *action* is a transformation of a previously constructed mathematical notion that a student perceives as external, in the sense that, due to the lack of connections to other mathematical knowledge, the transformation is carried out only by memorization or following explicit instructions. When an action is repeated and reflected upon, it might be interiorized into a process. The *process* is perceived as internal since connections to other knowledge have been established, allowing students to anticipate results, generate dynamical imagery, make justifications, and perceive it as independent of representation. Different processes can be coordinated to form new processes. When a student can think of a process as an entity in itself and perform actions on it then one says that the process has been encapsulated into an *object*.

An important idea in APOS is that of a genetic decomposition (GD). This is a model of how a student may construct a specific mathematical notion. A GD is expressed in terms of the theory's structures (actions, processes, objects) and mechanisms (interiorization, encapsulation, coordination). A GD of a given notion starts as an initial hypothesis that must be tested with student data. Experimentation can lead to its verification, refinement, or rebuttal. An initial GD for SoR proposed by Borji and Martínez-Planell (2023) considered only the disk and washer methods. We will briefly summarize it here, and then, at the end of this paper, we propose a revised version that also applies to shells. The GD suggested by Borji and Martínez-Planell, which we take as our initial GD, includes prerequisites (i.e., understanding of function, area of a circular region, geometry and volume of a right circular cylinder) and has three parts:

- GD1: Geometric SoR constructed by rotating points, curves, and regions around a given axis; this allows imagining and visualizing SoR
- GD2: Geometric approximation constructed by forming small partitions, rotating the rectangles in the partition, drawing the resulting disks or washers, and comparing with the SoR being approximated; it allows imagining the approximation of the volume with an increasingly large, but finite, number of disks or washers
- GD3: Geometric-analytic volume of the SoR constructed by small partitions expressing the corresponding volume numerically, then symbolically as an extended sum, using sigma notation, then using an infinite number of disk or washers to determine the volume by taking limits and finally expressing the limit as an integral; it allows relating different representations for volume including the limit of Riemann sums and integrals

Methods

The three authors collaboratively designed an activity (https://bit.ly/4a9UNOX) with tasks 1-4 involving rotation only (GD1); tasks 5-8 involving rotation and rotation corresponding to a single subinterval of a partition for volume approximation (GD1, GD2); and tasks 9-11 involving rotation of a curve, volume approximation with limits, and volume with integration (GD1, GD2, GD3). The activity was administered in a multivariable classroom in a Southwestern US university for bonus points with 22 students who were allowed to consult with others on the assignment and to take it home to complete, if needed. The first two authors individually combed through the activity data and scored each task part using 0 (completely incorrect), 1 (somewhat correct), 2 (mostly correct), and DNA (did not attempt). The scoring agreement was 96%; the research team discussed scoring until a consensus was reached.

The data from the activity tasks were used as a first pass to study the students' understanding of SoR. Next, the first two authors jointly revisited the description of the GD of SoR found in the Borji and Martínez-Planell (2023) paper and then individually reviewed the quiz and exam task data making notes of their observations.

We also considered quiz and exam tasks (https://bit.ly/4a9l0wZ) that dealt with SoR that required the use of washers and disks. The coding scheme for the quiz and exam tasks with descriptions of observed student struggles appears in Table 1. The codes were developed jointly by the first and second authors as they combed through the data. The correctness scoring for all tasks in the activity and GD coding for the quiz and exam student data were used to complete qualitative profiles in terms of GD1, GD2, and GD3 for each student's understanding of SoR. The third author observed the students in class, reviewed all data individually, and provided input on the reported results.

Student Performance

Comparison of Performance on Activity Tasks with Quiz and Exam Tasks

First, we compare student performance scores on the activity tasks divided into GD1 (imagining SoR), GD2 (imagining rotated rectangles approximating volumes of SoR), and GD3 (converting from geometric imagery to integral expressions). Students tended to perform better with higher average scores on tasks that required GD1 only (mean: 92%, median: 100%) than on tasks requiring GD1 and GD2 (mean: 81%, median: 92%). Similarly, they performed better on tasks requiring GD1 and GD2 than those requiring GD1, GD2, and GD3 (mean: 45%, median: 41%). Activity tasks 10 and 11, which required GD1, GD2, and GD3, had 8 parts each. Of the 352 possible responses for all 22 students on the 16 parts of these two tasks, 46% were left blank (coded DNA) and another 19% were scored 0.

GD Coding

Since the activity allowed for collaboration on its tasks, the quiz and particularly the exam task data (since the exam was administered at the end of the semester) were primarily used to create a profile for each student in terms of the GD of SoR. Each researcher independently recorded this in individual notes for the student data. Upon inspection of the student responses to the tasks, the research team noted that students often struggled with prerequisite skills that had not been identified earlier. The data prompted discussion to conclude that there were additions needed to the original GD of SoR. Table 1 below summarizes the most prevalent student struggles observed on quiz and exam tasks by GD 1, GD2 and GD3.

Table 1. Codes Used for Prevalence of Student Struggles on Quiz and Exam Tasks by GD1, GD2, GD3

	Codes for Student Struggle (prerequisites denoted)	Description of Student Struggle	Frequency of Struggle
GD1	*Rotation*	Difficulty rotating or missing rotation of curve or region	20
	Axis (Prereq)	Rotation around incorrect line or axis	2
	Region (Prereq)	Difficulty determining region to be rotated when given equations that defined region	2
	Graph (Prereq)	Difficulty graphing a curve given as an equation	1
GD2	*Partitions*	Difficulty creating or failure to create rectangles to partition region	20
	Orientation	Incorrect orientation for partitions	3
GD3	*Cross-sect Area (Prereq)*	Incorrect cross-sectional area of disk or washer	24
	Radius (Prereq)	Incorrect radius for disk or washer	21
	Differential	Difficulty with differential or lack of correspondence between differential and partition	17
	Limits of Int	Difficulty with limits of integration	9

The different coded observations suggested needed refinements for the GD. For GD1, the *rotation* code is not useful since a student might not produce an image on a task and yet be able to imagine the rotated solid. While this would be less of an issue in an interview, it was a limitation of having data collected via paper-and-pencil instruments. So, not drawing a rotation might mean that the student could visualize the drawing mentally; however, it might also indicate a reticence of the student to use a geometric argument. The *axis* code denoted observations suggesting lack of a prerequisite construction to identify the axis of rotation. This prerequisite was not mentioned in the initial GD by Borji and Martínez-Planell (2023). The codes *graph* and *region* seem to result from a lack of prerequisite knowledge. Students did not correctly draw the graph of one-variable functions or determine regions bounded by one-variable functions and lines. As with the *axis* code, this was not mentioned as a prerequisite in Borji and Martínez-Planell's GD. For GD2, the *partitions* code may tell us that a student did not draw correct rectangles to partition the region but might be able to imagine the approximating rotated rectangles. It may also signal students' reticence to use geometric methods. There were also observations with *orientation* codes that all occurred when the students tried to use shells rather than washers for the problem. The initial GD did not consider shells. The revised GD describes the constructions needed for the shell method, although we are not considering such data in this paper. The many observations with the *radius* and *cross-sect area* codes moved us to add prerequisite constructions for radius and area as well as to also change GD2 to include new constructions that will have students rotating "thin" rectangles of infinitesimal width *dx* or *dy* around horizontal or vertical axes, as well as computing volumes of the disks, washers, or shells that are formed. These changes in GD2 can also help students avoid issues described in the *differential* code, and together with changes in GD3, they can also avoid those in the *limits of int* code. Now, we consider an example to show how the coding was assigned.

Figure 1 shows Student 14's work to determine the volume of the SoR obtained by rotating the region bounded by $y = -x^2 + 9$, $y = 4$, $x = -1$, and $x = 2$ about the x axis. We coded his work GD1, since he drew the boundary curves and lines as well as a rotation symbol to the right of the x axis suggesting he imagined the SoR. His work was coded GD2 since his sketched rectangle suggests he imagined rotated rectangles approximating the volume of the SoR. His work was coded no GD3 since there were errors converting from geometric imagery to an integral

expression. He used $(-x^2 + 5)$ rather than 4 for the inner radius; so, it was coded *radius*. Consequently, the cross-sectional area is also wrong and thus coded *cross-sect area*. It was also coded *limits of int* since the limits of integration were 4 to 9 rather than -1 to 2.

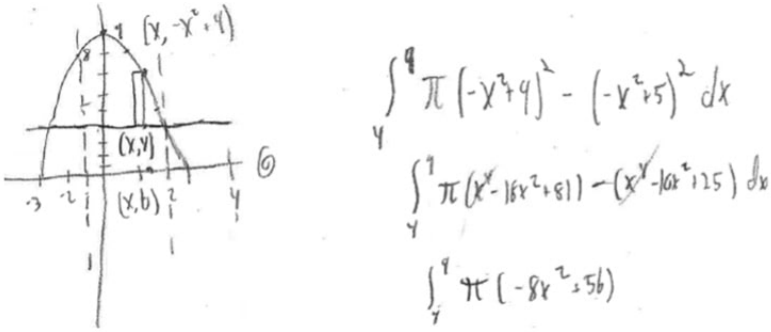


Figure 1. Student 14's response to one of the quiz tasks.

See Figure 2 for Student 15's work on the same problem. His work was coded as no GD1, and given the *region* code, for choosing the wrong region to rotate. It was coded as no GD2 and given the *orientation* code for drawing the approximating rectangle with the wrong orientation. It was coded no GD3 and assigned both the *radius* and *cross-sect area* codes since the written integral misses the radii and cross-sectional area of a washer and instead refers to a disk that is unrelated to the drawing.



Figure 2. Student 15's response to one of the quiz tasks.

Implications for Teaching Practice

Codes no GD1 and no GD2 might point to some student reticence to use geometric methods to find volumes of revolution. This may be a result of inconsistent participation in the GD-based homework tasks. Many students did not attempt the totality of the assigned tasks and may not have taken the tasks that required drawings seriously, instead producing very small, incomplete, seemingly careless sketches. This suggests that the students might have benefitted from the incorporation of the ACE (**A**ctivity in collaborative groups-**C**lassroom discussion-**E**xercises at home) pedagogical strategy that is an integral part of APOS theory. This pedagogical strategy foments student reflection, which is needed to interiorize actions into processes and encapsulate processes into objects. Awarding more credit than just a few bonus points to students for their work might have a positive effect on their motivation to complete GD-based tasks. So, the way in which the GD-based tasks are implemented is likely to play an important role.

The results also allow us to revise the genetic decomposition to include more detail. As observed in the results, we need to revisit the prerequisite knowledge of students to construct processes to identify axes, graph one-variable functions (including vertical and horizontal lines), and identify regions in the plane that are described in terms of graphs of functions and lines. The description of GD1 needs to include constructing a process to communicate (i.e., produce words or drawings) to describe a rotated planar region. Regarding GD2, a prerequisite process to

recognize cylinders, washers, and shells (obtained from rotating "thin" rectangles) is needed, as is distinguishing between the area and volume of common geometrical shapes. GD2 must be revised to make explicit the construction of processes to visualize approximating the area of a planar region with rectangles, all of which go from a given curve to another and account for the entire region, visualize the rotated rectangles as disks, washers, or shells; and recognize that the rotated rectangles are an approximation of the corresponding rotated region. Regarding GD3, even though most students did not attempt many of the corresponding tasks, students' work on the quizzes and exams suggests that the GD needs to be explicit about constructing prerequisite processes for the radius, circumference, area of a circle, as well as the volume of disks, washers, and shells. The description of GD3 also needs to be explicit about constructing processes to determine the radius of disks, washers, and shells; determining the height of shells; thinking of *dx* or *dy* as an infinitesimal thickness related to a partition of an interval, computing cross-sectional areas, relating the partitioned interval to the limits of integration, and in general, bridging the geometric and analytical representations from a finite to a generic Riemann sum and from this to imagining and expressing the limit of Riemann sums as an integral.

Revised Genetic Decomposition

The study of students' work suggests refinements are needed for Borji and Martínez-Planell's (2023) GD. The resulting revised genetic decomposition (see Table 2) addresses disks and washers as well as extending to also include shells.

Table 2. Revised Generic Decomposition for Solids of Revolution

GD0: Prerequisite ideas	• Understanding[1] of one-variable functions including algebraic and graphical representations • Understanding of horizontal and vertical lines including axes • Ability to identify planar regions defined by functions and lines whether given graphically or algebraically • Understanding of the difference between length, area and volume • Understanding of circles, circular regions, and cylinders, including related features such as radius, circumference, and surface area (including lateral area) • Understanding of volumes of disks, subtraction of volumes of figures when one is "inside" another (such as is the case with washers), volumes of shells • Understanding of sigma notation as a representation of summation • Introductory understanding of a limit of a length being zero
GD1: Geometric solid of revolution	• The action of rotating a single point around an axis is interiorized into a process of point rotation that allows imagining the resulting circle without having to do the rotation explicitly. The need to rotate curves and regions (viewed as collections of points) around an axis foments the encapsulation of the point rotation process into a point rotation object. • The action of rotating a curve around an axis and drawing the resulting surface can be interiorized into a process of curve rotation. The need to draw or imagine surfaces of revolution foments the encapsulation of the process of curve rotation into an object of curve rotation. • The action of rotating regions bounded by one or more curves of the form y=f(x) or x=f(y), one complete cycle around an axis to form a solid, drawing the resulting solid, and describing it verbally, is interiorized into a process that allows the student to imagine the formation of such solids of revolution and describe it verbally and with a drawing. The need to do actions on solids of revolution, such as computing their volume, foments their encapsulation into objects.

[1] By "understanding" a notion, we mean having constructed at least a process of the notion.

GD2: Geometric approximation by disks, washers, or shells	• The action of rotating a "thin" vertical (horizontal) rectangle of infinitesimal width *dx* (or *dy*) and with length that is either a number or a symbolic expression, rotating it around a given axis (any vertical or horizontal line not intersecting the interior of the rectangle), drawing the resulting "thin" solid and describing it verbally, can be interiorized into a process that allows imagining the formation of infinitesimally thin disks, washers, and shells. • This process can be coordinated with prerequisite processes in order to compute the resulting volumes (numerically and symbolically). • Coordination of these processes results in a process of disk, washer, shell formation that interrelates geometric, algebraic, numerical, and verbal representations. • Given a planar region defined by one or more curves of the form y=f(x) or x=f(y) and given (or having the student produce) the corresponding solid of revolution, the action of determining an interval and forming a partition of the interval with a small number (an explicit number, not a generic symbol) of subintervals and drawing the corresponding rectangles, disks, washers, or shells to approximate the volume of the solid of revolution is interiorized into a process that allows a student to imagine approximating a region with a small number of rectangles, imagine the rotated rectangles as disks, washers, or shells, and compare the volume of the rotated rectangles to that of the solid of revolution. • This process can be coordinated with the process of disk, washer, and shell formation to form a new process that allows the student to imagine partitions of an interval and corresponding planar region into many thin rectangles and approximate the volume of a solid of revolution with an increasingly large corresponding number of solid rings, washers, or shells without having to do so explicitly.
GD3: Geometric-analytic volumes of revolution	• The chain of actions of first writing the numerical approximation given by the Riemann sum corresponding to the solid obtained with a small partition in GD2 (for example, taking $\Delta x = 1$, for the solid obtained by rotating $y = x^2$ from $x = 0$ to $x = 3$ it could be something like $\pi(1^2)(1) + \pi(4^2)(1) + \pi(9^2)(1)$), then introducing variables and labeling a drawing of the previously obtained solid and writing the Riemann sum in extended form (for example, something like $\pi({x_1}^4)\Delta x + \pi({x_2}^4)\Delta x + \pi({x_3}^4)\Delta x$), then writing the Riemann sum using sigma notation (for example, something like $\sum_{i=1}^{3} \pi x_i^4 \, \Delta x$), before generalizing and finally taking limits ($\lim_{n\to\infty} \sum_{i=1}^{n} \pi x_i^4 \, \Delta x$) and representing and computing the limit as an integral ($\int_0^3 \pi x^4 \, dx$) may be interiorized into a process. • This process allows imagining the sum of volumes of a finite number of disks, washers, or shells as a numeric approximation to the volume of revolution (bridging the numeric with the geometric); allows imagining a partition of the region into rectangles of width Δx and use variables x_i rather than numbers (bridging the numeric and algebraic) to obtain and sum the corresponding volumes. • When this process is coordinated with the process resulting from GD2, the resulting process allows to imagine that a general partition into an increasing number *n* of thinner and thinner rectangles results in a better approximation (bridging the algebraic with the geometric), allows to express general Riemann sums in sigma notation and recognize the limit as the value of the volume of revolution, and allows to express the limit of Riemann sums as an integral. • The need to take the limit of such finite sums foments the encapsulation of the process into an object that identifies a limit of Riemann sums for the volume of a solid of revolution with the corresponding volume integral.

The GD-based tasks used should be revised to reflect the new genetic decomposition. They need to be class-tested using the ACE pedagogical strategy, and a new research cycle needs to be undertaken. Interviews are needed to help capture students' thinking about SoR, their meanings for the different notions involved, and the reasons for their problem-solving actions.

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The Complexity Coordinating the "Product Layer" and Definite Integrals in Introductory Physics

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Research on students' construction and interpretation of integrals has grown significantly over the past decade. The present paper contributes to recent research on challenges which students experience while coordinating the "product layer", $f(x) \cdot dx$, and the definite integral in applied contexts. In particular, we demonstrate a case study which examines how an undergraduate student, Matthew, linked the "product layer" and definite integral in an introductory physics task by blending mathematics, physics, and visualization knowledge resources. We analyzed Matthew's moment-to-moment reasoning by utilizing a modified dynamic conceptual blending diagram (MDCBD), coming from the dynamic conceptual blending perspective, and an expanded Visualization/Analysis/Physics (VAP)-model, as detailed in (Sus & Izsák, 2024). Our findings indicate that students' understanding of multiplication with quantities in applied integration contexts is highly complex and that neglecting the dynamic interactions of knowledge resources represents a significant gap in integration research.

Keywords: Integration, Product Layer, Knowledge-in-Pieces, Expanded VAP-Model, Dynamic Conceptual Blending

Introduction and Literature Review

Students often struggle to understand and apply the concept of definite integrals in applied contexts, particularly when reasoning about the "product layer," which involves interpreting the integral as a multiplicatively-based summation of quantities (e.g., Ely, 2017; Jones, 2013; Sealey, 2014). These difficulties become apparent when students attempt to model quantities such as distance and work using rectangular areas or other visual representations (e.g., Thompson et al., 2013; Nguyen & Rebello, 2011). While past research has explored students' general understanding of integrals (e.g., Meredith & Marrongelle, 2008; Cui et al., 2006; Yeatts & Hundhausen, 1992), less attention has been given to how they reason about this specific "product layer" within integrals.

Sealey (2014) identified the "product layer" as a complex area for students, and Oehrtman and Simmons (2023) analyzed how students construct definite integrals in applied problems, identifying three emergent quantitative models: basic, local, and global. However, their work did not examine closely how students' understanding of multiplication (in particular, knowledge resources) influences their construction of local models, or how their use of drawings affects their reasoning.

Further research by Izsák and Sus (2024) shows that students rely on various mathematical and physics knowledge resources when coordinating multiplication with definite integrals, and that these resources fluctuate based on the problem context. Visual representations, as noted by Berry and Nyman (2003), play a crucial role in guiding students' reasoning about definite integrals and are essential for reasoning about the "product layer." Research consistently suggests that integrating visual and analytical reasoning is key to fostering conceptual understanding in mathematics (Aspinwall & Shaw, 2002; Zazkis et al., 1996).

Our study aims to address these gaps by combining the approaches of Izsák and Sus (2024) with a focus on visual reasoning. We examine how students apply knowledge resources, including drawings, when coordinating multiplication with definite integrals in applied contexts.

To explore this, we use a novel methodological approach that integrates conceptual blending theory and visualization theory (Sus & Izsák, 2024). Our research questions are: *What knowledge resources do students use in their moment-to-moment reasoning about the "product layer" construction in introductory physics? How do students apply drawings in their moment-to-moment reasoning about the "product layer" construction in introductory physics?*

Theoretical Perspectives and Methodological Approach

In this section, we present theoretical perspectives and a new methodological approach which best spans the results obtained in the study. It includes three components — the knowledge-in-pieces cognitive perspective, the modified dynamic conceptual blending diagram, coming from the dynamic conceptual blending perspective (Van den Eynde et al., 2020), and the expanded VAP-model diagram (Zazkis, 2013) related to the visual conceptual blending perspective (Cunha et al., 2020).

The Knowledge-in-Pieces Epistemological Perspective

The perspective was first developed in science education research on conceptual change (diSessa, 1993, 2006) and has since been applied to topics in mathematics, such as whole-number multiplication (Izsák, 2005), functions (Izsák, 2004; Moschkovich, 1998), and integrals (Jones, 2013). It posits that reasoning is supported by diverse, fine-grained knowledge resources, which evolve from novice to expert through the construction, refinement, and reorganization of these resources. In this study, we focused on the knowledge resources students use when coordinating multiplication with a definite integral in introductory physics.

Dynamic Conceptual Blending, Modified Dynamic Conceptual Blending Diagram

Fauconnier and Turner (1998) introduced conceptual blending to model how people create new meaning by combining information from different experiential domains. Studies (e.g., Bing & Redish, 2007; Hu & Rebello, 2013; Wittmann, 2010) have applied this framework to describe students' use of mathematics in physics. In this study, we take the conceptual blending perspective (Van den Eynde et al., 2020) to analyze students' dynamic reasoning about the "product layer" in applied integration. We construct the modified dynamic conceptual blending diagram (MDCBD) through the following steps: (1) identifying and coding elements (knowledge resources); (2) listing selected elements chronologically; (3) assigning them to mental spaces — mathematics (mathematical concepts, including drawings and equations), physics (physical concepts and quantities, including drawings), symbolic (technical roles of mathematics), and blended (combinations of symbolic, mathematical, and physical ideas, including equations and drawings) and denoting them by M1, S1, P1, B1, etc.; (4) connecting selected elements with coded links; and (5) coding activated/deactivated elements and revisiting step (2).

Elements and connections are coded as more confident (e.g., "I remember") or more tentative (e.g., "I guess"), with more confident elements/connections shown as solid dots/lines and more tentative as empty dots/dotted lines. Activated/deactivated elements are marked with squared blue/red boxes, while connections are marked with blue/red solid or dashed lines, respectively. Figure 6a illustrates the MDCBD and its coding scheme.

By analyzing the elements and connections of these mental spaces, and their dynamic changes, we captured students' moment-to-moment reasoning about the "product layer" as they blended ideas from symbolic, mathematical, and physical input spaces.

The Expanded VAP-Model

Our approach builds on the VAP-model (Zazkis, 2013), which examines students' transitions between visual (V), analytical (A), and physical (P) reasoning modes in relation to graphs, functions, and physical concepts. We extended this model by incorporating five reasoning spaces: symbolic (S), mathematical (M), physical (P), blended (B), and visualization (V).

In our expanded model, the spaces S, M, and P are represented by the vertical edges of a tetrahedron, with the blended space occupying the interior. The altitude of each element within the tetrahedron reflects its activation time, moving from bottom to top. The coding of elements and their connections in spaces S, M, P, and B follows the same scheme as the modified dynamic conceptual blending diagram (MDCBD) described earlier.

Under visualization space V we consider a visual blended space complemented by a conceptual layer developed through elaboration (e.g., Cunha et al., 2020). It is represented by elements located at the base of the tetrahedron, arranged chronologically (V1, V2, etc.). These drawing elements are connected to the blended elements in space B, with coding consistent across all spaces. Figure 6b provides a schematic representation of the expanded VAP-model diagram.

Data

We conducted three one-hour, one-on-one, semi-structured think-aloud interviews (e.g., Bernard, 1994; Ginsburg, 1997) with three students enrolled in a second-semester calculus-based physics course at a selective university. The goal was to assess how these students reasoned about the integral concept in various problem situations, focusing on whether and how they constructed a "product layer."

The first interview explored students' reasoning about multiplication in four applied problem situations, none of which mentioned integrals but required multiplication followed by addition to construct a total. The second and third interviews focused on connecting definite integrals to three and four applied problem situations, respectively. We recorded the interviews with two cameras — one capturing the student and interviewer, and the other capturing the student's written work. After synchronizing the videos and transcribing the interviews verbatim, we reviewed them side-by-side with the transcripts to ensure accuracy. Throughout, we closely examined the knowledge resources students used in building the "product layer."

All students struggled at times to coordinate multiplication with the "product layer" in different problem situations. Matthew, who self-reported strong calculus skills (Calculus BC, 5), stood out. In the second interview (Water Pressure task), he demonstrated two types of reasoning about the "product layer," initially arriving at a normatively correct structure before shifting to an incorrect one. His drawings, which illustrated these shifts, helped us to visualize the connections between his analytical work and the physics context. We summarized his reasoning by focusing on his spoken language, hand gestures, and drawings.

Results

Analyzing Matthew's Moment-to-Moment Reasoning in the Water Pressure Task

In this section, we discuss Matthew's work on the Water Pressure task (adapted from Sealey, 2014) from Interview 2 (see Fig. 1). Notably, this task was novel to Matthew, minimizing the likelihood that he relied on recalled formulas or procedures from his physics or calculus courses. We first present the data Matthew produced and then analyze it using the methodological approach described earlier.

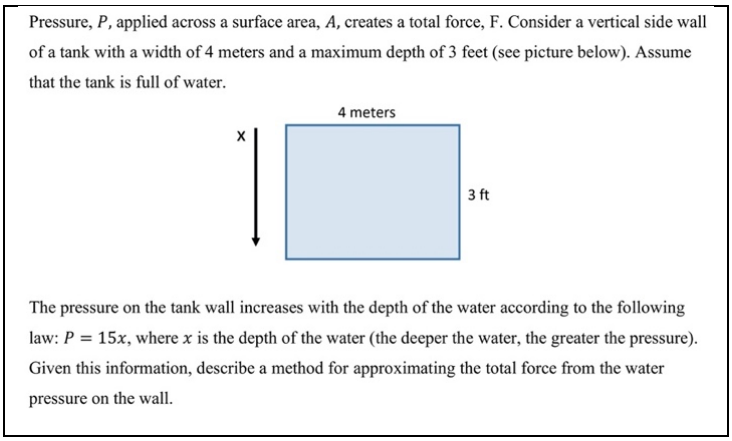


Figure 1. A Water Pressure task statement, Interview 2.

Let us remark that a normatively correct integral construction in this task would have the following representation $\int_0^3 15x \cdot 4\, dx$.

Matthew constructed an initial structure of the "product layer". Matthew began by writing down the given data which led him to think that "Each P is like the pressure on like a little bit of the area." He claimed that he did not know the actual relation between the quantities in the given problem situation: "I don't know the actual relation, so I'm just guessing." Then he stated: "I would say force is the integral of the pressure on a little bit of area (Fig. 2a). And umm the area is, well the area is constant essentially makes sense (Fig. 2b)."

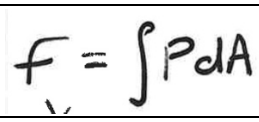
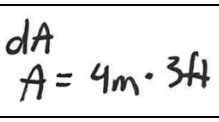
	
(a)	*(b)*

Figure 2. Matthew's initial thoughts to approach the task.

Matthew stated that he did not see why the equation $P = 15x$ is relevant because "you would just plug, you would just maximize the pressure by putting in the maximum depth of the tank in for x." Hence, he obtained: "So, I guess P would just be 15 times 3, which would be 45."

Analysis. Initially, Matthew applied a set of knowledge resources to reason through the problem. He first considered pressure as a set of values acting on small areas, suggesting he saw pressure in relation to these areas, though it wasn't clear how he combined the pressure values. He then expressed uncertainty about the relationship between force, pressure, and area, and, based on a guess, produced an integral expression for force (Fig. 2a). This expression was tied to his initial reasoning about pressure acting on a small area, potentially aligning with the PdA notation. Finally, by examining the given data, Matthew concluded that (a) the area was constant (Fig. 2b), and (b) the depth variable x was irrelevant to the pressure formula, as knowing the maximum depth allowed him to maximize the pressure. This progression of knowledge resources showed how Matthew initially generated normatively correct reasoning (Fig. 2a) but later introduced additional reasoning (a) and (b) that competed with the correct interpretation.

Matthew shifted his reasoning about the "product layer" structure. Matthew continued hesitating about the relationship between the pressure, area, and force quantities. He claimed: "I do not know if it's going to be area dP, like a little bit of pressure over the entire area, or it's this total pressure over the entire space of the area." Assuming that the area quantity is constant, and

the pressure quantity changes, Matthew produced another integral expression to estimate the total force (see line 2 from the top in Fig. 3):

Matthew: I think I would actually not go with this [integral expression in Fig. 2a] and do, maybe, dP times A. And then you get, because then you would pull out the area, which would be 4 meters times 3 feet and dP would be 15 dx, I guess, from 0 to 3. And I guess that would be my estimation of the total force.

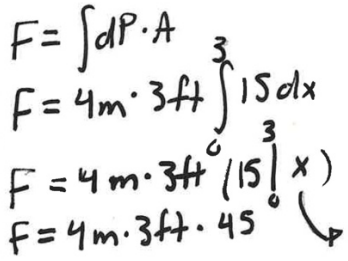


Figure 3. Matthew shifts his reasoning about the "product layer" structure.

Additionally, Matthew calculated the integral value in the force expression, arrived at $F = 4m \cdot 3ft \cdot 45$, and stated: "I am not super confident in this answer because I'd say the main problem is that I'm not understanding the relationship between force and pressure and area."

The interviewer asked Matthew if he had any other thoughts or ideas to approach the given problem. Matthew answered: "I'd say that's my final one. The reason I said F equals dP times A instead of $P\ dA$, which is area, is because the area isn't changing, like the size of the wall isn't changing, but the pressure is with the depth of the water."

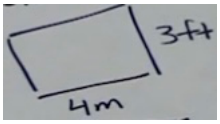
The interviewer next asked Matthew about the units attached to the 45 value in his force expression. Initially, Matthew stated he did not know the units for pressure but then provided $P = \frac{kg}{s^2 ft}$, which he justified by coordinating units for area and force.

Matthew explained his shift in reasoning using drawings. The interviewer asked Matthew whether he could explain his shift in reasoning using a picture. Matthew responded:

Matthew: So, if this is like your dA (Fig. 4a), you would have the pressure value acting on that little piece of area. And then when you sum up all the little dAs, you get this (Fig. 4b). But the reason I chose not to do that is because you already know the area in terms of constants. So, I'm assuming that what it really is, is that you have like your whole area, and you just have these little bits of pressure, these dPs acting across it. And when you sum up all these dPs, you would get the pressure (Fig. 4c).



(a)



(b)



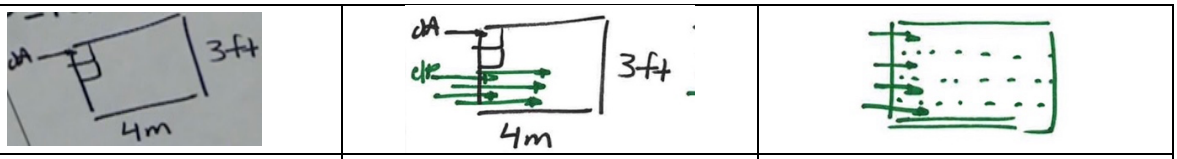
(c)

Figure 4. Matthew's drawings for his reasoning about the definite integral.

Following his reasoning, Matthew also produced a picture for little dAs by drawing little squares on the wall (Fig. 5a) and claimed: "I wouldn't use dA here because you already know the dimensions, it's not like something unknown."

Correspondingly, the interviewer asked Matthew to produce a picture for a term dP on the wall. Matthew drew green arrows pointing to different locations on the wall and stated that each

arrow represented a bit of pressure (Fig. 5b). In few seconds, he stated that it was not the best way to represent little bits of pressure. Then he produced a picture (Fig. 5c) and added: "It [pressure] is a function of depth. I would say that you would kind of have like this different sort of like sections, I guess you could say, of the area. And like each of these sections has a dP. So then when you sum up the pressure as like a function of the depth, you would get the total."



(a) (b) (c)

Figure 5. Matthew's graphical representations of dA and dP terms.

Following Matthew's responses, the interviewer highlighted differences in his drawings of the dP pieces. Matthew responded:

> *Matthew:* So, originally, I just think about [pressure] in terms of the fact that it's defined by a function of the depth. But then when I thought about that, it makes more sense to draw it like this [strips, Fig. 5c] because each bit of pressure is dependent on some sort of depth amount, like value. So, like dP is equal to $15dx$, so each little bit of depth contributes to that little bit of pressure.

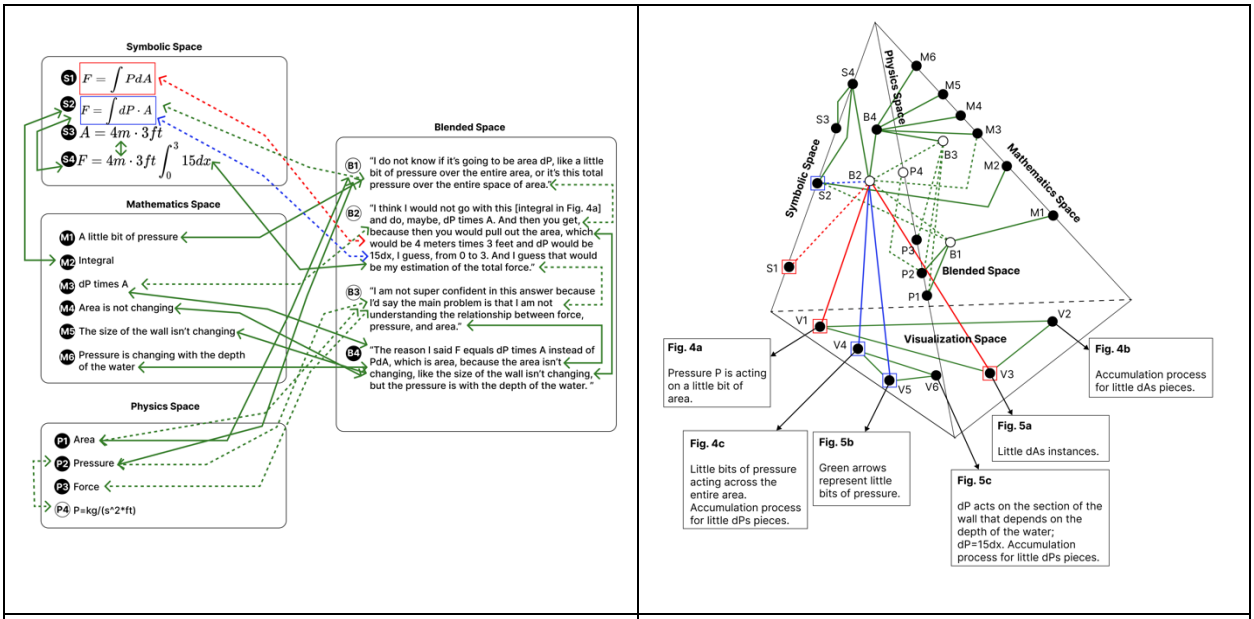
After this discussion Matthew confirmed that his final graphical representation for a dA term was given by picture (Fig. 5a) and for a dP term by picture (Fig. 5c). Also, he stated that the integral expression $F = \int dP \cdot A$ made more sense to him.

Analysis. We constructed a MDCBD (Fig. 6a) to better capture a shift in Matthew's moment-to-moment reasoning about the "product layer". The diagram organizes selected knowledge resources and their connections related to the shift in the order they appeared in the transcript. This diagram is a subset of the expanded VAP-model diagram (Fig. 6b). The expanded VAP-model diagram includes drawings that illustrate Matthew's shift and will be discussed further.

The MDCBD diagram illustrates Matthew's initial uncertainty (**B1**) about his integral model in Fig. 3a. This led him to reconsider how to calculate force (**B2**), prompted by doubts about the relationship between force, area, and pressure (**B3**). By integrating knowledge resources — specifically, the constancy of the wall's area and the varying pressure — Matthew revised his integral expression to $\int dP \cdot A$ for calculating force (**S2**). His reasoning shift centered on the observation that area was "constant" while pressure changed with water depth (**B4**). Matthew also drew on knowledge of antiderivatives and definite integrals. Notably, his reasoning evolved through three models of calculating force: (a) viewing pressure as values applied to small areas, (b) considering pressure as a single value, and (c) applying small amounts of pressure to the entire area of $4m \cdot 3ft$. Finally, Matthew employed a units' coordination approach to determine the correct units for pressure, addressing an initial uncertainty about them.

The expanded VAP-model diagram illustrates a shift in Matthew's reasoning about the "product layer" structure and its connection to drawings. It clearly identifies the moment of transition (**S1→S2**) and the blended element (**B2**) that prompted the shift. During this process, Matthew produced drawings (**V4**, **V5**) that clarified to us his understanding of the expression $dP \cdot A$ and provided evidence for rejecting his initial construction, $P \cdot dA$ ("the reason I chose not to do that is because you already know the area in terms of constants") and the corresponded drawing (**V1**, **V3**). The model also reveals Matthew's transition from visualizing pressure

distributed across the entire wall (**V5**) to focusing on the pressure acting on a horizontal section (**V6**). These drawings (**V4**-**V6**) ultimately led Matthew to conclude that $dP \cdot A$ was the more appropriate structure.



(a) *(b)*

Figure 6. The MDCBD and the expanded VAP-model diagram of Matthew's reasoning about the "product layer."

The MDCBD and the expanded VAP-model diagram in Figure 6 illustrate the complexity of Matthew's reasoning and visually track the evolution of his thinking about the "product layer" structure, linking elements from five mental spaces.

Discussion and Conclusion

This study builds on the work of Oehrtman and Simmons (2023) by showing that students' reasoning about the "product layer" in definite integrals is highly complex. Our findings reveal that students integrate mathematical and physics knowledge, supported by visual representations, when reasoning about this concept.

We observed that a student, Matthew, generated two different local models from the same basic model. Initially, he constructed a normatively correct local model $P \cdot dA$, which he then extended to the global model $\int P \cdot dA$. However, he later abandoned this and developed a new local model $dP \cdot A$ with the corresponding global model $\int dP \cdot A$. This shift was guided by drawings and the physical and geometric properties described in the problem. Notably, the $dP \cdot A$ model would have been valid if Matthew had considered the pressure acting on the tank floor.

By analyzing Matthew's transitions between models and his use of drawings, we gained deeper insights into his reasoning process. While based on a single case study, these findings suggest that research on integration should pay closer attention to the diverse knowledge resources, visual representations, and their interactions that students use in reasoning about multiplication and definite integrals in applied contexts.

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Understanding-Oriented Lectures in Calculus: Practical Strategies for Enhancing Students' Mathematical Learning

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Many instructors observe that students often rely on memorization in introductory college mathematics. However, existing literature on effective teaching strategies does not fully address the need to cultivate both conceptual understanding and deeper orientations towards understanding in students' learning. This study presents a case analysis of calculus lectures delivered by an experienced instructor at an R1 institution, revealing significant opportunities for enhancing conceptual learning and promoting stronger orientations towards understanding among students. The analysis identified four key strategies employed by the instructor: 1) framing instructional activities as a pursuit of understanding; 2) examining underlying meanings while connecting students' intuition; 3) providing motivations, justifications, and multiple perspectives; and 4) creating cognitively challenging tasks and encouraging students to embrace discomfort in learning. These simple, practical strategies do not impose significant demands on instructors yet hold great potential for helping more students approach mathematics through deep understanding.

Keywords: Conceptual Understanding, Orientations, Calculus, Lectures

Effective teaching in undergraduate mathematics classrooms is often associated with clear explanations, approachable instructors, prompt feedback, high expectations, fair assessments, interactive teaching methods, well-organized course structures, and the use of relevant examples and applications (Chickering & Gamson, 1991; Harper et al., 2019; Mesa et al., 2015). While these strategies are crucial, they may not fully address the need to foster both conceptual learning and orientations towards understanding (Huang, 2022, 2023) in students' mathematical learning. Despite recent advances in undergraduate mathematics education research (Artigue, 2016), many instructors continue to observe that students tend to rely heavily on memorizing formulas and procedures in introductory college mathematics (Burn & Mesa, 2015). This reliance on rote learning undermines students' ability to engage meaningfully with mathematics and develop mathematical proficiency (Schoenfeld, 2022).

Such a persistent issue raises a critical question: How can we better support students in developing both conceptual understanding and understanding-oriented approaches to learning mathematics? Given that most undergraduate mathematics courses continue to be delivered in lecture formats (Artigue, 2016; Johnson et al., 2018), it is unlikely that this mode of instruction will disappear anytime soon. Rather than dismissing lectures entirely, this study seeks to identify simple, practical techniques that instructors can integrate into their daily teaching practices without imposing significant additional demands. While transformative teaching methods, such as inquiry-based instruction (Laursen & Rasmussen, 2019), have made significant contributions to the field, they often require substantial resources and institutional support for large-scale implementation (Weaver, 2016). This paper specifically explores opportunities within calculus lectures, delivered by an experienced and well-recognized instructor at an R1 institution, that promote students' conceptual understanding and understanding-oriented approaches to learning mathematics. These opportunities offer great potential for low-cost, large-scale implementation

that could effectively support more students in approaching mathematics through deep understanding and enhancing their learning outcomes.

Literature Review

Mathematics education researchers have conceptualized mathematical understanding in various ways (Hiebert & Carpenter, 1992; Perkins & Blythe, 1994; Piere & Kieren, 1994; Schoenfeld, 1985; Shapiro & Stolz, 2019; Sierpinska, 1994). In this paper, I integrate Hiebert and Carpenter's (1992) view of understanding as a well-connected web of knowledge with Schoenfeld's (1985) operational definition, which emphasizes the ability to solve challenging problems. This paper focuses on the first two strands of mathematical proficiency identified by the National Research Council (2001)—conceptual understanding and productive disposition—while acknowledging the importance of the other three strands: strategic competence, adaptive reasoning, and procedural fluency.

Schoenfeld (2010) introduces the notion of orientation, which includes beliefs, values, dispositions, and biases that fundamentally shape people's behaviors in well-practiced domains such as teaching and learning. Building on this work, Huang (2023) elaborates on the construct of *orientations with respect to understanding in mathematics*, defining it as "the degree to which students exhibit an inclination towards and demonstrate an earnest concern for understanding in mathematics learning" (p. 49). Huang (2023) develops a methodological tool for measuring this construct with established validity and reliability. According to Huang, understanding-oriented students "care deeply about how things work, prioritize goals and actions for developing understanding, and are willing to put in much effort to understand underlying principles" (2023, p. 49). It is important to note that an orientation towards understanding is closely associated with a growth mindset (Dweck, 2006, 2013; Dweck & Yeager, 2019).

Previous work on orientations with respect to understanding has focused primarily on students. However, to design interventions that support the development of students' orientations towards understanding, we must first describe the types of opportunities available in various learning structures (e.g., lectures, curricula, small-group interactions) and analyze the nature of these opportunities. As a first step, this paper provides a case study of a semester of promising calculus lectures that appear to offer substantial opportunities for both enhancing students' conceptual understanding of core calculus concepts and fostering strong orientations towards understanding in mathematics learning.

Methods

This study is part of a larger investigation into students' calculus learning in a large-enrollment, first-semester calculus course at a public R1 institution. The course consists of three hours of lecture and three hours of discussion per week over a 16-week semester. Lectures are typically delivered by an instructor in a large auditorium twice a week, while graduate teaching assistants lead discussion sections of approximately 30 students on other weekdays. Each semester, over 1,400 students enroll in the course. The larger study involves observing all lectured-based instruction and three randomly selected discussion sections weekly, conducting surveys on students' learning experiences, interviewing the course instructor and graduate teaching assistants, and carrying out three rounds of task-based interviews with 57 students to gain an in-depth understanding of their calculus learning in Fall 2023. The primary data sources for this paper are 48 hours of video recordings of lecture-based instruction, supplemented by my field notes. The course instructor is a tenured mathematician who has been teaching mathematics at the university since earning his Ph.D. 12 years ago.

I used grounded methods (Charmaz, 2006; Vollstedt & Rezat, 2019) to explore the opportunities in calculus lectures that promote students' conceptual understanding and foster their orientations towards understanding in mathematics learning. Given that the connection between mathematics lectures and students' orientations towards understanding is largely unexplored in existing literature, grounded methods allowed for an iterative and flexible process where empirical data significantly informed the development of new concepts and categories (Vollstedt & Rezat, 2019). To distill video recordings into analyzable segments, I created activity logs at 20-minute intervals, partitioned the recordings into episodes, and developed verbatim transcripts with notes on the mathematical ideas discussed. In the next phase, I wrote analytic memos for each transcript, documenting emerging themes that could support students' conceptual understanding or orientations towards understanding, along with their related mathematical discourses in the classroom. I also connected these themes to relevant literature in this phase. Based on an initial list of themes developed from the previous phase, I coded all the transcripts and trained an undergraduate research assistant to independently code them as well. Disagreements in coding led to discussions on how to distinguish different codes and cluster similar ones, prompting us to revisit all transcripts to refine our code list and definitions. The coding process was iterative, requiring us to re-analyze all transcripts as new codes or revised definitions emerged. We resolved all disagreements in the process. In the final phase, we conducted a comprehensive review of all video recordings and field notes to ensure that the emerging categories aligned with our initial impression of the data and captured the whole story. This review guided our revisions to the categories to ensure accuracy and thoroughness.

Results & Discussion

The analysis revealed four key strategies that the calculus instructor employed to foster students' conceptual learning and nurture their orientations towards understanding in mathematics. These strategies are: 1) explicitly framing instructional activities as a pursuit of understanding; 2) examining underlying meanings while actively connecting students' intuition; 3) providing motivations, justifications, and multiple perspectives; and 4) creating cognitively challenging tasks while encouraging students to embrace discomfort in learning.

Explicitly Framing Instructional Activities as a Pursuit of Understanding

A central strategy employed by the instructor was the consistent and explicit framing of instructional activities as a pursuit of understanding. By using phrases such as "let's try to understand" and "I really want to understand what's going on," the instructor signaled that the goal extended beyond procedural mastery to cultivating both deep conceptual understanding and orientations towards understanding in students' mathematical learning.

This emphasis was evident in a lecture on September 12, 2023, when the instructor encouraged students to think beyond simply obtaining the correct answer. He remarked, "Many of you might know the answer, okay? But that's not the point here. *I want to really understand* how the answer relates to the definition and how we could maybe solve other problems based on this example" (emphasis added). This framing aligns with Huang's empirically based characteristics of students strongly oriented towards understanding, such as "spending time understanding confusing ideas even after getting an answer" (2023, p. 51).

In another lecture on September 21, 2023, the instructor challenged students to understand the limit of $arctan\ (x)$ as x approaches infinity without relying on the graph of $y = arctan\ (x)$. He said, "*I really want to understand* what's going on geometrically. I would like to get an explanation of what this limit is, without having to know the graph of the function. Based on

what you've learned about how we typically deal with trig expressions, what is something we can draw *to understand this*?" (emphasis added). When a student suggested drawing a right triangle, the instructor responded, "Very good idea, *let's try to understand* what $arctan\ (x)$ is in terms of a right triangle. How can we construct a right triangle using the given information?" (emphasis added). The conversation led to the creation of a visual representation, shown in Figure 1, which the class then used to explain what happens when x approaches infinity.

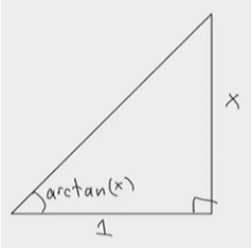


Figure 1. Understanding $arctan$ x with a triangle

This deliberate focus on understanding was also evident in a lecture on October 10, 2023, before introducing logarithmic differentiation. The instructor invited students to explore how logarithms might help finding the derivative of $f(x) = (x+1)(x+2)^5(x+3)^7(x+4)^9$ and to understand why the strategy worked. He emphasized,

> *I want to understand* why this is working. This looks like magic. It's so much easier than the product rule—but if something in math seems like magic, that just means we don't understand it well enough. So *let's really try to understand* what's going on here and why this method is working. *To make sure we understand* this method, let's go one step further and really see what's happening. (emphasis added)

The consistent and explicit framing of instructional activities as a pursuit of understanding fostered a learning environment where students were encouraged to engage deeply with the material, thinking critically, and build a strong conceptual foundation in mathematics. This approach aligns with what Huang (2023) observes to be key indicators of strong orientations towards understanding, supporting students in viewing mathematics as a sense-making activity and approaching mathematical learning through more understanding-oriented methods.

Examining Underlying Meanings While Actively Connecting Students' Intuition

The second strategy employed by the instructor involved examining underlying meanings of mathematical concepts while actively connecting with students' intuition. For instance, in a lecture on October 17, 2023, the instructor modeled digging deep into the underlying meanings of complex formulas and searching for intuitive explanations. He remarked:

> *I want to think about what this means and how we could explain this formula in a more intuitive manner.* Yes, you could use the methods from the previous class to derive this formula, but it's complicated and maybe not very intuitive. So I want to show you a way to think about it in terms of differential equations—you'll see that *this actually makes sense* and that it has to be the result. (emphasis added)

This approach aligns with Bruner's (1960) notion of uncovering "fundamental structures" in

learning. By focusing on the underlying meanings of mathematical concepts, the instructor fostered meaningful connections and supported students in seeing ideas through core disciplinary principles, both of which are crucial for developing mathematical expertise (Bransford et al., 2000; Hiebert & Carpenter, 1992; Schoenfeld, 1985). The practice of searching for intuitive explanations made abstract concepts accessible to students while nurturing their mathematical intuition, leading to more connected learning and the development of flexible mathematical thinkers (Agarwal & Sengupta-Irving, 2019; Fischbein, 2005; Schoenfeld, 2020). In the next example, the instructor clarified the meaning of derivatives using Leibniz's notation and connected it to students' visual and physical intuition in a lecture on October 5, 2023. He said:

> Let me say one more interpretation of *what this actually means*. The idea here is that dy/dx is a quotient. We can think of dy/dx as very small quantities, like Δx or Δy. Imagine you're standing on the point (x, y); then dy/dx tells you how much you need to change the y-coordinate slightly if you change the x-coordinate by one infinitesimal unit. (emphasis added)

This explanation exemplifies how the lectures supported both genuine sense-making and conceptual understanding of abstract concepts. Relating derivatives to students' common experiences, such as standing on a point and looking towards a particular direction, aligns with Boaler et al.'s (2016) work that emphasizes making abstract mathematical ideas tangible and relatable. Additionally, the instructor's use of phrases such as "I want to give you a bit more intuition of what this means," "we should see what it means visually," "I would like to find, in the real world, intuition for this theorem," and "put all the theorems apart; just think about your intuition—what would it say?" underscores his commitment to connecting mathematics with students' repertoires of knowledge (Bernstein, 1996) and fostering genuine sense-making of challenging concepts.

Providing Motivations, Justifications, & Multiple Perspectives

The third instructional strategy centered on offering motivations, justifications, and multiple perspectives to enrich students' conceptual understanding and encourage deeper engagement with mathematical ideas. Research suggests that presenting the motivations behind definitions, theorems, and proofs enhances students' ability to see how these ideas fit into the broader mathematical landscape, fostering more meaningful connections and deeper reasoning (Abramovich et al., 2019). In a lecture on September 14, 2023, the instructor exemplified this approach by motivating the formal definition of limits as well as its core components. He explained:

> I've explained the limit by saying that $f(x)$ is approximately equal to L as long as x is sufficiently close to a. But this is not a mathematically precise definition. This is not how you can base your mathematical theory on. We want to have a definition that, whatever you prove later on based on this definition, is going to be 100% true, guaranteed. So we need to make this more precise. First of all, we should probably talk about what we mean by *approximate*. Well, we can think of an error in an approximation, which is basically the difference between $f(x)$ and L, which is just $|f(x) - L|$. The smaller the error, the better the approximation. Then we also need to precisely define what is *sufficiently close*. Well, closeness—we talked about this in relation to x being approximately a. We could

> also call it approximation, but somehow it's often called closeness. It's the distance between x and a, which is $|x - a|$. Okay, these are intermediate steps. Now we are ready to make a mathematically precise definition of limits.

By breaking down the abstract concept into comprehensible parts and providing step-by-step rationales, the instructor transformed a potentially daunting definition into a logical sequence of ideas. This process of demystifying abstract concepts helped students appreciate the rigor and necessity of precise definitions, which are crucial for developing positive disciplinary identities and a sense of belonging to the broader mathematics community (Huang, 2024).

Moreover, the instructor consistently emphasized the importance of justifying ideas and understanding them from multiple perspectives. For instance, in a lecture on October 3, 2023, after algebraically proving the product rule and connecting it to more basic principles, the instructor proceeded to justify the rule geometrically. This approach aligns with research showing that connecting different mathematical representations—such as algebraic and geometric—reinforces students' conceptual understanding and encourages flexible thinking in mathematical problem-solving (Schoenfeld et al., 2023). The instructor reiterated the significance of multiple perspectives in a lecture on October 31, 2023, stating: "As I always say, it's always good to get several perspectives on the same problem because sometimes some perspectives are more useful than others. And that's the case here. So let me explain, one more time, why this limit is 2, using a different method." This emphasis on multiple perspectives and justifications reflects evidence-based practices in mathematics education, where students benefit from seeing problems approached in multiple ways (Schoenfeld et al., 2023).

Creating Cognitively Challenging Tasks & Encouraging Embracement of Discomfort

The fourth instructional strategy involved designing cognitively challenging tasks and encouraging students to embrace the discomfort inherent in deep learning. Research shows that engaging with tasks of high cognitive demand fosters deep learning and robust understanding (Schoenfeld, 2017). The instructor implemented this strategy by regularly presenting tasks that pushed students beyond their comfort zones, leading to some concerns about the difficulty of assignments. In a lecture on September 14, 2023, the instructor addressed these concerns:

> There are problems on the homework where you cannot apply the solution strategy that I told you, where you have to think deeper and somehow come up with your own solution strategy based on the intuition that I try to convey in lectures. It's not that we're giving you homework problems and we kind of forgot to give you the solution strategy. We do it on purpose. These are problems where you will actually learn something because you really have to go back to the material and learn the concepts—even though, of course, they may take a much longer time. But the goal should always be to learn the concept, not just finish the exercise.

This response illustrates the instructor's deliberate intent to challenge students and promote deep engagement with mathematical concepts. By designing tasks that required students to think deeply and critically, the instructor supported students' robust conceptual understanding.

In addition to presenting challenging tasks, the instructor emphasized the importance of embracing discomfort as a natural part of the learning process. In a lecture on August 24, 2023, he remarked:

> Mathematics can be tough, and it's normal to feel frustrated when you're trying to understand an abstract concept or working through complex problems. Don't shy away from that discomfort; instead, use it as a signal that you're pushing the boundaries of your understanding. Sometimes, the most significant learning happens when you persist…. Some students work on hundreds of problems but fall short of expected success because they don't try to really understand the main concepts. Doing more and more problems is sometimes easier than getting at the root of a misunderstanding. My recommendation to you is to embrace the feeling of unease—this is when you really learn.

This emphasis on discomfort as a catalyst for growth aligns with Dweck's (2006, 2013) work on growth mindset, which suggests that viewing challenges as opportunities for growth can significantly enhance students' learning outcomes. By framing discomfort as a natural and valuable part of the learning process, the instructor cultivated a classroom culture of resilience and growth, fostering stronger orientations towards deep understanding (Huang, 2023) among students.

Concluding Remarks

This paper has examined the opportunities within calculus lectures, delivered by an experienced and well-recognized instructor at an R1 institution over a semester, that promoted students' conceptual learning and orientations towards understanding. The findings highlight four key strategies that the instructor used to achieve these goals: 1) explicitly framing instructional activities as a pursuit of understanding; 2) examining underlying meanings while actively connecting students' intuition; 3) providing motivations, justifications, and multiple perspectives; and 4) creating cognitively challenging tasks while encouraging students to embrace discomfort in learning. Importantly, these strategies are not isolated but can be employed simultaneously within lectures and potentially other instructional structures.

This study contributes to the broader discourse on mathematics education by demonstrating that, even within traditional lecture formats, there are opportunities that instructors can leverage to support more meaningful and understanding-oriented mathematics experiences for students. Notably, the identified strategies do not require significant changes to existing instructional structures or significant time commitments from instructors. Instead, they are practical, low-cost techniques that instructors can easily learn and incorporate into their everyday teaching practices.

While this research focuses on calculus instruction in a large-enrollment setting, the strategies identified have broader implications for undergraduate mathematics education. Whether delivering lectures, following inquiry-based instruction, or employing a flipped classroom model, instructors can adapt these strategies to their contexts. These strategies offer a pathway towards transforming students' mathematical experiences by fostering both robust conceptual learning and stronger orientations towards understanding, which can directly address the concern raised by Burn and Mesa (2015) that students often rely heavily on memorization in introductory college mathematics. Future research could explore how these strategies can be scaled across different mathematical contexts and course structures, as well as how institutional support can be leveraged to sustain these instructional improvements. By implementing these strategies and cultivating a classroom culture that prioritizes understanding, we can help students develop deeper mathematical understanding and move away from unhealthy habits of memorization in favor of meaningful learning.

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Examining the Relationship Between Key Points in College Students' Drawn Life Stories in Mathematics and Their Relationship With Mathematics

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This research investigates significant moments in college students' drawn life stories in mathematics with the aim of examining implications for STEM engagement and potential reasons for gender-related differences in participation in mathematics intensive STEM fields. My research employs drawn life stories in mathematics specifically focused on students' experiences with mathematics to understand students' evolving relationships with mathematics, encompassing positive and negative experiences. Employing mixed methods analysis of visual and written mathematical narratives combined with qualitative interviews, my research explores the effectiveness of allowing participants to visualize their mathematical journeys, and quantitatively assesses the impact of recurring themes in students' drawn life stories in mathematics. This research advances our understanding of factors that influence the nature of students' relationships with mathematics, and potentially inform strategies for enhancing STEM participation and gender inclusivity, using a unique approach in mathematical narrative research.

Keywords: Mathematics Identity; STEM Participation; Gender

The United States Bureau of Labor Statistics (BLS) expects a nearly 11% increase in occupations in science, technology, engineering, and mathematics (STEM) fields from 2021 to 2031 (BLS, 2022). To ensure effective preparation of a STEM workforce that can meet these growing demands, it is necessary for both researchers and educator practitioners in STEM education to consider factors which may help students develop an interest in STEM and a belief in their STEM abilities. Thus, it is critical to avoid perpetuating historically established barriers to pursuit of STEM degrees and careers that limit the growth and diversity of the STEM workforce (Grossman & Porsche, 2014; Feder & Malcolm, 2016).

Mathematics is a useful area for this inquiry. It has been frequently referred to as a 'gatekeeper' to STEM degrees as many STEM degrees require prerequisite courses in mathematics (Bryk & Treisman, 2010; Norton et al., 2019). Additionally, most STEM fields draw on important mathematical foundations and skills. To better understand the factors that influence students' relationship to mathematics over the course of their lives, in my research I utilized an established life story narrative approach to examine students' experiences as learners of mathematics, through their early undergraduate studies (McAdams, 2001; John et al., 2020).

The process of creating and sharing life story narratives allows individuals to make sense of their past and present in a way that enables them to meaningfully anticipate the future (McAdams, 2001; John et al., 2020). Past research in mathematical narratives has been conducted primarily with preservice teachers (Towers et al., 2017). My research built on past studies by expanding the population of interest to undergraduate students more broadly as I sought to understand the factors which drive society to and away from mathematics.

A recent study in mathematical narrative research, conducted by John et al. (2020) focused on "turning points" that individuals describe in their prior histories with mathematics. In their research, they characterize turning points as either being a form of "redemption" in which individuals describe their relationship with mathematics as shifting from negative to positive, or

"contamination" in which individuals describe their relationship with mathematics as shifting from positive to negative. In my research, the term "turning points" was extended to "key points" to include both turning points and instances where an individual describes an experience that makes their existing relationship with mathematics even stronger - more positive or more negative.

While there are multiple examples of studies that utilize narratives to study relationships with mathematics, there are several ways in which this area of inquiry has had shortcomings, and thus, can be improved. For instance, one key finding from John et al. (2020) was that over half (53%) of the narrative sequences described by participants were redemptive in nature. The researchers suggest that this is likely due to a prevailing master narrative in the U.S. that places a large value on redemption (John et al., 2020; Syed, Pasupathi, & McClean, 2019). It is worth considering, then, whether there are other ways of more authentically allowing students to showcase their relationship with mathematics.

Additionally, John et al. (2020) did not provide insight into the experiences or other factors that coincided with these turning points and did not assign any sort of magnitude to identify strengths of the patterns in these factors or experiences. To address this limitation, my research utilized a novel approach to narrative research, adapted from Guillaume & Kirtman (2010), in which students have the opportunity to draw their mathematical narratives. By using an open-ended drawn approach to mathematical narratives, individuals can illustrate as many key points as they can remember and the magnitude of the impact of these key points on their relationship with mathematics. The opportunity to quantify these experiences is particularly notable as most narrative research in mathematics education has been purely qualitative in nature (Towers et al., 2017).

The potential for drawn life stories in mathematics to provide insight into critical aspects of students' experiences and resulting relationships with mathematics can also be valuable for understanding factors that have historically limited the diversity and inclusivity of advanced studies and careers in mathematics and STEM. Women in the United States have historically been and continue to be underrepresented among degree recipients in math-intensive fields such as computer sciences and engineering. For instance, while women make up 51% of the overall workforce in the United States, women comprise just 26% of the mathematics and computer scientists and 16% of engineers (NCSES, 2023).

To address this underrepresentation, more efforts to understand and mitigate barriers that prohibit women from pursuing STEM pathways are needed. Thus, this research also examined what insights drawn life stories in mathematics can provide to better understand and address gender-related differences in mathematics.

The research questions for my research are as follows:

- **Research Question 1:** To what extent do recurring themes account for key points in undergraduate students' life stories in mathematics?
- **Research Question 2:** What insights can drawn life stories in mathematics provide to better understand and address gender-related differences in STEM participation?

Literature Review

Mathematics Identity

Widespread use of the term identity among researchers in multiple fields has led to variations in how it is defined (Radovic et al., 2018; Bishop, 2012; Darragh, 2016; Grootenboer et al., 2006; Grootenboer, 2013). Given the variation in its meaning, social science researchers have

criticized the concept of identity for having a “slippery” and confusing nature (Wetherell, 2010; Radovic et al., 2018).

At the same time, identity has been characterized as the “missing link” between learning and its sociocultural context (Sfard & Prusak, 2005; Hong & Perez, 2024). From this perspective, learning extends beyond the construction of understanding and consists of becoming a new person with respect to cultural norms, practices, and interaction (Lave & Wenger, 1991; Wenger 1998). What students learn in and out of school extends far beyond the subject matter, students develop social and cultural perspectives on who they should be. The identities which students acquire then influence not only what students will learn, but what they will pursue and what they will become. Furthermore, as identities have been observed to potentially change over the course of students’ lives, identities are not considered to be static, but rather malleable (Andersson et al., 2015; Gonzalez et al., 2020).

Research on identity in mathematics education has been a growing field of study since the turn of the century (Radovic et al., 2018; Darragh, 2016); Simpson & Bouhafa; 2020). For the purposes of my research, I have adopted the definition of Boaler & Selling (2017, p.82) for mathematics identity as “the ways in which students think about themselves in relation to mathematics and the extent to which they have developed a commitment to, are engaged in, and see value in mathematics and in themselves as learners of mathematics". Furthermore, my research takes a Meadian conception of identity (Mead, 1913). Contrary to an Eriksonian conception of identity, which tends to view identity as singular and static, a Meadian conception suggests that a person's identity are multiple and constantly evolving (Machalow et al., 2020; Darragh, 2016; Lutovac & Kaasila, 2019).

Gender Identity Research in Mathematics Education

The exploration of gender research in mathematics education dates to the prolific work of Fennema and her colleagues in the 1970s (Fennema & Sherman, 1976; Fennema & Sherman, 1977). Research in the late 1970s following Fennema’s work would examine gender disparities in involvement and achievement in mathematics, along with investigations into potential factors that promote or hinder these differences (Leder, 2019). Fennema’s work was undoubtedly impactful on the field of mathematics education. As Walberg & Haertel (1992) reported, the Fennema & Sherman (1972) article was among the most frequently cited articles in the Social Sciences Education Index from 1966-1988.

Given that 72% of the STEM workforce are males (NCSES, 2023), research on the intersection between gender identity and mathematics identity has often focused on exploring the nature of persistent inequities in the STEM workforce. Some studies have shown that, in-general, males tend to have higher or more positive mathematics identities than females (Eccles & Wang, 2016; Lock et al., 2013; Cass et al., 2011, Nosek et al., 2002). Thus, recent research has explored a few different explanations for gendered differences in mathematics identity and in-turn gendered differences in the STEM workforce (Leder, 2019; Stoet & Geary, 2018; Rodriguez et al., 2020).

Mathematical Narrative Research

Narratives refer to the retelling of autobiographical accounts in a manner that conveys significance to the narrator and the listener (Bruner, 1990). Thus, the phrase “mathematical narratives” is often used synonymously with “mathematical autobiographies”. Mathematical narratives are a versatile data collection tool, allowing researchers to explore trends across a learners’ lifespan. As such, researchers have used mathematical narratives to explore several

constructs relevant to my research including affective measures such as motivation (Harkness et al., 2007; Phelps, 2010), beliefs (Eaton et al., 2012; Dewi et al., 2022), and attitudes (Ellsworth & Buss, 2000; Russo et al., 2021).

One of the largest subfields of interest in the study of mathematical narratives has been the study of turning points, aimed at understanding the events or factors that lead to shifts in individuals' autobiographies (Towers et al., 2017). The present study uses a more inclusive term "key point" to expand the definition of turning points to also include instances where an individual describes an experience that makes their existing relationship with mathematics even stronger - more positive or more negative.

In a particularly relevant case, Guillaume & Kirtman (2010) had participants chart graphs of their attitude and achievement in mathematics, in-order to analyze the turning points that individuals drew. In their research, Guillaume & Kirtman (2010) focused on the themes that emerged from their work and particularly focused on out-of-school connections that individuals discussed. For example, they found that only 42% of individuals discussed a home connection to mathematics and that of those who *did* note a home connection to mathematics, 65% fell under school-related categories such as workbooks/flashcards and tutoring/help with homework. While this study is interesting, the graphs that individuals drew were not used to their full potential. A major benefit to having individuals draw their life stories in mathematics, which my research capitalizes on, is that drawn life stories in mathematics offer an easy way for researchers to quantify turning points. In fact, individuals noted instances of turning points on their timeline, yet Guillaume & Kirtman did not classify these events in the traditional turning points framework, instead opting to simply observe patterns among the themes. Furthermore, one can measure how far a jump or drop key points cause because of having individuals draw their mathematical timelines. My research capitalizes on this potential by both examining the key points on individuals drawn life stories in mathematics, as well as quantitatively analyzing how large of shifts these key points cause.

Methods

Participants

This research was conducted at a large university in the Southeastern United States. Participants included 254 undergraduate students pursuing a variety of majors, enrolled in either MAC 2311: Analytic Geometry and Calculus 1 or MAC 1147: Precalculus Algebra and Trigonometry. In total, 214 of the participants were enrolled in MAC 2311 and 39 were enrolled in MAC 1147. These courses serve as prerequisites and general education courses for a variety of majors and hence were strategically selected as they include students pursuing both STEM and those pursuing other fields. Conducting this survey with students currently enrolled in an undergraduate mathematics course ensured that participants' mathematics experiences included postsecondary studies in addition to any insights they offered about their mathematical experiences in primary and secondary school.

Instrumentation

Data was collected using an instrument designed for participants to draw how their relationship to mathematics has changed over the course of their lives. An example of a completed version of this instrument in shown in Figure 1.

Participants were asked to think as far back in time as they can remember, about both in-school and out-of-school experiences, including people and experiences that have had a strong

effect (either positive or negative) on their relationship to mathematics and then to map out how these experiences have shaped their relationship to mathematics over time, while labeling any key points. Participants were then asked to expand on what those key points entailed on the next page and whether they would be willing to participate in a follow-up conversation to expand on their responses.

Finally, participants were asked a few demographic questions including what gender they identify as. The research question addressing gender identity gives the options of male, female, non-binary, and genderfluid; however, it also includes an open-ended option which has been recommended in recent research (Puckett et al., 2020).

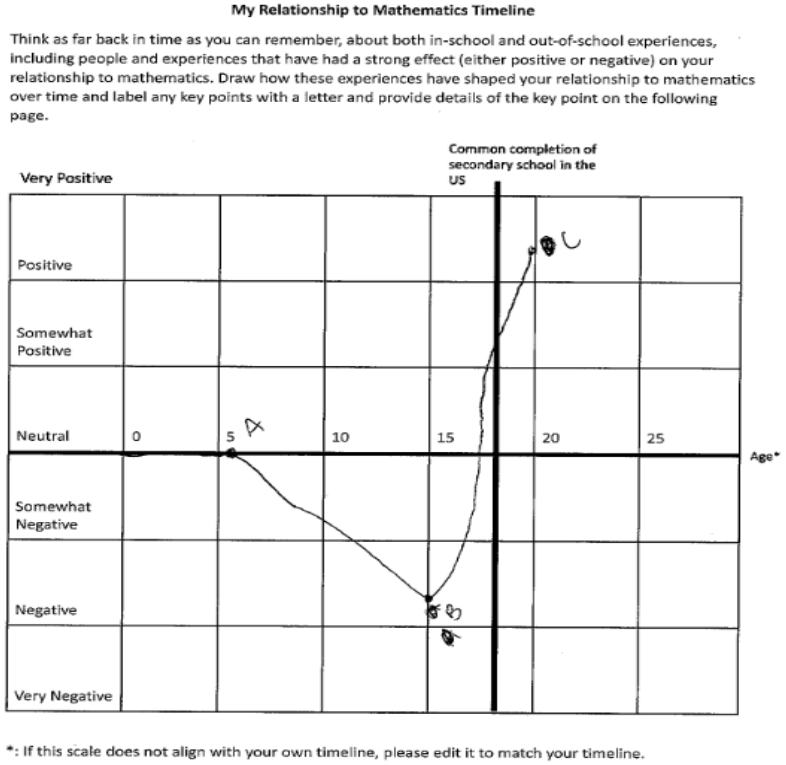


Figure 1. Example of a Drawn Life Story in Mathematics

Coding

Key points that participants described were each coded in three distinct ways. First the key points were coded deductively as either contamination, redemption, enhancement, or deterioration. These deductive codes stem from past narrative work indicating that turning points can often be seen as redemption or contamination sequences (McAdams & Bowman, 2001) and this framework has been used in the mathematical narrative space as well (John et al., 2020).

Additionally, each key point was inductively coded in two rounds, guided by the recommendations of Saldaña (2021). In the first round of coding, the descriptions of key points provided by participants were summarized. Notably, depending on the length of the participants' description of their key points, this first round of coding was not necessary in every case. The key points were then coded using pattern coding to identify emergent subthemes that underlie key points given by participants (Saldaña, 2021; Miles et al., 2014). Once subthemes were gathered, subthemes were grouped together into larger themes encompassing them. To help

ensure reliability of deductive codes, the themes and subthemes were continuously refined with the assistance of another researcher.

Finally, guided by past literature illustrating a tendency for mathematical narratives to overutilize qualitative research (Towers et al., 2017), data given by participants were analyzed quantitatively using a form of magnitude coding (Saldaña, 2021). For example, in Figure 1, key point B was coded as -1.5 as it moved the respondents relationship with mathematics from neutral down 1 to somewhat negative and then half way between somewhat negative and negative. This is one of the largest potential research benefits of having participants draw their life stories in mathematics instead of verbalizing them, as drawing allows for a more detailed and consistent way of analyzing how impactful each key point was.

Results and Conclusions

First, all 254 participants' graphs were analyzed based on the structure of the graph. The graphs were coded as "Positive" if the graphs were strictly neutral or positive, "Mixed" if the graphs ever crossed the x-axis – that is, if the participants relationship from mathematics ever went from positive to negative or vice versa, and "Negative" if the graphs were ever strictly negative. Notably, the overwhelming majority of graphs changed from positive to negative or vice versa at some point (76.77%). Additionally, it is worth noting that while there were many participants (23.23%) whose graphs were strictly positive, there were 0 whose graphs were strictly negative.

Themes among Key Points

Using emergent coding, 24 subthemes were identified among the key point responses, in addition to a robust "Other" subtheme.

Magnitude coding for each key point was broken down by themes and subthemes, and then further analyzed. Each subtheme was treated separately as positive (if the key point received an "Enhancement" or "Redemption code) and negative (if the key point received a "Deterioration" or "Contamination" code). Figure 2 shows a breakdown of the positive subthemes, by both the count and the average magnitude code applied within the subtheme.

The results from the magnitude code analysis of the drawn life stories in mathematics clearly showed two recurrent subthemes that were impactful for students both in positive and negative ways: "Teacher" and "Achievement". Beginning with the "Teacher" subtheme, participants explained a wide variety of reasons for which teachers were influential (both in positive and negative ways) on their relationship with mathematics. Some participants focused on ways in which teachers impacted their cognitive development citing reasons such as their teacher being "thorough" or "unhelpful". However, many participants spoke about affective ways in which their teachers impacted their relationship with mathematics. For example, participants described positive impacting teachers for reasons such as them being "fun", "caring", and "kind" and negative impacting teachers for reasons such as them being "not welcoming" and "rude".

Research tends to consider to what extent mathematics identity may be a predictor of future mathematics achievement (e.g. Bohrnstedt et al., 2021; Gonzalez et al., 2020); however, my research adds evidence to the inverse relationship that achievement may influence students' mathematics identities. One explanation for this may be that grades serve as an indicator for students of the extent of their belonging within the field. As such, a positive grade may strengthen or help recover a students' mathematics identity while a negative grade may deter or further damage students mathematics' identity.

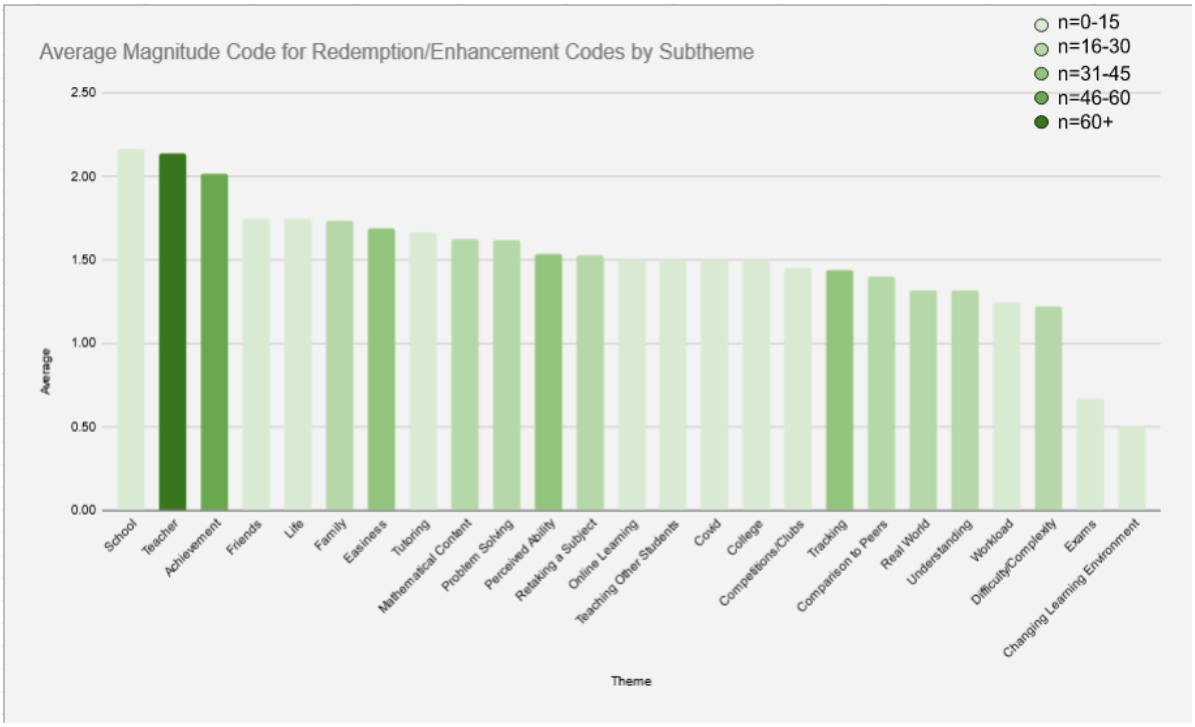


Figure 2. Average Magnitude Code for Redemption/Enhancement Codes by Subtheme.

Gendered Findings

Gender-related differences are explored in this study by examining both the prevalence and magnitude of positive (redemption or enhancement) or negative (contamination or deterioration) experiences in participants' drawn life stories in mathematics. Taking findings of both prevalence and magnitude together, with statistical significance it was shown that while positive key points were more prevalent for males than females, the positive key points that existed tended to be more impactful for females than males. This suggests that in-order to address gender-related differences in STEM participation, it may make more sense to allocate resources more towards increasing the number of positive events for female students rather than working to ensure events are more impactful for female students. For example, this finding suggests that additional celebration of achievement moments for women may not be as impactful as simply ensuring that women have more opportunities to experience positive achievement.

On the converse, insufficient evidence was found to support the notion that there was a statistically significant difference in the frequency or magnitude of negative experiences drawn on mathematical life stories between men and women. While this finding seems inconsequential, past sociological research indicates that women report more frequent negative feelings than men and that men are somewhat more likely to report that they suppress their emotions than women (Simon et al., 2004). Coupled together, the lack of statistical significance may suggest that this tool has helped male participants to more authentically share their negative emotions than a traditional survey may. There is some prior research to back this idea as drawing has shown to be a useful tool among adults as an emotional regulation technique (Drake et al., 2016; Dalebroux et al., 2008). However, in these studies drawings were used following a specific procedure with a much more in-depth protocol than my research required, so further research should be done to verify the extent to which the tool used in my research can aid in emotional regulation.

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What Makes Content Knowledge *Specialized*: A Case Study of Graduate Teaching Assistants' Mathematical Knowledge for Teaching

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As the push for active learning pedagogies in undergraduate mathematics continues, research into the teaching practices of graduate student instructors (GSIs) has expanded significantly. This paper presents findings from a multiple-case, descriptive case study of the demonstrated mathematical knowledge for teaching (MKT) of GSIs within the context of active learning Precalculus lessons. This study aims to bridge the theoretical and practical distinctions between the MKT knowledge domains as they are used by GSIs in their teaching practice. Practical implications for the professional development offered to GSIs, as well as theoretical implications for studying the MKT construct in post-secondary settings, are discussed.

Keywords: Mathematical Knowledge for Teaching, Graduate Student Instructors, Precalculus

Several decades of research comparing active learning to traditional lectures in STEM classrooms has shown significant improvements in student learning outcomes in classrooms where active learning pedagogies are used (Freeman et al., 2014). As a result, researchers and educators have increasingly called for the implementation of active learning in undergraduate mathematics courses (Akin et al., 2023). Active learning is an umbrella term for classrooms where students engage in mathematical activities that promote higher-order thinking, such as group problem-solving and investigations (Conference Board of Mathematical Sciences, 2016). Active learning is often viewed as more difficult to implement than traditional lecture because the instructor must give over some control to allow their students to engage meaningfully with the mathematics. When student thinking is centered in the classroom, the instructor must balance opportunities for inquiry with structured guidance toward correct mathematical understandings (Akin et al., 2023). At Ph.D. granting universities in the United States, it is common for mathematics Ph.D. students to receive funding for their studies by teaching introductory undergraduate mathematics courses, such as Precalculus (Beisiegel et al., 2019). As a result, many of the introductory mathematics courses at Ph.D. granting institutions are taught solely by graduate student instructors (GSIs) (Deshler et al., 2015). It can be daunting for novice instructors, such as GSIs, to implement active learning pedagogies in their courses without clear guidance and support (Beisiegel et al., 2019). As the push for active learning in undergraduate mathematics increases, there is a concurrent push to understand the knowledge that GSIs need to implement active learning pedagogies in order to develop effective teaching supports (Akin et al., 2023). The purpose of this study is to examine the knowledge that GSIs use within their teaching practice and identify the purpose(s) within their practice that their knowledge serves.

Theoretical Framing

It is commonly accepted that instructors need content knowledge to teach effectively, but that content knowledge alone is insufficient (Ball et al., 2008). From analyzing teachers' practice, Ball et al. (2008) hypothesized that six distinct but related domains of knowledge "made up" MKT. Following Ball et al.'s (2008) seminal piece defining MKT, research about teacher knowledge has revolved around refining the theoretical definitions and creating clear delineations between the domains (e.g., Copur-Gencturk et al., 2019). Copur-Gencturk and her

colleagues (2019; 2022) conducted several quantitative studies to disentangle the subdomains of knowledge in Ball et al.'s (2008) framework. Copur-Gencturk and Tolar (2022) refined their definition of MKT to include "teachers' understanding of the mathematics, their pedagogical content knowledge (i.e., knowledge of students' mathematical thinking, their instructional practices, and the representations they use), and their ability to notice content-related issues in the moment of mathematics teaching" (p. 2). Their findings concluded that, statistically, content knowledge, pedagogical content knowledge, and content-specific noticing skills are distinct elements of MKT among their participants (See Figure 1). Copur-Gencturk et al.'s (2019; 2022) work is important in pushing forward the field's understanding of teacher knowledge from a theoretical perspective. Methodologically, however, these studies employed large-scale written assessments and quantitative analyses rather than using Ball et al.'s methodology of qualitatively examining teachers' practices, which may limit the applicability of the MKT framework for informing efforts to develop teaching supports for GSIs.

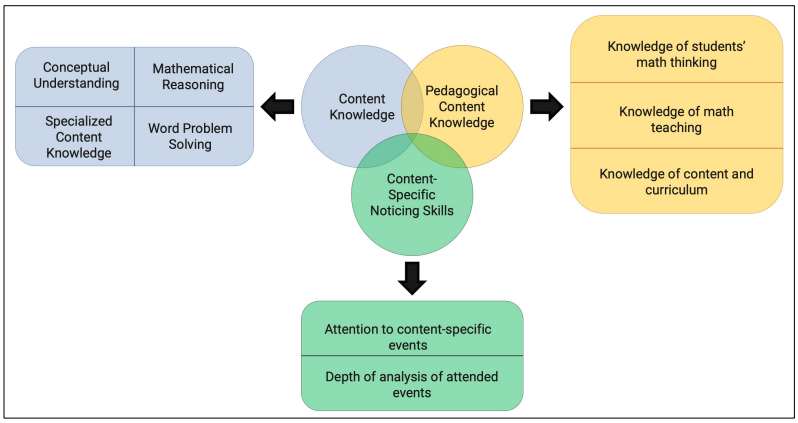


Figure 1. Mathematical Knowledge for Teaching as Conceptualized by Copur-Gencturk and Tolar (2022)

When extending ideas of MKT to post-secondary contexts, the theoretical distinctions between knowledge domains become more unclear (e.g., Howell et al., 2016; Speer & King, 2014; Speer et al., 2015). A question raised in Speer et al.'s (2014; 2015) work refers to how common content knowledge (CCK) and specialized content knowledge (SCK) are currently defined. Theoretically and practically, the concern is that knowledge considered as common for mathematics Ph.D. students versus those with only a bachelor's is most likely different in nontrivial ways. While assessing the mathematical validity of a student's response is unanimously accepted as SCK for an elementary teacher, this ability and knowledge may be considered common within the work of a Ph.D. student. For theoretical purposes, Copur-Gencturk and Tolar (2022) acknowledge that both domains overlap and, therefore, their definition for SCK is content knowledge "situated within the task of teaching, whereas the other components of [CCK] are not dependent on the teaching context" (p. 4). In other words, SCK can be operationalized as the mathematical knowledge that is used for purposes such as assessing students' mathematical reasoning, evaluating students' (potentially unconventional) solution strategies, and providing explanations of concepts to students. The study presented here is positioned to address concerns such as those raised by Speer et al. (2014; 2015) and to add to the current theoretical understanding of MKT by incorporating distinctions that come from the examination of GSIs' teaching practices. Specifically, this study is positioned to (1) identify the domain(s) of knowledge that GSIs use when using active learning pedagogies and (2) identify the purpose(s) for which their knowledge is employed when using active learning pedagogies.

Methods

This study followed a descriptive, multiple-case study design (Baxter & Jack, 2008) of mathematics GSIs for Precalculus at a large research institution in the Southeastern United States during the Fall of 2023. The study design is foremost descriptive in nature because the goal was to examine the real teaching practices of the GSIs in context and come to a more nuanced understanding of how their MKT was utilized in practice. The mathematics department at this university had recently undertaken several initiatives to improve student learning outcomes in introductory undergraduate courses. Such initiatives included creating and facilitating GSI professional development (PD) for Precalculus to support the implementation of active learning pedagogies. Participants in this study were three current Ph.D. students in the mathematics department who were GSIs for Precalculus. All of the participants in this study had completed at least two years of their programs and had successfully completed their qualifying exams. This paper presents the findings from one participant, Leona. At the time of data collection, Leona was a third-year Ph.D. candidate in Applied Mathematics. She had taught Precalculus in the year prior to data collection during the first semester when the new Precalculus units and PD were integrated into the curriculum. Leona was chosen as the first case because she was the only participant with previous experience as an instructor of record for this course.

Data Collection

MKT is employed in all aspects of a teacher's work; therefore, collecting multiple data sources is necessary to create a holistic view of the components of MKT as they are being used together (Meijer et al., 2002; Park & Oliver, 2008). To this end, data for this study were collected from two semi-structured interviews at the beginning and end of the semester, five PD meetings, and four classroom observations. The classroom data were split between two lecture classes and two active learning lessons focused on exponential functions. Data collected during the active learning lessons were the primary source used to address the following research question: How are the domains of Mathematical Knowledge for Teaching demonstrated in the teaching practices of graduate student instructors during active learning Precalculus lessons focused on exponential functions?

Data Analysis

All data were audio- and video-recorded before being transcribed using Temi. The researcher edited transcriptions for clarify and accuracy. The transcript data from the active learning lessons were chunked into "exchanges," defined by the GSIs' interactions with one or more students. Each exchange captured one distinct interaction between the GSI and their students. Exchanges would begin and/or end when the GSI stopped interacting with one group of students and began interacting with another. Each exchange served as the unit of analysis for data coding procedures. Following the chunking of the lesson transcripts, the first data analysis phase consisted of re-reading the lesson transcripts while re-watching the instruction videos. During this review, I wrote detailed descriptive memos about the exchanges. These memos focused on summarizing the interactions between the GSI and their students. The next data analysis step was assigning knowledge domain codes to each exchange. MKT knowledge domain codes were assigned based on whether the GSI demonstrated the domain of knowledge during the exchange. The codebook for the domains of MKT was created through the incorporation of Copur-Gencturk and Tolar's (2022) Content-Specific Expertise for Teaching Mathematics framework with the content-specific definitions for exponential functions from the Exponential Growth Learning Trajectory (Ellis et al., 2016). The second round of data coding

addressed the link between domains of knowledge and teaching practices. Units of analysis continued to be the exchanges. The teaching purpose codes were developed through a literature-informed, inductive method. In other words, the literature on teacher knowledge highlighted several purposes for which teacher knowledge could be used. These purposes included selecting and sequencing classroom tasks, assessing student understanding, and informing in-the-moment classroom decisions (e.g., Park & Oliver, 2008). While these teaching purpose codes were not defined in an a priori codebook, the code names and definitions were developed using a combination of language from the literature and the participant's words and actions.

Findings

I found that Leona most commonly demonstrated a combination of content knowledge and knowledge of the curriculum for the purpose of building mathematical connections between student thinking and key learning goals. Specifically, she demonstrated knowledge of exponential functions and the lesson's learning goals and task sequence when responding to students' questions and providing guided explanations (see Figure 2). Within these exchanges, Leona appeared responsive to her students' thinking and flexible in her responses or explanations. The curriculum was designed to elicit specific ways of reasoning about exponential functions. Teacher notes were embedded within the Desmos lessons, which were discussed with the GSIs during the PD meetings. The teacher notes integrated examples of common student responses with pedagogical suggestions designed to align with the goal of promoting productive covariational reasoning and building on current student understanding. Leona's interactions with her students often parallel the suggested pedagogical moves from the curriculum or suggestions discussed during the PD meetings. While her content knowledge was demonstrated throughout these exchanges, it was especially evident when she would bridge concepts and representations in ways that were not previously discussed with other GSIs or suggested by myself or the course coordinator.

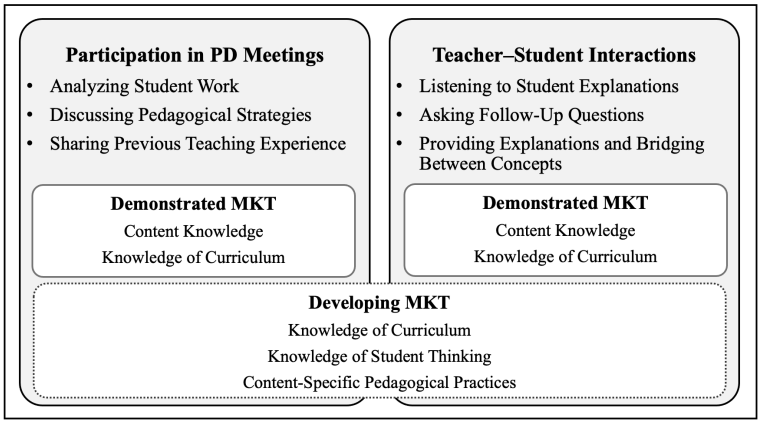


Figure 2. Visualization of Primary Finding for Leona's Case

Knowledge of Curriculum: Learning Goals and Task Sequence

One such student interaction occurred in the first of two active learning lessons, where a student was working on the questions from Figure 3. In these questions, students are first asked to identify two unknown heights (y values), given information about the amount of potion that Alice consumed (x values), and explain how they found one of the height values. Students are then given how Alice's sisters, Kadesha and Susana, found each of the values and asked to compare their own explanations with those given by Kadesha and Susana. In this one-on-one

interaction, a student asks for Leona's input on their solution because their method does not match either Kadesha's or Susana's method. The student wants Leona to check her work and help her understand the methods shown in the task.

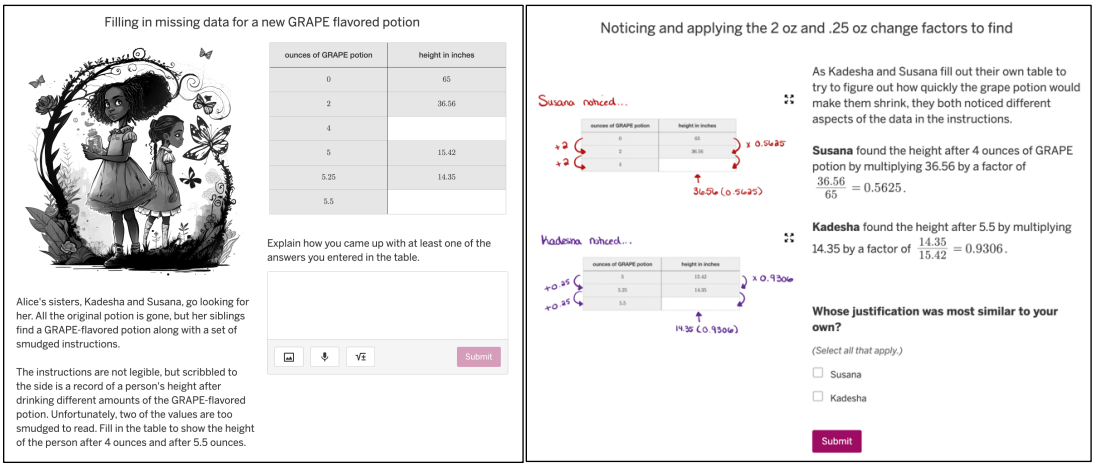


Figure 3. Student Facing Questions in Desmos Lesson

During this interaction, Leona asks the student to explain their reasoning so that she can first understand what the student has done before providing any other explanations or asking other questions. The student solved this task by substituting values from the table into the standard exponential function formula ($y = ab^x$) and solving for the base of the exponential. This lab's learning goal is for students to coordinate successive additive increases in ounces of potion with proportional multiplicative changes in height. The student's solution, while mathematically correct, relies on an algebraic representation of exponentiation, which does not push the student's understanding of the covariational relationship forward in the way that this task was intended to. Leona uses her knowledge of this learning goal and how the tasks are sequenced to inform her decision on how to structure the questions that she asks this student. Specifically, her questioning builds key mathematical connections between the student's solution strategy and those presented in the task from Susana and Kadesha. Additionally, Leona returns to the algebraic representations to help the student make another connection between the standard representation of exponential functions and a representation using the rate found by Susana. This further connection demonstrates Leona's knowledge of the task sequence because she is setting up the foundation of ideas that will become relevant in the second lesson on exponential functions.

Content Knowledge: Exponential Functions

The student in this interaction used the standard exponential function equation to find the unknown height values. Leona demonstrates her content knowledge of exponential functions throughout this interaction. Specifically, her content knowledge is being transformed into specialized content knowledge through her interaction with this student because she first must assess the mathematical validity of the student's solution and then use a series of prompts to draw mathematical connections between the student's thinking and the covariational relationships meant to be explored in this lesson. In the first part of this interaction, Leona is using her content knowledge of exponential functions and general algebraic rules to assess the validity of this student's method. There is a minor error in the student's algebra when solving for

the base x in the equation: $36.56 = 65x^2$. Leona notes that the student seems to have reversed the order of operations. In other words, the student applied a square root to the equation prior to dividing by 65.

Following this comment, Leona continues to demonstrate her content knowledge in conjunction with her knowledge of learning goals by guiding the student to draw important connections between her algebraic representation and the strategies presented by Susana and Kadesha in the task prompt. Leona draws on her understanding of exponential covariation defined by constant multiplicative proportions for successive n-unit additive changes and its connection with the algebraic representation of exponential functions throughout this interaction. In the following excerpt from the lesson transcript, Leona guides the student to make the connection regarding constant multiplicative proportions by highlighting how Susana's method finds a 2-ounce scale factor and the student's method finds a 1-ounce scale factor. Additionally, they verify that both of these scale factors will arrive at the same answer for the unknown height at four ounces (see Figure 4 for Leona's written work accompanying her verbal explanation).

Leona: But what is x changing by for you personally? For [Susana,] she did 36.56 over 65, right? Every time she does the multiplication…her original answer is 65 times this [ratio], right? This is what happens when x is increased by two. So, she starts at zero [ounces], then x equals x plus two, that's gonna be 65 times (36.56 over 65) times [36.56 over 65] again. But this ratio [36.56 over 65] is for how many?

Student: Two?

Leona: Yeah, so [Susana's ratio] is for two ounces.

Student: Okay. Oh, so there oh, oh, oh, oh, okay. Okay. Because [Susana's] is two. So then [multiplying again] would make it four. This would get you to four [ounces]

Leona: Which would get you your answer. But you are multiplying 65 by 0.75. How many times do you have to multiply by 65 to get [Susana's] same answer?

Student: Four.

Leona: Right? So, you have to do this [multiplication by 0.75] a total of four times, right? So that means every [multiplication of 0.75] is for how much?

Student: One [ounce].

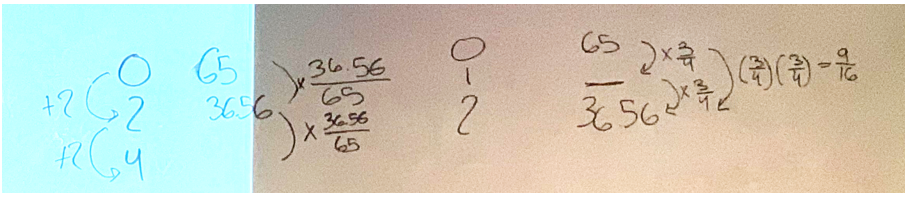


Figure 4. Leona's Written Annotations from Desmos Slides

Discussion

The overview of this finding is that Leona's content knowledge of exponential functions is used in conjunction with knowledge of the curriculum for the purpose of scaffolding interactions with students and building mathematical connections between student reasoning and key learning goals of the lesson. Copur-Gencturk and Tolar (2022) define specialized content knowledge as content knowledge used in teaching practices. In other words, content knowledge that is used for a pedagogical purpose is considered *specialized,* whereas general content knowledge refers to any content knowledge that does not rely on a teaching context. In the example above, Leona's use of content knowledge to make in-the-moment teaching decisions changes her general content knowledge into specialized content knowledge because she employs

it for a clear pedagogical *and* mathematical purpose. It is within the moments of teaching and interacting with her students that Leona uses her content knowledge, as well as knowledge of the curriculum, to inform her responses to students and how she builds key mathematical connections with students.

An important contextual piece to this finding is that the GSIs are provided with PD accompanying each active learning lesson. In these PD meetings, the GSIs are introduced to the lesson's learning goals and how the tasks are set up. They are given opportunities to work through the lessons themselves and discuss areas where they think students might need additional support. Lastly, they are introduced to common strategies that students may use and guided through collaborative brainstorming on ways to respond to students that align with the learning goals for the lessons. Leona actively contributed during these PD meetings by sharing her experiences with students from the prior semester and her strategies for implementing the active learning lessons. This is important because knowledge of the curriculum and content knowledge specific to these lessons was not something Leona necessarily possessed prior to her teaching. The PD providers specifically targeted this knowledge to support GSIs in their teaching practices. Additionally, the PD meetings provide a space for GSIs to develop knowledge related to teaching, such as specialized content knowledge, when they are participating in acts such as anticipating student thinking. The knowledge gained during these PD meetings and demonstrated in Leona's teaching is important when considering how the field makes theoretical distinctions between knowledge domains, such as content knowledge versus specialized content knowledge. Research focusing solely on the theoretical distinctions between knowledge domains may muddy our understanding of MKT as knowledge that is inherently demonstrated through actions within teaching practices. Specialized content knowledge has a history of being defined by a demonstration of specific teacher actions such as "building or examining alternative representations, providing explanations, and evaluating unconventional student methods" (Hill et al., 2004, p. 16). From my analysis of Leona's teaching practice, I believe that developing specialized content knowledge comes from her use of content knowledge in conjunction with other domains for specific pedagogical and mathematical purposes. In this case, her purpose was to understand her student's reasoning and respond to them in a productive way that guided them toward the mathematical learning goals of the lesson. These actions necessarily require content knowledge so Leona can provide clear explanations and correct mathematical reasoning. This study has practical implications for GSI PD because it offers insights into the knowledge that GSIs draw on and how it is used in their teaching practices. Specialized content knowledge can be developed or created by GSIs when they have been given time and space to think deeply about the mathematical content, learning goals, student thinking, and how these come together in the act of teaching.

This study aimed to understand GSIs' MKT by analyzing their teaching practices. The findings provide a lens into the overlapping and interactive nature of GSIs' MKT. The findings presented here support the implication that content knowledge becomes specialized when it is used in concurrence with other domains of knowledge for pedagogical and mathematical purposes. These findings have limitations related to the theoretical conceptualization of MKT and methods of inferring knowledge from observed practices. Future research should consider these limitations when designing studies. Further research should continue to refine the theoretical underpinnings of MKT and explore practical implications for GSI PD and related areas of active learning pedagogy in undergraduate mathematics education.

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Reconceptualizing Smartness: Student Perspectives in the Mathematics Community College Classroom

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This study investigates how students at a Southern California Hispanic Serving Institution (SCHSI) perceive and define smartness in the mathematics classroom. We employ a qualitative approach, analyzing narratives from an open-response survey question that asked students, "What do you think it means to be smart at [SCHSI] in math?" The findings challenge the traditional binary categorization of dominant and counter narratives, revealing a more nuanced understanding of smartness. The analysis identifies a spectrum of student perspectives, ranging from those emphasizing performance outcomes (e.g., grades, passing) to those prioritizing process-oriented approaches (e.g., understanding, effort, seeking help). The results underscore the importance of recognizing and valuing diverse forms of smartness in mathematics education, particularly within community college settings serving underrepresented student populations. By fostering an inclusive learning environment that acknowledges and supports various conceptions of smartness, educators can empower students to develop a positive mathematical identity.

Keywords: Smartness, Mathematics, Community College, Hispanic Serving Institution, Dominant Narratives, Counter Narratives, Student Perspectives

Vignette

> In Ms. Fernandez's precalculus class, students were engrossed in a test. One student finished early, standing up and turning in their paper. The teacher looks at the student and says "nice work!" This action triggered a wave of anxiety in other students, who interpreted the early completion as a sign of superior intellect and began to question their own abilities. Many students began to focus on the unfinished portion of their own test, feeling increasingly overwhelmed and doubting their own smartness.

Introduction

As educators, we possess the authority to shape or influence students' perceptions of themselves and their relationship with mathematics (Turner et al., 2013). Considering the relationship a student has with mathematics is very important to a student's identity and the role colleges serve in giving access and opportunity to STEM careers. Investigating how students learn mathematics in college, community colleges, in particular, is especially relevant to how college students are enculturated into mathematics.

Within the field of STEM specifically, the President's Council of Advisors on Science and Technology reported in 2012 that fewer than 40% of students in U.S. colleges who begin a STEM degree graduate with a STEM degree; for women and students of color who entered college as STEM majors, this number was only 20%. Indeed, community colleges alone educate half of all undergraduate students in the United States (Down et al., 2006, as cited in Mesa et al., 2014) and nearly half of all undergraduate mathematics students (Rodi, 2007, as cited in Mesa et al., 2014). Given the community college enrollment of minoritized learners and their experiences

in mathematics classrooms, more broadly, understanding these students' perceptions and experiences about what it means to do math and who can do math is valuable.

Of particular interest in this study is how students at a community college perceive smartness in their mathematics classes. In this study, we analyze narratives in an open response question from a survey meant to capture students' experiences at a Southern California Hispanic Serving Institution (SCHSI) to address the following research questions: The research question that guides this study is: *How do SCHSI students define smartness in mathematics?*

Literature Review

The way we position students within mathematics learning environments directly impacts their learning process and their self-perception as mathematics learners. Recognizing students as holistic individuals with lives extending beyond the classroom is important for understanding their identity, which is crucial for effectively supporting their success both within and outside these learning environments (Wilkes & Ball, 2020). Positive dispositions towards mathematics– where one's identity is in harmony with the field as opposed to clashing with it– can lead to persistence, agency, and confidence, all of which are useful qualities for navigating everyday life (Bishop, 2012; Brown, 2004). A key part of one's identity in mathematics is understanding the role of intelligence.

Historically, intelligence was quantified psychologically through IQ tests, which were developed in the early 1900s to identify individuals who might need additional support (Carroll, Kramer, & Dringenberg, 2019). Once adopted in the US, intelligence became a construct that was viewed as innate and fixed meaning it couldn't be changed over time (Dweck & Molden, 2000). Though not initially intended, this ultimately became a way for white privileged people to uphold their power and create and reinscribe beliefs and narratives about the intellectual inferiority of African Americans (Hatt, 2012; Darby & Rury, 2019). This perspective reinforced the idea that to be smart, used interchangeably with intelligence, means to perform identities that meet the standards of white middle-class males (Carrillo & Rodriguez, 2016). We use smartness to contrast the idea that intelligence is something that is innate as a social construct.

Smartness: A Theoretical Framework

Though smartness is something that is often perceived as "non relational, impersonal, rational, and scientifically objective," (Hatt, 2012, p. 456), many researchers consider smartness to be a social construct that give rise to the existence of its "denigrated other" (Leonardo, 2013, p. 2008) through the process of stratification and oppression of marginalized groups. When an individual is assigned smartness by an institution, it automatically affords them human capital (Hatt, 2012). Furthermore, because being good at mathematics is so intrinsically linked to smartness, and smartness is a prerequisite for personhood under the racial contract, the perception and beliefs surrounding smartness in the math classroom is of the utmost importance. One way to disentangle the features of smartness as a cultural construct is to explore the narratives surrounding it (Leonardo, 2013).

Smartness can be categorized as a dominant narrative or a counternarrative. We define dominant narratives to be the ways in which smartness often gets defined narrowly and often negatively affects learning outcomes. One dominant narrative found commonly in the literature surrounding smartness in math is that students must quickly and correctly identify solutions (Dunleavy, 2018; Boaler & Staples, 2008; Wickstrom, 2015). The most common dominant narrative is the attainment of good grades (Boaler & Staples, 2008; Louie, 2018; Wickstrom,

2015); Berry, 2008; Chang, 2017). Bishop (2012) and Louie (2018) found that a common narrative linked to smartness in mathematics is that of students not seeking help. Reinholz (2021) also found that it is common for students to perceive intelligence in mathematics to be linked to innateness.

Counternarratives are defined as the ways through which historically oppressed populations tell their own stories in order to challenge dominant narratives (Delgado, 1998, as cited in Reinholz, 2021). Dunleavy (2018) identified the ability to help one another as a disruption to dominant narratives. Bishop (2012) found that "Needing help, making mistakes, and asking questions can be perceived as face-threatening acts if those actions are seen as inconsistent with mathematics identities of being smart" (p. 53). Reinholz (2021) and Wickstrom (2015) found that some students were able to shift narratives of mathematics being innate toward the recognition of the importance of hard work.

We argue that dominant and counter-narratives are not only instantiations of the ways in which smartness is enacted, but they can also be defined as attending to process and/or outcome, which nuances how smartness is enacted. While the construct of intelligence or smart is not new, there is a dearth of research that investigates the construct of smartness in higher education explicitly and particularly in community college. In studies that do, they tend to focus on engineering and other STEM-related fields. This work contributes to the field by focusing on smartness in mathematics in the context of community colleges and addresses the nuance of how students describe what it means to be smart in mathematics classes.

Methods

Participants and Context

The study took place at SCHSI, a comprehensive two-year community college (Blind, 2018). The community college serves an ethnically, racially, and culturally diverse community in which almost 85% of the student body is from minoritized groups and Latinx students make up over 60% of the total enrollment. Despite the large number of Hispanic students, 244 STEM degrees were awarded, with only 127 (52%) of them being awarded to Hispanic students. This study is part of a larger project that aims to understand the impacts of gateway math course reform at SCHSI as a result of state legislation.

Participants included students taking mathematics classes taught by five instructors who participated in PD sessions facilitated by the [blinded] professional development team. The students were enrolled in one of the following courses: elementary statistics, pre-calculus, college algebra, and mathematics for general education majors. 103 students completed the survey. Sixty percent of the respondents identified as male and 40% as female; 63.8% of the respondents identified as Hispanic, 16.3% identified as Asian, 11.4 % identified as White, and 7.6% identified as Black; 49% of participants identified as first-generation (i.e., neither parent nor guardian completed a Bachelor's degree); and 40% of the participants declared or intended to declare a STEM major, 39% of the participants were non-STEM intending majors, and 19% were unsure.

Data Collection

Students were given a survey designed to capture their experiences in math classes. The initial survey used was modified from the Student Postsecondary Instructional Practices Survey for Mathematics (SPIPS-M). To increase students' response rates, the first author modified the initial survey used on the project to clarify questions that were ambiguous and to include open-

ended questions that focus on students' identity. The free response question being analyzed from this survey is "What do you think it means to be smart at SCHSI?" The five instructors who participated in professional development were asked to distribute the survey via email to their students. Each instructor determined incentives for students to complete the survey. Some instructors provided no incentives. Other instructors awarded points to students (i.e., extra credit on their test score). The exact choice of incentive depended on each instructor's preference, and some instructors encouraged students to complete the survey during class time, which was intended to attain a high response rate. All surveys were administered online through Qualtrics.

Data Analysis

Coding. Open coding in MaxQDA 2024 (VERBI Software, 2024) was used to analyze all data. New codes were added and old codes were refined as coding proceeded through multiple rounds. The development and use of a coding scheme were organized around the themes of smartness. Three coders collaborated, utilizing a spreadsheet to manage the data throughout seven rounds of coding. Some examples of codes that emerged from the data were: (a) good grades; (b) participation; (c) making mistakes; (d) memory; and (e) application. Some of these codes were in alignment with dominant narratives such as good grades and memory. In contrast, codes such as participation, making mistakes, and application were in alignment with counternarratives.

As we delved deeper into the analysis, we observed greater nuance than anticipated within the binary categorization of dominant and counternarratives, specifically with respect to process and outcome. To better capture this complexity, we created an axis to map the range of responses, expanding our understanding beyond a simple dichotomy of dominant and counternarratives. We then created a heat map to visualize the density and distribution of student responses across the different categories and identify areas of concentration or overlap (Figure 1).

Creation of a coordinate system. In our analysis, we found that responses involved process and/or outcome. Thus, we established a two-axis coordinate system, one of which was a continuum of process (e.g., helping others, working hard) and the other a continuum of outcome (e.g., good grades, passing class). We defined process as a dynamic and active nature of mathematical learning and outcome as a means to an end. We created a coordinate system to visualize this, with the horizontal axis representing the process spectrum from dominant to counter-narratives – left to right, and the vertical axis representing the outcome spectrum in the same way – bottom to top.

Along the process continuum (horizontal axis), the far end on either side represents the most extreme narrative associated with either dominant or counternarratives on process. For example, on the left extreme (dominant narrative) along the process axis are codes associated with learning quickly and memorizing. On the far right side along the process axis (counter narrative) are codes associated with making mistakes and helping others. Similarly, the outcome continuum (vertical axis) is structured the same way. For example, on the top of the outcome axis (counter narrative) are codes associated with applying concepts. At the bottom of the axis (dominant narrative) are codes associated with getting good grades and innateness of mathematical ability.

Each response was plotted in this coordinate system using a specific method. First, the core code of each response was identified and classified as either process or outcome-focused, and whether it aligned with dominant or counter-narratives. This initial classification determined the starting point on the coordinate system. The placement was then refined by considering

additional codes present in the response, which could shift the point along either axis. For example, the response "Passing your classes with high grades, being able to conceptualize problems and follow along with lectures" was coded as the core code of good grades (dominant outcome), but then was also coded with understanding and participation (both of which were counternarrative process). This ultimately landed the response in the lower third quadrant. To ensure consistency, we continually referred back to previously plotted responses in the same area, comparing their codes to maintain a cohesive interpretation of the data.

Findings

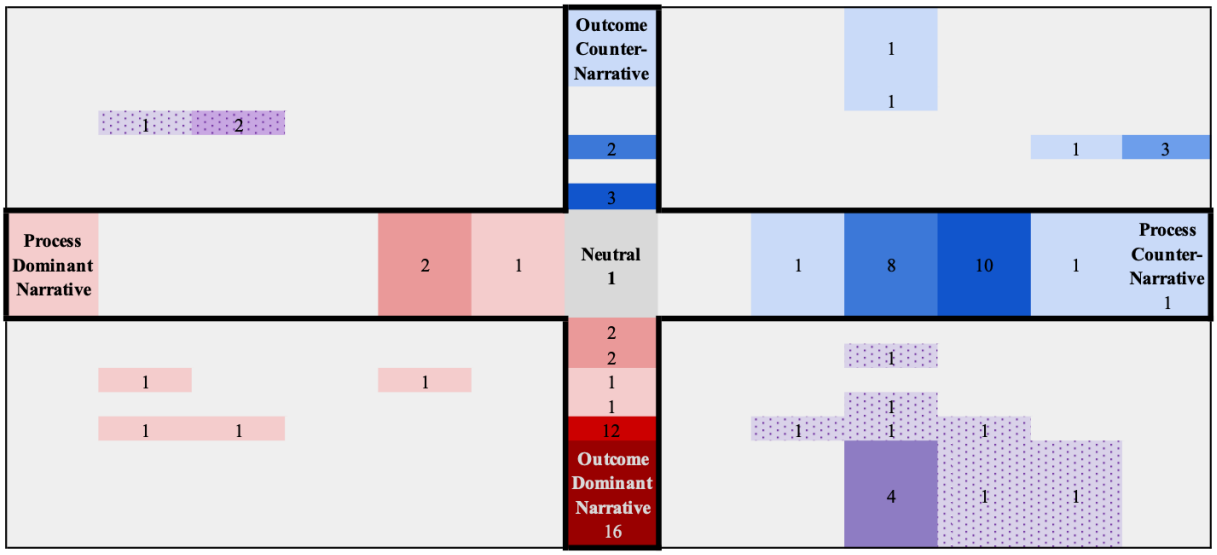


Figure 1. Heat Map generated from the responses

Axis

The highest concentration (N=28) of responses clustered along the axes themselves, rather than within the quadrants. This indicates a prevalence of simpler responses that primarily focused on either process or outcome, but not both in a complex way. Notably, the lower half of the outcome axis, towards the dominant narrative, had the highest density of responses. This signifies that the majority of responses were solely outcome-oriented and aligned with dominant narratives. For example, "getting good grades," or "Being able to get straight A's with ease." Additionally, the right side of the process axis was populated (N=18), signifying that many of the responses were solely process-oriented and aligned with counter narratives. For example, "To be willing to put in the work and learn."

Quadrant I

Quadrant I aligns with responses that contain elements of counter process narratives with outcome counter narratives. For example, one response was "To be smart in math means to understand math and be able to teach math to others." In this response, understanding math is a process toward ultimately being able to teach math to others (the outcome). Another student stated, "Be able to understand concepts and use those concepts in combinations with others to solve problems." Here, understanding concepts is the process toward the outcome of application. Quadrant I was the second most populated quadrant (beside quadrant IV), indicating that several

of the responses were indicative of alternative conceptions of smartness from both the process and outcome lens.

Quadrant II

Quadrant II aligns with responses that contain elements of dominant process and counter outcome narratives. For example, one student stated that smartness was "memorizing equations and how to apply them." This student seems to think that memorization of math is not a means to an end (an outcome) but more so is a process toward application. This quadrant had very few responses highlighting that students who tended to have dominant process narratives were unlikely to have a counternarrative outcome narrative.

Quadrant III

Quadrant III aligns with responses that contain elements of dominant process narratives and dominant outcomes narratives. There were only four responses that were placed in quadrant III. An example of one of the responses is "What I believe what it means to be smart in math is that the student is able to learn quickly and adjust quickly and be able to get the correct answers without needing homework." Here, the process is quick comprehension and adaptation of ideas and concepts with minimal help or practice (process) that leads to getting the correct answer (outcome).

Quadrant IV

Quadrant IV aligns with responses that contain elements of the counter process narratives and dominant outcome narratives. This was the most populated quadrant. Examples of responses in this quadrant are "Being able to grasp and understand the material being taught and retaining it," and "Passing your classes with high grades, being able to conceptualize problems and follow along with lectures." In the last example, the process was grounded in understanding and following along with lectures, but it seems that the outcome was to ultimately pass the class. Responses here indicate that students had counternarrative processes with respect to their ideas of smartness, however they still relied upon dominant outcome narratives.

Discussion

The analysis of student responses with respect to process and outcome revealed a significant concentration along the axes, suggesting a predominance of simpler views of mathematical smartness, focused primarily on either process or outcome. The lower half of the outcome axis, aligned with dominant narratives, was particularly populated, indicating a widespread focus on achieving specific results, such as passing classes or getting good grades. However, a notable presence on the right side of the process axis demonstrated that several students valued alternative processes like understanding concepts, putting in the work, asking for help, or helping others.

Interestingly, Quadrant I, representing counter narratives for both process and outcome, was the second most populated. This suggests that a considerable number of students held alternative views of smartness, emphasizing understanding and application over rote memorization and high grades. Quadrant II, combining dominant process narratives with counter outcome narratives, had the fewest responses, highlighting the rarity of students associating traditional learning processes with alternative notions of smartness.

The sparse population of Quadrant III further indicates the lack of an association between dominant process and dominant outcome narratives. Finally, Quadrant IV, the most populated, revealed a tension between counter process narratives and dominant outcome narratives. This suggests that while many students may appreciate alternative learning approaches, they still may feel confined to dominant conceptions of smartness. This analysis provides valuable insights into the complex and often conflicting perspectives on mathematical smartness held by students.

Conclusion

The way that educators implicitly and/or explicitly talk about smartness in the mathematics classroom is deeply influential in the way that students then conceptualize their own smartness. As illustrated in our initial vignette, the way we communicate about smartness in the math classroom, whether implicitly or explicitly, holds significant influence over students' self-perception and learning experiences. Our research underscores the importance of recognizing and valuing the diverse forms of smartness that students bring to the mathematics classroom. With an awareness of the often conflicting narratives students have of their own smartness in the mathematics classroom, educators can empower students to develop a positive mathematical identity. This is particularly crucial within community college settings, which serve a large proportion of underrepresented students in STEM fields.

By actively desettling the traditional idea of smartness in math and emphasizing the value of hard work, conceptual understanding, collaboration, and application, educators can help students move beyond narrow definitions of smartness. This shift empowers students to develop a more inclusive and expansive understanding of what it means to be mathematically capable. This approach fosters a more equitable learning environment where every student feels recognized and supported. Furthermore, we recognize that neither process nor outcome is inherently good or bad. Students operate within a system where achieving high scores is often necessary for advancement. However, integrating process-oriented approaches can enrich their experiences, broadening their understanding of smartness beyond a singular, potentially limiting perspective.

Limitations

A key limitation of this study lies in the one-dimensional nature of the data, stemming from its reliance on a single open-response question. Without the opportunity to probe for deeper insights or follow-up with participants, our analysis relied heavily on inference and interpretation. This constrains our ability to fully capture the rich complexities and nuances of student perspectives on smartness in mathematics. Future research could address this limitation by incorporating multiple data sources or employing methods that allow for more in-depth exploration of individual experiences and beliefs.

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Mathematicians' Values and Norms Related to Algorithms

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Algorithms play increasingly important roles in mathematics as they facilitate solving complex problems both in teaching and research contexts. Nevertheless, limited research has examined mathematicians' views of algorithms or the broader goals of the mathematical community that relate to algorithms or work therewith. Based on interviews with nine discrete or computational mathematicians, we highlight three values upheld by a variety of norms related to algorithms. Of note, some of these values and norms appear to be in tension.

Keywords: norms, values, algorithms, mathematical culture

The mathematical community holds many views about what it means to do mathematics. While research has previously examined individual mathematicians' beliefs about what it means to do and teach mathematics (e.g., Rupnow, 2023; Woods & Weber, 2020), recently new attention has been paid to the values and norms of the mathematical community (Dawkins & Weber, 2017; Rupnow & Randazzo, 2023; Vroom et al., 2024). These works have focused on mathematicians' perceptions of how particular tools like proof (Dawkins & Weber, 2017) and definitions (Rupnow & Randazzo, 2023; Vroom et al., 2024) permit particular values to be upheld via norms of practice in the mathematical community. To build on this work, we examine a third tool, algorithms, which arguably is more central to K-14 mathematics instruction, though it remains important for research in discrete mathematics, applied mathematics, and various applications. Moreover, deductive logic plays a key role in both proof and algorithm creation, suggesting the interwoven nature of these practices. We thus address two research questions:

1. Which values and norms are revealed through mathematicians' purposes for creating and using algorithms in research?
2. Which values and norms are revealed through mathematicians' characterization of instruction around algorithms?

Background Literature

Due to the ubiquity of technology of modern life, computing and computational thinking have received increased attention in curricula. Computational thinking originally centered on problem solving in computer science contexts (e.g., Wing, 2006) but has since been expanded to consider other mathematics and science contexts in which computation is relevant. Weintrop et al. (2016) created a taxonomy of computational thinking in mathematics and science practices with four categories: data practices, modeling and simulation practices, computational problem solving practices, and systems thinking practices. Within mathematics, Lockwood et al. (2019) defined computing as a mathematical disciplinary practice, specifically "the practice of using tools to perform mathematical calculations or to develop or implement algorithms in order to accomplish a mathematical goal" (p. 4). Moreover, they highlighted many ways that interviewed mathematicians use computing, including for facilitating experimentation, testing conjectures, approximating, and visualizing. While we note computing and computational thinking are not exclusively focused on algorithms, algorithms are a component of these related constructs.

While many definitions of algorithm exist in the mathematics education literature, Fan and Bockhove (2014) combined many to state: "an algorithm is a fixed set of step-by-step procedures

for solving a (mathematics) problem" (p. 483). Of note, this definition suggests a procedural aspect, but leaves the relevant problem open-ended. Relatedly, algorithms have not always been viewed positively in mathematics education. Kamil and Dominick (1997) characterized algorithms as procedures that obscured conceptual understanding and argued against teaching arithmetic algorithms like borrowing and carrying for subtraction and addition, respectively. In contrast, Rasmussen et al. (2005) focused on the activity around the use of algorithms when characterizing 'algorithmatizing' as a mathematical practice. In particular, they characterized this activity in terms of how students learn to use and understand a generalizable procedure that is useful across varied contexts, and, in positioning this as a mathematical practice, attributed value to this activity. Nevertheless, limited work has attended to how mathematicians characterize algorithms or how they relate algorithms and algorithmatizing to the mathematical community's expectations about the goals of mathematical activity and how mathematics is done.

Theoretical Perspective

The Triadic Network of Justification (Laudan, 1984) is a theoretical framework that characterizes the conceptual connections among axiology, methodology, and factual claims in a community. Axiology relates to ethics and values, methodology relates to commonly accepted norms of practice, and factual claims relate to what is accepted as settled fact in a community. While this framework was originally designed to conceptualize the nature of paradigmatic changes in the scientific community, this framework has been applied in mathematics education to characterize values (axiology), norms (methodology), and theorems (factual claims) of the mathematical community with respect to proof (Dawkins & Weber, 2017) and definitions (Rupnow & Randazzo, 2023; Vroom et al., 2024). In this study, we build on Laudan (1984) in a similar way to conceptualize values and norms related to algorithms.

Methods

Mathematicians at R1 institutions in the United States who were listed on their departmental website as researchers of applied/computational mathematics or combinatorics (or a similar description) were emailed to find participants. The second author conducted 60-minute remote, screen-recorded interviews with nine such mathematicians. Five participants' research aligned more with applied/computational mathematics and four aligned more with combinatorics. All names are gender-neutral pseudonyms. Interview questions focused on participants' views of algorithms and their use for different populations in both research and teaching contexts.

Using MAXQDA 2020 software (VERBI Software, 2019), all three researchers initially open-coded two transcripts to create norm codes and value categories for algorithms in alignment with Laudan (1984). After discussing our open codes, we created a codebook of norms. We then re-coded at the paragraph level (i.e., came to agreement on the norms present in each paragraph of the transcript) and came to consensus on the first two interviews. Finally, two researchers individually coded each of the remaining seven interviews and then discussed and came to consensus on the norms present in each paragraph. This process aligns with codebook thematic analysis (Braun et al., 2019) in that we entered analysis with some idea of the types of codes we would use and desired to reach consensus on the norms in each transcript but did not enter analysis with specific norms (codes) or values (themes) in mind.

Results

After analyzing the data through open-coding, we obtained a variety of norms (codes, italicized) which we gathered into three values (themes): algorithms are useful for solving

problems, correctness of algorithms needs to be verified, and algorithms need to be clear to people and communicable to others. These values with supporting norms are shown in Table 1.

Table 1. Algorithm Values, Norms, and Frequencies.

Values	Norms	Frequency (n = 9)
Algorithms are useful for solving problems	Algorithms are a tool	9
	Problems/contexts drive the use/creation of algorithms	9
	Algorithms can be used to scaffold thinking/learning	6
	Students need support in writing/implementing algorithms	5
	Coding knowledge helpful for using/proving properties of algorithms	5
	Utility of implementation of algorithms is more important than proving correctness	4
	Using/creating algorithms drives research on convergence/stability properties	3
Correctness of algorithms needs to be verified	Deductive logic is important for algorithms	9
	Proving correctness of algorithms is important	8
	Students need support in proving aspects of algorithms	4
	Students need to develop intellectual need for proving aspects of algorithms	2
Algorithms need to be clear to people and communicable to others	Algorithms need to be understood (enough, depending on context)	9
	Algorithms should be presented/tested with examples	8
	Assumptions/clarity of an algorithm (to humans) is important	7
	People should know the limitations of algorithms	6
	Algorithms should be presented in pseudocode	3

Algorithm Values

Algorithms are useful for solving problems. This value looks at the utility of algorithms for accomplishing a variety of tasks. The first norm, *Algorithms are a tool* encompasses a variety of ways that algorithms are useful for accomplishing some other purpose, such as for solving a numerical problem, generating data or objects, or helping prove a theorem. We observed references to this norm by all nine participants. For example, Ashton described how algorithms could be used to generate useful objects: "[Algorithms] have several purposes. One is to construct mathematical objects or examples." Similarly, Emery stated that algorithms may be used to create an object that was known to exist but whose form was unknown: "For example, if a mathematical result proves the existence of some object or structure in a given context, then an algorithm may be devised to produce it." Foster, who works in applied mathematics, described one way algorithms can be used for proving: "A lot of the ways you prove existence of solutions is to use an algorithm. So the granddaddy one is the Contraction Mapping Principle, but there are lots of them." In particular, the Contraction Mapping Principle, also known as the Banach fixed point theorem, guarantees the existence and uniqueness of fixed points in a metric space and has varied applications such as finding solutions to differential equations and root finding.

Connected to the idea that algorithms can be used to solve problems, is the norm *Problems/contexts drive the use/creation of algorithms*, which was also coded for all participants. This norm highlights how the need to solve a problem in a specific context may promote the creation or use of an algorithm. For example, Gideon stated:

> Monte Carlo is anything that involves randomness, which includes applications in mathematics…And so the early pioneers in the Monte Carlo method were the math physicists who worked on the atomic bomb. And so the Monte Carlo simulations were going to give an indication of whether the damage was going to be astronomical or under control. And so that was one of the first users of the Monte Carlo sim[ulation]. But since then, the subject has exploded with millions of other applications.

Notice the need to examine random processes in a nuclear reaction led to the creation of new a new type of algorithm, which is now applicable in many other settings. While all but one participant described the idea that *proving correctness of algorithms is important* (discussed further below), four of the participants also stated that this proof may not be the most important aspect of some algorithms. The norm *Utility of implementation of algorithms is more important than proving correctness* specifies that the usefulness of an algorithm may be more important than definitive proof that it works as expected. For example, Foster said:

> Sometimes those are actually very hard problems to show that a certain algorithm, which seems to produce an answer, actually produces the answer you want it to produce. And it's not always the case it does.…I mean, there are algorithms in numerical analysis, for example, that I know of, where nobody knows that it produces the answer you want. It appears to, but there's no theorem that says, "Okay, you follow this algorithm, and you will get closer and closer to the answer you want."

Of note, even mathematicians will use algorithms without rigorous proofs of convergence or of other properties that would assure correctness of results.

The norm *Using/creating algorithms drives research on convergence/stability properties* spotlights that using or creating algorithms may lead to new research on proving properties of these algorithms. Three participants had statements aligned with this norm, such as Becker:

> One of our recent papers where we had an algorithm that was basically designed to describe how to find a stability region that's very complicated to get… Other authors had given condition[s] for determining the region, but they couldn't actually do the computations to find it because it involved a good bit of ergodic theory.…So we actually give the algorithm in the paper, but the proof of the result that the algorithm actually does that was the real meat of the paper and showing that we got the region that we claim we got, that this thing was actually doable.

Five participants discussed students needing support with algorithms as described with the norm *Students need support in writing/implementing algorithms*. For example, Harley said:

> In some ways, algorithms can be more concrete for students to understand in theory. But in practice, as I say, students will struggle with algorithms. Even though it's something that you can actually compute through steps, it's hard for them to grasp the whole thing and it gets very tedious if you actually want to work out the mechanics.

The final two norms look at how algorithms can also be helpful for learning, and types of knowledge that may be useful in working with algorithms. The norm *Algorithms can be used to scaffold thinking/learning* was coded in six respondents' interviews and describes how algorithms are helpful for scaffolding the learning process. For example, Devin stated:

> What the algorithm does, the algorithm is going to allow you to compartmentalize what is

> happening, what is it going to, how is it getting there, and where are you going. It helps you with creating that pattern. It helps you to create that sequence of understanding so that what you want to happen, there is a logical process to it, right?

Here, Devin described how algorithms can scaffold an approach to a problem by breaking it into smaller parts that can be more easily understood and connected to each other.

Finally, the norm *Coding knowledge is helpful for using/proving properties of algorithms* was part of four participants' responses. For example, Devin said:

> This is when our plan meets the real world….So this is when we use our software, right? So we translate our algorithm into a mathematical software, right? So I may use MATLAB or I may use Mathematica or something like that. And I may find these numerical [simulations] that approximate the clinical information.

Here Devin emphasized the importance of being able to translate a mathematical algorithm into computer code so it could be solved in a large-scale, real world application.

Correctness of algorithms needs to be verified. This value focuses on the importance of deductive logic and aspects of proof to algorithms. The first norm, *Deductive logic is important for algorithms*, was part of every participant's response. For this norm, participants described how algorithms can intersect with deductive logic. Ashton provided the following example:

> I want to construct a big data set....So without thinking about what's going to happen at each individual step, I'm going to think like, "First, I need a function that does this, and then the output of that function is going to go into another function that does this. And I need to range over this set of inputs or whatever," and you put it together kind of in this modular way.

Here, Ashton described building an algorithm, step by step, following a deductive process.

While some norms of the prior value especially centered utility, participants also said that it is important to prove correctness of algorithms, coded with the norm *Proving correctness of algorithms is important*, which was mentioned by 8 participants. Ira described this in the following quote: "A new algorithm has got to be analyzed for stability and accuracy before it's at all useful. I think that's just one of the essential steps in developing an algorithm.

Similar to the norm about needing to support students in writing algorithms, the norm *Students need support in proving aspects of algorithms* centers the proving aspects. Four participants were coded with this norm, such as Emery, who said:

> When students are in a mathematics course in graph theory, what they're having difficulty with is the proofs, generally. And running through an example subject to an algorithm is not so much where they find the difficulty but rather something that they use to help them with difficulties in other aspects of proof.

Notice that Emery viewed proving as generally more difficult for students than understanding algorithms, at least in their context of graph theory.

Although the previous norm noted that students need support in proving, two participants expanded on this idea in the norm *Students need to develop intellectual need for proving aspects of algorithms*. Becker described:

> …when [computer science students] started having to prove the correctness of the algorithms, they didn't like it at all which I was like, "This is a senior-level math class. Mathematicians prove things, you've got to—because I knew I had CS majors, I was like, You're going to have to prove things in here just to help you understand."

Here, Becker described computer science students as not being as interested in proving properties of algorithms, and thus a need to communicate the importance of proof to algorithms.

This speaks to the potential tension between communities in the role of proof in algorithms.

Algorithms need to be clear to people and communicable to others. The final value focuses on the clear presentation and communication of algorithms. The first norm supporting this value is *Algorithms need to be understood (enough, depending on context)*, which was part of every participant's interview. Gideon illustrates this norm in the following:

> In a calculus course, the best you can hope for is illustrate an algorithm with some pictures such as gradient descent or tangent slopes or splines. And so the picture is often the link between the theory on one side and the program on the other side.

Here, Gideon described students as unlikely to fully understand an algorithm, but with the aid of illustrations, able to understand it enough for the course. In research, sometimes mathematicians could also limit which parts of an algorithm they understand, as Devin described:

> Anytime I am into a system, I try to break it because when you figure out a problem, then you can figure out why is this a problem. And I could create more algorithms from understanding that problem. And that's how you get an algorithm from an algorithm because there [is] embeddedness in things, right? And so why was that embedded? And that's how you unfold the different models. What I'm saying is someone else may create a model which you have no idea why they created that model.

Here Devin described the importance of understanding why aspects of an algorithm work while acknowledging that not all details of the design process need to be known. In contrast, Cooper set a high standard for creating new algorithms in mathematical research:

> A colleague of mine [is] designing spectral methods that give better results for certain stiff differential equations. That's where the ball game's totally different because that's where you really have to analyze how does my code [do] what it does, why does it do it? How is it superior to what the status quo is? And is it only superior for these equations, or is it really something that could potentially be the next upgrade for MATLAB or something like that?

Cooper noted that when generating a new algorithm, it is important to establish whether and how the new algorithm is better, as well as determine some sense of the contexts in which it is better.

All but one participant had responses coded with the norm *Algorithms should be presented/tested with examples*. Ira explained this as follows:

> Oh, you can never build a reliable code without testing it on a problem where you know the answer. This is something I drum into my students. We need analytical solutions as benchmarks for the code that we're developing in order to be able to solve systems where we don't know what the answer is.

Here, Ira described the need for testing an algorithm with examples and known answers, so it may be used in cases where the answer is not known. Others highlighted the utility of examples for understanding an algorithm, such as Ashton:

> If something's just described verbally in a paragraph, that's always a real pain. And lots of mathematicians who don't do so much computation will describe things just like in this big block of a paragraph or two that's just impossible to read. So the best thing is if you put it in pseudocode, and then if the authors or if wherever you're getting the algorithm from hasn't provided an example, then you start with the simplest example you could possibly come up with and then just slowly ratchet up the level of complexity until … what should happen as you do that is that each step of the algorithm becomes clear why it's needed. So sometimes you'll have certain loops in an algorithm or certain branches that don't kick in unless you encounter a certain situation …And so for a while,

> you'll be like, "I don't know why this part's here," and you just need to find an example where it triggers it. And a good writer will give you those examples.

Here Ashton noted that good writers assist readers by providing examples that illustrate why all parts of the algorithm are useful. This segment also was coded with the norm *Algorithms should be presented in pseudocode*, which was coded in three participants' interviews, because it notes that pseudocode is easier to translate to a working algorithm in the computer language of one's choice (or to understand for oneself) than a large block of text.

Another norm, the *Assumptions/clarity of an algorithm (to humans) is important*, was observed in seven participants' interviews. Becker for example, said:

> So you have to know the constraints upon the problem itself for which this will actually give you the solution because it's generally not always going to give you all possible solutions.…So it's the same thing with algorithms. You have to understand the conditions upon which classes of problems this algorithm will actually solve.

Here Becker described the importance of looking at the assumptions of an algorithm, to make sure your system satisfies the given conditions for which the algorithm works.

The final norm is *People should know the limitations of algorithms*, which was coded in six participants' interviews. Ashton illustrates this norm with the following:

> So sometimes people will just be like, oh, okay, great, we can solve this with this algorithm. And they don't think about the fact that the algorithm doesn't see the quality of the data. And oftentimes, the algorithms that are used in science and public policy and stuff really are biased by not just the data, but by the weights that are chosen, like, you set these parameters.

Ashton noted that the quality of the outputs of an algorithm depends on the quality of the inputs as well as the weights given to various criteria, all of which can fundamentally impact the validity of results.

Discussion

This study examined nine mathematicians' norms and values related to algorithms in the contexts of research and teaching. We observed three values: algorithms are useful for solving problems, the correctness of algorithms needs to be verified, and algorithms need to be clear to people and communicable to others. Throughout the interviews, the utility of algorithms for a variety of purposes was very clear. Many uses were highlighted including using algorithms to prove or test conjectures, create examples, or find specific solutions. Participants also expected algorithms to produce correct outcomes, both based on their deductive structure and additional work focused on providing proof that algorithms have desirable properties. Clarity and communication were also viewed as important aspects of algorithms, though participants did vary in how much understanding or clarity is deemed sufficient, especially depending on context.

However, unlike prior research examining mathematicians' values and norms related to proofs and definitions (Dawkins & Weber, 2017; Rupnow & Randazzo, 2023), we saw some disagreement on the importance of some values. While some participants centered the utility of algorithms as most important, with some suggesting proof was desirable but not necessary (especially for solving real-life problems), others believed proof of accuracy or particular properties was paramount. This may in part stem from mathematicians viewing "algorithms" as encompassing different things, based on their mathematical training, common uses for algorithms in their work, and the expectations of the stakeholders they serve. Nevertheless, more research is needed to better understand mathematicians' values related to computing or algorithmatizing and how they interrelate with their view of other mathematical practices.

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Undergraduate Students' Statistical Literacy in Media Contexts

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As the uses of data in society continue to expand, it is vital to prepare students to engage with data in meaningful and critical ways. This study used a course-wide assessment in an introductory statistics course to explore undergraduate students' statistical literacy in media contexts based on Watson and Callingham's (2003) six-level construct. Statistical analysis found student performance on non-media tasks had limited association with performance on media-based tasks. Students demonstrated higher levels of statistical literacy when provided with specific prompts. Findings suggest the need to develop students' propensity for critical questioning when engaging with statistics outside classroom settings through intentional incorporation and discussion of media items in introductory statistics courses.

Keywords: Statistics, Introductory Courses

For most adults, the news media is the primary source of information about social and scientific issues (Bissonnette et al., 2021). Statistical and mathematical products in the news take on many forms, requiring multiple inter-related skills to appropriately comprehend and evaluate (Gal & Geiger, 2022). As the pervasiveness of data in everyday life continues to increase, statistics educators are exploring how to prepare students to engage with data in meaningful and critical ways (Burrill & Pfannkuch, 2023). Undergraduate introductory statistics courses provide a natural place where the development of these skills could occur. Driven by this need, the purpose of this quantitative study was to understand the statistical literacy of students in an introductory statistics course within media contexts. Statistical literacy is defined as a multi-leveled construct building from idiosyncratic task steps (Level 1) where no mathematical or statistical thinking is occurring, to critical mathematical task steps (Level 6) which involve critical engagement and questioning with the statistics in context (Watson & Callingham, 2003).

Background

While early statistics education research focused on resolving persistent challenges in developing reasoning about specific statistical ideas (Garfield & Ben-Zvi, 2007), current trends looking to the future of statistics education research include broader issues of content, pedagogy, and ethics (Burrill & Pfannkuch, 2023). One of the four emerging trends Burrill and Pfannkuch identified is social statistics, which encompasses the importance of attending to risk, critical literacy, and communication in contexts such as the media and advocacy. Burrill and Pfannkuch noted that while statistical literacy from the perspective of societal needs was recognized by Gal (2002) as well as Watson and Callingham (2003), the ideas gained limited attention. Existing research on statistical literacy spans a variety of methods and foci. Previous quantitative studies typically used multiple choice tests to evaluate the statistical literacy of students in introductory statistics courses (e.g., Chance et al., 2022). Such studies frequently show students struggle to demonstrate mastery of topics such as sampling, variation, and study design even after completing a statistics course (delMas et al., 2007; McLauchlan & Schonlau, 2016).

Many statistics education researchers intentionally investigate statistical literacy in media contexts. Watson (1997) advocated for the use of media-based tasks in courses to assess statistical thinking in social settings outside the classroom. Other studies utilized media-based

tasks to investigate student statistical literacy in general (e.g., Kaplan & Thorpe, 2010) or understanding of specific topics such as sampling (Watson & Moritz, 2000). Budgett and Pfannkuch (2010) found incorporating media into classroom activities and assessments to be effective for college students in non-quantitative majors.

Simulation-based curricula (SBC) have been growing in popularity in introductory statistics courses (Tintle et al., 2015). SBC provide an alternative to using theoretical distributions to calculate confidence intervals and p-values by simulating repeated samples to generate bootstrap and randomization distributions (Lock et al., 2021). Advocates of SBC assert they allow students to see the logic and power of statistical inference and find a balance between statistics as proof and statistics as magic (Tintle et al., 2015). Researchers identified improvements in students' statistical literacy when they engage with SBC compared to engaging in more theory-based curricula (e.g. Chance et al, 2022; Maurer & Lock, 2016). Other studies noted how simulation tools contributed to student understanding of sampling (Findley & Lyford, 2019), p-values (Reaburn, 2014), and inference (Noll et al., 2023).

While current research calls for increased use of simulation-based methods in introductory statistics courses (Tintle et al., 2015), research is less clear about how SBC help prepare students to engage with statistics in the increasingly complex media environment. This study brings together research on statistical literacy, SBC, and statistics in news media to understand how students in introductory statistics courses utilizing SBC are prepared to engage with statistical information in media settings. Specifically, this study addresses the research questions:

1. At what level of statistical literacy do students in an introductory statistics course apply statistical topics when critiquing statistics in the news media?
2. What relationship exists between students' scores on non-media-based tasks and their levels of statistical literacy on media-based tasks?

Conceptual Framework

This study adopted Watson and Callingham's (2003) six-level construct of statistical literacy, grounding statistical literacy in the need for understanding, interpreting, and communicating in real-word contexts such as the news media. Individual levels are characterized by the types of task steps students take while engaging with statistical tasks. Progressing to higher levels of the construct statistical constructs requires more nuanced statistical thinking, clearer justifications or reasoning, and deeper engagement with the task's context. Level 1 and Level 2 involve mostly non-statistical thinking, instead relying on personal experiences and intuition. Level 3 and Level 4 utilize statistical terminology and arguments with increasing amounts of justification and connections to the task's context. Level 5 and Level 6 incorporate critical engagement and quantitative interpretations concerning the data and conclusions.

This study's conceptual framework draws on Gal and Geiger's (2022) categories of statistical and mathematical products (StaMPs). Gal and Geiger conducted a content analysis of over 300 media items and identified nine categories that encompass entities appearing in media items. This study focuses on five of the nine categories to investigate students' statistical literacy of statistical topics supported by SBC: (a) descriptive quantitative information; (b) models, predications, causality, and risk; (c) data quality and strength of evidence; (d) demographics and comparative thinking; and (e) critical demands. The nine categories provide a basis for what students should be prepared for when engaging with statistics in news media contexts.

Methods

This study took place at a mid-sized, regional university in the Rocky Mountain Region of the United States. The university is an emerging Hispanic Serving Institution with a student population that is 62% White and 24% Latine/Hispanic with 42% of incoming students being first generation college students. Introductory statistics courses utilize a textbook and materials which address simulation-based methods for calculating confidence intervals and p-values (Lock et al., 2021). All 189 students enrolled in seven sections of introductory statistics were asked to complete an assessment as part of the course for a completion grade. Overall, 123 students (65.1%) completed at least one part of the assessment and consented for their responses being included in the study.

Each participant completed a three-part assessment using Qualtrics during the 12th week of the semester. Part 1 consisted of six multiple-choice questions taken from the Basic Literacy in Statistics (BLIS) assessment (Ziegler & Garfield, 2018). The BLIS assessment was selected because it was developed for introductory statistics courses at the postsecondary level that include SBC. Selected items were not embedded in media contexts and assessed participants' understanding of three statistical topics that are encountered in the news: sampling, variation, and confidence intervals. Variation and confidence intervals were selected as they are topics where SBC have been shown to be particularly effective (Chance et al., 2022). Parts 2 and 3 were media-based tasks asking participants to read a headline and short article containing information related to the same three statistical topics from Part 1. Part 2's article discussed results from a recent drug trial and asked participants to evaluate the presented argument but gave no additional prompts about what to talk about (i.e., unprompted). Part 3's article reported results of a poll and asked participants to respond to prompts referring to specific features and statistics within the article (i.e., prompted).

Participant responses were scored to facilitate statistical analysis. Multiple choice items were scored on correctness and given a zero or one, forming six variables (MCQ1, MCQ2, etc.). Topic scores (i.e., MC_{Samp}, MC_{Var}, and MC_{CI}) were calculated based on the number of correct responses on each topic (out of two) and an overall multiple-choice score (i.e., $MC_{Overall}$) indicated the total number of correct responses (out of six). Scores for the unprompted (UP) and prompted (P) media tasks were determined using Watson and Callingham's (2003) six-level construct of statistical literacy. For each task, participant responses were assigned a level score from 0 to 6 for each of the three statistical topics based on the statistical literacy level their response addressed (i.e., UP_{Samp}, UP_{Var}, UP_{CI}, P_{Samp}, P_{Var}, and P_{CI}). A score of 0 was assigned to responses that did not address the statistical topic at all, either because it was not mentioned for the unprompted task or left blank for the prompted task. Based on the three topic scores, each participant's response was assigned an overall level of statistical literacy (i.e., $UP_{Overall}$ and $P_{Overall}$). In total, there were 18 variables: (a) six individual multiple-choice question scores; (b) three topic scores for sampling, variation, and confidence intervals on each part (nine total); and (c) three overall statistical literacy scores (one for each part).

Response scores were imported into R for statistical analysis. Summary statistics were calculated for each variable. Since scores were not normally distributed, variables were treated as ordinal when conducting statistical inference. Non-parametric tests such as the Wilcoxon Signed-Rank test and Kruskal-Wallis Rank test were performed to compare scores. Correlational analysis was performed to see how different variables related to each other. Spearman's rho correlations were used due to the non-continuous nature of the variables (Agresti, 2007). Additionally, analysis was performed to determine if multiple-choice questions were strong

predictors of statistical literacy level on media tasks. Ordinal logistic regressions were used to incorporate the ranked property of the variables and explore how multiple-choice responses predict the odds of higher statistical literacy (Osborne, 2017).

Results

Performance on the multiple-choice questions ranged from 69.1% correctly answering MCQ1, which dealt with identifying a sample and population, to 16.3% correctly answering MCQ6, which asked participants to interpret the center of a confidence interval. Most students answered one to three questions correctly with the mean $MC_{Overall}$ score being 2.285 correct out of six. No participant answered all six questions correctly. Participants had the most correct responses on sampling questions, followed by variance and, lastly, confidence intervals.

Table 1provides a summary of participant scores on the unprompted and prompted media tasks. For both tasks, participants demonstrated higher mean statistical literacy scores related to sampling than for variation or confidence intervals. The highest mean was P_{Samp} (M = 3.546) and the lowest mean was UP_{CI} (M = 1.822). Overall, participants demonstrated lower levels of statistical literacy on the unprompted task compared to the prompted task on all three statistical topics and the overall score. Wilcoxon signed-rank tests were performed to assess whether the medians of the distributions of scores on the unprompted and prompted tasks were equal. Results of the tests indicated significantly greater scores on the prompted tasks for sampling ($p < .001$), variation ($p < .001$), confidence intervals ($p = .002$), and the overall score ($p < .001$). Participants also had more zero scores on the unprompted task compared to the prompted task, indicating participants were not addressing certain pieces of information within the unprompted task. For example, 40 participants received a zero on UP_{CI} as their response did not include the comparison of values discussed in the article.

Table 1: Summary of Statistical Literacy Scores on Media Tasks

Task	Sampling		Variation		CI		Overall	
	Mean	*SD*	Mean	*SD*	Mean	*SD*	Mean	*SD*
Unprompted [a]	2.237	1.616	2.051	1.524	1.822	1.583	2.839	0.961
Prompted [b]	3.546	1.314	2.944	1.478	2.537	1.632	3.389	1.134

[a] n = 118. [b] n = 108.

To investigate the relationship between participant scores across topics and task type, Spearman rho correlations were calculated for all scores (see Table 2). Analysis focused on the correlations between non-media task scores (i.e., multiple choice) and media task scores to see what relationship existed. All correlations were weak, with the strongest correlation being $\rho = .260$ between $MC_{Overall}$ and UP_{CI}. All correlations were positive, indicating greater scores in one area were associated with greater scores in other areas. Eight of the 32 correlations were significant. MC_{Var} was not significantly associated with any of the media-tasks scores. $MC_{Overall}$ had many of the strongest correlations, including significant correlations with the $UP_{Overall}$ ($\rho = .216$) and $P_{Overall}$ ($\rho = .226$). The only multiple-choice topic score significantly correlated with a corresponding media-task topic score was for MC_{CI} and UP_{CI} ($\rho = .193$).

Due to potential similarities between the prompts in a multiple-choice setting and the prompted media task, additional analysis focused on the relationships between those sets of questions. For example, MCQ1 asked students to identify the sample and population of a provided context. Similarly, the first prompt of the prompted media task asked students to

identify the sample and population. Identifying both correctly would correspond to a statistical literacy score of at least Level 3 for P_{Samp}. While participants who answered MCQ1 correctly had a higher mean P_{Samp} score (M = 3.688 vs. M = 3.143), a Kruskal-Wallis rank-sum test found no significant difference in the medians of the two groups (p = .058).

Table 2: Spearman Correlation Matrix of Multiple-Choice Scores and Media Task Scores

		Unprompted				Prompted			
		Samp	Var	CI	Overall	Samp	Var	CI	Overall
Multiple Choice	Samp	.156	.144	.157	.192*	.145	.065	.218*	.234*
	Var	.139	.092	.147	.124	.113	.071	.043	.063
	CI	.179	.003	.193*	.073	.087	.068	.083	.157
	Overall	.248**	.127	.260**	.216*	.186	.118	.169	.226*

*p < .05. **p < .01.

The significant correlation between $MC_{Overall}$ and $P_{Overall}$ was explored further to understand the relationship between students' scores on non-media versus media-based tasks. A Kruskal-Wallis rank-sum test was performed to determine if there was a difference in the median $P_{Overall}$ score between groups based on $MC_{Overall}$ score. The test reported no significant difference (p = .131), suggesting scoring greater on the multiple-choice section did not correspond to greater distributions of statistical literacy scores on the prompted media task. An ordinal logistic regression model was fit to the two scores which similarly found different $MC_{Overall}$ scores were not significant predictors of $P_{Overall}$ scores. These results are reflected in participant scores as there were 21 participants who had lower $MC_{Overall}$ scores (score of two or less) but demonstrated greater levels of statistical literacy in the prompted media task (Level 4 or greater). Similarly, six participants had higher $MC_{Overall}$ scores (score of three or more correct) but did not demonstrate higher levels of statistical literacy in the prompted media task (Level 2 or lower).

Although $MC_{Overall}$ was not found to be a good predictor for the prompted media task, another ordinal logistic regression was performed to assess the predictive power of the individual multiple-choice questions. A binary variable for each question (correct or incorrect) was used in a model to predict overall prompted score. Table 3 includes the fitted coefficients for the regression. MCQ1 (sample and population) and MCQ5 (confidence interval estimation) were significant predictors of overall prompted score (p = .027 and p = .048 respectively). The odds ratio for MCQ1 indicates people who answer MCQ1 correctly were 2.569 times more likely to be in the next higher level of statistical literacy in the prompted task (95% CI [1.118-6.012]). Similarly, people who answer MCQ5 correctly were 2.053 times more likely to be in the next higher level of statistical literacy in the prompted task (95% CI [1.013-5.223]).

Table 3: Ordinal Logistic Regression for Multiple Choice Questions Predicting Prompted Task Scores

						95% Confidence Interval	
Question	Estimate (β)	*SE*	*t*-score	*p*-value	Odds Ratio	Lower	Upper
MCQ1	0.943	0.428	2.205	.027	2.569	1.118	6.012
MCQ2	0.603	0.458	1.317	.188	1.828	0.745	4.515
MCQ3	0.295	0.389	0.759	.448	1.343	0.627	2.893
MCQ4	-0.297	0.379	-0.782	.434	0.743	0.352	1.562
MCQ5	0.719	0.363	1.979	.048	2.053	1.013	4.223
MCQ6	-0.439	0.536	-0.820	.412	0.645	0.222	1.840

Discussion and Conclusion

The study found participants in introductory statistics courses applied statistical topics at varying levels of statistical literacy when engaging with the news media. Participants had difficulties correctly answering multiple-choice questions from a standard assessment (i.e., BLIS). The mean overall multiple-choice score was 2.285 out of six and only 16.3% selected the correct interpretation of a provided confidence interval. These results are consistent with ongoing challenges in introductory statistics courses McLauchlan and Schonlau (2016) identified where students do not always demonstrate improved statistical literacy after completing a course. While Chance et al. (2022) found SBC improved student outcomes related to topics such as variation and confidence intervals, this study shows challenges persist. Participants performed better on the sampling topic questions with more than two-thirds correctly identifying a sample and population from a provided context.

Results from the media-based tasks indicated that participants typically demonstrated significantly higher levels of statistical literacy when receiving specific prompts compared to an unprompted, open-ended question. Mean scores for the prompted task were mostly at Level 3 or higher, indicating students were more often engaging in statistical thinking and justification. However, mean scores for the unprompted task were mostly around Level 2, which corresponds to more intuitive thinking and/or incompleteness in statistical engagement. The results indicate a gap that needs to be addressed. Students have knowledge and abilities they are capable of applying but are not always doing so at the same level on their own (unprompted). For example, without prompts, several students (about one-third) did not regard key statistical information such as the comparison of the effectiveness of the two drugs being discussed in the article. In most real-life situations, such as engaging with the news media, information does not come with prompting questions. The results emphasize the need to facilitate students' development of what Gal (2002) described as *worry questions* to raise when assessing the quality of information.

Results of correlational analysis on the different task scores found only weak, positive correlations between scores on non-media multiple-choice questions and the level of statistical literacy on media-based tasks. The results indicate that demonstrating statistical knowledge in one type of task was not associated with demonstrating the same knowledge in other types of tasks. One specific example was that there was no significant difference in the distribution of sampling scores on the prompted task based on whether a participant correctly identified the sample and population in MCQ1. More broadly, about 30 students scored notably higher on one type of task compared to the other. Watson (1997) noted difficulty in students demonstrating greater levels of statistical literacy in multiple-choice settings. This study's findings indicate that multiple-choice questions may provide a limited picture of students' knowledge and abilities.

Ordinal logistic regression indicated correctly answering MCQ1 (identifying a sample and population) or MCQ5 (identifying what a confidence interval was estimating) were significant predictors in demonstrating higher overall levels of statistical literacy in the prompted media task. MCQ1 being the strongest predictor could indicate the importance of the concept. Understanding the difference between a sample and a population is a critical component of statistical inference and may contribute to students' ability to demonstrate greater levels of statistical literacy with the variation and confidence interval aspects of the task.

Limitations

The study has limitations to consider when interpreting these results. The assessment took place during Week 12 of the semester. While students had been assessed on the relevant content, students may have improved their statistical literacy related to the topics in this study by the end

of the semester. Furthermore, the study only assessed students' understanding of sampling, variation, and confidence intervals which may not represent the entirety of their statistical literacy. The assessment contributed to participants' course grade, but only as a completion grade. Therefore, not all students may have responded at a level representative of their highest ability. There are possible limitations related to decisions about scoring such as the inclusion of the zero score. Importantly, zeros indicated the topic ideas students did not engage with for the tasks; zeros did not indicate participants had no statistical literacy related to the topic.

Implications

The study emphasizes the importance of supporting students' applications of statistics in real-life settings and the practice of critical questioning. Instructors and course developers should consider how to incorporate media tasks into class activities to provide opportunities for students to practice working in these contexts. Such tasks allow for discussions surrounding how statistics are presented in the news media and how the presentation can differ from textbooks and academic journals. These practices can help prepare students to engage with statistics in social settings, such as the news media (Burrill & Pfannkuch, 2023). When assessing students' statistical literacy, instructors should consider multiple forms of assessment and contexts to provide students with different opportunities to demonstrate what they know.

Further research should continue to explore how students approach statistics in media contexts. Qualitative research such as task-based interviews can provide additional insights into students' thinking and approaches to media items. In particular, research can investigate what students pay attention to or find important in these settings and explore the kinds of resources students draw on when evaluating statistical claims. Teaching experiments or other methods could be implemented to see what kinds of tasks, prompts, and activities are most effective in promoting critical questioning and higher levels of statistical literacy.

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Students' Perceptions of Differences Between Physics and Mathematics Classes

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Student learning of physics is built on a foundation of mathematical knowledge. Accordingly, physics education research has devoted much study to students' use of specific mathematical content in specific physics contexts. More generally, though, students also experience various "cultural" differences between the (mathematics) courses where they learn this content and the (physics) courses where they apply it. Researchers have discussed these cultural differences, but to our knowledge they have not investigated whether students are aware of them, or students' beliefs about how these differences may affect their learning. We used a survey questionnaire and individual interviews to study this in a group of math-physics double majors (or majors in one with minors in the other). These students are indeed aware of many such cultural differences, cite many classroom examples, and have strategies to cope with them.

Keywords: physics education, mathematics, disciplinary cultures

Introduction

The physics education literature has extensively studied students' use of mathematics in their physics coursework (Caballero et al., 2015). Often this focuses on specific mathematical techniques in specific physics contexts, such as students' use of vector calculus in electromagnetism (Bollen et al., 2015). However, some authors have also pointed out more general "cultural" differences between mathematics and physics which likely affect students' ability to make use of mathematical knowledge in a physics context (Redish and Kuo, 2015). We interpret this idea broadly to include, for example, the standards of proof or derivation of mathematical facts, the ways that mathematical notation is used, how approximations are used and justified, and the use of potentially ill-defined concepts such as the Dirac delta function.

An illustration of such a cultural difference, in the use of function notation, is provided by "Corinne's shibboleth", attributed to Corinne Manogue (Dray and Manogue, 2002). Mathematicians and physicists are claimed to give different answers to this question: If $F(x, y) = x^2 + y^2$, then what is $F(r, \theta)$? Mathematicians should treat the function arguments as dummy variables and answer $r^2 + \theta^2$, while physicists typically assume that F represents a physical quantity that can be expressed in either rectangular or polar coordinates, and answer r^2.

Our interest is in whether students themselves are aware of such cultural differences and how they may impact their learning, and whether they have coping strategies to deal with them as they apply their mathematical knowledge in physics. We studied these questions through a survey of double majors in mathematics and physics at our institution, plus interviews with a subset of the participants.

Purpose and Theoretical Background

The purpose of this exploratory study was to learn whether undergraduate students are aware of the cultural differences in the treatment and use of mathematical content between their mathematics and physics courses, how they see these differences as affecting their learning, and what coping strategies they may employ. Our understanding of cultural differences here is consistent with that of Rabin et al. (2021), which explored obstacles to interdisciplinary communication between STEM fields arising from different practices around notation,

terminology, modeling, and rates of change. On the other hand, it is distinct from the notion of disciplinary cultures in Reinholz et al. (2019) which refers to the administrative organization of the disciplines within the university setting. We were particularly inspired by the work of Redish and Kuo (2015), who point out that "Math in science (and particularly math in physics) is not the same as doing math. It has a different purpose – representing meaning about physical systems rather than expressing abstract relationships" (p. 563). We interpret the construct of cultural differences very broadly, and indeed our students discussed a wide range of topics ranging from specific mathematical content to the teaching styles and assignments in their mathematics and physics courses. One could frame our questions in terms of transfer of learning from mathematics to physics, but we think the issues are broader than that, and not unidirectional.

The authors are two physicists and one mathematician having a postgraduate physics background. In formulating our survey and interview questions, we drew upon our own experience and the literature cited above.

Our research questions were:

1. Are undergraduate double majors aware of cultural differences between mathematics and physics, and differences in the presentation and usage of mathematical content between their physics and mathematics courses?
2. What do they cite as major differences in "common" mathematical content?
3. How do they perceive these differences as affecting their learning?
4. What coping strategies do they employ?
5. What specific classroom experiences do they cite?

Participants and Methods

We designed our study to include all undergraduate double majors in mathematics and physics at our institution. The rationale was that such students would have taken enough high-level courses in both disciplines to have experienced a range of cultural differences, content areas, and teaching styles. These would include proof-based mathematics courses beyond the basic calculus sequence. There were fewer double majors than we had hoped, so we decided to add students majoring in either discipline and minoring in the other. By email we invited each of them (36 in all) to complete a survey questionnaire in exchange for a gift card, obtaining 20 completed surveys. Following an initial analysis of these, we invited a subset of students to be interviewed in person, with another gift card as compensation. There were 11 completed interviews in all, which were audio recorded and transcribed.

The surveys asked students' opinions about specific mathematical topics that we suspected might be problematic, such as Fourier series/transforms, infinitesimals, and linear algebra; differences in notation between students' physics and mathematics courses; and differences in the standards of proof or justification between the two fields. We asked whether they have strategies for coping with differences in presentation of "the same" topic in their physics versus mathematics courses, and whether they have advice for mathematics or physics instructors about presenting such cross-disciplinary content. We also included the "Corinne's shibboleth" task since we have not seen published data on it in the literature.

Interviews were semi-structured, following a general script but allowing for detours to explore student comments in more depth. In addition to asking students to expand on their survey answers, we asked what they thought the role of mathematics in physics was; how mathematics and physics classes "feel" different; and whether seeing the same topic presented in both a mathematics and a physics class had facilitated or impeded their learning of it. We tried to keep

our questions unbiased, not showing favoritism to either mathematics or physics, and open-ended to allow students to bring up topics we might not have expected.

After an initial analysis of the surveys, we selected students to interview whose comments we wanted to explore in more depth. All surveys and interview transcripts were read multiple times by all authors, and discussed as a group. Surveys and interviews were analyzed thematically (Terry et al., 2017), coding ideas that multiple students brought up and consolidating them into general themes. Our results below also include student comments that were less frequent but worth exploring in future research.

Results

Theme 1: The Nature of Mathematics and Physics

Our students perceive mathematics to be deductive, standardized, and highly structured, whereas physics to them is more inductive, flexible, and eclectic. This applies to the nature of argumentation or proof in each field, the presentation of material in lectures, and the homework and exam problems posed. They find mathematical proofs highly structured, with logical steps spelled out in detail and named theorems cited as justifications. Derivations in physics look more like calculations, relying on known techniques from algebra or calculus without explicit justifications. When words appear in a physics derivation, they may actually interrupt the flow of the reasoning to introduce a physical assumption or approximation. Students feel that mathematical proofs contain all the reasoning needed to understand them, whereas they may have to stay after class to ask their physics instructor to explain the steps of a derivation if they want to fully understand it.

Topics presented in mathematics courses are fully developed from definitions and axioms, whereas mathematical methods used in physics may be introduced without complete justification and simply illustrated through specific examples. Students understand and accept that mathematics is used as a tool in physics, and this tool need not be fully justified from first principles.

> My math classes have been, they have, like a familiar structure … they have some motivation, they have some definitions, a theorem, and a proof … Whereas in physics I feel like it's kind of been more all over the place. Like the transitions between when somebody's giving like a heuristic argument or an actual proof of what they're trying to show, it's a bit blurry. [Interview, double major]

Students find homework and exam problems in mathematics to be quite standardized and predictable in terms of what a correct solution should contain, even across different courses. In contrast, they see wide variation in expectations across physics courses, and are unsure what methods or approximations are allowed in homework solutions:

> I can get full credit on the assignment even though I don't know why the thing I just applied works in this instance…you can BS the physics assignment and still get 100 on it, even though you don't understand why the stuff you're writing works…in a math homework assignment, if you BS the homework assignment you're not getting 100 on it…because you're going to be missing some crucial steps in your proof, and you know

you're missing those crucial steps, and when a proof is complete you feel it. [Interview, physics major]

Theme 2: Physics "Tricks"

Students referred to some uses of mathematics in physics as "tricks," which includes two distinct categories. Some tricks, such as interchanging the order of derivatives and integrals, could be mathematically justified but this justification may be omitted in class. Others are physically motivated approximations. Both were sources of confusion for students.

We were surprised that physical approximations were considered problematic by many students. Often these rely on Taylor series expansions, a topic that many students felt was insufficiently emphasized in their calculus courses. But even the small-angle approximation for a pendulum raised questions. Why not give an exact solution instead? Won't the error accumulate over time as the pendulum swings? Why not give a rigorous error bound, as would be the case in a mathematics class? Why not retain additional terms in the Taylor expansion? We suspect that students' prior mathematics classes have contributed to a general preference for exact solutions and an aversion to discarding information by approximating.

> I've taken real analysis, when you do real analysis and you're giving the approximation functions for equations in that class like the Taylor expansions, they of course give you the error and they prove why the error works and they are like, this is how we minimize the error to make sure that there is as little error as possible. We weren't given anything like that [in the physics course]… I personally find it a little bit confusing to have that stuff and then not have any explanation for why these work the way they do and like what the margin of error is and how we can minimize that margin of error, especially since it feels like it would be important. [Interview, mathematics major]

Theme 3: Preparatory Mathematics Courses

In general, students felt that it was most helpful to learn specific concepts first in mathematics courses and then to see them applied in physics. The sequencing of their mathematics and physics courses did not always allow this, however. Vector calculus in mathematics, followed by its use in electromagnetism, was cited as a successful example of this synergy. Those students who took complex variables in math prior to quantum mechanics were also pleased with the foundation they had gained. In contrast, students felt that their introductory linear algebra course did not provide enough background for quantum mechanics, and that their physics instructors were not always aware of this. Our basic linear algebra course typically gets to eigenvalues at the very end, and does not cover Hermitian operators or complex vector spaces. Also, eigenvalues in quantum mechanics are often not calculated by the standard characteristic polynomial method taught in linear algebra. Most students had not taken a second linear algebra course, which does cover spectral theory in more depth.

Many physics departments provide a "mathematical methods" course that can function as a bridge between mathematics and physics. Mathematical topics are covered in more depth than in other physics courses, but still with physical context provided for them. Along with more applied courses in the mathematics department, students did find these useful for connecting physics and mathematics. We view such courses as "trading zones," a term from anthropology that has been adapted to describe interdisciplinary settings where mathematicians and physicists, for example, might interact using a common language (Galison, 2010). Such courses have considerable potential to help students coordinate their mathematical knowledge with its physical applications.

Other Topics, Not Part of Major Themes

Infinitesimals

We anticipated that the common use of infinitesimal amounts of charge, displacement, or work in physics might conflict with students' prior experience in calculus, where limits are emphasized and infinitesimal or differential quantities may be considered meaningless on their own. Students did express some confusion about the ways in which differentials were *manipulated* in physics, which they included in their lists of "tricks," but were generally not bothered by their supposed infinitesimal nature. Some recognized infinitesimals as an alternative way of expressing the concept of limit, which could be translated into that language if required.

> I know that if I cancel them [infinitesimals] then I would be able to get an answer pretty easily, but there's a reason that I was stuck on this problem for an hour and it wasn't that I didn't see that I could cancel them, it was that I kept looking at it and saying, well why can I do that? [Interview, physics major]

Fourier Transforms

Many students cited Fourier transforms (or series) as a particularly challenging mathematical technique they encountered in physics. This seemed to be because they encountered them early, in a lower-division optics course, without prior exposure in any mathematics course. They felt that the technique was presented simply as a tool without a clear mathematical derivation. Also, although a Fourier transform has the form of an improper integral, it is rarely evaluated from the definition of improper integrals provided in calculus. Rather, "tricks" such as completing the square in the exponent are used. Some students were unfamiliar with complex numbers and confused by the switch from sines and cosines to complex exponentials. In quantum mechanics the Fourier transform converts between the position and momentum representations, but students did not always see how this interpretation was justified. Its use in quantum mechanics is also linked to the Dirac delta function, which has problematic aspects of its own for students. A mathematics major commented that since quantum mechanics violates our physical intuition, it makes sense that its mathematical tools (for example, the delta function) would also violate our mathematical intuition!

> I personally felt like there was a lot of like tricks that are used in the Fourier transform that are just kind of well-known and you just have to look up … I didn't know they were tricks, so I would like try to actually solve them. And it would take me so long, and I would like get together with friends, try to like, solve these problems … And then we just look it up, and then it's actually quite easy and it's just like a well-known fact, or whatever. [Interview, physics major]

Corinne's Shibboleth

We asked students this question in the "bare" form cited in the Introduction, without placing it in a physical context as did Dray and Manogue (2002), thinking that this might eliminate possible bias toward the "physics" response. However, we did add this follow-up question: could you imagine a physicist or mathematician giving a different answer than you did? We found that *every one* of our students gave the physics answer r^2 (or, in two cases, simply r – probably an algebraic error rather than a distinct interpretation), indicating that they interpret the variables meaningfully as rectangular and polar coordinates. Only one (a double major) answered the

follow-up by saying that the mathematics answer was possible, and that the physics answer is technically an abuse of notation. We are not surprised that undergraduate students would interpret the variables concretely as coordinates. However, even mathematics majors did not suggest the standard mathematical function syntax when prompted for an alternative answer.

Notation

The only specific example of conflicting notation (really a convention) between mathematics and physics classes that students cited was the reversal of the symbols θ and φ for the three-dimensional spherical coordinate angles between the mathematics and physics literature. They got used to it, however. Other conventions that were mentioned were the use of complex exponentials versus sines and cosines as well as the placement of factors of 2π in Fourier analysis.

Discussion

We found that our students are indeed aware of many cultural differences between physics and mathematics, and how these impact their learning. Sometimes they found it beneficial to have both a mathematical and a physical perspective on common topics, sometimes lack of coordination between courses in the two disciplines impeded their synthesis of these topics. Students' strategies for coping with these differences included consulting online resources, asking professors for clarification, and mentally separating their "physics understanding" from their "math understanding."

Our study is subject to several limitations. Although we contacted every student at our institution who was a double major in mathematics and physics, or majoring in one and minoring in the other, our sample was still small: 20 surveys and 11 interviews. The students we interviewed all had senior standing at the time, and therefore might have taken many of the same classes with the same instructors. Some of our findings may reflect characteristics of those instructors rather than the course content or culture as such. We are also on the quarter system and our curricula may differ from semester-based institutions. We did not observe systematic differences in the responses of our subpopulations of double majors, mathematics majors, and physics majors, but our small sample size would not support such claims in any case. It would be desirable to conduct a similar study at multiple institutions, with more students, allowing for some quantitative analysis of the results.

Recognizing the small sample size, we cautiously make some pedagogical suggestions. Greater awareness of the cultural differences identified here would be beneficial to both physics and mathematics instructors. Both should be familiar with what is taught in the other discipline and the relative sequencing of prerequisite courses. Students themselves suggested that physics instructors make supplementary mathematical resources available for topics that are taught rapidly, or not in depth, in class. Physics instructors might pay more explicit attention to motivating and justifying approximations and other "tricks".

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Examining Ways in Which Students Engage With Sets of Outcomes in Combinatorics

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Lockwood (2014) articulated the importance of sets of outcomes, suggesting that a "set-oriented perspective" is a productive way of thinking for students as they solve counting problems. While this construct has indeed proven useful, it remains quite broad. In this report we aim to provide more detailed insight into the variety of ways students engage with outcomes while counting, exploring ways that we might elicit robust set-oriented perspectives in students. We present data from student interviews that were designed to elicit engagement with sets of outcomes, and we report on a variety of ways in which students worked with outcomes as they solved problems. We highlight contrasting cases that emerged in these interviews, making the case that there is room for nuance (and differences) in students' engagement with sets of outcomes in counting.

Keywords: Sets of outcomes, Combinatorics, Discrete mathematics

In combinatorics education, researchers have drawn upon the notion of a set-oriented perspective, which is "a way of thinking about counting that involves attending to sets of outcomes as an intrinsic component of solving counting problems" (Lockwood, 2014, p. 31). This perspective has been used in a variety of combinatorial contexts, including combinatorial proof (e.g., Lockwood, Reed, & Erickson, 2021) and computing (e.g., Lockwood & De Chenne, 2020), and it has been highlighted as a potentially productive way of thinking that may help orient students toward successful counting (e.g., Lockwood & Gibson, 2016; Soto et al., 2022).

In such work, the set-oriented perspective itself has remained rather broad, a high-level construct that promotes a general approach toward attention to outcomes. However, studies have (even implicitly) revealed variety in students' use of outcomes. Students may largely attend to specific outcomes (e.g., Lockwood & Reed, 2020), students may create partial lists of outcomes, or they may partition entire outcome sets (e.g., Lockwood, 2014). Based on such observations, we were motivated to better understand and explore ways in which students might, in practice, engage with sets of outcomes. Doing so would help paint a more nuanced picture of students' engagements with outcome sets, and might eventually enable us to elaborate and clarify the construct of a set-oriented perspective. We are broadly trying to answer the following research question: *In what ways do undergraduate students engage with sets of outcomes as they solve counting problems?* More specifically, in this report, we address the question: *In what different ways do students leverage sets of outcomes to solve counting problems?*

Literature Review and Theoretical Perspectives

Sets of Outcomes and a Set-Oriented Perspective

We frame our work within Lockwood's (2013) model of combinatorial thinking. When solving counting problems, students variously attend to *formulas/expressions, counting processes,* and *sets of outcomes*. Formulas/expressions are numbers, operations, variables, or relationships that yield the cardinality of some outcome set. Counting processes are real or

imagined procedures for enumeration, and outcome sets are the elements being enumerated by a counting problem (Lockwood, 2013). In this paper we focus on the ways that students engage with the set of outcomes while solving various counting problems. Aligned with Lockwood's model is the notion of a set-oriented perspective. Harel (2008) described a *way of thinking* as a cognitive characteristic of a mental act; we take a set-oriented perspective (Lockwood, 2014) to be a way of thinking about the mental act of solving counting problems. This construct builds upon Lockwood's (2013) model, in which sets of outcomes are a central component. Lockwood and others have used this set-oriented perspective to frame students' combinatorial reasoning.

Prior work shows evidence of students reasoning about particular outcomes as they solve counting problems. Describing an equivalence way of thinking in combinatorics, Lockwood and Reed (2020) gave instances in which students compared and contrasted particular outcomes with the goal of determining whether two outcomes should be considered equivalent. When solving a problem about distributing identical lollipops to children, the students contrasted outcomes {1,2,3} and {3,2,1} (which children received a lollipop). They realized the two were equivalent, saying, "So you would be counting this [{1,2,3}] in a different combination, but really it wouldn't be an additional combination, because it's all the same, so this [{3,2,1}] is pretty much the exact same thing. It's just 123. […] because all the lollipops are the same" (p. 11). This is one way in which students may engage with outcomes and was integral to Lockwood and Reed's (2020) students' developing a set-oriented perspective.

In another example from Lockwood (2014), we see a contrasting way in which students reason about sets of outcomes. Here, a student was trying to count 8-letter passwords with at least 3 Es, reasoning that he could count all possible 8-letter passwords and subtract those with 0, 1, or 2 Es. We infer that the student partitioned a set of all possible passwords into subsets of desirable and undesirable passwords. In this common "total-minus-bad" approach, students engage with the set of outcomes by conceiving of the desired set in relation to some broader set, a different usage than the identification of particular outcomes in the previous example.

Other studies have made the case for the value of focusing on sets of outcomes and a set-oriented perspective (e.g., Erickson, 2022; Soto et al., 2022; Wasserman & Galarza, 2019). However, none of these studies specifically identified qualitative differences in how students reasoned about sets of outcomes. Our goal in this report is to detail some of the different ways in which students may engage with sets of outcomes when solving a variety of counting problems.

Local vs. Global Perspectives toward Sets of Outcomes

In framing our results, we also draw upon a distinction between local and global engagements with sets of outcomes (cf. Hamdan, 2006 and Cook et al., 2022). We characterize a local perspective as focusing on one or more particular outcomes, without necessarily taking into account the entire set of outcomes. It is local in that the focus is on particular elements rather than the broader structure. In contrast, we characterize a global perspective on a set of outcomes as a focus on the set as a whole, rather than emphasizing only specific elements.

In Lockwood and Reed (2020), the student's comparison of outcomes 123 and 321 as being equivalent suggests a local perspective on the set of outcomes; the students focused on two particular outcomes and drew a conclusion about what was being counted and which formula to use. In contrast, Lockwood's (2014) student's use of the total-minus-bad strategy suggests a global perspective on the set of outcomes, not appealing to particular elements of the set but rather to some overall structure of the entire set. This distinction between local and global

perspectives toward outcomes is one way to characterize students' engagement with the set of outcomes; it is one dimension across which we analyzed our data.

Engaging with Outcomes with Numerical vs. Structural Areas of Focus

We also highlight a distinction reflected in students' engagement with outcomes that relates to the focus of the outcome-related activity. In working with outcome sets, one may emphasize the cardinality of the sets, emphasizing or exploring the numerical properties, relationships, or patterns in those cardinalities. For example, Lockwood et al., (2015) reported students creating lists of outcomes among several small cases and noting numerical patterns among them. This suggests what we could call a *numerical focus* as they engaged with sets of outcomes. In contrast, students may focus not on cardinalities and numerical patterns, but rather on features or structural aspects of the set of outcomes. Students in De Chenne and Lockwood (2019) noted structure in lists of outcomes to determine the *nth* outcome in a given list. Here, a focus was on structural regularity in the set, and not on any numerical patterns. This distinction represents another dimension across which we analyzed our data.

Methods

The data we present is from a series of task-based clinical interviews (Hunting, 1997) conducted with 9 students (3 graduate and 6 undergraduate) at a large public university in the United States. We present results from interviews with Kyle and Erin (pseudonyms) – undergraduate students who had taken or were taking discrete mathematics at the time of the interview. These were part of a broader study exploring student uses of equivalence in counting.

The students solved various counting problems on an iPad and described their work, and both the students and the screen were videorecorded. During and after their problem-solving activity, the interviewer asked hypothesis-confirming questions to enable post-interview inference into the students' understandings of formulas, concepts, and strategies. If the students were unsure of a solution, or were unable to proceed after some initial work on the problem, the interviewer often encouraged them to try smaller examples of the same problem. To provide opportunities for students to engage with sets of outcomes, the interviewer would at times suggest exploration of a pre-generated outcome set (of a smaller case). For instance, the interviewer might have provided a pre-generated set of arrangements of 1, 2, 3, and 4, and the student could use the GoodNotes app on an iPad to select and drag any outcome to a different location on the screen, creating their own organizations of the provided outcome set.

Data Analysis

To identify ways in which students engaged with sets of outcomes, the second author reviewed all of the interviews and flagged episodes in which students specifically leveraged outcomes or outcome sets in their problem-solving process. (An example of students *not* leveraging outcome sets would be a student identifying the wording of a problem with a particular solution type, making statements like "This is a combination problem.") This researcher articulated broad themes and categories for the flagged engagements, drawing upon the local/global and numerical/structural distinctions. The first author then also reviewed all of these episodes that featured sets of outcomes. Together, via discussion, the first and second authors chose representative episodes to highlight illustrative distinctions among students' engagement with sets of outcomes.

For this paper, because of space constraints, we limit ourselves to presenting just a handful of episodes to highlight ways in which students engaged with outcomes. Also, especially given the

fact that students were at times encouraged to work with whole sets of outcomes, we are not making claims about their spontaneous use of outcome sets, nor are we reporting on frequencies in a meaningful way to suggest that some are much more common than others. Rather, we seek to make the point that, even within this interview context where they were at times encouraged to engage with outcomes, students' means of leveraging outcome sets were far from monolithic. This all highlights the point that there are multiple ways in which students may engage with sets of outcomes, and allows us to progress towards elaborating nuance about what might be entailed in the way of thinking that underlies a set-oriented perspective.

Results

In this section, we provide some key distinctions in students' engagement with sets of outcomes. Because of space limitations, the goal here is to provide contrasting, detailed examples of students engaging with sets of outcomes.

Distinction 1: Local vs. global engagement with outcomes in a single student

We begin by contrasting two cases with Kyle, focusing on his work on two problems.

Kyle's work on the Groups of Students problem. First, we see Kyle's work on the Groups of Students problem: *How many ways are there to split a class of 20 into four groups of 5?* Kyle initially considered a process of successively selecting five students, reflected in the expression C(20,5)*C(15,5)*C(10,5)*C(5,5). But he noted that this process would yield redundancies, saying "That feels like just kind of the obvious way to go about it, but I feel like doing it this way is where I'm going to miss out on the redundancies." He went on to specify, "If we chose the first five, and then the second five, third five, and the fourth five. Yeah, and but then, just reversing the order where it's fourth five, third five, second five and first five, then that's redundant in that sense where it's not unique for groups of five, but it would be included in this [expression]." We interpret here that Kyle articulates one specific outcome generated by the process of successively choosing students, namely, one which partitions the students into 1-5, 6-10, 11-15, and 16-20. But he realized that the process would yield the same partition in different orders.

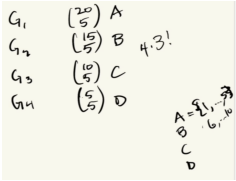


Figure 1. Kyle's work on the Groups of Students problem, in which he specifies a particular outcome

He continued working and specified the outcomes, assigning labels to the groups of A, B, C, D, as shown in Figure 1. He said, "Like if I said, uh, I would say A, B, C, D, then that's like, uh, one through five and then totally same for six and so on, so forth." Ultimately, Kyle realized he could arrange the groups of that partition in 4! ways and so the final solution would be C(20,5)*C(15,5)*C(10,5)*C(5,5)/4! (we omit details due to space). We claim Kyle's work here represents a local engagement with sets of outcomes. We infer this since he articulated specific outcomes - at no point attending to some broader organization of the entire set of groupings. Even more, he first explicitly articulated groups as a particular outcome – namely, splitting into four groups of 1-5, 6-10, 11-15, and 16-20, as groups A, B, C, D. So, he mediated attention to potential redundancies in the counting process by locally attending to specific outcomes and whether different arrangements were distinct.

Moreover, Kyle leveraged this local engagement to then generate the cardinality of the duplication, noting what counting processes might lead to similar arrangements of the outcomes. Notably, he did not just engage in numerical pattern matching, but considered what 3! and ultimately 4*3! might mean in terms of arranging specified groups of his specific outcome. With this local perspective, Kyle focused on forming and organizing iterations of a particular outcome, which he extended to the rest of the outcomes without explicit reference to a global structure.

Kyle's work on the Decreasing Numbers problem. In a contrasting example, Kyle was solving the Decreasing Numbers problem: *How many ways are there to form decreasing sequences of 3 numbers from the numbers 0-9?.* He did some initial work establishing cases based on the leading digit, but was overwhelmed by the number of cases. The interviewer then gave Kyle a set of outcomes to work with (as noted, this was an intentional intervention) – he was provided a written set of arrangements of the numbers 1-4 (Figure 2a), and encouraged to try and solve a smaller version of the problem: *How many ways are there to form decreasing sequences of 3 numbers from the numbers 1-4?* Kyle began by partitioning the set into four (ignoring the first digit), and highlighting the desirable decreasing 3-digit numbers, as seen in Figure 2b. This process led him to arrive at an answer of 4!/6, which yields the four desirable outcomes that he highlighted. In observing the set of outcomes, he noted, "Within each one of those permutations, only one of them is a decreasing sequence."

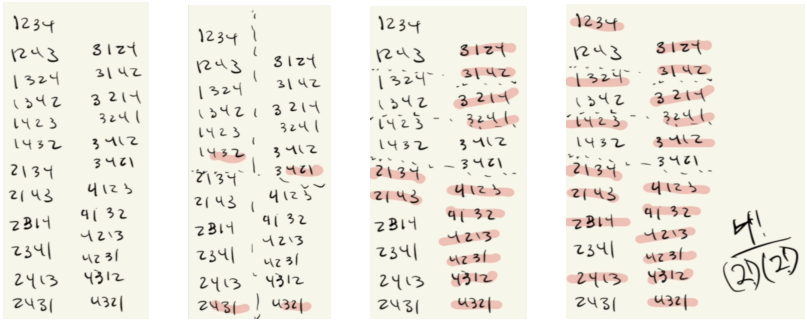


Figure 2a-d. Kyle's manipulation of sets of outcomes that were provided for him

Kyle expressed a desire to consider what might happen in other cases, so the interviewer prompted him to stay in the smaller 1-4 case but now consider decreasing sequences of two numbers. With a fresh list of outcomes, he again began to work with the set of outcomes. In Figure 2c, he partitioned the set according to the first two starting numbers, and he realized that he could exclude some outcomes. Here, he decided to highlight outcomes that were excluded, "this time I'm going to highlight the ones that I'm excluding." In Figure 2c, he first excluded outcomes that had the same two starting digits in different orders (so, for example he kept 1234 and 1243, and he excluded 2134 and 2143). Then, in Figure 2d we see within the remaining pairs that started with the same two digits, he excluded those whose last two digits were not in decreasing order (he excluded 1234 and kept 1243, for example). He related this process to his expression of 4!/(2!2!); the first division by 2! was for the arrangements of the first two digits, and the second division by 2! was for the second two digits. Kyle arrived at a correct expression having partitioned the set of outcomes and made note of equivalent outcomes in the set.

We feel that this episode shows a different approach to outcomes for Kyle, where he immediately started to partition the set into outcomes in which he was interested. Notably, though, he considered the desirable outcomes as they related to the entire set, and he then used equivalence and division to figure out a more general formula. We highlight this episode to show

the same student reasoning about sets of outcomes in a qualitatively different way. Here, Kyle was adopting a global perspective – he started with the whole set and partitioned it, connecting the operation of division to his partitioning activity. We maintain that his treatment of the entire set of outcomes is different from his approach on the Groups of Students problem, where he built up the set of outcomes and considered particular elements.

Distinction 2: Different Global Perspectives Among Students

It may be natural to think that, if students had access to an entire list of outcomes, they would use them in similar ways. However, in our data, we found that even though we gave students entire sets of outcomes, they did not always use them in the same way. In contrast to Kyle's previous example, we look briefly at Kyle's work on the Towers problem and then Erin's work on the Passwords problem. In addition to noting the problem-by-problem variety of ways that students engaged with outcome sets, we also make the point that there was variety even in the context of the same intervention. Said differently, students did not use entire sets of outcomes for the same purposes.

Kyle's work on the Towers problem. The Towers problem states: *Silas is making "towers" from 3 green, 4 red, 2 yellow, and 8 orange blocks. Using all 17 blocks, how many different "towers" could Silas make?* Kyle became stuck initially, and the interviewer suggested he try a smaller case with 2 green and 2 red blocks. Kyle was provided the same list of arrangements of 1-4 as in Figure 2, but 1 and 2 were labeled red and 3 and 4 were labeled green. Seen in Figure 3a, Kyle first used the provided outcome set to pick out the 6 desired outcomes in the subcase, and then proceeded to hypothesize a formula/expression to generate the 6 outcomes (Figure 3b).



Figure 3a-c. Kyle's use of outcomes in the Towers Problem

In contrast to his work on the decreasing sequences problem, here the provided set of outcomes was used as a source from which the desired outcome set was generated. Then, the desired outcome set was used for its cardinality, and the first outcome "RRGG" was used as a representative from which Kyle conjectured a counting process (and the paired 2*C(3,1) expression) to generate the representative outcome. In this episode, we see Kyle using the outcome set to generate a known cardinality (6) from which counting processes and formulas/expressions could be hypothesized.

Erin's work on the Passwords problem. Similar to Kyle's work on the Towers problem, Erin engaged in numerically-motivated uses of the outcome set for the Passwords problem: *A password consists of four upper-case letters. How many such 4-letter passwords contain at least one E?* Erin initially approached this problem by first selecting a position for the E to go (C(4,1)) and then filling in any remaining position with 26 letters; thus her answer was C(4,1)*26^3. This overcounts, as passwords with more than one E are counted multiple times by this approach.

The interviewer again suggested a smaller case: *How many 3-letter passwords of A, B, C contain at least one A?* The interviewer provided an entire set of outcomes for this smaller case

(Figure 3a). Erin used this list as a means to count a total of 19 outcomes, and then spent an extended period of time attempting to find known counting formulas that calculated to 19, noting that 19 being prime provided a difficulty for her. After some time, the interviewer drew her attention to the case of AAA being generated three times in her original solution, and she solved the problem by generating the cardinality of each sub-case (passwords with just 1, 2, or 3 As), connecting the subsets of outcomes and the expressions that generated those subsets. She was encouraged that her listing activity and expression resulted in a count of 19.



Figure 3a-b. The set of outcomes provided to Erin, and her work on the problem

In this final case, Erin used sets of outcomes in multiple ways. First, she used the provided set of outcomes only for its cardinality, before attending to formulas/expressions exclusively. We note this not to criticize Erin's work, but to emphasize that simply giving students outcome sets is not sufficient for them to partition the set, manipulate it, seek some structural regularity, or even focus on particular elements within that set. Ultimately, Erin then produced her own full list of outcomes that reflected her imagined counting processes for the subcases. We see this as Erin taking personal ownership of the outcomes as relevant to her process of construction.

Discussion, Conclusion, and Avenues for Future Research

We have provided contrasting cases that illustrate different ways in which students engage with outcome sets. This highlights distinct approaches to outcome engagement that ultimately suggest a diversity of experiences students might have when developing a set-oriented perspective. We now discuss some potential implications of our findings.

One point of discussion is that much of the research has made a case for the importance of having students engage with sets of outcomes, but we have shown evidence that more nuance and distinction is needed to elaborate more specific ways in which students engage with outcomes and with sets of outcomes. We have shown some potential axes of distinction in engagement with outcomes, but more work is needed to operationalize distinctions between engagement types, particularly in pursuit of ways to elicit a set-oriented perspective.

We have also shown that simply providing students with opportunities to manipulate outcome sets is not sufficient to engender certain kinds of engagement with outcomes - having them partition, identify structural regularity, or even focus on certain outcomes did not always necessarily occur even when students were provided with organized lists of outcomes. More work can explore what kinds of supporting directions or activities might be effective in helping students to engage in productive outcome-related activities. Additionally, in providing outcome sets, we note that there are some limits to how many outcomes can be presented to students; this emphasizes the importance of leveraging smaller cases.

Further, we have in this paper mostly shown students' engagement with outcomes in just a few problems. There is more to be explored about how engagement with outcomes more broadly develops into a stable way of thinking and thus a set-oriented perspective (Lockwood, 2014).

Acknowledgments

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Pedagogical Moves Related to Analogy that Support a Unified Understanding of Eigenequations in a Quantum Mechanics Class

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We analyze the pedagogical moves related to analogies used by a quantum mechanics instructor to support a class community in developing a shared unified understanding of eigenequations across the contexts of linear algebra, spin, energy, and position. We characterize an instructor's pedagogical moves as he engaged students in analogical reasoning. Some moves include posing tasks conducive to analogizing; preparing, soliciting, and scaffolding students' participation in analogical reasoning; using deictic gestures and inscriptions; juxtaposing symbols representing the analogized concepts; and explicitly highlighting the sameness of the analogized concepts. We exemplify these pedagogical moves with analytical descriptions of illustrative class episodes. We discuss how these moves can support the class community's expansion of their common ground by fostering the development of the class's shared unified understandings of eigenequations.

Keywords: Pedagogical Moves, Analogy, Eigentheory, Linear Algebra, Physics

Quantum mechanics students reason about eigentheory in various contexts, including spin, energy, and position, with eigenequations forms such as $\hat{x}|x_i\rangle = x_i|x_i\rangle$, $\hat{S}_z|+\rangle = \hbar/2\,|+\rangle$ and $\widehat{H}|E_n\rangle = E_n|E_n\rangle$. It is nontrivial to abstract the shared structure of these eigenequations and recognize them as instantiations of the same concept with analogous interpretations. Reasoning about a concept across contexts entails having a *unified understanding* of the concept: a coherent awareness of similar structure, a recognition of distinct instantiations as examples of the concept, and an ability to reason about the concept without reference to specific contexts (Lee & Heid, 2018; Serbin, 2023). Researchers have characterized students' unified understandings in various math contexts (e.g., Bagley et al., 2015; Cook et al., 2023; Melhuish et al., 2020), but there is little research on how instructors can help students develop unified understandings in physics. Recognizing eigenequations from varied contexts as instantiations of the same concept can be facilitated by analogizing (Gentner, 1983; Glynn, 1989; Richland et al., 2004). By analogizing across spin, energy, and position, physics students can generalize the structure of eigenequations that correspond with the mathematization of those quantum mechanical systems. In this study, we investigate how an instructor's analogizing might contribute to the class's taken-as-shared (Saxe & Farid, 2021) unified understanding of eigenequations. Our research question is: What pedagogical moves related to analogies can be used by a quantum mechanics course instructor to support the class community in developing a shared unified understanding of eigenequations?

Literature Review

Reasoning about eigenequations and their objects (operator, eigenvectors, eigenvalue, equal sign) and operations (matrix-vector multiplication and scalar-vector multiplication), individually or in coordination, is conceptually complex (Thomas & Stewart, 2011; Wawro et al., 2019). For instance, Wawro et al. (2024) found that for $Ax = \lambda x$, quantum mechanics students held either a *relational view* of the equal sign (the resulting object Ax on one side of the equal sign is the same as the resulting object λx on the other side) or a *functional view* of the equal sign (the matrix A acted on or transformed an input x and produced λx). Physics students must make sense of these coordinated constituent objects in varied quantum mechanical systems (Karakok, 2019; Wawro

et al., 2024) and interpret eigenequations in terms of their physical referents (e.g., Gire & Manogue, 2012; Her & Loverude, 2020). This may involve recognizing eigenequations like $\hat{S}_z|+\rangle = \hbar/2|+\rangle$, $\widehat{H}|E_n\rangle = E_n|E_n\rangle$, and $\hat{x}|x_i\rangle = x_i|x_i\rangle$ as all sharing a structure or interpreting the symbols associated with that structure as meaning "transformation that reproduces the original" (Dreyfus et al., 2017, p. 10) or "to operate is to act" (Pina et al., 2024, p. 7). Physics students sometimes perceive conceptual incompatibility between the meanings they hold for eigenequations in different contexts. For example, interpreting an eigenvector as being stretched by λ is nonsensible in a spin context where eigenstates are of unit length (Wawro et al., 2024).

There is little research on how instructors can support students in cross-contextual reasoning about eigenequations. Karakok (2019) demonstrated a physics professor's focus on supporting students' understanding of objects within an eigenequation by frequently asking students, "What kind of a beast is this?" (p. 9) while pointing to symbols in $A\vec{v} = \lambda\vec{v}$. The professor's explicit focus on symbols seemed to foster students' understandings of them. Attending to meanings of symbols in eigenequations may support students in developing unified understandings of them.

Theoretical Perspectives

Having a *unified understanding* of a concept involves recognizing seemingly disparate notions from different contexts as instantiations of the same concept with a shared overarching structure (Lee & Heid, 2018; Zandieh et al., 2017). Researchers have characterized students' unified understandings of mathematical concepts across contexts in linear algebra and abstract algebra, such as functions, linear transformations, homomorphisms, inverses, identities, and factorization (Bagley et al., 2015; Cook et al., 2023; Lee & Heid, 2018; Melhuish et al., 2020; Serbin, 2021, 2023; Zandieh et al., 2017). These studies documented how students reason about the relationships, similarities, or shared properties among instantiations of a concept (e.g., additive or multiplicative inverses) and recognize them as the same concept with the same overarching structure that holds regardless of context. Such reasoning can develop as students "unify their existing understandings of seemingly unrelated mathematical concepts by identifying structural similarities among the concepts, thereby increasing the extent to which the relations among the concepts in their minds become more coherent" (Serbin, 2023, p. 4).

Developing unified understandings of a concept may involve *analogical reasoning*, "the ability to perceive and operate on the basis of corresponding structural similarity in objects whose surface features are not necessarily similar" (Richland et al., 2004, p. 37-38). Gentner (1983) characterized analogy as a mapping from a source domain to a target domain that preserves relations between objects and attributes in them. A *source domain* is a collection of knowledge that can be applied in a new domain, the *target*, to develop new knowledge. *Objects* are whole entities, parts of entities, or a collection of entities. *Attributes* are characteristics or properties of objects, and r*elations* are associations between objects, attributes, or other relations. When analogizing, one maps objects, attributes, and relations from the source by corresponding them with objects, attributes, and relations in the target and drawing inferences about them (Holyoak & Thagard, 1989; Novick & Holyoak, 1991; Richland et al., 2004). Hicks (2020) classified analogical reasoning activities as: exporting aspects from the source, importing aspects to the target, recalling knowledge about the source, distinguishing differences between the source and target domains, and adapting the target to accommodate for those differences. Glynn's (1989) teaching with analogies model has six instructional strategies: "(1) introduce target concept, (2) recall analog [source] concept, (3) identify similar features of concepts, (4) map similar features, (5) draw conclusions about concepts, and (6) indicate where analogy breaks down" (p. 208). Instructors can use these strategies as they engage students in analogizing.

Finally, *common ground* (Clark, 1996; Saxe et al., 2015) includes the, at best, taken-as-shared "knowledge of word meanings and norms for communication, shared knowledge that enables successful communication and coordinated action" (Saxe & Farid, 2021, p. 640). Common ground "gets built up in strata" (Clark, 1996, p. 120), where "every new piece of common ground is built on an old piece" (p. 119). It is constituted over time via communicative displays by class members, thereby expanding the class's taken-as-shared understandings.

We coordinate the theories of unified understandings, analogical reasoning, and common ground to characterize how a professor's pedagogical moves facilitate a class in developing a taken-as-shared unified understanding of eigenequations. Analogizing instantiations of a concept is a way for instructors to help expand the class's common ground and develop a taken-as-shared unified understanding of that concept. We consider an analogy's source domain to be a subset of a class community's common ground. An instructor can make analogies by mapping objects, attributes, and relations from that source to a target. In mapping these objects, attributes, and relations to the target, new objects, attributes, and/or relations are created and can become part of the common ground. The common ground thus expands, and the target becomes part of a new source from which further analogical mappings can be made. This common ground expansion by analogizing contributes to the class's development of a taken-as-shared unified understanding.

Methods

Data for this study come from transcribed videos of whole-class discussions and lectures in a senior-level quantum mechanics course at a public university in the northeastern US. The class met for 50 minutes and used a spins-first textbook (McIntyre, 2012). We analyzed 22 class days. We found segments in which the instructor analogized eigenequations and enacted pedagogical moves that could support the class in developing their understanding of eigenequations. We took such instances in the data as indicative of the class community expanding its common ground for eigenequations, such as by recognizing different forms as instantiations of the same overarching concept while attending to any shared property, use, or aspect of structure. We analyzed the episodes in terms of Gentner's (1983) structure-mapping theory for analogies. We identified the given analogy's source domain, target domain, and what objects, attributes, relations, or higher-order relations were being mapped between to the two domains. Such analyses helped us answer our research question by indicating the class community's progress toward developing a taken-as-shared unified understanding of eigenequations, which is part of the class's common ground.

We used deductive and inductive coding (Miles et al., 2013) to identify pedagogical moves in the segments. *Pedagogical moves* are "discursive acts an instructor performs during instruction to accomplish an instructional goal" (Serbin et al., under review). We focused our analysis on instances when the instructor gave vocal, written, or gestural communicative displays in which he described at least two different concepts with similar features. We made an a priori codebook containing pedagogical moves for supporting analogical activity from Hicks (2020) and Glynn (1989), from which we deductively coded. We then, through inductive coding, created new codes to describe other pedagogical moves in the data. These new pedagogical moves are given in Table 1 and are exemplified in the Results. We wrote memos (Maxwell, 2013) on how those moves seemed to support the class's development of a unified understanding of eigenequations.

Results

We summarize two class episodes from Day 5 and Day 19 that illustrate Dr. Holmes's pedagogical moves that occurred while the class engaged in analogizing eigentheory concepts.

Table 1. New Pedagogical Moves for supporting Analogical Activity

Pedagogical Move	*Description*
(1) Posing a task conducive to analogizing	Assigns a task or problem that prompts students to engage in analogizing two or more concepts.
(2) Preparing students to look for similar objects, attributes, or relations in source and target	Primes the students to identify features shared by two concepts, perhaps by suggesting there is a similarity that the students should find.
(3) Scaffolding students in identifying and mapping objects, attributes, and relations	Provides support to students as they analogize two concepts, e.g., asking guiding questions that direct students' attention to the shared features of the concepts.
(4) Juxtaposing symbols representing objects in the source and target	Writes symbols that represent concepts being analogized in a way that the symbols are aligned or near each other.
(5) Using deictic gestures	Uses their body to make gestures (McNeill, 1992) that emphasize corresponding aspects in the source & target.
(6) Using deictic inscriptions	Writes an inscription that points to or emphasizes corresponding aspects in the source and target domains.
(7) Soliciting student participation to map objects, attributes, and relations from the source to the target	Invites students to verbally contribute their ideas in the class discussion as they engage in analogizing aspects in the source and target domains.
(8) Explicitly highlighting sameness of aspects in the source and target	Verbally claims that two aspects in the source and target domains are equivalent or the same.

Analogizing the Eigenequation $Av = \lambda v$ with the Spin Eigenequations $\hat{S}_z|\pm\rangle = \pm\frac{\hbar}{2}|\pm\rangle$

On Day 5, the instructor, Dr. Holmes (pseudonym), used several pedagogical moves to guide the class in analogizing the canonical eigenequation $Av = \lambda v$ and the spin eigenequations:

Dr. Holmes: When you have a matrix [points to A, see Figure 1a], you have eigenvectors [points to v on the right side], right. Do people remember what what's special about an eigenvector, for a matrix, for a given matrix? What makes it eigen-esque-ish?

Student: When it's operated on by A, it only scales.

Dr. Holmes: That's right… eigenvectors don't rotate at all, they only scale. If you operate A on this particular eigenvector [points to Av], all you get is some scaling factor times the original vector [points to λv], alright. That's the magic. Those are magic vectors.

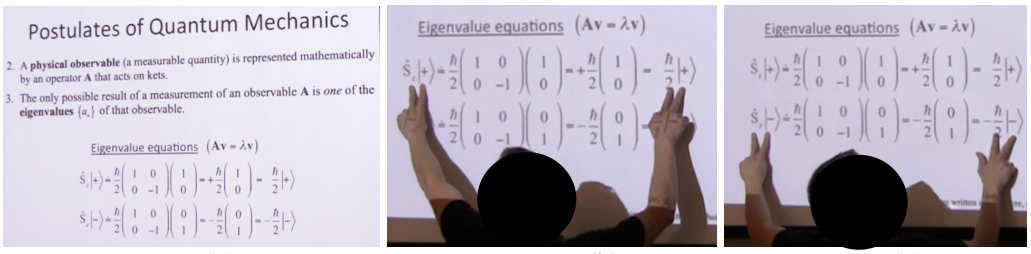
Dr. Holmes *recalled aspects of the source domain*, which included the canonical eigenequation and its operator, eigenvectors, and eigenvalue. He *solicited student participation* to *recall* these objects, relations among them (the matrix-vector relationships), and their attributes ("it only scales," "magic vectors"). He called out the ubiquity of $Av = \lambda v$, claiming, "we can write these for our operators," *explicitly highlighting the sameness* of the eigenequations' existence for operators in the source and target domains. He *introduced the target domain*, which included the spin eigenequations and their components. These pedagogical moves of *recalling, soliciting, highlighting*, and *introducing* prepare the class community to expand its common ground by setting the stage to introduce eigenequations for the first quantum mechanical context of spin.

Dr. Holmes: This equation [points to $Av = \lambda v$] is sort of canonical, right. People write these all the time for linear algebra, for matrix-vector relationships. And we can write these for our operators. I have the $S\ z$ operator … we're gonna have the value, we're gonna have the matrix, and we're gonna have the operators … You operate the $S\ z$ [points to $\hat{S}_z$ then $|+\rangle$] on the up spin state. Right, we have this matrix [points to

matrix]…and you operate it on the 1, 0 state [points to $\binom{1}{0}$]… if you do that, you get h bar over 2 times the vector you started with. So this is the eigenvalue part [points to two ends of equation (Figure 1b)]. Same thing for this, you get a negative h bar over 2 on the bottom [points to two ends of equation (Figure 1c)] for the down state.

He *identified* and *mapped (exported and imported)* similar objects (e.g., value, matrix), relations, and attributes from the source to the target domain. He *mapped the relation* between the operator and the eigenvector by claiming, "You operate the $\hat{S}_z$ on the up spin state," which was parallel to what he said about the canonical eigenequation: "You operate A on this particular eigenvector." He also *mapped a relation* by saying, "you get h bar over two times the vector you started with" which was similar to his claim: "All you get is some scaling factor times the original vector."

Dr. Holmes also used the moves of *gesturing* and *juxtaposing symbols* while analogizing. On the slide, he *juxtaposed the placement of the symbols representing objects in the source and target*. He vertically aligned the two spin eigenequations underneath the canonical eigenequation (see Figure 1a), which had the potential to support the class in recognizing the shared structure of the different eigenequations as matrix-eigenvector product on the left and eigenvalue-eigenvector product on the right. Dr. Holmes accompanied this with *using deictic gestures.* He used two fingers on each hand to simultaneously point to each of the two objects ($\hat{S}_z$ and $|+\rangle$) on the left and the two objects ($\hbar/2$ and $|+\rangle$) on the right (see Figure 1b). He then lowered his hands to point to the corresponding objects in the other spin eigenequation (see Figure 1c). These deictic gestures helped *map the relations* between the objects in one eigenequation to the other, so the class could see that those relations were preserved. We posit these moves of using juxtaposition and deictic gestures could serve to expand the class's common ground, thus contributing to the class's development of a unified understanding of eigenequations. The class had an opportunity to recognize different eigenequations as instantiations of the same concept with the shared structure regarding the objects within the eigenequations and the relations among those objects.



(a) (b) (c)

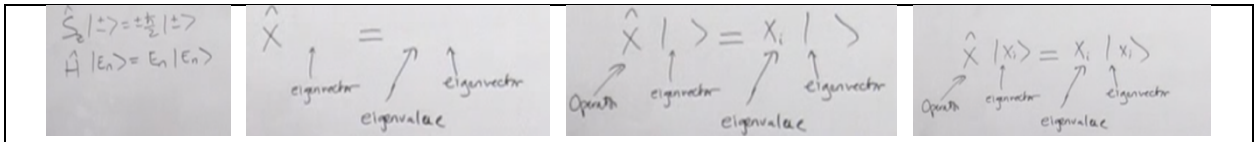
Figure 1. (a) Slide projected during class on day 5. (b, c) Dr. Holmes's deictic gestures juxtaposing eigenequations

Analogizing the spin and energy eigenequations with the position eigenequation

On Day 19, Dr. Holmes used various pedagogical moves to help the class analogize the source spin and energy eigenequations $\hat{S}_z|\pm\rangle = \pm\frac{\hbar}{2}|\pm\rangle$ and $\hat{H}|E_n\rangle = E_n|E_n\rangle$ with the target position eigenequation $\hat{x}|x_i\rangle = x_i|x_i\rangle$. He first *posed a task conducive to analogizing*: "Write down an eigenvalue equation for an operator $\hat{x}$ that represents (1-D) position." He launched the task by explaining: "We've measured all these other things… spins…and energies, right? But let's say we want to measure the position of a particle… If it's a thing we're measuring, then it has all the properties of the other things we're measuring." He *recalled the source domain* and *prepared students to look for similar attributes* of objects in the source and target domain. After students worked on the task, Dr. Holmes wrote $\hat{S}_z|\pm\rangle = \pm\frac{\hbar}{2}|\pm\rangle$ and $\hat{H}|E_n\rangle = E_n|E_n\rangle$ (Figure 2a)

and continued: "I want an equation like this...We're talking about positions, by saying operator $\hat{x}$...If I want to write an eigenvalue equation where that's my operator, what is it going to tell me? What do these eigenvalue equations tell me, in general?" Dr. Holmes *identified and mapped (exported and imported) similar features of the concepts*, mapping "what these eigenvalue equations tell me" from the source to the target. He also *provided scaffolding for identifying and mapping objects and attributes* in the source for students to map to the target. They continued:

> *Dr. Holmes:* If I operate on an eigenvector [points to $|E_n\rangle$ next to $\widehat{H}$] with that operator, what is this [points to E_n in $\widehat{H}|E_n\rangle = E_n|E_n\rangle$] then? [*Student:* A scalar]. What do I need here? [draws an arrow to the right of $\hat{x}$, writes "eigenvector"]. I need an eigenvector, and I need that same eigenvector here [writes "=", leaves a space, draws arrow to a third space, writes "eigenvector"]. What do I need there? [points to space after "="] (see Figure 2b).
>
> *Students:* A scalar. The position.
>
> *Dr. Holmes:* The position. But yeah, that's the eigenvalue. [writes x_i after "="] (Figure 2c).



(a) (b) (c) (d)

Figure 2. Dr. Holmes's symbol forms written on the board during the Day 19 episode.

Dr. Holmes first *mapped the operator and eigenvector $\widehat{H}|E_n\rangle$ objects* from the left side of the source eigenequation to the objects $\hat{x}__$ in the target, with the space labeled as "eigenvector." In doing so, he *juxtaposed the operator and eigenvector symbols in the source and target*. This was supported by his *use of deictic gestures and inscriptions* (arrows) to highlight the correspondence between operator, eigenvector, and eigenvalue objects in the source and target eigenequations. He again *used deictic gestures and inscriptions,* saying "I need an eigenvector, and I need that same eigenvector here" and drawing arrows. This *mapped the relation* between the eigenvectors in the source eigenequations to the relation between those in the target eigenequation. By asking "what do I need to go there?" he directed the class's attention to the eigenvalue in the source and to its position in the target. He affirmed the student responses. He *solicited student participation to map objects* and *juxtaposed eigenvalue symbols in the source and target* eigenequations.

Dr. Holmes continued, "Now we're going to be in space space, as opposed to energy space or spin space, so you really have to keep these things straight in your head." This *distinguished the attributes of objects in the source domain and target domain.* Dr. Holmes continued by next focusing on how to symbolize the eigenvectors in the position eigenequation they were creating:

> *Dr. Holmes:* What should I use as how to represent an eigenvector of position? Like given some of the conventions we have for writing stuff [points to Figure 2a], what would be- [points to space for left eigenvector in target eigenequation]?

Students: x_i. x ket.

> *Dr. Holmes:* Yeah, well first, it better look like that, right [writes kets $\hat{x}|\ \rangle = x_i|\ \rangle$, Figure 2c], and what do I want to put in here? [*Student:* x_i] x_i right? [writes $|x_i\rangle$, Figure 2d]. That's fine. Because what the convention was, this is the eigenvalue for the eigenvector, right?

In asking students how to write a position eigenvector given conventions they had for writing spin and energy eigenequations, Dr. Holmes *used deictic gestures* to direct the class's attention to the source eigenequations, *recalled the source attributes* of how eigenvectors are written, and *prepared students to identify similar attributes of eigenvectors in the source and target*. He also

solicited student participation to map the attributes of eigenvectors from the source to the target. Considering the student responses, Dr. Holmes *exported source attributes* and *imported them to the target* that eigenvectors are symbolized as kets and labeled by corresponding eigenvalues. This *mapped the relation between eigenvectors and eigenvalues from the source to the target*. Dr. Holmes used a variety of pedagogical moves to engage the class in analogical reasoning as they created an eigenequation in their new position context, which we posit contributed to the class developing a unified understanding of eigenequations as they came to recognize them from three quantum mechanical systems as instantiations of the same overarching concept. Dr. Holmes used gestures and arrows to juxtapose $\hat{S}_z|\pm\rangle = \pm\hbar/2\,|\pm\rangle$ and $\hat{H}|E_n\rangle = E_n|E_n\rangle$ from the source with $\hat{x}|x_i\rangle = x_i|x_i\rangle$ in the target. Together the class mapped objects (operator, eigenvectors, and eigenvalue) with their attributes and relations among the objects from the source contexts of spin and energy to the target context of position. His moves fostered the class's awareness of the shared properties among three distinct eigenequations, which expanded the common ground and contributed to the class's development of a unified understanding of eigenequations.

Conclusion

Engaging students in analogical reasoning can foster their unified understandings of a concept. Thus, we investigated the pedagogical moves a quantum mechanics instructor used to engage students in analogical reasoning. We identified eight pedagogical moves (Table 1) and described their potential to foster a class's development of unified understandings by expanding its common ground. Our characterization of pedagogical moves is the primary contribution of our study. The moves that stand out as most salient overall in our data set include juxtaposing the symbols that represent objects in the source and target domain, using deictic gestures, and using deictic inscriptions. These moves fostered the class's awareness of the shared structure among eigenequations in three different contexts and contributed to the class's development of a unified understanding of eigenequations. Given their productivity, we encourage instructors to use these pedagogical moves to analogize concepts and guide students to develop unified understandings.

Another contribution of this study is its connection of the theoretical constructs of analogical reasoning (Gentner, 1989), unified understandings (Lee & Heid, 2018; Zandieh et al., 2017), and common ground (Saxe et al., 2015). Studies have focused on individuals' unified understandings of mathematical concepts (e.g., Bagley et al., 2015; Cook et al., 2023; Melhuish et al., 2020; Serbin, 2023) but not yet on a class community's unified understanding. We posit that a class community can develop a unified understanding of a concept and thus expand its common ground by analogizing different instantiations of the concept. The class maps objects representing that concept, its attributes, and relations among that concept and others from the source to the target domain. Then, the target becomes part of the common ground, and the class comes to recognize those different examples as instantiations of the same overarching concept. When students engage in analogical reasoning, they can make connections between concepts, which is conducive to developing unified understandings of mathematical and physical concepts.

Lastly, we acknowledge some limitations to our methodology that could impact the interpretation of our findings. We cannot explicitly link the instructor's pedagogical moves with development of each student's unified understandings. Thus, we can only claim that his pedagogical moves related to analogizing were potentially conducive to guiding the class's shared unified understandings. It is outside the scope of this study to analyze individual students' unified understandings of eigenequations, so we suggest this as a direction for future research.

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Examining Undergraduate Students' Perceptions of Service Learning in Upper-Division Mathematics Courses for K-12 Teacher Candidates

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Prospective teachers need hands-on experience in the field before they can independently manage their own classrooms. Although experiential learning is widely acknowledged for its benefits, its implementation in content-focused mathematics courses for prospective K-12 teachers remains limited. To address this gap, mathematics content courses for future teachers should offer experiences that extend beyond traditional classroom settings. One effective approach to achieving this is through incorporating service learning experiences. This paper explores the integration of service learning into three distinct mathematics courses for prospective K-12 teachers, including two upper division mathematics courses, highlighting how this approach was implemented in each course. Results suggest service learning in upper-division mathematics courses benefits prospective teachers by fostering experiential learning, enhancing collaboration, communication, and career skills, as well as deepening engagement with course content and the community.

Keywords: prospective K-12 teachers, service learning, geometry, capstone courses

Introduction

Prospective teachers (PTs) must gain experience in the field before taking charge of their own classrooms (Jackson et al., 2018). Service learning (SL) has been incorporated into teacher education programs around the globe as a teaching strategy to allow PTs to connect theory, practice, and community (Bates & Lin, 2015; Jackson et al., 2018). SL is "a pedagogical and learning strategy used in schools and communities around the world and has produced positive social, emotional, and cognitive growth among students" (Clark, 2017, p. 2). Further, it integrates community service and academic learning, allowing students to contribute to the needs of the community while applying content learned in the classroom (Carrington & Saggers, 2008; Mortari, Silva, & Ubbiali, 2019; Verducci & Pope, 2001). SL methodologies strive to inspire students, offer valuable services to others, establish links between universities or schools and their communities, and offer genuine incentives for introspection (Jackson et al., 2018; Lee & Statham, 2010). Many studies found that participating in SL components of a course enhanced students' understanding of necessary skills and their ability to perform tasks and skills aligned with course learning outcomes (Anderson et al., 2019). Engaging with K-12 students via SL deepens PTs' capacity for empathy, understanding, and encourages them to view students as community members with whom they have a responsibility and desire to connect (Chambers & Lavery, 2012).

Despite the generally favorable reviews of experiential learning, researchers have observed there is a notable scarcity of literature on its application within content-focused mathematics courses designed for K-12 teachers. This gap highlights the need for further research and exploration to better understand how experiential learning can be effectively incorporated into mathematics courses for students majoring in mathematics with a teaching emphasis.

In this study, we explore the following two research questions: (a) What do PTs perceive as the most valuable learning opportunities and skills gained from their involvement in SL activities within content-focused mathematics courses designed for K-12 teachers? (b) How do PTs view the relevance and usefulness of SL experiences in their mathematics courses on their future careers?

Conceptual Framework

Our conceptual framework is based on Carver's (1997) model for experiential education, in which student experience is central. Carver defined this experience through three interconnected facets: agency (A), belonging (B), and competence (C). Agency involves students' self-awareness and their ability to create growth opportunities. Belonging refers to students' integration into a community where they contribute and feel connected. Competence involves acquiring and applying new knowledge and skills. Additionally, the student experience can be seen both as a process and an outcome of the SL experience. Past experiences influence current experiences, making student experience a process and in the end, an output.

The development of ABC occurs as students face challenges, learn from mistakes, and grow as a person. The outcomes include acquiring new skills; becoming more adaptable; and gaining self-respect, peer respect, and a sense of safety. The learning environment supports ABC development through both setting characteristics (resources, behaviors, and shared values) and program characteristics (active learning, authenticity, future connections, and drawing from past experiences). Active learning directly involves students in the learning process while authenticity guarantees that the learning experience is genuine and relevant. Connecting to the future allows students to apply their knowledge to long term goals. These elements contribute to a positive student experience, encouraging PTs to implement similar experiential approaches with their students. We expand on Carver's model by incorporating the future intention of using similar activities or pedagogies as those used during their SL experience (see Figure 1). With positive student experience, we believe that teacher candidates are more likely to view the relevance and usefulness of the SL experiences on their future careers.

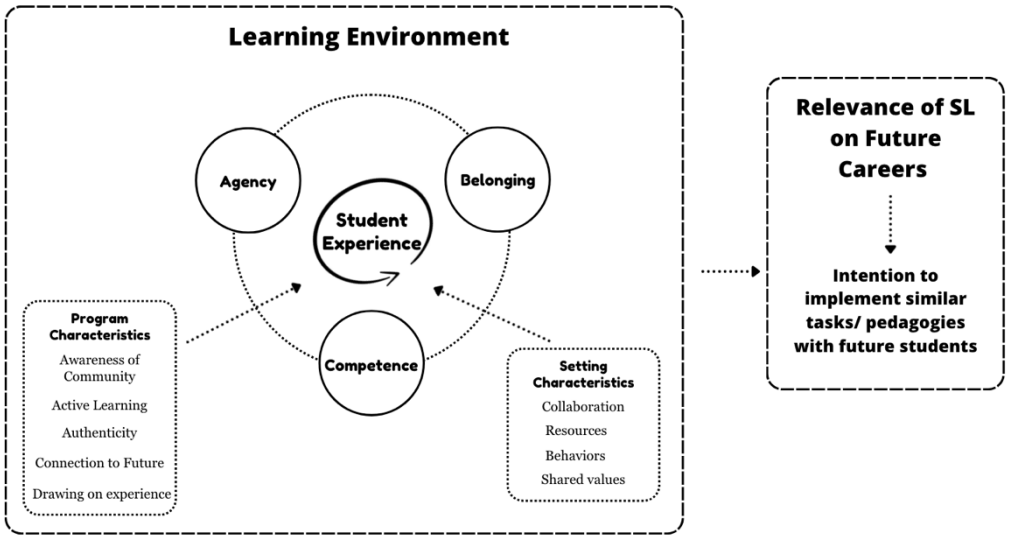


Figure 1. A conceptual framework for experiential learning for PTs (adapted from Carver (1997)).

Methodology

Study Participants

This study was conducted at a public institution on the West Coast of the United States. Participants included 38 students from three SL designated mathematics courses; 36 of whom were planning to become future K–12 teachers. The three courses included in the study were: College Geometry (eight students enrolled), an Elementary Mathematics and Learning Process course (10 students enrolled), and a capstone Mathematics course (20 students enrolled). Both College Geometry and the capstone courses were for math majors in the Teaching Emphasis track.

Course Context

In the College Geometry course, students examined the axioms and theorems of Euclidean geometry and compared various geometric axioms and theorems of Euclidean, non-Euclidean, and finite geometries. In the Elementary Mathematics course, students explored the relationship between mathematics and how it is learned. They expanded their knowledge of K-8 mathematics and explored how to positively impact children's attitudes toward and understanding of the subject. The capstone course provided a culminating experience that bridged the gap between the advanced mathematics studied in college and the mathematics taught in secondary schools. Throughout the course, students explored the nature of mathematics, with the overarching goal of connecting their experiences as learners to effective mathematics teaching.

Service Learning

Each student completed 10 hours of SL in the College Geometry course, five hours in the Elementary Mathematics course, and 26 hours in the capstone course. The SL activities involved (co)facilitating hands-on mathematics sessions with K-12 students, teachers, or seniors, solely based on content learned in the class. In the capstone course, 20 of the 26 SL hours were dedicated to tutoring middle and high school students.

In the College Geometry course, students learned about various finite geometries, such as 3-point geometry, Fano's geometry, and Young's geometry. They applied these concepts to create Spot-it games with three symbols per card, four symbols per card, five symbols per card, etc. Students also explored the application of Young's geometry in the SET card game. They developed presentation materials and hands-on manipulatives and facilitated sessions at a Wellness Program for seniors aged 65 and above, a Senior Living community, and a local Conference for middle school girls.

In the Elementary Mathematics and capstone courses, PTs facilitated various non-curricular, hands-on math activities for high school students from two local schools. They first presented the topics or worked on them with their peers in class, made modifications, and planned their sessions. Some of the topics included applications of number theory, logic, and geometry. Activities included Gerrymandering, Rook's move, etc.

Data Collection

Data was collected through pre- and post-surveys, and beginning- and end-of-semester reflections on students' SL experiences. The pre- and post-surveys included several Likert scale questions, ranging from "strongly agree" (5) to "strongly disagree" (1), along with open-ended questions. The pre-survey questions were adapted from Manolis (2004), while the post-survey

questions were adapted from the University of Georgia - Office of Service-Learning survey (2015). The end-of-semester reflection was adapted from the Campus Compact's "The What? So What?? Now What??? Reflection Model" (Rolfe et al., 2001). "The What?" component describes the event, i.e., facilitating the activities. The "So What" component examines the event's significance in relation to classroom concepts and personal experiences. The "Now What" component reflects on future actions that relate to the broader context of using mathematics in the "real world."

Data Analysis

We used descriptive statistics and an a priori and descriptive coding process (Miles et al., 2014) to interpret the data. The first round of coding involved members of the research team identifying and categorizing codes guided by Carver's conceptual framework on student experience. One PT was counted twice, as they were enrolled in both College Geometry and the Capstone course in successive semesters. In the beginning semester reflection, none of the students had prior experience with SL in a mathematics course before taking the SL courses associated with this study. Four students self-reported having previous SL experience in their education courses, and four other students reported having SL experience through Child and Adolescent Development classes, an art class, or through tutoring students in English.

Research Question (a): What do PTs perceive as the most valuable learning opportunities and skills gained from their involvement in SL activities within content-focused mathematics courses designed for K-12 teachers?

Out of 35 responses in the post-survey, 15 students strongly agreed, and 12 students agreed that the SL component of this course helped them develop their citizenship skills. Eighteen students strongly agreed, and 15 students agreed that the SL component made them more aware of their responsibilities as community members. Seventeen students strongly agreed, and 15 students agreed that the SL component helped them gain more knowledge about the community they worked with and the issues it faces. Twenty-two students strongly agreed, and 10 students agreed that the SL component helped them see how the material covered in the course could be useful in everyday life or in other situations. Nineteen students strongly agreed, and 14 students agreed that the SL component enriched their application of classroom knowledge. Fifteen students strongly agreed, and 18 students agreed that the SL component helped them to develop functional life skills, such as problem-solving, critical thinking, communication, and assertiveness.

From the end-of-semester reflections, student experiences are highlighted through three interconnected facets: agency, belonging, and competence. For agency, a majority of PTs felt a stronger connection to their future roles as teachers or envisioned themselves pursuing teaching in the future after participating in the SL sessions. PTs found it valuable to gain even a small glimpse of what being a mathematics teacher might involve. One PT wrote, "My goal is to become a math teacher and just getting a small insight to what that would be like is nice. It feels good to help people understand even if it is a small game like Spot it." PTs also gained confidence in their ability to teach and support other students within their peer group, with one PT noting that, "a service learning project is not an easy event to execute and through my performance as an individual I was able to not only regain confidence in myself, but also aid others in my group." Most importantly, PTs reflected on issues facing their communities. A PT in the capstone class wrote,

> From my hometown community, there were issues with lack of educational resources and education as a whole (especially mathematics as a subject) was not taken seriously. I

> intend to solve these issues to allow proper education be easily accessible to ensure the success of the students I teach in the future.

For belonging, PTs valued getting to know members of their community and showed a genuine interest in connecting with others on a personal level. One PT enrolled in the College Geometry course wrote, "I feel there was also merit in simply being there and interacting with people, listening to their stories and conversing about a myriad of topics, the simple act of interacting was welcomed." Through community involvement, PTs were also able to engage in generational sharing. During the SL events, PTs were eager to share what they learned in the classroom and became more open to learning from community members. A PT in the College Geometry class working with seniors wrote,

> I want to help others, I want people to learn from me as much as I want to learn from others. I want to be a part of my community, working with others and lending a helping hand in any way I can.

In terms of competence, several PTs felt a sense of accomplishment through teaching. One PT wrote, "Teaching has a way of being rewarding that goes beyond simple thank you's. Seeing those you have mentored grow as learners and then in return want to be a positive influence themselves is amazing." Another PT mentioned the relevance to the application of mathematics, saying, "Something new I learned about my service topic is that it is [not] just about math or geometry. It's about how we can use this math or geometry and apply it to real-world problems to solve them." Other PTs claimed that this experience enhanced their communication skills, with one PT writing that "facilitating the event allowed me to hone my communication and leadership skills, as I guided students through the various activities and encouraged their participation. I also was able to modify the games for students who needed it." Another student reflected on how the SL experience challenged them to make impromptu decisions and produce solutions to challenges they experienced during the SL events, saying,

> There were a lot of English Learning students that came to visit and finding ways to communicate the rules and get them involved was truly a challenge. In my group we did not have a native speaker to assist the students the best we could, but we managed to get by and adapt. I think I personally grew because of this as I was forced to find ways to communicate the material I was teaching to my visitors.

For student experience (as output), it was observed that students found SL to be rewarding and useful for them. Two students enrolled in the College Geometry course stated, "It has really benefited me as a person so doing the service learning with the seniors […] was very rewarding because it's a completely different age group than I'm used to." and "It's very rewarding, seeing people understand things, and knowing that you were a big part of that." Additionally, students asserted that they felt fulfilled (output) by participating in SL. One student wrote, "Learning new things is extremely fulfilling and helpful to people." Furthermore, many PTs expressed that SL allowed them to actively learn the course content. The SL experience aligned well with the content students were learning in their SL mathematics course. One student wrote, "This service-learning experience complimented with what I am learning in the class because the activities always revolve around geometry models and games." The same student then stated that "One new information that I learned about geometry during service-learning was the icosahedron, a platonic solid, because I've never heard of the shape, and I was able to make a model out of paper strips." In addition, students reflected that they experienced personal growth as a learner by interacting with individuals from different age groups and diverse backgrounds. A student in the College Geometry class wrote,

> I grew as a learner as I was able to interact with people who were out of my age range and people who came from different backgrounds and different world[s]. Meeting people who are younger than me […] gave me a brand-new perspective on how they act and view education.

While another geometry student shared, "My motivation has definitely changed from one of completing these projects for credit to completing them to better the community I am now part of." Students were also given the chance to reflect on their past experiences as students and how they can make an impact. One student wrote in their reflection,

> It is important to me that those teaching have a passion towards teaching and care about their students. I've had teachers in the past who just show up and don't care about what their students are learning. This made learning in the classroom miserable. I want students to feel heard and understood. I want them to care about what they are learning so that they can grow up and pass it on. I hope to be a passionate teacher who cares about her children.

Research Question (b): How do PTs view the relevance and usefulness of SL experiences in their mathematics courses on their future careers?

Many PTs shared that they plan to adopt some of the practices they learned through SL in their future careers as teachers. Among the courses from which we collected data, 28 out of 38 participants expressed that their experience with SL has provided them with insights into how they want to teach in the future. The students developed clear and concrete ideas about how they would apply the tools they learned through SL, with one student reflecting, "In the future, I want to use what I learned from these outreach activities in my teaching. I will be adding fun, hands-on games and puzzles to my lessons to get students more involved and help them engage in math."

Another student shared that their experience with SL has better prepared them to obtain their teaching credential, saying,

> This experience taught me the value of collaboration and the importance of working together towards a common goal. As I prepare to start my credential program at [University] and begin my placement at [High School], I am excited to apply the insights and skills I gained from this event. I am eager to create lesson plans that are engaging, culturally responsive, and collaborative. I want to ensure that my students see the relevance of math in their daily lives and feel confident in their ability to succeed.

Through SL, the undergraduate students could see themselves as future and current educators, gaining valuable tools and experience that will help them succeed in the future by making mathematics engaging and relevant to their future students. One student reflected,

> Participating in this event was a significant learning experience for me as a teacher. It taught me the importance of making math engaging and relevant to students' lives. Through hands-on activities and interactive games, I witnessed firsthand how students' enthusiasm for math grew when they saw its real-world applications. This helped me grow in my ability to create inclusive and interactive learning environments. I learned to tailor activities to students' individual needs and interests, ensuring that every participant felt valued and motivated to learn.

Some students demonstrated an eagerness to share mathematics with their future students in a way that combats negative stereotypes and seeks to avoid issues like math anxiety or other negative experiences with mathematics, with one student sharing that, "By attending this event, I learned much about how I can diverge from traditional mathematics education and teach it in a

more exciting and grounded way, which I hope will help absolve mathematics of its negative connotation." It is known that teachers pass math anxiety to students, so having teachers be more cognizant of this and how they can tailor their classroom to their teaching styles will benefit their future students greatly.

Additionally, of the remaining 10 of 38 students who did not clearly indicate an intention to use what they learned through SL in their future classrooms, seven of these students expressed that SL confirmed their desire to teach in the future. Therefore, 35 of the 38 study participants reported that SL gave them confirmation that they want to teach mathematics in the future and gave many of them ideas about how they will be able to engage their future students in mathematics.

Conclusion and Discussion

Written reflections from PTs offered valuable insights into their perspectives on their SL experiences in the (upper-division) mathematics courses. Our results provide support for incorporating SL into upper-division mathematics courses, offering significant benefits for students planning to become teachers, whether at the elementary, middle, or high school level. Through SL, students can engage in experiential learning within their upper division mathematics courses and develop essential work skills, including collaboration, communication, and practical career competencies. Students developed a stronger connection to their future roles as teachers or envisioned themselves pursuing teaching in the future. They strengthened their collaboration and interaction with others in the community. Many students expressed that SL allowed them to actively engage with the course content and complemented what they were learning in class.

Well-designed SL experiences enhance classroom learning and help students not only grasp the content but also develop career-ready skills. This is especially important for undergraduates in upper-division mathematics courses who are preparing to become K-12 teachers. There is no need for PTs to wait until the credential program or student teaching, where the focus is primarily on curricular instruction, to gain teaching experience. SL can demonstrate how and why upper-division mathematics can be meaningfully used in lesson planning for middle and secondary students. Further research is needed to explore the development of upper-division mathematics SL courses that can offer valuable experiences for PTs.

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From "Struggle" to "Acceptance": Andy's Narrative of First-Time Teaching and Her Frames

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Using narrative analysis on experiences from Andy, a mathematics graduate student instructor, teaching for the first time, I address the question: How do frames of teaching and learning compare before, during, and after teaching for the first time as the main instructor? I focus on Andy's frames while reconstructing her semester-long experience, attending to the events that shaped her perception of these frames. For one, her "business" frame of teaching that she entered the semester with was at odds with her newfound responsibilities as the content instructor. This and other findings discussed have implications for the professional development and mentoring of future mathematics content instructors, including how newer instructors might be better oriented to manage and attend to classroom events.

Keywords: frames, noticing, graduate student instructors, narrative

The landscape of research on frames and teaching has no doubt explored the influence that frames have on the practices instructors enact. For one, the way instructors frame mathematics as a discipline and the learning of mathematics can result in responding to students in drastically different ways (Cristobal, 2024; Louie et al., 2021). These differences in frames may also result in perpetuating a culture of exclusion (Louie, 2017; Thanheiser, 2023). As a response, there have been strides, through professional development, in nudging educators towards asset-based frames of their students' contributions to the learning process (Bastian et al., 2024; Scheiner, 2023).

One corner of this landscape which requires more exploration is to understand how frames change outside of programs designed to nudge them. In other words, what requires more exploration is how do frames change as an instructor teaches day-to-day (or course-to-course). Therefore, the research question guiding this study was: *How do frames of teaching and learning compare before, during, and after their first-time teaching as Instructors of Record?*

This contributed report stems from a larger study that aimed to understand the experiences of mathematics graduate student instructors (GSIs) before, during, and after their first semester of teaching as the instructor of record. The perspective of GSIs is important for mathematics education to know and understand (Miller et al., 2018). In particular, for this study, the transformative experience of teaching for the first time provided a research space to explore how frames change. I report on one GSI's experience teaching for the first time which provides insight into how frames of teaching and of learning change in teaching.

There are multiple contributions drawn from this report. For practice, professional development (PD) leaders can develop resources to preemptively address difficult situations drawn from these experiences. Relatedly, I have hypothesized that graduate school is where future college faculty's teaching philosophy and practice begin to form. Therefore, a theoretical contribution of this study is adding to the array of experiences that future instructional faculty may bring when they enter the profession.

Motivation from the Literature

From a brief search, there appears to be little literature on how mathematics instructors' frames change. Two exceptions are Scheiner (2023) and Thanheiser (2023). Scheiner (2023) analyzed written reflections from prospective teachers before and after their experience in a

semester-long course designed to shift their frames with regards to student thinking. He found that within intentional design towards critical reflection of how instructors framed students' thinking, the prospective teachers were able to shift from deficit-based to strength-based frames. Thanheiser (2023) discussed three frames that mathematicians may have about the subject of mathematics itself. These three frames coincided with her career-long changing perception of what mathematics is.

This study is motivated to further the understanding of how instructors' frames change. Whereas Thanheiser provided a glimpse into how frames of mathematics can evolve, her anecdote happens over a large timescale. Whereas Scheiner demonstrated that an intentional PD program can indeed shift instructors' frames, he did not demonstrate such shifts or changes in the absence of intentional PD. The study did not intend to change instructors' frames, only to understand that if frames changed during an instructor's first semester of teaching, how do the frames compare. However, by asking instructors to reflect on their teaching and learning of mathematics, their frames changed. Fortuitously, this report filled gaps from Scheiner (2023) and Thanheiser (2023) by demonstrating that frames can change during a semester of teaching when instructors are asked to reflect on their teaching and learning of mathematics.

Theoretical Perspective

The theoretical focus of this narrative exploration is on frames. Frames (Goffman, 1986) are the interpretive contexts that participants of a given situation use in order to quickly interpret information, filter the details, and decide how to appropriately proceed. In addition, frames are conceptualized as a facet of teacher noticing, where the way one frames the teaching or learning mathematics influences the ways in which they attend, interpret, or respond to situations in the classroom (Louie et al., 2021; Scheiner, 2021). As exemplified by Jessup (2023), a situated and socially constructed conceptualization of noticing, and also frames, allows for the consideration of the instructors' histories that shape how they notice.

This study focused on instructors' frames of their teaching and their students' learning by building upon the analytic framework in Cristobal (2024). In this framework, frames of teaching could be understood as interpretive contexts that support instructors' perception and enactment of (1) what their role is in the classroom as the instructor, (2) what professional knowledge (Kunter et al., 2013) is relevant or valuable in the act of teaching mathematics, and (3) what type of interactions are favored or useful in fulfilling their role and achieving the intended mathematical learning goals.

Similarly, an instructor's frames of (student) learning can be understood as interpretive contexts that support instructors' perception and enactment of (1) what instructors must do for students to learn intended content, practices, and orientations, (2) what content, practices, and orientations should instructors attend to in the classroom, (3) what type of interactions are favored or useful in fulfilling their role and achieving the intended mathematical learning goals, and (4) where the class fits into students' schedules (or career paths) and what challenges competing responsibilities could pose to their learning.

To address the research question, change is conceptualized as change in or across one or more of these seven aspects of the framework.

Data and Method

This report focuses on the results of narrative analysis of Andy, a GSI, during her first semester of teaching College Algebra. Andy was chosen because her narrative illustrated the possibility of how frames change during a semester of teaching and when asked to reflect on her

frames. The data used in this report comes from a larger study of mathematics GSIs that aimed to understand the experiences of mathematics GSIs before, during, and after their first semester of teaching as the instructor of record. In this paper, "instructors of record" are responsible for the content-delivery and assessments of an undergraduate mathematics course.

To reconstruct narratives surrounding frames, I conducted multiple interviews before, during, and after each participant's first semester of teaching, conducted weekly observations, and collected weekly journal reflections. I followed a grounded theory approach during collection (Charmaz, 2006), meaning that subsequent collections were informed by the previous. For example, each new observation began to focus on aspects of teaching and learning that I had noticed in the prior weeks' data, such as GSI's ways of responding to student questions.

To organize all these forms of data, I utilized the three-dimensional inquiry space (Clandinin & Connelly, 2000). The three-dimensional inquiry space involves *interaction* (encompassing personal and social themes), *continuity* (encompassing past, present, and future), and *place* (encompassing physical context and space). These dimensions are used to organize the narratives towards restorying – the process of reorganizing narratives to some chosen structure (Creswell & Poth, 2018). I expand this structure to organize analysis towards both reconstruction of the narrative and theoretical saturation (Miles et al., 2019) about the nature of how frames change.

Within each dimension of the inquiry-space, I continued the grounded theory approach through multiple rounds of constant comparative analysis (Glaser, 1965). First, I focused on themes that relate to frames of teaching, then I focused on themes that relate to frames of student learning. This allowed for an understanding of each dimension for a robust reconstruction of the narrative. After the initial reconstruction, I met with each participant to discuss the accuracy of the narrative.

Findings

In these findings, I illustrate Andy's change in frames by narratives from before, during, and after Andy's first semester of teaching.

Before the Semester

Andy's story starts two weeks before the summer teaching orientation for mathematics GSIs. Andy and I discussed her experiences as a student and her understanding of what it meant to teach and learn mathematics. Then, I asked her to recall a difficult experience as an undergraduate student. She shared that she struggled in a particular physics course that had multiple, seemingly disconnected topics, and expressed a desire for a thread to tie the topics of a course together. For her, instructors should attend to some thread that ties the topics of the course together for the student, an aspect of her frame of student learning.

For Andy, learning mathematics is a collaborative, social endeavor. She expressed that "because you remember certain things that stood out to you as elegant and useful, and other people remember different things from lecture," it was important for students to work and learn together. Thus, student-to-student interactions were highlighted in how she discussed learning.

The process of learning, to her, involves repetition and seeing the "underlying" structure of a problem, filling this "skeleton" with substance from the given information. Therefore, for her, the instructor is there to introduce the structure and choose examples that have clear substance.

When I asked Andy to complete the sentence, "Teaching mathematics means…" She bluntly answered "struggle" and explained further, "[Students] are expecting you to teach them a useful skill which to them will always only be a useful skill…knowing in your head [that] there is like a beautiful structure to it that they don't want you to give them… And you kind of have no right to

make them want more from it, and it's like a struggle." This spoke towards an interpretive context for where her teaching fits into students' career paths, grim as that context may be. Thus, as an incoming instructor, she conceptualized that teaching will be a struggle.

Focusing more on teaching, she believed that teaching is about passing down knowledge from the past, inviting students to participate and appreciate the subject. A skillful teacher, to her, is one with sufficient content knowledge that builds the cohesive story amongst the different topics in a mathematics course. Further, a skillful teacher can encourage students to see multiple points of view to help develop understanding of some mathematical structure. Andy believed that she had the necessary content knowledge. While she admitted that she lacked the necessary confidence, she was willing to try and motivate why we learn algebra to future students.

This was where Andy's frames of teaching and of learning were at, before entering her classroom. It is important to reiterate that she had never taught before, but with her experiences as a student, she was able to voice her conceptualizations of teaching and possible practices she can aspire to enact.

During the Semester

A few weeks passed and it was time for the first day of class. Andy walked into a tight room full of 40 students with varying levels of comfort with the subject. One student already expressed concern about failing and Andy quickly responded with assurance that they won't fail if they put in the work. Throughout the 75-minute class, Andy primarily lectured and often asked if explanations made sense or if anything needed to be reiterated.

In our discussion immediately afterward, she informed me that it didn't feel right to ask them to come to the board and present on the first day. Then, I asked Andy why she allowed some students to leave early when they asked for her permission. She responded that, "I'm providing a service, I'm not their mother," and she didn't feel it was her place to enforce them. This sentiment of providing a "service" was echoed again in her journal reflection at the end of the week, where she wrote: "I felt like I wasn't useful to my students" and "I felt like I didn't know what I was teaching." She wanted to be more useful to them through having clearer definitions and explanations. We agreed that perhaps it was just a consequence of the first day of classes.

However, as the weeks went by, it became clear that her students leaving early and not fully participating in the course was a reality attached to this teaching. The observation notes, the journal reflections, and our conversations mentioned this in some form or another. In our following interview, I asked her how she viewed her role in this classroom, and she shared:

> *Andy (Interview, Week 5)*: My role in the classroom is as someone who's seen the material before and is comfortable with it to try to relay what I know about it in a more approachable way to people who will need to make use of it in the future… I kind of feel like I am providing a resource by structuring the information in a way that I think helps get the idea across. But if they don't want to pay attention or do the work that I'm asking them to do, ultimately, that's them refusing a service they're paying for.

Andy's frame of teaching from before was still there (passing down knowledge), but it had gained a dimension that I labeled as "business." This "business" frame of teaching was apparent in almost all aspects of her teaching. To Andy, her role was still passing down past knowledge, but there is a pressure to package that knowledge neatly and succinctly. Her professional knowledge was focused on content knowledge, and how she can deliver this service with little confusion to the consumers. During observations and our conversations afterwards, it was clear

that she preferred to interact with students individually. She could provide the best service when she only had to attend to one person, rather than multiple, varying levels of proficiency. In the structured interviews throughout the semester, Andy expressed regrets about not being able to provide the best service. For example, after the second exam, she shared that she didn't know what old review material was allowed to be shared with her students. She expressed that it would not have been too much work to make her own review materials but, given her other responsibilities as a graduate student taking graduate-level courses, it was not doable.

Similar to her frames of teaching, Andy's frames of student learning also began to crystallize under a label. I labeled her semester's frame of learning as 'encode-decode'-ing languages and communication which she elaborated on multiple occasions:

> *Andy (Interview, Week 5)*: So, after some weeks teaching this class, I would say that learning math is learning how and why information is *encoded*, and how you would unambiguously *decode* that information.
>
> *Andy (Interview, Week 12)*: They should see like a conventional system for communication. And after seeing that convention in some context, see how it can... uh... illuminate, for lack of a better word, extra information and structures hidden in the information.

Since the goal of learning is to understand encoding and decoding communication, Andy would explain every step of a solution and being as unambiguous as possible whenever she wrote up a solution on the board. She shared in an interview that she hoped that her students learned from her that "you can express the same idea to as many people as possible" while being unambiguous.

In her journals, she consistently set "encourage students to talk to each other more" as a weekly goal. Andy began lessening the time she spent presenting on the board while students sat silently, and she shifted towards discussions at the individual or table-group level. Through this encode-decode frame, she would walk around the room addressing student questions without a writing utensil. When we discussed this, she shared that she wanted the students to be the ones writing as she facilitates their thinking aloud. Andy emphasized that if there was no one asking questions during group work time, she would slowly approach the table that had gone silent.

> *Andy (Interview, Week 5)*: I think usually that there's one of two things happening. The one that is more applicable, I think, to the "what do you look for if you don't hear obvious signs of help" is that both of them are stuck on the earlier question. But one of them moved forward rather than continuing to read onto the question that they didn't know how to do. The other one would be that the student that's ahead knew how to answer that question, but not how to explain it to their partner, or their partner didn't feel comfortable asking them to explain it.

With this 'encode-decode' frame of student learning, Andy also understood that learning is not a trivial task, and that language can be "ambiguous" and confusing. She was also aware of how college algebra is a graduation requirement for her students, and that the necessary skill this course imparted on students is the ability to decode seemingly ambiguous language. Another orientation Andy attended to is students' confidence to succeed. She recalled telling a student, "Frustration is allowed, but shame is not," when the student expressed feeling discouraged about their intelligence.

After the Semester

In light of her experience, Andy said she would like "to honor students experiences as people learning the material" as opposed to how she would have preferred to experience the material. Her 'encode-decode' frame of learning was not faulty, but rather she realized her explanations were too rigorous for what one might expect from a college algebra course. She still views mathematical learning as similar to learning a language and disambiguating information. However, because of her teaching and a particular experience in her own coursework this semester, she realized that there is value in adhering to some mild lack of rigor.

> *Andy (Interview, Post-Semester)*: [talking about squishing, lassos, and ropes] Seeing that picture just helped me like on a gut instinct. I see this picture and know what I have to write down… I think I've been overzealous rather than fundamentally wrong. Like, I do think that at the end of the day it's nice to arrive at a point where you can say this works because of XYZ… But I think I should have given more patience and importance to, "Here are some gut feelings that you can have about this."

On the other hand, Andy viewed her 'business' frame of teaching in a less favorable light. While framing her teaching as providing a well-crafted service did help her interact with individual students, it became stressful to manage her classroom. A consistent theme throughout the semester had been her apprehension in demanding students to engage, pay attention, and not leave class early. Andy unpacked where this 'business' frame stemmed from:

> *Andy (Interview, Post-Semester)*: For me, I had the luxury of being able to spend time on my schooling if I wanted to… and so, of course, it's easy for me to be like "oh, this is a business. If you want to learn what I'm teaching you in class, then you will dedicate the time to or you won't."

Andy does not believe this 'business' frame of teaching will change since, to her, this was a consequence of the current attitudes towards what a college education is. However, similar to her frames of learning, Andy realized that she needed to motivate her teaching, not just through her content knowledge, but by also drawing on more pedagogical content knowledge (Kunter et al., 2013). While she believed it was too late to change her classroom's dynamic, she realized it was part of her role as instructor to ask for students' engagement and attention.

Before the semester had begun, Andy shared that, to her, teaching mathematics meant "struggle" between wanting to show the beauty of mathematics and how students might only want to be shown the useful skills from mathematics. I did not remind her of this, but asked again, "Complete the sentence: teaching mathematics means…" She responded:

> *Andy (Interview, Post-Semester*): Acceptance, I think, like you're always gonna want your students to see the beautiful thing... I think I said something similar, but maybe a little bit more cynical at the beginning. It's like you want them to see the beautiful thing, they don't necessarily care about the beautiful thing. But you have to accept that you're gonna try to show them something nice, and maybe they won't get it, but they'll at least have had to have thought about it for a little bit, and that's something.

Discussion

This report aimed to gain insights into how frames change in a shorter time scale and in the

absence of intentional PD. I also aimed to address the question: *How do frames of teaching and learning compare, before, during, and after their first-time teaching as Instructors of Record?*

Andy's frame of teaching began as a 'I am passing down knowledge from the past to students, inviting them to appreciate the subject.' Then, as the semester progressed, her frames of teaching met with the realities of teaching, mostly the unexpected responsibilities with classroom management. While this new frame still had the essence of giving knowledge, it was packaged in this interpretation of teaching as a business. After the semester, Andy accepted her 'business' frame of teaching, while updating her understanding of what her role is in the classroom. She is both passing down knowledge and enforcing some classroom management.

Andy's frames of student learning began with three distinct themes: (1) there is a thread connecting the course topics together, (2) understanding an underlying mathematical structure comes from repetitive collaborative problem-solving, and (3) the 'struggle' between the beauty and immediate usefulness of mathematical concepts. Then, as the semester progressed, these three themes shaped or combined to become her 'encode-decode' frame of student learning.

Thus, this report reaffirmed that, even in the absence of intentional PD, frames of teaching and of student learning may still change within a semester. Andy's frames before and during teaching coalesced and crystallized because of her experience in the classroom. While she felt herself beholden to her early-formed frames during the semester, she also knew how to better them for her future courses by adhering less strictly to rigor and more on intuition building. At least in Andy's narrative, frames of teaching and of learning do not *drastically* change, but they do strengthen in their clarity and how it informed and will inform Andy.

Implications and Future Direction

One lesson learned from following Andy's semester is towards PD of novice instructors. Andy's main struggle in the semester had been classroom management, and how she was not aware that it was part of her responsibility as a college mathematics instructor. It is tempting to simply tell GSIs that it is 'par for the course,' but this lack of awareness may stem from a disconnect between instructors' expectations and realities of teaching. Therefore, PD might serve novice instructors like Andy better if PD includes stories from previous instructors that demystify the realities of teaching at their institution. In addition, asking instructors to reflect on their teaching and learning is one way to shift their frames.

This study was in a particular setting and context of a new instructor. Thus, exploration of experiences from different settings is necessary to grasp how frames change. By understanding how day-to-day or course-to-course experiences (re)shape instructors' frames of teaching and of learning, we may also further understand how teaching practices perpetuate. For example, Andy accepted that her 'business' frame was her reality because of larger societal forces, which may speak to instructors believing they have no power to change attitudes towards mathematics.

I mentioned how Andy felt regret about not providing students with more review materials before their exam because she had to attend to her own coursework. Andy was not the only GSI in the larger study who felt the pressures of competing responsibilities of a GSI. While GSI PD does prepare new GSIs with ample structure for content, materials, and assessment for their teaching responsibilities, it might better serve the GSIs if PD also included ways to juggle the competing roles of being an instructor and being a graduate student.

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Aspects of Culture and Context which Shape Frames of Teaching and Learning

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Sociopolitical lenses applied to frames research have affirmed that frames are culturally constructed. A subsequent curiosity, then, is to uncover what aspects of culture and context shape an instructor's frames with respect to teaching and learning mathematics. To this end, this study followed three mathematics graduate student instructors during their semester of teaching for the first time. Through a grounded theory design, informed by critical theories, the author found themes within beliefs, values, and norms were salient in these teaching experiences. One consequence of this study is giving names to the aspects of culture and context which shape frames. This insight is valuable for understanding how to shape asset-based frames.

Keywords: frames, culture, sociopolitical, critical lens, graduate student instructors

Culture is ever-present in our lives. The collection of cultures we attached ourselves to, both willingly and by birth, informs how we move through society. It ranges from the small things to the large, like what utensils are used to eat or how language is used. Culture is typically an unspoken force that informs ways of living, with its major elements being symbols, language, norms, values, and artifacts (e.g., Conerly et al., 2021). In addition, the context which an individual is in may dictate what aspects of their culture they draw upon. For example, a school cafeteria, a fancy restaurant, and one's kitchen all dictate different behaviors and norms.

Consequently, culture and context also shape instructors' ways of teaching (e.g., Hofstede, 1986). A review of college mathematics teaching in proofs-based courses found that whether an instructor used instructor-centered or student-centered approaches was largely informed by the instructors' beliefs of what is the best form of teaching (Melhuish et al., 2022). In addition, the norms instructors enact may also differ if the type of course they teach is a computational course or a proof-based course (e.g., Dawkins, 2009). Focusing on culture and context together, Dreher et al. (2020) explored how East Asian (Taiwan) and Western (German) cultures approach teaching and found that different cultural norms informed their different ways of teaching.

This report uncovers the aspects of culture and context which shape an instructor's way of teaching. This study followed three mathematics graduate student instructors (GSIs) teaching as the main instructor for the first time and focused on their frames of teaching and of learning as a proxy to understand their ways of teaching. Teaching for the first-time is a formative period of one's instructional career, where students become educators (e.g., Smollin & Arluke, 2014). This period provided a space for me to notice the aspects of culture and context which inform these GSIs' understandings of what it means to teach and learn mathematics. Thus, this report is guided by the research question: *What aspects of culture and context were salient in the experiences of mathematics graduate students teaching as instructors for the first time which could be attributed to influencing their frames?*

The influence of culture and context on frames has been demonstrated and exemplified (e.g., Jessup, 2023; Scheiner, 2021). However, these studies discussed how the general concepts of "culture" and "context" shape instructors' teaching practice. This contributed report named aspects of culture and context which shape frames of teaching and of student learning. This report also aims to address the culture of exclusion (Louie, 2017) by unveiling and naming parts of *any* culture which could pervade mathematics classrooms.

Theoretical Perspective

The theoretical stance of this study is that an instructor's ways of teaching can be understood from the ways the instructor has framed their teaching and their students' learning. In general, frames are used by individuals to understand a given situation and quickly interpret information, filter the details, and decide how to appropriately respond (Goffman, 1986). In one way, frames have been understood as a facet of noticing – the ways in which instructors attend, interpret, and respond to teaching situations (Louie et al., 2021; Scheiner, 2021).

This study focused on instructors' frames of their teaching and their students' learning by building upon the analytic framework in Cristobal (2024). Frames of teaching can be understood as interpretive contexts that support instructors' perception and enactment of (1) what their role is in the classroom as the instructor, (2) what professional knowledge (Kunter et al., 2013) is relevant or valuable in the act of teaching mathematics, and (3) what type of interactions are favored or useful in fulfilling their role and achieving the intended mathematical learning goals.

Similarly, an instructor's frames of (student) learning can be understood as interpretive contexts that support instructors' perception and enactment of (1) what instructors must do for students to learn intended content, practices, and orientations, (2) what content, practices, and orientations should instructors attend to in the classroom, (3) what type of interactions are favored or useful in fulfilling their role and achieving the intended mathematical learning goals, and (4) where the class fits into students' schedules (or career paths) and what challenges competing responsibilities could pose to their learning.

Inspired by Louie, Adiredja, and Jessup (2021), this study drew upon critical race theory (CRT; Delgado & Stefancic, 2001; Ladson-Billings & Tate, 1995) in order to understand the culture and context which entrench teaching. I further drew upon two CRT-informed frameworks: community cultural wealth (Yosso, 2005) and critical whiteness study (Harris, 1993). In this view, culture referred to "behaviors and values that are learned, shared, and exhibited by a group of people" (Yosso, 2005, p. 76). Furthermore, this study was primarily concerned by the context that these GSIs were teaching in a predominantly white institution (PWI). More specifically, this study aimed to understand how the four functions of 'whiteness as property' (Harris, 1993) surfaced in the ways GSIs framed their teaching and their students' learning. See Figure 1 for a summary of each function. The study was concerned if these four functions materialized in how GSIs viewed mathematical knowledge.

Rights of Disposition Whiteness can be transferred ownership.	**Right to Use and Enjoyment** Whiteness can be used to accomplish and exercise power and enjoyed through affordances of privilege.
Reputation and Status Property Whiteness has a reputation, a thing of significant value, and having that meant having a certain status.	**The Absolute Right to Exclude** Whiteness can be interpreted, not by an inherent unifying characteristic, but by the exclusion of others who are "not white".

Figure 1. Summary of the four functions of 'whiteness as property' from Harris (1993)

CRT afforded the understanding that culture, of whiteness or of students of color, can either empower or discourage individuals. As a lens for empowerment, community cultural wealth extended the notion of 'wealth' which members of communities (of color) may draw upon for empowerment (Yosso, 2005). As a lens for discouragement, the culture of exclusion (Louie, 2017) can be understood as parallel to whiteness as property wherein mathematics knowledge is treated as property which can be transferred, used to exercise power, enjoyed for its privileges, has a reputation in society, and which individuals may be excluded from receiving.

Data and Method

The interview and observation data used in this report came from a larger study of three mathematics GSIs teaching for the first time as instructors of record: Andy, Carlos, and Jacob (pseudonyms chosen by the participants). "Instructors of record" are responsible for the content-delivery and assessments of an undergraduate mathematics course. These GSIs taught at a large, Midwestern, public predominantly white research university. Interviews were conducted before, throughout, and after the fall semester, a total of five interviews with each GSI. Weekly classroom observations were gathered throughout the semester for each GSI. The GSIs taught intermediate algebra, college algebra, or a mathematics course for liberal arts.

Following a grounded theory format (Charmaz, 2006), multiple rounds of coding were done to address the research question. The first rounds identified salient themes inductively (Miles et al., 2019) within each GSIs' interviews, with close attention to how each discussed teaching and learning mathematics. Using the observation data, further trends were identified, either substantiating a theme found in the interviews or becoming new themes. Then, the themes were grouped under categorical labels aligned with the aspects of culture which surfaced. To address context and whiteness as property, the themes were coded as either treating or not treating mathematics knowledge as property, using the guiding question: "does this theme speak towards one of the functions of property? If so, is it challenging the notion or perpetuating it?" These themes were further labeled using the four functions identified in Harris (1993). Finally, using the analytic framework, connections were made between themes and the supports frames of teaching and of learning provided to the instructor.

Results

In these findings, I name the aspects of culture (behavior, values, and norms) and aspects of the context (of being at a PWI and the lens of 'mathematical knowledge as property') which surfaced in the analysis of GSIs' experiences with teaching for the first time.

Aspects of Culture

Behaviors refer to how individuals act. Across these GSIs, it was an unsurprising theme that they behaved like novice instructors such as with classroom management. Furthermore, this behavior materialized in a fear of appearing incompetent or unprepared while they teach. Another behavior that was salient across was attending to students who were silent during group work. When asked how they decide which table group to approach during their group work sessions, all interpreted the silence as not knowing what to do or ask. For Carlos, silence even took precedence over vocalized struggle.

> *Carlos (Week 5 Interview):* Something that would take precedence is if… they're just like looking at it like this, staring without writing anything. At least the other people… they're talking and like sort of trying. And so, a table that can't even start a problem… that would take precedence over a table that is like struggling with a problem.

Values refer to what individuals consider to be important or beneficial. In this case, the focus was on what the GSIs valorized about teaching, learning, and overall experience during their semester which informed their teaching practice and the beliefs about their practice. Here, each GSI spoke of instances when their expectations and beliefs about teaching were challenged by the realities of teaching. This could be a positive challenge. For example, Jacob found the experience fun as opposed to scary.

Jacob (Week 15 Interview): It was less scary than I expected. I think that they [Course Convener] did a good job of like giving us support… I was sort of unsure about [teaching mathematics for liberal arts], but it was actually a lot of fun.

But it could also be a negative challenge. Since he valued providing extensive feedback, Carlos found that grading and providing written feedback took up more time than anticipated. Andy found that students preferred the quick ways of solving a problem rather than what she preferred: the most general (and sometimes technical) way of solving a problem. Across all of them, they discussed the challenge of being both a graduate student attending to their coursework and an instructor attending to their students' learning.

Norms largely refer to the rules and expectations with respect to behaviors (Homans, 1966). For this context, norms referred to rules and expectations which govern discussion in the classroom (Cobb, Yackel, & Wood, 1992). All these GSIs faced challenges on what to attend and emphasize in discussions: algorithms versus conceptual ideas and rigor/technical language versus intuition. In Jacob's case, he preferred to interpret students' full understanding of a concept in the class; always responding with "explain your thought process" to a student's answer. But the online exams of his convened course emphasized correct execution of algorithms to input answers in a box. As mentioned before, Andy preferred rigorous explanations as opposed to quick shortcuts, and she learned quickly that intuition is as important as rigor was. As for Carlos, his students expected him to expand on student's answers. For example, Carlos' students might present their final answer to the class, but it became an expectation (according to Carlos) that Carlos would expand on the process laid out by the student.

One salient theme that overlaps behavior, value, and norm within the data was the difference between lecturing on the board and talking with individual students during class or in office hours. Each highlighted an awareness that they can explain things better when they do not have to account for the varied array of learning styles across the classroom. Within individual conversations, Andy, Carlos, and Jacob were able to tailor how they spoke to a student. This related to how each GSI valorized discussion in the classroom, and Andy even considered individual conversations as being where true learning happens.

Andy (Week 7 Interview): I tend to rush when I'm at the board, and I think I tend to do things in less detail at the board, as a sort of appetizer, so to speak… It's for people who want to meet some degree requirements, and then, like small groups is actual learning, I suppose.

Analysis resulted in an understanding that, broadly, each cultural aspect of behavior, value, and norm had influence on each GSIs frames in one way or another. For one, being novices influenced their frames of teaching. For example, Andy struggled with her role as the instructor with respect to classroom management, because the behaviors she relied on as a novice instructor focused on lecturing and attending to students' questions. The GSIs' particular attention to silence in the classroom also informed their frames of students' learning. An important aspect of learning, to them, was discussion between students and between instructor and individuals. They all understood that their teaching and their students' learning benefited from smaller group interactions.

The conflicts raised from expectations versus reality, algorithms versus conceptual ideas, rigor versus intuition also informed the GSIs' developing frames. For Carlos, he may have

valued giving extensive feedback to students as part of his frames, but the reality of being a GSI meant he needed to prioritize other aspects of teaching and learning that might be more beneficial to his students. For Andy, coming to terms with the reality that students do not want rigor meant changing what she must do as the instructor in order for students to learn the content promised by the course syllabus. For Jacob, algorithms became something that needed attention alongside the conceptual ideas because of the nature of assessment in his course.

Aspects of Context

Here, I report on the findings which address the concept of 'mathematics knowledge as property' as a way to understand the context of teaching mathematics at a PWI. I have opted to keep the discussion and quotes in this section unnamed so as to prevent singling out a GSI. This discussion is not meant to vilify the GSI and their possible contribution to a culture of exclusion. Instead, I focus on how insidious and pervasive these functions or aspects of 'whiteness as property' may be in the context of teaching.

In this light, 'rights of disposition' materializes in the context of mathematics as treating mathematics as a static body of knowledge which can be given to a person willing to receive it. Therefore, an education in mathematics is simply the passage of knowledge. This was somewhat salient in some GSIs discussion of their role in the classroom as the expert, and skillful instruction entailed flexibility in being able to pass down abstract concepts. However, one GSI added that mathematics as a subject encapsulated many ideas and concepts which cannot be easily essentialized under one definition. They further shared that, beyond just receiving a portion of this large body of knowledge, they hoped their students also learned that:

> "[Mathematics] is sort of like a living subject... It wasn't handed down to us from God, that it was constructed by humans to solve real world problem."

This spoke towards the collaborative view of teaching and learning these GSIs have, countering 'the absolute right to exclude' function of mathematics as property. These GSIs spoke against this function that mathematics could be used to exclude (i.e., a math person versus not). To all of them, learning mathematics involved discussion, participation, and engagement among *all* the students in the classroom. Observations of all GSIs also confirmed their attempt and insistence on every student participating and engaging. This spoke towards their frames of learning in that in order to learn, students must join in the conversation. One GSI made it a goal to teach students how to be "a good classmate" through the mathematics group work they do.

The discussions surrounding their status as novice instructors involved their fear of being perceived as an incompetent instructor. Though it does not directly speak to mathematics as property which afforded them 'reputation and status property,' this fear could be interpreted as not having a certain reputation because of mistakes in lecture or answering students' questions. This spoke towards their frames of teaching; in how they understood their role in the classroom or the expectation of it: a perfect instructor.

Lastly, the function of 'right to use and enjoyment' where mathematics can be used to exercise power and enjoy privileges did not surface in the analysis. Instead, it was counteracted by multiple experiences. For one GSI, they wanted to create an "egalitarian" classroom, where everyone is equal and can contribute to how they want to learn. This also spoke against the previous function of property as regardless of one's previous mathematical background, they were treated as equal to their peers. Another example of countering this function of property was seen in all GSIs' discussion of honoring students' preferred way of learning. All GSIs made

changes to their teaching practice as the semester progressed to accommodate students' preferences. One GSI even dedicated time in their time-constrained class to explicitly address the mid-semester evaluations and how changes would be made going forward in the semester. The changes they made also changed their frames of teaching and of student learning, as they learned what interactions or practices they should enact based on students' input.

Discussion

This contributed report aimed to name aspects of culture and context which shape an instructor's way of teaching. As a proxy to understand an instructor's ways of teaching, this report focused on instructor's frames of teaching and learning. Thus, I asked: *What aspects of culture and context were salient in the experiences of mathematics graduate students teaching as instructors for the first time which could be attributed to influencing their frames?*

This report shared some aspects of culture (behavior, value, and norm) which were salient in these GSIs experiences. Behaviors centered on their novice identities and how silence from students was a sign of confusion during work sessions. Values focused on their expectations versus the realities of teaching and how reconciliation had to be made to accommodate these realities. Norms surrounded their conflicts about what to emphasize in discussion. I found that these aspects of behavior, value, and norm overlapped when each GSI discussed their preference in teaching to smaller groups or individuals as opposed to the full classroom.

Moreover, through a lens of critical whiteness study, this report also shared how aspects of context (teaching at a PWI) surfaced in these experiences, either perpetuating or countering the treatment of mathematics knowledge as property. Some GSIs might have conceptualized mathematics teaching as a passing of knowledge, somewhat perpetuating a view of mathematics knowledge as property. However, all of them fervently countered the notion that mathematics knowledge should be used to exclude through their belief that true learning of mathematics occurred in discussions with peers. All the GSIs professed the value in discussions and working together with peers to learn mathematics and enacting practices that valorized collaboration amongst students. I hypothesize that the view of mathematics as a body of knowledge perhaps stemmed from the convened structure of the courses. In other words, the GSIs were handed and asked to teach a course, its curriculum, and its materials with very little say on what is taught and when in each course. There was no evidence from these GSIs of perpetuating the notion that mathematics can be used for reputation or as a conduit to exercise power and privilege.

These aspects of culture and context both influence these GSIs' frames of teaching and of learning. Behaviors, values, and norms shaped the actions they understood as appropriate in the classroom and what their role was in fulfilling their desired learning outcomes for their students. Furthermore, though this analysis did not address all supports of frames, I infer from these findings that GSIs' understandings of the professional knowledge relevant to their course (frames of teaching) and how their course fits into students' academic career (frames of students learning) may also be informed by the values and culture instructors bring into their classroom.

Implications and Future Direction

The main concern of the study was to uncover aspects of culture and context which shape an instructor's way of teaching in order to understand how frames may be shaped or changed. In that light, behaviors, values, and norms were salient aspects of culture which informed instructors' frames of teaching and frames of student learning. Then, attending to inclusive, collaborative practices were salient aspects of context which informed these frames. With these insights, professional development leaders may focus on a combination of behaviors, values, or

norms which shape future instructors' understanding of what their role is in the classroom and how to better attend to equitable learning experiences for their students. Andy, Carlos, and Jacob were privileged in that the courses they taught were convened which provided structure to their teaching, which may have had its impact on the understanding that teaching is an act of passing down a body of knowledge. But this convened structure may have also forced them to attend to more collaboration as that was the emphasized method of instruction during their department's new instructor orientation. Therefore, a departmental or professional development emphasis on collaboration in mathematics courses may be one way to counteract the pervasive nature of the culture of exclusion of mathematics.

The PWI context where this study occurred motivated the exploration of 'context'. In particular, the study approached 'context' through the specific avenue of how 'whiteness as property' or 'mathematical knowledge as property' surfaced in these GSIs experiences. More research in different regions and contexts should be conducted in order to further understand aspects of context which shape frames. These contexts do not necessarily need to use lenses from CRT, perhaps more simple aspects of context may be explored such as the type of course taught (e.g., computation-heavy versus proof-based) or logistical constraints (e.g., 16-week courses versus 10-week).

Lastly, this study built on the growing literature of mathematics GSI development as instructors and their overall experience in graduate school. While the perspective of the GSI is important, it is also not the only one. Experienced instructors, who have been entrenched in their local community and culture, may have drastically different opinions on what aspects of culture and context have shaped their frames of teaching and of student learning. There is much more ground to explore in this area of frames, culture, and context.

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How do Introduction-to-Proof Textbooks Explain Conditionals? Mathematical, Natural-Language and Hybrid Examples

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Mathematics defines the propositional conditional 'if P then Q' as the truth-functional material conditional: it is false if P is true and Q is false, and true otherwise. This is counterintuitive with respect to natural language in that 'missing-link' conditionals (with true but unconnected P and Q) are considered true, as are conditionals with false antecedents. How is this reflected in the examples used by introduction-to-proof textbooks? Do these use natural-language conditionals in their explanations, or avoid them in favour of mathematical conditionals? What characteristics do their chosen examples share, and where do they diverge? We address these questions using a theoretically informed qualitative analysis of content on conditionals in 17 commonly recommended introduction-to-proof textbooks. We consider advantages and disadvantages of different approaches in relation to research in philosophy and psychology as well as in undergraduate mathematics education.

Keywords: conditional, example, implication, logic, reasoning.

Introduction

Undergraduate mathematics students encounter numerous mathematical conditionals, in theorem statements and in more informal claims and conjectures. In content courses, most of these are *universal conditionals* (also called *generalized conditionals*, Durand-Guerrier, 2003) of the form 'if $P(x)$ then $Q(x)$', interpreted to mean '$\forall x$, if $P(x)$ then $Q(x)$'. It is important for students to distinguish such conditionals from their converses and inverses, so that they can avoid errors in constructing and validating proofs (e.g. Selden & Selden, 2003).

In introduction-to-proof courses, students might also encounter *propositional conditionals* of the form 'if $P(x_0)$ then $Q(x_0)$', where x_0 is a specific object. They learn that these are treated according to a *material* interpretation, so that 'if $P(x_0)$ then $Q(x_0)$' is false if $P(x_0)$ is true and $Q(x_0)$ is false, and true otherwise. This gives the truth conditions that we want because it means that a universal conditional is refuted only by examples that we understand as counterexamples. However, it is peculiar in relation to natural language. In everyday settings, people tend to think that 'missing-link' conditionals – e.g. 'if bats have wings then Paris is in France' (Bourlier et al., 2023) – are false or nonsensical, not true. They also usually think that for cases where $P(x_0)$ is false, 'if $P(x)$ then $Q(x)$' is irrelevant rather than true (Newstead et al., 1997).

How is this reflected in the examples used by introduction-to-proof textbooks to explain conditionals? Do texts use natural-language conditionals to support understanding of the material conditional? Or do they avoid natural language, giving up possible analogies in favour of more clear-cut mathematical examples? Do they include missing-link or false-antecedent conditionals? To address these questions, we analysed examples used to explain conditionals in commonly recommended introduction-to-proof textbooks. We consider advantages and disadvantages of different approaches in relation to research in philosophy and psychology.

Theoretical Background

The mathematical conditional has a stipulated meaning: the propositional connective 'if-then' conforms to the standard truth table for the material conditional (Table 1, with illustrative

content). This connective is truth-functional, meaning that the truth value of 'if $P(x_0)$ then $Q(x_0)$' depends only upon the truth values of $P(x_0)$ and $Q(x_0)$. As noted above, it underpins the mathematically desirable interpretation for universal conditionals: the universal conditional 'if $P(x)$ then $Q(x)$' is not falsified by cases where $P(x_0)$ is false or $Q(x_0)$ is true (Edgington, 2020). For instance, the conditional 'if x is a circle then x is an ellipse' is universally true – it is not falsified by circles, non-circular ellipses, or squares (see Table 1).

Table 1. Truth table for the material conditional with 'if x is a circle then x is an ellipse'.

	circle	ellipse	if circle then ellipse
circle	T	T	T
(impossible)	T	F	F
non-circular ellipse	F	T	T
square	F	F	T

However, adopting the material interpretation requires sacrifices because we usually subject natural-language conditionals to an *inferential* reading, assuming that the consequent is caused by the antecedent or otherwise follows reliably from it (Douven et al., 2020). This makes missing-link conditionals sound false or nonsensical. Also, we are usually uncomfortable with with false-antecedent cases, considering conditionals irrelevant in these (Newstead et al., 1997). The picture is further complicated by the fact that natural language often allows 'if' to be interpreted as 'if and only if'. This is fairly standard in conditional promises, threats, and statements about permission (Nickerson, 2015): from 'if you mow the lawn, then I will give you \$5', a person could reasonablly to infer that if they do not mow the lawn, they will not receive \$5 (Cummins et al., 1991).

In mathematics, we carefully distinguish 'if' from 'if and only if'. But otherwise the pragmatics is similar. In theorems and other claims, we usually assert 'if $P(x)$ then $Q(x)$' or '$P(x) \Rightarrow Q(x)$' only when proposing that for all relevant x, $Q(x)$ follows deductively from $P(x)$. Thus, we would happily say that 'if n is divisible by 6, then n is divisible by 3' is true and 'if n is odd, then n is prime' is false. However, we would not usually consider the corresponding propositional conditionals individually. These sound natural when a universal conditional is true and the truth value of the antecedent of a corresponding propositional conditional is unknown, as in 'if 4,686 is divisible by 6 then 4,686 is divisible by 3' (Epp, 2004, p.17). But the natural-language use of conditionals to indicate uncertainty (Grice, 1989) means that they sound unnatural where the truth values are known:

- if 12 is divisible by 6 then 12 is divisible by 3 (true-true case so true);
- if 9 is divisible by 6 then 9 is divisible by 3 (false-true case so true);
- if 8 is divisible by 6 then 8 is divisible by 3' (false-false case so true).

They sound even more unnatural when a universal conditional is false:

- if 3 is odd then 3 is prime (true-true case so true);
- if 9 is odd then 9 is prime (true-false case so false);
- if 2 is odd then 2 is prime (false-true case so true);
- if 8 is odd then 8 is prime (false-false case so true).

These sound strange because we cannot infer primeness from oddness; the '2' and '8' cases sound additionally awkward due to their false antecedents.

Introduction-to-proof texts therefore have issues to consider in the examples that they use. They could engage with the peculiarities of the material conditional, highlighting missing-link

and false-antecedent cases. Or they could focus primarily on conditionals that are more 'natural' in everyday language and in mathematics in that they are universal and submit to inferential interpretation. These choices will have consequences: students' understandings of mathematical concepts are heavily influenced by their experience with examples (Sinclair et al., 2011). Even if a text defines the material conditional clearly, the examples it uses for illustration will influence students' conceptual understanding. In the study reported here, we investigated what introduction-to-proof texts do by asking, for their example conditionals:

1. What balance is struck between mathematics and natural language?
2. What proportions have inferential versus missing links?
3. What other features are evident (propositional vs. universal, specifc content, etc.)?

Method

We assembled a list of introduction-to-proof texts commonly recommended in the US and the UK. The US list was compiled by David & Zazkis (2019); it contained one book per course. The UK list we compiled by identifying introduction-to-proof courses and using websites or contacting lecturers to acquire book lists. Of 66 courses identifiable via the UCAS website of UK university degree programmes, we acquired lists for 52 (nine did not have a list, and five did not respond to enquiries or follow-ups). The US and UK lists were not equivalent because UK courses do not usually have a single 'course book' or identify a main book on their list. We do not consider this a methodological problem because we were not trying to conduct a representative survey; we simply wished to select texts that students were likely to consult.

From the two lists, we identified all books recommended at five or more institutions across the two countries. One of these had no specific content on conditionals, one had minimal content, and four discussed only universal conditionals, usually with a focus on distinguishing a conditional from its converse (Alcock & Sa, in press). We do not discuss these further. The 12 texts that discussed both universal and propositional conditionals are listed in Table 2, together with recommendation numbers in the US and the UK.

Table 2. Texts Analysed, with Numbers of Recommendations.

Author(s)	Title	UK	US	Total
Houston	How to think Like a Mathematician	25	2	27
Chartrand, Polimeni, Zhang	Mathematical Proofs: A Transition to Advanced Mathematics		17	17
Smith, Eggen, St Andre	A Transition to Advanced Mathematics		15	15
Velleman	How to Prove it: A Structured Approach	6	7	13
Eccles	An Introduction to Mathematical Reasoning	6	4	10
Rosen	Discrete Mathematics and Its Applications	3	7	10
Hammack	Book of Proof	3	6	9
Epp	Discrete Mathematics with Applications	1	8	9
Stewart & Tall	Foundations	6		6
Lay	Analysis with an Introduction to Proof	1	4	5
D'Angelo, West	Mathematical Thinking: Problem-Solving and Proofs		5	5
Sundstrom	Mathematical Reasoning: Writing and Proof		5	5

For each text, we listed all example conditionals in the section(s) where conditionals were explained, regardless of whether these were expressed using 'if-then' or 'implies'. We also included biconditionals and conditionals deliberately formulated differently, e.g., 'You can run for president only if you are a citizen' (Velleman, 2006, p.49). Counting was not straightforward because some texts used the same conditional multiple times to make or clarify different points. Because we were interested in the content used, we decided that a single conditional used multiple times would count once. Across the 12 books, we identified 162 conditionals in total.

Analysis and Results

To address Research Question 1, we classified each conditional as *mathematical* or *natural-language*; we also introduced a third category, *hybrid*, because some texts included conditionals involving content of both types. Illustrations appear below.

- *Mathematical:* If x is even then x^2 is even (Houston, 2009, p.64).
- *Natural:* If you pass the final exam, then you will pass the course (Hammack, 2013, p.43).
- *Hybrid:* $\pi = 3 \Rightarrow$ Paris is the capital of France (Smith et al., 1997, p.11).

This classification revealed considerable differences in approach, as documented in Table 3. Some texts used very few conditionals, sticking to mathematical content; some used mainly natural-language conditionals; some used lots of each. Only three used hybrid conditionals.

Table 3. Counts of mathematical, natural-language and hybrid conditionals in the analysed texts, with sub-classification as having inferential or missing links.

	Houston	Chartrand et al.	Smith et al.	Velleman	Eccles	Rosen	Hammack	Epp	Stewart & Tall	Lay	D'Angelo & West	Sunstrom	Total
Mathematical													
Inferential	21	3	13	1	3		7	7	1	7	2	7	72 (45%)
Missing	1	5	7		2					1			16 (10%)
Natural													
Inferential	12	1	2	16	1	8	2	16		3		4	65 (41%)
Missing						1		1					2 (1%)
Hybrid													
Missing			3			1		1					5 (3%)

To address Research Question 2, we classified each conditional as having an *inferential* or *missing* link. All hybrid conditionals had missing links – these were used to make the point that the truth value of a propositional conditional does not require a link between its antecedent and consequent. Some texts made the point explicitly; others relied on readers to notice what they were supposed to learn. Illustrative mathematical and natural-langauge conditionals with missing links appear below (these contrast with the inferential links above).

- *Mathematical:* $(\pi < 3) \Rightarrow (1 + 1 = 2)$ (Eccles, 2007, p.12).

- *Natural:* If computers are machines, then Babe Ruth was a baseball player (Epp, 2004, p.26).

Table 3 shows that the texts' example conditionals overwhelmingly had inferential links[1]. Five texts included none with missing links, and those that did use them included only small numbers. This presumably reflects the fact that while introduction-to-proof courses commonly cover the material propositional conditional, mathematicians are primarily interested in developing students' understanding of universal conditionals.

To address Research Question 3, we sliced the data differently, classifying the conditionals as *propositional* or *universal* with additional (sub)classifications to distinguish features relevant to the theoretical background. Four conditionals were peculiar in some way and did not fit our classification scheme[2]. The remaining 158 did, and we discuss each classification in turn.

Twenty condtionals (~8%) were unambiguously *propositional*. Of these, four were hybrid and had missing links (like the example above). Two were natural-language; these had a missing and an inferential link respectively (again, see above). The remaining 14 were mathematical; nine had missing links and four had inferential links either with a false antecedent or in a false claim. One (from Epp, repeated below) had an inferential link but uncertain truth values of the antecedent and consequent, so worked particularly well with both the logic of the material conditional and the pragmatics of natural language.

- If 57 is prime, then 3 is odd (Chartrand et al., 2018, p.20).
- $-1 = 1 \Rightarrow 1^2 = 1$ (Houston, 2009, p.65).
- If 4,686 is divisible by 6, then 4,686 is divisible by 3 (Epp, 2004, p.17).

Thirty-four conditionals (~22%) had *ambiguous* status in relation to the propositional/ universal distinction because they used natural language (one was hybrid) and invoked a named person (sometimes 'I') or a physical situation. Six of these were promises and two were threats; the remainder were more general assertions. Asserted with clear specific referents, these would be propositional. However, all had inferential links between antecedent and consequent, and readers were usually asked to consider how they would apply in multiple situations, so they were effectively treated as more general claims. Some would naturally be interpreted in line with the logic of the material conditional, others less so; as noted above, everyday promises and threats would often be interpreted as biconditionals.

- If it stops raining by Saturday, then I will go to the football game (Lay, 2014, p.4).
- If you do not get to work on time, then you are fired (Epp, 2004, p.20).
- I am not American implies that I am not Texan (Houston, 2009, p.73).
- If John didn't go to the store, then we're out of eggs (Velleman, 2006, p.44).

The remaining 104 conditionals were unambiguously *universal*.

Of the mathematical universal conditionals, 50 (~33% of the total) were *universal assertions*. Two involved more than one variable; we classified these as having missing links but note that a link is possible with additional specification. The remaining 48 were mathematical claims that could be theorems or conjectures; two were clearly intended to have uncertain truth value, nine were clearly false, the remaining 36 were clearly true. Readers were often invited to consider these in relation to specific cases and thereby to consider propositional conditionals, but their statements were universal.

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[1] Two conditionals did not fit this classification: Lay's (2014, p.7) 'If roses are red, and violets are blue, then I love you' and D'Angelo & West's (1999, p.32) 'If S is "This book was published in the year 73', then $S \Rightarrow P$ is true'.

[2] Those in footnote 1, plus Smith, Eggen & St Andre's (1997, p.11) 'If $4n > 10$ then 1 is prime' and Rosen's (2007, p.7) 'If today is Friday, then $2 + 3 = 5$'.

- $f(a) = 0 \Rightarrow a > 0$ (Eccles, 2007, p.14).
- If $x^2 > 5$, then $x^3 + 12x + 4 < 3$ (Houston, 2009, p.64).
- If p is a prime number, then p is odd (Hammack, 2013, p.57).
- $x > 0 \Rightarrow x + 1 > 1$ (D'Angelo & West, 1999, p.32).

A further seven mathematical universal conditionals (~4%) referred in their antecedents to *specific objects*. These are interesting from a theoretical perspective because such statements are common in mathematics and are technically universal but represent a sort of extreme case – universal statements 'normally' allow multiple different cases for a true antecedent. For some, a student's prior experience might lead to natural consideration of false antecedent cases; for others, it is harder to imagine that it would.

- If $x = 3$ then $x^2 = 9$ (Sundstrom, 2016, p.58).
- If $x = 2$, then x is a solution to $x^2 = 2x$ (Smith et al., 1997, p.14).

Finally, ten mathematical universal conditionals (~6%) captured *universal definitional* relationships, either standard definitions or local definitions constructed for a specific task. A further six (~4%) were definition-like in using categorical *inclusion* relationships. We note that standard definitions are biconditionals, but this and their status as definitions was only sometimes highlighted. Categorical inclusions clearly conform to the logic of the material conditional (in a sense, all mathematical conditionals capture categorical inclusions, but this was more explicit here).

- $a < 0 \Rightarrow |a| = -a$ (Smith et al., 1997, p.14).
- If $g(m,n) = n^2 + n + m$, then $g(n,n+1) = (n+1)^2$ (Lay, 2014, p.22).
- If a real number is an integer, then it is a rational number (Epp, 2004, p.82).

Of the natural-language universal conditionals, five (~3%) captured *universal inclusions* due to category membership or geographic location. These correspond clearly to the logic of the material conditional, so it is perhaps surprising that they were not more often used. Eleven (~7%) we classed as *universal associations* because they had inferential connections, but these arose from human conventions or plans or observations rather than category inclusions or physical causes. Fifteen (~9%) were *universal permissions, promises or threats*. As noted before, these would not always be interpreted according to the logic of the material conditional. Often, they would support pragmatically valid *denial of the antecedent* inferences: for the last two below, the hearer might reasonably infer that if there are not ten people, the lecture will not be given, and that if they do not clean their room, they will not be allowed to watch TV. Such pragmatic issues were sometimes but not always discussed.

- Being an article of clothing is necessary for being a glove (Houston, 2009, p.72).
- If x is a student in this class, then x has visited Mexico or Canada (Rosen, 2007, p.43).
- You can work here only if you have a college degree (Lay, 2014, p.5).
- If at least ten people are there, then the lecture will be given (Velleman, 2006, p.49).
- If you do not clean your room, then you cannot watch TV (Sundstrom, 2016, p.57).

Discussion

This work addressed three research questions about examples used in introduction-to-proof texts' discussion of conditionals:

1. What balance is struck between mathematics and natural language?
2. What proportions have inferential versus missing links?
3. What other features are evident (propositional vs. universal, specifc content, etc.)?

In answer to Question 1, the balance of mathematical versus natural-language conditionals varied dramatically by text: authors do not agree on how to use different content. In answer to Question 2, a large majority of conditionals had inferential links. In line with our theoretical framing, we suggest that this reflects a mathematical interest in true universal conditionals: these can be treated inferentially, and mathematicians are interested in the corresponding proofs. In answer to Question 3, only about 8% of the conditionals used were unambiguously propositional; a further 22% could potentially be propositional but were arguably treated as universal claims. Of the unambiguously universal conditionals, a large proportion (33% of the total) were mathematical claims that could form theorems or conjectures. Smaller proportions of the mathematical conditionals had specific-object antecedents or involved defining or categorical inclusion relationships. Universal natural-language conditionals were spread across inclusions, associations, permissions, promises and threats. Of these, the inclusion relationships most naturally conform to the logic of the material conditional.

This analysis reveals the complexity of choices facing textbook authors and instructors. What influence might their choices have on student learning?

Using conditionals with inferential links has advantages and disadvantages. Students will likely find these 'natural', but they obscure the fact that the truth value of the material conditional depends only upon the truth values of its antecedent and consequent. So their likely utility depends upon the goal. If we want to teach students how to to interpret the conditionals that they will encounter in content courses in undergraduate mathematics, then the collective emphasis is probably reasonable. At least, it is reasonable if we assume that students will overlook the peculiarities of the material conditional. For students who notice those, explanations are somewhat thin on the ground (see also Alcock & Sa, in press).

If the aim is to teach the material conditional as part of mathematical logic, then the examples offered provide only partial exposure to relevant issues. Only some texts confronted the peculiarities head on, explicitly writing out propositional conditionals in order to reflect upon how they relate to the material conditional interpretation. Most avoided this by presenting universal conditionals and inviting thought about how they apply to particular cases. As we know that people often consider conditionals nonsensical when they have missing links (Bourlier et al., 2023) and irrelevant in false-antecedent cases (Newstead et al., 1997), more substantive discussion would likely be needed in order to help students connect their understanding of familiar mathematics or everyday situations to formal logic.

We conclude with two issues worthy of note. First, those conditionals that most naturally conform to a material interpretation were mathematical and natural-language conditionals capturing inclusions in the form of categorical or geographical relationships. Permissions, promises and threats are riskier as these are often interpreted as biconditionals; this does not mean that we should avoid them, but we suggest that they might need more in-depth discussion. Second, we saw only one use of a simple and clearly true inferential universal conditional applied in cases where the truth value of the antecedent would be unknown (Epp, 2004, p.17, 'if 4,686 is divisible by 6, then 4,686 is divisible by 3'). This seems particulary useful in linking mathematical logic with everyday pragmatics, and we suggest that the educational effects of exposure to this and other types of example merit theoretical consideration and empirical testing.

Acknowledgments

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Learning Approaches in an Introduction to Proofs Course

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We take a situated cognitive perspective to better understand how undergraduate students utilize study resources in their introduction to proofs (ITP) course, particularly compared to their use of these resources in pre-proof courses. Prior research has shown that the transition to proof contains an increase in expectations of rigor compared to previous undergraduate mathematics courses. However, less is known about how students select particular study resources to meet these new expectations. We conducted task-based interviews with four undergraduate students enrolled in an ITP course to better understand the study resources they employed in both pre-proof and proof-based contexts, and the potential motivating factors behind adjustments to study methods. We found that students attempted to migrate previously used resources as well as adopting new resources and were motivated to adjust some study habits due to the novelty of the ITP context, a lack of relevant resources, and instructor modeling.

Keywords: situated cognition, transition to proof, study resources

Studies on the *transition to proof*, or the transition of undergraduate students from pre-proof courses into upper-division undergraduate and graduate proof-based courses, have pointed to difficulties students often face during this transition due to "an expectation of greater student autonomy, a change in the nature of the objects studied, and the seeming unapproachability of professors" (Selden, 2012, p. 39). We hypothesize that this transitory period in students' mathematics learning could affect the resources they access for studying and/or the ways they use particular resources. In this study, we wish to better understand (a) in what ways student learning approaches may differ between pre-proof and proof-based mathematics courses, and (b) what aspects of their introduction to proof (ITP) course potentially motivate these adjustments.

By *learning approaches*, we refer to both (a) the materials, tools, and/or relationships that individuals use for learning (*resources*), and (b) how individuals use these resources or the actions the individual takes for the purpose of learning (*resource uses*). This construct is similar to that of study habits (Alcock et al., 2020; Hora & Oleson, 2017) in the literature, but with an emphasis on both the resource *and* its use. By doing so, we hope to go beyond broad categorizations for study resources used by students and better understand the nuances of why they choose particular resources to use in their ITP course. Specifically, our guiding research questions are:

(1) What learning approaches do students in an ITP course utilize when presented with proof-based tasks?
(2) When students express differences in learning approaches between pre-proof and proof-based contexts, what motivating factors are associated with these differences?

Understanding what motivates students to adjust their learning approaches during their transition to proof will help researchers and educators better support future students in this transition.

Relevant Literature

Research on students' achievement in ITP courses has found that the transition from pre-proof to proof-based mathematics courses is not seamless for undergraduate students (Moore, 1994; Smith et al., 2017; Stylianides & Stylianides, 2009). Students often must reconceptualize intuitive mathematical understandings into formal definitions (Selden, 2012) and struggle to develop self-regulated (or independent) learning strategies in ITP contexts that allow for personal reflection and improvement of learning strategies when current methods are ineffective (Talbert, 2015). We conjecture that student difficulty in transitioning from pre-proof to proof-based mathematics courses may be exacerbated by students struggling to either adopt new learning approaches or adapt old ones within ITP contexts. This may be aggravated by the diversity of content and approaches in ITP courses (David & Zazkis, 2020) which could make many of the resources students may have previously used, such as standardized mathematics help websites or online tutoring services, less applicable or helpful.

Given the unique and varied nature of ITP courses and the challenges students face in adapting to them, it is valuable to further understand the learning approaches students employ as part of this transition. Robbins et al. (2004) conducted a meta-analysis of higher education studies on psychosocial and study skill factors, finding that college students' academic-related skills impacted their retention and academic goals. In STEM contexts, research on undergraduate students' use of learning approaches is still mostly exploratory, with studies largely working on characterizing students' digital resource use (Aguilar & Puga, 2020; Ogan et al., 2015) or broadly identifying their study habits through self-reports (Alcock et al., 2020; Hora & Oleson, 2017). We intend to contribute to this area by focusing on learning approaches students adapt during their transition to proof, to enable future work in determining which are effective in meeting the unique challenges of ITP.

Theoretical Perspective

We take a situative view of learning in this research project, meaning the environment in which mathematics is learned is inextricably linked to the knowledge that is created (Lave & Wenger, 1991). This perspective acknowledges learning as fundamentally distributed, enacted, and social (Roth & Jornet, 2013). This understanding of cognition leads us to an interest in learning approaches that includes the physical materials referenced and used by students, as well as the actions they take during their learning process and the individuals they work with. In particular, we highlight the following underlying assumptions within this theoretical framework (based on the work of Greeno, 1998): (a) both behavioral skills and conceptual understandings are aspects of participation within a social practice, and (b) changes in the environment (e.g., physical/temporal location, participants present, etc.) lead to distinct conceptual understandings. We use these key assumptions to orient our study design and methodology, as explained below.

Proof-based course environments have unique expectations for the skills and conceptual understandings the students possess, as demonstrated in the literature (Selden, 2012; Stylianides & Stylianides, 2009). This observation, combined with assumptions (a) and (b) from our situated cognitive perspective, leads us to believe that students' learning will change because the situations are unique. That is, since the content and expectations in ITP are distinct from previous math classes, it is reasonable to assume that student approaches to learning will vary within this new context. We assume that since there are some similarities between pre-proof and proof-based courses, including lecture-based instructional style and references to foundational mathematics skills such as algebra and arithmetic, students will initially attempt to adapt their already generated math class "tools" to a new context (Brown et al. 1989).

Methods

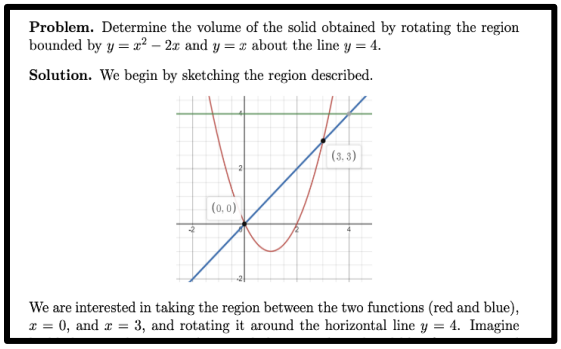
Study Design and Participant Selection

Task-based interviews were conducted with four volunteer students enrolled in a single ITP course at a large public university in the southwestern United States. This research project began as a class project, and the ITP course selection was predetermined for the class project. Students were recruited who had completed a calculus course within the last two years to prioritize recent experiences with the two contexts of interest. The four focal students were interviewed mid-semester, between the first and second summative assessments (Exams 1 and 2).

Informed by our situative perspective on learning, task-based semi-structured interviews (Mejia-Ramos & Weber, 2020) were utilized to explore the learning approaches used by students in pre-proof and ITP courses, as well as motivations for these choices. While our theoretical perspective lends itself to physical classroom observations, practical constraints led us to leverage individual task-based interviews to simulate pre-proof and proof-based classroom environments, an approach with precedent (Roorda et al., 2015; Ticknor, 2012). Ultimately, we assumed that presenting students with pre-proof and proof-based mathematics contexts through these different tasks would encourage students to reflect on different mental processes and potentially unique learning approaches.

Task Design and Data Collection

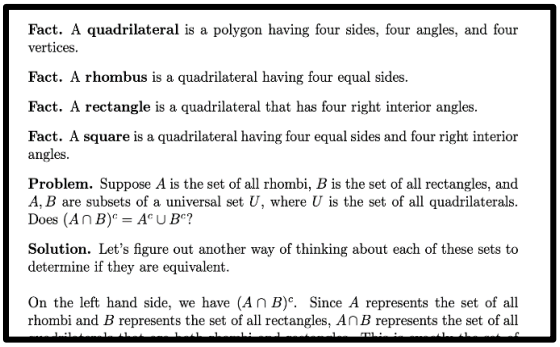
The interviews contained both contextual and task-based questions. The contextual items included general questions about the participant's goals and experiences in recent math classes (both pre-proof and their ITP course). Four tasks were created, two proof-based and two pre-proof, to simulate the contrasting environments. The tasks were presented as problem statements with worked solutions, as shown in Figures 1 and 2, so the student was not expected to produce an answer to the prompt.


Problem. Determine the volume of the solid obtained by rotating the region bounded by $y = x^2 - 2x$ and $y = x$ about the line $y = 4$.

Solution. We begin by sketching the region described.

We are interested in taking the region between the two functions (red and blue), $x = 0$, and $x = 3$, and rotating it around the horizontal line $y = 4$. Imagine

Figure 1. Pre-proof task example.


Fact. A **quadrilateral** is a polygon having four sides, four angles, and four vertices.

Fact. A **rhombus** is a quadrilateral having four equal sides.

Fact. A **rectangle** is a quadrilateral that has four right interior angles.

Fact. A **square** is a quadrilateral having four equal sides and four right interior angles.

Problem. Suppose A is the set of all rhombi, B is the set of all rectangles, and A, B are subsets of a universal set U, where U is the set of all quadrilaterals. Does $(A \cap B)^c = A^c \cup B^c$?

Solution. Let's figure out another way of thinking about each of these sets to determine if they are equivalent.

On the left hand side, we have $(A \cap B)^c$. Since A represents the set of all rhombi and B represents the set of all rectangles, $A \cap B$ represents the set of all

Figure 2. Proof-based task example.

After being presented with each task and given time to review them, the participants were then asked the primary orienting question: *Imagine this was a type of problem you are expected to know how to complete in your class. What study resources might you use to prepare for this type of problem?* Follow-up questions were then asked for each resource mentioned by the student, having students (a) reflect on their use of this resource in their pre-proof and ITP courses; and (b) note any factors that influenced their adoption, adjustment, or removal of learning approaches between contexts. One researcher interviewed the students, while the other researcher(s) present acted as witnesses, keeping memos and supervising the protocol execution

for consistency and quality. These interviews were video- and audio-recorded using Zoom conferencing software.

Data Analysis

Thematic analysis (Braun & Clarke, 2012) was used to systematically organize the data set into groups based on students' learning approaches, then compare these groups to identify themes about the factors motivating any changes in these learning approaches. Following Braun and Clarke's six-phase approach, interview recordings were auto-transcribed by Zoom, then manually cleaned and reformatted by a research member. During this process, data familiarization (phase 1) also occurred as the team made memos regarding notable learning approaches. The research team developed initial codes (phase 2) for one transcript across three domains: (a) *Resources*, or specific tools/processes mentioned by the student; (b) *Resource Use*, or ways of implementing a Resource mentioned by the student; and (c) *Motivators*, or explanations given by the student when noting Resources or Resource Uses that changed between pre-proof and proof-based contexts. Resources were generally given shorter descriptions and corresponding tags for easier sorting; Resource Uses and Motivators were generally given longer, low-inference descriptions that might include direct quotes from the text.

After consensus was reached on the initial transcript, the remaining three transcripts were coded by individual team members, adding new tags and revising prior descriptions, which were reconciled and then reviewed through member checking. Each resource tag was re-checked for internal consistency once all transcripts were coded. The research team then developed and reviewed themes (phases 3 & 4) related to the Resource Uses and Motivators mentioned by students. This was a recursive process of moving between the whole dataset and individual coded chunks while discussing patterned or unique findings with the research team. This process led to the development of three themes (phase 5) that illustrate different Motivators mentioned by students for their change in resource use between pre-proof and proof-based contexts. These are further elaborated in the Results below (phase 6).

Results

Results are presented in two parts as relevant to our research questions. First, we discuss the various learning approaches (Resources and their associated Resource Uses) mentioned by students in ITP contexts. Then, we outline the three themes that illustrate common Motivators for change in learning approaches. Student quotes will be used throughout, with the four student names being anonymized with the following pseudonyms: *Aaron, Basil, Chris,* and *Dana*.

Learning Approaches in ITP

Students mentioned a variety of resources and resource uses that they leveraged when presented with proof-based problems. Table 1 below displays a selection of tagged Resources that were mentioned by at least one student while reflecting on a proof-based practice problem, alongside the different Resource Uses students mentioned for these resources in the proof context. Some Resource categories are specific actions taken by students, such as attending instructor Office Hours. Other categories, like Brainstorming and Visual Aids, are more broadly defined, as students would provide different levels of specificity for more general actions, like thinking over a problem statement or visualizing different aspects of the statement. As for the Resource Uses, students would occasionally mention using certain resources simply to find other, more relevant ones. In particular, students often mentioned using the resources of Internet Search and Homework Assignments to find additional practice problems.

Table 1. Selection of resources students mentioned in proof contexts, with associated uses.

Resource	Description	Associated Uses in ITP
Homework Assignments	Reviewing past (assigned or unassigned) homework problems, including review of instructor comments and/or online homework site feedback	• To find practice problems • To identify patterns in problems
Peers	Working with other students outside of class time, whether online or in-person (or unmentioned)	• To compare answers with others • To "double-check" work with another
Practice Problems	Using practice problems that are similar to the current problem	• To get familiar with similar types of problems • To see a given solution path for a similar problem
Visual Aids	Drawing pictures, diagrams, or other visual aids to represent mathematical ideas	• To visualize geometric objects, function graphs, sets, or set relationships mentioned in problem context
Internet Search	Searching for information about similar content as the problem, with or without specifying an online search engine	• To find similar problems • To review a concept from ITP
Brainstorming	Thinking about or mentally reviewing material; writing down ideas, thoughts, processes, or scratch work; and/or verbalizing ideas to self before formally writing down solution steps	• To prepare oneself and take in the information

We also note that these are not all of the Resources mentioned by the students. Some resources mentioned in proof contexts, like Online Messaging and Question Patterns, were cut for space. Students also mentioned using some resources, including Computational Devices and Online Text Material, only in pre-proof contexts. This list is not meant to be an exhaustive list of resources used by students in ITP courses, but rather a starting point, developed from interviews with four students.

Motivating Factors for Changes in Learning Approaches

Three themes were developed to highlight consistent factors that students mentioned as Motivators for change in their Resource Use between pre-proof and proof-based contexts: *Novelty of Proof Content, Relevance of Resources,* and *Instructor Modeling & Course Structure*.

Theme 1: Novelty of proof content. Students mentioned changing their learning approaches in light of the novelty of the new content in their ITP course. Generally, participants noted a difference in how problems were structured in their course compared to pre-proof courses. As

stated by Dana, "[ITP course] isn't really, you know, solving problems. It's more just finding relationships between things […] they're just completely different kinds of problems." These differences were often characterized as an increased level of complexity or abstraction in the problems. In response to these differences, some students would alter their Resource Use while retaining the Resource itself between pre-proof and proof contexts. For instance, Chris noted that due to concepts in ITP being "fairly abstract," he had increased use of office hours compared to pre-proof courses. Similarly, Basil noted a decrease in his use of online materials, stating that "you can't check a proof like that online [*laughs*] that's not, that's not how that works."

The most common motivation for change due to this new math context was the ability for the same problem to be answered sufficiently with different approaches. All four participants mentioned this as a motivation for change. Dana, Chris, and Aaron cited increased difficulty in using the Resource of Question Patterns, noting that "[ITP] is just harder, because, like, it's more than two patterns for more questions," and "some things I can't [do based on memory], just because again, there's not always one correct answer for everything." Dana particularly mentioned that this decreased the effectiveness of the Practice Problems Resource for her because the variety of solutions made it so she was unable to check her work. In sum, the novelty of the mathematical context in ITP, in terms of both the complexity of questions and multiplicity of solution paths, was related to students' adjustment of their learning approaches.

Theme 2: Relevance of resources. The second theme highlights how some students utilized learning approaches based on the perceived relevance of these approaches to their learning needs. For instance, one student did not use office hours in their pre-proof course, because all lectures were recorded, making additional in-person sessions unnecessary. In cases where students would try to implement a previously used learning approach in ITP, multiple participants commented on struggling to find resources that matched the nuances of their proof problem statements. Basil shared his reluctance to use internet searches, stating, "I would not look up on the internet for this problem, 'cause there'd be way too many miscellaneous things that I would not want to look at. I'd get some third-grade geometry stuff that would show up." This illustrates a common challenge mentioned by students of sifting through irrelevant material, which can overwhelm students with content that doesn't directly help them achieve their learning goals.

Students conjectured that this irrelevancy of material was related to the diverse and inconsistent ways ITP courses are taught. Aaron expressed,

> I think the material is not maybe taught exactly the same between different colleges and even different classes in every college. I feel like each professor's ideas about proofs and kind of the important groups to teach are different […] I think it would be hard to find specific videos on exactly what we were learning.

This mirrors what David & Zazkis (2020) found when noting the various ways ITP courses are implemented in the U.S. The wide variety of online resources available for pre-proof courses contrasts with the specificity required in proof-based learning, where students struggled to find resources that they found relevant to their ITP course content.

Theme 3: Instructor modeling & course structure. Finally, we noted that change was sometimes motivated by the format of the ITP course itself. The *Instructor Modeling* portion reflects moments when students explicitly mentioned adopting a resource after seeing their instructor using or advocating for the resources during instruction in their ITP class. Aaron and Dana mentioned instructor modeling as a reason for their adoption of certain resources in ITP. When it came to Brainstorming, Aaron noted, "our professor makes us think a lot before we talk

to our group mates. She's like, 'look at it for like 3 or 4 min.' So, then I just do that. And then I guess that kind of just stuck." Aaron mentioned not using this resource previously in calculus contexts, instead choosing to "start writing stuff down almost immediately." Similarly, Aaron mentioned using set diagrams and drawings of geometric shapes in ITP explicitly because the instructor used that resource in class while not finding a need for visual means in calculus. Dana had a similar relationship to the Visual Aids Resource, finding that set diagrams were a great way to visualize material in ITP, despite preferring equations to graphs in calculus contexts.

Generally, this theme represents the recurring instances where students mentioned the overall structure and content of the course (and the implementation of that content) as a motivator for resource adoption in their ITP course. One such motivator was the allotment of class time to use particular resources. Chris used the Peers resource in ITP but not in calculus contexts, noting that, "I think [ITP course] is special, because we had peer discussion. We, I also wanted to, like, note down my thought process during it. But in Calculus 2, [it] is pretty much like taking notes from the professor." Because Chris was explicitly given the time to talk with peers in their course, he utilized that resource. Other such motivators include being given additional unassigned homework problems that could serve as Practice Problems, accessing course slides that served as a notetaking guide for Class Notes, and leveraging handouts given as Course Materials. Though these aren't motivators specific to ITP courses, they seem to indicate a necessary condition for the use of some resources: to be allowed access and permission to use said resource in the course.

Discussion

Overall, the themes noted in these students' motivators for changes in learning approaches paint a picture of the transition to proof as being one of uncertainty and novelty, echoing what has been previously found about the challenges of the transition to proof (Selden, 2012; Stylianides & Stylianides, 2009). The students reflected on using a variety of learning approaches in their ITP course, both adapting resources from pre-proof math classes and adopting new resources. They also commented on the factors that motivated some of these changes in learning approaches. We noted the novelty of the material leading to difficulty and struggle, the resources lacking relevance to their course context, and a reliance on instructor modeling for effectively studying in the ITP course as themes in motivation. The overall student discomfort in the midst of transition is exemplified by Chris, who said, "Sometimes [in ITP course], like, the concept is fairly abstract. So, I would have to, like, go to office hours to ask for, like, more detail [...] but in calculus, everything that I learned is pretty much, like, computable. Like, you can calculate it easily. So, it's more thorough." While this is a single example, the comments and tone of the students we interviewed indicated a general discomfort and feeling of uncertainty, especially noting the loss of calculation, a solving technique they felt comfortable with.

This research project serves as a starting point for better understanding the types of resources used by students in ITP courses, particularly compared to previous learning approaches they applied in pre-proof contexts. Future work may expand on this study by increasing the number of participants or leveraging the list of student-mentioned learning approaches to run a broader survey-based study asking students to reflect on their use of some of the mentioned learning approaches. In light of our last theme, if instructor modeling and course structure are motivating to adopt new learning approaches, these components could be used intentionally to foster effective approaches and support students as they are introduced to the novel and abstract world of proof.

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"I Only Have One Black Student… With a Sample Size of One, It's Hard to Say": A Critical Analysis of Epistemology in Mathematicians' Racial Sensemaking

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Mathematics departments nationwide are tasked with addressing racial inequities, including students' racialized experiences in calculus courses. Faculty must interpret racial information critically to advance equity through change initiatives. However, mathematicians often hold colorblind views of instruction that are reinforced by mathematics epistemological values, such as neutrality and objectivity. It is, thus, unclear whether mathematics epistemology inhibits the advancement of racial equity within mathematics departments. This analysis reports on racial sensemaking among 22 mathematicians at a research-intensive university amidst a reform effort to improve Black and Latin students' calculus experiences. Guided by theories of epistemic culture, I compare mathematicians' mathematical interpretations of racial phenomena with those proposed by critical race theory. Findings exhibit how mathematicians conflated contextualizing racial information with stereotyping, and focused on quantities without recognition of students' lived experiences or the social justice implications of their arguments. Implications are provided for using epistemology as a lever for change.*

Keywords: epistemology, mathematics faculty, critical race theory

Study Purpose & Background

Postsecondary mathematics is rife with racial inequity. For example, at the entry level, racialized pressure and isolation in undergraduate calculus can demand additional labor from Black and Latin*[1] students to persist in science and mathematics fields (Battey et al., 2022). At the faculty level, racial dynamics in mentorship and collaboration can contribute to differences in early-career support and faculty development (McNeill & Jefferson, 2024). While these racial issues are, arguably, produced by mathematicians, mathematicians are also tasked with remediating such inequities. For example, research aiming to advance calculus instruction recommends that mathematics departments collect and interpret local data to guide improvements (e.g., Zazkis & Nuñez, 2015). Research focused on practitioner inquiry into higher education inequities emphasizes that faculty must approach such work with equity-mindedness to resist colorblind or deficit interpretations of racial phenomena (Dowd & Bensimon, 2015). However, faculty in mathematics often hold colorblind interpretations of instruction (McNeill et al., 2022; King et al., 2023). It is unclear, then, whether mathematics faculty are prepared to critically interpret racial data when these same racial phenomena may escape their awareness in their teaching and other professional work.

How one curates and interprets information to generate knowledge is a question of epistemology. Disciplinary epistemologies can influence how faculty consider racial issues in their professional work. For example, in a study exploring equity in doctoral admissions, Posselt (2015) found that disciplinary theories, epistemologies, and methodologies shaped committees' perceptions of applicants' qualifications. For example, in the highly quantitative field of

[1] The term, Latin*, is inclusive of all genders (Salinas, 2020). Our use of Latin* avoids the patriarchal use of "Latino" to describe people of Latin American descent.

economics, "economists not only looked for quantitative expertise; they also relied primarily on quantitative measures to gauge quantitative expertise" (p. 818), while philosophers were "deeply uncomfortable with the false precision" (p. 821) of quantitative rankings. These different epistemological orientations played a role in committees' reliance on racialized metrics for evaluating applicants. Similar evidence of interplay between disciplinary epistemology and racial equity has also been found in mathematics departments. For example, mathematics departments can employ seemingly-neutral quantitative metrics for faculty evaluation that invisibly reward traditionally white achievements while neglecting Black women faculty's less traditional scholarly work (McNeill & Jefferson, 2024). Similarly, mathematicians' views of mathematics as fixed and pre-existing parallel common perceptions of students' mathematics ability as fixed and innate, reinforcing inequitable practices like tracking (Burton, 2004).

In the social sciences, the epistemologies promoted by critical race theory (e.g., Delgado & Stefancic, 2023) have become an accepted standard for antiracist thinking. However, mathematical thinking holds a privileged position in society (Gutiérrez, 2013) and many mathematicians believe that mathematical thinking is the purest form of reason (Burton, 2004). For this reason, developing mathematicians' capacity for addressing racial inequity may require unsettling deeply entrenched ways of thinking that are believed to be superior.

The present analysis uses the context of a calculus program's equity-oriented reform effort to examine mathematics epistemological values (MEVs) in mathematicians' racial sensemaking. To inform mathematics faculty development initiatives that seek to advance racial equity, the present analysis explores the following questions. What role do MEVs play in mathematicians' sensemaking about racial (in)equity in undergraduate calculus? How do these mathematical interpretations of race compare to those established through critical race epistemologies?

Epistemic Cultures

Knorr Cetina's theory of epistemic cultures (1999) guided the analysis. Epistemic cultures are "cultures of knowledge societies" (p. 241). More specifically, epistemic cultures arise when an "internally referential" (p. 2) community generates and codifies knowledge based on a largely shared set of epistemological values. Disciplinary communities, such as mathematicians or critical race theorists, are examples of such communities as they each hold values and practices related to the creation of knowledge that are largely distinct from everyday thinking. I hold these two epistemic cultures as the subject of examination and summarize their epistemologies.

Mathematics Epistemological Values

Although there is no universal definition of mathematics, mathematics can be characterized as relating either to proof or to application (Hacking, 2014). Mathematicians generate knowledge from the epistemological practice of proving claims via arguments based in formal logic and reliant on axioms. Although applied mathematics is intertwined with proof-based mathematics, applied mathematics is often distinguished by its use of computations which "proceed in a mechanical way by set rules" (Hacking, 2014, p. 145).

The notion that mathematical arguments constitute "proof" highlights common perceptions of mathematical claims as unequivocally, universally true or false (Burton, 2004; Hacking, 2014). Indeed, when studying how mathematicians generate knowledge, Burton (2004) found that 90% of mathematicians held an objectivist, positivist view of mathematics as existing independent of human thought. Beyond this, mathematicians tended to view mathematics as "the highest form of rationality" (p. 6), and as governing the nature of reality, even beyond the scientific. However, as with all human endeavors, the mechanisms for building mathematical

knowledge are socially constructed (Knorr Cetina, 1999). For example, there are ambiguities related to what constitutes formal logic and what constitutes an axiom (Dutilh Novaes, 2011; Shapiro, 2004). Furthermore, despite framings of mathematics as objective, aesthetics, affect and intuition play centrally in mathematicians' knowledge generation (Burton 2004).

Critical Race Theory & QuantCrit as Epistemological Values

CRT originated in legal studies and theorizes how race and racism operate in U.S. society. Characterizations of CRT vary, but share a core set of beliefs including: (1) CRT recognizes that racism is endemic to American life; (2) CRT expresses skepticism toward dominant legal claims of neutrality, objectivity, colorblindness and meritocracy; (3) CRT challenges ahistoricism… [and] presumes that racism has contributed to all contemporary manifestations of group advantage and disadvantage along racial lines; (4) CRT insists on recognition of the experiential knowledge of people of color; (5) CRT is interdisciplinary; and (6) CRT works towards the end of eliminating racial oppression as part of a broader goal of ending all forms of oppression (Mastuda et al., 1993, p. 6). CRT can also be seen as a set of epistemological values describing ways that knowledge about race and racism is generated. For example, CRT suggests that the lived experiences of students of color are valuable sources of knowledge about race and racism, while ahistorical, decontextualized comparisons of mathematics achievement among white students and students of color are less robust sources of knowledge.

CRT has primarily been engaged in qualitative contexts and some scholars "remain skeptical about the possibility that numbers can ever fully capture the 'material impact' of intersectional racism" (Gillborn et al., 2018, p. 169). However, given the power and utility of numerical information in American society, a quantitative extension of CRT, known as QuantCrit, has been developed that incorporates similar principles. In addition to CRT principles of advancing social justice and understanding racism as ordinary in American society, QuantCrit takes as given that: (1) Numbers are not neutral; (2) Racial categories are socially constructed; and (3) Data cannot speak for itself and must be contextualized by the experiential knowledge of people of color.

Methods

Research Context and Participants

This study took place at Wesselman University, a large, elite, predominantly white, research-intensive university in the southern U.S. During the 2022-2023 academic year, the mathematics department had approximately 30 tenure-track faculty, 20 postdoctoral faculty, 10 senior lecturers, and 25 doctoral students. The demographics of the entire teaching staff were approximately 70% white and 85% male. During the same year, the domestic undergraduate population was roughly 40% white, 20% Asian American, 10% Black, 10% Latin*, 5% multiracial, and 5% unknown race.

In 2020, university administration identified a need to improve the calculus program, particularly to ameliorate racial disparities in calculus grades and attrition from STEM majors. A research team began a collaboration with the Wesselman mathematics department in Spring 2021 to support this reform effort. Our partnership took three phases: (1) Documenting Black and Latin* students' experiences in calculus; (2) Exploring instructor and stakeholder sensemaking about racial equity, and (3) Facilitating professional development with calculus staff. The present analysis is based on data from the study's second phase.

Prior to the beginning of the Fall 2022 semester, instructors teaching calculus I courses were invited to participate in the study at an in-person meeting of calculus staff, and again through

email. Faculty who were stakeholders in the calculus reform effort, such as the department chair, the coordinator for calculus courses, the director of undergraduate studies, and members of the "rethinking calculus" committee, were also invited to participate through email. All who expressed interest participated. In this report, I describe participants in aggregate to preserve anonymity given the unique departmental context. The 22 participants consisted of 11 calculus instructors (five lecturers and six doctoral students) and 13 stakeholders (seven tenure-track faculty members, five lecturers, and a postdoctoral scholar), with two participants serving in both instructor and stakeholder roles. Participants were majority white and men.

Researcher Positionality

In alignment with CRT's rejection of objectivity, I do not detach myself from the present analysis, but instead harness my personal experiences at the juncture of mathematics and social science contexts as a source for knowledge production. I am a white transgender person that holds doctoral training and faculty experience in both mathematics and education fields. While I perceive myself as belonging to the epistemic culture of critical race theorists, my past and present experiences immersed in the culture of mathematics have allowed me to engage either set of epistemic machinery with relative ease. I leveraged this ability while coding to critically evaluate the epistemological underpinnings of participants' reasoning.

I spent three and a half years in partnership with Wesselman mathematics, serving as a researcher, a calculus instructor, and an instructor educator. I frequently experienced the epistemological conflicts that this analysis formally characterizes first-hand. This first-hand experience, combined with my prolonged collaboration with Wesselman mathematics, contribute to the study's trustworthiness.

Data Collection and Analysis

Data consists of interviews which were semi-structured, audio recorded, and transcribed verbatim. Interviews averaged one hour in length, with stakeholders participating in one interview and instructors participating in four. Interview questions solicited participants' perspectives on the role of race in calculus courses and related changes to the calculus program. As examples, stakeholders were asked, "How do you think proposed changes to calculus courses would impact Black and Latin* students' experiences?" Instructors were asked, "How do you see race playing a role in students' experiences or opportunities for success in your course?"

The theory of epistemic cultures guided a data analysis that followed an open, axial, and selective coding scheme (Strauss & Corbin, 1998). Two sets of open codes were used to flag data for implicit epistemological values. One set of open codes captured participants' use or rejection of MEVs. This included instances where participants discussed numbers, used mathematical language (e.g., "sample size," "it follows that"), or otherwise employed mathematical reasoning (e.g., testing a generalization by looking for counter examples), as well as instances where participants framed mathematical thinking as irrelevant or unhelpful. These codes helped to understand how participants did and did not use MEVs in their racial sensemaking, addressing the first research question. Another set of open codes flagged epistemological orientations to race, including excerpts that agreed or contrasted with the CRT and QuantCrit.

Axial coding found connections between the two sets of open codes, addressing the second research question comparing mathematical and CRT interpretations of racial phenomena. Specifically, axial codes identified categories at the intersection of these open codes, where included excerpts engaged or rejected MEVs in similar ways, and shared a common relationship

with CRT epistemologies. Lastly, selective coding extracted thematic relationships between MEVs and CRT epistemological values.

Findings

Although data contained instances in which participants rejected MEVs or engaged MEVs in ways that aligned with CRT epistemologies, participants most often engaged MEVs uncritically in ways that contrasted with CRT epistemologies. In what follows I demonstrate three themes that arose: (1) Using quantitative arguments to naturalize racial inequality; (2) Focusing on representation over experiences; and (3) Conflating contextualization with stereotyping.

Using Quantitative Arguments to Naturalize Racial Inequality

Participants engaged epistemologies of mathematics in their interpretations of large-scale information such as racial disparities in students' grades across disciplines. It was common for participants to question the methods by which large-scale statistics were calculated, as well as their interpretation. Through this, participants framed racial inequity as minimal and exempted themselves from responsibility for remediating racial inequity. For example, some participants denied that racial disparities in grades were cause for concern in mathematics, arguing that grade inflation in other departments produced compression that simply made racial disparities appear less pronounced: "Once you normalize students' GPAs by GPA average in departments, it's not clear how much the problem is centered in math." Similarly, another participant argued,

> This [racial] disparity [in grades]… the statistics were not done right because you have to divide by the variation, the sigma... mathematics classes could have a huge variation, while, let's say English can have a very small variation. If you do that, then everybody is actually falling correctly more or less.

The use of the word "correctly" is notable as it naturalizes the unequal distribution of grades among racial groups. However, its juxtaposition with statistical computations makes this subjective naturalization of racialized grades appear objective. Other participants similarly used quantities to naturalize racial disparities.

> I don't know what the expectation is from the university for the calculus classes… Less people who are Black, Latinos, to fail?... All of them, they don't want to fail?... Let's say you have all of your Black and white students get a similar SAT score, [but] Black students [are] doing worse in calculus. Then that, to me, that's a great problem. But if students with around the same SAT score stay around the same, more or less, then to me that's a lesser problem.

Expecting students with similar standardized test scores to have similar levels of achievement in calculus is a mathematical framing that may seem neutral because it suggests that Wesselman mathematics courses do not exacerbate racial disparities. This frames the perpetuation of preexisting racialized disparities in mathematics achievement as natural.

These interpretations contrast with the epistemological values of CRT and QuantCrit. The idea that any mathematical framing of racial issues is "correct" relies on notions that mathematics is objective. This directly opposes QuantCrit's position that numbers are not neutral and contrasts with CRT's rejection of objectivity. Furthermore, these examples argue against remediating racial inequities in students' grades, conflicting with CRT's social justice aims.

Focusing on Representation Over Experiences

Participants' focus on racial representation to the exclusion of acknowledging students' racialized experiences also illustrated the influence of MEVs. For example, when I shared preliminary findings with the department showcasing narratives from Black and Latin* students,

I heard that faculty had "disagreements with your presentation" because "the data was not quantitative." Instead, participants often focused on percentages and probabilities, neglecting the human experiences that accompany these figures. For example, when interpreting a Black student's narrative of racial hypervisibility when participating in a class where white students dominated discussion, one participant reasoned,

> If there is one dominant student in the class, race aside, your odds of being the dominant student in the class are one in however big the class is. And if you are already in a minority of the people in that class, your odds of being the dominant... it's like these differences are exacerbated by the very fact that they're already in a minority.

The participant uses the mathematical meaning of "minority" (a number that is less than half of the total) and frames racial inequity as a byproduct of the class's racial representation rather than resulting from issues of power and privilege. Participants focused on probabilities and representation even when describing their goals for advancing racial equity. When responding to the same prompt about racialized participation dynamics, another shared,

> I would guess that I'm fitting into fairly standard trends in terms of participation. I think I can get my minority students to speak up a little bit… Yesterday, I think I had two Black students speak up, actually, when I asked questions. So, I'm getting some in there. But... even accounting for probabilities, I would say my white students are much more willing to speak up, even compared to Asian students.

The participant's reflection implicitly frames Black students speaking in class as the goal and does not consider the quality of their learning experiences. Their remark that two Black students spoke, combined with their consideration of probabilities, frames racial equity as numerical.

Both CRT and QuantCrit emphasize the importance of lived experience as a source of knowledge. The QuantCrit value of contextualizing numerical information within experiential knowledge directly contrasts with participants' inattention to students' affective experiences.

Conflating Contextualization with Stereotyping

Participants were extremely hesitant to provide racial interpretations of their instructional observations. Although this was not explicitly mathematical, participants' engagement of mathematical language such as "sample size" and "conjecture" suggested that their sensemaking was influenced by MEVs. For example, when asked their perception of whether race played a role in a successful white student's performance, one participant shared, "Everything I'm about to say is conjecture… but I think she's Russian. Russia has a strong mathematical culture… maybe there's something there." The term "conjecture," in the context of mathematics, describes an act of generalizing from one or more cases to make a universal claim. The participant's hesitation to provide a racial interpretation, combined with their use of "conjecture," implicitly suggests that they equate interpreting racial observations with generalizing or stereotyping. This is also evidenced in the following excerpt, in which a participant was asked to describe their racial observations related to participation in their classroom. The participant, unprompted, voiced skepticism about the validity of a racial generalization one might draw from their observation, using MEVs as justification: "I only have one Black student. He doesn't participate a ton… With a sample size of one, it's hard to say. Is that all of the Black students, or that this one guy is kind of quiet?" The participant's consideration of the "sample size" suggests that knowledge of race should rely on statistically significant trends. Given that generalizing is a common and valued mathematical practice while placing information in a sociohistorical context is relatively less common, it is possible that MEVs shaped participants' misunderstanding of tasks which asked them to contextualize their instructional observations.

CRT and QuantCrit presume that racism is ordinary in American society and plays a role in producing present-day inequities that fall along racial lines. This includes inequities in prior educational opportunities or access to classroom participation. Furthermore, CRT's refusal of colorblindness encourages consideration of racial elements of events that may be seemingly neutral at face value. Hence, a CRT perspective would presume that race plays a role in students' calculus experiences while holding an antiessentialist assumption that race influences students' learning in different ways.

Discussion & Implications

K-12 mathematics education scholars have long critiqued school mathematics as grounded in white, middle-class values that perpetuate racial oppression (e.g., Ladson Billings, 1997). Postsecondary mathematics has been historically shaped by the same global forces of white supremacy and antiblackness (McNeill & Jefferson, 2024; Bishop, 1990; Hottinger, 2016). However, interrogation of the mathematics discipline at this level has been much more limited. Some scholarship has illustrated how dominant epistemological values in mathematics can reinforce the oppression experienced by students of color (Adiredja & Zandieh, 2020; Street 2024). The present analysis extends these findings, demonstrating that dominant epistemological values may reinforce oppression, not just when students' engage with mathematics, but in any context where mathematical thinking and racial issues collide. For example, Adiredja and Zandeih (2020) describe how linear algebra examples produced by women of color and based on their everyday experiences could readily be dismissed as "idiosyncratic" (p. 254) in the dominant mathematical view. The present analysis illustrates that mathematicians may readily dismiss the racial experiences of individuals as idiosyncratic more generally, potentially because dominant epistemologies of mathematics assign little value to individual cases.

Implications for Developing Mathematicians' Critical Consciousness

Faculty development programs focused on issues of equity are becoming more commonplace (Castillo-Montoya et al., 2023). However, equity-focused development programs for mathematics faculty demonstrate mixed efficacy (e.g., Ching, 2018). The present findings demonstrate that mathematicians' sensemaking about race exhibits mathematical features. This suggests that developing racial consciousness may require mathematics faculty to critically re-evaluate the epistemic machinery they employ to consider racial issues. However, given the privilege that mathematical thinking receives in society, it may be unlikely that mathematics faculty doubt their epistemological approaches. Those engaged in developing mathematics faculty racial consciousness should consider making critical epistemological approaches explicit for faculty and addressing epistemological conflicts that faculty raise.

Implications for Research: Conceptualizing a Critical Mathematics Epistemology

The present analysis demonstrates ways that mathematical and antiracist thinking conflicted. However, this is not to say that mathematical thinking is inherently oppressive. A body of research in K-12 education contexts has worked towards reconceptualizing mathematics as critical and liberatory (e.g., Gutiérrez, 2018; Gutstein, 2012; Martin, 2019). Despite the fact that people of color practice mathematics at all levels, critical postsecondary mathematics is undertheorized. Critical race epistemologies suggest that the lived experiences of mathematicians of color may be a valuable resource for such knowledge generation. Future research should explore how mathematicians of color engage mathematics in critical sensemaking to support an epistemic culture change in mathematics.

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Identifying and Categorizing Instructor Considerations for Active Learning Implementation

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In this paper, we discuss our experience collaborating with mathematicians to increase their use of active learning pedagogy in a proof-based linear algebra course. We use this experience to attend to three primary research objectives. First, we identified three primary categories of instructor considerations that would determine whether or not they would incorporate a proposed strategy. Second, we observed and made sense of which of these were most prominent for these mathematicians. Third, we determined what combination of considerations needed to be satisfied to warrant the implementation of a strategy by these mathematicians.

Keywords: Active learning; changing pedagogical practice; linear algebra; university mathematics instruction

When Michelle Artigue (2016) reflected on her career as a researcher in undergraduate mathematics education, she lamented that one of the field's biggest weaknesses was its lack of impact on how undergraduate mathematics courses were taught. Specifically, Artigue was alarmed by the "insufficient dissemination of research results towards the relevant communities or practitioners, and the very limited influence of our research on university teaching practices." (p. 12). Indeed, it is still the case that most university mathematics courses are taught predominantly by lecture (Fukawa-Connelly et al., 2017; Johnson et al., 2018; Johnson, 2019), even though effective, research-based, alternatives are available and have shown positive effects on student learning outcome (see Freeman et al., 2014).

To redress the lack of impact of undergraduate mathematics education research, Artigue (2016) recommended that mathematics educators form "collaborative projects, building and negotiating, jointly with mathematicians and other university teachers, problématiques that make sense for all those involved, and meet their respective interests and needs" (p. 12). In this study, we follow Artigue's suggestion of utilizing collaborative teams between mathematics education researchers and mathematicians to offer insight into instructional change. We describe a collaborative project between mathematics education researchers and mathematicians in which we together attempted to integrate active learning pedagogy into mathematicians' lectures in a proof-based linear algebra course.

In conducting this study, we found that some active-learning activities that seem easy to integrate into a lecture from the perspective of mathematics education researchers were rejected as infeasible from the perspective of the mathematician. Our research aim here was to better understand features of active learning strategies that contribute to their feasibility. In order to do so, we needed to identify considerations, such as obligations and goals, held by the mathematicians we were working with. We then considered the features of the instructional strategies in relation to these considerations to determine what combination of criteria needed to be met for a suggested instructional strategy to be feasible.

Literature Review

Fukawa-Connelly et al. (2016) found that among lecturers "56 percent agreed (and 26 percent more slightly agreed) with the statement I think students learn better when they do mathematical work (in addition to taking notes and attending to the lecture) in class" (p. 279). Yet when mathematics educators have observed class meetings in university mathematics courses, active learning strategies were rarely used (e.g., Fukawa-Connelly et al., 2017). It may be the case that some mathematicians feel that the questions that they ask during class time are sufficient to engage students and have them actively think about the material, or it may be the case that some mathematicians believe the limited active learning that teacher questioning provides is all these teachers can offer if they have the obligation to cover the course material (Woods & Weber, 2020). Alternatively, some mathematicians might be unaware of plausible alternatives to lecturing and still others might feel they lack the time to learn a new way of teaching due to other professional obligations (Johnson et al., 2018). Or perhaps, mathematics educators do not fully understand what it means, to mathematicians, for students to 'do mathematical work' in class. Perhaps this is partially due to "active learning" itself not being well-defined.

Braun et al. (2017) observed that, although recommendations to introduce active learning into university mathematics classrooms are commonplace, there is not a shared understanding for what active learning pedagogy is. In this paper, we follow the Conference Board of the Mathematical Sciences in defining *active learning pedagogy* broadly, where active learning pedagogy refers "to classroom practices that engage students in activities, such as reading, writing, discussion, or problem solving that promote higher-order thinking" (as cited in Braun et al., 2017, p. 124). In other words, active learning in which students are engaged in some activity involving mathematical content that does not involve passively listening to a lecture or practicing procedures. One approach is to use short active learning activities, such as think-pair-share questions or clicker questions, which can augment lectures. While these activities have been used in primary and secondary mathematics classrooms, as well as university physics classrooms (e.g., Crouch & Mazur, 2021), they have been the subject of limited research in undergraduate proof-based mathematics courses (Melhuish et al., 2022, but see Alcock, 2018, and Iannone & Miller, 2019, for exceptions).

Herbst and Chazan (2003) observed that when mathematics teachers do not adapt reform-oriented recommendations for changing instruction, some mathematics educators attribute this to the teachers suffering from a "lack of knowledge or a paucity of vision" (p. 3). Herbst and Chazan argue that this stance is counterproductive. Teachers are not free to teach in any manner that they choose. Rather, they chose to honor obligations to their discipline, to their students, and to their institutions (Chazan et al., 2016). Many instructional decisions, especially in choosing whether to use, or how to adapt, reform-oriented instruction, involve managing conflicting obligations. Here, in understanding our mathematicians' reactions to suggested instructional strategies, we are looking to understand their obligations and their goals for instruction.

Context and Methods

This study took place at a large northeastern university in the United States and involved a collaboration between mathematics educators and mathematicians, where both groups worked together in six weekly meetings to incorporate active learning strategies into a proof-based linear algebra course. The course presented an axiomatic treatment of abstract vector spaces and linear transformation on those spaces and covered topics such as eigenvectors and eigenvalues, the

Jordan canonical form and its associated theorems, and inner product spaces. The research team included the three final authors and four mathematicians agreed to participate in this study. All meetings were recorded over Zoom and transcribed. Only three of the four instructors could fully attend meetings regularly, so only three were included in this analysis.

During each meeting, the mathematics education research team proposed an active learning pedagogical strategy that mathematicians were asked to incorporate into their lessons. After a strategy was agreed upon, its implementation was discussed. The implementation involved choosing specific questions (e.g., if an exit ticket was to be used, what would the specific prompt be?) and the practical issues of how the strategy would be employed in the classroom context (e.g., if an exit ticket was to be used, how would the prompt be given to students and what student artifacts would the instructor collect?) In general, the mathematics education research team would provide mathematicians with an array of questions which mathematicians could then choose to use or adapt. In following meetings, the groups would debrief on how the active learning strategy went, what benefits and shortcomings this might have had from the mathematicians' perspectives, and what changes might be needed for this to be viable in the future.

In analyzing transcripts from the six collaborative planning meetings, we used the method of constant comparison through the grounded theory approach (Strauss and Corbin, 1994) to form a preliminary codebook. First, we began by looking at the data through the lens of Obligations, Instructor Goals, and Features for Implementation. Then, we flagged instructor comments which aligned with these larger themes, resulting in a codebook including these three categories as parent codes. Subcodes were identified by looking for commonalities and recurring themes within each parent code.

The Obligations category included *Time*, *Institutional*, and *Student Preference* as subcodes. The Instructor Goals category included *Building Math Understanding* and *Attending to Social/Emotional Needs of the Student* as subcodes. The final category, Features for Implementation, included *Ease of Use*, *Features of Question Posed*, *Mathematical Features of Question*, and *Student Cognitive Resources and Perspective* as subcodes. Then, we checked for co-coding to observe potential relationships between (1) obligations and features and (2) features and goals for these three linear algebra instructors. Below are three tables defining the subcodes from each category with examples and occurrence counts between all three instructors.

Table 1. A table of Obligations subcode definitions with examples and counts of their emergence.

Obligations			
Subcode	Definition	Example	Count
Time	Available time for class meetings and to cover topics.	"My biggest adversary in the classroom when I'm teaching is the clock on the wall."	74
Institutional	University-set expectations for the course	"The material should really emphasize vector spaces which are not $\mathbb{R}^n$	50

Table 1. A table of Obligations subcode definitions with examples and counts of their emergence.

	and students' prior knowledge.	because otherwise it's 250 material."	
Student Preference	What students are likely to do or how they will react.	"If you ask them to watch a video, I suspect a lot of them will not watch. That is the problem."	37

Table 2. A table of Instructor Goals subcode definitions with examples and counts of their emergence.

Instructor Goals			
Subcode	Definition	Example	Count
Building Math Understanding	Developing students' mathematical rigor and intuition.	"There are students who understand the mechanics and the concepts, but what they have difficulty with is the flow of the logic."	84
Social/Emotional Needs of the Student	Engaging with students, attending to what they want to learn, and potential social influences.	"One thing that I'm trying to get out of active learning is a chance to talk to the students one on one… they'll ask questions they're embarrassed to ask."	44

Table 3. A table of Features for Implementation subcode definitions with examples and counts of their emergence.

Features for Implementation			
Subcode	Definition	Example	Count
Ease of Use	Planning required to implement.	"This is something I should have in my pocket. I need to know what I'm giving at the end."	74

Table 3. A table of Features for Implementation subcode definitions with examples and counts of their emergence.

Features of Question Posed	Easy to comprehend or short questions.	"I wanted to make these so that students don't need a lot of time and can just get to what they want to decide."	38
Mathematical Features of Question	Questions emphasizing particular math concepts.	"What's wrong if you just define the distance as the inner product of a vector within itself?"	85
Student Cognitive Resources and Perspective	Instructor beliefs about students' abilities.	"If you want them to create something, that's impossible for the beginning. So I have to give them an example."	37

Results and Discussion

Using Atlas.ti software, a table of code occurrences was developed to identify the prominent Obligations, Instructor Goals, and Features for Implementation influencing the use of suggested strategies by the three linear algebra instructors. The predominant Obligation from the collective analysis was *Time* which was referenced 74 times between all three instructors. The predominant Instructor Goal was *Building Mathematical Understanding* which was discussed 84 times. The predominant Feature for Implementation was *Mathematical Features of Question* which was mentioned 85 times. Although these aspects were identified to be most significant in the collective analysis, other key factors surfaced when investigating the respective profiles of the three instructors.

Time was the most referenced Obligation and *Building Math Understanding* was the most prominent Instructor Goal for all three instructors respectively; however, variation emerged between instructors regarding the Features for Implementation category. The primary Feature for Implementation for both Instructor C and Instructor Y was *Mathematical Features of Question*. Of all three categories, Instructor C spoke most often about concerns regarding the *Mathematical Features of Question* which was followed by *Time*. Instructor Y also spoke most often about the *Mathematical Features of Question* and *Time*. However, Instructor S primarily discussed *Ease of Use* and *Building Math Understanding*. *Mathematical Features of Question* was one of the least mentioned considerations for Instructor S.

In addition to how frequently certain topics were mentioned, the instructors expressed their openness to more engaging practices such as student discussions but would conclude that there was simply not enough time allotted. This underscores the need to align with institutionally imposed obligations and its precedence over instructor interest. Similarly, instructors' need to attend to specific mathematical features stems from this dominant institutional priority to cover required content and distinguish the course from its calculation-based predecessor. Instructors at

times chose to modify questions to attend to these considerations and reinforce their goal of building students' intuition and rigor. With these findings, the first two authors constructed Figure 1 to showcase the collective impact of the most commonly discussed institutional requirements and student-related considerations on the implementation of proposed strategies.

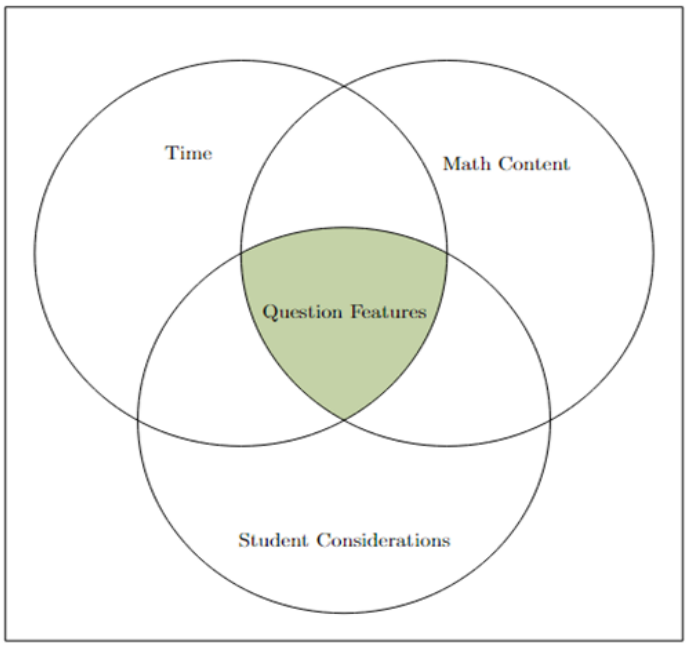


Figure 1. A three-circle Venn diagram illustrating the prominent considerations for implementing a proposed strategy.

Considering its prevalence from the collective analysis, Time reserves its own circle of the diagram. Time impacts some previously mentioned considerations for implementation, such as needing to satisfy university-set course expectations and the planning required to do so while incorporating a new strategy. Math Content was assigned its own circle given the emphasis placed on Building Math Understanding and Mathematical Features of Question. Finally, Student Considerations encompass the collective influence of Student Preference, Social/Emotional Needs of the Student, and Student Cognitive Resources and Perspective. Although these may not have been as commonly mentioned compared to the other two circles, each instructor did attend to how their students may respond or did respond to their efforts to implement a strategy or expressed beliefs concerning their students' mathematical or cognitive abilities.

This Venn diagram also provides the criteria these instructors deemed important to satisfy when considering implementing a proposed strategy. More specifically, the nature of the questions being used for a proposed strategy sits at the intersection of all three circles. Short, easy to comprehend, and mathematically precise questions seemed to be perceived as the most feasible option for instructors to adapt and incorporate into their lessons and planning. The instructors expressed that shorter and easy to comprehend clicker questions would meet their need for questions that could be answered quickly to alleviate timing concerns and reduce anticipated difficulty unrelated to the mathematical content. Mathematical precision was also

emphasized, especially by Instructor Y and Instructor C, to mitigate potential issues regarding term definitions and emphasize the significance of aspects such as vector spaces.

To distinguish this course from the previous, conceptual questions seemed to be of greater interest than calculation-based questions, which may also be due to interest in quicker questions. Additionally, such questions which also pinpoint whether a common misconception has been addressed or not was observed to be an important feature. A final component which resonated with all instructors was planning certain questions to be given after students had time to reflect on what they had learned outside of class time. The instructors expressed apprehension toward providing Exit Tickets on concepts students had just learned based on their beliefs that students would not fully understand the concepts that soon. Instructor S, for instance, instead chose to incorporate Do Now activities in the following class to check students' understanding of previous concepts before introducing new material.

Conclusion

Due to our small sample, we cannot claim that our results generalize to other mathematicians or to a context beyond a proof-based linear algebra course. In particular, we do not have grounds to claim that the obligations or goals that we outlined would be shared by other mathematicians or that others would agree with the features that made instructional strategies more feasible would be valued by others. However, we are able to make some observations about what appeared to matter most to these mathematicians as they considered implementing different active learning strategies.

In essence, the interaction between time spent covering the mathematical content and addressing student understanding is important as a constraint for the group of instructors we assessed. It was important for researchers to find the most time-efficient way to determine student understanding of the mathematics involved if they are hoping for their strategies to be implemented. This suggests that research requires the input of instructors who are aware of the most common misconceptions in the material to bridge the gap between research and what happens in the classroom. These findings also suggest that instructors may have found Do Now activities more time-efficient than Exit Tickets based on their intuition regarding students' window of retention.

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Students' Perceptions of Their Professors' Integration of Computation in Their Probability and Statistics Courses

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Technology has become an integral part of undergraduate mathematics, particularly the use of technology to solve problems (i.e., the use of computation). In probability and statistics, this push has resulted in several projects designing and assessing tools that are conjectured to be advantageous to students and their learning. Despite this trend, minimal research exists on how students perceive the use of computational tools in their courses. As such, we designed a brief survey for students enrolled in introductory probability and statistics at a university in the Northeastern United States. Using thematic analysis, we qualitatively analyzed these survey responses to explore their perceptions of the integration of computation into their courses. Three themes were identified, relating to features of tools, augmentation of actions, and long-term benefits. This exploration of students' perceptions allows us to better understand their views on computation and the need for professors to make instructional goals explicit.

Keywords: Computation, Probability and Statistics Education, Student perceptions

There have long been pushes for the further integration of technology into educational settings (e.g., Bray & Tangney, 2017). While technology has been implemented in education in several ways (e.g., instructional videos), we are focused on computation (i.e., technology as a tool for solving problems) and computational tools (i.e., devices with computational capabilities). To that end, we follow diSessa's (2018) framing of the use of computation, as a form of literacy (i.e., computational literacy), shaping communities' activities.

According to diSessa (2018), mathematics is particularly well positioned to incorporate the use of computation. Indeed, there have been a number of studies probing the use of computation in mathematical contexts (Chan et al., 2023). These studies have identified a growing interest in the use of computation (Weintrop et al., 2016) and a range of tools being used within mathematics education (e.g., Scratch, MATLAB; Chan et al., 2023). This is further evidenced in the context of probability and statistics through calls to promote the inclusion of computation as a means of exploring and visualizing concepts (e.g., Carver et al., 2016).

Despite the growth in the use of technology and computation, specifically, in mathematics education, a question emerges as to students' perceptions of their use of computation. Namely, past research has identified the positive effects of using computation on learning statistics (Wilensky, 1995), features that computational tools should include (e.g., interactive capabilities; Johnson & Berenson, 2019), and a breadth of reasons for using computation (e.g., career preparedness; Nolan & Temple Lang, 2010), but it is not clear what students think about their professor's decision to use computation in their courses. In this study, we were guided by the following research question: *What are undergraduate mathematics students' perceptions of their professor's reasons for integrating computation and specific computational tools into their probability and statistics course?*

Literature Review

The origins of the value of modern computation in mathematics trace back to the work of Papert (1980) and the creation of Logo, a programming language that has been argued to offer unique perspectives and learning opportunities in mathematics. Since Papert, many new computational tools have been designed and the positive potential of computation in mathematics has been demonstrated (Bray & Tangney, 2017).

For example, Wilensky (1995) presented a study where computation led their students to understand an apparent mathematical paradox. Specifically, Wilensky had students use the computational tool of Starlogo, a programming language. The paradox explored in this study arises when tasked to find the probability that a chord of a circle, chosen at random, is longer than the radius of the circle. Mathematically, one's method for picking a random chord determines the solution and there are at least four valid methods (i.e., the answers to this question include ½, 2/3, ¾, and $\sqrt{3}/2$). In the study, after a student produced a solution (e.g., by constructing a chord through a process of fixing a point on the circle and letting the chord's endpoint vary), the researchers presented a different method for selecting a random chord and the resultant solution. After wrestling with the apparent paradox (i.e., two different, mathematically correct, solutions to the problem), the students were prompted to use programming to simulate the problem context. Through the process of writing the program, the students came to a crossroads about how to construct the chord. For many students, this process of programming resulted in the realization that one's decision on how to construct a random chord was the determining factor in the probability you derived. Wilensky argued that the use of programming was fundamental to this discovery and allowed students to gain new insights.

In another study, Basturk (2005) identified a difference in learning outcomes in a study that compared the performance of graduate students enrolled in an introductory statistics course with a computation lab to students who took the same course without a lab. In the treatment, the SPSS statistical software was used by students who met for an extra 40 minutes a week to work on computer exercises and use real data sets to apply concepts learned in lecture. Researchers found that students who used computation scored statistically higher on midterms and final exams than control students. Basturk concluded that SPSS was a useful tool for introductory statistics.

Beyond students' mathematical performance and thinking, the integration of technology has been conceptualized as an influential part of probability and statistics instruction (Nolan & Temple Lang, 2010). Nolan and Temple Lang (2010) argued that the use of computation is an important aspect of preparing students for their future careers because practicing statisticians are continually using and being influenced by advances in computation. As such, it is important for students to learn statistical content using computation and learn broader computational skills (e.g., data structures, debugging) that allow them to adapt to future tools. Despite the influence computation can have on students, it is unclear whether students or even instructors are aware of these advantages to using computation in introductory statistics courses.

Several studies have assumed the importance of computation in introductory statistics courses and sought to identify what tool best serves students. For example, Johnson and Berenson (2019) designed a study which ranked seven different computational tools along 11 different categories. The goal of this project was to identify specific aspects that made certain tools better for teaching and learning statistics. Three of the categories were classified as utility related (e.g., availability in the workplace, cost) and the remaining eight were classified as performance related (e.g., simplicity of access and downloading, user friendliness). Johnson and Berenson concluded that JMP, a menu-driven statistical software, served these criteria the best

and was a strong option for introduction to statistics. While some professors may have a similar list of considerations when choosing a computational tool for their courses, it is unlikely that these considerations are made explicit and that students are aware of the benefits of a given tool.

While several projects have identified reasons supporting the use of computation, it is possible that students do not perceive these affordances. For example, Povey and Ransom (2000) analyzed their undergraduates' journals kept over the course of a semester using computational tools like graphing calculators, Logo, and Excel and identified several surprising perceptions their students held about computation. Within these journals, "tales of resistance" (p. 48) toward these tools were identified. Specifically, students expressed concerns about being overly reliant on the tools, a lack of motivation in mathematics stemming from the use of the tools, and a lack of reflection produced by the speed and black-box nature of tools. While this study is from over two decades ago, it is reasonable for a disconnect to be present between the instructional goals of the use of computation and how students perceive the goals of computation as a whole.

Indeed, students' perceptions of their professors' goals do not always align with professors' goals. Lew et al. (2016) conducted a study comparing real analysis students' perceptions of the key points of a lecture to their professor's intended key points. Lew and colleagues reported a disconnect between what the professors of the study intended for their students to attend to and what students made note of as important. Similarly, this disconnect between objectives and students' perceptions has been identified in the use of instructional videos (Weinberg & Thomas, 2018). As such, inquiry into students' perceptions of why their professors integrated computation into their course is a needed lens for the use of computation in mathematics education.

Methods

Participants and Survey Design

To understand student perceptions of instructor goals for integration of computation into their probability and statistics course, we designed and administered a survey to students in an introductory probability and statistics course at a large, private, university in the Northeastern United States. This course was selected because the official course description mentions the use of computation. Each semester, this course is taught by five to eight different professors and serves over 400 students. Professors are given full authority to decide which computational tool they will use and how they will use it. A recruitment email was sent to eight professors who were teaching or recently taught the course, asking them to send out the survey to their students.

In the survey, students were provided a definition of computational tools, which we described as "a piece of technology used to solve problems and draw conclusions," and then they were asked who their professor was, to describe their background experience using computation, and to list the computational tools that were used in their class. Finally, two open-ended questions were posed: (a) "Why do you think your professor incorporated computation, **generally**, in your course?" and (b) "Why do you think your professor had you use that **specific** tool in your course?". We refer to these as the general and tool specific questions, respectively.

We received 106 responses from students from six different professors' classes. While no computation was required as a prerequisite, most students reported some background experience with Excel or Python. All students reported using at least one of three computational tools in their course: Minitab, R, or Python. Minitab is a menu-driven software built for statistics. R is a programming language used for statistical computation and visualization. Similar to R, Python is a programming language frequently taught in introductory computer science courses.

Thematic Analysis

As described by Braun and Clarke (2006), the six phases of thematic analysis were used to identify patterns in our students' responses. Phase 1 began by reviewing student responses and summarizing each response into short low inferential phrases. For instance, one student said, "[computation was used] in order to help visualize certain concepts through digital means." We captured this as *tools can be used to visualize concepts/graphs/data.* During this phase, several responses were omitted because the students did not respond to the question posed. For example, some students took the survey as an opportunity to complain about their professor. For the general question, 13 total responses were omitted, and two responses were omitted from the specific tool question. In Phase 2, initial codes were induced. To do this, the brief summaries from Phase 1 were grouped in Lucidchart according to patterns of meaning. After these groupings were established, initial code names and definitions were determined. Next, all survey responses were coded. Importantly, codes were non-exclusive and multiple codes were applied when survey responses contained multiple perceptions. To ensure inter-rater reliability, Phase 2 culminated with all data being coded by two of the authors. Any discrepancies were discussed and negotiated. Phase 3 and Phase 4 consisted of rounds of thematic mapping of the induced codes, which allowed us to identify themes present in the data. Themes were then defined in Phase 5. This entire process was iterative, with the authors refining definitions, changing code names, and re-clustering themes to ensure an accurate description of student responses.

Results

Through our thematic analysis, three themes and 14 codes were induced (Table 1). In this section, we define our identified themes, outline each of the codes within the respective themes, and provide characteristic student responses for each code. Codes are italicized for clarity.

Table 1. Themes and codes inducted from students' perceptions

Themes	Codes	
Consideration of a Tool's Features	Preference of Professor Student Background Ease of Access Ease of Use	Built for Statistics Availability of Resources Versatility
Actions Augmented Using Computation	Ease of Calculation Exploration of Parameters and Data	Use for Visualization Collaboration
Longer-Term Benefits of Using Computation	Conceptual Understanding Use for Applications	Career Readiness

Consideration of a Tool's Features

Consideration of a tool's features captures the perception that computation and computational tools were used because of the affordance of the attributes related to computation, generally, or of a specific tool. Most occurrences of this theme came in responses to the specific tool question (71% of responses to the specific tool question contained at least one of the codes for this theme, compared to just 9% of the general question responses). In total, this theme has seven codes (see Table 1).

The first two codes of this theme, *preference of professor* and *student background*, relate to users' previous experiences with computation. *Preference of professor* captures students' perception that the tool used was preferred by or familiar to the professor. For example, one student said, "[Python] might just be the tool the professor likes working with." Similarly, *student background* captures students' perception that their or their peers' previous experiences with computation influenced the integration of computation. For instance, several students felt their professor was aware that many students were computer science majors. One student claimed, "a significant portion of us have experience working in Python through our majors."

Our next two codes focus on the perception that their professor chose computational tools that were easy to get ahold of and use. *Ease of access* captures the perception that computation or a specific tool was used because it was easy to find online, freely available through the university, or conveniently integrated into specific types of notebooks (e.g. Google Colab). For example, one student stated, "R was used in the Google [Colab] application, which …[made] it accessible to all students." Alternatively, *ease of use* captures the perception that a tool was easy to learn, approachable, and intuitive. One student said, "Minitab is relatively simple and easy to understand even in your first session." 47% of students evidenced *ease of use* in their responses to the tool specific question, which was the most common perception evidenced in our data. Furthermore, 67% of the responses coded as *ease of use* were from Minitab users.

The final three codes of this theme center on attributes of the design of a specific computational tool. The code *built for statistics* captures the perception that a tool was used because it had features which made it particularly suitable for a probability and statistics course (e.g., having built in statistical functions). For instance, a student said, "R is built for statistical analysis and computation… R just has the functions already built in". Similarly, *versatility* captures the perception that a tool was adaptable to other contexts (e.g., other problems, classes, academic settings). One student claimed, "[Python] is a very versatile tool and can be adapted for usage far outside of this course." For this student, the use of Python was a strategic one, which we interpreted as being a part of teaching students a tool that has a breadth of uses. The last code of this theme, *availability of resources*, captures the perception that a tool was selected because information for its use was online. For example, a student said, "R had many features as well that were easily researchable." This suggested the perception that professors care about the students' ability to locate information and learn how to best use a tool through out-of-class resources.

Actions Augmented Using Computation

Actions augmented using computation captures the perception that computation and computational tools were used because of the tasks that can be completed and processes that can be improved upon with computation. The student responses that evidenced this perception documented actions made possible and easier through computation and specific tools. This theme was identified more frequently in response to the general question (55%) than the tool specific question (34%). In total, this theme has four codes (see Table 1).

The first code, *ease of calculation*, captures the perception that computation allowed for efficient and accurate analysis, particularly in comparison to doing calculations by hand. *Ease of calculation* was the most common perception evidenced in the general question (cited by 34% of students). One student said, "[Computation is] useful for solving stats problems quicker and more accurately than solving by hand." Going a step further, many students who evidenced a perception that *ease of calculation* was a part of their professors' motivation, appreciated this benefit. For example, a student stated, "Computation is a very useful and efficient tool for statistics and statistical analysis. Not including [computation] would prove a detriment."

Beyond calculations, computation's use was perceived to be motivated by the augmentation of exploration, visualization, and collaboration. *Exploration of parameters and data* captures the perception that computation allowed for quantitative manipulation of data and experimentation with varying parameters. For example, a student claimed, "Computation was used to demonstrate how different variables impact distributions. This can easily be done with computation." *Use for visualization* captures the perception that computation allowed for the creation of graphs and figures which allowed students to qualitatively see something important. One student said, "[R helped] with visualizing distributions through plotting data." The final code of this theme, *collaboration*, captures the perception that the ability to work together and easily share or export data, was also a factor in their professors' consideration. One such student stated, "R is…easy to setup & collaborate on through notebooks, which makes it ideal for a group project."

Longer-Term Benefits of Computation

Longer-term benefits of using computation captures the perception that computation and computational tools were used as a mechanism for delving deeper into the material or as a steppingstone for something students may need later in life. We use "longer-term" in the sense that the perceived reasons for using computation have an impact that extends beyond its immediate application (i.e., one assignment). Once again, this theme was more common in responses to the general question (77%) than the tool specific question (40%). In total, this theme has three codes (see Table 1).

Our first code, *conceptual understanding*, captures the perception that computation allows for better comprehension of course concepts. One student said, "[Computation] allows for greater understanding and practice of the material." Analogously, another student said, "[computation] allows us to think more broadly about the concepts." For these students, computation was perceived to be a part of how their professor was improving their teaching. As such, computation's use in their course was perceived to be motivated by the longer-term benefit of computation's influence on learning and understanding of the course's content. For the general question, Minitab users' responses were most frequently coded for *conceptual understanding* (50%), and R users were the least frequent (18%).

Use for applications captures the perception that computation allows students to gain authentic experience with statistical problems and see the applications of their learning. For these students, computation allowed them to use pre-existing data sets or generate data for applying statistical concepts in an authentic way. In one instance, a student claimed, "[Computation was used] to assist us in applying what we learn to real world examples of raw data." For students who expressed this perception, computation was a way for their professors to give them "real" experience using statistics.

Finally, *career readiness* captures the perception that exposure to computation was useful for preparing students for future academic or professional settings. Students who evidenced this code focused on the idea that computation will be used in their careers, rather than focusing on the general benefit of computation in their current course. One student said, "in real life computation is more likely to be used in the statistics field so by giving us experience with it we are more prepared." Another student's response, to the tool specific question, was coded for *career readiness* because they claimed, "[R was used] to help us when modeling statistical information in our careers and in academic pursuits." These responses shared a perception that their professors were focused on the use of computation in the field of mathematics or statistics and, generally, that computational skills will be needed later in life. Compared to the Minitab/Python users (31%), the R users most frequently evidenced this perception (51%).

Conclusions

In this study, we identified 14 perceptions our students held about their professors' implementation of computation and computational tools in their undergraduate mathematics course. While there are several reasons why a professor may use computation or a specific tool, this work explores the different perceptions students hold of their professors' goals. The 14 perceptions were further clustered into three main themes. The first theme, consideration of a tool's features, centered on the affordances of different attributes of computation or of a specific computational tool. Several codes in this theme align with the criteria used by the statistics instructors in Johnson and Berenson (2019) to identify the best computational tool for introductory statistics courses. For example, Johnson and Berenson considered the criteria of the availability of the tool and the cost. Similarly, our students perceived *ease of access* as a part of their professor's decision to use computation or specific computational tools. Furthermore, Johnson and Berenson's criteria included whether a tool was user friendly and had features that were simple to use (e.g., simplicity in data importing). Likewise, our students expressed the perception that *ease of use* was a part of their professor's implementation of specific tools.

The second theme we identified, actions augmented using computation, centered on the tasks that can be completed with the use of computation. Again, this theme appears to align with real considerations when integrating computation into mathematics courses. For example, Bray and Tangney (2017) categorized the use of technology in mathematics education research as either transforming or enhancing a professor's course. In our theme, the codes of *ease of calculation* and *use for visualization* capture ways in which our students perceived computation to be about enhancing their work. Similarly, *exploration of parameters and data* described students' perceptions of activities that might not be possible without computation and, thus, were transformed by its inclusion in their course.

The last theme, longer-term benefits of computation, focused on the affordances of using computation that temporally extends beyond its in-the-moment use. Once again, this perception can be supported by the field's conceptualization of computation. The first code we identified, *conceptual understanding*, is grounded both by the influence computation has on students' mathematical thinking (Wilensky, 1995) and the learning gains associated with the use of computation (Basturk, 2005). Relatedly, *use of applications* is firmly established in the calls made by introduction to statistics course materials (e.g., GAISE; Carver et al., 2016). Finally, Nolan and Temple Lang (2010) made theoretical claims about the use of computation for analyzing real/complex data and the importance of preparing students for the evolving landscape of tools used by statisticians. This supports our students' perceptions of *career readiness*.

This study is suggestive of several avenues of future work. Foremost, while it was beyond the scope of this research report, the data collected from our survey offers the opportunity to compare, quantitatively, how students' perceptions differed based on the tool used in their course, their background experiences, and their professor. For example, Minitab users simultaneously perceived *ease of use* and *conceptual understanding* most frequently, but R users' perceptions more frequently aligned with *use for applications*. This emergent result warrants further inquiry. Additionally, several studies have identified a difference between professors' intentions and students' perceptions (e.g., Lew et al., 2016). As such, our identified themes offer a framework to compare professors' motivations behind their use of computation, generally, and a given tool, specifically, to how their students responded to our survey. While many of the perceptions we identified are grounded in the literature, it remains an open question whether they align with the professor's goals or are a product of a broader narrative.

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"When I Write in English, It Doesn't Touch My Heart": Bilingual Graduate Students' Language Switching in Mathematical Understanding

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Bilingual students are an integral part of the mathematical graduate education system, so understanding their perspective on learning mathematics is important to ensure they are provided proper support. While many students learn undergraduate mathematics in English, some graduate students' first exposure happens at the upper undergraduate or graduate level, introducing additional challenges. The purpose of this study is to discover reasons for bilingual graduate students' language switching when understanding and experiencing mathematics. All four bilingual participants were in a math PhD program and acquired their undergraduate degrees outside of the United States in their respective languages. The results revealed that students switch languages not only between reading textbooks or during lectures, but also when working on solving mathematical problems and internalizing difficult concepts.

Keywords: code switching, graduate students, bilingualism

Bilingualism is a complex concept that does not have a universal definition. Although a consistent component of the term is the use of two or more languages, there are slight differences in definitions seen in the literature. Grojean (1997) defined bilingualism as "the ability to produce meaningful utterances in two (or more) languages" (p. 164) and bilinguals as "people who use two (or more) languages (or dialects) in their everyday lives" (p. 164). On the other hand, Planas and Setati (2009) added a focus on comprehension of language describing a bilingual as "someone who has acquired or learned to speak and understand the world by means of two languages, though the contents of her/his understanding may not be the same depending on the language that s/he uses" (p. 38). This highlights a potential discrepancy between using or speaking a language and the application of that language in different settings.

The science and engineering fields, many of whom utilize mathematics, report an increase in foreign-born students. In 2017, almost 60% of the workforce in computer science and mathematics were foreign-born (National Science Board, n.d.). Because of the almost universal use of English in higher level mathematics, many of these students need to shift their mathematical thinking from their native language to find success in advanced education programs. This exploratory study seeks to understand the language shifts of students with a bachelor's degree in mathematics achieved predominantly in their native language. Therefore, our research question is: For those students, what are the reasons behind switching languages when seeking help in understanding mathematical concepts?

Literature Review

Analogous to the myriads of definitions of bilingual people, language switching, or code switching, has been defined in numerous ways (Green & Wei, 2014; Hofweber et al., 2020; Moschkovich, 2007). Zeller (2020) claimed that code switching "ranges from inserting single words into the main language of discourse to alternating languages for larger segments of a discourse and can occur at various syntactic positions and with various word classes" (p.1). Alternatively, Bosma and Blom (2019) adopted a definition of code switching as an "ability of a bilingual to alternate effortlessly between two languages" (p. 1432). Both definitions have one

common element – code switching is a dynamic process of integrating one language into another and should not be regarded as a lack of language proficiency in any of the languages, but rather as a sign of bilingual competence (Bosma & Blom, 2019). When considering code switching, the asymmetry between language switching sometimes corresponds to the notion of minority-majority language pair, where the majority language is used by most inhabitants of the country (Bosma & Blom, 2019).

Despite the existent notion and practice of language switching, the approach in classrooms mathematics bilingual students follow may be more intuitive. In a report by Clarkson (2005), eighty Vietnamese students were studied while solving mathematical problems in English and 51% described switching to Vietnamese for at least one problem, which supports the use of code-switching as one of the strategies for problem solving. Bernardo (1999) discovered that the success of Filipino-English students in solving word problems in Filipino was higher than in English due to a better comprehension of the stated problem. According to Bernardo and Calleja (2005), during their study of eighty-five Filipino fifth graders, when mathematical problems were presented in English, students reverted to their native language for several reasons. One reason was the social aspect of the classroom which allowed students to seek help from other classmates in Filipino (Bernardo & Calleja, 2005). Another circumstance of language switching occurred when students perceived a problem to be too hard, whether it was a written word problem in English or what may be regarded as straight processing of algorithms or counting. Students were less likely to produce solutions when the problems were presented in their non-dominant language (English) and were more successful in applying appropriate arithmetic procedures to problems written in their dominant language (Filipino).

The research shows that bilingual students tend to use more advanced strategies for solving complicated mathematical problems in their native language (Domínguez, 2011). According to Domínguez (2011), "if bilinguals are strategic language users, they may also be strategic problem solvers, especially if learning to problem solve is viewed as a communication-based experience" (p. 309). While many bilingual students use their native language outside of school, different language of instruction might provide additional cognitive costs which could result in lower accuracy and processing speed (Saalbach et al., 2013). Domínguez argued that integrating students' everyday experiences into mathematical problems could blend two languages together, which was an approach that has been posited for equitable mathematics education (Adiredja et al., 2020). Participants tended to have more discussions, arguments, and risk taking along with mutual learning, which appeared during solving problems in Spanish, their native language, while reverting to traditional approaches when working on problems in English.

The findings from the Dominguez study were further supported by Ní Ríordáin and Flanagan (2020). Two undergraduate Irish-English students, whose dominant language was Irish, chose to interact in English, as it was the dominant language of instruction in their university. This reflects the impact of the social context on students' choice of language. That social context was also apparent in the study of Catalan-Spanish speaking students (Planas and Setati, 2009), where students used Catalan in a socialization setting while briefly addressing the whole group and Spanish when acting like mathematical learners in their small groups (Planas & Setati, 2009). Other studies (e.g., Planas and Civil, 2013) showed that students' overall demeanor changed to more relaxed when using their home language in mathematical explanations. In contrast, when using their non-native language, students' presentation style changed to less natural and non-engaging.

According to Ní Ríordáin and Flanagan (2020), alternate use of two languages is not only a demonstration of proficiency of the languages but of the social circumstances of the languages in use. The dependence of bilingualism's expression on the immediate surrounding makes this process a highly socially contingent phenomenon. In academic settings, the shift to a common, or a higher status, language may occur more often as it pertains to students' perception of access to social goods, while lower status language may bear the perception of being related to segregation and marginalization (Ní Ríordáin & Flanagan, 2020). It is crucial to note that in these settings, the social process is integrated in social interaction where issues of identity and power are structural (Ní Ríordáin & Flanagan, 2020), so the freedom of expressing oneself is often constrained by a social aspect or the surroundings.

The reviewed literature demonstrates the occurrence and reasons for language switching in mathematics, and it describes reasons and effects of language switching. However, to our knowledge, examining language switching among mathematics graduate students has not been investigated. With that, we hope our findings will contribute to the evolving literature and bring awareness to the experiences and learning approaches of bilingual graduate students in mathematics.

Methods

The aim of this exploratory study is to acquire insights on reasons behind bilingual graduate students' language switching when working on mathematical problems. To achieve that, four graduate students were recruited from a central, high-research, U.S. university's doctoral program. All participants were approached in person and a set of qualifying pre-screening questions was asked. Among ten reached graduate students, four agreed to participate in this study that earned their bachelor's degrees outside of the United States (Table 1).

Table 1: Participants and Backgrounds

Participants	Bachelor's Degree	Master's Degree
Bach	Vietnam	U.S.
Ergun	Turkey	Turkey
Maria	Colombia	U.S.
Miguel	Mexico	Mexico

The semi-structured interviews were video- and audio-recorded in a seminar room ranging between thirty and sixty minutes. Open-ended interview questions were developed with a purpose to learn each student's educational background along with their mathematics-related studying and learning practices. For example, participants were asked, "Can you describe your transition to all-English lectures?" or "Did you ever use online or offline resources in your native language to understand some mathematical concepts better?". Additional follow up and clarifying questions were posed.

Once data was collected, the interviews were transcribed. Since the study is exploratory and no known coding scheme was used, all transcripts were coded using a hypothesis coding approach (Saldaña, 2013). The hypotheses were "developed from a theory/prediction about what will be found in the data before they have collected or analyzed" (Saldaña, 2013, p. 147) and additional codes were identified and classified as "other". The coding was done individually by interviewers, compared, and finalized based on mutual agreement: "[they] made no attempt to

develop a numerical reliability rating, because our goal was consensus, with each point of difference being debated and clarified until the group agreed on appropriate usage of the set of codes" (Harry et al., 2005, p. 6).

Results

During the interview analysis, and after the original hypotheses, the research team discovered three main categories of language switching: external factors (during lectures and reading textbooks); internal processes (processing mathematical information and solving problems); and social influences (driven by social or environmental factors).

External Factors

As all four participants earned their undergraduate degrees outside of the United States, their initial learning of mathematics primarily happened in their native language until higher mathematical content was introduced. This equipped them with the ability to understand mathematics in multiple languages, widening their pool of resources. For example, Ergun's current interest is p-adic groups, so he discovered a Turkish professor's work on that topic in a Turkish mathematical digital magazine, explaining the concept clearly. Miguel has searched for recent Spanish dissertations written on topics of interest to receive the most recent information available in his own language. Although helpful, online sources are generally not frequently used by the participants because of the lack of these resources online. Maria stated that finding advanced math-related resources online in Spanish was not easy compared to those in English and Ergun stopped consulting Turkish resources altogether due to their scarce numbers. Similarly, Bach found it disadvantageous to consider online sources in Vietnamese because questions he might seek answers for are already stated in English:

> And I am Googling something like that... Becauseif you look at the words in the textbook, you will type that in English, right? You don't type it in Vietnamese. So, you type that and it will show many English textbooks so that's... Obviously, I have to look at English only.

While most online resources were deemed insufficient, in some situations, participants were able to find complement material to available resources including books and hand-written notes in their native language. For example, Miguel kept Spanish copies of "Topological Groups" and "Introduction to Topological Transformation Groups" as well as Spivak's Calculus, and Maria continuously referenced a Spanish copy of Munkres' "Topology" textbook. Similarly, personally written notes in participants' native language were regarded as a valuable resource. Both Maria and Miguel tend to consult their own previously taken notes in Spanish due to familiarity with notations or simply because the content of those notes was once challenging. Interestingly, Maria reports that the frequency with which she refers to her old notes changed as she learned more mathematics in English throughout her academic career, identifying her knowledge replacement in a new language.

The externally influenced language shifts during lectures were beyond the control of our participants, but they seem to have had a positive impact. During Bach's senior year of undergraduate study, mathematical lectures were held primarily in English, which provided early exposure to mathematics in English. Likewise, Miguel's early introduction to mathematics in English happened during conferences:

> Some people came to the National University of Mexico, they did their lecture, and it's, in some sense, I believe a really safe environment because it's not like they're going to question you if you understood the material or not. So, that helped a lot, I think.

The notion of gradually easing into learning mathematics in English appears to have a positive effect on Bach and Miguel's perception of language switching in a math setting. Beyond lectures, textbooks written in English were often used during undergraduate education. Bach shared that the number of comparable mathematical textbooks written in Vietnamese was scarce, so while some texts were recommended by his professors, he often searched for relevant English literature on his own. The practice of favoring English texts also influenced Maria's undergraduate experience, where the textbooks were in English or Portuguese, but the proofs were still expected to be written in Spanish, even if she referenced the non-Spanish textbook.

Internal Processes

The translation is not foreign for mathematical graduate students and, rather, at some stages of their academic careers, is integral. Translation can be considered as an internal factor of language switching controlled by the student compared to external factors which are outside of participants' control. Bach shares his communication experience while using internal translation for double switching between receiving information in English, generating a response in Vietnamese, and translating it back to English: "So, before we actually speak that one, we think about it in Vietnamese and we translate words by word, word by word," which is not applicable solely to verbal communication, but to written, too. Maria shared that, when solving math problems stated in English, she began by "thinking in Spanish and then trying to write it down in English when I first came here," but then by "one semester after my first semester I started to think more in English". For Ergun, reading math problems is not the challenge, "the difficulty starts from generating arguments. And so, to attack the problems... my brain automatically goes to Turkish." This presents a vivid example of an internal language switch, where participants unintentionally default to their native language for response creation or problem solving due to sufficient mathematical knowledge acquired in their native language.

The notion of mental internalization and translation we label as "inner voice", which entailed mental work participants conducted while learning or solving mathematics. For instance, as Ergun worked on a math problem stated in English, he wrote questions and keywords to himself in Turkish:

> Yeah, I have some English statement, mathematical, maybe proposition. So, now, I'm working on some bilinear forms. So, I kind of check myself and I see that these questions… I'm asking lots of questions to myself, and writing lots of keywords, and it's kind of a puzzle and makes sense after some point if I work on it in one hour or two hours. But if I get some new good proposition, write in English. But then, still the teaching part is always Turkish to internalize, to understand some deep, deepness... To have some deepness still. My voice, inner voice is Turkish.

There is a strong connection of Ergun's mathematical work to his language, as he shared:

> So, some writings are Turkish, and some write things are English, and it's a mixture… When I write in English, it doesn't touch my heart. So, if I write in Turkish, I feel much more at home and just... I get more motivation... The pop-up questions are much more fruitful for me. Sometimes English is mechanical. I don't know. Maybe it's a linguistic issue. I mean, the words always trigger something, and they are connected to each other.

> And, in my native language, it's connecting very quickly in my mind, in brain, but English words is much more slow works for me.

It is prominent that Ergun's linguistic attachment to his native language plays a significant role in solving mathematical problems. While participants' current lectures are purely in English, there are times, like with Ergun, their notetaking is still in their native language. For example, Maria finds herself jotting down quick notes in Spanish to provide herself a gist of presented material. The language switch during note taking emphasizes the connection of our participants to their native language, as it is an important aspect of their identities.

The thought process around problem solving in different languages is unique and intriguing. As it can be observed, Ergun integrates Turkish and English as his problem-solving strategy where the translation process is no longer obvious but, rather, fluid and continuous. Ergun has a well-established practice that intertwines the language of the problem and language of mental processing. However, Maria shared a different experience: "when I translated from Spanish, there is where I have those problems. Worse. They are, like, worse if I translate. No, I need to think in English and try to write it down in English." This indicates that language switching when solving problems for some students is not a helpful strategy.

While language can be tied to previous experience, it can also serve as a helpful resource. Maria described a time during her topology class where a notion of universal set was introduced. At first, she encountered difficulties with conceptualizing the definition, so the professor provided an alternative name – ambient set. Since Maria took topology class in Colombia, she immediately recalled the familiar Spanish word "ambiente" which triggered previously acquired knowledge about ambient sets. Similarly, while working with his advisor, Miguel encountered concepts stated in Spanish and struggled to understand their differentiation. When he attended to his professor, he was given English alternative definitions of these concepts that immediately distinguished them from one another.

Social Influences

Another type of language switching identified seemed to be dictated by social norms and a strive to learn English. For example, Maria shared her early attempts to write proofs in English, saying: "When I was finishing my undergrad, I was... trying to write down the statements of the homework and try to write down the proofs in English… it's easier in English to find a master's, with a scholarship of course." The described strive to improve proof-writing in English was driven by a notion of socially accepted high status of the English language in a mathematical setting. Similarly, Ergun stated that, "Mostly I gave up to look up Turkish resource maybe at the beginning, after one year in my master's program, because I was feeling that I need to learn to read English, just need to improve myself more." The idea of integration is perceived by participants to relate to improvement of their English skills, which is what Bach shared about:

> Later I figured out that this is not convenient to just use Vietnamese. So, in general, I want to improve my English, so I stop thinking as Vietnamese. So, then I stopped using... Googling or finding this term in Vietnamese… I will say I preferred English. Yeah, because now I am in a completely 100% English program, so I have to do more with English. So, instead of using these sources in Vietnamese, I just start reading English.

The data shows that our participants view English as either higher status language or majority language (Bosma & Blom, 2019), which could skew their perception of their native language's importance or relevance.

Discussion

Our goal in this paper was to learn why bilingual graduate students shift between languages when working through mathematical problems to further understand how language and problem solving intertwines as well as to highlight potential hurdles bilinguals might face during their graduate careers. Another objective of this study was to determine what impact transitioning from (mostly) non-English lectures to purely English mathematical courses had on our participants.

In the light of our research question, we conducted four semi-structured interviews with mathematics graduate students who earned their bachelor's and, sometimes, master's degrees outside of the United States with the primary language of instruction being not English. During their bachelor's or master's experience, all participants consulted English advanced mathematics textbooks based on professor's reference and for the expansion of available information and resources. We also found that our participants' personal notes taken in their native language served as a valuable resource as they contained familiar mathematical notation and content viewed as challenging in the past. Even though online resources were not indicated to be in the highest demand, two graduate students used queries in their native language to widen the pool of information or simply because they prefer a particular non-English author. We also discovered that internal language switching while solving math problems was not uncommon among our participants, as some switch to their native language when producing mathematical arguments which enhances their understanding and success in proving problems.

To our knowledge, there are no studies done with bilingual graduate students pursuing a PhD in mathematics on language (code) switching, so we strive to contribute our results to the currently growing literature. Almost all, if not every, department of mathematics have graduate students with similar backgrounds as our participants, and while carrying multiple languages for these students can be a tool for studying mathematics, lack of certain vocabulary in their second language can impose difficulties during their studies. We believe it is important to be aware of additional support graduate students might need to succeed in graduate programs, even if it is a simple recognition and understanding of potential language barriers.

While participants were representing four different continents and three different languages, our results were limited to a small number of perspectives. A broader study to include experiences of those from other countries and continents would gain insight into the global needs of different students.

Implications and Future Research

The results of our findings reveal a potential need to improve integration of bilingual students into English-led courses. This may lead to professors and educators being more mindful of a language gap in not only mathematical domain, but also in everyday language. As the results showed, some students found success in intertwining both their native language and English, while others were convinced switching to English was the best choice for them. Instructors should encourage graduate students to practice different problem-solving techniques which could involve language switching. Future work in this area should consider increasing participants' pool to diversify participants' experience, which would enhance the knowledge and awareness of various graduate students' practices and difficulties and could more broadly be done for other domains of graduate programs. Also, interviewing participants in their native language could reveal more insights to the researcher while circumventing a language barrier.

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Inquiry into Suitability of a Qualitative Framework about Scaffolding Moves to Support Mathematical Modeling for the Purpose of Quantitative Analysis

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An important goal in research on mathematical modeling is to understand the influence of teacher scaffolding moves on students' model construction. This paper validates the use case of two common frameworks (one for scaffolding and one for student modeling processes) with STEM undergraduates. Study I investigated whether variability in scaffolding strategies could predict variability in participants' modeling processes. Findings suggested that the initial assumption—that scaffolding moves articulated by the framework are clearly aligned with specific modeling processes—was not supported. Study II further explored how scaffolding moves related to distinct stages of model construction and sought to improve cohesion of the categories of scaffolding moves. Our results challenge simplistic expectations about the relationship between scaffolding strategies and learner engagement, highlighting the need for more nuanced approaches to using these frameworks. The studies also demonstrate the importance of qualitative follow-up to refine our understanding of how scaffolding supports complex model construction processes.

Keywords: mathematical modeling, mixed methods, scaffolding

For STEM majors, mathematical modeling (hereafter: modeling) is a major instructional goal because it can authentically promote interaction between scientific and mathematical knowledge domains (Hallström & Schönborn, 2019). For research with a focus on blending knowledge (Hu & Rebello, 2013), a cognitive approach to studying students' engagement in mathematical modeling is often adopted (Kaiser, 2017). Over several decades, this has led to a fairly stable *descriptive* account of the processes attendant to modeling – called a mathematical modeling cycle (MMC) (Blum & Leiß, 2007) - which has, in turn, motivated research agendas like articulating features of learning environments that elicit those processes (Hankeln, 2020; Maaß, 2010) and improving modeling competence (Ärlebäck & Albarracín, 2019; Durandt et al., 2021). Naturally, that work has extended to understanding what teachers can do to support students' progress in constructing models. However, the descriptive accounts of student engagement are not predictive. On its own, an MMC does not indicate how an individual might engage in a specific task (Ärlebäck & Bergsten, 2010; Czocher, 2016) and it leaves out the role of the teacher. Separately, there are frameworks that describe teacher knowledge (e.g., Greefrath et al., 2021) and frameworks that hypothesize potential teacher moves that would be supportive of students' engagement in modeling (e.g., Stender & Kaiser, 2015).

Ideally, scaffolding moves would be linked to student engagement in modeling. To date, the existence and extent of these associations have not been systematically explored despite their heavy usage for describing student modeling engagement and teacher scaffolding moves separately. In this paper, we investigate whether a qualitative framework describing scaffolding moves aligns functionally with a framework describing engagement in modeling and justify modifications needed to use the frameworks to carry out studies capable of predicting which moves promote which processes. We report a pair of mixed methods studies that interrogate the utility and promise of two frameworks by addressing the following questions: (1) Are the descriptive categories of the frameworks naïvely associated? (2) Do theoretically similar codes

from the scaffolding moves framework function similarly? (3) What other considerations are there for using these frameworks to study the influence of scaffolding on modeling processes?

Theoretical and Analytic Frameworks

We conceive of a laboratory-based learning environment as a task-modeler-interviewer triad (Koichu & Harel, 2007), and in this study we focus on the modeler-interviewer interactions. We are interested in particular interactions that serve as *scaffolding* for the modeler's *engagement* in modeling. This means we required a framework for engagement, a framework for scaffolding, and a means for relating the two. To describe participant engagement in mathematical modeling, the field uses a mathematical modeling cycle (MMC) (Blum & Leiß, 2007). The cycle decomposes the mathematical modeling process into six sub-processes which connect stages of model construction: a modeler conceives a real-world problem, *understanding* it as a (mental) situation model, *simplifies* it to an idealized real model, *mathematizes* that to produce an externalized mathematical model, *works mathematically* to obtain a mathematical result, which is then *interpreted* in terms of the real-world context to yield real results, and then *validated* against the real-world situation.

Sub-Processes	Definition	Connects Stages
Understanding	Forming an initial idea about what the problem is asking for	real world → situation model
Simplifying & Structuring	Identify (un)important real-world entities and relationships	situation model → real model
Mathematizing	Represent idealized version of the real-world problem using mathematical conventions	real model → mathematical model
Working Mathematically	Analyze or solve mathematical problem	mathematical model → mathematical results
Interpreting	Re-contextualize mathematical results	mathematical results → real results
Validating	Verify results against constraints	real results → real situation

Table 1 Stages and processes of model construction from Blum and Leiß (2007)

Scaffolding is proffered support that aids a student "when performing a task that the student might otherwise not be able to accomplish" (van de Pol et al., 2010, p. 274); we focus on *contingent* scaffolding in-the-moment rather than longer-term scaffolding processes like fading and transfer of responsibility. In the context of modeling, an interviewer needs to provide that support contingent to the modeler's current mathematical reasoning and progress in model construction/evaluation. Stender (2016) developed a framework for scaffolding during modeling that aims to describe moves an educator can make to respond to students' reasoning and further their progress through the modeling cycle that retain student autonomy. Among the categories of moves in the framework are those labeled "B3" – these classify contingent moves targeting the cognitive dimensions of modeling (as opposed to moves that target affective factors). For example, the code B3.6 Prompt to Include Aspects labels moves in which the modeler is asked or encouraged to include something from the real-world she has not yet included in her model, for example posing the question "Do you think maybe there should be another [separate] population, like people who are recovered?" during a disease transmission task when the modeler was struggling with two meanings for the idea of "healthy people." Due to space constraints, it is not possible to include in this manuscript the 26 B3 codes and 6 MMC codes we used, along with rules and examples. The codebooks are available in the following publications: the MMC

codebook was originally published in Czocher (2016) and the B3 codebook was originally published in German in Stender (2016).

Methods

Our interrogation of the analytic frameworks spans multiple mixed methods studies (Creswell, 2014). We used cognitive task-based interviews to generate data, cluster analyses and logistic regression to develop variables and explore associations among them, and then followed with interpretive methods to validate, refute, and explain patterns in the results.

Data were generated through a set of individual, task-based clinical interviews (Clement, 2000; Maher & Sigley, 2014) with 24 undergraduate STEM majors at a large southern university in the US who were studying differential equations. Each participant worked on between four and eleven modeling tasks across 10 hour-long sessions. The modeling tasks were based on canonical problems from differential equations and dynamical systems that call on feedback loops (e.g., disease transmission; predator-prey). Of the 11 tasks, the final six were open problem statements and contained only contingent scaffolding provided by the interviewer. The interviewer accepted all student productions as provisionally correct (Goldin, 2000) and attempted to make sense of why the participant reasoned as they did. The interviewer intervened in the participants' modeling process if the participant requested help, if it seemed that the participant got stuck, or to generate and test conjectures about the participants' ways of reasoning about the mathematical or real-world aspects of the task. Interviewer turns were formulated so they could map to the scaffolding moves framework.

In this report, we consider only the tasks without embedded scaffolding to isolate the influence of contingent scaffolding on participants' modeling processes. The dataset for this study comprised 52 events completed by 22 (distinct) participants working on six modeling tasks. Qualitative analysis proceeded with deductive coding procedures (Saldana, 2016), using the pre-defined, published codes for engaging in mathematical modeling processes (MMC codes), model construction stages (Stage codes), and contingent scaffolding moves for modeling (B3 codes). Due to the complexity of the codebooks, total duration of the 52 events, and planned quantitative models, our primary concerns about reliability were the chance of missing codable segments and consistent application of the codebooks across participants and tasks. Thus, two analysts independently coded each event. Pairs met regularly to reconcile codebook application to their assigned events and to resolve disagreements based solely on code definitions. In total there were 3,478 instances across 26 B3 codes and 5,596 instances across 6 MMC codes. Due to the interview protocol, *understanding* (233) and *working mathematically* (261) processes were rarely elicited compared to the others, and so were excluded from further analysis.

Study I: Are the Two Frameworks Naïvely Associated?

By meaningful association between frameworks, we mean that variability across the set of MMC codes should be accounted for by variability across B3 codes, when aggregated across the events. At each instance of a B3 move and for each MMC process, we determined if the target process was observed within a two-minute window subsequent to the B3 move. If the process was observed at least once in the window, we said the B3 move was taken up by the participant. Combining results across events, we estimated the probability of uptake for each process and set of B3 codes. To account for variation in base rate of occurrence of MMC codes, we estimated a base probability under the null or no effect model, where the null assumption is that MMC codes are uniformly distributed across the interview sessions. Under the null model, we let X_k be the number of instances of MMC code k in a given two-minute window. Then X_k follows a binomial

distribution with size equal to the total number of instances of code k and probability equal to 2 divided by the total combined time of the sessions. The base probability is $p_{b_k} = P(X_k > 0)$. We then normalized the probability of uptake, by computing an odds ratio:

$$OR = \frac{(p/(1-p)}{(p_{b_k}/(1-p_{b_k})}$$

If the uptake probability equals the expected value under the null model, then $OR = 1$, and indicates the contingent scaffolding move is not associated with an increased uptake of process k. On the other hand, $OR > 1$ indicates that the contingent scaffolding move promotes process k. The results table was a daunting $18 \times 4 = 72$ rows long without clear patterns emerging. While the reader might be muttering "no duh," we note that this initial analysis was necessary for exploring whether each B3 category had a cohesive nature with respect to the MMC's. There were some pairs of B3 and MMC codes with a strong association, for example, B3.11u Attention to Units was often followed by *validating* (OR = 1.463), but in general, the B3 codes were not clearly tied to specific phases of modeling, which washed out any aggregate impact of scaffolding moves on participant modeling processes. We took this result as evidence against the naïve hypothesis that the scaffolding moves framework was aligned with the modeling processes framework. We determined that the hypothesis rested on an implicit assumption that, in the aggregate, each B3 code would contain scaffolding moves that were largely homogenous with respect to the modeling process induced in the participants. This called for qualitative follow-up.

Upon inspecting the instances of B3 codes, some of the codes seemed to operate in a general descriptive way, supporting a wide range of participant modeling activities. Such B3 codes had very high usage and because the data corpus was quite large, we had opportunities to see more specific usage patterns arise. For example, the code B3.12 Interviewer Explaining or Instructing had 254 instances, but the nature of what was being explained or instructed was not uniform across all of the instances. An inductive coding process within B3.12 yielded 6 subcodes (instructing real world, instructing mathematics, instructing modeling, considering alignment between model and reality, explaining distinctions and similarities between scenarios, explaining meaning of representations). We carried out a similar process for 3 other B3 codes. This yielded a new total of 47 B3 codes. Again, inspecting instances of B3 codes, most of the B3 categories did not correspond to a single aspect of model construction. This explained our results from Study I. If the categories of scaffolding moves are not homogenous with respect to what aspects of model construction they are targeting, then we ought not expect a consistent response from the modeler in terms of what process is elicited. Thus, we further hypothesized that the proportion of uptake of scaffolding moves depended on which stage of model construction the B3 codes referred to. Our next step was to examine the extent to which each B3 code could be said to refer to a single stage of model construction or a Profile of stages.

Study II: How Does the Scaffolding Framework Connect to Model Construction?

Our next step was to explore the distribution of B3 codes across model construction stages. We conducted a deductive qualitative coding procedure inspired by Çakmak Gürel (2023) and Tropper et al. (2015) that mapped each instance of a scaffolding move to (1) the stages of model construction presented in the modeling cycle it referred to (see Table 1, third column) or (2) to no stage. For example, the move "Let's work on just the susceptible and infectious. And we'll pick back up the removed later" used with a participant who was working on modeling disease transmission was coded as B3.23 Narrowing Scope because the interviewer suggested the participant to ignore the removed population. Because the move referred to the distinct

populations identified by the participant (susceptible, infectious, recovered), we inferred it to refer to the Real Model stage of the MMC. For example, the code B3.24 Documenting Writing was an interviewer move that merely sought to clarify what the participant wrote down (e.g., Does that say "x" or "α"?) and was not coded as referring to a stage of model construction.

This qualitative analysis along with descriptive statistics enabled us to examine the characteristics of the B3 scaffolding moves in two ways. First, for each scaffolding move, independently from the B3 code it is assigned to, reflects a combination of model construction stages it refers to. These combinations can be aggregated across B3 codes to give a sense of the relative frequency each stage is referred to within a given B3 code. For example, the proportion of B3 instances referring to both the Real Model and the Math Model was 39.31%. Second, each B3 code can be said to have a *profile* for the stages of model construction its instances refer to. For example, the code B3.7 Encourage to Make Assumptions or Simplify contained instances that typically referred to Real stages of model construction. We noticed that some profiles referred to stages of model construction associated with the Real World (i.e., situation model (SM), real model (RM), and real results (RR)) others referred only to stages of model construction associated with the Math World (i.e., mathematical model (MM) and mathematical results (MR)), and some profiles referred to both. Hence, we used the analysis of B3 code instances in terms of model construction stage to collapse to four broad categories: Real (scaffolding move refers to situation and/or real model), Math (scaffolding move refers to mathematical model), Both (scaffolding move refers to both Real and Math), and Neither (scaffolding move refers to neither Math nor Real). To fully characterize the differences in uptake of MMC codes according to whether each B3 category contained instances that primarily referred to Real, Math, Both, or Neither stages of model construction, we developed a set of regression models. For each process we fit a mixed logistic regression model with a participant random effect, u_i, to account for dependence (Agresti, 2012):

$$\log\left(\frac{p}{1-p}\right) = \beta_0 + \beta_1 \text{Real} + \beta_2 \text{Math} + \beta_3 \text{Both} + u_i$$

To test our hypothesis that the induced MMC fingerprint would depend only on the stage profile of the B3 codes (i.e., that in the aggregate, the B3 codes have clear associations with sets of elicited modeling processes) we conducted a two-step cluster analysis. We first clustered the B3 code instances based on their Stage profiles. Then, within each Stage Profile cluster, we clustered the B3 codes according to the MMC processes elicited by the codes in that sub-cluster.

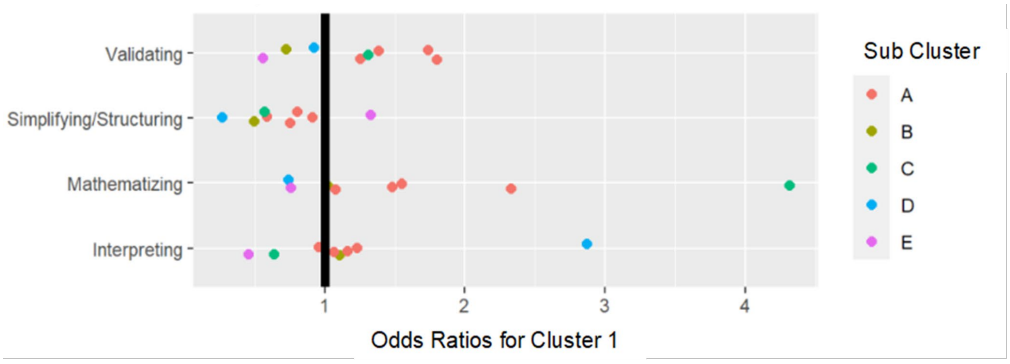


Figure 1 Sample of odds ratios for Cluster 1, from the two-step cluster analysis. Each dot represents a B3 code category. Cluster 1 contains B3 codes whose instances tend to target Both mathematical and real-world stages of model construction. Subcluster 1-A tended to promote validating, mathematizing, and interpreting while suppressing simplifying/structuring.

We refer to the elicited MMC processes as the induced MMC Fingerprint associated with a given B3 code's Stage Profile.

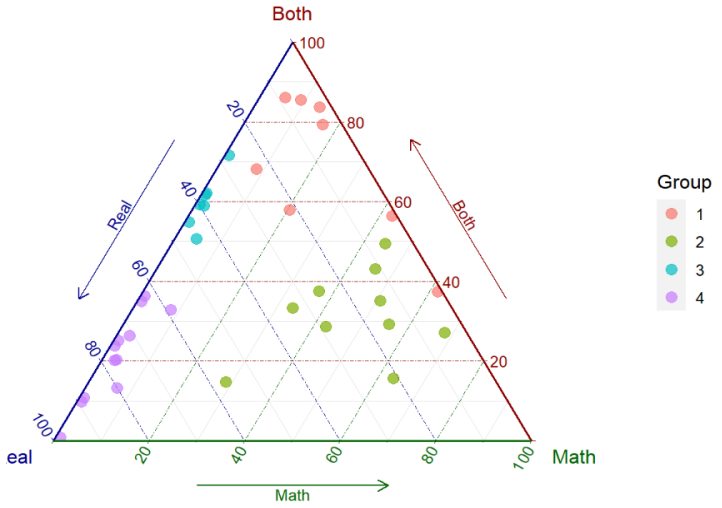


Figure 2 Ternary graph depicting clusters of B3 codes according to Stage Profile. Colors are re-used and do not indicate the same cluster membership as the colors in Figure 2. Figure 1 contains only codes from Group 1 in Figure 2.

The result was four clusters each containing between three and five sub-clusters. Odds greater than 1 indicated that, in the aggregate, moves in the given B3 code elicit the given modeling process at a higher rate than the null model would predict (see Figure 2 for Cluster 1 results). For example, the code B3.14 Suggestion for action, related to content (next mathematical step or analysis) strongly promotes validating ($\text{odds} = 2.21$) while the code B3.11p suppresses simplifying/structuring ($\text{odds} = 0.14$). We observed good separation between the clusters based on Stage profile, as illustrated in the ternary graph in Figure 1. Each point within the ternary graph can be thought of as a triple corresponding to a Stage profile (% Real, % Math, % Both). For example, the apex which is labeled Both corresponds to a (fictional) B3 code whose moves were always classified as Both and never classified as either Real (only) or Math (only) and the Cluster 3 codes tend to contain moves that do not refer to Math (only).

This seemed a good sign and so we qualitatively characterized all clusters and all 16 subclusters (impossible to report here and it would bore the reader if we did). We found that while quantitative analysis grouped B3 codes together based on the similarity of their profiles of model construction stage, moves within each cluster induced dissimilar MMC fingerprints and moves within the subclusters were often difficult to characterize. This suggested we should further analyze the similarities and differences of B3 codes within each Stage Profile cluster as well as examine the expression of B3 moves within each B3 code to explain each cluster's capacity to elicit (or suppress) multiple modeling processes. For example, both B3.22 Modify Scenario and B3.23 Narrowing Scope are B3 moves which were grouped into the same Stage Profile, and both moves elicit the *simplifying/structuring* and *mathematizing* modeling processes. Thus, while Study II did indicate that the modeling processes elicited by the various scaffolding moves depended, at least in part, on the Stage the moves in the category tended to target, the grouping criteria Real/Math/Both/Neither was too coarse to facilitate theory-driven interpretations for many of the sub-clusters. We discuss the next steps in the discussion, next.

Discussion

Our investigations confirmed that the MMC Fingerprint induced by categories of scaffolding moves does depend on model construction Stage. This finding partially supports previous assertions (see, for example Schukajlow et al., 2015; Stender & Kaiser, 2015) that successful scaffolding depends on which stage or phase of the modeling cycle a move is relevant to. However, Stender and Kaiser (2015)'s assertion implies that the key to understanding successful scaffolding lies in diagnosing which phase of the modeling cycle the modeler is *in* and conditioning the contingent facilitator response on that information. In contrast, we observed that many of the interviewer's scaffolding moves -- when contingent to the modeler's mathematical reasoning and accounted for what had been produced so far by the modeler – may or may not be related to the stage or phase of modeling cycle the modeler finds themselves in by the time the interviewer intervenes.

Our analysis challenges two other naïve expectations about associating interviewer moves and modeler engagement through the modeling cycle. First, one might expect that if the interviewer provides scaffolding for phase N of the modeling cycle that the modeler would respond by progressing to phase $N + 1$. That does not seem to be the case. For example, there were sub-clusters whose MMC Fingerprints were characterized by *interpreting* that followed from interviewer moves that focused on the situation model. Underlying the first expectation is a deeper assumption that the appropriate thing to do if a modeler needs scaffolding for stage M of model construction that the proffered support should target stage M. This also seems not to be the case. Our analysis of uptake of scaffolding moves was based on the antecedents to participants' elicited modeling processes. The moves were contingent to the participants' in-the-moment needs and often targeted Stages of model construction that did not correspond to the contemporaneous stage that the participant was in. Thus, paralleling well-known results that students' routes through modeling cycles are nonlinear and idiosyncratic (Ärlebäck & Bergsten, 2010; Borromeo Ferri, 2006; Czocher, 2016), it seems that one should also not expect that promoting student progress in their model construction would be as simple as targeting specific stages or phases of the cycle.

The quantitative analysis established associations between scaffolding moves and the modeling processes they elicited. Some interviewer moves that mapped to the same sub-cluster elicited very similar MMC fingerprints suggesting that the moves either functioned similarly or co-occurred frequently enough that their MMC uptake was similar. We also found that interpreting these associations was complicated by a lack of theoretical coherence within some clusters. In particular, moves that seemed similar based on the code descriptions or move expression may not have similar Stage profiles and therefore did not cluster together. This observation knocked down a third naïve hypothesis that moves that are behaviorally described similarly would have a similar impact on modeling processes. Additionally, our results suggest that the number of B3 codes could be streamlined by examining co-occurrences of similarly functioning code categories. As our investigation has laid the groundwork for the two-step clustering approach, future work can examine the uptake resulting from a streamlined B3 codebook. This next step would bring the goal of estimating the likelihood that a given scaffolding move would induce a particular modeling process within reach.

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How Course Coordinator Values and Practices Align With the Needs of Instructors

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Course coordination in high enrollment undergraduate mathematics classes in the United States is a common practice that seeks to provide consistency in learning opportunities for students while balancing instructional support and pedagogical autonomy. In this study we investigate how the needs of instructors who teach in a coordinated course align with the values and practices of course coordinators. Analysis of 44 survey responses from both coordinators and instructors who teach in a coordinated course identified several coordinator actions and corresponding orientations that align with instructor needs as well as actions that instructors find valuable but that are not frequently occurring. These findings are significant because they can lead to better articulation of the knowledge, skills, and dispositions needed to be an effective course coordinator, which could then lead to the design of professional development aimed at helping coordinators be better able to meet the needs of their co-instructors.

Keywords: Coordination, orientation, calculus, professional development

It is common in the United States for high enrollment undergraduate mathematics courses required of most STEM majors to consist of multiple sections taught by several different instructors and for those sections to be coordinated. Coordination typically consists of uniform course elements (e.g., common text, exams) and regular instructor interactions around issues of instruction to support each other in delivering high quality courses. A key characteristic of effective course coordination is the delicate balance between enacting policies and practices that provide consistency across sections while at the same time maintaining instructor pedagogical autonomy (Rasmussen et al., 2015; Rasmussen et al., 2021). This can be challenging given the increasingly wide variation in who teaches calculus and their sometimes-contingent status.

While previous studies have documented the different elements of course coordination and the different roles that course coordinators play (e.g., Adeyemi et al., 2023; Rasmussen et al., 2015), what is currently missing from the literature is a close investigation of how the values and practices of course coordinators align with (or do not align with) the needs of instructors who teach in a coordinated course, particularly given the change in who is teaching these courses and the variation in their background and experiences. The goal of this study is to begin to fill this gap, and we address the following two research questions:

- What are the needs and experiences of coordinated faculty?
- To what extent do the values and practices of coordinators, coordinated contingent faculty, and coordinated permanent faculty align with one another?

Insight into the needs of the varied instructors who teach calculus and the extent to which the values and practices of coordinators meet these needs is significant because it can lead to better articulation of the knowledge, skills, and dispositions needed to be an effective course coordinator, which could then lead to the design of professional development aimed at helping coordinators be better able to meet the needs of their co-instructors.

Theoretical Background

Central to our research goal is the need for insight into the practices and values of course coordinators. An overarching construct that encompasses both what coordinators do (i.e., their practice) and what they value is that of *orientation*. A person's orientation to a particular task or job not only provides a sense of direction for what one does but also reflects what that person deems important (Schoenfeld, 2010). As such, the orientation that a coordinator takes to their role shapes what their goals are and how they interact with their colleagues. In the context of classroom teaching, an orientation is a disposition or tendency one takes toward some aspect of their job as a teacher (Thompson et al., 1994; Philipp, 2007).

The work of Martinez et al. (2022) provides a comprehensive analysis of course coordinator orientations that we heavily draw upon in this study. Their analysis revealed two overarching course coordinator orientations. They refer to these two orientations as a Resource-Managerial orientation and a Humanistic-Growth orientation. A Resource-Managerial orientation refers to values and practices that leverage the knowledge, experience, and logistical management skills of the coordinator to facilitate consistency across multiple sections and different instructors. They identified the following five themes that characterize a Resource-Managerial orientation: a) draws on knowledge of the course, department, and institution, b) demonstrates experience teaching and knowledge of teaching the course, c) provides material related to content and curriculum, d) communicates clearly, and e) exhibits strong administrative skills. In contrast to a Resource-Managerial orientation, a Humanistic-Growth orientation refers to values and practices that facilitate the personal and professional growth of the instructors teaching in the coordinated course. A Humanistic-Growth orientation is manifested in the following five themes: a) attends to student experience, b) concerned about others, c) takes action to build community, d) attentive to instructor differences, and e) provides instructor support.

Part of the significance of this work is that the two orientations offer different but complementary approaches to professional development. More specifically, Martinez et al. (2022) argue that a Humanistic-Growth orientation outlines how course coordinators can provide professional learning for instructors in a way that is flexible, attentive to individual differences, and builds community. On the other hand, a Resource-Managerial orientation is a leadership approach that can promote instructional innovation by offering instructional materials, insights and knowledge about student difficulties, and relevant department and institutional policies. As such, the job of a coordinator is (or can be) complex and multi-faceted.

In this study we further this line of inquiry into course coordinator orientations by investigating the extent to which course coordinator orientations align with the needs and experiences of the coordinated faculty at one institution. This particular institution is an ideal case for such analysis as it has a long history of coordination and a growing number of teaching-focused faculty that serve as course coordinators.

Methods

Top Applied University (TAU) is a public engineering school with just under 6,000 undergraduate students, all of whom complete Calculus I, II, and III and at least one additional math course. Coordinated courses within the department are typically structured in sections of about 40 students, which are taught by a mix of adjunct faculty, Postdoctoral Teaching Fellows, and Graduate Teaching Fellows and are coordinated by a Teaching Faculty member. At TAU, Teaching Faculty are largely comparable to Tenured/Tenure-Track Faculty (T/TT) in how they are viewed and in how they can participate on campus but are generally only evaluated on teaching and service. Teaching Faculty are eligible for promotion but not tenure at TAU.

Currently, the department is almost evenly split between Teaching Faculty and Tenured/Tenure-Track Faculty. For this study's purposes and given the context at TAU, we define adjunct faculty, Postdoctoral Teaching Fellows, and Graduate Teaching Fellows as contingent faculty. We define Teaching Faculty and Tenured/Tenure-Track Faculty as permanent faculty given their nearly identical status to that of Tenured/Tenure-Track Faculty.

We developed a survey with two versions, one for coordinators and one for coordinated faculty, both of which focused on actions that coordinators take in their roles that could contribute to an effective and supportive environment for coordinated instructors. Of the 12 total invited coordinators, 11 responded (91.7% response rate), and of the 47 invited coordinated instructors, 32 responded (68.1% response rate). Among the coordinated faculty, 19 (59.3%) were contingent instructors and 13 (40.6%) were permanent faculty. We based the actions on previous research (Martinez et al., 2022; Rasmussen et al., 2015), and feedback from department faculty. The finalized 14 coordinator actions for these surveys is listed in Table 1:

Table 1. Coordinator actions organized by HG (Humanistic-Growth) and RM (Resource-Managerial)

HG1	The coordinator attended to student experience.	RM1	The coordinator provided clear, consistent information about what content was covered.
HG2	The coordinator provided opportunities for professional development.	RM2	The coordinator provided common written homework.
HG3	The coordinator made me feel welcome in the department.	RM3	The coordinator maintained common online homework.
HG4	The coordinator valued my input and feedback in the course.	RM4	The coordinator provided course content materials.
HG5	The coordinator allowed for some degree of instructional autonomy.	RM5	The coordinator exhibited strong administrative skills.
HG6	The coordinator provided regular meetings throughout the semester to ask questions about teaching the course and students.	RM6	The coordinator maintained a common Canvas course or Canvas Blueprint for the course.
HG7	The coordinator demonstrated knowledge of pedagogical and evidence-based practices.	RM7	The coordinator provided guidance on how to evaluate exams.

For each action, we asked participants to identify how frequently they implemented those actions (coordinators) or how often those actions occurred in their coordination experiences (coordinated instructors) and how important these actions were for the role (coordinators) and to have a positive experience in a coordinated setting (coordinated instructors). Both sets of questions were asked on a three-point scale.

We also asked both groups two open-ended questions: what worked well about their experience in coordinated courses and what challenges they encountered in coordinated courses. The first two authors used thematic analysis (Braun &Clarke, 2006): they read through and considered patterns in the responses, taking an inductive approach to identifying themes. They each reviewed the entire set of responses, coding responses with one or more themes. Finally, they met to discuss the coding and reconcile any differences until complete agreement was reached.

Results

We averaged the ratings on the three-point survey questions to create heat map visualizations to compare perceptions of frequency/importance of each action among the groups of participants: coordinators, coordinated permanent faculty, and coordinated contingent faculty (see Figure 1).

Humanistic-Growth
IMPORTANCE
Ranges: ≤2.0, 2.01-2.5, 2.51-3.0

	Coordinator	Contingent	Permanent
1. Attends to student experience	2.64	2.53	2.69
2. Provides PD opportunities	1.45	2.16	1.62
3. Makes me feel welcome	2.45	2.68	2.31
4. Values my input	2.55	2.74	2.38
5. Allows for instructional autonomy	2.00	2.58	2.23
6. Provides regular meetings	2.27	2.11	1.85
7. Demonstrates pedagogical knowledge	2.00	2.42	1.62

Humanistic-Growth
FREQUENCY

	Coordinator	Contingent	Permanent
1. Attends to student experience	3.00	2.74	2.46
2. Provides PD opportunities	1.82	2.00	1.62
3. Makes me feel welcome	2.82	2.95	2.92
4. Values my input	2.91	2.79	2.85
5. Allows for instructional autonomy	2.45	2.89	2.69
6. Provides regular meetings	2.55	2.47	2.23
7. Demonstrates pedagogical knowledge	2.27	2.68	2.38

Resource-Managerial
IMPORTANCE

	Coordinator	Contingent	Permanent
1. Provides consistent info about content	2.91	2.95	2.92
2. Provides common written HW	2.82	2.74	2.85
3. Maintains common online HW	2.64	2.63	2.58
4. Provides course materials	3.00	2.68	2.69
5. Exhibits strong administrative skills	3.00	2.95	3.00
6. Maintains a common Canvas course	2.82	2.74	2.77
7. Provides guidance on evaluating exams	2.55	2.63	1.75

Resource-Managerial
FREQUENCY

	Coordinator	Contingent	Permanent
1. Provides consistent info about content	3.00	2.89	2.85
2. Provides common written HW	3.00	2.89	3.00
3. Maintains common online HW	2.73	2.84	2.54
4. Provides course materials	3.00	2.84	2.92
5. Exhibits strong administrative skills	3.00	2.95	2.69
6. Maintains a common Canvas course	2.82	2.79	2.85
7. Provides guidance on evaluating exams	2.91	2.84	2.50

Figure 1. Average perceived importance and frequency of different coordinators actions by faculty group. Scores of 1 = Not important/In few coordination experiences, 2 = Somewhat important/About half of coordination experiences, 3 = Very important/In most or all coordination experiences

We first consider research question 1: What are the experiences and the needs of coordinated faculty? We use the ratings about the frequency of coordinator actions to explore coordinated instructors' experiences and the ratings about importance of these actions to examine coordinated instructors' needs. We also use coded responses to the open-ended questions.

What are the Experiences of Coordinated Faculty?

Finding 1a: **Coordinated faculty agree that most coordination actions are occurring frequently and more consistently agree about the frequency of Resource-Managerial actions as opposed to Humanistic-Growth actions.** In looking at the two frequency heat maps on the right of Figure 1, 71% (10 of 14) actions have averages of at least 2.50 for both coordinated contingent and coordinated permanent faculty, suggesting general agreement that many actions are occurring most or all of the time. The four actions that coordinated faculty feel are happening less often are all in the Humanistic-Growth category: 'attends to student experience' (HG1), 'provides PD opportunities' (HG2), 'provides regular meetings' (HG6), and 'demonstrates pedagogical knowledge' (HG7). We observe that there is much more variability in the HG frequency table compared to the RM frequency table. Additionally, faculty commented more frequently about Resource-Managerial actions compared to Humanistic-Growth actions when asked about what is working well for coordination.

What are the Needs of Coordinated Faculty?

Finding 1b: Coordinated faculty believe that nearly all Resource-Managerial actions are important and report them as more important than Humanistic-Growth actions, although most Humanistic-Growth actions are rated as at least somewhat important. The importance heat map averages for RM statements for coordinated faculty are all 2.58 or above (see bottom left heat map) apart from 'provides guidance on exams' (RM 7) for permanent faculty (average 1.75). By comparison, if we look at the heat map averages for HG statements for coordinated faculty (see upper left heat map in Figure 1), only one Humanistic-Growth action, 'attends to student experience' (HG1), was rated as important by both coordinated contingent and coordinated permanent faculty with an average of 2.53 and 2.69 respectively. Additionally, about half or more of all coordinated faculty in each group rated each of the RM and HG coordination actions as at least somewhat important. This suggests that even though there may be differences in importance ratings between RM and HG actions, all coordination actions have some value to many faculty.

Finding 1c: Coordinated contingent faculty have different needs from the coordinated permanent faculty, primarily with respect to Humanistic-Growth actions. Most differences in needs between coordinated contingent and coordinated permanent faculty were with the HG actions, with coordinated contingent faculty rating all HG actions, except 'attends to student experience' (HG1), as more important than the coordinated permanent faculty did (see upper left heat map in Figure 1). It could be that these actions provide a greater benefit to contingent than permanent faculty due to the nature of their employment. All these actions focus on helping instructors feel included and supported in growth, something that permanent faculty may already get elsewhere, but what contingent faculty are primarily obtaining through their coordinated experience.

Another difference between coordinated contingent and permanent faculty had to do with the need for consistency across sections. In responses to the open-ended question about challenges with coordination, 64% (9 of 14) of coordinated contingent faculty expressed concerns about providing a consistent experience for students compared to only 22% (2 of 9) of coordinated permanent faculty. We believe there is a good reason more contingent faculty brought up the consistency issue. Namely, for coordinated contingent faculty members, teaching is the entirety of their job, so consistency issues over which they have little to no control can cause anxiety over whether they are adequately performing the position requirements.

We next consider research question 2: To what extent do the values and experiences of coordinated faculty align with what coordinators value and practice? To answer this question, we focus our attention on all three columns of the heat maps shown in Figure 1 and the responses to the open-ended questions for both coordinated faculty and coordinators.

Alignment between coordinators and coordinated faculty.

Finding 2a: The needs of coordinated permanent faculty generally align with the values of coordinators, with a few exceptions. By examining columns one and three on the left two heat maps, we can see that for most coordination actions, the importance averages are similar for coordinators and coordinated permanent faculty. This is perhaps unsurprising, because the coordinators are all permanent faculty, although neither group was a subset of the other when considering survey respondents. In Finding 1c, we addressed the ways in which coordinated contingent faculty importance ratings aligned with those of coordinated permanent faculty in RM categories as well as the ways they did not align in HG categories. The alignment of coordinators with coordinated permanent faculty means that there is also alignment between coordinators and

coordinated contingent faculty in importance for RM actions and more misalignment between coordinators and coordinated contingent faculty in importance for HG actions. We believe that those actions for which coordinators and coordinated permanent faculty did not align in terms of importance reflect a lack of clarity around expectations for the coordinator role.

Alignment between needs and experiences.

Finding 2b: Most of the needs of the coordinated faculty (related to the RM and HG actions) are being met by the coordinator practices. We can compare the importance ratings of the coordinated faculty to the experience ratings of the three groups (which are generally aligned) as a measure of how these needs are being met. For RM actions, the needs of coordinated faculty are mostly being met in their coordination experiences. It is interesting to note that for (RM5) 'exhibits strong administrative skills' the coordinated permanent faculty unanimously report it as 'very important' with a 3.0 importance rating, yet only 69% (9 of 13) report the action as always happening in their coordination experiences.

In looking at the ratings of importance and experience for HG actions, there are only two actions for which the frequency rating was lower than the importance rating. We highlight 'provides PD opportunities' (HG2) for coordinated contingent faculty (2.16 importance, 2.0 experience). While this is not a large difference, we have previously discussed the importance of this action to some contingent faculty. The following was expressed by one of the coordinators in the open-ended questions and relates to the lack of clarity around professional development: "It is also challenging to gauge how involved I need to be with each individual coordinated instructor and what my role is helping/mentoring/evaluating them. It would be nice to have some more clarity about what, exactly, my role is for them as professionals."

Discussion

We first consider the question, "How are the needs of coordinated faculty being met?" The results show that RM actions were valued more than HG actions in general by coordinated faculty (Finding 1b) and there are different needs for the coordinated contingent faculty and coordinated permanent faculty (Finding 1c). About half or more of all coordinated faculty in each group rated each of the RM and HG coordination actions as at least somewhat important (Finding 1b). Additionally, given the concern that coordinated contingent faculty had around consistency, it seems that coordinated faculty in general need a coordinator who performs all coordination actions while also addressing consistency concerns across sections. We also observed that some needs were clearly not being met (Finding 2b), which suggests a need to revisit the coordinator's duties and expectations list adopted by the department. Furthermore, our data supports Martinez et al.'s (2022) thesis that RM actions serve as a baseline for coordinator expectations and that HG actions are advanced practices.

We next consider the question, "How should the coordinator role be defined so it meets the needs of coordinated faculty while also supporting coordinators?" Our findings indicate that in addition to ensuring that coordinators perform RM actions, it is also valuable to support coordinators in performing HG actions. Given that almost all of the coordinated courses at TAU utilize a mix of contingent and permanent faculty, each coordinator action is valuable to at least some cohort of coordinated faculty. The three HG actions that were not rated as 'very important' were still rated as 'somewhat important' by contingent faculty (Finding 1b) while coordinators viewed these actions as either unimportant (HG2, HG7) or only somewhat important (HG6). The data suggests that coordinated contingent faculty view coordinators as mentors, whereas coordinator actions are not always fulfilling that role.

We hypothesize that the variations in which actions a coordinator chooses to perform may occur because the coordinator is unsure of the required list of duties, does not have time to attend to some actions, or is not sure how to perform some of the actions. Additionally, some coordinators may not believe HG actions should be part of the job. While we do not know the exact reasons, we conjecture that all the above are factors to at least some coordinators. To help coordinators appreciate the value of HG actions, we believe that greater uptake would occur if we were able to demonstrate improved student learning and increased sense of belonging of contingent faculty when coordinators embrace an HG orientation and attend to HG actions.

Conclusion

TAU is not unique in employing both contingent faculty and permanent faculty to teach in coordinated courses, although the ratios and types of contingent faculty may vary at other institutions. Findings suggest two needs for coordination more broadly. First and foremost, there is a need for our professional societies to provide guidance on what the role of a course coordinator should be and to provide recognition for this challenging and important role. Identifying evidence-based coordination best practices would support departments and institutions in defining the coordinator role more locally. Coordinators have both direct and indirect impact on all the students enrolled in coordinated courses and therefore have the potential to make a significant impact on student learning (Bressoud et al., 2015; Rasmussen et al., 2019).

In addition to identifying the coordinator role and best practices, there is also a national need to provide professional development for coordinators to develop both the dispositions and skills necessary to be effective in their roles. The different needs of coordinated faculty pose a particular challenge for coordinators. While many coordinators may feel more comfortable performing RM actions, the differing values for and uncertainty around HG actions imply that coordinators may need more support in performing these actions. Our professional societies could also lead this charge by providing workshops, materials, or even certification for coordinators. Given the impact a coordinator has on instruction and student learning, it is imperative for departments and professional societies to provide guidance on the role, identify best practices, develop professional development programs, and to provide recognition for coordinators.

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Innovation Configuration Maps: A Tool for Guiding Team-Based Change Work in Higher Education

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DAT Grant Project Team

The Department Action Team (DAT) model supports groups of individuals in creating sustainable improvements to education within their department. A DAT, consisting of department members and external facilitators, works collaboratively towards a shared goal. Using innovation configuration (IC) maps we study three DATs, exploring members' perspectives about their goals, work, and the process of actualizing goals. We examine the degree to which these groups perceived their alignment with the DAT model, and close with a discussion of how IC maps can be used as a tool to assess the alignment of team-based change work to change models.

Keywords: innovation configuration maps, department change, principles for change.

Meaningful and lasting change is a complex process that goes beyond the efforts of a single individual (Fisher & Henderson, 2018) and can often call into question strongly held beliefs and ingrained practices. Because deep-rooted institutional structures and cultural norms complicate educational transformation (Austin, 2011; Kezar et al., 2015), effective and lasting change must involve stakeholders collectively developing and working towards a shared vision to support and assess innovation efforts to improve STEM education (Henderson et al. 2011; Palmer & Giering, 2023). The Departmental Action Team (DAT) model is such a model that engages multiple stakeholders, taking into account both context and culture to enact and sustain change at the department-level (Reinholz et al., 2019).

The work presented in this paper was part of a larger project aimed to develop collaborative teams (DATs) to work on a local problem in undergraduate STEM education[1]. As part of this work, Innovation Configuration (IC) maps in the form of a survey were developed to support DAT members to come together on the key characteristics of the DAT model. Assessing each DAT's alignment with the DAT model is essential for evaluating their change efforts. In this manuscript we explore the work of three STEM DATs and use IC maps to answer the following question: *What are DAT members' perceptions of alignment with the DAT model?*

Theoretical Framework: The Department Action Team Model and Guiding Principles

A DAT is a type of working group typically consisting of 4-8 department members (e.g., faculty, non-tenure track faculty, students, staff) and 1-2 external facilitators. External facilitators are typically individuals trained in group facilitation and may be part of an institution's Teaching and Learning Center. DATs work to make sustainable change by addressing an educational problem of broad-scale importance. Past DATs have focused on developing program-level outcomes, assessments, and studying student experiences for program improvements. A DAT's goal is developed through a shared visioning process (Reinholz et al., 2021), and the DAT meets regularly to enact a project related to the identified goal. The work of the DAT is guided by six core principles which reflect a synthesis of research grounded in organizational change and higher education literature (Quan et al., 2019, p. 010141-2):

[1] Go to https://dat-project.org/ for more information about the DAT project.

1. Students are partners in the educational process (P1).
2. Work focuses on achieving collective positive outcomes (P2).
3. Data collection, analysis, and interpretation inform decision-making (P3).
4. Collaboration among group members is enjoyable, productive, and rewarding (P4).
5. Continuous improvement is an upheld practice (P5).
6. Work is grounded in a commitment to equity, inclusion, and social justice (P6).

Although Quan et al. note that "fostering departmental change requires more than principles alone" (e.g., readiness, resources, etc.; p. 010141-18), they acknowledge that when coupled with an effective model, such as the DAT model, a change initiative can be more effective. External facilitators use these principles to guide a DAT in their work as well as formatively assess a department's shift in culture (Ngai et al., 2023; Reinholz et al., 2019).

Innovation Configuration Maps as an Analytical Framework

An IC map is an instrument intended to document and potentially quantify particular aspects of a change effort. Specifically, an IC map illustrates "different ways of getting from A to B" by summarizing "in words of the different ways the key components of an innovation can be made operational" (Hall & George, 2000, p. 3). In this manuscript we focus on an IC map of the six core principles of the DAT model. Each principle makes up a component of the IC map, and each component has variations that illustrate how that principle may manifest in a DAT. These variations illustrate degrees of alignment with the DAT model. For example, the P1 component is illustrated with its five variations in Figure 1[2]. Variations of P1 that are aligned with the DAT model are listed first and are black, ■. The ideal manifestation of P1 is the first option, and the second is an acceptable variation that is still in alignment with P1. Those below are partially aligned (gray, ■) or unaligned (red, ■).

P1 - Students are partners in the educational process

■ The DAT has student members. Student members participate meaningfully and see themselves as partners.

■ The DAT has student members that contribute to decision making to an extent, and see themselves as partners in the DAT some of the time.

■ The DAT seeks input from students to inform its work.

■ The DAT has student members, but they do not see themselves as partners in the DAT.

■ The DAT does not have student members, and the DAT does not seek input from students to inform its work.

If you have additional comments, please explain:

Figure 1. Principle 1 and its corresponding variations.

Methods

This study focuses on the participant perspectives from three STEM DATs: Herbs, Summoning, and Prophecy (all pseudonyms). Data collection occurred during the last phase of the Herbs and Summoning DAT work as external facilitation began to diminish, and immediately after the Prophecy DAT concluded their work. Thus, we deemed them as well-positioned to pilot the DAT Core Principles IC Map, which has gone through subsequent revisions. The current version can be found in *Facilitating change in higher education: The Departmental Action Team mode* (Ngai et al., 2020). Twenty-nine DAT members were invited to

[2] All principles and their corresponding variations can be found in this document: https://tinyurl.com/3zaphsbm.

participate, and 19 responses were received. Respondents included department members as well as facilitators, many of whom were involved in the development of the IC Maps.

We used a Google form to administer the IC Map as a survey and synthesized perspectives from DAT members to establish a holistic picture of the level to which the DAT was embodying each of the six principles. First, DAT members were asked open-response questions about their DAT's goals and the DAT experience, and then multiple-choice questions about the six principles followed where respondents were prompted to choose variations of each principle that matched their DAT's implementation. Following each multiple-choice prompt, respondents were also asked if they had additional comments, which were provided as open responses. The survey ended with two closed-form responses about department structures, however, the focus of this manuscript is on how the DAT engaged with the six core principles.

Participants' responses were categorized (aligned, partially aligned, unaligned) by the authors based on the variation selected for each core principle. Responses from facilitators triangulate and add context to broader results.

Results

In the following sections we describe three DATs (Herbs, Summoning, and Prophecy) and how they focused their work based on their survey responses. We discuss members' perspectives of how their respective DAT's work aligned with the six principles.

Herbs

The Herbs DAT assembled for about two academic years with the primary aim to assess the alignment between the department's desired student outcomes and the skill development from targeted coursework for undergraduate majors. At any given time, the Herbs DAT was composed of up to seven members: up to four faculty members, one graduate student, one undergraduate student, and up to two facilitators, with members cycling on or off the DAT due to other obligations. Seven Herbs DAT members responded to the IC map survey: one student, three faculty members, and three facilitators.

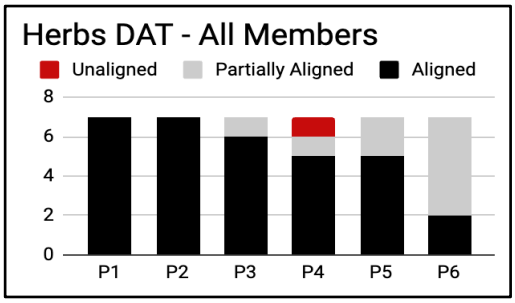


Figure 2. Herbs DAT responses to IC map.

The Herbs DAT members were in complete agreement with regard to alignment with P1 and P2 (Figure 2). Although in agreement on P2, there was some tension, with a facilitator noting that there were times when a couple of DAT members had "outsized influence" on the work. For the most part members were also in agreement with regard to their use of data (P3). The one dissenting respondent, a facilitator, commented that "data comes from their choice of assessments. [It is] unclear whether they will engage in pure bias prediction activity. [Institutional Research] and course data has not been included."

While five of the seven respondents characterized Herbs' collaboration (P4) as ideal, a few members noticed certain areas that did not align with the DAT model. One member indicated

that they perceived the group collaboration to be overall positive, but questioned whether the DAT would actualize their goals. One facilitator chose an unaligned option, noting that there were some members with "lower attendance." However, another facilitator highlighted the DAT's success in fostering collaborative learning about assessment.

For P5, two members selected a partially aligned statement expressing concern with specific details of how and when assessments would be conducted and how results would impact future work. Two faculty expressed uncertainty about lasting impact, with one indicating it would be dependent on whether faculty continue the work, and another noting the need for establishment of an ongoing committee if the work was to continue.

Regarding P6, the DAT respondents were mostly in agreement that their work was partially aligned with a commitment to equity, inclusion, and social justice, with two members feeling that the work was aligned. For example, one facilitator acknowledged the diversity in the membership of the Herbs DAT but expressed some reservation about the degree to which Herbs' work was grounded in equity, inclusion, and social justice. They explained, "I wouldn't say inclusion is routinely considered, but they have done so unprompted."

Summoning

The Summoning DAT operated as a DAT for about two years, honing on the issue of cultivating community among the undergraduate majors. Summoning began their work with six members: three faculty members, two graduate students, and one facilitator. The membership expanded to include up to two undergraduate students and two external facilitators. Six of the ten Summoning members completed the IC map survey, including three facilitators, one faculty member, and two students.

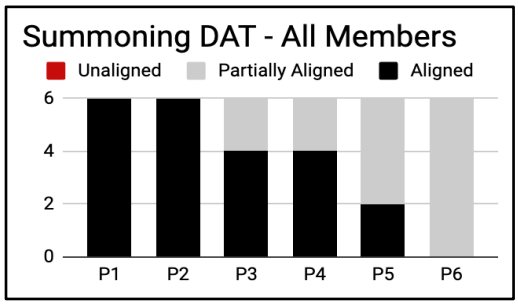


Figure 3. Summoning DAT responses to IC map.

The Summoning DAT was in agreement with perceived alignment with P1 (Figure 3). Summoning members saw the value that students brought, noting that they were "driving forces" for change and "accomplishments would not have come to fruition without them." One student member expressed that they were able to develop communication skills and a sense of belonging:

> I have been able to work with my professors in the DAT and meet with some amazing people, learn how to communicate effectively and hear so many different ideas. I really was able to learn the beauty of collaboration from a socially diverse group.

One facilitator responded, "this is the DAT that showed me how friendly and supportive a group can be toward its undergraduates."

The DAT members also were in agreement about alignment with P2. While the DAT felt that their work was primarily consensus-driven, some members felt that there were some instances when work was driven by a few more vocal members - thus categorized as aligned, but not ideal.

Disagreement among members crops up more with regard to P3 and P4, with four members perceiving their work as aligned with the DAT model and two members selecting partial

alignment. Members who felt that they were aligned with P3 noted that they used student survey data, department statistics, and faculty and student opinion polls, noting specifically that their work was "motivated by student voices". The two respondents who perceived the work as partially aligned, one of whom was a facilitator, did not elaborate on their selection. There was however further elaboration with regard to the partial alignment characterization of P4. Two DAT facilitators chose partial alignment here, with one adding "they have been stronger as a group around their process skills, than in executing projects that will be big/structural/sustainable." While the group collaborated well, it struggled to achieve its goals.

Four respondents' selections were coded as partially aligned with P5; these respondents acknowledged that change may occur, but two of them did not believe that Summoning considered nor developed an assessment to guide the change, and the other two did not believe the group planned for sustainability.

The IC map prompt for P6 asked respondents to measure the extent to which their work is grounded in equity, inclusion, and social justice with choices pertaining to both 1) the group composition and 2) the DAT's work. All six respondents indicated the DAT partially aligned with the ideal of P6. One facilitator stated,

> [The group was diverse] in role and gender. They have considered how language and scheduling impact [undergraduates] vs. other roles in the department. They have considered issues of [universal design for learning] and accessibility, but they didn't apply these strongly to event planning. They struggled over how to advertise inclusively for various reasons. One member requested inclusion be put in the agenda, once.

One member stated, "We lack diversity in our membership, but I think it is something that we realize and we are trying to think about it with respect to our goals." So, while there was agreement regarding group composition, there was some tension around whether or not DAT work was grounded in P6.

Prophecy

The Prophecy DAT was made up of four faculty members, one graduate student, and two external facilitators. Membership remained unchanged for the Prophecy DAT during its one-year tenure. Of the seven DAT members, three faculty members, one graduate student, and two facilitators responded to the IC map survey. Responses on the IC map survey indicated that the Prophecy DAT struggled in identifying a singular goal for their work.

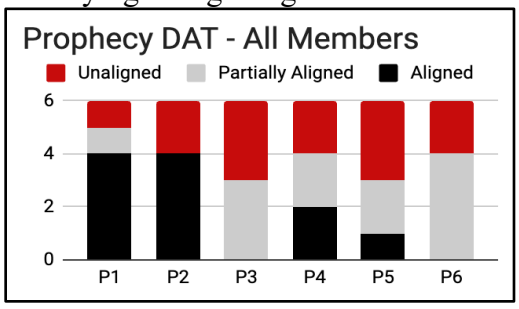


Figure 4. Prophecy DAT responses to IC map.

Figure 4 illustrates the mixed perceptions Prophecy DAT members had with regard to the six principles. Although P1 and P2 had some agreement of perceived alignment, there were still notable differences. For instance, the graduate student member stated finding it "difficult to contribute [her] opinion." Other DAT members felt that their student member was valued and

had meaningful participation opportunities. Further, both facilitators and the graduate student stated that having an undergraduate student on the DAT would have been valuable.

The divide with P2 was between department DAT members and facilitators. The facilitators did not see the work as having a meaningful positive outcome. In contrast, the department DAT members selected statements coded as aligned with P2. These selections seem to contradict these members' statements that they had difficulty identifying a goal.

None of the Prophecy DAT members selected choices coded as aligned with P3. A facilitator pointed out that data was examined early on when the DAT was looking into doing work associated with Prophecy majors and research experiences. However, the facilitator also stated in their response that when they brought up the need to examine data connected to Prophecy's final goal, some DAT members pushed back, telling the facilitator "this is not an education research project." This indicates that in addition to not coalescing around a distinct goal, these DAT members did not recognize data's role in decision-making and their change efforts more broadly.

P4 and P5 had the least agreement among DAT members, with members' responses ranging from aligned to unaligned. While some found the meetings generally enjoyable, some were frustrated with the presence of facilitators, with one member stating that they felt like the facilitators were trying to "move in a direction" that was different from what the department DAT members wanted. There was a lack of agreement regarding sustainability (P5), most members viewed the work as a temporary "fix". One department DAT member noted that faculty are already overburdened with service, so the DAT work was not a priority and "a permanent DAT would be unwelcome both for the DAT members and for other department members."

The department DAT members were in agreement regarding P6, with all members selecting the partial alignment option, "The group exhibits diversity within its membership. The group can consider how decisions impact diverse populations when prompted." One pointed out that diversity was a challenge as members of the Prophecy Department were predominantly White and male. However, another DAT member felt that their group "could have been more diverse". The student member noted the lack of diversity, being the only student and female member, apart from facilitators. One department DAT member felt that the DAT did "indeed frequently consider the effects of [their] work on the diversity of our major population". However, a facilitator DAT member, in contrast, selected the unaligned option and added a note in their open response that they did not see any commitment by the DAT to equity, inclusion, or social justice.

Discussion

Responses to the IC Map survey yielded varied perceptions of alignment with the Core Principles of the DAT model. However, across all three DATs, P6 was the principle where most respondents did not perceive alignment with the DAT model. Historically, oppression and power imbalance have been prevalent in academic structures and across roles. Deep change work must attend to academic systems, structures, and roles (e.g., Fisher & Henderson, 2018; Henderson et al., 2011; Kellam et al., 2020; Malachowski et al., 2020; Reinholz & Apkarian, 2018), with consideration given to understanding how change impacts marginalized groups. A DAT could attend to this in several ways, including recruiting diverse (e.g., in role, background, gender, race, etc.) members when establishing a DAT, soliciting input and feedback from a diversity of stakeholders, and critically attending to issues of equity, inclusion, and social justice as it pertains to the DAT's overall goals. While Herbs and Summoning exhibited diversity in their membership, external facilitators noted the lack of a grounding or execution of DAT work attending to equity, inclusion, and social justice. Department DAT members of Prophecy, on the

other hand, acknowledged a lack of diversity in their membership. Further a facilitator questioned Prophecy's commitment to these principles in their DAT work altogether.

DAT Goals correlated with the enactment of six principles

We noticed a relationship between clearly articulated goals agreed upon by all members and better alignment with the principles. Summoning and Herbs were exemplars of how clear goals were associated with (perceived) alignment with the six principles. In contrast, Prophecy struggled to identify and state a goal and achieve alignment with the principles. This association may occur when a clearly articulated goal facilitates a DAT's embodiment of the six principles. For instance, a goal can guide a DAT's approach to stakeholder input, data collection, and feedback, which are necessary for informing their work.

Herbs. The work of the Herbs DAT focused on student outcomes and skill development in targeted coursework. They believed that including a student voice (P1) was crucial to achieving their student-centered goal. In addition, Herbs' clearly articulated goal may have better positioned them to focus on achieving positive outcomes as a collective (P2). Short-term success was perceived, but the long process to actualizing the stated goals raised concerns about the DAT's productivity (P4) and sustainability of efforts (P5). Ultimately, members acknowledged that institutionalization was necessary for long-term sustainability. Overall, the Herbs DAT mostly agreed on alignment (or in some cases partially aligned) with the six principles and considered themselves successful in pursuing their goal.

Summoning. Summoning aimed to foster community among undergraduate students in their program. To incorporate the student voice, which was crucial in discussing student community aspects, Summoning recruited and actively engaged student members in their DAT work (P1). Both department DAT members and facilitators emphasized the value of the student voice, and targeted data sources that would help them capture it (P3). Although Summoning focused on collectively achieving positive outcomes (P2) and collaborated in a rewarding way (P4), they struggled to have a clear assessment plan to guide ongoing work and change efforts (P5) focused on the undergraduate student community in their program.

Prophecy. Unlike Herbs and Summoning, Prophecy struggled to define a clear goal and also struggled with uncertainty about which student voices should be heard and debates over student inclusion (P1). The absence of a focus hindered discussions on data-driven decision-making (P3). Furthermore, some department members felt that facilitators were steering the work in a direction not chosen by the DAT members, causing collaboration difficulties (P4). Overall, this lack of clarity made it hard for the Prophecy DAT to focus their meetings and work in a meaningful way, which negatively impacted long-term efforts and sustainability (thus Prophecy's lack of alignment with P5). It is possible that for the Prophecy DAT the absence of a clear goal contributed to varying perceptions of alignment with the six principles.

Conclusion

The DAT model provides a framework for anyone wishing to pursue department-level change with the IC Map acting as a research-based tool for professional development facilitators or teams engaged in change work. The six core principles of the DAT model can guide teams to actualize sustainable and meaningful change within their communities. DAT facilitators and team members can use the IC Map to regularly evaluate the alignment of their work with the six core principles and make adjustments as needed. However, the analysis of the three DATs underscores the need for clearly articulated goals towards pursuing sustainable departmental change.

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Equity Does as Equity Feels: Perspectives on Equity and Active Learning in Collaborative, Problem-solving Courses at an HSI

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In this study, we explore notions of HSI servingness and equity in practice as perceived by instructors, facilitators, and students in active learning collaborative, precalculus and calculus problem-solving courses. These courses were part of a grant-funded, scholars program for minoritized student populations. While program outcomes on student achievement, engagement and confidence (discussed elsewhere) are promising, our intent here is to identify and characterize how equity was understood and experienced in practice by those taking and facilitating the courses. Our analysis suggests that instructors and students agree on the learning themes where equity played a role within these active learning courses, but differ in their conceptions of equity within these themes, and in how they saw equity being enacted in practice.

Keywords: active learning, precalculus, calculus workshops, equity, HSI servingness

Purpose of the Study

As Hispanic/Latine populations grow, more universities are being designated as Hispanic-Serving Institutions (HSIs). Yet, Latine representation in STEM remains stagnant, urging HSIs to prioritize making mathematics learning more equitable for Latine and other minoritized students (e.g., Lozano et al., 2018). One popular response approach is scaling student-centered, evidence-based practices, like active learning (Theobald et al., 2020). But how does equity take form in active learning spaces? And to what extent does it serve minoritized populations?

As active learning uptake increases (Patrick, 2020), more needs to be learned about how it is experienced and enacted in practice to promote equity, particularly in gateway mathematics courses. One approach to do this is learning how different stakeholders—researchers, institutions, instructors, and students—understand equity and perceive it in practice. While equity can mean enhancing student achievement and learning, it is also tied to enriching students' experiences, affirming their identities and values, and promoting human flourishing.

In this paper, we seek to understand equity in practice from the perspective of different stakeholders, in the context of active learning, problem-solving (PS) courses (henceforth, PS courses) intended to support the transition to the university for particular minoritized populations. We analyze how facilitators and students conceptualized and experienced equity in the (precalculus and calculus) PS courses at a 4-year HSI.

The PS courses centered in this study serve to support an NSF-funded PS Scholar program designed to enhance equity in the transition to the university. As such, the program ran at three sites, including two 2-year HSIs. This paper focuses on the experience in the PS courses at the 4-year HSI. While results on the program's impact on student achievement, engagement and confidence have been preliminarily reported elsewhere and are promising, we focus here on qualitatively understanding how equity was seen and felt in the PS courses. We address two research questions: (RQ1) *In what ways did facilitators see students as equitably supported (or not) in the PS courses and what conception(s) of equity underlie their perspectives?* (RQ2) *In what ways did students see themselves equitably supported (or not) in the PS courses and what conception(s) of equity underlie their perspectives?*

Relevant Literature

Active learning practices have been shown to improve outcomes for traditionally minoritized populations by reducing achievement gaps (Laursen et al., 2014; Theobald et al., 2020). Successful examples include the Emerging Scholars Program (ESP) calculus workshops, inspired by Uri Triesman's work with Black and Latine students at UC Berkeley (Hsu et al., 2011). These workshops, now implemented across the U.S., support minoritized students by creating small, diverse classrooms where students work collaboratively on rich mathematical problems, just as in the PS Scholar program grounding this study. Research indicates that these workshops can lead to improved course grades and increased persistence in STEM (Duncan & Dick, 2006; Moreno & Muller, 1999). One qualitative study by Oppland-Cordell and Martin (2015) highlights the positive impact the workshop had on Latine students' mathematical and racial identities. However, McNeilly (2021) notes a need for deeper understanding of how students experience these active learning workshops and how different practices promote equity.

In everyday classrooms, active learning practices are not always inherently equitable (e.g., Johnson et al., 2020; Reinholz et al., 2022; Rios, 2023). For instance, Rios (2024) demonstrates how multilingual students of color at HSIs were often positioned in deficit ways during group work, while Leyva et al. (in press) demonstrate that group work could either support or hinder Latine students' access to supportive, "racially-affirming communities" (p. 1). Additionally, there are varying perspectives among mathematics instructors on the role of equity within active learning. Bennet et al. (2022) found that instructors often view equitable teaching as a subset of active learning, yet Williams et al. (2022) reported that instructors did not explicitly address equity in their understanding of active learning. Moreover, research also suggests varying perspectives on the relevance of social identities, like race, when conceptualizing equity (Garcia et al., 2020). Some studies suggest that color-neutral instructor attitudes are linked to less effective equitable practices (Aragón et al., 2017; Garcia et al., 2020). Addressing these gaps, our study explores how students and instructors perceive and enact equity in active learning classrooms, focusing on critical notions of equity that center the whole learner.

Theoretical Framing

Several theoretical frameworks distinguish between critical notions of equity—centering identity, cultural capital, and lived experiences—and equity in academic achievement. Gutiérrez's (2007) equity framework in mathematics education, for example, comprises two axes: the dominant axis, which focuses on access and achievement, and the critical axis, which emphasizes identity and empowerment. The dominant axis conceptualizes equity as granting access to resources—such as quality teaching, instructional materials, and a supportive classroom environment—to ensure student learning and achievement (Gutiérrez, 2011; Rubel, 2017). The critical axis highlights the importance of considering students' identities and the power dynamics within the classroom. It acknowledges that students' cultural, racial/ethnic, gender, and linguistic identities play an important role in their educational experiences. Gutierrez (2007) argues that to champion equity, classrooms must increase access *and* create spaces that affirm and empower students' identities, allowing thriving without sacrificing who students are.

In the context of HSIs, Garcia (2017) introduces a framework describing four institutional identities HSIs may embody in serving students. In alignment with Gutiérrez's (2007), Garcia's framework is built upon two key dimensions. The *outcomes* dimension captures Latine retention and graduation rates. The *culture* dimension reflects how the campus environment affirms Latine students' identities and enhances their educational experiences. Garcia's term, *servingness*—an

institutional commitment to advancing both of these dimensions—connotes the embodiment of support for Latine student flourishing alongside academic achievement (e.g., Garcia, 2017).

We argue that the alignment between Gutierrez's and Garcia's frameworks suggests that when equity beliefs and practices include aspects of both the dominant and critical axis, the classroom environment mirrors and supports an institutional identity of HSI servingness.

Methods

The PS courses centered in this paper were taught at a large, urban 4-year HSI in the Southwestern U.S. They were open to all students concurrently enrolled in main precalculus or calculus course, including student participants in the NSF-funded PS Scholars program grounding this study (see Study Purpose). PS Scholars participants at the 4-year HSI included 2-year HSI transfer students and students with STEM majors belonging to certain specific minoritized populations: first-generation students, Latine women, and Veteran students.

The PS courses were interactive, co-curricular problem-solving courses utilizing evidence-based practices, including active and collaborative learning. Students met once a week and worked collaboratively (in groups of 3-4 students) directly on whiteboards for the two-hour class period, discussing and eventually co-solving 3-4 challenging problems shared with the class a week prior to each course meeting. Students were expected to think about/sketch ideas for solving each problem ahead of class. During class, instructors and facilitators actively listened, observed, asked questions and offered as-needed feedback as students worked on the boards.

The participants in this study were ten course facilitators (faculty, graduate and undergraduate student teaching assistants) and nine students enrolled in the program. Nine of the facilitators were either inferred as being white or self-identified as white during the interview. In contrast, student participants self-identified as being Mexican, Mexican American, or Hispanic (five), and African American (one). Three other students self-identified as White. Semi-structured interviews were conducted using two different interview protocols–one for facilitators and one for students. The protocols focused on understanding participants' takeaways from the course, and perspectives on equity. Interviews were recorded and lasted approximately one hour.

Transcripts were analyzed using thematic analysis (Braun & Clark, 2022), read multiple times, then coded in two rounds using deductive and inductive coding schemes (Fereday & Muir-Cochrane, 2006). The first round of coding was deductive and yielded two principal codes developed based on Gutiérrez's (2007) notion of dominant and critical equity. This first round served to parse out our understanding of the various meanings of equity discussed by students and instructors. The second round of coding was inductive and focused on analyzing the data to identify emergent codes. Specifically, we were interested in developing codes that captured the course features and teaching practices students and instructors identified as relating to equity (both dominant and critical). Using an iterative open coding scheme, emergent codes were identified and refined. Once the final codes were developed and defined, the data was carefully re-coded to ensure consistency and accuracy. The final stage of analysis organized the data into a content log based on three organizational categories: (1) student vs. facilitator data, (2) dominant vs. critical equity notions (3) equity themes tied to course features and teaching practices.

Findings

We identified three themes surrounding student and instructor perceptions of equitable support in the co-curricular courses: (1) access to facilitator support, (2) engaging with problems geared towards critical thinking, and (3) collaborating with others in groups. However, our analysis reveals differences in whether students and instructors felt that the practices described

by each theme were, in fact, equitable, as well as differences in their underlying conceptions of equity (i.e., equity understood from the dominant or critical axis). For example, while both facilitators and students felt that theme (1), access to facilitator support, enhanced student learning, students expressed wishing they could interact with faculty who shared their cultural and language identities. Similarly, while both students and facilitators appreciated the rich problem sets in the course (theme 2), one student expressed wanting the problem contexts to be more relevant to everyday life.

In this paper, we focus on exploring theme (3), group work (Figure 1). Facilitators' perceptions about equity in group work were primarily aligned with dominant conception of equity, that is with notions of equity tied to access and achievement. Most Facilitators did not

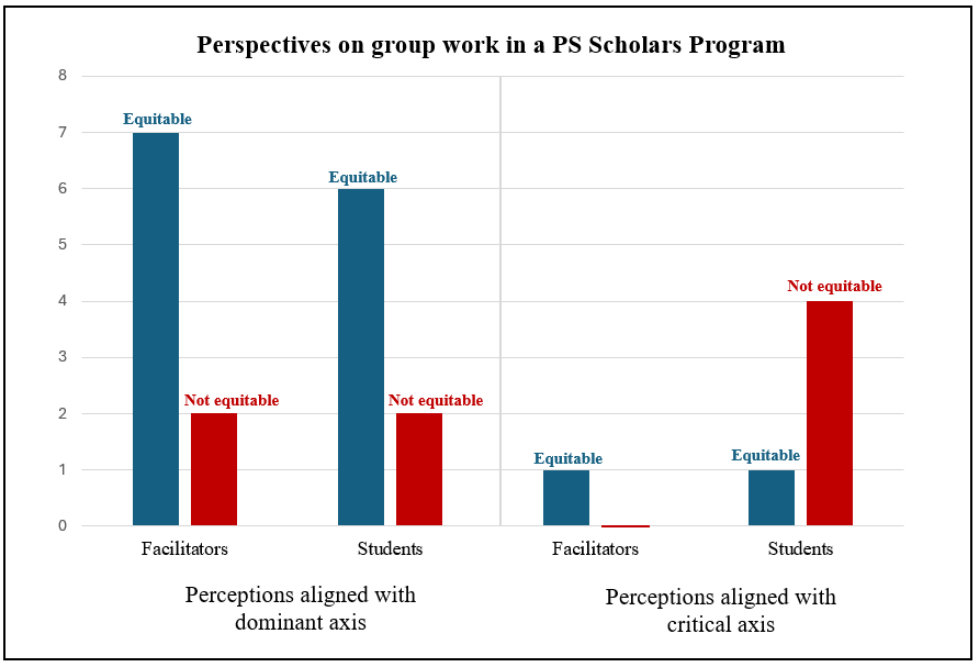


Figure 1. Number of facilitators and students who discussed group work as equitable (or not equitable). The left shows examples coded as aligning with the dominant axis, the right shows examples coded as aligning with critical.

perceive group work as connected to critical dimensions of equity. In contrast, students' perceptions of equity in group work aligned both with dominant and critical notions of equity, hence touching on aspects of what Garcia calls a Latine-serving HSI identity. While both students and facilitators saw group work as largely supporting equity in access and achievement in alignment with Garcia's Latine-producing HSI identity (blue bars vs. red bars, Figure 1, left), students also saw group work as undermining critical dimensions of equity (hence servingness), including identity and power (blue bars vs. red bars, Figure 1, right). We expound on the facilitator and student perspectives underlying these findings next, parsing them out in terms of Gutierrez's framework of dominant and critical equity. We come back to both Gutierrez's and Garcia's frameworks in the discussion.

Dominant Notions of Equity

Most facilitators and students perceived group work as aligned with equity in access and achievement and as mostly supporting (rather than undermining) equity in access and achievement (blue bars, Figure 1, left). Two facilitators and two students also perceived aspects of group work as undermining equity in access and achievement (red bars, Figure 1, left).

Facilitator Perspectives: *Group Work as Equitable*. Most facilitators viewed group work as a "key feature" of the course that supported equity. When discussing this, facilitators made arguments related to access and achievement. For example, they felt that group work gave students access to peers with different ideas and ways of thinking about mathematics, which supported their learning and success in the course. Several facilitators also mentioned group work naturally facilitated participation among students. One instructor noted that students "worked together beautifully" and that it was easier for them to "make mistakes" among peers than "in front of the whole class." Instructors also felt the challenging problems encouraged equitable participation. James explained, "everyone is struggling together," so there no dominant voice over the group work.

Facilitators also shared two practices they often used to ensure participation during group work was equitable: (1) monitoring student participation and stepping in as needed to ensure all were "actively participating," and (2) "mixing up" the groups frequently to ensure that "stronger" and "weaker students" worked together and helped each other learn. Some instructors described the utility of using colored markers to see that everyone was participating; others shared ways of bringing all students to actively participate by asking directed questions like "How would you explain that?" This was seen as helping everyone learn, by everyone sharing what they knew.

Several instructors commented on how regrouping allowed "students that did well" to share their knowledge with different student cohorts and benefit most of the class. Garrett sensed some students disliked regrouping, but felt it amplified students' access to knowledge and learning:

> For equity, one of the most important things was to make sure the groups were different every time so they all had access. /…/ I don't think students loved randomized groups, there was pushback on that. /…/ Of course, students who speak Portuguese will naturally gravitate to students who speak Portuguese or we have a lot of students from China who tend to stick together because they have a very shared experience and they can help each other survive through the process.

Garret acknowledges that students may naturally gravitate toward peers who share aspects of their identity, but believes promoting equity involves focusing on students' mathematical backgrounds to ensure equal access to learning.

Facilitator Perspectives: *Group Work as Not Equitable.* Two instructors raised concerns group work did not equitably support all students. James and Ashley observed that some students were "resistant" to group work. Despite the strategies used, "weaker" students at risk of failing tended to "hide in the background," showed less progress, and were reluctant to participate. James noted some students prefer to "lead" while others "follow," and the followers often fall into hard to break habits, likening it to an elastic band that wants to snap back to their "comfort zone". Other instructors perceived some students put minimal effort and let those who seemed more capable do the work. Overall, instructors felt all students gained something from group work, with James commenting that "just being there was enough" to support their learning.

Student Perspectives: *Group Work as Equitable*. Several students noted that group work equitably supported them in the course by enhancing their access to learning opportunities. They explained that collaborating in groups improved their understanding of mathematics by exposing them to different ways of thinking about and solving problems. For these students, the advantages of group work align with the dominant axis of equity. Alberto commented:

> To work in a collaborative setting is being able to get different perspectives. Since not every professor teaches the same way, each student will understand it differently. Having your peers explain it in different ways, that was really helpful.

Another student, Lucía, notices the presence of Hispanic peers yet seems to value the benefits of group work in terms of seeing different mathematical backgrounds:

> We were a pretty diverse set of individuals and we were mostly made up of Hispanic students, so it was nice being able to share ideas and different ways of learning things.

Student Perspectives: *Group Work as Not Equitable.* Two students felt that aspects of group work did not equitably support their access to learning and achievement. For example, Chantel described the problems she was expected to collaborate on as feeling "like a different language." She did not feel like she "understood" enough math or had the confidence to engage with problems, especially in front of peers. Chantel shared that if the goal of the course was to help students "get better at math", it was "helping her [group members] get greater at math" while she felt that she could not take "advantage" of the course.

Critical Notions of Equity

All but one facilitator perceived group work as neither supporting nor hindering critical dimensions of equity. Students, however, talked about identity and power when addressing group work, and saw this practice as mostly undermining critical dimensions of equity (Figure 1, left).

Facilitator Perspectives: *Group Work as Equitable.* One facilitator, Hannah, provided examples of equitable teaching practices related to group work, which suggest a conception of equity that also considers identity and power. Hannah described "intentionally" grouping women together and Veteran students together. Hannah worried that female students might feel "super overwhelmed and walked over" in groups with only men. Additionally, coming from a military family, Hannah recognized "nontraditional students" may not "relate" to "students in their 20s". Thus, having another Veteran student to work with would help them feel "validated".

By and large, facilitators did not seem to consider students' racial identities in their decisions or perspectives toward equity during group work. We see this in Garrett's quote described above. Similarly, Tamra commented there "was a relatively diverse ethnic group in the class," but added, "I don't feel like any of those differences [impacted] how well [students] participated in the classroom and how effective the classroom was for them."

Student Perspectives: *Group Work as Equitable.* For Stacie, group work affirmed her identity as a Veteran, in alignment with critical dimensions of equity—she felt "engaged and connected," less isolated when tackling problems by having other Veteran students in her group. Stacie appreciated sharing military collaborative practices with her group and felt her instructor, raised by military parents, created a space that felt "comfortable" and supportive for Veteran students.

Student Perspectives*: Group Work as Not Equitable.* Other students, primarily of color, felt group work interfered with supporting identity and power in the classroom. Several students felt the course lacked opportunities to meaningfully connect with peers who shared their cultural backgrounds. Anisia, for example, valued connecting with other peers from Mexico who understood her culture, language, and music, but struggled to "find [her] community" in class. She suggests group work can be challenging for connecting with peers when you are "forced to discuss math".

Other students, like Chantel, a first-generation "African American student" studying STEM, felt disempowered during group work. Raised by a single mother who was "always afraid of math" and with "parents who could not afford tutoring" Chantel felt overwhelmed and "dumb" when working with peers who appeared more confident and knowledgeable:

> When it comes to doing math in front of other people who seem to have it all together, like they really understand it conceptually, it makes me not want to do it because I feel dumb /…/ even if I think I have the right answer, I don't have the confidence to believe in myself.

Chantel shared that "where I come from, like my background, I think that was a hard factor for me in that class".

Similarly, Mateo noted "power dynamics" between "minority students" and "white students", who have had more "exposure" to advanced math courses at "wealthy high schools". He shared:

> Even though we are HSI, I feel like most Hispanics come in with the idea that there's definitely a power difference. If you go into a classroom, who is most likely to speak up is white students /…/ I feel like that's something that's kind of hindering this process of making everyone feel like they're part of the classroom.

Mateo wished that facilitators in the course would have focused on "trying to get rid of that power dynamic and making sure everyone's on the same playing field".

Discussion

Facilitators and students identified three themes that felt relevant to equity in the courses (facilitator support, rich problems, group work). These have been noted as core pillars of active learning and IBL (Apkarian et al., 2024; Laursen & Rasmussen, 2019). Focusing on equity in practice, however, our findings suggest differences in how facilitators and students viewed equity within these themes. Facilitators tended to focus on dominant notions of equity that center access and achievement. In contrast, students expressed a more holistic view of equity aligned with access and achievement as well as identity and power. Facilitators and students primarily agreed that the courses supported dominant notions of equity, which is consistent with traditional active learning design and measures of active learning success. Students' reflections on critical dimensions of equity—like identity and power—suggest the PS courses (which preceded the HSI designation by several years) embodied a Latine-producing HSI identity (Garcia, 2017) not quite yet fully embracing *servingness*. In other words, instructor and student perceptions suggest more can be done to embrace servingness at the classroom level in active learning PS courses, in ways that center identities, values, and lived experiences of all students.

The field's emphasis on dominant dimensions of equity when it comes to active learning may have prevented some instructors from thinking more critically about why students might choose not to participate or may not have access to participation. Rubel (2017) reports that the myth of meritocracy, feelings of "I can't relate", and colorblindness were three reasons White school teachers struggled to engage with critical equity-oriented practices. These reasons also emerged as themes in our data. For example, the perception that students who do not participate simply need to put in more effort aligns with ideas of meritocracy. These suggest an area for professional development (PD) to better support HSI faculty in cultivating equitable practices. Such PD could include facilitating conversations about the role of identity in students' STEM experiences (Adiredja et al., in press) and designing and implementing place-based, culturally responsive curriculum (Lozano, 2023).

We close by noting that, historically, much of the work on calculus workshops has used quantitative methods to assess normative outcomes as measures of success (McNeilly, 2021). While we are not questioning the impact of these programs, our work suggests that focusing exclusively on dominant understandings of equity—aligned with access and achievement—may overlook how equity is intended, perceived, and felt by minoritized students in the classroom, and shortchange opportunities for instructors to learn about and enact critical equity practices.

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Coordinate Representation of Vectors and Matrix Representations of Linear Transformations in a DGS Assisted Linear Algebra Learning Environment: The Case of Polynomial Vector Space

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The contributed report highlights the understanding of matrix representations of linear transformations mapping finite-dimensional polynomial vector spaces among university mathematics majors within a dynamic geometry software (DGS)-assisted linear algebra learning environment. Through an analysis of data from in-depth qualitative interviews, framed by the theories of representations in linear algebra (Dorier & Sierpinska, 2001; Hillel, 2000) and the concreteness principle (Harel, 2000), the findings reveal a range of innovative and creative ways in which the students integrated visual and analytical approaches to understand, produce, and interpret matrix representations of linear transformations. The results contribute new insights into the pedagogy of university-level linear algebra, highlighting the growing understanding of the role of visualization in the teaching and learning of the subject. The study also adds to the increasing body of evidence supporting the importance of visualization in linear algebra education.

Keywords: matrix representation; linear transformation; polynomial vector space; differential operator; dynamic geometry software

Researchers widely focused on the pedagogy of linear algebra (Dogan-Dunlap, 2010; Dorier, 1991, 1995, 1998; Dreyfus, Hillel & Sierpinska, 1998; Gueudet-Chartier, 2006; Harel, 1987, 1989, 1990; Parraguez & Oktac, 2010; Pavlopoulou, 1993; Robert & Robinet, 1989; Rogalski, 1994; Sierpinska, 1995, 2000; Sierpinska, Dreyfus & Hillel, 1999; Sinclair & Gol Tabaghi, 2010; Thomas & Stewart, 2011; Zandieh, Wawro & Rasmussen, 2017) along with representations in R^n and the multi-representational aspect of linear algebra (Dorier, Sierpinska, 2001; Hillel, 1997, 2000). Harel and Kaput (1991), and Harel (1990) formulated the concreteness principle, as a fundamental approach for the teaching and learning of linear algebra. He recommends MATLAB as a tool that would help students visualize vectors and matrices as concrete mathematical objects, in accordance with the concreteness principle. Aligned with the concreteness principle (Harel, 2000) and in search for a desirable matrix representation for a linear transformation as brought forward in Hillel (2000), this study focuses on linear algebra students' understandings of the matrix representations of linear transformations mapping finite dimensional polynomial vector spaces in a dynamic environment.

Representations in Linear Algebra

Describing linear algebra as an explosive compound of systems of representation, Dorier and Sierpinska (2001) classifies the representations used in linear algebra as the graphical, the tabular, and the symbolic representations (p. 270). Hillel (2000) views basis-dependent representations in linear algebra as the major cause of difficulties for students of introductory linear algebra courses. Basis-dependent representations are not unique; these representations are contingent upon the particular basis or bases used. The two major challenges for linear algebra students are the understanding of such basis-dependent structures along with the understanding of representational invariants, that is, the understanding of what stays invariant irrespective of the bases that are being employed in representing linear algebraic structures (p.199). In general,

when representing linear transformations like matrices, students often have difficulties in interpreting the columns of these matrix representations. A matrix representation itself is used to represent a linear transformation, yet, students are tempted to think of columns of this matrix as the images of the corresponding basis vectors under the linear transformation: "It stems from thinking that the columns of the matrix representation of an operator are, so to speak, images of the basis vectors, not representations of these images relative to a basis" (p.203-204). In their exploration of linear transformations, the mathematics majors in this study creatively and innovatively integrated visual and analytic approaches to construct meaningful matrix representations. Thus, the study is framed within the guiding theoretical perspectives of the concreteness principle (Harel, 2000) and the multi-representational context of linear algebra (Dorier & Sierpinska, 2001; Hillel, 2000).

Table 1. Interview tasks: Find the matrix A' for $T: \mathbb{P}_n \to \mathbb{P}_m$ relative to the bases B and B'.

Task 1	♦Multiplication by x Operator $\begin{cases} T: \mathbb{P}_2 \to \mathbb{P}_3 \\ p \mapsto xp \end{cases}$	$\mathcal{B} = \{1, x, x^2\}; \mathcal{B}' = \{1, x, x^2, x^3\}$
Task 2	♦Multiplication by x^2 Operator $\begin{cases} T: \mathbb{P}_2 \to \mathbb{P}_3 \\ p \mapsto x^2 p \end{cases}$	$\mathcal{B} = \{1, x, x^2\}; \mathcal{B}' = \{1, x, x^2, x^3, x^4\}$
Task 3	♦Differential Operator $\begin{cases} D_x: \mathbb{P}_2 \to \mathbb{P}_1 \\ p \mapsto p' \end{cases}$	$\mathcal{B} = \{1, x, x^2\}; \mathcal{B}' = \{1, x\}$
		$\mathcal{B} = \{1 + x, x - x^2, x + x^2\}; \mathcal{B}' = \{1 - x, 1 + x\}$
Task 4	♦Differential Operator $D_x: \mathbb{P}_3 \to \mathbb{P}_2$	$\mathcal{B} = \{1 + x^2, x - x^3, 1 - x^2, x + x^3\}; \mathcal{B}' = \{1 + x, x - x^2, x + x^2\}$
Task 5	♦Integral operator $\begin{cases} J: \mathbb{P}_2 \to \mathbb{P}_3 \\ x^k \mapsto \int_0^x t^k dt \end{cases}$	$\mathcal{B} = \{1 - x, x - x^2, 1 - x + x^2\}; \mathcal{B}' = \{1, x, x^2, x^3\}$
		$\mathcal{B} = \{1 + x, x - x^2, x + x^2\}; \mathcal{B}' = \{1, x, x^2, x^3\}$
		$\mathcal{B} = \{1, x, x^2\}; \mathcal{B}' = \{1 + x^2, x - x^3, 1 - x^2, x + x^3\}$
Task 6	♦Integral operator $\begin{cases} J: \mathbb{P}_3 \to \mathbb{P}_4 \\ x^k \mapsto \int_0^x t^k dt \end{cases}$	$\mathcal{B} = \{1, x, x^2, x^3\}; \mathcal{B}' = \{1, x, x^2, x^3, x^4\}$
		$\mathcal{B} = \{1 + x, x - x^2, x + x^2, x^3\}; \mathcal{B}' = \{1, x, x^2, x^3, x^4\}$
Sample Interview Outline: Is T/D/J a linear transformation? What does it mean for T/D/J to be a linear transformation? What is the standard matrix representation? Is it possible to have a square matrix as the matrix representation of T/D/J? Is it possible to use non-standard bases in the expression of the matrix representation of T/D/J? How would you multiply/differentiate/integrate a polynomial using the matrix representation of T/D/J? How would you compare the two approaches? How does the matrix representation you proposed act on the polynomial on the DGS ? How would you distinguish between the actual object from its representation?		

Context and Method

Qualitative-descriptive interview data were collected at a U.S. university as part of a research project aimed at enhancing math and math education majors' understanding of advanced mathematics, with a particular focus on the connections between geometry and linear algebra using DGS-MATLAB technology. The qualitative interviews followed a semi-structured interview model (Kvale, 2007), where the interviewer used follow-up probes and questions to delve deeper into the interviewees' responses. Sixteen math majors participated and answered a series of questions centered around matrix representations (see Table 1). The tasks were designed to be explored through both visual and analytic approaches, in line with the study's theoretical

framework. Although optional, the math majors were eager and enthusiastic about using MATLAB/DGS, primarily to check their work, visualize analytic solutions, test conjectures, and provide examples or counterexamples.

The analysis of the videotaped interviews, along with MATLAB-DGS inscriptions, was conducted using thematic analysis (Boyatzis, 1998) and the VA (visual-analytic) strategy coordination method (Zazkis, Dubinsky, & Dautermann, 1996). This methodology was employed to describe the innovative and creative ways in which participants coordinated visual and analytic approaches to make sense of matrix representations of linear transformations. After transcribing the interview videos, the author reviewed the videos and original transcripts line by line, using thematic analysis in coordination with the interview analysis method described by Zazkis et al. (1996), to identify the strategies (visual and analytic) employed by students in exploring matrix representations. The final stage of data analysis involved a comprehensive review of the entire dataset multiple times, following the constant comparative methodology (Glaser & Strauss, 1967).

Results

This section considers the polynomial multiplication operator (Tasks 1-2), the differential operator (Tasks 3-4), and the integral operator (Tasks 5-6). In each task, students were asked to find the matrix representation A' for the given linear transformation T relative to bases $\mathcal{B}$ and $\mathcal{B}'$. The analysis that follows focuses primarily on student-generated approaches and strategies.

```
>> t1=[0;0;1;0;0];t2=[0;0;0;1;0];
>> t3=[0;0;0;0;1]; A=[t1 t2 t3]

A =

     0     0     0
     0     0     0
     1     0     0
     0     1     0
     0     0     1
```

```
>> p=[2;-3;1]

p =

     2
    -3
     1
```

```
>> w=A*p

w =

     0
     0
     2
    -3
     1
```

Figure 1. S1's analytic-arithmetic representation for Task 2.

The Polynomial Multiplication Operator

Most of the research participants thought of Tasks 1-2 illustrating standard matrices of linear transformations that act as a polynomial-multiplication operator as straightforward and sufficed with the analytic representation by expressing $T(\mathbf{e}_1), T(\mathbf{e}_2), T(\mathbf{e}_3)$ in terms of $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ accompanied by the MATLAB verification process in obtaining the matrix representations. S1, for instance, first expressed the image vectors $T(1) = x, T(x) = x^2, T(x^2) = x^3$ in terms of the vectors of the standard basis $\mathcal{B}'$ of the codomain vector space $\mathbb{P}_3$ as $T(1) = 0\mathbf{w}_1 + 1\mathbf{w}_2 + 0\mathbf{w}_3 + 0\mathbf{w}_4, T(x) = 0\mathbf{w}_1 + 0\mathbf{w}_2 + 1\mathbf{w}_3 + 0\mathbf{w}_4, T(x^2) = 0\mathbf{w}_1 + 0\mathbf{w}_2 + 0\mathbf{w}_3 + 1\mathbf{w}_4$. She then wrote the 4×1 coordinate matrices $[T(\mathbf{e}_1)]_{\mathcal{B}'}, [T(\mathbf{e}_2)]_{\mathcal{B}'}, [T(\mathbf{e}_3)]_{\mathcal{B}'}$ using the coefficients in each linear combination in the previous step. Upon juxtaposing these coordinate matrices she obtained the desired 3×4 matrix $A = [[T(\mathbf{e}_1)]_{\mathcal{B}'} \quad [T(\mathbf{e}_2)]_{\mathcal{B}'} \quad [T(\mathbf{e}_3)]_{\mathcal{B}'}]$. To illustrate how the multiplication-by-x-operator works via the matrix representation, S1 first converted the polynomial $\mathbf{p} = 2 - 3x + x^2$ to its column vector representation $[\mathbf{p}]_{\mathcal{B}} = [2 \quad -3 \quad 1]^T$ relative to the standard basis $\mathcal{B}$ of the domain vector space $\mathbb{P}_2$. Upon applying the matrix A on $[\mathbf{p}]_{\mathcal{B}}$, she obtained $A[\mathbf{p}]_{\mathcal{B}} = [T(\mathbf{p})]_{\mathcal{B}'}$. Finally, she retrieved the image polynomial $T(\mathbf{p})$ from its coordinate representation $[T(\mathbf{p})]_{\mathcal{B}'}$ via the linear combination expression $T(\mathbf{p}) = 0\mathbf{w}_1 + 2\mathbf{w}_2 - 3\mathbf{w}_3 + 1\mathbf{w}_4 = 2x - 3x^2 + x^3$. S1's approach for the other warm-up example was similar, as illustrated in Figure 1.

The Elementary Differentiation Operator

Interviews on differential operators (Tasks 3-4) began with the warm-up activity in which math majors explored the standard matrix of the differential operator $D_x: \mathbb{P}_2 \longrightarrow \mathbb{P}_1$. The common approach was the analytic one: (i) Math majors first determined the 3 image vectors $D_x(1), D_x(x), D_x(x^2)$ by applying the linear transformation D_x to each vector of the standard basis $\mathcal{B} = \{\mathbf{e}_1 = 1, \mathbf{e}_2 = x, \mathbf{e}_3 = x^2\}$ of the domain vector space $\mathbb{P}_2$; (ii) They then expressed each image vector $D_x(1) = 0, D_x(x) = 1, D_x(x^2) = 2x$ in terms of the vectors of the basis $\mathcal{B}' = \{\mathbf{w}_1 = 1, \mathbf{w}_2 = x\}$ of the codomain vector space $\mathbb{P}_1$ as $D_x(1) = 0\mathbf{w}_1 + 0\mathbf{w}_2, D_x(x) = 1\mathbf{w}_1 + 0\mathbf{w}_2, D_x(x^2) = 0\mathbf{w}_1 + 2\mathbf{w}_2$; (iii) They then wrote three 2×1 coordinate matrices $[D_x(\mathbf{e}_1)]_{\mathcal{B}'} = [0 \quad 0]^T, [D_x(\mathbf{e}_2)]_{\mathcal{B}'} = [1 \quad 0]^T, [D_x(\mathbf{e}_3)]_{\mathcal{B}'} = [0 \quad 2]^T$ using the coefficients in each linear combination in the previous step; (iv) Finally, they combined the coordinate matrices from the previous step to form the sought 2×3 matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

Students then switched to the MATLAB approach to visualize what has been analyzed, and to use the 2×3 matrix A to evaluate $D_x(\mathbf{v})$ where $\mathbf{v} = 2 - 3x + x^2$. For that purpose, the common approach was to enter the 2×3 matrix A and the vector $\mathbf{v}$ by typing **D1=[0;0]; D2=[1;0]; D3=[0;2]; A=[D1 D2 D3]** and **v=[2;-3;5]**, respectively. Finally, typing **w=A*v** resulted in a column vector with components -3 and 10 (fig.2a). S2, for instance, stopped at this stage and remarked that the vector's components were the same as the coordinates because she was dealing with standard basis representations. Although several students stopped at this stage; some others further indicated that $[-3 \quad 10]^T$ was the coordinates of the vector sought, and that the additional step of moving back from coordinate vector representation $[-3 \quad 10]^T$ to the vector itself was necessary, whereby deduced the answer as $\mathbf{w} = -3 + 10x \in \mathbb{P}_1$. Students who felt the need to take this additional step of mapping the coordinate representation back to the actual vector itself could be thought of as being aware of the distinction between the components of a vector and the coordinates relative to the standard basis, even though the same set of numbers were used to express them (Hillel, 2000, p.201).

In another task, students were asked to determine the matrix representation of the differential operator $D_x: \mathbb{P}_3 \longrightarrow \mathbb{P}_2$ using the nonstandard bases $\mathcal{B} = \{\mathbf{v}_1 = 1 + x^2, \mathbf{v}_2 = x - x^3, \mathbf{v}_3 = 1 - x^2, \mathbf{v}_4 = x + x^3\}$ and $\mathcal{B}' = \{\mathbf{w}_1 = 1 + x, \mathbf{w}_2 = x - x^2, \mathbf{w}_3 = x + x^2\}$. Unlike the previous task in which students had begun with the analytic approach, in this task all students began with a visual approach primarily for the purpose of verifying that the given sets $\mathcal{B}$ and $\mathcal{B}'$ were indeed bases for $\mathbb{P}_3$ and $\mathbb{P}_2$, respectively (fig.2b-e). One group of students verified this via the traditional rref procedure on MATLAB (fig.2b,d). Those who embraced the DGS approach created sliders in an attempt to demonstrate that any polynomial of degree ≤ 3 (or degree ≤ 2) could be written as a linear combination of the basis vectors of the set $\mathcal{B}$ (or $\mathcal{B}'$), respectively (fig.2c,e). Upon establishing $\mathcal{B}$ and $\mathcal{B}'$ as bases, students switched to the analytic procedure in order to obtain the matrix representation of the differential operator. S3 expressed each image vector $D_x(1 + x^2) = 2x, D_x(x - x^3) = 1 - 3x^2, D_x(1 - x^2) = -2x, D_x(x + x^3) = 1 + 3x^2$ in terms of the vectors of the basis $\mathcal{B}' = \{\mathbf{w}_1 = 1 + x, \mathbf{w}_2 = x - x^2, \mathbf{w}_3 = x + x^2\}$ of the codomain vector space $\mathbb{P}_2$ as $D_x(\mathbf{v}_1) = 0\mathbf{w}_1 + 1\mathbf{w}_2 + 1\mathbf{w}_3, D_x(\mathbf{v}_2) = 1\mathbf{w}_1 + 1\mathbf{w}_2 - 2\mathbf{w}_3, D_x(\mathbf{v}_3) = 0\mathbf{w}_1 - \mathbf{w}_2 - \mathbf{w}_3, D_x(\mathbf{v}_4) = 1\mathbf{w}_1 - 2\mathbf{w}_2 + 1\mathbf{w}_3$. In the third stage, S3 wrote the corresponding 3×1 coordinate matrices $[D_x(\mathbf{v}_1)]_{\mathcal{B}'} = [0 \quad 1 \quad 1]^T, [D_x(\mathbf{v}_2)]_{\mathcal{B}'} = [1 \quad 1 \quad -2]^T, [D_x(\mathbf{v}_3)]_{\mathcal{B}'} = [0 \quad -1 \quad -1]^T, [D_x(\mathbf{v}_4)]_{\mathcal{B}'} = [1 \quad -2 \quad 1]^T$ using the previously determined coefficients in each linear combination from the second stage. Finally, he obtained the matrix representation of

the differential operator by juxtaposing the four column vectors from the third stage as the 3×4 matrix $A = [[D_x(\mathbf{v}_1)]_{\mathcal{B}'} \quad [D_x(\mathbf{v}_2)]_{\mathcal{B}'} \quad [D_x(\mathbf{v}_3)]_{\mathcal{B}'} \quad [D_x(\mathbf{v}_4)]_{\mathcal{B}'}] = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & -1 & -2 \\ 1 & -2 & -1 & 1 \end{bmatrix}$.

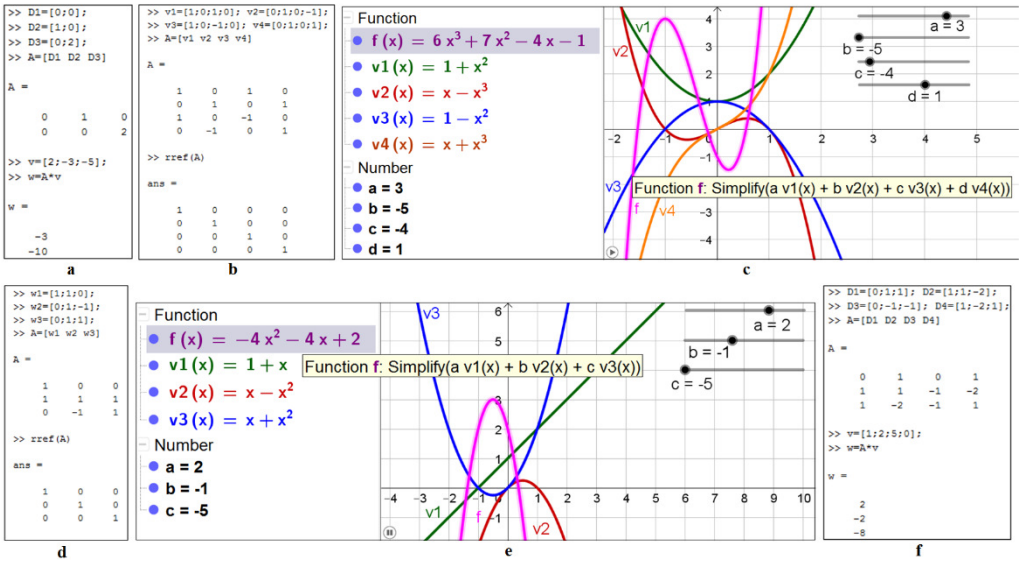


Figure 2. Analytic and visual approaches.

The interviewer then asked the research participants to evaluate the derivative of the cubic polynomial $\mathbf{v} = 6 + 2x - 4x^2 - 2x^3$ via the two approaches and then to compare the two answers. Using the power rule, students easily determined the derivative of the cubic polynomial as $D_x(\mathbf{v}) = 2 - 8x - 6x^2$. In an attempt to see how the matrix approach works, S4 first expressed the cubic polynomial $\mathbf{v}$ in terms of the vectors of the basis $\mathcal{B} = \{\mathbf{v}_1 = 1 + x^2, \mathbf{v}_2 = x - x^3, \mathbf{v}_3 = 1 - x^2, \mathbf{v}_4 = x + x^3\}$ of the domain vector space $\mathbb{P}_3$ simply as $\mathbf{v} = 6 + 2x - 4x^2 - 2x^3 = 1\mathbf{v}_1 + 2\mathbf{v}_2 + 5\mathbf{v}_3 + 0\mathbf{v}_4$; then obtained the coordinate vector representation of the cubic polynomial as $[\mathbf{v}]_{\mathcal{B}} = [1 \quad 2 \quad 5 \quad 0]^T$. She then switched to MATLAB and followed a procedure similar to S2's (fig.2a) and determined the matrix applied coordinate vector $A[\mathbf{v}]_{\mathcal{B}} = [2 \quad -2 \quad -8]^T$ as shown (fig.2f). Upon the interviewer's probing whether she would want to stop or continue, S4 responded that she should continue, explaining that what she had was not the vector itself but its image, specifically its coordinate vector representation according to the basis B prime. She added that she still needed to find the vector itself, or the image polynomial. Using the linear combination analytic approach, she determined the image polynomial as $2\mathbf{w}_1 - 2\mathbf{w}_2 - 8\mathbf{w}_3 = 2(1 + x) - 2(x - x^2) - 8(x + x^2) = 2 - 8x - 6x^2$.

In another task of determining the matrix representation of the differential operator $D_x: \mathbb{P}_2 \longrightarrow \mathbb{P}_1$ using the nonstandard bases $\mathcal{B} = \{\mathbf{v}_1 = 1 + x^2, \mathbf{v}_2 = x - x^2, \mathbf{v}_3 = x + x^2\}$ and $\mathcal{B}' = \{\mathbf{w}_1 = 1 - x, \mathbf{w}_2 = 1 + x\}$, upon establishing the basisness of $\mathcal{B}$ and $\mathcal{B}'$ in a similar manner as in the previous task via DGS-MATLAB approach, S5 first obtained the matrix representation of the differential operator as a 2×3 matrix D via the analytic-arithmetic procedure in a manner similar to S4's, then demonstrated how this matrix representation actually acts on the coordinate representation $[\mathbf{v}]_{\mathcal{B}}$ of the previously explored polynomial $\mathbf{v} = 2 - 3x + 5x^2$ in the DGS. Upon graphing all the key actors (fig.3a-c); S5 concluded her exploration by verifying that the image polynomial (fig.3d) obtained both ways coincide via the linear combination syntax in a similar manner as before.

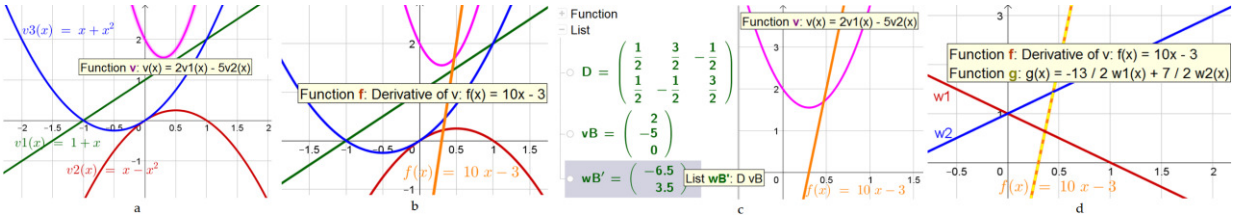


Figure 3. S5's visual approach for Task 3.

The Elementary Integral Operator

In the previous tasks on the elementary differential operator $D_x: \mathbb{P}_n \longrightarrow \mathbb{P}_{n-1}$, math majors made use of both analytic and visual approaches by involving DGS and MATLAB in their explorations. Such a visual-analytic coordination proved to be the case in the investigation of the integral operator mapping polynomials vector spaces as well. In the task of finding the matrix representation of the integral operator $T: \mathbb{P}_2 \longrightarrow \mathbb{P}_3$ defined by the equation $T(x^k) = \int_0^x t^k dt$ using a nonstandard basis $\mathcal{B} = \{\mathbf{v}_1 = 1 + x, \mathbf{v}_2 = x - x^2, \mathbf{v}_3 = x + x^2\}$ for $\mathbb{P}_2$ and the standard basis $\mathcal{B}'$ for $\mathbb{P}_3$, S6 first analytically verified that $\mathcal{B} = \{\mathbf{v}_1 = 1 + x, \mathbf{v}_2 = x - x^2, \mathbf{v}_3 = x + x^2\}$ was actually a basis for $\mathbb{P}_2$ via the straightforward rref procedure. She then applied the analytic procedure via which she obtained the coordinate vector representations $[T(\mathbf{v}_1)]_{\mathcal{B}'} = [0 \quad 1 \quad 1/2 \quad 0], [T(\mathbf{v}_2)]_{\mathcal{B}'} = [0 \quad 0 \quad 1/2 \quad -1/3], [T(\mathbf{v}_3)]_{\mathcal{B}'} = [0 \quad 0 \quad 1/2 \quad 1/3]$; then switched to the MATLAB approach by introducing the three 4×1 coordinate matrices by typing **T1=[0;1;1/2;0]; T2=[0;0;1/2;-1/3]; T3=[0;0;1/2;1/3]**. She then thought aloud and stated that because $\mathcal{B}' = \{1, x, x^2, x^3\}$ for $\mathbb{P}_3$ was a standard basis, she could simply combine the three 4×1 column vectors to form the desired matrix representation via the syntax **A=[T1 T2 T3]**.

To test her visualization, she first obtained the coordinate vector representation $[\mathbf{v}]_{\mathcal{B}} = [1 \quad -1 \quad -1]^T$ of the polynomial $\mathbf{v} = 1 - x$ relative to the basis $\mathcal{B} = \{\mathbf{v}_1 = 1 + x, \mathbf{v}_2 = x - x^2, \mathbf{v}_3 = x + x^2\}$ in the domain vector space $\mathbb{P}_2$. Then she applied the matrix A to $[\mathbf{v}]_{\mathcal{B}} = [1 \quad -1 \quad -1]^T$, resulting in $A[\mathbf{v}]_{\mathcal{B}} = [\mathbf{w}]_{\mathcal{B}'} = [0 \quad 1 \quad -1/2 \quad 0]^T$ (fig.4a). She interpreted this as the coordinate vector representation of the polynomial $\mathbf{w} = x - (1/2)x^2$ relative to the standard basis for the codomain vector space $\mathbb{P}_3$. She concluded by indicating that the real things are related to each other via integration, that is, the fact that the integral of $\mathbf{v}$ polynomial would result in the $\mathbf{w}$ polynomial under the T operator. In a similar task of finding the matrix representation of the integral operator $T: \mathbb{P}_3 \longrightarrow \mathbb{P}_4$ defined by the equation $T(x^k) = \int_0^x t^k dt$ using a nonstandard basis $\mathcal{B} = \{\mathbf{v}_1 = 1 + x, \mathbf{v}_2 = x - x^2, \mathbf{v}_3 = x + x^2, \mathbf{v}_4 = x^3\}$ for $\mathbb{P}_3$ and the standard basis $\mathcal{B}' = \{\mathbf{w}_1 = 1, \mathbf{w}_2 = x, \mathbf{w}_3 = x^2, \mathbf{w}_4 = x^3, \mathbf{w}_5 = x^4\}$ for $\mathbb{P}_4$, S7 followed almost the same approach as S6 with the slight difference in her concluding steps where she demonstrated the action of the matrix representation on the polynomial $\mathbf{v} = 2 - x + 3x^2 + 4x^3$ with the coordinate vector representation $[\mathbf{v}]_{\mathcal{B}}$ (fig.4b).

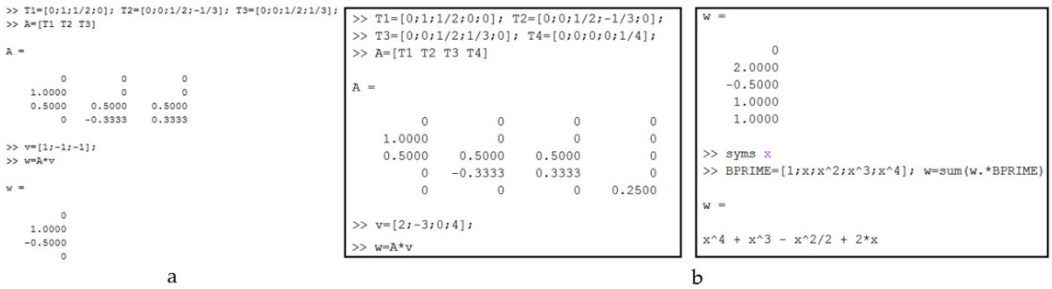


Figure 4. Students' MATLAB approach for Task 5.

In another version of the same task for finding the matrix representation of the integral operator $T: \mathbb{P}_2 \longrightarrow \mathbb{P}_3$ defined by the equation $T(x^k) = \int_0^x t^k dt$ using the standard bases $\mathcal{B} = \{\mathbf{v}_1 = 1, \mathbf{v}_2 = x, \mathbf{v}_3 = x^2\}$ and $\mathcal{B}' = \{\mathbf{w}_1 = 1, \mathbf{w}_2 = x, \mathbf{w}_3 = x^2, \mathbf{w}_4 = x^3\}$, all math majors embraced a straightforward step-by-step approach similar to the integral operator one previously analyzed. Upon exploring the matrix representation both analytically and in MATLAB in a similar manner as before, math majors came up with the matrix representation A; they also felt the need to explore the integral operator and its representation further with the DGS. For that purpose, S8 entered the basis vectors of $\mathcal{B}$ as functions by respectively typing **v1(x)=1; v2(x)=x; v3(x)=x^2**; in the DGS. S8 referred to $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ as a constant, a line with positive slope, and a parabola, respectively. Upon the interviewer's probing how she would visualize the vector $\mathbf{v}$ whose coordinates relative to the basis is given by $[4 \quad 0 \quad -3]^T$, she entered the desired vector by typing the linear combination syntax **v=4v1-0v2-3v3**, whose graph S8 referred to as an upside-down parabola (fig.5a). She then typed the command **f=integral(v)**, resulting in the curve with purple color (fig.5b). Next, she focused on the representations. S8 introduced the matrix A of the integral operator T which she had previously obtained algebraically by typing the syntax **A={{0,0,0}, {1,0,0}, {0,1/2,0}, {0,0,1/3}}**. She then entered the coordinate representation of vector $\mathbf{v}$ relative to the standard basis $\mathcal{B} = \{1, x, x^2\}$ for $\mathbb{P}_2$ using the syntax **vB = {{4}, {0}, {-3}}**. Next, she typed **wB'= A*vB** and explained that wB' represented the coordinate representation of the sought vector relative to the standard basis $\mathcal{B}' = \{1, x, x^2, x^3\}$ for $\mathbb{P}_3$ which should coincide with the graph of the purple curve when plotted (fig.5c). In the last step of her visualization, she entered the sought vector by paying close attention to its coordinate representation by typing **g=0*1+4*x+0*x^2-1*x^3** and commenting that the two graphs should match, a result she indicated she was expecting (fig.5d).

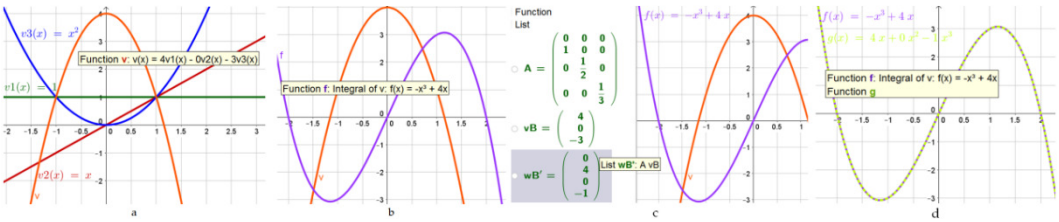


Figure 5. S8's visual approach for Task 6.

Conclusion

The findings highlighted in the present report are in agreement with Hillel's (2000) observations regarding the importance of correctly finding a matrix representation of a linear transformation relative to a basis; the report's findings are also in agreement in that math majors were also "able to think about matrix representations of linear operators as objects of inquiry in their own right" (p.199). Furthermore, math majors demonstrated representational creativity which enabled them "consider the general conditions under which a linear operator can have a particularly desirable representation" (p.199). In order to help linear algebra students understand how objects and their representations interact in analogy, a CAS such as MATLAB or DGS such as GeoGebra could be very beneficial. This has a pedagogical implication that aligns with the Linear Algebra Study Group (LACSG) recommendation for use of technology and computer software: "Faculty should be encouraged to utilize technology in the first linear algebra course" (Carlson, Johnson, Lay & Porter, 1993, p.45). Because basis-dependent representations are not unique, generating object-representation analogy maps in a DGS-MATLAB facilitated learning environment could help linear algebra learners make sense of what stays invariant.

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Student Explanation Strategies in Postsecondary Mathematics and Statistics Education: A Scoping Review

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This scoping review examines the use of student explanation strategies in postsecondary mathematics and statistics education. We analyzed 46 peer-reviewed articles published between 2014 and 2024, categorizing student explanations into three main types: self-explanation, peer explanation and explanation to fictitious others. The review synthesizes the theoretical underpinnings of these strategies, drawing on the retrieval practice hypothesis, generative learning hypothesis, and social presence hypothesis. Our findings indicate that while self-explanation and explaining to fictitious others foster individual cognitive processes enhancing generative thinking, peer explanation have the potential to combine these benefits with collaborative learning. However, explanation to fictitious others have the potential to mitigate some of the negative impacts that may occur in peer explanation, such as more knowledgeable students dominating peer discussions. The efficacy of the methods varies based on implementation, duration, and context. This scoping review contributes to the growing body of literature on generative learning strategies in postsecondary education and provides insights for optimizing the integration of student explanation techniques in within mathematics and statistics.

Keywords: Student explanation, Self-explanation, Peer explanation, Peer instruction, Explanation to fictitious others

Introduction

In mathematics and statistics education, explanations serve as a critical tool for conveying complex concepts and fostering deep understanding (Rittle-Johnson & Loehr, 2017). These explanations can be broadly categorized into two types: teacher explanations and student explanations.

Teacher explanations traditionally dominate both face-to-face and online video-based instruction in postsecondary education, often criticized for fostering student passivity. In response, recent research has increasingly explored strategies to reduce the reliance on teacher-centered methods, particularly following Freeman et al.'s (2014) meta-analysis, which found that active learning significantly boosts student performance in Science, Technology, Engineering, Mathematics (STEM) fields. From a cognitive perspective, meaningful learning and long-term knowledge retention are shown to require active engagement in the process of knowledge generation and construction, rather than passive consumption (Fiorella & Mayer, 2014; Freeman et al., 2014; Rittle-Johnson et al., 2017)

Student explanations, on the other hand, involve learners articulating their understanding of concepts, processes, or problem-solving steps, either to themselves or others. Hodds et al. (2014) demonstrated that prompting undergraduate students to elucidate mathematical concepts enhances their focus on analyzing and explicating the learning content, thereby facilitating improved learning outcomes. In general, there is a consensus that student explanations play a crucial role in shaping understanding and engagement (Bisra et al., 2018; Fiorella & Mayer, 2014; Rittle-Johnson et al., 2017).

Despite the recognized importance of student explanations, current research still exhibits several gaps in integrating their theoretical bases. Additionally, the specifics of implementation

processes of student explanations in mathematics and statistics remain relatively vague. This investigation aims to contribute to the growing body of literature on generative learning strategies in postsecondary educational contexts, by categorizing the literature about student explanation in mathematics and statistics education at the tertiary level.

Theoretical grounding

The cognitive processes underlying student explanation can be understood through the lens of Human Cognitive Architecture (Atkinson & Shiffrin, 1968). This model posits that information processing occurs in stages: from sensory memory (visual, auditory, and tactile inputs) to working memory (where information is manipulated) and finally to long-term memory (where knowledge is stored and retrieved). These processes are fundamental to understanding how student explanations facilitate learning in mathematics and statistics education.

The retrieval practice hypothesis, proposed by Roediger & Karpicke (2006), aligns with the role of long-term memory in this cognitive model (Karpicke & Roediger, 2008). It posits that the act of explanation enhances learning through active retrieval of previously acquired knowledge. In mathematics and statistics education, this can be applied by encouraging students to explain complex formulas (Lyle et al., 2020), statistical concepts (Hun Lim et al., 2015), or problem-solving procedures without referring to notes or textbooks (May, 2022). This process strengthens memory traces and improves long-term memory, as demonstrated by Lachner et al. (2020, 2021).

The generative learning hypothesis (Lachner et al., 2018) relates to both working memory and long-term memory processes. It suggests that student explanation involves complex cognitive processes beyond mere retrieval. In higher-level mathematics and statistics courses, this can be implemented by having students create explanations that connect different mathematical concepts or apply statistical principles to novel problems. This approach emphasizes active knowledge construction and the development of meaningful connections between concepts, engaging both working memory for immediate processing and long-term memory for knowledge integration (Fiorella & Mayer, 2014).

The social presence hypothesis (Short et al., 1976) also involves working memory and long-term memory processes, but through an interpersonal dimension. In mathematics and statistics education, this can be applied through peer teaching activities or group problem-solving sessions. This theory suggests that anticipating an audience's knowledge leads to continuous adjustment of explanatory content, enhancing learning (Mills et al., 2020). Empirical evidence shows that learners engaged in student-to-student explanation reported higher levels of social presence (Nasir et al., 2023), motivation (Morris et al., 2023), and increased mental effort (Alegre et al., 2019), indicating enhanced engagement of working memory and potential for long-term memory consolidation.

These three hypotheses collectively offer a comprehensive framework for understanding the efficacy of student explanation in mathematics and statistics education, highlighting the interplay between sensory input, cognitive processing, and social factors in enhancing learning outcomes.

Methods

The approach for the scoping review is underpinned by Arksey & O'Malley's (2005) five-stage framework, which adopts a rigorous process of transparency, enabling replication of the search strategy and increasing the reliability of the study findings. The five stages of the framework are: (1) identifying the initial research questions; (2) identifying relevant studies; (3) study selection; (4) charting the data; and (5) collating, summarizing and reporting the results. These stages were utilized in this review of the student explanation literature.

Identifying the initial research questions

The focus of our review was to examine the specifics of the use of student explanation in mathematics and statistics postsecondary education, thus we proposed the following research question:

How can literature on student explanation strategies in postsecondary mathematics and statistics education be categorized to reveal current research patterns and trends?

Identifying relevant studies

All publications included in this review were published in English and indexed in Eric, Scopus, or Web of Science databases. Additionally, we utilized websites (i.e., Google Scholar and Research Rabbit https://www.researchrabbit.ai/) and conducted citation searches to ensure comprehensive coverage. The linked descriptive key search terms that were developed to guide the search are outlined in Table 1. Research prior to the past decade, when educational reforms have incorporated principles of active learning into higher education curricula, is unlikely to reflect these specific teaching techniques in higher education. Therefore, the time period 2014 to 2024 was considered appropriate. The search strategy's exclusive focus on the term 'explanation' may have inadvertently excluded relevant studies that incorporated explanatory activities as intervention components but utilized different terminology in their documentation. A full list of inclusion and exclusion criteria is outlined in Table 2.

Table 1. Search string

Structured combination of keywords, phrases, and Boolean operators
(“Higher Education” OR “Postsecondary” OR “Tertiary Education” OR “Undergraduate” OR “College”) AND (“Mathematics” OR “Statistics” OR “Science education” OR “Engineering science” OR “STEM”) AND (“Learning by explaining” OR “Explaining-based learning” OR “Explanation-driven learning” OR "Self-explanation" OR "Self-explaining" OR “Learning by teaching” OR “Learning-by-teaching” OR “Team-based-explanation” OR “Peer tutoring” OR “Peer instruction” OR “Peer assist” OR “Peer-led” OR “Explanation to fictitious others” OR “ Non-interactive learning”)

Table 2. Inclusion and exclusion criteria

Criterion	Inclusion	Exclusion
Time period	2014 to 2024	Studies outside these dates
Language	English	Non-English studies
Type of article	Original research, published in a peer review journal	Articles that were not peer-reviewed or published elsewhere

Literature focus	Articles where the overwhelming theme relates to student explanation	Articles that made a passing or token reference to student explanation
Population and sample	Undergraduate or college students enrolled in a recognized program of study	All other students not enrolled in undergraduate study

Study selection and data charting

The process of study selection followed the Preferred Reporting of Items of Systematic Reviews and Meta-Analyses (PRISMA) protocol (Moher et al., 2009). We implemented a search strategy across Eric, Scopus, and Web of Science databases, which yielded 759 initial results. After removing duplicates and screening titles and abstracts, 102 publications progressed to the full-text evaluation stage. Based on our predefined criteria, we excluded 56 publications. Ultimately, 46 studies were included in the final analysis (the full list is available at figshare: https://doi.org/10.17608/k6.auckland.28202357). Summaries were developed of each study related to the author, year, location of study, study design, study methods, and sample size, and a brief comment on the limitations and recommendations of the individual selected study.

Findings

Based on the analysis and synthesis of the results of selected literature, the student explanation strategies can be categorized into three broad types: self-explanation, peer explanation and explanation to fictitious others. Of the 46 studies reviewed, 15 focus on self-explanation (12 on mathematics, 3 on statistics), 26 discuss peer explanation (9 on STEM, 14 on mathematics, and 3 on statistics), while 5 mainly focus on explanation to fictitious others (3 on STEM, 2 on mathematics).

Self-explanation, peer explanation, and explanation to fictitious others

Self-Explanation (SE) is a cognitive process wherein individuals elucidate to themselves the meaning of new information, often integrating it with existing cognitive schema (Bichler et al., 2022). This strategy facilitates learners' active participation in knowledge construction (Hall & Vance, 2010). SE are generated for the learner, not intended to teach the content to other people (Rittle-Johnson et al., 2017). Specifically, these methods can be broadly categorized into three main forms: written (e.g., documenting explanations on paper or in digital text boxes) (Salle, 2020), oral (e.g., recording verbal explanations via audio devices or self-narration) (Hodds et al., 2014), and gestural (e.g., conveying information through body language) (Salle, 2020).

The 15 SE studies we reviewed primarily explore the efficacy of SE. These studies examined various aspects, including promoting learning outcomes (e.g., Atisso et al., 2023; Rittle-Johnson & Loehr, 2017; Ryan & Koppenhofer, 2022), self-explanation for proof comprehension (Hodds et al., 2014; Maarif et al., 2020; Wirth et al., 2024), the impact of self-explanation quality on learning (Bichler et al., 2022), types of self-explanation and their impact on learning (Salle, 2020), and influence on cognition (Hadfield, 2021; Miller-Cotto & Auxter, 2021).

As postsecondary education has evolved, peer tutoring and cooperative learning approaches have been increasingly integrated with explanatory techniques in university settings (Naughton et al., 2021; Pollock et al., 2023). This study employs the term "Peer Explanation" (PE) to refer specifically to the explanatory process within peer instruction at undergraduate and postgraduate levels. PE focuses on explanatory interactions among students, emphasizing the impact of collaborative dynamics on the quality and effectiveness of explanations. At the same time, PE seems to include the process of SE, since students convey their own understanding to their peers. Unlike the independent cognitive process of SE, PE requires real-time bi-directional or multi-directional communication, demanding immediate question-and-answer interactions between the explainer and the audience, where listeners can raise questions, request clarification, or seek additional information (Maxwell & Wiles, 2022). PE typically involves a formalized process where students engage in mutual explanation within learning groups of at least three participants. In classroom settings, PE typically operates in pairs or fixed groups (3-5 members), with clearly defined roles (explainer and listeners) and structured time allocations and rotation mechanisms.

The 26 PE studies we reviewed examine various aspects, including impact on learning motivation (Mayerhofer et al., 2023), improvement of academic achievement (e.g., Biju, 2019; Corbo & Sasaki, 2021; Han & Thiel, 2022; Kishbaugh et al., 2018; Maxwell & Wiles, 2022; Morales et al., 2014; Nasir et al., 2023), effect of mixed-gender activities (Dumitru et al., 2022; Powell et al., 2023), cognitive development (Kustina & Meredith, 2023), the relationship with self-work during lectures (Leitner & Gabel, 2024), and the development of self-efficacy (Mills et al., 2020; Morris et al., 2023; Naughton et al., 2021). Advantages and Disadvantages of Online PE (Campbell, 2019).

With technological advances transforming postsecondary education, innovative methods for capturing student explanations through audio and video recordings have emerged. These developments have given rise to a new learning activity known as Explanation to Fictitious Others (EFO) (Lachner et al., 2018). EFO refers to students generating explanations and then explaining them to non-present, fictitious others via video, audio or text. While EFO also involves student-generated explanations, it differs from SE in that the explanatory content must be recorded through media tools (video, audio, or text), making it available for sharing and self-refinement (Hoogerheide et al., 2016). Generally, it includes three stages (Fiorella & Mayer, 2014): first, students prepare their explanations; second, they have to explain the subject matter to potential others either orally or in writing; third, students answer subsequent questions in interactive settings or receive feedback from an instructor. EFO triggers distinct cognitive processes that are essential for students' knowledge acquisition (Lachner et al., 2018).

The 5 EFO studies we reviewed mainly focus on comparing the effects of explaining through writing and explaining via video (Hoogerheide et al., 2016; Lachner et al., 2018), the impact of EFO on learning achievements and students' self-perceptions (Fiorella & Mayer, 2014; Hoogerheide et al., 2014), the effects of the timing of EFO implementation on students (Lachner et al., 2020), and the impact of EFO on students' problem-solving performance (Hoogerheide et al., 2019). EFO typically yields the most significant results when students record videos explaining problems (Hoogerheide et al., 2016, 2019).

Theoretical underpinnings of SE, PE and EFO

The Retrieval Practice Hypothesis and Generative Learning Hypothesis provide theoretical support for the efficacy of SE and EFO in enhancing students' long-term memory through retrieval and more advanced generative learning mechanisms such as organization and integration (Lachner et al., 2021). This is particularly evident in the learning performance of

students with different levels of prior knowledge (Hodds et al., 2014; Maarif et al., 2020; Ryan & Koppenhofer, 2022; Leppink et al., 2012). Low prior knowledge students perform worse on challenging questions due to insufficient knowledge reserves, which hinders their working memory. Therefore, PE can offer a useful knowledge link to address this disparity. Because of differences in students' knowledge, PE could be beneficial to further balance their learning. The Social Presence Hypothesis provides additional theoretical support for PE. The students' interactions and discussions with each other are able to some extent promote their cognitive thinking and build their understanding (Mills et al., 2020; Morris et al., 2023). This collaborative aspect of PE could help bridge the gap between students with different levels of prior knowledge, facilitating a more balanced learning environment where students can benefit from each other's strengths and compensate for individual weaknesses.

However, PE may also have limitations. Students with lower levels of understanding might unintentionally reinforce each other's misconceptions, leading to the spread of incorrect information (Kishbaugh et al., 2018). Additionally, the quality of explanations can vary significantly, as not all students possess the ability to clearly articulate complex concepts (Bichler et al., 2022). Without proper guidance and oversight from the instructor, peer-led discussions may sometimes fail to address critical gaps in understanding, potentially hindering overall learning outcomes (Smith et al., 2009). Furthermore, students who are more confident or knowledgeable may dominate the discussion, limiting the contributions of others and reducing the collaborative benefits intended by peer explanation (Mills et al., 2020).

Thus, EFO, by eliminating real-time peer interactions, mitigates some of the potential negative effects that the presence of peers can have in peer explanations. Although the target audience for the explanation is non-present fictitious others, it still involves the stages of preparation and explanation. In the preparing stages, EFO encourages learners to engage in a multi-faceted cognitive process that seems to differ to SE. With the expectation to teach others, this process involves selecting salient information to be conveyed, organizing it into a coherent mental representation, and integrating it with existing knowledge structures. This aligns with the teaching expectancy effect, as described by Fiorella and Mayer (2013), which posits that anticipation of teaching enhances one's own learning.

The explanation stage leverages the power of perspective-taking. Learners, when constructing explanations for their imagined audience, are compelled to consider diverse viewpoints. This cognitive exercise not only enhances the quality of their explanations but also reinforces their own understanding. As Hoogerheide et al. (2016) suggest, the belief that one's explanations may influence others can serve as a potent catalyst for self-directed learning. Moreover, the concept of non-present others in EFO serves a dual purpose. It heightens learners' awareness of their potential audience, thereby increasing their level of arousal and engagement (Hoogerheide et al., 2019). This heightened state of cognitive alertness may contribute to more effective information processing and retention.

Conclusions

This scoping literature review synthesizes the theoretical underpinnings of student explanation and categorizes student explanations into three main types: SE PE, and EFO. While SE and EFO foster individual cognitive processes that support effective generative thinking, PE combines these benefits with collaborative learning. However, EFO have the potential to mitigate some of the negative impacts that may occur in PE, such as more knowledgeable students dominating peer discussions. The retrieval practice, generative learning, and social presence hypotheses underpin the effectiveness of these approaches. However, the varying

efficacy of SE, PE and EFO warrants further exploration, as current research reports mixed outcomes depending on factors such as implementation, duration, and context. Future research should focus on consolidating findings through systematic or integrative reviews to better understand and compare the impact of SE, PE and EFO on student learning outcomes.

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The Impact of Questions on Students' Independent Reading of Mathematics

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This study contributes to research on the impact of questions on students' reading of mathematics and informs design considerations for questions included in workbooks. We used eye-tracking complemented by open-text questions. Participants were engineering and science students from the first to the third year of their studies. We considered proceduralness, conceptualness, and difficulty as dimensions of the workbook questions. We found that what we call a 'conceptual question' triggered more attention to the explanation than the example part of the workbook text, but so did what we call the 'procedural question'. In addition, relative question difficulty may have influenced our results. An implication of our findings is the need to investigate the interrelations of the conceptualness and proceduralness of mathematical questions while also considering the role of questions' relative difficulty.

Keywords: Reading Attention, Questions, Conceptual Knowledge, Eye-Tracking

Questions are used for both assessment and as learning tools, with one aim being to help students attend to important content and prepare for exams (Quiroz et al., 2022) . Educators often develop mathematical questions integrated within worksheets or workbooks, which can be studied independently, with expectations that tertiary students should develop as autonomous learners. However, students have been observed engaging only minimally with the content of mathematical texts (St Clair-Thompson, Graham, & Marsham, 2018). Therefore, research is needed that investigates the impact of questions on independent reading in undergraduate mathematics, specifically regarding questions' effectiveness in directing attention to particular parts of a mathematical text.

In this paper we report outcomes from a study that aimed to provide evidence of variations in where students direct their attention while reading mathematics with the goal of answering questions. Our specific focus is on comparing the impact of questions targeting explanations versus those targeting examples. Findings may enhance our understanding of how different question types influence students' reading of mathematics.

Literature Review

We provide a theoretical framing for the study by synthesizing the educational psychology literature on relevance manipulation and the mathematics education literature on conceptual and procedural knowledge. We also draw on the educational assessment literature to consider the relative difficulty of mathematics questions.

Relevance Manipulation

The directive to read with a specific purpose acts as a form of relevance manipulation (McCrudden & Schraw, 2007). Relevance is the degree to which a text segment is germane to a reader's goals. Authors assign importance to text segments according to the degree to which they are essential to understanding the ideas intended to be communicated in the text (McCrudden & Schraw, 2007). However, readers often prioritize relevance over importance (McCrudden & Schraw, 2007; Pichert & Anderson, 1977). Even when in-text cues indicate importance, readers

are more likely to focus on and recognize the significance of text segments that align with their perceived relevance (Sperber, 1995).

Relevance has been associated with optimal cognitive efficiency in processing the information presented in the text (Sperber, 1995). Moreover, questions can be designed to evoke particular kinds of reflection. For example, the Cognitive Reflection Test (CRT) (Frederick, 2005) comprises three questions that activate what are called System 1 cognitive processes, which are fast and generate intuitive answers with little conscious deliberation. To answer the questions correctly, these processes must be overridden by what are called System 2 cognitive processes, which are slower and more reflective. Accordingly, we are interested in how questions can be utilized as an integral part of mathematical texts to facilitate reading through relevance manipulation and to instigate cognitive reflection.

Mathematics Questions

We consider three dimensions of questions from the assessment literature and the mathematics education literature.

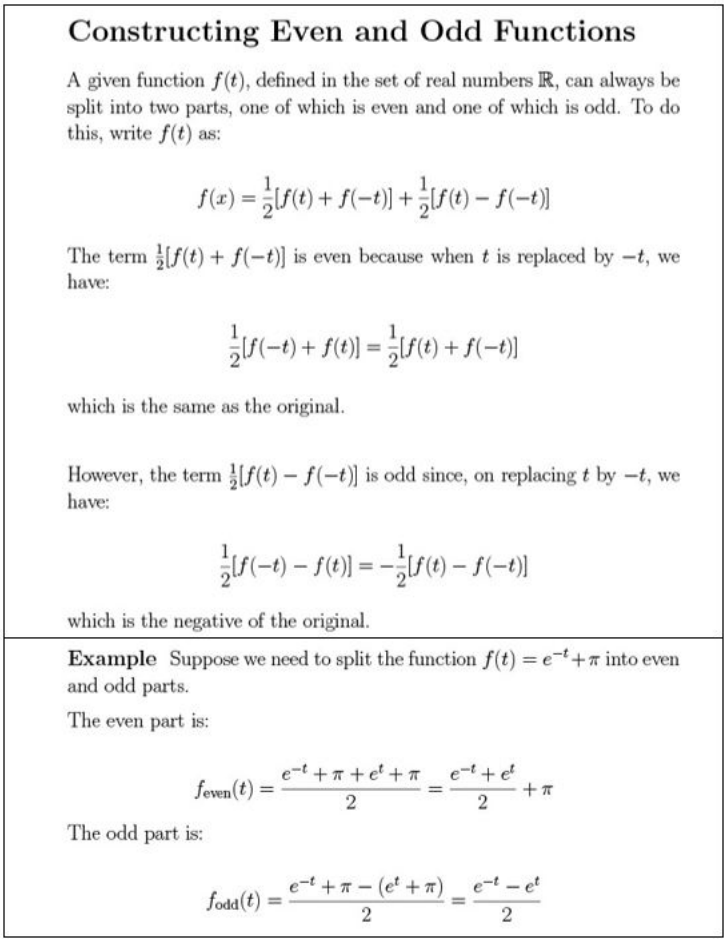
Difficulty. Some questions are more difficult to answer than others due to various factors including requisite background knowledge, length or complexity of the question, whether it is open or closed, use of context, and so on (Holmes et al., 2017). In the assessment literature, difficulty, and its converse facility, are typically measured in terms of the proportion of accurate responses by a sample of students (Bond & Fox, 2007). Questions of different difficulty have been found to impact student reading comprehension differently in other subjects (Cerdán et al., 2009), and this might be expected to be the case for mathematics too (e.g. Engelbrecht , Bergsten & Kågesten, 2009).

Conceptualness. Some questions, often used in research studies or to generate discussion in lectures, are intended to assess conceptual knowledge. Well-known examples include carefully designed and validated multiple-choice items such as those in Epstein's (2007) Calculus Concept Inventory. Conceptual knowledge has been defined in terms of understanding a domain's definitions, representations, rules and categories, as well as comprehending underlying principles and the connections between them (Crooks & Alibali, 2014). We might expect what we call here *conceptual questions* to direct students' attention to explanatory parts of a text.

Proceduralness. Routine questions, such as those that appear in common mathematics examinations and tests, are often considered to be procedural (Sangwin & Jones, 2017). Such questions tend to be closed and require students to apply one or more steps to produce a number or transformed expression or equation. We might expect what we call here *procedural questions* to direct students' attention to worked examples within a text.

The Study

We conducted a study designed to identify variations in students' attention while reading mathematics with the goal of answering questions, specifically comparing the impact of a question focusing on the explanatory part versus one targeting the example part of a text. In particular, we were interested in whether answering a conceptual question results paying greater attention to the explanation part of reading material, and whether answering a procedural question results in greater attention to the example part. Seeking differences in how conceptual or procedural questions drive attention can shed light on how question types might affect the depth of their comprehension of mathematical ideas.



Constructing Even and Odd Functions

A given function $f(t)$, defined in the set of real numbers $\mathbb{R}$, can always be split into two parts, one of which is even and one of which is odd. To do this, write $f(t)$ as:

$$f(x) = \frac{1}{2}[f(t) + f(-t)] + \frac{1}{2}[f(t) - f(-t)]$$

The term $\frac{1}{2}[f(t) + f(-t)]$ is even because when t is replaced by $-t$, we have:

$$\frac{1}{2}[f(-t) + f(t)] = \frac{1}{2}[f(t) + f(-t)]$$

which is the same as the original.

However, the term $\frac{1}{2}[f(t) - f(-t)]$ is odd since, on replacing t by $-t$, we have:

$$\frac{1}{2}[f(-t) - f(t)] = -\frac{1}{2}[f(t) - f(-t)]$$

which is the negative of the original.

Example Suppose we need to split the function $f(t) = e^{-t} + \pi$ into even and odd parts.

The even part is:

$$f_{\text{even}}(t) = \frac{e^{-t} + \pi + e^{t} + \pi}{2} = \frac{e^{-t} + e^{t}}{2} + \pi$$

The odd part is:

$$f_{\text{odd}}(t) = \frac{e^{-t} + \pi - (e^{t} + \pi)}{2} = \frac{e^{-t} - e^{t}}{2}$$

Figure 1. The reading text: explanation on the top and example on the bottom.

Methods

Participants. We recruited 45 engineering and science students from various year levels through flyers, announcements, and class presentations. Participants received a £10 voucher as a gesture of gratitude for their contribution.

Materials. For the reading material, we adapted page 12 from *Helping Engineers Learn Mathematics* (HELM) Workbook 6, as shown in Figure 1. The HELM series, available online, was developed from 2002 to 2005 by a consortium of five UK universities led by Loughborough University and have been widely adopted. The page we used comprises two parts. The first introduces and explains the concept that every function can be written as the sum of two components: one symmetrical to the *y*-axis (the even part) and one symmetrical to the origin (the odd part). The second part illustrates how we can find the odd and the even part of a function. We call these the *explanation text* and *example text* respectively.

The first author developed two questions as shown in Figure 2. The conceptual question was intended to relate to the ideas presented in the explanation text, and the procedural question was intended to relate to the example text. We then developed two visual stimuli for eye-tracking. Both contained explanations and example texts. On the right of this text, one of the mathematical questions was presented.

Conceptual Question

Suppose $f_{even}(t)$, and $f_{odd}(t)$ are respectively the even and the odd part of a function $f(t)$ defined in the set of real numbers.

If $f_{even}(\pi) = f_{odd}(\pi) = 2023$, then $f(-\pi) = 0$.

a) Correct

b) Wrong

Procedural Question

To find the odd part of the function $f(t) = t^3 + 3^t$, we write:

$$f_{\text{odd}}(t) = \frac{1}{2}\left(t^3 + 3^t - \left((-t)^3 + 3^{-t}\right)\right) = \frac{1}{2}\left(t^3 + t^3 + 3^t - 3^{-t}\right)$$

Therefore:

$$f_{\text{odd}}(t) = \frac{1}{2}\left(3^t - 3^{-t}\right) + t^3$$

a) Correct

b) Wrong

Figure 2. The conceptual and procedural questions

Eye-tracking measures. Eye-tracking determines gaze point coordinates and eye movement velocity when a person views a visual stimulus (Bojko, 2013), which are then classified into fixations and saccades. Fixations are continuous low-velocity data points, while saccades are high-velocity sequences of eye movements between fixations. Researchers define regions of a visual stimulus to obtain eye-tracking metrics that address their research questions. These regions are called Areas of Interest (AOIs). The interpretation of these metrics is based on the Eye-Mind Assumption (EMA), which posits that eye movements and visual focus are aligned with attention and cognitive focus (Just & Carpenter, 1980).

Here, we used as our metric fixation counts, where a high fixation count serves as a proxy for increased attention directed towards an AOI (Skaramagkas et al., 2023; Strohmaier et al., 2020). We defined two AOIs: one for the explanation text and one for the example text (Figure 1).

Open-text questions. The post-eye-tracking questions included "Did you find the questions you had to answer complex or difficult?" and "Were the questions anyhow unexpected to you?"

Procedure. The study followed a within-subjects experimental design. Participants were randomly allocated into two groups. The participants in one group encountered first the visual stimulus with the procedural question, and then the visual stimulus containing the conceptual question. For the other group the sequence was reversed, i.e. conceptual then procedural question. Participants completed the eye-tracking and then the open-text questions one by one in an eye-tracking lab where the researcher unobtrusively observed through a second screen behind the participant.

Results

The distributions of fixation counts across question type (conceptual, procedural) and text type (example, explanation) are shown in Figure 3. As can be seen the data are skewed, with some participants not looking at some text types at all for a given question type (zero fixation counts), and a few participants with much higher fixation counts than most.

Overall, the conceptual question resulted in higher fixation counts than the procedural question, and this was the case for both the explanation and the example text. For the explanation text the median fixation count when participants were shown the conceptual question was 57, and when shown the procedural question the median was 24. For the example text these medians were 16.5 for the conceptual question and just 3 for the procedural question. Therefore, the conceptual question resulted in more attention in general than the procedural question.

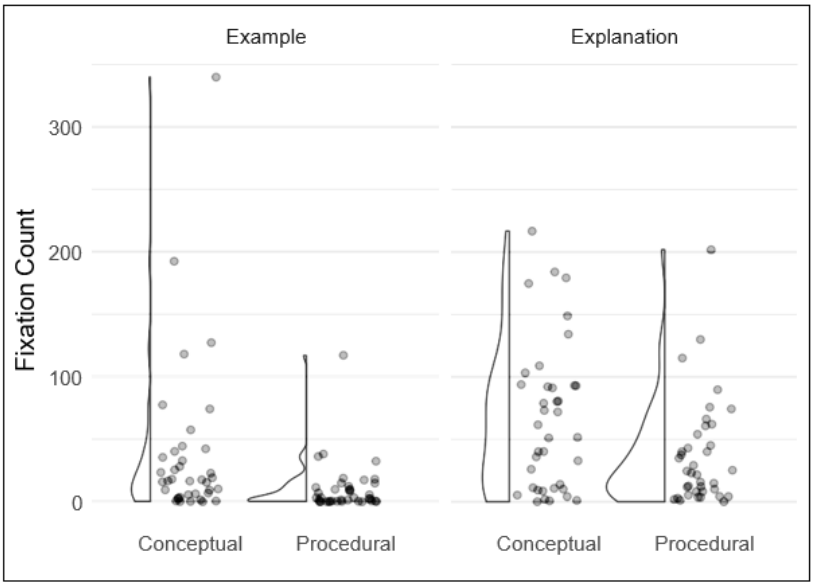


Figure 3. Jitter and raincloud plots of fixation counts. Each point shows the mean fixation count for each participant within each question type (conceptual, procedural) for each text type (example, explanation). The raincloud represent the density of points along the fixation count axis.

These medians also show that the conceptual question followed the pattern we expected: higher fixation counts for the explanation text than the example text, and a paired Wilcoxon test revealed that this difference was significant, $W = 558.5$, $p = .002$. In other words, when presented with the conceptual question the participants paid more attention to the explanation text than the example text. However, the medians also contradicted our expectations for the procedural question that participants would pay more attention to the example than the explanation text. In fact, they paid more attention to the explanation text, and the difference was again significant, $W = 649.5$, $p < .001$. Therefore, for both question types, fixation counts were higher for the explanation text than the example text.

This greater attention on the explanation text might reflect that it is larger than the example text, as shown in Figure 1. We therefore considered the ratio of the median fixation counts. For the conceptual question the ratio of median fixation counts for the explanation to example texts was 3.5, and for the procedural question it was 8.0. Therefore, the procedural question resulted in attention being paid to the explanation rather than the example text relatively more than the conceptual question, again contradicting our expectations.

Discussion

We conducted an eye-tracker experimental study in order to understand how relevance manipulation might be productively applied to independent study materials. Specifically, we investigated how what we called conceptual and procedural questions might direct attention to the explanation and example parts of a mathematical text.

The conceptual question was associated with higher fixation counts on the explanation text than on the example text as we expected. However, many participants still paid reasonable attention to the example text when presented with the conceptual question. Part of the reason for this may be that participants were looking for a comparison example to help them answer the question. Some of the responses to the open-text questions supported this, with one participant stating, “There was not an example to compare with” and another stating “No example given for a case like that”.

We were surprised that the procedural question was also associated with higher fixation counts on the explanation text than the example text. We wonder if part of the reason is that the true-false format may have resulted in participants attempting to validate the question with reference to the explanation text. In other words, what we intended as a question requiring calculation may not have been perceived as such by the students. Some participants' open-text comments supported this, for example "I was expecting something to work out, not just mental thinking" and "There was no calculation".

Another consideration is the relative difficulty of the two questions. The participants in our study were more successful with the procedural question (51% answered correctly) than the conceptual question (21% answered correctly). There may be something more inherently difficult about conceptual questions and, as discussed earlier, this may arise due to procedural questions being more common in tests and exams. Consistent with this one participant commented "I have not encountered a question like this".

It is possible that the unfamiliarity some participants stated, which, as aforementioned, may have contributed to the question's perceived difficulty, also reduced their appreciation of the relevance of the explanation text. The conceptual question required participants to combine pieces of information from the explanation text and make inferences based on these, rather than directly referring to any single text segment that is immediately recognizable as corresponding. Some participants did not recognize any relevance in the specific parts of the explanation text that we intended as pertinent. Evidence of this appears in some participants' open-text comments, such as "The text did not mention such a scenario", and "Nothing similar in the text". Therefore, the ability to appreciate relevance when reading, particularly for answering conceptual questions, may need to be cultivated in undergraduate students.

Our findings also bring forward the complexity of designing questions to function as conceptual or procedural and the confounding role of difficulty. In a follow up study, we are using the method of comparative judgement (Jones & Davies, 2023) to investigate whether the conceptualness or proceduralness of a question are independent or belong to the same spectrum. We are also investigating the extent to which the relative difficulty of a question relates to conceptualness and proceduralness. This will enable us to better interpret the results reported here, and also to conduct future studies to investigate further the role of relevance manipulation in the design of mathematical texts and questions for students.

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Pre-Service Teachers' Strategic Noticing During Modeling Through Centers of Focus

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Mathematical modeling tasks require modelers to consider both real-world conditions and mathematical properties while constructing models that represent the given real-world phenomenon. While the field is matured with research on the processes involved in constructing models, little research has explored what modelers center their focus on (i.e., notice) while solving modeling problems and how those centers of focus inform the strategies for solving modeling tasks. Through analyzing data from two pairs of undergraduate pre-service teachers' work on modeling tasks, we found three cases in which PSTs' centers of focus informed their strategies for model construction. In this report, we illustrate two of those cases and discuss implications for research and instruction on modeling.

Keywords: modeling-oriented noticing, mathematical modeling, strategic noticing

To enhance student experiences in mathematical modeling, the mathematical modeling education community has undertaken numerous research programs. For example, scholars have conducted research on student thinking in modeling (e.g., Borromeo-Ferri, 2010), teacher knowledge for modeling (e.g., Greenfrath et al., 2022), scaffolding strategies that advance students in their modeling process (e.g., Czocher et al., 2024) and designing modeling tasks and learning environments through employing theoretical frameworks (e.g., Lesh and Doerr, 2003). The majority of these works have been built on a simplified framework of the modeling process, commonly referred to as modeling cycles or modeling process diagrams (e.g., Blum and Leiß, 2007; Niss 2010). From the cognitive perspective of modeling, a modeling cycle usually consists of a real-world domain, a mathematical domain, and modeling competencies that are unique to solving modeling tasks (usually depicted as transition phases from one domain to another or within domains). Examples of micro-level modeling competencies include systematizing, mathematizing, validating, interpreting, and understanding the situation (e.g., Blum and Leiß, 2007). Examples of global competencies include metacognition, reflection, and anticipation (e.g., Stillman, 2011). In some modeling cycles, there is a clear separation between the real-world domain and the mathematical domain (e.g., Niss, 2010), while in others, these domains are interwoven (e.g., Czocher, 2018). Regardless of the extent to which the real-world domain and the mathematical domain are integrated, Stillman and Brown (2014) argue, the identification of the two domains as distinct is based on the epistemological foundation that real-world and the mathematical world "co-construct each other" (Doerr and Pratt, 2008, p. 260). This separation between the two domains, is at the "heart of modeling" (Stillman and Brown, 2014, p. 3). This is because the modeler has to *notice*—a global competency—both real-world conditions and mathematical elements and transition between these domains simultaneously (Blum, 2011) to construct a model that is viable to represent the experienced phenomena. However, little research has been conducted on how modelers' noticings of real-world and mathematical properties enable or constrain their modeling process. Knowing what students notice and how they attend to what they notice is instrumental for designing modeling tasks that can elicit certain modeling competencies and for contingent scaffolding—skills necessary for teaching modeling. In light of this, the objective of this paper is to shed light on learners' modeling-oriented noticing as they constructed models to represent real-world situations.

Background and Theoretical Framework

Noticing refers to intentionally focusing on particular phenomena so as to inform one's decision-making in the future (Mason, 2008). Noticing became popularized within the field of education through the domain of "teacher noticing" (Jacobs et al., 2010; Sherin & van Es, 2005). Teachers notice phenomena in their classroom, and these noticings inform their decision-making. There are three elements of teacher noticing: attending, interpreting, and deciding how to respond (Jacobs et al., 2010). First, teachers attend to aspects of students' mathematical strategies and interpret students' mathematical understanding based on those "attendings." Then, teachers make decisions about how to respond to students based on their interpretation of students' mathematical understanding. Scholars have explored *what* teachers notice, *how* they notice, and productive ways to notice. Within the field of mathematical modeling, among other competencies required for teaching modeling (For example, see Borromeo Ferri and Blum, 2011), modeling-specific noticing has been explored as an avenue to support students' learning of modeling. For example, Alwast and Vorhölter (2022) conceptualized teacher noticing within modeling as teachers' skills in perceiving, interpreting, and decision making based on students' difficulties, metacognitive strategies, and approaches to solving modeling task (pp. 167-268). On the other hand, Glabraith (2015) identified three classes of modeling specific noticing: Noticing as a mathematician (determining the situations that can be modelled using mathematics), noticing as an educator (designing tasks that support modeling competencies), and noticing as a student mentor (noticing students' in-the-moment modeling-oriented thinking) (p. 11).

Recently, scholars have expanded the noticing literature to examine how and what students notice to inform their mathematical thinking (Campbell & Yeo, 2023; Hohensee, 2016; Lobato et al., 2012; Lobato et al., 2013). In this study, we build upon recent literature on students' modeling-oriented noticing (Galbraith et al., 2017), that extends Galbraith's (2015) classes of modeling-specific noticing to also include *noticing as a student*. Here, the focus is on the specificities of what students notice as relevant in order to progress in constructing models of real-world situations. Galbraith et al. (2017) included modeling-oriented noticing as a component of anticipatory metacognition, a global modeling competency, which "shifts the emphasis towards reflection that points forward to actions not yet undertaken (p. 84)." The authors argue that successful modeling not only requires 'noticing' but also discerning the relevance of what has been observed (p. 85). Therefore, the authors defined a new construct, productive modelling-oriented noticing as, "the process of modelers noticing in a productive manner what is important and what is not in order to generate strategies" for successfully carrying out a modeling activity (p.84). Within modeling-oriented student noticing, Galbraith et al. (2017) proposed two ways in which noticing can be expressed: strategic noticing and explanatory noticing. We interpret strategic noticing as noticing the steps required, including macro and micro-level steps, to achieve modelers' modeling goals (e.g., decision to include air-resistance to model the speed of a falling ball). We interpret explanatory noticing as noticing that serves as an explanation of their approach to solving the task (e.g., because air-resistance can slow the speed of a ball). Since the strategies generated inform modelers' next steps, it can be said that *strategic noticing* is a key driver of modelers' progress through the modeling process.

The cognitive processes of sorting relevant information from competing information are not new; it has long been used to study how individuals construct knowledge (e.g., Piaget, 1952). Typically, what surfaces as relevant from the rest are shaped by the goals the learners set for themselves (Simon et al., 2004; von Glasersfeld, 1995). Lobato et al., (2012) refer to these significant aspects of noticing as 'centers of focus,' defining them as "features, regularities,

properties, or conceptual objects to which individual students pay attention" (p.439). Within a mathematical modeling context, where the modeler is required to transition between the real-world and mathematical domains simultaneously, either the real-world conditions, the mathematical elements, or both become the centers of focus that inform the modelers' strategies. Therefore, in this report, we address the following research question: *How do real-world and mathematical centers of focus inform modelers' strategic noticing*?

Methods

Data were collected by conducting task-based interviews (Goldin, 1997) with two pairs of undergraduate pre-service teachers (PST) enrolled in a public university in the western United States. The first pair consisted of Gwen and Arya and the second pair consisted of Luna and Sahana, with all names being pseudonyms. We chose to conduct the interviews in pairs to create an environment conducive to observing PSTs' modeling-oriented noticing due to their communication with each other as they co-constructed mathematical models. Each pair participated in 8 interview sessions. In the first 7 sessions, the PSTs worked on 11 mathematical modeling tasks, which were adapted from prior literature. These tasks were selected due to their heterogeneity of modeling characteristics (see Maaß, 2010) and their appeal to socio-justice and environmental issues. The last session was used for reflection and member-checking. In this paper, we report on Luna's and Sahana's work on the tasks shown in Table 1.

Table 1. Tasks from which findings are reported

Tasks	**Description (abridged versions)**
Hospital Task (Zbiek and Connor, 2006)	Boise (Idaho), Helena (Montana), and Salt Lake City (Utah) are three large cities in the northwestern part of the United States. The map shows the locations of the three cities as well as other large cities in the area. Suppose you are hired to determine the best location for the hospital. Where would you locate the hospital?
Piano Tuners Task	How many piano tuners are there in the city of New York?

The interviews were conducted in a laboratory setting where a video camera was placed to record PSTs' written work and gestures. The PSTs were given freedom to interpret the tasks in any way they chose, and they were not given any time restrictions for completing the tasks. The interviewer's questioning mainly focused on unpacking PSTs' reasoning and uncovering the aspects of PSTs' modeling-oriented noticing. For example, to evoke justification of their strategies, the interviewer asked questions such as "why was it important to consider this?" "why did you make that assumption?," and "how reasonable is the assumption that you made?." The interviewer maintained an ongoing research journal where she recorded her reflections and possible conjectures of PSTs' reasoning at the end of each interview.

Since the overarching goal of the study was to investigate PSTs' modeling-oriented noticing, we used an a priori framework for our initial coding. Two members of the research team coded the videos for strategic noticing (Galbraith et al., 2017), where the unit of analysis was chunks of text that discuss a single idea. Table 2 provides a list of indicators that were developed using techniques from constant comparison methodology (Strauss and Corbin, 1990) to identify PSTs' strategic noticing. The research team met and discussed any discrepancies in the coding until we reached complete agreement.

Finally, each instance of the strategic noticing was coded for real-world and mathematical centers of focus. If the strategic noticing was explicitly grounded in real-world considerations, it was coded as real-world. If the strategic noticing was based on information given in the task,

simplification for mathematical convenience, mathematical strategies (i.e., writing an equation, finding a quantity to proceed), and mathematical properties, those instances were coded as mathematical center of focus. The explanations and justifications the PSTs provided for their strategies were used to determine the centers of focus.

Table 2. Indicators for strategic noticing

Indicator	**Definition**
I1: Taking stock	PSTs review key points through *reiterating*, *underlining/circling/highlighting,* or *recalling* previously provided information from the modeling task.
I2: Researching information	PSTs search and gather relevant data and information through technology.
I3: Identifying parameters	PSTs identify specific parameters or variables necessary to progress
I4: Identifying the need to make assumptions	PSTs acknowledge and express the need for making assumptions to progress.
I5: Simplifying the situation	PSTs simplify complex situations, making model progression possible. Examples include making necessary assumptions, prioritizing and sorting the most critical assumptions from less important ones, rounding values and/or purely guessing the values of a parameter.
I6: Proposing strategies and considerations	PSTs brainstorm and consider various approaches, strategies, and factors for developing and solving the model. This phase often involves uncertainty or hesitation by students as they deliberate on an approach to adopt.
I7: Implementation of a strategy	PSTs execute the planned steps or approach for constructing a model by breaking the problem into smaller steps, assigning known values to parameters, and making decisions for choosing certain computations. This phase typically involves greater certainty in the adopted approach and less hedging or hesitation.
I8: Verifying/ validating the model or approach	PSTs check the accuracy, reliability, and effectiveness of the implemented model or approach. This includes ensuring that the model's outcomes align with expected results, testing against known data or information, and validating assumptions and calculations.
I9: Noticing patterns and characteristics of the model	PSTs observe and recognize patterns, mathematical properties, or features inherent in the model. Examples include identifying relationships between variables, observing trends in data, or noticing underlying principles that affect the model's behavior

Findings

We present our findings through three cases: (1) Noticing of mathematical elements foregrounded; (2) Noticing of real-world conditions foregrounded; and (3) Noticing of both mathematical elements and real-world conditions foregrounded. In case 1, PSTs' strategic noticing was informed primarily through mathematical elements as their centers of focus. In case 2, PSTs' strategic noticing was informed through real-world conditions as their centers of focus. That is, the centers of focus driving each strategy were grounded in real-world conditions. In case 3, PSTs strategic noticing was informed either by real-world conditions, or mathematical

elements, or both. Due to space constraints, we illustrate two of the cases below. In our illustrations, we identify the strategic noticing (I#), and the centers of focus for each strategic noticing (R for real-world, M for mathematical elements). Concluding each case, we describe how the students' centers of focus informed their modeling strategy.

Case 2: Noticing of Real-World Conditions Foregrounded

We illustrate this case using Luna and Sahana's work on the hospital task. After reading the task, Luna suggested that she wanted to connect the three cities and make a triangle and proposed that the hospital would be somewhere within the triangle (I6, R). She made this proposal because she wanted to speculate which other cities would also use the hospital, justifying her strategy based on real-world considerations. Luna justified her strategy as:

> Luna: If it's the three points, it's the areas surrounding it. A lot of areas in Utah especially that's shown here, has a high population. Montana's a bit more spread out. People from Wyoming might be coming, depending on where it is. Northeastern Idaho is the more populated area of Idaho…So I'm just thinking this is (referring to her triangle), it just gets a rough estimate of where there's more people and who all is going to that hospital.

Next, Sahana expressed her dilemma in choosing a hospital location near the I-15 freeway (I6, I1, R) for better accessibility versus choosing a location based on the population in the major cities and surrounding areas (I6, R) because she stated more population implies more use of the hospital. Luna suggested that Salt Lake and Boise are more concentrated therefore it would make sense to put the hospital closer to Boise and Salt Lake area, for example either in Idaho Falls or Pocatello, rather than closer to Helena (I6, R), once again prioritizing real-world conditions.

The interviewer gave additional information regarding the population of each city. The data given was consistent with PSTs' conception of the population of the three cities. The PSTs decided that the hospital should be placed either at Pocatello or Rexburg (I6, R) because the two cities are closer to Salt Lake City and are on the highway, therefore easily accessible, justifying their proposal through real-world conditions.

Based on her personal experience living in Pocatello, Luna suggested Pocatello as a great place for hospital (I6, R) because it is the junction for many other popular highways (such as I-86 interchange). Sahana concurred as well. Both PSTs agreed that Pocatello was the next best city for a hospital after Salt Lake City and drew a star on the map to finalize their solution. Furthermore, the PSTs agreed that Pocatello was a good city for a hospital as evidenced in the exchange below:

> *Luna:* We want to go where there's a higher concentration of people because that makes it more accessible if there's more people. Pocatello has, it's just where all of the major roads or highways and interstates combine. And so it's just a good meeting spot.
>
> *Sahana:* …Yeah. Realistically Pocatello is a better place to be equipped for building a hospital, funding a hospital, maintaining a hospital.
>
> *Luna:* Yeah. Staff in the hospital. Yeah.
>
> *Sahana:* Relative to main roads. Real world ability.
>
> *Luna:* It's like the ability to support or community's ability to support… a hospital. To be able to staff it and maintain it. And it has to be in a place that it's also going to draw people to want to work there…
>
> *Sahana:* And live there.

In this vignette, we presented six instances of strategic noticing, all of which were informed by PSTs real-world considerations as their centers of focus. That is, the PSTs approached the modeling task by strategically by prioritizing only real-world conditions (over mathematical

elements) by attending to the following: the population of each city, highway accessibility, the city's capacity to fund and maintain a hospital, and its ability to attract people to relocate for work.

Case 3: Noticing of Mathematical and Real-World Elements Foregrounded

To illustrate this case, we use Luna and Sahana's work from the Piano Tuners task. After reading the task, Sahana said that they need to make assumptions (I4, **M**) because "we don't know anything, so we literally, we're going to fill it," addressing the scope of the task. The PSTs suggested that they need to identify the number of people in New York city (NYC), and how many of those people have pianos (I3, **M**). They further proposed that whether a person plays a piano or not would not affect their model (I5, I6**, R**) because "some people will just get it tuned", "some people play the piano and don't get it tuned", and that "some people get it tuned way after they should have. It's not a consistent amount of time." Here, the PSTs were justifying their approach through real-world considerations. The PSTs researched the population of New York as 8.468 million, which they approximated to 8.5 (I5, **M**).

Next, Luna suggested that there were not very many pianos in NYC that need tuning (I6**, R**) because NYC apartments are small. Luna further suggested that that there are probably more electric keyboards as they are easier to move into smaller apartments, justifying her approach through real-world considerations. Therefore, the PSTs assumed that there were 956,000 pianos in total (those that need tuning and those that do not) (I5, **R, M**). Sahana commented that 956,000 seemed like a lot of pianos but expressed that she can commit to it (I7, **R**) since "they (NYC) do music-y things", have "big college that does music stuff", and that there is Broadway, where her commitment was informed through her noticing of real-world considerations.

Next, The PSTs wanted to determine "how many of them [pianos] need to actually be tuned?" (I3**, R, M**). Here, the PSTs decision to consider only a portion of the pianos as needing to be tuned was informed through their real-world knowledge and their mathematical goal towards estimating the number of piano tuners. The PSTs assumed "one third of all pianos are classic, regular, pianos [that need tuning]" (I5, **R, M**) and evaluated the number of piano tuners that need tuning as $315,480$. The PSTs proceeded with determining "how many pianos can a tuner tune in a year?" (I3, **M**). To accomplish this, Sahana simplified the situation (I5, **M**) by assuming that all pianos need to be tuned once a year. She commented on the reasonableness of the assumption as "I don't think they actually do, but we're just going to do it for our sake," indicating her decision was driven primarily for mathematizing purposes. Sahana proposed that a piano tuner can tune a piano in 30 mins (I6**, R**). This decision was based on two real-world considerations: "piano tuners are insanely good. They can do it really quick" and "you have to factor in travel time." Next, the PSTs assumed (I5**, R**) that a piano tuner's work week consists of 5, 8-hour workdays. The PSTs justified their assumption as, "he's a busy piano tuner, got to put food in the table." Given these conditions, the PSTs computed the number of pianos tuned by a piano tuner during a week as 80 (see Figure 1).

Having determined the number of pianos tuned in a week, the PSTs proceeded with considering the number of piano tuners tuned in a year (I3, **M, R**). While the PSTs assumed that there were 54 weeks in a year (I5**, R**), Luna then brought up that the piano tuners will not be working all 54 weeks (I6**, R**) because piano tuners are "probably not going to work on Christmas." The PSTs decided that there 52 working weeks within a year (I7, **R**). Under these assumptions, the PSTs found that a piano tuner can tune 4,160 pianos in a year (see Figure 1). Finally, Luna proposed (I5**, M**) taking "how many pianos there are (315,480), and divide[ing] it by 4,160 (i.e. number of pianos needing tuning per year) [to] give the total number of piano

tuners". The result was 75.8, which the PSTs approximated to 76 (**I5, R**) because the PSTs argued that there can't be partial piano tuners. However, the PSTs proposed (**I6, R**) that 75.8 could be accounted for by the fact that some piano tuners may work part-time.

In this vignette, we presented 20 instances of strategic noticing. In particular, to find the number of piano tuners in NYC, the PSTs noticed that the following parameters first need to be determined: the population of NYC, the number of pianos in NYC, the number of pianos need tuning, and how many pianos can a piano tuner tune in a year. Each of these parameters were consequences of the PSTs considering both mathematical centers of focus (building on prior quantities to arrive at the final quantity) and real-world centers of focus (piano tuners do not work during Christmas). As illustrated in the vignette above, there were some strategies, where the centers of focus were both math and real-world elements which primarily served the PSTs to consider a new quantity as a result of a real-world constraint (for example only considering a subset of the number of pianos because not all pianos need to be tuned).

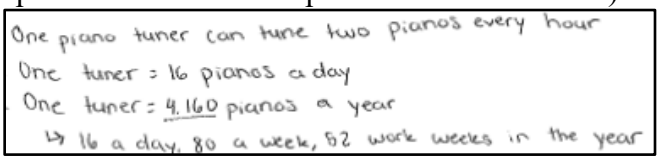


Figure 1. Luna and Sahana's work for the piano tuners task

Discussion

Our paper has three primary contributions. First, we contribute to the existing research on PSTs and modeling by positioning PSTs as *learners of modeling* through a novel cognitive lens, rather than focusing solely on typical modeling competencies. Second, our findings have important implications for design and implementation of modeling tasks. For example, in the piano tuners task, the PSTs considered both real-world conditions and mathematical strategies when developing a model for the situation, at times alternatively and at times simultaneously. The characteristics of PSTs' centers of focus may be attributed to the nature of the piano tuners task, which was open ended, artificial, and therefore allowed for more creativity. In contrast, in the more realistic hospital task, the PSTs prioritized real-world conditions over the use of mathematical strategies (e.g., measuring distance proportional to population size). This may be attributed to PSTs' local knowledge about the cities and the surrounding areas. Therefore, we ask: *How real do we want modeling tasks to be if we want student to develop certain modeling competencies such as using mathematical approaches to solving a modeling task*? Finally, our study contributes theoretically and methodologically to existing frameworks on *noticing*. Theoretically, we build on noticing as a teacher competency to also include *noticing as a student* specific to modeling. Our findings can be used to make inferences about what students attend to, how they interpret those attendings, and what decision they make based on their interpretations depending on students' centers of focus. For example, when the center of focus is real-world conditions, students attend to the real-world conditions that may affect the model (*attending*), analyze the perceived information based on personal or real-world experience (*interpreting*), and place emphasis on the significance of the real-world conditions over mathematical strategies in model development (*responding*). Methodologically, we generated a list of indicators that can be used to identify strategic noticing (Galbraith et al., 2017). Research within the cognitive perspective of modeling has largely stagnated over the last decade, where the emphasis is placed on phases of the modeling process (such as those depicted in the modeling cycles). Following Galbraith et al (2017), we see our work as advancing the modeling field in a newer direction through introducing a fresh perspective for analyzing student thinking in modeling.

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Assessing Mathematical Knowledge for Teaching College Algebra: Differences Between Community College Instructors and Math Majors Without Teaching Experience

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This study compares mathematical knowledge for teaching college algebra at community colleges between instructors and math majors without teaching experience, aiming to show that the instrument assesses more than mathematical knowledge. We used an instrument designed to assess knowledge used in doing two tasks of teaching across three function types taught in college algebra, linear, rational, and exponential functions. The Multiple Indicators and Multiple Causes (MIMIC) model was applied to 416 community college mathematics instructors and 85 college students to estimate mean difference in the knowledge between the groups. The higher scores in the MKT-CCA suggests that instructors possess higher MKT than the students. As such, this instrument can be used to identify areas for focused faculty development.

Keywords: mathematical knowledge for teaching, college algebra, teaching experience, MIMIC

Our research group has been working on the development of an instrument to assess mathematical knowledge for teaching college algebra at community colleges. To measure this knowledge, we use an instrument organized by two tasks of teaching (understanding student work and choosing problems) with three function types (linear, exponential, and rational). An earlier study (Mesa et al., 2024) provided evidence supporting the validity of the inferences that can be made about instructors' knowledge, particularly, regarding its teaching-related features. The instrument provides a measure that differentiates the knowledge among community college instructors and the measure is correlated with both experience in teaching advanced college algebra and the frequency of doing a specific task of teaching-- choosing problems, a unique task teachers engage in on a routine basis.

To further support that the instrument assesses knowledge of teaching and not just mathematics, we compared the performance of two groups: math major students, who may have content knowledge but lack teaching experience, and community college instructors, who can reasonably be assumed to have both content knowledge and teaching experience. The analysis can further give evidence that the instrument is a viable tool for measuring this knowledge across the combined groups and to understand the characteristics of the knowledge it assesses. Specifically, we test whether the instrument can reliably differentiate knowledge levels between students and instructors, and whether instructors, those with teaching experience in the subject matter, perform better on the measured knowledge than students, those without such experience.

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After confirming the instrument's ability to distinguish knowledge levels among both groups, we compare their knowledge on a common scale. The significant difference in knowledge between instructors and students who lack teaching or tutoring experience provides additional evidence that our instrument measures an aspect of knowledge related to teaching. This led us to pursue the following question: How does the measured mathematical knowledge for teaching college algebra differ between community college instructors and college students who are majoring in mathematics?

Theoretical background and literature review

Central to the conceptualization of the instrument used in our work is that mathematical knowledge for teaching can best be elicited through the mathematical work embedded in activities that teachers do when they teach, and that are directly related to how they organize instruction (the interactions between content, students, and teachers) (Cohen et al, 2003). Herbst and colleagues developed an instrument measuring teachers' mathematical knowledge, structured by the types of work teachers do across different instructional situations (Herbst & Kosko, 2014; Ko et al., 2016). The data collected from their instrument have provided evidence that, at least in the context of teaching of high school geometry, it is possible to differentiate two distinct dimensions of the knowledge: one related to knowledge needed to choose givens for a geometry problem and another related to understanding students' work (Ko & Herbst, 2020).

The distinction of knowledge by different tasks of teaching is grounded on the concept of instructional exchange, which describes how knowledge is "transacted" in classrooms and the role of the teacher managing the exchange (Herbst & Chazan, 2011). Two key tasks – providing students with activities that would enable and support mathematical work and interpreting the work to assess student learning – represent a teacher's role in managing the exchanges. To manage this exchange effectively, teachers need to draw on their mathematical knowledge to make decisions guided by the instructional goals. By focusing on these instructional exchanges, Herbst and colleagues provide a unique lens for examining teachers' mathematical knowledge in key tasks of teaching that are applicable to any teacher, subject matter, and context.

As our instrument is grounded on specific activities that the instructors are expected to do as they teach, we ascertain this knowledge is unique to those who teach the specific content. Consequently, people who presumably know the content (because they took the courses and have been using that content for other more advanced math courses) but who do not have teaching experience with the content would not perform well on an instrument assessing such mathematical knowledge for teaching, particularly in the context of community college algebra instruction.

Research on mathematical knowledge for teaching has consistently documented that as teachers teach more advanced mathematical courses (e.g., middle teachers with experience with high school mathematics; community college teachers teaching courses beyond calculus), they perform better on instruments measuring mathematical knowledge for teaching than those teachers with experience only on the mathematical content assessed by the instruments (Ko et al., 2021; Hill, 2007). Thus, we hypothesized that people who knew the content, but had not taken too many advanced mathematical courses, would not perform well on an instrument assessing such mathematical knowledge for teaching community college algebra. Consequently, we anticipated that a sample of people who could be seen as having the knowledge of the mathematical content that the instrument assesses but without teaching experience on that content and without knowledge of more advanced mathematics, would score lower on our instrument compared to instructors teaching the content in community colleges.

Methods

For the comparison, we chose undergraduate mathematics majors and a sample of community college instructors. The student sample was recruited from 47 large, public, and four-year colleges across all states that offered a mathematics degree and that had a high percentage of admitted students. Students were eligible for participating in the project if they had no tutoring or teaching experience beyond middle school. Of the 191 eligible students, 149 started the questionnaire and 110 completed the instrument. Out of 110 participants, 85 students' responses were used in the analysis after confirming their eligibility for participation. The majority (57%) were between 18 and 22 years old, 51% identified as male and 42% as female, and a minority (13%) identified as Hispanic or Latino, while the majority (76%) identified as White. All the participating students indicated that they took a calculus course, and 80% of the students indicated that they took a college algebra course. The instructor sample was recruited from two-year degree granting institutions in the United States by inspecting community college websites and then sending direct invitations to instructors for participation in this project. Among the respondents, 556 instructors from 314 institutions participated in the project. Among the 556, 416 instructors completed the instrument, and their responses were used in the analysis. Of the 416 instructors, 48% identified as male and 46% as female; in terms of race, 76% identified as White, 11% as Asian, 4% as Black, 2% as mixed, and 4% chose Other. Seventy-eight percent of the participants said they held full-time positions; 9% were on tenure track. The average number of years of teaching experience was close to 17 years (mean = 16.76, SD = 9 years; range: 1.5 to 47 years). The majority (68%) held a master's degree and 20% held PhDs in mathematics, mathematics education or another mathematics-related field (about 5% were PhD in Mathematics Education).

For this study, we used items of the MKT-CCA instrument designed to measure mathematical knowledge for teaching used in two tasks of teaching (understanding student work, choosing problems) college algebra in three function types (linear, exponential, rational) (Mesa, Ko, Boeck, et al., 2023). The instructors and students completed the exact same items. Following the prior studies that validated the instrument regarding the dimensionality of the measured MKT (Mesa, Duranczyk, et al., 2024; Mesa, Ko, et al., 2024), we used a unidimensional model estimating the amount of MKT-CCA as a single score. However, given the error correlations among the testlet sub-items that have the same stem and the possibility of the effect of different item formats (multiple choice or testlet), we used only multiple-choice items (19 items) in this study. A sample item is presented in Figure 1.

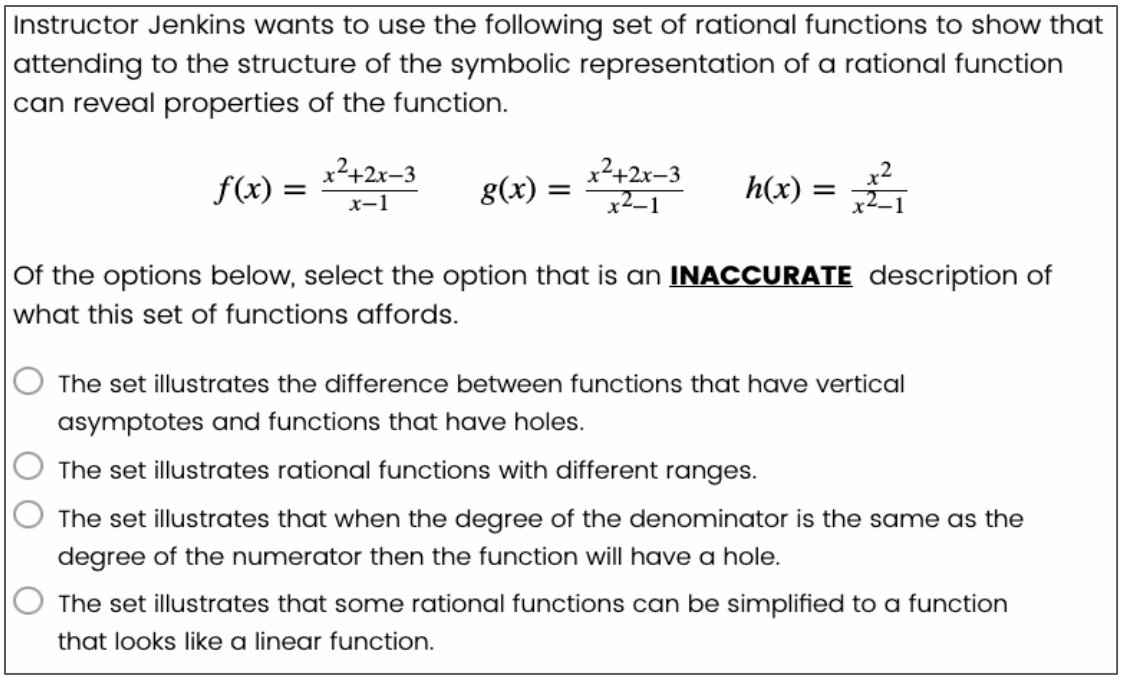


Figure 1. A sample MKT-CCA item involving Choosing problems in Rational function

To compare MKT-CCA scores between instructor and student groups, we applied the Multiple Indicator Multiple Cause (MIMIC) model that compares latent mean between the groups while detecting item bias. Specifically, in this model, we estimated the effect of the group variable on MKT-CCA while estimating the unidimensional MKT-CCA score for both instructors and students. The group variable indicated whether a sample is from a group of instructor (0) or student (1).

Results

Establishing unidimensional construct MKT-CCA

A one-parameter (1PL) and two-parameter (2PL) item factor analysis was conducted on the 19 items using the weighted least squares with mean and variance adjusted (WLSMV) estimator for the whole sample, including instructors and students. Both models fit the data well. In the 1PL model, factor loadings of all 19 items were set to be the same, while in the 2PL, they were freely estimated. The fit indices for 1PL and 2PL were as follows: 1PL (RMSEA=0.038, CFI=0.941, TLI=0.941l) and 2PL (RMSEA=0.011, CFI=0.996, TLI= 0.995). In all models, factor mean and variance was set to 0 and 1, respectively. All the analyses were performed using Mplus v.7.4 (Muthén & Muthén, 1998–2015). All item loadings were significant and greater than 0.3. A comparison of the 1PL and 2PL models using DIFFTEST suggested that 2PL is significantly better than 1PL (DIFFTEST(18) = 71.016, $p<0.001$), indicating that item discriminations are significantly different across different items. Consequently, in the subsequent analyses, we used a 2PL approach.

Comparing MKT-CCA scores between instructor and student groups

The MIMIC model fits the data well (RMSEA=0.021, CFI=0.979, TLI=0.977) and the result suggests that being in the student group has a significantly negative effect on MKT-CCA. Specifically, on average, the student group scored 0.40 standardized deviations lower on the

MKT-CCA than the instructor group. This effect size can be considered as medium according to Cohen's criterion (small, medium, and large for values of 0.20, 0.50, 0.80) (Cohen, 1988). The estimated model diagram is presented with standardized estimates in Figure 2.

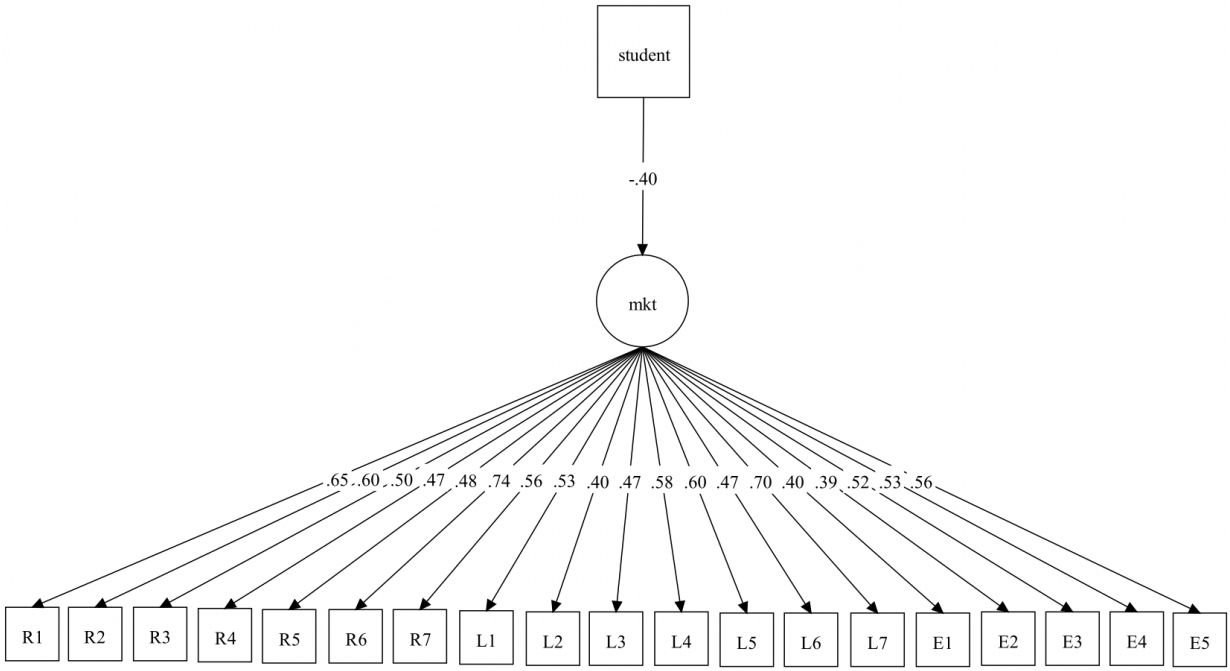


Figure 2. MIMIC model with student group indicator (standardized & significant estimates)

Identifying differential item functioning (DIF) items

Next, we used the MIMIC model to identify potentially biased items across the two groups in how each item measures the MKT-CCA. The difference in MKT-CCA score should stem from the differences in their MKT-CCA, not from items favoring one group over the other. In other words, items with significantly different item intercepts, even after adjusting for latent mean differences, indicate bias that may not relate to the knowledge commonly measured by the 19 items. We tested whether the two groups responded similarly to each item measuring MKT-CCA while controlling for their MKT-CCA. Those items showing differences after adjusting for latent mean differences are considered differential item functioning (DIF) items (Kaplan, 2008). This analysis showed that instructors are more likely to answer correctly than students for two items Item L7 and Item R1 after controlling for their MKT-CCA. Item L7 is an item about choosing problems in linear functions. The stem of the item stated a professor's intended goal for the students and asked participants to choose the option that presents a form of linear equation that will enable his or her students to achieve the learning goal. The correct answer is one form of a linear equation that is not as commonly introduced in textbooks as other forms like slope-intercept or point-slope form. The form is more likely to be found in advanced mathematics courses or specialized texts dealing with analytic geometry. Thus, students might be less familiar with the form than instructors, who have more experience teaching various courses related to algebra, and it might have caused students to score lower than instructors at the same level of MKT-CCA. Item R1 provides an example of incorrect student work and asks the respondent to select the option that is likely to be produced by the student who solved the problem. To choose the correct answer, the participant must be able to identify the error the student made in the

example and choose the option that shows the same error. Thus, familiarity with similar kinds of errors or experience in seeing the error would help a participant in choosing the correct answer. As instructors have more experience in analyzing a variety of student errors, the experience may benefit them in getting a correct answer, even if their mathematical knowledge level is the same as a student's.

We conducted an adjusted MIMIC model that allows estimating the effects of the two items, which are suggested to have DIF, on MKT to account for group differences in the measurement of biased items. Specifically, we regressed MKT-CCA on the group indicator while also adding suggested regression paths from the group indicator to each of the two items. This approach allowed us to estimate the mean difference while also determining the remaining item variance explained by the group indicator. The estimated diagram with significant effects is presented in Figure 3.

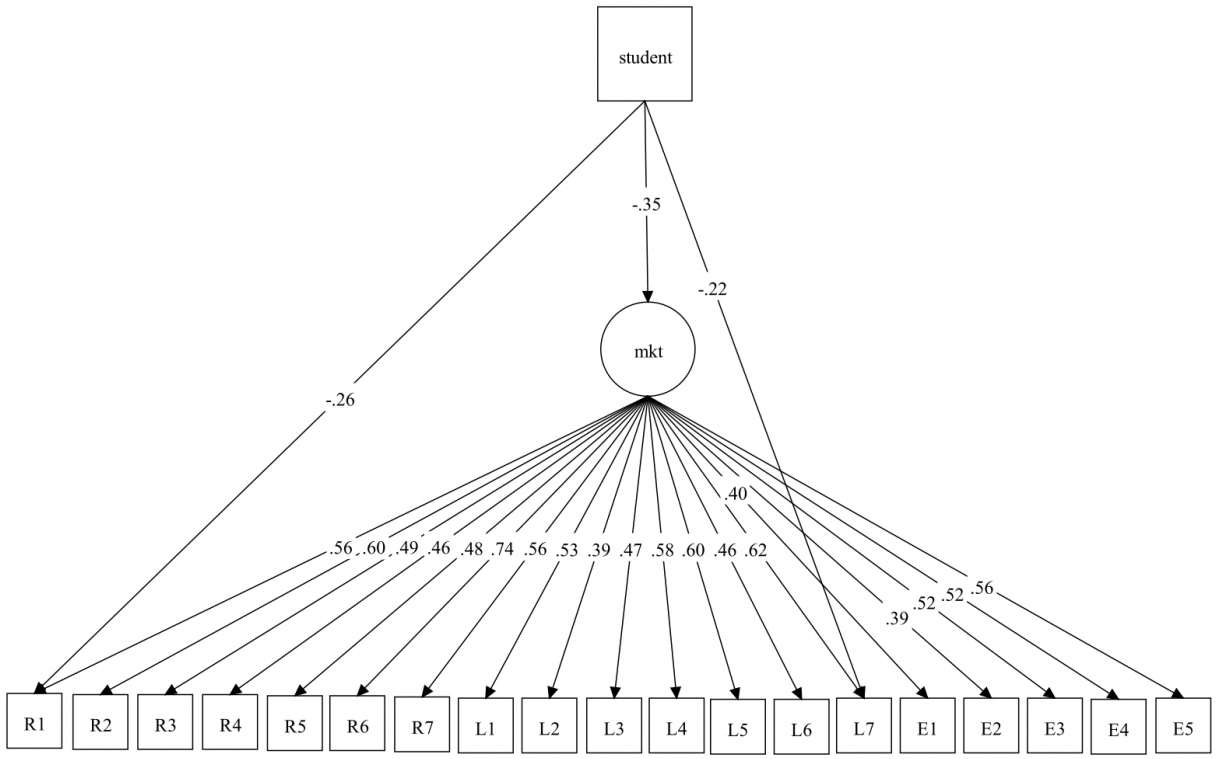


Figure 3. Adjusted MIMIC model with student group indicator and DIF effects (standardized & significant estimates)

The adjusted MIMIC model yielded improved fit statistics (RMSEA=0.012, CFI=0.993, TLI=0.993). As expected, considering the direction of DIF items, the estimated difference in MKT-CCA between instructors and students remained positive and significant (standardized beta=-0.35). This indicates that instructors tend to possess higher MKT-CCA than students, after controlling for DIF items.

Discussion

This study examined the differences in performance on developed instruments consisting of 19 items designed to measure mathematical knowledge for teaching college algebra in community colleges between community college mathematics instructors and college students.

We hypothesized that students would score lower because the participants in the student sample included in this study did not have teaching or tutoring experience. The results confirmed this hypothesis – estimated MKT-CCA was lower for students, and this pattern was consistent across all items. Our hypothesis was based on the characteristics of the knowledge construct we intended to measure through our instrument. As our instrument items were designed to capture the mathematical knowledge used in teaching mathematics, higher scores for instructors with teaching experience imply that the knowledge measured by our instrument is related to teaching. However, it remains unclear whether the instructors' higher scores are due to knowledge acquired from teaching experience or from taking more advanced mathematics courses. Future analysis could examine item-by-item response patterns among instructors and identify qualitative features of the items significantly affected by teaching experience versus those affected by advanced mathematics coursework.

In this study, we applied a MIMIC model rather than multiple group models given that it had a lower sample size requirement, and it is suggested to be an efficient way of comparing latent constructs with adjusting item intercept bias (Diemer, 2023). However, as we did not apply a measurement invariance model that tested the extent to which each of the different components of item estimates across groups, including invariance in factor structure and item discrimination, there is a possibility that we are missing some differences in how each item works for the groups regarding the strength in discriminating the knowledge. In a future study, we will apply a full measurement invariance model and test equal factor loadings for the groups and examine whether the model provides the same result.

By providing a tool that can be used for both instructor and student (including graduate students interested in teaching) populations, our study contributes to the field in identifying potential areas for interventions, training, and curriculum development to enhance not only current community college instructors' mathematical knowledge for teaching college algebra, but also that of students, who might eventually be interested in teaching mathematics at the community college or possibly at other post-secondary institutions.

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Symbolic Forms for Inverse Relationships? Calling for Clarification on the Construct

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Symbolic forms is a theoretical construct that has been used to study the meanings students have for symbols. One promising application of symbolic forms is to describe how students construct mathematical equations to represent real-world relationships, a skill needed across STEM disciplines. As more scholars use symbolic forms as an analytic tool across multiple contexts, they make adaptions to the theoretical construct by expanding existing lists of symbolic forms. In this paper, we continue this line of work by documenting undergraduate STEM majors' symbolic forms corresponding to inverse relationships that arose as they modeled dynamic scenarios. We end the discussion with theoretical considerations researchers must make to continue to use symbolic forms to describe students' equation construction.

Keywords: symbolic forms, inverse, differential equations, representations

Scholars across multiple disciplines are investigating how students understand and use symbols in a variety of contexts including chemistry (Becker & Towns, 2012; Rodriguez et al., 2018), physics (Dreyfus et al., 2017; Meredith & Marrongelle, 2008), and mathematics (Dorko & Speer, 2015; Jones, 2013). One thing these scholars have in common is their use of Sherin's (2001) theoretical construct of symbolic forms to operationalize how students understand patterns and meanings of symbols. One promising line of inquiry using symbolic forms is to describe students' construction of mathematical equations intended as representations of real-world scenarios (Schermerhorn & Thompson 2023; Roan & Czocher, 2024), a key skill needed across STEM disciplines. To work towards a fuller description of how students coordinate their knowledge of real-world relationships with symbolic forms to create mathematical equations, we used symbolic forms as a malleable tool of inquiry (see Moore et al., 2024). That is, we used symbolic forms to describe students' construction of equations to represent a predator-prey scenario and disease transmission scenario (Roan & Czocher, in press). That work allowed us to describe how students utilized symbol templates related to the mathematical concept of inverse, a type of mathematical concept that is important not only to modeling but to mathematics as a whole. We end the paper with theoretical considerations researchers must face to move forward in using symbolic forms as a theoretical construct to describe equation construction.

Theoretical Background and Literature Review

Symbolic forms (Sherin, 2001) are a theoretical construct used to describe how students think about equations. Symbolic forms are thought of as a blend (see Schermerhorn & Thompson, 2023) of two components: a symbol template and the associated conceptual schema. The symbol template is the pattern of the symbols. The conceptual schema associated with the symbol template is the idea being expressed by the symbols (Sherin, 2001). For example, a common symbol template used in mathematics is ■ + ■. A conceptual schema associated with the symbol template ■ + ■ is *a whole comprised of two parts* (Sherin, 2001). Conceptual schema are intended to be "relatively simple structures, involving only a few entities and a small number of simple relations among those entities" (Sherin, 2001, p. 491). The term conceptual schema is akin to diSessa's (1993) *phenomenological primitives* (p-prims), that is "elements of intuitive knowledge" like "increased effort begets greater results" (diSessa, 2018, p. 69). Further work has

described the conceptual schema as the "underlying mathematical justification of the symbol template" (Schermerhorn & Thompson, 2023, p. 12). Conceptual schema correspond to the students understanding of relationships and structures (Sherin, 2001), thus conceptual schema may not align to normative characterizations of structures. At the heart of the theoretical construct is the hypothesis that "students learn to express a moderately large vocabulary of simple ideas in equations and to read these same ideas out of equations", where the "elements" of the vocabulary are symbolic forms (Sherin, 2001, p. 482). Researchers who use symbolic forms as a theoretical construct typically provide undergraduate students with pre-existing mathematical equations and ask them to interpret those mathematical equations. For example, Becker and Towns (2012) used symbolic forms to document how students understand mathematical equations in different physical chemistry contests like thermodynamics. Their findings indicate that students start to make sense of unfamiliar mathematical equations using heuristics that can be described by symbolic forms. In their study, the symbolic forms originally found in Sherin (2001) were sufficient to describe the students' reasoning. In other cases, researchers adapt the theory to include novel symbol template – conceptual schema pairs. For example, Dreyfus et al. (2017) investigated two students' sense-making while working on a quantum mechanics task. They used symbolic forms as an analytic tool to describe how the students made sense of the task. In doing so, they observed students using a symbol template – conceptual schema pair they called an eigenvector-eigenvalue symbolic form. Scholars have also used symbolic forms, commonly together with conceptual blending, to describe students' construction of mathematical equations (Ryan & Schermerhorn, 2020; Schermerhorn & Thompson, 2019, 2023). Schermerhorn and Thompson (2019) described how undergraduate physics students constructed a differential length vector for a spherical coordinate system. Part of this work was to add new symbolic forms to the list, and another part was to describe how students coordinated concepts from physics with symbolic forms to create their mathematical equations. They were able to identify what concept, like direction, motivated the student's use of a symbolic form, like parts-of-a-whole.

We situate our work within a similar line of inquiry as Schermerhorn and Thompson (2023); our work focuses on students' construction of differential equations that represent canonical scenarios students tend to encounter when in a differential equations class such as a predator prey scenario or a disease transmission scenario. A key mathematical relationship underlying those two scenarios is *inverse*. This is, because the predator – prey and disease transmission scenarios afford students the opportunity to think of flux to and from different populations (predator-prey) and/or sub-populations (disease transmission), students must coordinate their understanding of *additive inverse* with symbolic forms to create their models. To work towards a fuller description of how students coordinate their understandings of real-world relationships with symbolic forms to create mathematical models, we answer the research question: *what symbolic forms do STEM undergraduates use to represent inverse relationships between quantities that form their models for dynamic scenarios?* To answer this question, we share examples symbolic template conceptual schema pairs our participants used when representing scenarios that mathematically called for an additive inverse.

Methodology

The data for this paper draws from a larger dissertation study designed to characterize the cognitive obstacles students encounter while creating mathematical equations that represent dynamic scenarios. In this paper, we report on eleven participants' work on two modeling tasks: predator-prey and a disease transmission (Roan, 2024). The participants were high achieving

undergraduate STEM majors who self-reported familiarity with mathematical concepts like instantaneous rates of change and who had taken or were enrolled in a differential equations course. The data were collected via task-based interviews conducted and recorded using Zoom. Images of the participants' written work were collected at the end of each session. In both tasks, participants were asked to create a (system of) differential equation(s) that would model the given dynamic scenario. In the predator-prey task, participants were asked to model the species dynamics between cats and birds in a back-yard habitat. In the disease transmission task, participants were asked to model how quickly a disease, like tuberculosis, spread through a community of infected (sick) or susceptible (healthy) individuals.

The goal of data analysis was to document distinct symbolic forms participants utilized while creating their models. We documented the symbolic forms by identifying the symbol template and identifying the conceptual schema associated with the symbol template. Identifying the use of a symbol template is straight forward, identifying the conceptual schema associated with the template was more inductive. This analysis was aided by the interview protocol, that guided the interviewer to ask the participant to explain why they chose a symbol template. For example, when a participant wrote down a mathematical equation like $C(t) \times B(t)$, the interviewer would ask "why did you decide to use × here?". We then used this information to infer what simple mathematical relationship (i.e., conceptual schema) the participant was trying to express using the symbol template. In instances where participants did not explicitly express their reason for using a symbol template, we inferred what simple mathematical relationship they were trying to represent using the participants' explanations of what the mathematical equation represented. We matched the symbolic forms our participants used when creating their models to clearly analogous symbolic forms documented in prior literature (see Brahmia, 2019; Dorko & Speer, 2015; Dreyfus et al., 2017; Jones, 2013; Pina et al., 2023; Riihiluoma et al., 2023; Schermerhorn & Thompson, 2023; Serbin & Uscanga, 2023). For example, our participants used the symbolic forms extensive – extensive, intensive – extensive, base $\pm$ change, and more, all of which are documented in Sherin's (2001) initial list of symbolic forms. We do not provide evidence of this use and instead focus on providing more details about the [potential] symbolic forms our participants utilized when creating models of dynamic scenarios that focus on representing inverse relationships. That is, we focus on the instances where our participants used symbol template – conceptual schema pairs that depict inverse relationships that did not have clear analogous symbolic forms from the literature. That is, the conceptual schema and symbolic template have not been reported together as a symbolic form previously. In this report, we show data excerpts from Ivory, Yixli, and Rillir, whose work was selected because they concisely explicated their reasoning with symbol templates. Ivory and Yixli were physics majors, and Rillir was an electrical engineering major.

Results

In the following section, we describe three symbol template – conceptual schema pairs our participants utilized when representing relationships that mathematically called for an additive inverse. We note here that, from the researcher's perspective, these relationships called for an additive inverse. We report the conceptual schema, the underlying mathematical justification, our participants associated with the symbol templates they chose to use, which might not align with the researcher's perspective. For each pair, we will provide the symbol template, our inferred conceptual schema, and evidence of that inferred conceptual schema. Additionally, we will state which task the symbol template – conceptual schema pair was utilized for, as not all pairs were utilized across all tasks. An overview of our findings can be seen in Table 1.

Table 1 An overview of symbolic forms

Symbol Template	Conceptual Schema	Task Scenario
$x = \frac{1}{y}$	Quantity y is inversely related to quantity x.	Disease Transmission
$x = -y$	Flux in equals the flux out	Disease Transmission
$x = 1 - y$	Quantity x is the percent complement of quantity y.	Predator - Prey Disease Transmission

Inverse

Ivory (physics) was working on the disease transmission task. She had established that the instantaneous rate of change of the sick population would be represented by the symbol $\frac{dS}{dt}$ and the instantaneous rate of change of the healthy population would be represented by the symbol $\frac{dH}{dt}$. Ivory sought to create a mathematical equation that would represent the relationship between the instantaneous rate of change of the sick and healthy populations. To meet her goal, Ivory wrote down $\frac{dS}{dt} = 1 \Big/ \frac{dh}{dt}$. When explaining her model, Ivory stated

> Ivory: You could also do one over $\frac{dh}{dt}$, I think. [LAUGHS] Because you're either sick or not sick. So the inverse of sick people would be the healthy people. And the-- the inverse of healthy people would be the sick people.

A few moments later, when asked about her model she explained again that she had written $\frac{dS}{dt} = 1 \Big/ \frac{dh}{dt}$ because "it's just, you're either healthy or you're sick." We infer from Ivory's explanation that she envisioned that the entire population could be split into two subpopulations: the sick and the healthy. Further, if a person is sick, then they cannot be healthy; thus her logic goes, the sick population and the healthy population are inversely related.

The conceptual schema, the intuitive mathematical relationship being represented, is: one quantity is inversely related to another quantity. The symbol she associated with this symbol template was $x = \frac{1}{y}$. Mathematically, we would anticipate that the conceptual schema associated with such a symbol template would be something like one quantity is inversely proportional to another quantity. However, it is not the case that Ivory attended to proportionality when writing down her mathematical equation, leading us to believe that Ivory's conceptual schema paired with the symbol template $x = \frac{1}{y}$ is novel, though nonnormative. In fact, Ivory edited her mathematical representation and changed the symbol template multiple times before settling on a normatively correct mathematical representation for an SI model, $\frac{dS}{dt} = -\frac{dH}{dt}$. We will discuss the ramification of her edits in the discussion.

Additive Inverse

Like Ivory, Yixli (physics) was also working on the disease transmission task. He had previously created a mathematical equation that represented the change in the infected population, which he labeled as ΔI. A full breakdown of Yixli's model is given below in Table 2.

Table 2 A breakdown of Yixli's mathematical equation for the change in the infected population

Quantity	Symbol / Equation
Infected people	$I(t)$
Susceptible people	$S(t)$
Possible Encounters	$I(t) \times S(t)$
Exposures per Possible Encounter	α
New infection per exposure	τ
Change in the infected population	$I(t) \times S(t) \times \alpha \times \tau$

Yixli sought to create a mathematical equation that would represent the instantaneous rate of change of the susceptible population. To meet his goal, Yixli wrote $\Delta S = -I(t) \times S(t) \times \alpha \times \tau$. When explaining his model, Yixli explained that

> Yixli: OK, so I think this will just be inverted? [writes $-\tau \times \alpha \times S(t) \times I(t) = \Delta S$]. And that's it. You know, because if we're- if we gain two on the-the infection, then we lose two on the susceptible. Um, so it's just going to be the negative reciprocal. Not negative reciprocal, but, you know, um, this is going to be the negative of- of- of the rate of the infection.

We infer from Yixli's explanation that he envisioned that flux out of the susceptible population is equal in magnitude to the flux into the infected. That is, the conceptual schema being represented here is: flux in equals the flux out. The associated symbol template to this conceptual schema is $x = -y$. We used Yixli as an example because of his well-worded explanation, but many of our participants used this symbolic form during the disease transmission task.

Percent Complement

Rillir was working on the predator-prey task. He had previously created a mathematical representation for the number of interactions between a cat and a bird that do not result in death. The quantities and their corresponding symbols within his mathematical equation are shown in Table 3.

Table 3 A breakdown of Rillir's mathematical representation for the number of interactions between a cat and a bird that do not result in death.

Quantity	Symbol/Equation
Bird population	$B(t)$
Cat population	$C(t)$
Total possible interactions between species	$B(t) \times C(t)$
Percent of possible interactions that happen	α
Percent of actual interactions that do not result in death	β
Number of interactions between a cat and a bird that do not result in death	$B(t) \times C(t) \times \alpha \times \beta$

Rillir sought to create a mathematical equation representing the number of interactions between a cat and a bird that result in death. To meet his goal, Rillir wrote down $100 - \beta = G$, explaining that "since it's a β. Since it's a percent. You can do $100 - \beta$". He then replaced β with G in his mathematical representation so that $B(t) \times C(t) \times \alpha \times G$ represented the number of interactions between a cat and a bird that do result in a death. We infer from Rillir's

explanation that he thought that since there are two outcomes of an interaction, death and not death, the percentage of interactions that do result in death must be the percent complement of the percentage of interactions that do not result in death. That is, the conceptual schema being presented here is: one quantity is the percent complement of the other. The associated symbol template to this conceptual schema is $1 - x = y \; or \; 100 - x = y$, depending on whether the percentage is in decimal form or not. One other participant used this conceptual schema-symbolic template pair, and they also explained their mathematical representation in a similar way to Rillir.

Discussion

In this paper, we have presented three conceptual schema–symbol template pairs our participants utilized while modeling dynamic scenarios that have some connection to the idea of *inverse*. The pairs were observable because the dynamic scenarios the participants modeled lent themselves to consider flux within a population and between populations. Our work expands what is known about how students use of symbolic forms to represent inverses and also has implications for studying students' mathematical equation construction using symbolic forms.

Serbin and Uscanga (2023) used the symbolic form construct to discuss how pre-service mathematics teachers reasoned about additive, multiplicative, and compositional cancellation. The symbolic form construct focused their analysis on what kinds of symbolic templates (i.e., templates with or without explicit depictions of inverse or identity) their participants used when thinking of cancellation operations. We infer that Serbin and Uscanga's descriptions align with two of the three ways students' reason about inverses across contexts described by Cook et al. (2023): inverse as an undoing and inverse as a manipulated element. By studying students' construction of mathematical equations that represent dynamic scenarios, we have described three conceptual schema–symbol template pairs that align with the third way of reasoning Cook et al. (2023) described: inverse as a coordination of the binary operation, identity, and set. Our efforts add to the growing list of symbolic forms and further test the symbolic form construct in a new area and adding to its applicability in more contexts.

However, we have rhetorically referred to these as conceptual schema – symbol template pairs because there are alternative interpretations of each of these pairs that brings to light considerations for using symbolic forms to describe equation construction. First, we proffer alternative interpretations of each example and second, raise the difficulties of making inferences based on student reasoning.

First, Ivory used the conceptual schema "quantity y is inversely related to quantity x." paired with the symbol template $x = \frac{1}{y}$. She, in that moment, justified using symbol template $x = \frac{1}{y}$ using the mathematical idea of inverse. When students create mathematical equations to represent real-world relationships, the "underlying mathematical justification" (Schermerhorn & Thompson) for the symbol template may not be mathematically accurate. In prior literature, researchers have discussed how the conceptual schema might not accurately depict the phenomena of interest (see Schermerhorn & Thompson, 2023) but the schema and template still aligned mathematically. This raises the question, do cases where the conceptual schema and the symbol template do not mathematically align belong on the growing list of symbolic forms seen across contexts? We assert that they should belong on the list, because these cases are informative about what underlying justifications students associate with the symbol templates. Further, these indicate places where students might need support to disentangle nonnormative conceptual schema with symbol templates. Ivory later edited her model because she realized that

her mathematical equation was incorrect and that the conceptual schema she had associated with the symbol template $x = \frac{1}{y}$ did not match the relationship between her quantities as well as she originally thought. When students make edits to their mathematical equations the conceptual schema may change. When that change occurs, researchers must consider if theoretically that is that another instance of a symbolic form or a different manifestation of the same symbolic form.

Second, we consider Yixli's use of the conceptual schema *flux in equals the flux out* along with the symbol template $x = -y$. An alternative interpretation of Yixli's example is that he used the symbolic form Sherin (2001) called competing terms, where the conceptual schema was *influences in competition*, and the symbol template was $\blacksquare \pm \ldots \pm \blacksquare$. In Yixli's case, he simply had one term that was doing the influencing. From our perspective, this interpretation overlooks that Yixli created a mathematical equation that simultaneously described a positive influence on one quantity and a negative influence on another, a key aspect of the justification Yixli gave for using that symbol template. Thus, his conceptual schema was distinct from *influences in competition* since, from his perspective, no influences were competing.

Lastly, we consider Rillir's use of the conceptual schema *quantity x is the percent complement of quantity y*, along with the symbol template $x = 1 - y$. In our interpretation, the $1 - y$ or $100 - y$ is like an operator, a thing done to a percentage to get its complement. This aligns with other explanations of symbolic forms such as in Jones (2013), who described the parts of an integral as operators that acted on an original function. An alternative interpretation of Rillir's example is that he instead activated the whole - part symbolic form (Sherin, 2001) where the conceptual schema is taking away a part of an original whole and the symbol template is $[\blacksquare - \blacksquare]$. Sherin (2001) described this symbolic form as "a new net amount is produced by taking away a piece of an original whole" (p. 534). It is unclear the extent to which Rillir and other participants saw the 1 or 100 in the symbol template as a whole that the percentage is being taken from rather than an operator that does something to the percentage. When cases like this occur, researchers must decide if theoretically this is an instance of a singular symbolic form or an instance where the student is using a few symbolic forms waring a trench coat. Researchers need to take into consideration the loss or gain in explanatory power when making that decision. We infer the theoretical consideration is directly linked to the grainsize of students' justifications. Thus, researchers must consider the theoretical construct's ability explicate differing grain sizes of the mathematical justifications underlaying students' choice in symbol template. We acknowledge some of our difficulty classifying our participants symbol template – conceptual schema pairs as symbolic forms may lie with the interview protocol. However, our limitations to making more definitive claims have brought to the forefront important theoretical considerations. That is, the three examples reported in our paper bring to light considerations research must address if work is to move forward in using symbolic forms to describe students' construction of mathematical equations.

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Graduate Student Instructors' Mathematical Knowledge for Teaching Inverse Functions and Their Goals for Student Learning

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The mathematical meanings that instructors hold necessarily impact the ways that they teach and their goals for student learning. This study examined five graduate student instructors' (GSIs') mathematical meanings related to inverse function with the goal of exploring links between their meanings and the meanings they hoped for their students to construct. I found that the GSIs had multiple meanings for inverse functions that were evoked in different contexts, and that they had trouble drawing connections between these meanings when asked about relationships between them. Similarly, their goals for student learning did not include connections between meanings and were additionally constrained by factors such as the departmental curriculum used and the need to adhere to mathematical convention that students were accustomed to.

Keywords: Mathematical Knowledge for Teaching, Inverse Functions, Graduate Student Instructors

Researchers and teacher educators have long called for more attention to the mathematical meanings that teachers hold (Thompson & Thompson, 1996), arguing that without a coherent web of strong mathematical meanings, teachers are not best positioned to effectively work to foster strong understandings in their students (Tallman, 2021; Thompson, 2013). Graduate student instructors (GSIs) teach large numbers of undergraduate students in introductory math courses every semester (Blair et al., 2018), and often have little to no training (Belnap & Allred, 2009; Lewis & Tucker, 2009). Although there has been a tacit assumption that the mathematical content they teach is unproblematic for GSIs (e.g., Ellis, 2014) recent research focused on GSIs' mathematical meanings calls this into question and indicates a need for more attention to GSIs' mathematical meanings (Musgrave & Carlson, 2017; Rocha, 2022).

This proposal reports on a study designed to investigate the mathematical meanings related to inverse functions held by GSIs teaching Precalculus, and to gain insight into aspects of their mathematical knowledge for teaching (MKT) in this context. It is well known that students generally struggle with inverse functions (e.g. Bayazit & Gray, 2004; Brown & Reynolds, 2007; Engelke et al., 2005; Even, 1992; Paoletti et al., 2018; Vidakovic, 1996), but less is known about how teachers understand these ideas, especially at the postsecondary level. The broad goals of the study were to answer the questions:

1) What are graduate student instructors' meanings for inverse functions?
2) How do these meanings impact their goals for student learning related to inverse function?

Literature Review and Background

The knowledge required for teaching has been studied for several decades, beginning with quantitative analyses of teachers' education and their students' achievement (e.g., Ferguson & Womack, 1993; Guyton & Farokhi, 1987; Monk, 1994). When these studies began to shed doubt on the assumption that teachers with more mathematical knowledge would have higher student achievement, mathematics education researchers sought new methods for investigating the link

between teacher knowledge and student outcomes. Several frameworks for conceptualizing mathematical knowledge for teaching (MKT) have been proposed, with some focusing on measuring and categorizing that knowledge (e.g., Ball et al., 2008) and others working to provide explanatory mechanisms for how MKT might be developed (e.g., Silverman & Thompson, 2008). I draw on Silverman and Thompson's conceptualization of MKT for this proposal, as it focuses specifically on the mathematical meanings that teachers hold with respect to a given concept.

Silverman and Thompson's (2008) framework for mathematical knowledge for teaching is based in a constructivist perspective, where "knowledge" and "meaning" are thought of differently than how they are generally used elsewhere. Rather than knowledge referring to things that someone believes to be true, it instead refers to assimilation to a scheme, which entails assigning meaning to the object of assimilation by fitting it into an already developed cognitive structure and includes inferences based on this meaning (Thompson, 2013). These related inferences are critical in understanding someone's meanings, because they establish what it is that particular knowledge will allow someone to do. Silverman and Thompson (2008) argued that the first step in developing MKT is to first have a strong personal understanding of the mathematical concept. This understanding includes relationships between the concept in question and other content knowledge, creating a web of connections between seemingly disparate mathematical topics. Next, a teacher must transform this knowledge into knowledge with pedagogical power by understanding how students conceptualizing mathematical ideas in a similar way to the teacher will enable them to learn related ideas, and by imagining the actions that the teacher might engage in to help support students in developing this understanding. Silverman and Thompson (2008) claimed that this transformation of knowledge into knowledge with pedagogical power requires a teacher to consciously reflect on their understanding, and to reflect on how students might develop compatible understandings.

The productivity of instructor's mathematical meanings can be defined by how productive these meanings would be for students' long-term learning if they were to also hold those meanings (Byerley & Thompson, 2017). From this perspective, meanings that help a student to answer questions about the given topic correctly but that are not easily built upon to develop future mathematical ideas are less productive than meanings that provide a foundation for further learning. In addition, the purpose of knowledge is to help the knower maintain a state of equilibrium as they engage with the world (von Glasersfeld, 1995), and so meanings that are useful in a wider variety of contexts can be considered to be more productive than those that are useful only in limited contexts (Byerley & Thompson, 2017). Similarly, those meanings that provide important foundations for reconciling future perturbations that occur and engender learning are considered to be more productive.

Methods

This study was conducted at a large public university in the southeastern United States, and five graduate student instructors participated. The GSIs ranged from second to sixth year mathematics PhD students and had all taught Precalculus for at least one semester prior to the commencement of the study. At this university, inverse functions are taught in Precalculus as a prelude to exponential and logarithmic functions. Students are expected to determine whether a function is one-to-one algebraically and by examining the graph of a function, and they are presented with the standard composition definition of inverse functions. Students learn to find the inverse of a given function algebraically by switching x and y and solving for y (the "switch and solve" technique), and graphically by reflecting over the line $y=x$.

I conducted a series of clinical interviews (Clement, 2000) with participants individually. The first two interviews focused on the GSIs' goals for student learning in relation to inverse function, and the remaining interviews were task-based interviews aimed at developing models of the GSIs' mathematical meanings for inverse functions (Steffe & Thompson, 2000). The GSIs participated in between 6 and 9 interviews each, with the number of interviews determined by how long it took to develop stable models of each participant's mathematical meanings. I engaged in ongoing analysis between each task-based interview to develop and refine my models for how my participants understood inverse functions and developed tasks for future interviews to test these hypotheses. I then compared the goals that participants had for the meanings their students would construct to the models of my participants' own meanings that I developed to look for areas of agreement or disagreement.

Results

Throughout the task-based clinical interviews, it was clear that the GSIs held meanings that allowed them to productively engage with inverse function tasks. However, most of them drew on meanings that were disparate (from my perspective) to solve different types of tasks, and they often had trouble describing coherence or connections between their different meanings. Isabelle, a sixth year graduate student, consistently thought of inverse functions as functions that describe the same invariant relationship from two perspectives. With this meaning, one function is viewed as giving one quantity (Q_1) in terms of another quantity (Q_2), while the inverse function describes the same relationship between Q_1 and Q_2 but instead gives Q_2 in terms of Q_1. She described this as being one of the big ideas that she wanted students to understand about inverse functions, saying,

> So let's say if we have cosine theta is *a*, we can have cosine inverse *a* is theta. And what it tells us is this is theta, uh, *a*, 1, right? [pointing at the triangle in Figure 1] So both of them tell me the same story. So I have the same, uh, object or same thing in the real life and I can say that in a different way. And I can use whichever I want to use or whichever is useful for me in that situation.

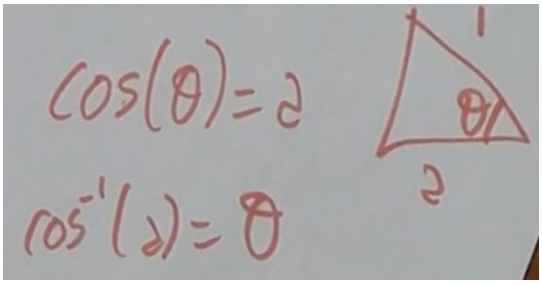


Figure 1. Isabelle 's work while discussing intended student meanings for inverse function.

At other times throughout the interview sequence, Isabelle also talked about wanting students to understand that inverse functions undo the original functions via function composition, and to think about inverse functions involving inverse operations from the operations in the original function. I was curious about the extent to which these two meanings were related for her, because she talked about undoing and composition in relation to algebraically defined functions without context, whereas she discussed the invariant relationship meaning when discussing trigonometric functions or functions with a real-world context. As an example of the latter, when presented with the following task (Figure 2), Isabelle almost immediately identified that the two graphs could be thought of as representing the same glass, and that they could be seen as inverse

functions. To explore the connections between her meanings for inverse functions, I asked her to clarify how a student might understand how these two ways of thinking about inverse functions might be related in this context.

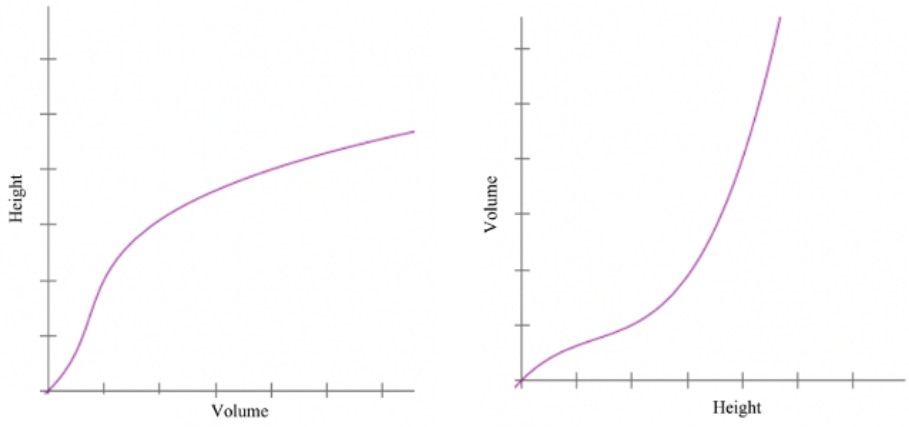


Figure 2. A task used in Isabelle 's interview.

Interviewer: What about if a student was like, "I thought inverse functions, like, undo each other, um, but these are just, like, kind of representing the same relationship. How do, how do I, like, make sense of that?" How would you respond?

Isabelle : For the undoing thing, I would say they are undoing in algebraic way.

Interviewer: Okay, what do you mean by that?

Isabelle : Like, when I'm adding two, for one of them, I'm subtracting two for the other one.

Interviewer: Okay.

Isabelle : But they represent the same thing in the real life. They represent the same glass.

She continued to describe how undoing can be seen algebraically when finding the inverse of a function, and I pushed her to explain conceptually how a student might connect the idea of undoing with the physical context of the glass.

Interviewer: If we're thinking about, like, the relationship between volume and height, or height and volume, is there a way of thinking about the undoing, in terms of, like, that relationship?

Isabelle : No, because I feel like they are doing the same thing. Like, again, I will just look at one part here [pointing to one graph], or here [pointing to the corresponding section on the other graph], and for both parts, when I look at, like, volume and height, let me say it like this, when I think, this is the height, this is the volume, this is the height, this is the volume here, so they give me this [pointing to her drawing of the associated glass]. They give me the same object. So because of that, in the real life explanation, there is no undoing. Because height is going, well, let's say, two units, but volume is going eight units in both of them. So, no. There is no undoing part.

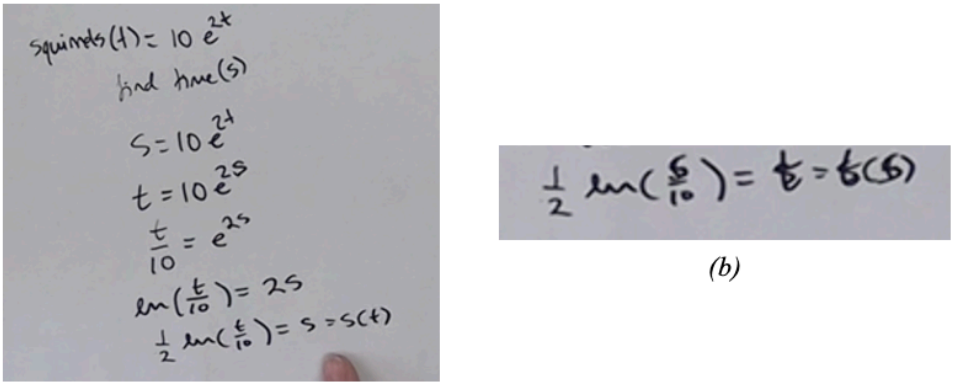
I interpreted this to mean that Isabelle thought about composition and "undoing" in the context of algebraically defined functions, but that these meanings were not explicitly related to her meaning for inverse functions in real-world contexts (i.e., a function and its inverse capturing an invariant relationship). When I asked whether she wanted students to see these different ways of thinking as related, she did not express a desire for them to make any connections. While I

have only presented an example of a lack of connections in one participant's meanings due to space constraints, other GSIs demonstrated similar differing meanings for inverse functions across various contexts, and these inconsistencies were reflected in the lack of specific connections between the ideas that they wanted students to develop.

However, the GSIs' teaching goals were not only constrained by their personal mathematical meanings, but also by their perceived constraints due to the curriculum used and mathematical convention that they perceived their students to be accustomed to. All the participants explicitly described a disconnect between finding the inverse of a given function in a real-world context and solving the same type of problems without context. When finding the inverse of a function expressed algebraically, participants generally used the "switch and solve" technique. When solving problems in context, however, participants did not swap the variables and instead simply solved for the other quantity in the problem.

One participant, Nadia, described a problem that she used in her class, in which the population of squirrels in someone's backyard was growing according to the relationship $S = 10e^{2t}$ where S represented the number of squirrels and t represented time. Students were tasked with finding a function that would represent the time elapsed as a function of the population of squirrels in the yard. When doing this problem in an interview, Nadia first employed the switch and solve technique and ended up with a function that was again in terms of t (see Figure 3a). Once she finished, she perceived that her solution no longer accurately described the original relationship and switched the variables again to fix it (Figure 3b), stating "if it were *x*'s and *y*'s, it would have worked out with inputs and outputs. Um, but I shouldn't have switched these because I wanted to keep time and squirrels different." To her, the "switch and solve" technique was appropriate to use when dealing with x and y because it maintained the convention that x is the input of a function and y is the output of a function, but was not appropriate to use when the variables in a function represented real-world quantities. When I questioned how she helped students understand when to swap variables versus when to leave them, she stated,

> I think it would have confused them more if I had talked about that, because then it would be like, why are *x*'s and *y*'s special where we want to switch them? And, and I mean, the answer to that is just because it would be more confusing for the students to be trying to graph, like, if you were, if you had a function *x* with *y* as the variable, and you were trying to graph that on your regular plane, that would just be more confusing for the students. As opposed to saying *y*'s are on the *y* axis, *x*'s are on the *x* axis.



(a) (b)

Figure 3. Nadia's work for the squirrel population problem.

Another participant that identified the same disconnect discussed that he would prefer to approach the topic differently than the given curriculum did in order to address the perceived disconnect. Jake reflected on how he would prefer to teach students to find the inverse of a given function, saying he would like them to solve for *x* in terms of *y* instead of swapping the variables. He noted that,

> Somehow that makes more conceptual sense to me, because *y*, kind of, you know, in this sort of picture [pointing to Figure 4], it's like, the *x*'s lived over here [pointing to the circled *x*], the *y*'s lived over here [pointing to the circled *y*], and then there is the function, which takes you from there to there [motioning from left to right]. And so this is a recipe which tells you, given an *x*, how do you get a *y*? So it's like, how do you go this way? And if you solve for *x* in terms of *y*, that's telling you given a *y*, how do you get an *x*?

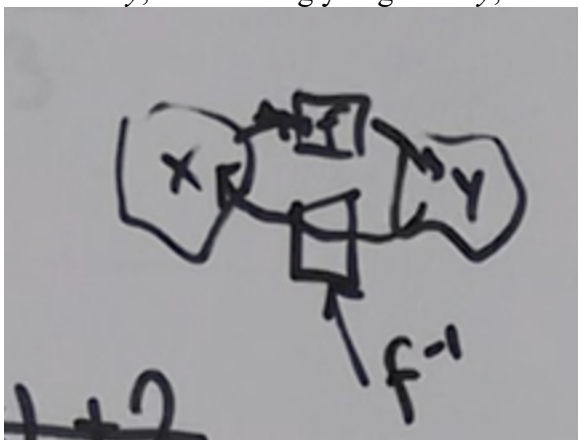


Figure 4. Jake's function diagram.

He went on to describe why he does not teach in this way, saying,

> I've tried that in like once or twice and I think that it's another inertia momentum thing where, you know, just solving, they're just so used to solving for *y*. Yeah, ideally, I would not do the swap. But, yeah, why do they do the swap in the process? I think it's just somehow to make it, um, yeah, because, like, they look at this kind of thing [pointing to the equation $y = 3x + 2$] and they're, like, not sure what to solve for somehow. Um, so once you make the swap, then it becomes clear. Like, uh, I wanted *y* equals some stuff, and now I don't have that situation anymore, so now I have to do algebra to get, you know, *y* equals some stuff in the process.

I interpreted this to mean that Jake prioritized maintaining the procedural habits that his students were accustomed to over teaching in a way that aligned with his personal meanings for inverse functions. He had tried to teach using his own meanings in the past, but was, in his opinion, unsuccessful, which could indicate that he had not transformed his personal meanings into meanings that were pedagogically useful in this context. While he had an image for the meanings he wished for his students to construct, he did not know how to support them in constructing these meanings. Nadia and Jake both compromised the coherence of their instruction to maintain habits that students were used to (such as graphing *x* on the horizontal axis and always solving equations for *y*), instead of working to help their students construct mathematical meanings that were compatible with their own.

Discussion and Conclusion

Through this study, I sought to investigate the coherence of participants' meanings for

inverse functions because, in accordance with my theoretical perspective, coherence in an instructor's mathematical meanings is a necessary condition for them to be able to intentionally design instruction to support students in constructing powerful, coherent meanings. I found that the meanings for inverse functions that the GSIs' drew on in different contexts were distinct from each other, and that they struggled to draw connections between these differing meanings. Some, like Isabelle , did not have a desire to draw these connections, while others actively searched for ways to connect their meanings during our interviews. One participant expressed her frustration when trying to understand how a function and its inverse can simultaneously represent the same quantitative relationship and can be seen as undoing one another, saying "I feel like I'm so close to getting this right. Um, like this connection between what we saw here, that like, it's, it's really the same relationship, and it being some inversion of the relationship. It's still not there." She sought to reconcile these ideas over the course of the interview sequence, and wanted to find a way to help her students understand the ideas as compatible as well.

The curriculum that the GSIs used in their courses centers actions for students to do, rather than meanings for them to construct. If student meanings for inverse function are limited to actions such as "I should compose functions to show that they are inverses", "I switch x and y and solve for y to find an inverse function", and "I reflect the graph over $y=x$", then each of these meanings is only useful in very specific contexts. If the productivity of meanings is determined by the range of situations that they allow a learner to handle, then these meanings are extremely limited. As the GSIs pointed out, the "switch and solve" technique has the potential to be conceptually limiting for students. The GSIs themselves understood why the technique works in some cases but not others, but had not operationalized these understandings in ways that made them useful pedagogically. Jake and Nadia, along with other participants, described their reasons for teaching this technique despite its limitations, which centered around wanting to match what students had seen in previous classes, wanting to align with the textbook and course materials, and not wanting to confuse students by altering the convention that x is always an input and y is always an output. These are all entirely valid reasons and underscore that shifting to teaching for mathematical meanings is incredibly difficult at the level of the individual instructor, and requires a broader scale effort at the level of course coordinators and curriculum designers.

It should not come as a surprise that the GSIs struggled to describe central meanings behind the concept of inverse functions when the curriculum they use has no such underpinnings. It would be unreasonable to expect GSIs to come into their graduate programs with fully developed sophisticated understandings of all the mathematical ideas they will teach, and often the materials that they are given to teach offer them no opportunities to further develop their understandings. While this study uncovered worrisome findings about GSIs' mathematical knowledge for teaching, I do not intend for it to be a critique of the GSIs, but instead of the systems within which they are expected to operate. The GSIs in this study are clearly mathematically capable and yet they hold limited meanings for these mathematical concepts and meanings that are not pedagogically powerful. This is an indication that they have not had opportunities that would lead them to develop stronger mathematical meanings, not that they are incapable of it. This has implications for both professional development programs for GSIs and curriculum design, as developing stronger meanings in GSIs will be of limited value if they are still constrained by action-focused curriculum. Future research should investigate how to help GSIs develop more coherent meanings, and how to help them transform their meanings into meanings that are pedagogically useful.

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Developmental Mathematics Student Motivation: A Mixed-Methods Study

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Failure and attrition are unfortunately common experiences for developmental mathematics students. These students also frequently experience course repetition and redundancy, though these phenomena are less well-understood. In this mixed-methods study, I aimed to investigate how developmental math students' attitudes and motivation are impacted by their experiences with relearning and course repetition. Additionally, I aimed to describe how these experiences influence shifts in motivation, resulting in changes to academic behaviors and, thus, performance. Quantitative analyses, though statistically insignificant, warrant future inquiry regarding the attitudinal impact of course repetition. Qualitative analyses reveal how students' particular aims and learning histories meaningfully mediate shifts in their motivation. The results of this study demonstrate the salience of investigating how *repetition impacts students' motivation, as well as the utility of theories of motivation in the study of developmental math students' perceptions.*

Keywords: Developmental math, motivation, student perceptions

Developmental mathematics education is a unique and complex system at the threshold of postsecondary mathematics education. Courses in this system–typically focused on algebraic topics–aim to ready underprepared students for college-level coursework through remediation and reinforcement of their mathematical knowledge and skills. Discussions regarding the high rates of failure and attrition documented in developmental mathematics education tend to seek remedies and improvements, rather than to more deeply understand the mechanisms and consequences of the outcomes themselves. For instance, the related phenomena of *course repetition*—when a student retakes a course that they previously withdrew from or failed—and *redundancy*—when a college student takes a course at the same or lower level than what they completed in high school—are common, though research is limited regarding these experiences (Ngo, 2020). Bicak and colleagues (2023) found that two-year to four-year institution transfer students who experienced math course repetition had worse outcomes–including lower cumulative GPA, lower rates of degree attainment, and longer time for degree completion–than students who did not experience course repetition. Ngo (2020) found that nearly 40% of community college developmental math students were in a redundant math course. Lastly, Ngo and Velasquez (2023) found that 48% of community college students experienced a math trap, which is "when students never go beyond their highest level of math in high school" (p. 1637). These findings, situated in the context of community colleges and within the broader scope of postsecondary math programs, warrant further inquiry into the influence of course repetition and redundancy with a specific focus on developmental math, its students, and their learning experience.

Ngo (2020) pondered the "psychic costs to redundant math experiences," speculating "implications for college persistence, degree attainment, and STEM participation" (p. 223). Pursuing this line of inquiry, I am interested in unpacking the influence course repetition has on students' attitudes and motivation, which impacts their performance via their academic behaviors. Research documenting developmental math students' perspectives is limited, despite

it yielding rich insight into the learning experience of these students. I use mixed methods to conduct this study and am guided by theories of motivation.

Theoretical Perspective

Amman (2022) defined *relearning* as "the experience of trying to learn content one has learned (or attempted to learn) before" (p. 1). Given the algebraic content of developmental math courses, it is likely that most students have taken an algebra course during their 6-12 math education. Additionally, as a student experiences course repetition, they are further engaging in relearning. Amman's research demonstrated how students' perceptions of relearning meaningfully shaped their learning in their developmental math classes. In this study, relearning is used to frame the unique context of developmental math students' learning experiences.

To guide my inquiry regarding students' motivation, I used Eccles and Wigfield's (2020) expectancy-value theory (EVT) as a foundational theoretical framework. Specifically, I was interested in students' academic self-concept, subjective task values (i.e., relative cost value), and expectation of success as psychological determinants of motivation (PDM), that is, as proximal influences on their "task and activity choice, performance, and engagement" (Eccles & Wigfield, 2020, p. 3). I am interested in leveraging the relearning framework and EVT together because they both acknowledge the salient accumulation of values and perceptions across time.

This mixed-methods study was guided by the following questions:

1. What is the relationship between measures of motivation and degrees of relearning?
2. Why do shifts in students' motivation occur and how do these shifts relate to experiences with relearning or course repetition?

In quantitatively addressing the first question, I hypothesize that positive measures of motivation would negatively correlate with degrees of relearning (e.g., students with more relearning experiences are less likely to agree with positively oriented motivation statements), while negative measures of motivation would positively correlate with degrees of relearning. In qualitatively addressing the second question, other theories of motivation—such as implicit theory (e.g., Yeager & Dweck, 2012), attributional theory (e.g., Graham, 2020), and control-value theory (e.g., Pekrun, 2019)—meaningfully informed my analysis of the interviews.

Methods

Data Collection

For this explanatory sequential mixed-methods (quan → QUAL) study, I conducted two phases of data collection in three consecutive semesters (Spring 2024 and Summer Sessions I and II) at Temple University, a large, urban research university on the East Coast. Because I was interested in students' degree of relearning—reflecting the extent of a student's experience with relearning, repetition, or redundancy—I sought participants enrolled in Temple University's second level developmental mathematics course, College Algebra.

The first phase of data collection involved a quantitative survey instrument with Likert-type items that measured students' mathematical PDM—that is, their academic self-concept (ASC) on a six-point scale, valuation of relative cost (RC; specifically, effort, emotional, and opportunity costs) on a six-point scale, and expectancies of success (ES) on a five-point scale. These scales were adapted from Marsh and O'Neill's (1984) Academic Self-Description Questionnaire III, Gaspard et al.'s (2014) expectancy-value instrument, and Pintrich and DeGroot's (1990) Motivated Strategies for Learning Questionnaire, respectively. The survey had 30 items, 10 ASC items that alternated between positive and negative orientation, 11 negatively oriented RC items, and 9 positively oriented ES items.

The results from the survey were used to identify participants of interest for the second phase of data collection, which involved a semi-structured interview focused on students' prior and current math learning experiences. Across the three semesters of data collection, 32 students consented and completed the survey, and 4 students took part in the interviews (Table 1).

Table 1. Interview Participants

Participant	Year of Study	Degree of Relearning
Samantha (she/her)	Second year[1]	First enrollment in College Algebra
Aniya (she/her)	First year	First enrollment in College Algebra
		One prior enrollment in Intermediate Algebra
Victoria (they/them)	Third year	Third enrollment in College Algebra
		Two prior enrollments in Intermediate Algebra
Robin (he/him)	First year transfer[2]	First enrollment in College Algebra
		One prior enrollment in Intermediate Algebra

[1] Samantha was a junior by credits.
[2] Robin was a sophomore by credits.

Data Analysis

I used R to conduct cluster and correlational analyses to probe the relation between degree of relearning and student motivation from the survey responses. Specifically, I conducted hierarchical clustering using Gower distance and found Spearman's correlations since I was analyzing Likert-type, ordinal data (Islam, 2023). I conducted a cluster analysis to group the participants according to the orientation and degree of their PDM. To analyze the interview data, I first conducted a thematic analysis of each participant's interview, and then conducted a cross-case comparison of these results to draw conclusions regarding the influence of relearning and course repetition on students' attitudes. During the thematic coding, I looked for descriptions of motivated behavior, that is, when participants explained why they behaved in a certain way. It was in this portion of data analysis that I leveraged different theories of motivation to generate a comprehensive understanding of each participants' experience.

Results

Quantitative. Five clusters emerged from the hierarchical cluster analysis. There were strong, statistically significant positive correlations between participants' cluster and their responses on the positive motivation items (ASC: ρ=0.72; ES: ρ=0.70) and strong, statistically significant negative correlations between participants' cluster and their responses on the negative motivation items (ASC: ρ=-0.79; Effort: ρ=-0.87; Emotional: ρ=-0.74; Opportunity: ρ=-0.77). The cluster analysis discerned two clusters of increasingly negative PDM (clust –1, clust –2) and three clusters of increasingly positive PDM (clust 1, clust 2, clust 3); Samantha was in clust –2, Aniya and Victoria were in clust –1, and Robin was in clust 1. There was a significant, moderate negative correlation between the participants' clusters and their degrees of relearning (ρ=-0.35, p=0.051). In terms of the correlations between participants' degrees of relearning and their responses to the motivation items, there were weak negative correlations for positive motivation items (ASC: ρ=-0.21; ES: ρ=-0.05) and moderate, statistically significant positive correlations for negative motivation items (ASC: ρ=0.38, p=0.034; Effort: ρ=0.36, p=0.046; Emotional: ρ=0.21; Opportunity: ρ=0.42, p=0.018).

Qualitative. All four of the interview participants are pursuing STEM majors: Samantha, Aniya, and Victoria need to pass College Algebra as a prerequisite for their major courses, while Robin needs to complete the calculus sequence for his engineering major.

Samantha. When I spoke with Samantha at the end of the spring 2024 semester, she described herself as never having had a good relationship with math and as being "very grade-motivated, extrinsically motivated." That said, in high school, Samantha successfully completed AP Statistics, which she viewed as one of her few positive math experiences. Samantha traced the shift in her feelings toward math back to elementary school when she struggled to learn fractions, saying:

> Everyone was like, 'Oh, you're just not that good at math,' and then I was like, 'Oh, I'm not good at math—why am I gonna try?' So, it was always just the thing where I'd get to math class be like, 'Oh, here we go.'

Thus, Samantha internalized others' conception of her (lack of) math ability and concluded that it was therefore not worth trying. The last part of Samantha's explanation demonstrates her negative fixed mindset regarding her math ability that she brings with her into the classroom.

Samantha's responses also demonstrate the dependency of her motivation on her grades. When asked to describe a time when she felt good in College Algebra, Samantha replied, "On the first quiz, the first quiz I got 100, and I was the only person in the class to get 100. But it's been downhill since then." She went on to say:

> I did well on the first quiz, and then as soon as I tried on the second quiz and didn't do as well, I was so disengaged in class. [...] There's been a lot of times where I've put effort in—like, trying to do the homework, trying to understand—and then I get an average grade back and I feel like I put above average work into this class, which is very frustrating.

This response underscores Samantha's belief in the reciprocity of effort and performance.

Aniya. When I spoke with Aniya at the end of the spring 2024 semester, she identified a drastic shift in her relationship with math that occurred in College Algebra. When reflecting on her prior math experiences, Aniya described herself as being "really good" at math and "extremely confident" in her skills. Therefore, she was surprised when she tested into Intermediate Algebra— Temple University's first level developmental course—because she was "used to a higher math," leading to her having a negative attitude. As a result, Aniya explained that she wasn't paying attention or engaging in class, and therefore initially performed poorly. However, due to this outcome and her instructor's support, Aniya "switched it and turned it positive" by engaging in class, acknowledging that she could use a "refresher." Ultimately, Aniya described Intermediate Algebra as a positive experience that meaningfully reinforced the strength of her mathematical understanding. This positive shift demonstrates Aniya's resilient and positive malleable mindset regarding her math ability.

Aniya's confidence and positive perception of her own math ability shifted toward the negative in College Algebra when she was accused of cheating on the first exam. Following this incident, Aniya said she "completely checked out," explaining that she was disheartened by the nature of the accusation, which, at its core, questioned her mathematical knowledge. When discussing how her survey responses would have "absolutely" been different if she had taken the survey prior to the spring semester, Aniya reflected:

> It is absolutely saddening to me, if I'm being completely honest, because I really used to enjoy math and it was one of my favorite subjects. [...] And, honestly, I can say now that if I could choose to never take a math course again, I would.

While reflecting on her new feeling of annoyance and inadequacy when doing math, Aniya stated, "Now, I don't feel smart anymore when it comes to math."

Victoria. When I spoke with Victoria at the end of summer 2024 session I, they described themselves as an overall anxious learner with particularly inhibiting test-taking anxiety. When asked how they felt about their math skills upon entering college, they said:

> I wasn't terribly excited about it—just because I know math isn't my strong suit—but I didn't expect it to be as hard as it is; I didn't expect to have to take Intermediate Algebra for two semesters, and then this being my third attempt at College Algebra.

The reality of repeating these courses multiple times weighed heavily on Victoria. They later affirmed that their survey responses would have been different prior to the current term, noting, "my answers kind of just progressively went down the more like times I took it." Specifically, they tied the connectedness of their PDM and course repetitions to their STEM major, saying:

> I guess it [(my survey responses)] kind of goes down more because I'm a STEM major. I feel like a lot of stuff I have to do is math stuff and it's like I don't really understand math a lot. Like, I'm doing a bit better, but I'm still having the same issues that I was having, like, not understanding them. It kind of makes me base my self-worth and academics on my math class and I'm not doing good, so it's like my [survey] answers are poorer.

When discussing their current attempt at College Algebra, Victoria described defeating anxiety about the ramifications of not passing including having to reluctantly change major or dropping out of college, "I don't want to have to drop out and then be like, 'Oh my God, I had to drop out because I couldn't pass College Algebra.' So, it's a lot mentally riding on me passing this time."

Robin. When I spoke with Robin at the end of summer 2024 session I, he brought a unique perspective as a non-traditionally aged student, returning to college after over a decade in the workforce to complete his degree, advance at his career, and continue to support his family. As such, Robin's motivations extend beyond academic credit. Prior to enrolling at Temple University, Robin pursued a degree at another local university, completing up through Calculus I after having taken it twice. He noted that he felt frustrated and disinterested in learning math at his previous institution.

After receiving a low score on the placement exam, Robin reported that he "didn't feel confident at all about his math skills" and explained that he was presently enrolled in College Algebra because he "needed to get caught up" to swiftly advance through his engineering math coursework. Speaking of his placement in Intermediate Algebra, Robin explained:

> I'm willing to just start over because I don't want to get to this level and then be stuck because I don't remember any of the basics. It was no big deal for me to just pay the extra money to start over and, you know, learn properly again.

Thus, looking to the future, Robin is motivated to shore up his mathematical fundamentals, knowing he will have to complete the calculus sequence. Furthermore, Robin noted his awareness of his academic financials, explaining his prior reluctance to complete his degree:

> I don't want anything that's going to stress me out: I don't get that [concept], so I have to take [the class] twice. Now, it's wasted money because I have to take it twice or because I failed.

Thus, reflecting on the past, Robin describes his academic behaviors as being motivated by a desire to avoid (the financial consequences of) repeating classes again.

Discussion

Quantitative

In exploring the relationship between measures of motivation and degrees of relearning, I hypothesized that students' motivation would decrease as their degree of relearning increased. Though not all the correlations were statistically significant, the directionality of the negative correlations between the positively oriented motivation items and degrees of relearning and of the positive correlations between the negatively oriented motivation items and degrees of relearning support this hypothesis.

There are two notable limitations to these findings. First, while the survey items used had strong internal validity (Cronbach's alpha = 0.966), the survey's external validity—for measuring the EVT constructs of ASC, RC, and ES in college math students—is questionable given its construction from different extant instruments. Second, this study's sample size is quite small (n=32), further limiting the validity of these analyses. That said, given the limited research focused on understanding the impact course repetition has on students, their non-cognitive factors, and their learning, these results warrant further study.

Qualitative

The conversations I had with my four participants underscored the unique and individual nature of student motivation and its sensitivity to individuals' particular aims and learning histories. For instance, Samantha and Aniya reacted completely differently in response to poor grades, which can be understood by leveraging attribution theory, implicit theory, and control-value theory. On the one hand, Samantha described feeling the demotivating negative emotions of frustration and helplessness when her "above average effort" yielded average grades. In this response, Samantha also expressed her belief that her other classes were "arguably harder" than College Algebra and reported exerting more effort and receiving higher grades in those other classes. Given Samantha's negative and fixed view of her math ability and her grade-based extrinsic motivation, it is likely that this experience, in some way, confirmed the notion that her effort could not compensate for her perceived lack of math ability. On the other hand, Aniya appeared to feel *motivating* negative emotions when she initially received poor grades in Intermediate Algebra, which prompted her to modify her behaviors to be more engaged in class. Implicit theory suggests that Aniya's malleable view of her math ability attributed to her response. While this has less to do with these students' experiences with relearning, the differences between Samantha and Aniya's responses underscore the salience of students' motivational characteristics to their course engagement and performance, as well as the importance of the perspectives they bring with them into the classroom.

Focusing on the influence of relearning and course repetition experiences specifically, Robin and Victoria's responses demonstrate how such experiences influence shifts in motivation. On the one hand, Robin's prior experience with course repetition—and desire to avoid its financial consequences—motivates his current approach to learning math. Specifically, knowing he has to complete the calculus sequence, Robin demonstrates diligence in solidifying his mathematical foundation. On the other hand, reflecting on their second attempt at College Algebra following their two enrollments in Intermediate Algebra, Victoria described being very conscious of their course repetitions, which prompted an adjustment of their behaviors in an effort to avoid future repetitions: "I realized that I can't be anxious to ask for help. Like, I'd rather ask for help than fail my class." Therefore, Robin and Victoria demonstrated how experiences with course repetition can serve as motivators, prompting positive behavioral changes. However, overall, Victoria's experience with multiple course repetitions appears to inhibit, more so than help, their

attitude, motivation, and academic behaviors. Specifically, Victoria described relating their course repetitions to their self-worth via their STEM major because their success on their third attempt at College Algebra would dictate the future of their academic trajectory. This generated a lot of anxiety for Victoria, which increased their test anxiety and likely impacted their memory and performance by depleting their cognitive resources (Ashcraft, 2002; Meinhardt & Pekrun, 2002). Relatedly, Victoria described struggling to remain optimistic about their likelihood of success and ability to continue their desired academic path. Therefore, Victoria's experience demonstrates how experiences with course repetition can accumulate and negatively impact students' attitudes and motivation, underscoring the importance of this investigation.

Lastly, Aniya's experience (i.e., positive prior math attitudes and outlooks, and then a hard shift during a developmental math course) might provide insight into the development of math traps, when students do not go beyond their highest-level high school math course. Though this is not to say that Aniya is or will get stuck in a math trap, it is important to highlight that Aniya proudly described passing Calculus "with flying colors" in high school and that, before College Algebra, she was extremely confident in her math abilities. When discussing her developmental math courses, she described two instances where her prior, more advanced mathematical knowledge caused her instructor to question the strategies she used. While one experience she described fondly as positive reinforcer of her math ability, the other (when she was accused of cheating) left her reluctant to engage with math in the future. I see Aniya's experience as a cautionary example of the negative consequences of redundant math.

This study is a first step in documenting the motivational, non-cognitive impact of course repetition. By capturing the perspective of students, we can better understand *why* and *how* developmental math courses suffer from high rates of failure and attrition. Future research warranted by this study includes a semester-long study of how course retakers' motivation shifts over the course of a semester.

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Does Mentoring Fill the Knowledge-GAP in Applying to Graduate School in Mathematics?

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Applying to graduate school in mathematics requires both a desire to attend graduate school, but also an understanding of the application process. The more students know about the process, the more successful they are in their pursuit of admission to a graduate program. We examined mathematics majors' knowledge of both graduate school and the graduate school application process as part of a larger study examining barriers to students applying to graduate school in mathematics. We also examined the impact of mentors on student interest in graduate school and whether having a mentor has an impact on student knowledge of graduate school. We frame and explain our results using the theory of social capital.

Keywords: Mentors, Graduate School Applications, Social Capital

In 2023, approximately 3.1 million students were enrolled in graduate-level programs and 15.2 million enrolled in undergraduate programs in the US (National Student Clearinghouse Research Center, 2024). A fraction of those 15.2 million will decide to apply to graduate or professional programs in the next few years. There are a variety of reasons for attending graduate school including increasing earning power, career options, depth and breadth of disciplinary knowledge, obtaining more satisfaction with their career and accomplishments, increasing self-esteem, access to professional leadership opportunities, being assigned more responsibilities, and intrinsic motivation for the challenge. However, Estien et al., (2023) points out that before students enter graduate school, they must overcome the obstacle of applying.

Researchers have investigated the graduate/professional admissions process in several disciplines (e.g., Chari & Potvin [Physics], 2019; Hadinger [Medical School], 2017; Landrum [Psychology], 2010; Sylvan et al. [Speech-language Pathology], 2020). Estien et al. (2023) states that researchers have found "several factors that shaped students' experiences applying to graduate school including: financial constraints (Hadinger, 2017; Sylvan et al., 2020); concerns regarding the use of admissions criteria such as the Graduate Record Examination (GRE) scores (Sylvan et al., 2020); a lack of support, guidance, and advising from faculty members (Hadinger, 2017; Sylvan et al., 2020); and a lack of transparency due to variation in admissions requirements across graduate programs (Sylvan et al., 2020)." (p. 56). Furthermore, the National Academies of Science, Engineering and Medicine (NASEM) argues that mentorship is one way to enhance students' graduate school aspirations (2019) and has multiple benefits for students like higher GPAs, completing more coursework each semester, stronger self-esteem and academic self-efficacy, higher career goals, less stress, better psychological health, improved attitudes, and better discipline-based skills (Luna & Prieto, 2009). We add to this by examining what mathematics majors know about the graduate school application process through the lens of the theory of Social Capital and assess the impact of mentoring on undergraduate mathematics students' knowledge of the application process. Specifically, we seek to answer the following:

1. *What do undergraduate mathematics majors know about applying to graduate school and about graduate school itself?*

2. *Is there a difference in students' knowledge between students who do and do not have a mentor?*

Literature Review

Poock & Love (2001) identify extensive literature examining variables that influence students' choice of colleges at the undergraduate level. These studies focus on which students go to college, the process of students' choosing which colleges to apply to and which colleges they attend, but not on the application process at the undergraduate level. Similar to the limited research on the application process at the undergraduate level, there is also meager literature on students' choice of graduate school. Kallio (1995) called for further investigations into why students apply and choose to enroll (or not) in graduate school once admitted, but there is still scant literature on the application process at all levels. There is specifically little information known about students' application to and knowledge of the application process for graduate or professional school.

Chari & Potvin (2019) is the first study investigating undergraduate students' perceptions about graduate admissions in physics. Their sample of 802 physics juniors and seniors ranked letters of recommendation, prior research experience, GPA or grades in physics and mathematics classes, reputation of recommender, and prior research publications as the top five most important admission criteria respectively when applying to graduate school. They compared these rankings to physics faculty's top five most important admission criteria from a prior study (Potvin, et al., 2017): GPA or grades in physics and mathematics classes, letters of recommendation, undergraduate courses taken, GRE physics subject test, and prior research experience. They found that there were statistically significant differences between faculty and student perceptions on prior research experiences, reputation of recommenders, prior research publications, personal statements, and familiarity with the department.

Landrum (2010) surveyed 348 students about whether they planned to apply to graduate school and knew about the application process and found that personal belief in how well they would do academically, on their preparation for graduate school, and available opportunities beyond the classroom were important factors in students' decision to apply. In addition, Sanders & Landrum (2012) surveyed undergraduate psychology majors assessing their understanding of the application process for graduate or professional school. Of the 134 students that responded, research experience, letters of recommendation, and GPA were rated as the most important components of an application for graduate school. Finally, Oakes et al. (1999) implemented a successful seminar series at Purdue University that consisted of seminars on graduate school, the General and Engineering GRE, and applying for fellowships. They found that students' understanding about graduate school increased substantially after participating in the seminars.

There is also little research on how academic advisors influence students' decisions to apply and attend graduate school. However, mentorship is a key aspect that has shown substantial impact on students' post-graduate pathways (Luna & Prieto, 2009). The literature documents different aspects of mentoring (or mentorship) that plays an important role in students' graduate school plans like faculty mentoring through research experiences (Adedokun et al., 2013; Eagan et al., 2013; Merolla and Serpe, 2013; Park et al.,2022; Pender et al., 2010), graduate school preparation programs (e.g., McNair Program [Renbarger & Beaujean, 2020] and EMERGE [Peteet & Lige , 2016]), mentoring programs (Meza et al., 2018), classroom experiences (Park et al., 2022), and faculty interactions (Espino, 2014; Hanson, et al., 2016).

Theoretical Background

We apply the theory of social capital to explore the impact of mentors on the graduate school application process. Coleman (1988) defines social capital by its function and states that it is a "variety of different entities all sharing two elements in common: they all consist of some aspect of social structures, and they facilitate actions of actors, whether persons or corporate actors, within the social structure" (p. S98). Furthermore, Coleman (1988) clarifies that "an important form of social capital is the potential for information that inheres in social relations" (p. S104). Knowledge is an essential resource necessary to apply to graduate school since students need to know it exists, why they would want to apply, and what is required to apply. In this paper, we focus on the relationships that undergraduate students have with their mentors that can provide multiple forms of resources to them because of social capital that are useful for applying to graduate school in mathematics. This social capital could be in the form of information on the graduate school application process or access to letters of recommendation. We add to the body of literature by examining what information undergraduate mathematics majors know about the graduate school application process primarily as a result of mentoring.

Methods

Instrument Development

The research team created a survey based partially on a survey developed and used to determine undergraduate physics majors' interest in graduate school and how important they believed different aspects of the application process were (Chari & Potvin, 2019).[1] Nineteen of our survey items were adapted from that instrument. A notable difference in our survey was that we provided an opportunity for participants to express their lack of knowledge about different parts of the application process. The final survey had 57 items separated into four categories: (a) knowledge about different aspects of the application process, (b) barriers to applying, (c) interest in graduate school and what students want in programs they apply to, and (d) demographic questions. Most items were Likert scale or multiple choice, though four were open-ended and some multiple-choice items allowed participants to type in a text response.

Data Collection

The research team sent the survey to department chairs and undergraduate program directors at all undergraduate mathematics programs at U.S. colleges and universities with at least 1000 students total ($N = 985$). Initial emails were sent in Fall 2022 through Spring 2023 with follow-up emails sent to encourage a greater response rate. The research team also posted the survey on social media, listservs, and in newsletters for several professional organizations in mathematics.

Data Analysis

We received 1090 responses from students at 181 colleges and universities, with 519 complete responses. Participants could skip part of an item and still have their response marked as complete. Thus, the *N*s for different items are not always equal. Descriptive statistics and Chi-squared tests of association were run in IBM SPSS. To address our research questions, we analyzed responses for five survey items, where two of those items had 8 and 6 subitems respectively.

[1] The full survey is available at this link: https://researchrepository.wvu.edu/faculty_publications/3291/

Results

Knowledge of Applying to Graduate School

Table 1 provides the overall participant knowledge of the application process for graduate school in mathematics.

Table 1. Reponses to "Please rate your familiarity with the following concepts about graduate school application process."

Sub-Item	*N*	Never heard this before	May have heard this before	Definitely heard this before
Requires a minimum GPA	586	5%	21.8%	73.2%
Requires transcripts	585	1.4%	7.5%	91.1%
Requires letters of recommendation	585	4.1%	12.1%	83.8%
Letters of recommendation should be from mathematicians	585	19.8%	35.2%	45%
Programs require application fees	585	8.2%	25.3%	66.5%
Some programs offer ways to reduce or eliminate application fees	585	26.7%	31.8%	41.5%
Often requires a personal statement	584	15.8%	23.1%	61.1%
Some doctoral programs do not require applicants to get a master's degree first	585	30.3%	18.8%	50.9%

The participants who responded "Definitely heard this before" to the sub-item asking about a minimum GPA for graduate school were then asked: "You have indicated that you heard that you need a minimum GPA to have a competitive graduate school application. What is that GPA?" The average of the responses to this follow up question was a GPA of 3.3.

There was a wide range of responses from participants depending on the sub-item. For some sub-items such as "applying to graduate school requires transcripts" nearly all participants chose the option "definitely heard of before". For other sub-items including "the availability of application fee waivers" participants were less certain that they had "heard of this before". Overall, most participants had heard of most of the sub-items in this item.

Knowledge of Graduate School

Table 2 provides the overall participant knowledge of graduate school in mathematics.

Table 2. Reponses to "Please rate your familiarity with the following concepts about graduate school in mathematics".

Sub-Item	*N*	Never heard this before	May have heard this before	Definitely heard this before
A master's degree usually takes 2 years to complete	586	3.6%	21.1%	75.3%
A doctorate usually takes between 5 to 7 years to complete	583	8.4%	25.9%	75.7%
Math graduate programs usually provide some sort of funding	584	18.2%	31.3%	50.5%
Students in doctoral programs with funding usually have tuition waivers	584	38.3%	24%	37.7%
Funded graduate students must maintain some minimum overall GPA to keep their funding	583	18.5%	40.8%	40.7%
Graduate students in some programs are unionized and are protected under a collective bargaining agreement	584	69%	13.2%	17.8%

In comparison to the previous item, participants were less likely to have heard information about graduate school itself than they were to have heard about the graduate school application process. Participants mostly knew the length of graduate school but knew less about graduate school funding structures.

Mathematical Mentors

We asked participants "Do you have someone you consider a mathematical mentor? (This is someone who has not necessarily been formally assigned to you by your institution but gives you advice on how to advance your career in mathematics.)" Of 584 participants, 327 (56%) said they had someone they consider a mathematical mentor. We used a Chi-squared test of homogeneity to see if there was an association between interest in graduate school and having a mentor.

Table 3. Reponses to "Which of the following describes your interest in graduate school in mathematics?" for Students With and Without Mentors.

Option	Mentor	No Mentor	*N*
Not interested in attending graduate school in mathematics	46 (52.3%)	42 (47.7%)	88

Interested in earning a master's degree only	54 (54%)	46 (46%)	100
Interested in earning a master's or attending a post baccalaureate program then earning a doctorate	76 (52.4%)	69 (47.6%)	145
Interested in earning a doctorate without earning a masters or attending a post baccalaureate program first	107 (70.4%)	45 (29.6%)	152
Not sure	43 (44.3%)	54 (55.7%)	97

There was an association in which participants with mentors were more likely to say they were interested in earning a PhD than their peers without mentors $\chi 2(4, N = 582) = 19.56$, $p = < .001$ ($V = .18$). This result had a small effect size. In addition, we ran a Chi-squared test of independence and found that participants with mentors were more likely to say they knew about the following aspects of graduate school and the graduate school application process (Table 4).

Table 4. Results for Chi-Squared on Student Knowledge For all tests df = 2.

Sub-Item	Mentor	Never heard this before	May have heard this before	Definitely heard this before	*N*	Values	*p*-value	*w*
Letters of recommendation should be from mathematicians	Yes	54	86	183	585	38.82	< .001	.26
	No	59	120	80				
Often requires a personal statement	Yes	44	65	216	584	8.77	.012	.12
	No	48	70	141				
Some doctoral programs do not require applicants to get a master's degree first	Yes	85	59	182	585	7.91	.019	.11
	No	92	51	116				
Math graduate programs usually provide some sort of funding	Yes	57	82	186	584	15.41	< .001	.16
	No	49	101	109				
Students in doctoral programs with funding usually have tuition waivers	Yes	109	80	134	584	7.95	.019	.12
	No	115	60	84				

There were no associations ($p > .05$) for all other sub-items in Tables 1 and 2 that were not reported in Table 4. For the five sub-items in Table 4, participants with mentors were more likely to respond that they had "definitely heard this before". The effect size is small for all sub-items

(all *w*'s between 0.1 and 0.3), but the effect size for "letters of recommendation should be from mathematicians" is approaching the upper bound for a small effect size.

Discussion

Overall, we found significant variability in student knowledge of applying to graduate school and graduate school itself. Some sub-items, such as the requirement for transcripts, were nearly universally known by our participants. Other sub-items including "students in doctoral programs usually have tuition waivers" were largely unknown to our participants. Some programmatic requirements and structures were more common to be well known, including the length of degree programs and parts of the application. However, the financial aspects of graduate school including funding and tuition waivers were less well known. This is problematic for students from low-income backgrounds, because they may believe they cannot afford to apply to or attend graduate school even though they may know how to apply.

Almost half of our participants did not have someone they considered a mathematical mentor. We found that students who had mentors were more likely to state that they were interested in directly earning a PhD in mathematics compared to their peers without a mentor. Since students who want to go to graduate school may be more likely to seek a mentor, we cannot say this association is necessarily causation. Still, participants with mentors were more likely to know more about specific aspects of the application process and graduate school than their peers without mentors. They were more likely to know about the norms in applying to graduate school in mathematics such as letters of recommendation should be from mathematicians and that you can be admitted to a doctoral program directly after undergraduate school. They were also more likely to know about norms in graduate student funding, including that mathematics graduate students are usually funded and receive tuition waivers. All these factors are items that faculty mentors, mathematicians with advanced degrees, are knowledgeable about given their own experiences in graduate school. The results of our study show that the social capital that mentors can and do provide benefit students seeking to apply to graduate school, especially through access to knowledge about the application process.

That said, there were a few aspects of the application process and graduate school that were not well known by our participants and having a mentor was not associated with increased knowledge for these. The two we believed were most important are the availability of application fee waivers and the graduate school GPA requirement to maintain funding. It is the case that not all programs may have something in place like fee waivers, so this piece of information may not be as readily available to mentors. In a previous study on this data set, we found that minoritized students were more likely to view application fees as a barrier to applying to graduate school (McEldowney et al., 2024). Therefore, mentors not knowing about or talking to students about application fee waivers is especially problematic for marginalized students. In addition, giving mentees an advanced warning of graduate school GPA requirements could help ensure that more students succeed after being admitted to graduate school.

An important consideration for these results is sampling bias. While we attempted to obtain a representative sample, participants were students who chose to take a relatively long survey. Thus, we can reasonably assume that the average mathematics major knows less about graduate school than our results indicate.

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Algebra Conceptual Understanding and Predicting STEM Degree Completion

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Algebra has been found to be a barrier to college and STEM-major completion, and a contributing factor to unequal access to STEM fields. Conceptual understanding has often been discussed as an important component of mathematics learning; yet more marginalized students often have less access to rich mathematics instruction and to algebra learning opportunities. However, to date no research has investigated the relationship between conceptual understanding and math or STEM outcomes in college. In this research, we explore the predictive validity of the recently validated Algebra Concept Inventory (ACI) to determine whether college students' ACI scores predict math course grades, STEM-major math course completion, and STEM vs. non-STEM degree attainment as well as whether ACI score explains outcome differences by race/ethnicity/gender.

Keywords: Algebra, conceptual understanding, college students, STEM majors, equity

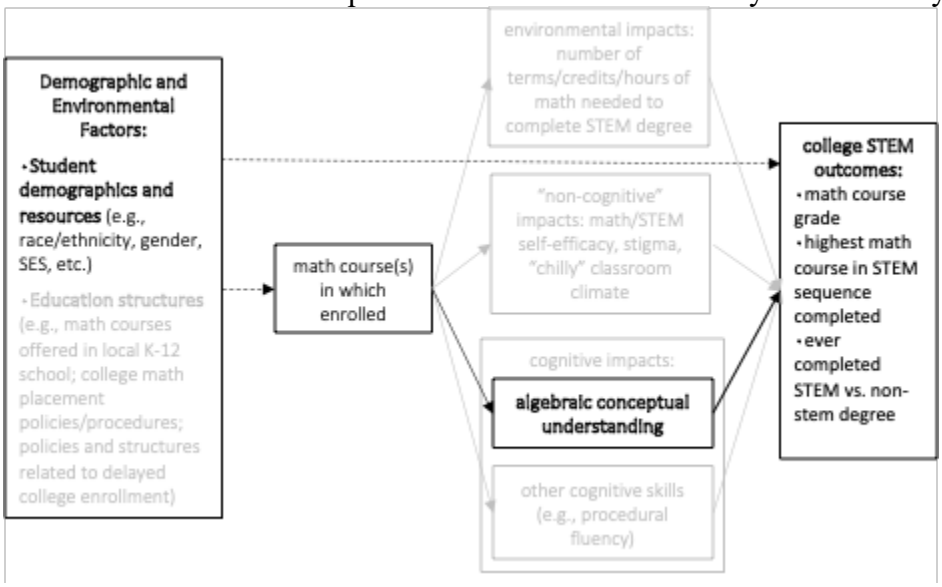
Students' prior mathematical preparation (e.g., Cohen & Kelly, 2020; Minaya, 2021) and their college performance (Chen & Soldner, 2013) influence whether they enroll in or persist in STEM degrees. Algebra serves as both a barrier to college degree completion (e.g., Bailey et al., 2010; Ganga & Mazzariello, 2018) and a requirement for earning a STEM degree (Bahr et al., 2017; Sithole et al., 2017). However, there is no large-scale research on whether algebraic conceptual understanding influences STEM degree completion. Algebraic conceptual understanding may also represent an equity issue, as research indicates that marginalized students have fewer opportunities for rich instructional experiences to develop higher-level skills, such as conceptual understanding (Schoenfeld, 2022; Yeh et al., 2020). Marginalized students are also less likely to enroll in and complete STEM degrees (e.g., Black et al., 2021; Hatfield et al., 2022; Wright et al., 2023).

Prior research on persistence in STEM degrees has explored various predictors, including environmental impacts from course sequences (e.g., Ryu et al., 2022), college math placement policies (e.g., Ngo et al., 2018), and general cognitive skills (Park et al., 2021). Yet, there is limited exploration of whether specific measures of conceptual understanding predict STEM outcomes. Existing models of STEM major retention that do include academic factors tend to use more diffuse assessments of general knowledge, such as GPA or standardized test scores (e.g., Alkhasawneh & Hargraves, 2014; Gipson, 2016; Wolniak, 2016). Additionally, models including specific math course completion as predictors may obscure the separate impact of different factors that are difficult to distinguish because they are correlated with course enrollment and completion (e.g., Crisp et al., 2009; Larnell, 2016; Quarles & Davis, 2017).

Our research is distinguished by a college STEM major retention model that includes algebraic conceptual understanding as a core factor (Figure 1). We use scores on the validated Algebra Concept Inventory (ACI; Wladis et al., 2024a) as a measure of algebraic conceptual

understanding to explore the following research questions:

- Can ACI scores predict: a) students' grades in a student's current mathematics course? b) whether they ever completed core math courses required for STEM majors? c) whether they ever completed a STEM vs. non-STEM degree?
- To what extent do ACI scores explain differences in outcomes by race/ethnicity or gender?



Notes: Grey arrows represent relationships that have been tested in other research. Thick black arrow at far right represents main relationship tested in this research. Thin black arrow represents fixed effects by course (controlled for in all analyses). Dotted black arrows represent control variables included in analyses that use controls.

Figure 1. Model of STEM Major Access and Retention as it Relates to College Math Course Placement

Algebraic Conceptual Understanding

Conceptual understanding is crucial in mathematics (Booth, 2011; Richland et al., 2012), though it is frequently neglected in algebra instruction. This is especially important for STEM students, who should be able to flexibly apply their algebraic knowledge in unfamiliar situations (Quarles & Davis, 2017). Access to effective and meaningful mathematics instruction is an equity concern since marginalized students often face fewer chances to develop advanced mathematical skills (e.g., Schoenfeld, 2022), and this access gap may significantly influence STEM degree progression and completion.

Algebraic conceptual understanding as measured by the ACI refers to understanding of mathematical concepts, which is defined as including mathematical reasoning about concepts. In the ACI, concepts are defined as well-defined mathematical processes, objects, or ideas accepted by the mathematical community (Sfard, 1991). The ACI emphasizes algebraic concepts that are accessible to students beginning in elementary algebra such as equivalence, syntactic structure, algebraic properties, functions/variation, and variable.

Below we present an ACI item as an illustrative example:

In the expression $-2x^3 + 5$*, which part is being cubed (or raised to the third power)?*

a. $-2x$
b. $2x$
c. x

Students conceptualizing the normative structure of the syntax of the expression would most likely choose C, the correct answer. Alternatively, students who choose incorrect options A or B

typically extract the structure of the syntax from non-mathematically salient features drawn from instructional experiences, like spacing, or how often instructors treat substrings of the form ax for some number a as a unified subexpression during transformation. These students may conceptualize the meaning and structure of algebraic syntax differently than those who link it to normative operational precedence (for more details, see Wladis et al., 2024b, 2024c).

The ACI's 402 multiple-choice items were developed and previously validated through a cyclical process using the Evidence-Centered Assessment Design (ECD) Framework (Mislevy & Riconscente, 2005) (see details in Wladis, et al., 2024a). The ACI is the first large-scale assessment to undergo mixed-methods validation for measuring algebraic conceptual understanding in college students, particularly for students across course levels. It was validated for use with students starting in elementary algebra. In this study we use previously collected ACI scores to explore the relationship between algebraic conceptual understanding and college outcomes such as math course grade, completion of STEM-major math courses, and STEM vs. non-STEM degree completion.

Method

Participants and Data Collection

The participants in this study were students enrolled in mathematics courses at the largest community college in the City University of New York (CUNY) between spring 2019 and summer 2023. The participants were enrolled in a wide range of math courses, from elementary algebra to advanced topics such as calculus, linear algebra, and differential equations. Data obtained from CUNY's Institutional Research (IR) Offices included detailed information on each student's course enrollment and grades, demographic characteristics (e.g., gender and race/ethnicity), academic background (e.g., G.P.A., credits earned, and major), as well as socio-economic status, which was approximated using household income data from the American Community Survey based on students' home zip codes. Institutional records were combined with the existing ACI dataset to explore relationships between students' algebraic conceptual understanding and their academic outcomes.

The ACI dataset included student scores generated during the creation and validation of the ACI. A total of 402 unique items were developed and tested; this study analyzes data from 6,582 students who took the inventory for whom CUNY then also provided additional administrative data. Math courses in which students were enrolled included elementary algebra, intermediate algebra, mathematics for elementary educators I and II, quantitative reasoning, various levels of statistics, mathematics for liberal arts majors, mathematics for health science majors, discrete mathematics, precalculus, Calculus I, II and III, advanced calculus, linear algebra, abstract algebra, and differential equations. For more information on the initial data collection and validation of the ACI see Wladis, et al. (2024a).

Data Analysis

A student's score on the ACI was the primary independent variable of interest. ACI scores are based on multiple test forms designed through common-item random groups equating design, where each student was given a different subset of approximately 25 of the 402 items. Analysis has demonstrated excellent reliability and validity for the inventory (Wladis et al., 2024a). Given its wide coverage of algebra concepts and limited prerequisite knowledge, the ACI is well-suited for investigating whether algebraic conceptual understanding is a predictor of math outcomes and STEM degree progress or attainment for students enrolled in a wide range of math courses.

ACI scores were generated using two-parameter logic (2PL) item-response-theory (IRT) models, which account for differences in item difficulty and discrimination. The models produced scores (θ values) representing how many standard deviations a student's performance was above or below the mean compared to others in the sample. For students who took the ACI more than once in a term (due to multiple math course enrollments), the mean ACI score for that term was used in the analysis.

Dependent variables included in this study are course grade (converted to GPA scale), successful course completion (defined as earning a grade of C or better), and STEM or non-STEM degree attainment during the study period. Control variables included gender, race/ethnicity, age, GPA, credits earned, and median income of home zip code. Missing data were minimal, affecting less than 2% of the dataset, so we used listwise deletion for these cases. We categorized math courses by their relevance to typical STEM major required math course sequences and by prerequisites. Fixed effects were added to account for the specific course taken, and/or course level or sequence.

We primarily employed fixed-effects linear and logistic regression models to predict math course grades and STEM degree attainment. For binary outcomes, both logistic and linear probability models were used, with linear models reported for easier interpretation. For mediation analysis, we used the KHB decomposition method (Kohler et al., 2011), to decompose disparities in ACI scores into "indirect" and "direct" components related to race/ethnicity and gender.

To address differences between the full student population, and students for whom we had ACI scores, we used entropy balancing (Hainmueller, 2012) to reweight data and improve covariate balance. This allowed ACI-taking to be independent of measured background characteristics. This step was important since students who consented to take the ACI may have been more likely to be higher-achieving, with better math self-efficacy and performance. They were more likely to be Asian, female, and slightly older, with fewer credits earned. Despite these differences, we maintained a wide representation of gender, GPA, and racial/ethnic groups. Entropy balancing resulted in a weighted dataset that closely reflected the full population of students enrolled in math courses, ensuring that our analysis would allow for inferences across the general population.

Results and Discussion

One goal of our analysis was to determine if ACI scores would predict course completion and STEM degree attainment. A one standard deviation (SD) increase in ACI score on average correlated with a 0.4 increase in grade points (scale 1-4) in the math course in which a student was enrolled ($p < 0.001$) (Table 1). This was reduced but remained highly significant ($p < 0.001$) in full models with covariates. For individual courses, ACI scores significantly predicted grades in most math classes, especially those that rely heavily on algebra: intermediate/college algebra[1], precalculus, and Calculus I ($p < 0.001$). ACI scores also significantly predicted course completion (if attempted) for each of the five core STEM major mathematics requirements, in base and full models ($\alpha = 0.001$ level for all except full model predicting Calculus II completion, where $\alpha = 0.008$). These differences were not due to correlations between ACI

[1] We were unable to include elementary algebra courses in this analysis due to small *n*; very few had grades beyond pass/fail designations.

score and currently enrolled course level, because all models included fixed effects for course level.

Table 1. ACI Score as Predictor of Math Course Grade, Completion of Core Math Courses for STEM Major, Completion of STEM vs. Non-STEM Major.

As a Predictor of Math Course Grade (Grade Scale: 1-4)								
	Base models				Full models[a]			
Fixed effects by course sequence number, regression results								
	Coefficient	SE	*p*	sig.	Coefficient	SE	*p*	sig.
ACI score	0.40	0.03	0.000	***	0.26	0.02	0.000	***
sequence	0.08	0.03	0.007	**	0.05	0.03	0.066	.
Separate models for each course type, regression results								
Coefficient for ACI score	Coefficient	SE	*p*	sig.	Coefficient	SE	*p*	
intermediate/college algebra	0.55	0.05	0.000	***	0.40	0.04	0.000	***
precalculus	0.51	0.04	0.000	***	0.36	0.04	0.000	***
Calculus I	0.46	0.09	0.000	***	0.24	0.05	0.000	***
Calculus II	0.28	0.10	0.006	**	0.06	0.11	0.600	
As a Predictor of Successful Completion of Core Math Courses in STEM Major Requirements								
Ever completed:	Coefficient	SE	*p*	sig.	Coefficient	SE	*p*	sig.
elementary algebra	0.16	0.04	0.000	***	0.15	0.05	0.001	**
interm./college algebra	0.14	0.02	0.000	***	0.11	0.02	0.000	***
precalculus	0.13	0.02	0.000	***	0.09	0.02	0.000	***
Calculus I	0.15	0.03	0.000	***	0.09	0.02	0.000	***
Calculus II	0.11	0.02	0.000	***	0.08	0.03	0.008	**
Calculus III	0.05	0.11	0.667		0.01	0.08	0.951	**
differential equations	0.62	0.38	0.105		0.90	0.33	0.006	**
mathematics education	0.38	0.08	0.000	***	0.23	0.06	0.000	***
statistics	0.34	0.05	0.000	***	0.20	0.05	0.000	***
quantitative reasoning	0.35	0.11	0.001	**	0.18	0.09	0.048	*
health	0.35	0.07	0.000	***	0.18	0.07	0.009	**
As Predictor of STEM vs. non-STEM Degree Completion								
ever attained STEM vs. non-STEM degree	Coefficient	SE	*p*	sig.	Coefficient	SE	*p*	sig.
ACI score	0.051	0.012	0.000	***	0.049	0.012	0.000	***
ACI score	0.040	0.013	0.002	**	0.040	0.013	0.002	**
course grade	0.178	0.014	0.000	***	0.188	0.015	0.000	***

Notes: Multi-Level Weighted Regression, fixed effects by course (first row) or separate models by courses (listed in first column), random effects by instructor, math course grade as dependent variable, ACI score as independent variable; *Sig. levels*: *** $\alpha = 0.001$, ** $\alpha = 0.01$, * $\alpha = 0.05$, . $\alpha = 0.10$

[a]Full models include control variables: GPA at start of term, number of prior credits earned by start of term, race/ethnicity, gender, age, and median household income of zip code

We also found that for students who completed a two- or four-year degree during the study period and who were in the same level mathematics course when taking the ACI, that those with

higher ACI scores were significantly more likely to graduate with a STEM degree. In fact, the ACI score predicted STEM degree completion more strongly than course grade alone. For every one standard deviation increase in ACI score, students were 5.1 percentage points more likely to complete a STEM degree when accounting for fixed effects for course level. In models with both ACI score and course grade, one standard deviation increase in ACI score and one letter grade increase corresponded to 4.0 and 1.8 percentage point increases in STEM degree completion, respectively; this illustrates the predictive power of ACI scores above and beyond the information provided by course grades.

Another goal of our analysis was to determine the extent to which ACI scores explain differences in outcomes by race/ethnicity and gender. For race/ethnicity, we found direct and indirect discrepancies related to course grades highly significant in both base and full models with controls (Table 2).

Table 2. ACI Score as Mediator Between Underrepresented Minority Status (URM)[a] or Gender and Math Course Grade

	Base models				Full models			
underrepresented minority status	Coef.	SE	*p*	sig.	Coef.	SE	*p*	sig.
total discrepancy (total "effect")	-0.65	0.05	0.000	***	-0.42	0.06	0.000	***
direct discrepancy (direct "effect")	-0.54	0.05	0.000	***	-0.36	0.06	0.000	***
indirect discrepancy (indirect "effect")	-0.11	0.01	0.000	***	-0.06	0.01	0.000	***
female gender	Coef.	SE	*p*	sig.	Coef.	SE	*p*	sig.
total discrepancy (total "effect")	0.24	0.04	0.000	***	0.23	0.05	0.000	***
direct discrepancy (direct "effect")	0.29	0.04	0.000	***	0.26	0.05	0.000	***
indirect discrepancy (indirect "effect")	-0.04	0.01	0.000	***	-0.03	0.01	0.001	**

Notes: Mediation using the KHB Method: Mediation of the Relationship Between Underrepresented Minority Status or Gender and Math Course Grade by ACI Score, fixed-effects by course sequence number, clustering by instructor; *Significance levels*: *** $\alpha = 0.001$, ** $\alpha = 0.01$, * $\alpha = 0.05$, . $\alpha = 0.10$

[a]Underrepresented minority indicates students of color who have been traditionally underrepresented in STEM fields, and includes Black, Hispanic, and American Indian/Native Alaskan students (vs. White and Asian/Pacific Islander students)

In Table 2 we see that ACI score accounts for 17% of the difference in course grades by race/ethnicity in base models and 13% in full models. We also found direct and indirect discrepancies related to STEM-major-math-course-sequence completion highly significant in base and full models (except full models for elementary algebra). ACI scores explain about 16-18% of the difference for intermediate/college algebra through Calculus I in base models and 7-11% in full models. Finally, for race/ethnicity, neither the total nor direct discrepancy related to STEM-vs.-non-STEM-degree completion is significant in base or full models, while the indirect discrepancy is significant. Interestingly, after accounting for the indirect relationship of ACI score as an explanatory variable for differences in STEM vs. non-STEM major attainment by race/ethnicity, underrepresented minorities are ***more*** likely to complete a STEM vs. non-STEM degree (although not statistically significantly so). Thus, ACI score explains 100% of the total discrepancy in STEM vs. non-STEM degree attainment by race/ethnicity.

For gender, exploring mediation models shows different trends for course completion. While both direct and indirect discrepancies are highly significant in base and full models, the direction of the relationship is different. Despite having lower ACI scores, women on average earn higher

grades in mathematics courses. Thus, considering a mediation model that includes ACI scores reveals a suppressor "effect" where women earn even higher grades than men with equivalent ACI scores. Looking at STEM-major-math-course-sequence completion, the models suggest a suppressor "effect" (significant for all courses except elementary algebra and Calculus II full models), where women are on average more likely to complete core mathematics course requirements for STEM majors, even though they score on average lower on the ACI. Finally, concerning degree attainment, exploring mediation models for gender shows the total, direct, and indirect discrepancies are all statistically significant, with ACI score explaining about 10% of the total discrepancy in base and full models (or 10% of differences in STEM vs. non-STEM degree attainment by gender).

Implications and Conclusion

This study found that ACI scores predict math course grades, successful completion of core math courses required for STEM majors, and STEM vs. non-STEM degree attainment. Notably, the ACI score was predictive of STEM degree completion even when controlling for course grades, showing a predictive strength comparable to or even stronger than the predictive power of course grades. In particular, a one-standard-deviation increase in ACI score was associated with a one-third to one-half letter grade improvement in courses such as algebra and calculus, an 8-16% increase in the probability of completing core math courses, and a 4-5% increase in STEM degree attainment.

While this study included a broad range of students, we note that those who completed the ACI differed slightly from the overall student population enrolled in math courses. Despite this, there was still strong representation across racial/ethnic groups, genders, GPA bands, and course levels. CUNY, where the study was conducted, is more diverse than the average U.S. college, with higher proportions of racial/ethnic minorities, first-generation students, lower SES students, and those requiring developmental coursework. While the findings may be generalizable to a national student population, CUNY's unique diversity makes it a valuable setting for studying the relationship between algebraic conceptual understanding and college math/STEM outcomes for groups that have been traditionally underrepresented in math and STEM.

These results suggest that algebraic conceptual understanding, as measured by the ACI, plays a crucial role in college math and STEM outcomes. The findings support calls to better integrate algebraic conceptual understanding into K-12 and higher education teaching and curricula. Moreover, the study highlights equity concerns, as ACI scores explained significant portions of racial/ethnic and gender disparities in math course grades, core course completion, and STEM degree attainment. Women, despite having lower ACI scores, were more likely to succeed in math courses, which may be related to other factors like dedicating more time to their studies (Wladis et al., 2021; 2024d; 2024e).

This research provides evidence that race/ethnicity and gender differences in access to high-quality K-12 math instruction likely contribute to disparities in STEM outcomes. Future research should investigate how improving algebraic conceptual understanding may enhance college math performance and STEM degree attainment, particularly for groups that have been traditionally marginalized and underrepresented in math/STEM. This study provides some of the first large-scale quantitative evidence linking algebraic conceptual understanding to STEM success, showing the need for targeted interventions to improve math outcomes and equity in STEM.

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Students' Proof Comprehension via Set-Based Reasoning in a Transition-to-Proof Course

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This study explores Transition-to-Proof (TTP) students' proof comprehension by analyzing written responses to proof intention and validation tasks. We examined students' use of set-based reasoning as they completed these tasks. Our analysis of 28 students' written work found that students used set-based tools for logic to support their reasoning. We also found that their explanations in their responses included discussion of the overall structure, individual lines or components, the mathematical veracity of the claim, and a walk through of the proof-text. We concluded that in these tasks students leveraged set-based reasoning for logic to make sense of the context and claim of the proof, supporting them in their overall proof comprehension.

Keywords: Transition to Proof, Set-Based Reasoning, Proof Comprehension

Reading, writing, and validating proofs are fundamental skills for participation in higher mathematics, so it is vital for undergraduate students to acquire these skills to learn and communicate within advanced mathematics (Tall et al., 2012). Because of this, understanding transition to proof (TTP) students' comprehension of proof, and providing interventions that meaningfully increase proof comprehension for students has been a focus within undergraduate mathematics research (Alcock & Weber, 2005; Selden & Selden, 2003; Dawkins & Zazkis, 2021; Hodds et al., 2014). These studies have demonstrated that interventions can improve students' performance on proof comprehension tasks.

Our study explicitly builds on the work seeking to improve students' proof comprehension by strengthening set-based reasoning for logic within conditional proofs (Dawkins et al., 2023; Dawkins & Roh, 2024). These works have provided evidence of components students deem important in proofs via validation tasks, along with presenting a variety of interventions tailored to this population of TTP students.

Building on the work of Dawkins et al. (2024), we explore how set-based reasoning for logic was evidenced in a class of students' written work on proof intention and validation tasks. While this study shares similarities with existing research on proof validation and set-based reasoning for logic within proof, we focus on analyzing the written artifacts of a TTP class rather than individual interviews or group discussions. In this study, we sought to answer the question: In what ways is set-based reasoning present in transition to proof students' proving activities?

Theoretical Framework and Related Literature

As we focused on student attention in their justifications surrounding proof validity and intention, we adapted existing research in the field and identified three key constructs for this study: *Overall Structure*, *Line-by-Line*, and *Mathematical Veracity*. *Overall Structure* refers to the structure or form of a proof (Selden & Selden, 1995, 2003; Tucci et al., 2023); Mejia-Ramos et al. (2012) emphasize this as a component of holistic proof comprehension. We take *Line-by-Line* to mean student attention related to local aspects of the proof text within their justifications surrounding proof activities, including meanings of terms and individual lines within the proof.

This includes what the existing literature has described as surface features (Selden & Selden, 2003), warrants between lines of a proof-text (Alcock & Weber, 2005), and comprehension of local aspects of a proof (Mejia-Ramos et al., 2012). By *Mathematical Veracity*, we refer to the degree to which students consider a mathematical statement to be proven as true. This concept is characterized by Tucci et al. (2023) as the alignment of the theorem, the proof, and the students' own understanding.

Proof Validation and Intention

Proof comprehension is complex (Mejia-Ramos et al., 2012), so we focus on two task types that provide connected but distinct insights into proof comprehension: Proof Validation and Proof Intention. Proof Validation, determining if a proof text is valid or invalid (e.g., Alcock & Weber, 2005), is a core component of proof comprehension. However, students may not necessarily consider what the proof intends to establish, or determine the Proof Intention (e.g., Roh & Lee, 2024; Tucci et al., 2023). As such, a validation task might overlook the importance of intention unless it specifically asks the student to articulate what the proof sets out to prove. Understanding proof intention involves recognizing the overall structure of the proof and how each part (line-by-line) contributes to what the proof is trying to establish (mathematical veracity). Because of this interrelated nature, both proof validation and proof intention are valuable for gaining insights into how students comprehend proofs.

Set-Based Reasoning for Logic

Finally, this work builds on prior work about set-based reasoning. Set-based reasoning is an approach to logic leveraging sets and set relations as a core tool (Hub & Dawkins, 2018) that has been applied to logic instruction in the TTP environment. To provide context for students' demonstrated proficiency in set-based reasoning, we adopted the codebook and approach of Dawkins et al. (2024), considering normativity and internal consistency of student work in tasks using multiple logical registers (diagrams, set relations, conditionals, etc.). We also developed a new coding set related to student annotations present in the margins, including their logic or set-based understandings of a proof task, detailed in the methodology section.

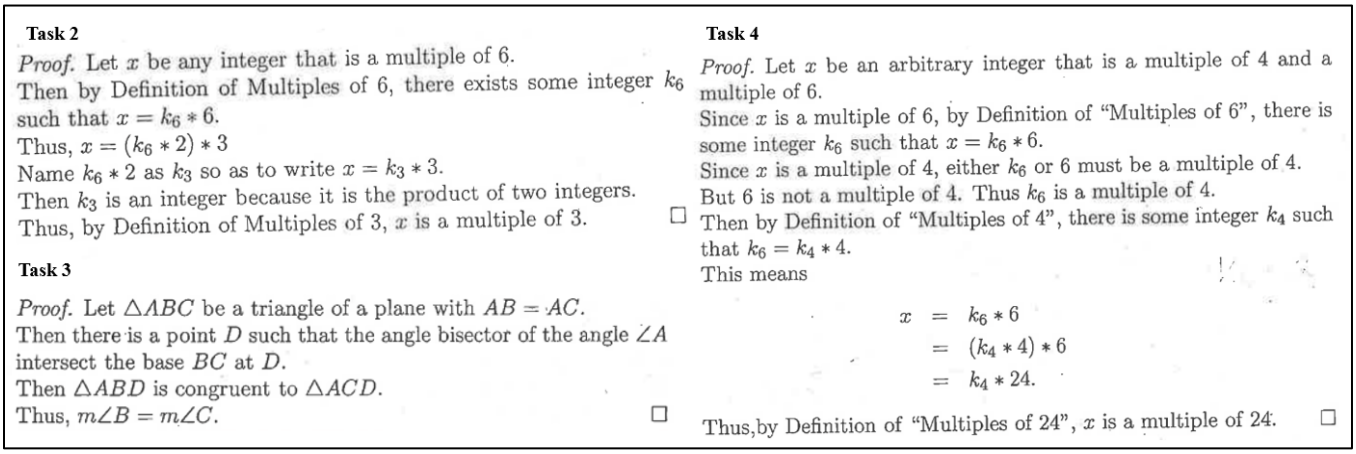
Methodology

Data Collection

This study is part of a more extensive project (NSF DUE #2141925) aiming at curriculum development for transition-to-proof courses at the undergraduate level. The larger project intends to implement a series of interventions, including set-based reasoning for logic, to support novice provers in their proof comprehension. This paper specifically analyzed part of the data collected from the project's second phase, which implemented the RAMP curricular materials in the classroom at a large Southwestern university.

In order to focus on students' use of set-based reasoning in proof comprehension, we chose tasks in homework and exams that contained problems centered on set-based reasoning, proof validation, and proof intention. We considered the written work of 28 students, and this analysis focused on the midterm, which was the first formal assessment that included proof tasks in the course. The midterm included three multipart tasks that asked for both proof intention and proof validation for provided proof texts with varied contexts and varied normative sufficiency (See Figure 1 below). This midterm occurred in week 7, after one unit of set-based reasoning for logic (weeks 1-3) and one unit of introduction to conditional proofs (weeks 4-6). To provide

context for the set-based reasoning of students, we also analyzed two preceding homework assignments from week 4 and week 6.



Task 2

Proof. Let x be any integer that is a multiple of 6.
Then by Definition of Multiples of 6, there exists some integer k_6 such that $x = k_6 * 6$.
Thus, $x = (k_6 * 2) * 3$
Name $k_6 * 2$ as k_3 so as to write $x = k_3 * 3$.
Then k_3 is an integer because it is the product of two integers.
Thus, by Definition of Multiples of 3, x is a multiple of 3. □

Task 3

Proof. Let $\triangle ABC$ be a triangle of a plane with $AB = AC$.
Then there is a point D such that the angle bisector of the angle $\angle A$ intersect the base BC at D.
Then $\triangle ABD$ is congruent to $\triangle ACD$.
Thus, $m\angle B = m\angle C$. □

Task 4

Proof. Let x be an arbitrary integer that is a multiple of 4 and a multiple of 6.
Since x is a multiple of 6, by Definition of "Multiples of 6", there is some integer k_6 such that $x = k_6 * 6$.
Since x is a multiple of 4, either k_6 or 6 must be a multiple of 4.
But 6 is not a multiple of 4. Thus k_6 is a multiple of 4.
Then by Definition of "Multiples of 4", there is some integer k_4 such that $k_6 = k_4 * 4$.
This means

$$\begin{aligned} x &= k_6 * 6 \\ &= (k_4 * 4) * 6 \\ &= k_4 * 24. \end{aligned}$$

Thus,by Definition of "Multiples of 24", x is a multiple of 24. □

Figure 1: Proof Texts Responded to in Midterm Tasks

Data Analysis

We began by developing an initial codebook for attention in proof validation explanations based on the existing literature. The next phase of the data analysis was to refine and finalize the codebooks by adding new codes for the data that did not fit into the initial codebook and grouping the codes into four categories. Through this process, we developed a new code for proof validation emergent from our data*, walk-through*, which will be detailed in the results section. In the final phase, we finalized the codebook (see Table 1).

Table 1: Finalized codebook for student explanation in verification/intention tasks

Code	Definition
Line-by-line	Discussion of strengths or weaknesses in the line-by-line argument (e.g., use of definitions, logical or mathematical steps, or arguments or reasoning connecting individual lines of the proof) or show overall concern for the process of the proof argument
Overall Structure	Discussion of the structure or framework of the proof, including explicit attention to the first and last lines of the proof as significant and explicitly stating proof structure (i.e., direct proof, proof by contradiction)
Mathematical Veracity	Discussion of the mathematical correctness of the statement being proven or provides a counterexample demonstrating mathematical falsehood
Walk-through	Evaluation includes a retelling of the proof logic, without other comment or evaluation, a walk-through of the steps taken by the proof-text author

In our initial pass of the data, we also noticed that many students had used the margins of the midterm exam to make notes about the proof validation tasks. We developed a codebook for these notes using open and axial coding (Strauss & Corbin, 1998), as seen in Table 2.

Results

Through this study, we characterized student comprehension of proof-text tasks by considering the written explanations they provided in their responses. We also saw students engage with set-based reasoning in support of their proof comprehension in these tasks. We

reflect on the apparent effectiveness of the approaches and variation in methods between the three different presented tasks.

Table 2: Margin Evidence Codebook

Code	Definition
Logical Statements	Annotations that are related to logical reasoning (i.e., logic statement types, generating a conditional statement, or identifying logical equivalence of multiple statements)
Set Based Reasoning	Annotations related to sets and set-based reasoning (i.e., definitions of sets, Euler diagrams, relations between sets)
Examples	Concrete examples or cases considered, such as diagram drawings or arithmetic with example values
Truth Value	Explicit annotation of provided potential conditional as mathematically true or false

Attention in Proof Comprehension

Through iterative refinement of our codebook, we identified four components of the proof text that students attended to when explaining their reasoning in these tasks. These categories are based on student written responses to the prompt "Evaluate if the proof-text properly proves what it attempts to prove. Explain why or why not." These categories are not mutually exclusive; students' written responses often contain multiple parts or arguments, so they often attend to multiple components in the same response.

Overall structure. Many responses contained explicit references to the overall structure of the proof-text. For example, some responses referenced common types of proofs (e.g., proof by contrapositive), content recently introduced in the course. For example, Bailey wrote,

> The proof-text properly proves that for any integer x, if x is a multiple of 6, then x is a multiple of 3. This is a direct proof. (iii) is the contrapositive of the direct proof and would result in the same truth table. However, the proof text does not use the contrapositive format.

Others showed awareness of the overall structure by commenting on the first and last lines of the proof, a technique introduced in a prior curricular activity. In Task 2, Adam wrote:

> The proof text is attempting to prove that for all integers x, if x is a multiple of 6, then x is a multiple of 3. The proof text does indeed prove this properly, as it starts by establishing that x is any integer that is a multiple of 6, breaks it down with the prime factorization, and ultimately concludes with x being shown to be a multiple of 3.

These were not isolated instances. In Task 2, 15/26 responded to the overall structure in their response. This was not consistent across all three tasks, however. 8/26 cited the overall structure in their written explanation for Task 3, and 10/25 cited it in Task 4.

Line-by-line. Students also provided arguments based on surface features or individual components of the proof, which we coded as line-by-line arguments. For example, Tucker's response to Task 3 reads,

> The proof-text does not properly prove what it attempts to because there is a gap in justification between line 2 and line 3. The proof-text does not point out the consequence of the angle bisector and quickly jumps to line (3).

Arguments in this category were cited as justification for the proof's validity. For example, Evan's response to Task 2 cited the internal components of the proof as the reason the proof

proves what it attempts to, saying: "Yes, because it logically states and justifies each step using either algebra or a definition that ultimately arrives at the desired conclusion."

Overall, 13/26 responses to Task 2, 19/26 to Task 3, and 10/25 to Task 4 included a line-by-line argument. Some responses included both line-by-line and overall structure components; 14/77 responses were coded as both. This shows that students attending to overall structure do not necessarily cease to use line-by-line reasoning in their validation explanations.

Mathematical veracity. Some explanations demonstrated attention to the correctness of the statement being proven, which we called mathematical veracity of their answer a claim. For example, John response reads:

> The proof tries to prove that multiples of 4 and 6 are always multiples of 24 but doesn't do it properly because they neglected the case where multiples of 4 and 6 are also every other multiple of 12 as well.

In Tasks 2 and 3, with 2/26 and 1/26 responses coded as attending to mathematical veracity, and in Task 4 8/25 responses cited mathematical veracity. This is not surprising; compared to the previous two categories, mathematical veracity doesn't directly connect to a recent curricular activity. Fewer responses coded as citing mathematical veracity likely do not have to do with what students think is important but rather what recent course activities have primed them to include in responses to proof-text tasks.

Walk-through. While most student explanations analyzed contained arguments we expected to see based on previous literature, we also noticed a new focus. Some responses included a retelling of the proof itself, or a walk-through of the reasoning of the proof as justification for the proof properly proving what it set out to prove. For example, Ben's response to Task 3 contains a walk-through of the proof-text:

> The proof-text attempt to prove that for any triangle ABC, if $AB = AC$, then $m\angle B = m\angle C$. The prove does not fully prove the prove properly. The statement "Then triangle ABD is congruent to triangle ACD" is not properly justified. The proof start state how the point D from the angle bisector creates line AD. Then, since $AB = AC$, $\angle CAD$, and $\angle DAB$ are equivalent by definition of angle bisector, and $AD = AD$ triangle ABD is congruent to triangle ACD. The proof is mostly correct and just would benefit from some clarity.

In most cases the walk-through was included along with another category of explanation. Walk-through was the only code applied to a written response only 4/19 of the times it was used.

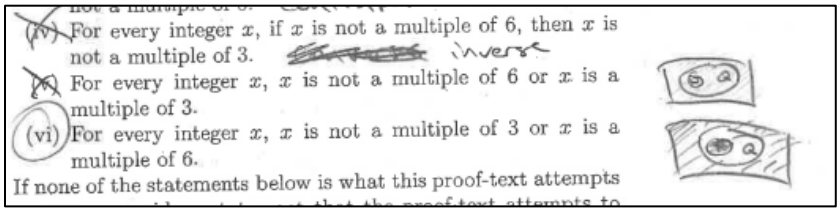


Figure 2: Example of student set-based margin annotations

Engagement in Set-Based Reasoning

In the process of our analysis, we observed margin annotations present (see Figure 2) on all three tasks, including a variety of annotations, some of which were set-based reasoning tools that had been presented earlier in the course. We had observed general proficiency, though not without error, with set-based reasoning for logic in the work of this same class on prior homework assignments, using the same approach and codebook as Dawkins et al. (2024), but

due to space constraints, we focus only on the set-based tools students used on these midterm tasks. However, it was because of this context that we attended to these margin annotations.

Evidence of set-based reasoning in proof comprehension contexts. Since the midterm questions did not require students to show their work in the question stem, we can conclude that these annotations were added because they were of value to the student in their problem-solving process on the task. Based on these observations, we developed the codebook in Table 3. These margin annotations demonstrate that students found set-based tools for logic useful as they solved proof intention and verification tasks (see Table 2).

Table 3. Number rates of occurrence in margins of midterm tasks

Task	Set-based	Logic	Example	Truth
2	12/26	14/26	1/26	1/26
3	4/26	6/26	21/26	2/26
4	3/25	12/25	4/25	1/25

The margin work also indicated attention to the logical reasoning skills previously introduced as part of the RAMP curriculum, showing student uptake of those concepts and skills. This is unsurprising, as these concepts were freshly introduced and directly associated with proofs in the way they were presented. However, this is also some indication of adoption, as students found these concepts valuable enough to make personal notes to support their proof comprehension on these tasks.

Diversity of use amongst tasks. The margin annotations were not uniformly present across the three tasks, as evidenced in Table 3. Two components likely contributed to this. First, the mathematical context of the task seemed to be associated with the margin notes used; specifically, the use of examples was much higher on Task 3, the only geometry-based task. It seems likely to us that students could run examples of the divisibility context in their mind's eye but that the more spatially challenging context of geometry led to the higher use of examples as a written aid. Also, set-based tools appeared at a much higher frequency on Task 2 than on Tasks 3 and 4. However, all students who used set-based tools on Task 3 or Task 4 had also used them on Task 2. This suggests that because of the similar setups of these three problems, some students used set-based tools to help them conceptualize the first task but then transferred their conclusions to the next two questions that were of a similar form, without a need to reconstruct the annotations they had previously made. Also, the question directly preceding Task 2 on the assessment asked about equivalent sets; it is possible that this prior task primed set-based thinking in the students. However, we posit that the students would not have used set-based annotations if they found them fully irrelevant to the task at hand.

Our analysis of the students' work on these tasks, in light of our observations of their work on prior set-based homework, suggests that while set-based reasoning might not be trivial to the students, it was accessible to them. In particular, the students' use of set diagrams, notations, and terminologies in their margin scratch work indicates that students found this set-based reasoning approach beneficial for reading and comprehension of proof texts on these types of tasks.

Normative Proof Validation

While we avoid making statistical claims past describing the proportion of the responses that fit in each category, as this was not a representative sample, we did see patterns in response categories of attention, margin note types, and the normativity of the response to the question of

the proof text's properness, which we considered a proof validation task. These observations, we believe, provide insights as to how these techniques connect to proof comprehension broadly.

Overall structure and mathematical veracity in validation explanations. Student attention to overall structure in their explanation of reasoning did not consistently co-occur with a normative evaluation of the proof text in these three tasks. In Task 2, all 15 responses citing overall structure correctly validated the proof as proper, and 8/9 overall structure responses validated the proof as proper in Task 3. However, Task 4 tells a different story. 9/10 responses coded for overall structure declared the proof "proper" even though its claim is mathematically false. This suggests that the responses that attended to overall structure, while relatively successful at the proof intention component of the task, were not consistently normatively validating the conditional they successfully identified.

In Task 4, many responses that correctly identified the proof as improper were coded as mathematical veracity, (see Table 1 for definition). For example, John responded:

> The proof tries to prove that multiples of 4 and 6 are always multiples of 24 but doesn't do it properly because they neglected the case where multiples of 4 and 6 are also every other multiple of 12 as well.

In Task 4, 8/25 responses cited mathematical veracity, among which 7/8 correctly declared the proof improper in proving what it attempts to prove. While attention to overall structure in the response seemed to help students complete proof intention tasks normatively, mathematical veracity most frequently co-occurred with a normative declaration of a proof as invalid.

Conclusion/Discussion

Through our analysis of these two interrelated proof comprehension task types, proof intention and proof validation, we found evidence of students' use of set-based reasoning tools from their TTP course. Students used overall structure arguments in their explanations of proof validation and included components of set-based reasoning and logical structures in their margin work on the tasks. Because margin work was not a required or graded component of the assessment, we take this to mean that the students found these tools relevant to the presented proof comprehension tasks. It is worth noting that these set-based tools were not self-emergent; the intention of the course was to teach intro to proofs using set-based reasoning for logic.

Many of the aspects of proof we saw students attend to were consistent with those described in prior work, including student focus on proof structures (Selden & Selden, 2003; Tucci et al., 2023), surface feature or line-by-line reading (Selden & Selden, 2003; Dawkins & Zazkis, 2021), and a focus on mathematical veracity of the claim made (Tucci et al., 2023). This demonstrates that these categories continue to be evidenced in full class as demonstrated on written proof comprehension tasks. However, we did observe a new component included in the written explanations, that of a walk-through of the proof text. Though this was a small set of tasks, it does seem from these written responses that walk-through occurs as a way for students to say it's right because it's right, since among these artifacts, a claim of improperness tended to include identifying a failure within the proof, leading to a code of line-by-line instead. It is also possible that this retelling action is a byproduct of the novelty of the context and the task; students may be retelling themselves the argument of the proof as they work to explain their reasoning. However, this is a conjecture, and so we believe that more investigation into this novel code of walk-through as an explanation is warranted.

Acknowledgments

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The Impact of an Upper Division Inquiry-Oriented Content Course on Prospective Teachers: Embracing a Critical Stance

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In this study, we analyze interviews with prospective secondary school teachers focused on how an upper division inquiry-oriented content course influenced their beliefs about learning and teaching mathematics. In particular, we examine how students describe their experiences in this course and the extent to which (and why) students see themselves using an inquiry-oriented instructional approach in their future teaching. Using thematic analysis, we found that the ways prospective teachers described their experiences in the course aligned closely with the four pillars of Inquiry-Based Mathematics Education, and the course empowered students to embrace a critical stance to self-reflect on their experiences and to raise their awareness of their agency to change things (for the better) through their own actions and future forms of instruction.

Keywords: critical stance, inquiry, prospective teachers, beliefs about teaching

The relevance and usefulness of upper division mathematics courses for prospective secondary school mathematics teachers has long been of concern (Begle, 1972; Klein, 1932; 2016; Wasserman et al., 2019). A number of studies document that teachers find their advanced mathematics courses have little relevance to their future professional life (e.g., Cofer, 2015; Wasserman, 2017, Zazkis & Leikin, 2010). One approach to addressing this issue, and the one taken in our larger study, is to redesign upper division math content courses to make intentionally strong connections to high school mathematics content and teaching.

A recent issue of *ZDM Mathematics Education* has made progress on the discontinuity (Klein, 1932) between university and secondary school by examining how the intersectional nature of mathematics educational content might be addressed in a wide range of university courses in order to prepare better secondary mathematics teachers (Wasserman et al., 2023). For example, Buchbinder and McCrone (2023) focused on bridging the disciplinary practice of proving and the teaching of secondary mathematics. In another paper in this special issue, one that is related to our larger project, Apkarian and colleagues (2023) investigated the impact of an upper division mathematics course on prospective teachers' knowledge of rate of change and their shifting beliefs about mathematics learning and teaching. In other related prior work that investigated the impact of this same course, Goodchild and colleagues (2021) took a community of practice perspective to document the adoption by these prospective teachers of a *critical stance* toward learning and teaching mathematics. In this report, we further this line of inquiry by examining prospective secondary school teacher experiences in an inquiry-oriented upper division math course and the extent to which they see themselves enacting an inquiry-oriented instructional approach in their future teaching. In particular, we address the following two research questions: *(1) How do students describe their experiences in an upper division math course and how are these experiences similar to or different from their previous college and high school math experiences?* and *(2) To what extent do students see their future selves using an*

inquiry-oriented instructional approach and what are their reasons for doing so (or not doing so)?

The work reported here is part of a larger NSF-funded project that is investigating the impact of an upper division mathematics course, Inquiry-Oriented Dynamical Systems and Modeling (IODSM), on prospective teachers' knowledge of content connected to high school mathematics, the ways in which they engage in the eight Standards for Mathematical Practice (Common Core State Standards Initiative, 2010), their beliefs about learning and teaching mathematics, and contributions to their emerging professional practices.

Theoretical Framework

As alluded to in the introduction, two central and related ideas to our line of investigation are inquiry-oriented instruction and critical stance. The upper division math course that we are redesigning for prospective teachers embraces and reflects the four pillars of inquiry-based mathematics education (IBME): (1) Students engage deeply with coherent and meaningful mathematical tasks, (2) Students collaboratively process mathematical ideas, (3) Instructors inquire into student thinking, and (4) Instructors foster equity in their design and facilitation choices (Laursen & Rasmussen, 2019). Our creation of coherent and meaningful tasks where students reinvent mathematics is guided by the instructional design theory of Realistic Mathematics Education (Freudenthal, 1991; Gravemeijer, 1999) and our perspective on the ways in which students and instructors inquire into others' thinking and reasoning is informed by the constructs of social and sociomathematical norms (Yackel & Cobb, 1996). In particular, the instructor sought to create a classroom environment that nurtured the following social norms: students routinely explain their thinking, however tentative, students listen to and try to make sense of others' reasoning, and indicate agreement or disagreement with reasons. In terms of sociomathematical norms, the instructor sought to foster student explanations grounded in mathematical relationships and meaning, as opposed to procedural accounts of their solution.

For his part, the instructor sought to foster the two student pillars by valuing and bringing out student ideas and then leveraging those ideas to move forward the mathematical agenda (Kuster et al., 2018; Rasmussen & Marrongelle, 2006). The instructor's attention to equity included specific teaching choices such as attending to group composition, explicitly crediting ideas to student authors to foster intellectual ownership, and regularly collecting and leveraging student reflections on each class session. Another choice made by the instructor was to assign rotating group roles. For example, one of the assigned roles was a Reporter who reports back to the whole class about their group's progress which contributes to the instructor's efforts to ensure equitable participation and minimize biases when asking groups to report out (Reinholz, 2023).

The four pillars of inquiry-based mathematics education are foundational for students to develop a *critical stance*. Our definition of critical stance draws on that put forth by Curzon-Hobson (2003, p. 201): A *critical stance* is "an attitude or disposition towards oneself, others and the object of inquiry that challenges and impels learners to reflect, understand and act in the milieu of potentiality." A critical stance therefore results in an awareness of and reflection on one's own experience, meanings, and knowing. In prior related work, Goodchild and colleagues (2021) distilled this characterization of critical stance into three key features: First, students must be aware of the possibility that their mathematical learning experience could be different, for this they need to experience variety in the way mathematical learning experiences are designed and implemented by the instructor. Second, students must be able to reach some judgment, based on their reflection and evaluation of their experience so that they might recognize when one experience of meaning, learning or teaching is better, deeper, or richer than another. Third,

students need to be aware of their agency to change things (for the better) through their own actions and future instructional practice as a secondary school teacher.

Methods

This report focuses on data collected during the Spring 2023 semester. The goal of this pilot study was to begin modifying an existing differential equations curriculum (Rasmussen et al., 2018) so that it explicitly deals with high school mathematics content and teaching and to investigate the impact of this course on prospective teachers' knowledge, beliefs, and their participation in the Standards for Mathematical Practices (Cruz et al., 2024).

We conducted semi-structured individual interviews with nine students from a class of 30 at a large, Hispanic-serving institution in the southwestern United States. Of the nine volunteers, five participants identified as women and four identified as men. One of the authors was the instructor of the class and they did not know who participated in the interviews until after the interviews and course were completed. In this report, we focus on interview questions related to their previous mathematics experiences and beliefs about learning and teaching mathematics as it relates to having a critical stance and their future professional practice. The relevant interview questions were:

- Have you had other math courses taught in a way similar to this class?
- Do you imagine yourself using this kind of teaching in your future practice? If so, how? If not, why not?
- In what ways, if any, have your experiences in this class affected your beliefs about teaching mathematics?
- Are there ways of experiencing or doing math in this class that have been powerful for your learning?
- How do these ways of experiencing or doing mathematics compare to your other undergraduate math courses? To your high school math courses?
- How about ways in which these ways of doing math in this class have affected your beliefs about what it means to learn mathematics?

The interviews took place in person with one of two interviewers participating over Zoom. Interviews were transcribed and anonymized. All names we include here are pseudonyms.

We first read the transcripts to familiarize ourselves with the data. We then used thematic analysis (Braun & Clarke, 2006) to identify, analyze, and interpret patterns within the data. During the initial coding process, we open coded the data with some a priori attention toward the components of a critical stance (awareness, self-evaluation, and agency). We coded in pairs and went through several rounds of reading and updating our codes until a 100% agreement was reached. During axial coding, we grouped codes according to the following five themes: (1) students' awareness of a different experience, (2) intellectual responsibility lies with students, (3) the role of social interactions, (4) instructor actions and decisions, and (5) course and facilitation design choices with an eye towards equity. In further axial coding, we related these themes to our theoretical perspectives of inquiry and critical stance.

Results

Research Question 1. Recall that our first research question is, *How do students describe their experiences in an upper division math course and how are these experiences similar to or different from their previous college and high school math experiences?* Our analysis of the interview data revealed that the pedagogical approaches experienced by students in the IODSM class are very different from their collective prior mathematics experiences. Seven of the nine

students indicated that their previous mathematics, high school or lower division mathematics undergraduate classes, were all or mostly lecture-based where they engaged in "mindless copying", "repetitive practice" and it was all very "fast paced" to keep up with the instructors. The newness of a different mathematical learning experience is addressed in the first of our five themes.

Theme 1: Students' awareness of a different experience. Recall one component of a critical stance is awareness: students must be aware of the possibility that their experience of the practice could be different. Three students in the IODSM course indicated that they were unaware that math could even be taught like this. For example, Chepi said:

> I never thought that a math class could be like this where you're just figuring it out for yourself and trying to make sense of it yourself with other people. You're getting new perspectives rather than your own, because sometimes you could think of things one way. There's no right way to do math, as we've seen and in a class like this, I've seen people do some weird things that you would never think for yourself and it works.

The remaining four themes further flesh out the various ways in which IBME offers a different, and, from students' perspective, more meaningful learning experience.

Theme 2: Intellectual responsibility lies with students. For the second theme students indicated overwhelmingly that they could develop mathematical ideas for themselves, with minimal input from the instructor. For example, George said, "I think that [the] process of having the students really dig into the material with minimal assistance from the instructor is very good." Students also reported feeling confident in reinventing mathematics where they have ownership of ideas. Here, the way this manifested for students was the importance of having intellectual responsibility for their own learning. For example, Hakeem said:

> Because, you're creating something on your own and obviously you're gonna take pride in that. When you have something that you created, you're very proud of it and you're very defensive of it also. So if someone says that you're wrong, you're definitely gonna want to counter argue that and defend yourself. And then they're gonna also, and it's that back and forth, that dialogue is where new ideas and true learning happens.

This theme is strongly connected to IBME Pillar 1, Students engage deeply with coherent and meaningful tasks, because of the intellectual ownership and need to "really dig into the material."

Theme 3: The role of social interactions. Another theme that emerged was the role of social interactions and the power of collaborative learning. The statements coded in this grouping were attributed to eight of the nine students. This theme resonates with the second IBME pillar, Students collaboratively process mathematical ideas, because students positively reflected on the importance of group work, group roles, and the opportunities for student-to-student discussion. For example, the collaborative processing of ideas is evident in Hakeem's reflection when he said "it's that back and forth, that dialogue is where new ideas and true learning happens." Hakeem also emphasized the value and benefit of students being the primary source of ideas:

> I think it's better to have those small groups and be able to talk to them about your understanding of where you're struggling with. I think that's way more beneficial and it just brings more comfort to the student with the material. And then eventually, I'm sure you could see it too, as the semester progresses, more students are being able to talk or come out on their own and being able to share their ideas. Whereas at the beginning of

> the semester, there's only probably like three people that would consistently be asking questions, consistently be sharing. So I definitely think having inquiry-based small group discussions where the lecturer is only there as a resource, not necessarily the source, is the best way to learn.

Theme 4: Instructor actions and decisions. Five students' descriptions of the IODSM course highlighted the instructor choices about the course design that indicated purposeful instructor inquiry into student thinking. For example, Jackie said the following about the end-of-class reflections: "...I think [it] is really nice that we get to see other students thinking in their writing as opposed to just presenting in class", whereas Hakeem commented on how he used the reflections: "I went back on my reflections and on the entire class's too, because [the instructor] posts them… gathering from their knowledge, adding it to my own and making it my own." We see this theme as implicitly aligning with the third IBME pillar, Instructors inquire into student thinking, because the instructor regularly used student thinking from the end of class student reflections to jump start the next lesson (i.e., move forward the mathematical agenda).

Theme 5: Course and facilitation design choices with an eye towards equity. Four of the nine students spontaneously commented that their new experiences in the IODSM course contributed to equitable learning experiences. It is notable that the interview questions did not directly ask about equity and hence student reflections related to this theme emerged organically from the interview data. For instance, Francis realized the facilitation effort that the instructor made to promote group diversity during collaborative working time:

> …I like how, as a student, you get to experience it. So I would want to give my students the same experience just because, like giving diversity, how he [the instructor] puts boys and girls in groups… So it's not just a boys club or a girls club. It's more of everyone just collaborating, working together, getting to know each other… It's not just like, oh, I don't know that person. That's more like, oh hey, how's it going?

Similarly, Jackie observed the impact of the course design in which each student in the group work has a role where they said, "We're in groups and each group member has a different role to help, I guess, function and make sure that everyone has a voice in their group." Gabby also addressed the value of group work when she said, "My favorite part of the class is working together as a class, where like we'll have our time to do our own group work and either some groups are excelling, some are struggling and everyone's kind of trying to keep each other in check." This theme aligns with IBME Pillar 4, Instructors foster equity in their design and facilitation choices, as the group work and group roles allowed for more equitable participation.

Research Question 2. We next address our second research question, *To what extent do students see their future selves using an inquiry-oriented instructional approach and what are their reasons for doing so (or not doing so)?* All nine students answered positively that they would like to use the inquiry-oriented instructional approach in their future practice. For example, Diem said "I like it… I would like to do the same format to teach in the future." Breanne similarly commented "I would want to… And the teacher that I've been shadowing in a high school, she kind of does similar things... everybody sits in little groups and like group presentations… but a little bit dialed down."

Three of the nine students saw themselves implementing similar instructional approaches, but with caveats that the "high school curriculum is packed" and "may not fit the school model." We view these awarenesses of the potential secondary school context as an indication of their critical stance, since they evaluated the practicability of the inquiry-oriented approach in a high school

setting. This shows their agency, transitioning beyond a learner perspective to a teacher perspective in their future professional life. For example, Gabby envisioned that:

> I would really like to. I know it takes a lot more time to do, and that's one of the bigger issues of doing this kind of thing in the classroom and just like the rest of the school not being set up this way or not being supported to set up this way. But ideally, yes. Even if I'm not able to fully do it, I have already started thinking about how I wanna do little things that we do in this classroom and incorporate that into a more traditional [classroom]…

Note that Gabby was explicitly aware that she might take a position in high school where "the rest of the school not being set up this way or not being supported to set up this way." She also started to brainstorm what she should change to integrate inquiry-oriented learning into her future teaching. Similarly, George embraced a critical stance by anticipating similar challenges. In particular, in the following excerpt, he recognized the trade off between covering a lot of material and the depth of understanding:

> But you have to think about it too in the sense that if you do try to fit in more material, it's less likely that the students will have as good of an understanding on the material that they can cover. So you kind of have to weigh how well these students really need to understand a concept versus how much material do they need to know.

These examples highlight how prospective teachers exercised their agency by hypothesizing the balance of time that an inquiry-oriented approach might require in their future teaching.

For the majority of students who see their future selves using an inquiry-oriented instructional approach to a great extent, we observed similar themes in their reasonings aligned with the pillars of IBME. For example, Chepi recognized the benefits and appreciated the IODSM course's collaborative thinking activities to "help kids learn to think for themselves rather than to be taught cookbook how to do anything in math" (Pillars 1 and 2). Many students, like Chepi, reflected on and evaluated their IODSM experiences to reason why they want to use this teaching from the perspective of learners. For example, George said,

> I think that the typical lecture while students just kind of watch model of teaching is outdated and not very beneficial to the students. So this class is just so reinforcing and I think it's very helpful to the students to acquire these ideas on our own and like really work our way through the concepts.

The students again embraced the agency to make the judgmental call that the intellectual responsibility of understanding and sense-making mathematics should lie within learners instead of teachers telling students what to do and how to think. This finding re-emphasizes the themes of students' awareness (Theme 1) concerning students' intellectual responsibility for learning (Theme 2).

Another reason why students favored inquiry teaching and learning is that the experience is "more class engaged, student engaged, rather than just everybody listening to one person talking and writing on the board" (Breanne). Christiano also elaborated on the engaging aspect stating that "it encourages my desire to teach. It makes me excited, because there's definitely other math classes that I've been in, and I definitely see, I think to myself, I don't want to teach like this and in this class… it kind of solidifies my desire." Breanne commented that learning from classmates with group work and the whole class was a "productive struggle" which helped her "retain

information better." In addition, students were well aware and highlighted the collaborative, safe, open, and supportive learning environment— "everyone kind of knows each other, so it's kinda open. No one's too shy to speak up" (Francis) and "I think that's way more beneficial and it just brings more comfort to the student with the material. And…. as the semester progresses, more students are being able to talk or come out on their own and being able to share their ideas" (Hakeem).

Discussion and Conclusion

Overall, results indicate that students' prior mathematics experiences were more different than similar to the pedagogical approach taken in the IODSM course. It is noteworthy that the pedagogical approach they experienced provided opportunities for them to increase their awareness that teaching and learning can be different, in positive ways, from the majority of their prior experience. Moreover, how the students embraced aspects of the pedagogical approach all align with the pillars of IBME. It should be noted that the pillars of IBME were in the mind of the instructor when designing the course structure, but these pillars were not explicitly discussed with students or brought to the attention of the students. Thus, their collective responses in the interviews about their intellectual responsibility, their collaboration with their peers, and their instructors' role in their learning and his attention to equity, all indicate that the IBME pillars were realized in this course.

In this report, we provided evidence that students embraced a critical stance—they were aware of the new mathematical experience with the IODSM course as compared to their previous experiences, were inspired to evaluate the experiences as a learner and a teacher, and recognized their agency to make changes for the better in their future teaching. Drawing from their experiences with the IBME-inspired instructional approach in the IODSM course, students reasoned positively about why they would see their future selves implement an inquiry-oriented instructional approach and the value that such an approach has for learners. Even if they had caveats, students explained their concerns and further proposed how they would modify the inquiry-oriented approach they experienced to adapt to their imaged secondary school settings. Further, student reflections on their experiences in this enactment of an inquiry-oriented course effectively empowered and fostered their embracement of a critical stance. Students were aware of the new mathematical experience, reflected on and evaluated whether the IBME experiences were more profound, and used their agency to imagine enacting an inquiry-oriented instructional approach in their future teaching. Our results provide an existence proof that, in this case, the particular enactment of IBME supported and nurtured a critical stance.

Going forward, in the context of the larger NSF study, we will go beyond interview data and collect classroom video data from other institutions with different instructors and contexts so that we can explore the potential for other IODSM class structures and specific teacher moves and decisions to positively affect future teachers' beliefs about mathematics learning and teaching and their embracement of a critical stance. The larger study also aims to understand how prospective teachers' knowledge of content related to high school mathematics deepens, the ways in which they engage in the eight Standards for Mathematical Practice (Common Core State Standards Initiative, 2010), and how approximations to their professional practices emerge and grow (Llinares, 2013).

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An Exploration of Productive Student Reasoning About Equivalence Relations

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Equivalence relations are a foundational concept in mathematics, yet productive student reasoning about equivalence relations remains largely understudied. Based on the descriptions of productive reasoning about equivalence that we have encountered in the literature, and the results of the data we collected from task-based interviews with advanced mathematics students, we argue that an essential aspect of productive reasoning with equivalence relations is the flexibility to transition between an element-wise, local perspective of the equivalence relation and an equivalence class level, global perspective of the equivalence relation.

Keywords: Equivalence Relations, Equivalence, Local Perspective, Global Perspective

Introduction

Equivalence relations are one of the most foundational concepts across the entire mathematics curriculum, ranging from numerical equality in early childhood (e.g., McNeil, 2014) to abstract algebra at the advanced undergraduate level (e.g., Cook et al., 2022). Additionally, as a prominent feature of many introduction-to-proof courses, various textbook authors describe equivalence relations as "a surprisingly powerful tool" (Dumas & McCarthy, 2007, p.59) "particularly valuable" (Smith et al., 1986, p. 145), "quite significant" (Hammack, 2013, p.184), and "extremely important" (Roman, 2006, p. 141). Yet despite their centrality in mathematics, student reasoning about equivalence relations, and in particular *productive* student reasoning about equivalence relations, remains largely understudied. In this report, we will synthesize the literature on students' reasoning about equivalence relations and use task-based interviews with an advanced mathematics student to provide an answer the question: What does productive reasoning about equivalence relations entail?

Based on the descriptions of productive reasoning about equivalence that we have encountered in the literature, and the results of the data we collected from the interviews, we will argue that an essential aspect of productive reasoning about equivalence relations entails a flexible transition between an element-wise local perspective and an equivalence class-based global perspective. These ideas are implicitly present in the existing literature about equivalence relations, but have not yet been explicitly articulated or empirically substantiated. The articulation and demonstration of reasoning from both local and global perspectives and the benefits of using these perspectives flexibly will be the primary contributions of this paper.

Literature Review and Theoretical Perspectives

This study is part of a broader effort to develop and refine a *conceptual analysis* (Thompson, 2008) that describes and accounts for students' reasoning about equivalence across mathematical domains. In alignment with the tenets of conceptual analysis, our efforts here aim to identify and describe elements of students' reasoning "that might be propitious for [their] mathematical learning" (Thompson, 2008, p. 60). The literature on students' reasoning about equivalence, broadly defined, is vast. For example, there is a wealth of research focused on how students productively reason about specific instances of equivalence in K-14 settings (e.g., Cook, Dawkins, & Reed, 2021; Jones & Pratt, 2012; Kieran, 1981; Smith, 1995). Here, however, we

aim to examine how advanced mathematics students might productively reason about tasks that involve an unfamiliar, formally defined equivalence relation. As suggested by quotations in our introductory paragraph above, these kinds of tasks are typical in proof-based mathematics courses. Indeed, while some of the challenges and difficulties of reasoning about such tasks have been reported in the literature (e.g., Brandt, 2013; Chin & Tall, 2001), descriptions and empirical examples of what constitutes productive reasoning has received substantially less attention. This is the primary gap we aim to address in this study.

The literature does provide some theoretically-based hypotheses about what productive reasoning about equivalence relations might entail, but they have not yet been empirically examined with students. Our primary argument in this paper (and the answer we offer to our research question) involves empirically examining one of these theoretical hypotheses: a key element of productive reasoning about equivalence relations entails viewing the relation through (and moving flexibly between) both a *local perspective* and a *global perspective* on the relation. A *local perspective* involves reasoning about whether or not specific pairs of elements are related with respect to a given equivalence relation. Researchers, however, have hypothesized that productive reasoning about equivalence relations requires going beyond this familiar element-to-element perspective to consider classes of related elements. Hamdan (2006), for example, argued that students' should be supported in "envisioning the equivalence classes corresponding to an equivalence relation, which requires going beyond the element-wise conception of a relation, to a more *global* perception of it" (p. 130-131, emphasis added). Similarly, Schumacher (1996) argued that moving flexibly between the relation (to focus on relationships between specific elements) and classes of related elements (the partition structure induced by the relation) "are two faces of the same problem. […] These two views complement each other, and it is useful to be able to cross over easily from one to the other" (p. 46). In both instances, we infer that the authors are acknowledging the importance of a local perspective on the relation but are also arguing that a focus on equivalence classes (or, more broadly, collections of related elements) is critical – other researchers have advanced similar theoretical arguments (e.g., Chin & Tall; Asghari; Asghari & Tall (2005)Therefore, following Hamdan's (2006) use of the term, we characterize a *global perspective* of an equivalence relation to involve reasoning that focuses on collections of related elements. We hypothesize that utilizing both perspectives is a key aspect of productive reasoning with equivalence relation tasks.

Methods

We employed the task-based clinical interview methodology (e.g., Goldin, 2000; Clement, 2000) in order to explore students' organic reasoning processes. Specifically, we conducted individual interviews with four graduate students studying mathematics at a large R1 university. Each student participated in two interview sessions conducted by the first author; each session was up to 2 hours in length. Since this project was largely exploratory in nature, we chose to work with students who would already have strong conceptions about or extensive experience with formal equivalence relations in order to observe and draw inferences about the nature of productive reasoning. In this short report, we focus on our analysis of the activity of one of the four students (pseudonym: Aspen). All interviews were recorded using an iPad application, which produced videos of students' written work with synchronized audio.

The interviews centered on a series of literature-informed tasks, each of which were derived from the exercises involving equivalence relations in proof-based mathematics textbooks. Per our research hypothesis, the tasks were designed to (a) be typical in form yet unfamiliar enough

to require legitimate mathematical reasoning to solve, (b) be open-ended (to encourage multiple approaches), and (c) elicit reasoning from both a local perspective and a global perspective. The task we focus on in this report is featured in Table 1.

To analyze the data, we conducted a theoretical thematic analysis (Braun & Clarke, 2006) that was organized around our characterizations of *local perspective* and *global perspective.* The interviews were transcribed, and the first author reviewed the transcripts, first classifying the approaches to each of the tasks as either productive (successful) or unproductive (unsuccessful), and then coding using notions of *Local Perspective*, *Global Perspective*, and *Transition*. Additional codes were applied to further describe students' approaches to each task. Throughout this process, the first author chose excerpts from the transcripts to discuss and review with the second author. These excerpts and the application and interpretation of the codes was discussed and refined, at which point the first author would revisit the data in light of any revisions or new insights that occurred.

Table 1. Statement of and rationale for one of the interview tasks.

Task:	Rationale:
Let $(x,y)R(w,z) \Leftrightarrow x^2+y^2=w^2+z^2$ be a relation that partitions the plane into curves. What is the maximum distance between two points on curves containing (1,2) and (0, 4)?	This is a modified version of a task from Smith, Eggen, and St. Andre's (2006) introduction to proof text. We included it because of its potential to elicit the use of a local perspective, a global perspective, or both.

Results

Our analysis and presentation of the results here focuses on the responses of one participant (Aspen) to the task in Table 1. We first discuss what we interpreted as Aspen's productive use of a local perspective on the equivalence relation and then do the same for her productive use of a global perspective.

Demonstrating Productive Use of a Local Perspective on the Equivalence Relation

Aspen began by checking to see whether (0,4) and (1,2) (the two ordered pairs given in the statement of the task) were related. She concluded that they are not by using the condition for the relation – see Figure 1 (left).



Figure 1. Aspen reasons that (1,2) and (0,4) are not related (left) and sketches various points related to (1,2) (in green) and (0,4) (in yellow), respectively (right).

Aspen then generated points that were related to each of these two points. She described her approach by directly referencing the statement of the equivalence relation and noting that, because $0^2 + 4^2 = 16$, the points related to (0,4) "have to be the x coordinate squared plus the y coordinate squared being 16." (She reached a similar conclusion about the points related to (1,2) and the sums of the squares of the coordinates as 5.) She proceeded algebraically by attempting to 'guess and check' some tractable values for x and y that might satisfy this condition. The elements she identified as related to (0,4) included $(\pm 4, 0)$, $\left(\pm 2\sqrt{2}, \pm 2\sqrt{2}\right)$, and $\left(\pm 2\sqrt{2}, \mp 2\sqrt{2}\right)$. Similarly, for (1,2), she identified $\left(0, \pm\sqrt{5}\right)$, $\left(\pm\sqrt{5}, 0\right)$, $(\pm 1, \pm 2)$, and $(\pm 1, \mp 2)$. We interpret that Aspen was demonstrating a local perspective here because she was focusing on relationships between pairs of points – specifically, (0,4) and whichever new point she had generated. This approach helped her develop familiarity with the equivalence relation and paved the way for her to turn her attention to collections of related elements, which we detail in the next section.

Demonstrating Productive Use of a Global Perspective on the Equivalence Relation

After generating these points by demonstrating what we interpreted as a local perspective, Aspen concluded that "it's not gonna be a ton of nice points," and then explained that she was "just gonna plot some points on this curve, see if I get anything that resembles a pattern." In her rough sketch, she used yellow to denote the points related to (0,4) and green to denote the points related to (1,2) – see Figure 1 (right). After she had sketched these points, she observed that "I think it's gonna be like a circle." When asked why, she explained:

> *Aspen*: I think this because I did 4 points on (0, 4) in the yellow one, um, and it happened to do that [referring to the four points on the yellow circle that intersect the axes]. […] This is $\left(2\sqrt{2}, 2\sqrt{2}\right)$, which also would work if it was $(-2\sqrt{2}, -2\sqrt{2})$. Same thing if one of them was negative more than the positive. […] If this, say, is my point that is $(2\sqrt{2}, \pm 2\sqrt{2})$, um, then I'll have the same exact point over here, over here, and over here, approximately [referring to the points of intersection of the circle and the lines $y = \pm x$]. […] So I think they're both gonna be circles.

Aspen's realization that the curves containing $(1,2)$ and $(0,4)$ were both circles was particularly propitious for her progress on this task, as she was then able to reason that the maximum distance between points related to (1,2) and points related to (0,4) could be quantified by reasoning about the radii of the two circles:

> *Aspen:* So really, if they're both circles, then the distance is gonna be the same at all the points. [...] I think *that* [referring to the length of the red line segment along the x-axis she drew from (-4, 0) to ($\sqrt{5}$, 0) - see Figure 2] would be the maximum distance. Because if I took any point on, say, the edge and took it to the opposite edge, that should give me [...] $2\sqrt{5}$ because that's the diameter of the [green] circle […], plus, uh, $4 - \sqrt{5}$.

In this way, Aspen (correctly) reasoned that the length of the red line segment could be computed by adding the length of the diameter of the green circle ($2\sqrt{5}$) to the difference between the lengths of the diameters of the two circles ($4 - \sqrt{5}$).

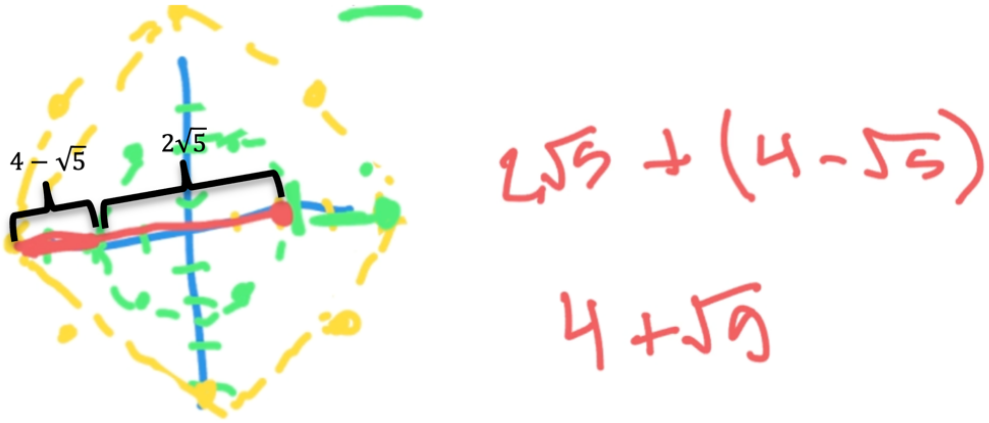


Figure 2. Aspen's red line segment, the length of which she reasoned was the desired maximum distance (we have labeled the two lengths she calculated), and her calculation of its total length.

In a subsequent interview session, the interviewer proposed two alternative line segments whose lengths, he suggested, might also represent the maximum distance (see Figure 3).

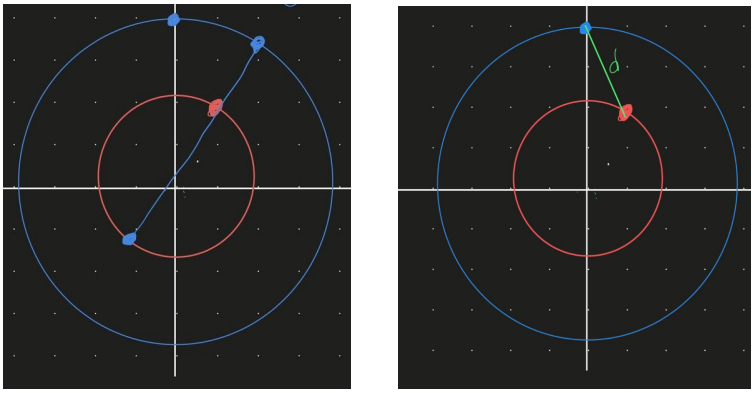


Figure 3. Interviewer's first (left) and second (right) alternative responses to the task.

After the interviewer prompted her to consider the image in Figure 3, left, Aspen responded that she also considered this a legitimate solution:

> *Aspen:* I would agree that this is the... It's the same concept that I have drawn here [referring to her own approach in Figure 2]. It's just using a different angle, but it's still gonna be the maximum distance. [...] The radius of the circle is always gonna be the same no matter where you're picking your point, so we are always gonna have this diameter of this circle in red, and then the difference between the, uh, [lengths of the diameters of the] blue circle and the red circle in this picture. [...] There are easier points to pick. I would say that this one is not the easiest point, um, but it doesn't matter which point you pick.

Aspen disagreed, however, that the second approach in Figure 3 was correct, arguing that the proposed length was not the maximum distance. She explained, however, how one could attain the maximum distance by 'moving' the red point along the red circle:

> *Aspen*: I would say that there can be a further distance. And, how I'd respond to that is say

your blue point on the blue [outer] circle is fixed. You can technically move this red point on the red [inner] circle as long as it stays on the red circle. [...] I can move this point anywhere on the circle […]. Is the distance getting bigger or smaller? And then, if it's getting bigger, then where's gonna be the biggest possible? And, hopefully, they would say that should be this point here [gesturing to $(0, \sqrt{5})$]. [...] You can't move in the between space.

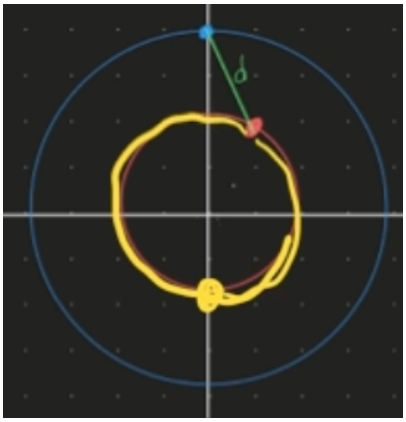


Figure 4. Aspen explains how to modify a proposed solution to attain the maximum distance.

We interpret that Aspen, after starting her response to this task by reasoning with a local perspective, had shifted to a global perspective because she began reasoning about collections of related elements. As evidence, consider that, after plotting specific instances of related points (that she had generated using a local perspective), she asserted that "they're both gonna be circles." Additionally, all of her subsequent comments about the maximum distance referred to these circles and specific distances between them (and not to specific pairs of related elements). Though Aspen identified specific locations on these circles (as in Figure 4), she did not specify coordinates for them, even arguing that "it doesn't matter which point you pick." This shift from local to global also involved a shift from algebraic reasoning (in which she used a 'guess and check' approach to identify pairs of values of x and y that would satisfy $x^2 + y^2 = 16$ or $x^2 + y^2 = 5$) to geometric reasoning (in which she plotted points in the Cartesian plane to observe graphical properties of the curves containing these points).

Discussion

In this report, we posed a question that has received very little direct attention in the literature: what does productive reasoning about equivalence relations entail? To answer it, we empirically examined and elaborated a theoretical hypothesis from the literature: that productive reasoning about equivalence relations entails attention to specific related elements as well as classes of related elements. We characterized the former as a *local perspective* and the latter as a *global perspective* on an equivalence relation and used these constructs to analyze the reasoning of an advanced mathematics student in task-based clinical interviews. We believe our analysis here supports and provides empirical substantiation for the literature-based hypothesis that, with regards to a local and a global perspective, "it is useful to be able to cross from one [perspective] to the other" (Schumacher, 1996, p. 46).

Use of both a local perspective and a global perspective supported Aspen's completion of these tasks in concrete ways. Using a local perspective, for example, enabled Aspen to develop

familiarity with the relation and generate a list of pairwise equivalent points via algebraic techniques. Generating these points was critical because it enabled her to identify the geometric patterns exhibited by the equivalence classes and thus facilitated the shift to a global perspective. Using a global perspective enabled Aspen to reason geometrically about the situation – specifically, she was able to use geometric properties of circles to efficiently identify that a way to compute the desired maximum distance without referencing any particular coordinates of points on those circles. This was a notable departure from her local approach, which centered on particular points and their coordinates.

Our primary contribution, therefore, is the description, analysis, and illustration of two perspectives that can support students in reasoning about equivalence relations. Through our conceptual analysis lens, in other words, we consider the local and global perspectives to "be propitious for students' mathematical learning" (Thompson, 2008, p. 60). This contribution addresses two gaps in the equivalence literature. First, while the equivalence literature is substantial, few studies have investigated students' reasoning with formal equivalence relations, and those that have are largely focused on the difficulties students encounter. This study contributes to the literature by provided theoretical descriptions of what students' *productive* reasoning could or should entail. Second, very little research had previously examined students' reasoning about equivalence from a global perspective, perhaps because all of K-14 mathematics approaches equivalence from a local perspective. This study therefore contributes to the literature by analyzing a student's use of a global perspective and how it supported her successful completion of a somewhat typical equivalence relations task; it also offers some insights into how a global perspective might be productively used in conjunction with a local perspective.

We acknowledge that our characterizations of a local perspective and global perspective – though useful classifications of students' reasoning – are very broad. They do not, for example, afford insight into *how* and *in what ways* students reason locally or globally. And while the literature does provide answers to such questions from a local perspective (e.g., Cook et al., 2022), much is still unknown about such specifics from a global perspective. We believe this is a fruitful avenue for future research. Our efforts here provide (at least) one promising suggestion in this respect: Aspen's use of the global perspective and her associated remark that "it doesn't matter which point you pick" aligns with researchers' suggestions in the literature that robust reasoning about equivalence relations and classes involves a certain indifference towards the specificity of elements in particular classes (e.g., Asghari, 2019). This could be a useful refinement of our characterization of a global perspective. Another limitation of this study is that we have only investigated the reasoning of advanced mathematics students who had already developed the capacities needed to reason productively with equivalence relations. How students might, for example, develop the capacity to employ and move flexibly between the local and global perspectives of equivalence relations remains unclear. Working with undergraduate students in similar problem contexts might provide insights into how productive reasoning about equivalence relations *develops* rather than only describing what it might entail.

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Undergraduate Student Conceptions of Mathematical Syntax Meanings

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This research investigates students' structure sense in algebra, focusing on how they conceptualize and interpret mathematical syntax. Here we focus on how students parse symbolic representations in algebra, in particular, how and why students recognize certain substrings to be unified subexpressions. We use prescriptive conceptions (desired mental images) to guide our noticing of students' descriptive conceptions (actual mental images) as they engage with conceptual algebra tasks from the Algebra Concept Inventory. The findings offer implications for instruction and curriculum, suggesting approaches that may better support students in developing a deeper understanding of algebraic structures and symbolic reasoning.

Keywords: Algebra concept inventory, student conceptions, syntactic structure, mathematical syntax, symbolic representations

Research in mathematics education shows that students often struggle with structure sense, or the ability to understand, recognize, and manipulate the underlying structure of algebraic expressions (Hoch, 2003; Linchevski & Livneh, 1999). Working with symbolic representations as objects (e.g., transforming expressions and equations), requires understanding of the structure of these objects. Existing research focuses on what students *do* with symbolic representations (e.g., Hoch & Dreyfus, 2006, 2010; Kirshner & Awtry, 2004; Vincent et al., 2017), but includes only limited analysis of *why* they perform specific manipulations, and what *conceptions* students have of syntactic structure in algebra. This research addresses this gap by investigating how students conceptualize the parsing and chunking of mathematical syntax.

Understanding specific student conceptions related to structure sense can help instructors develop effective teaching strategies (Hoch & Dreyfus, 2010), and address diverse learning needs (Tomlinson, 2001). It is also key to building curricula that enhance students' mathematical communication (Sfard, 2008), strengthen their foundation for advanced math (Taban & Cadorna, 2018), and improve problem-solving skills (Linchevski & Livneh, 1999).

We focus on two key aspects of mathematical syntax: (1) the extent to which students are able to conceptualize certain substrings of symbolic representations as unified objects (subexpressions), which relates to operational vs. structural views of syntax; and (2) which criteria they use to determine that structuring of symbolic representations, which relates to student definitions around syntax. More simply put, we are interested in *how* students parse symbolic representations, and *why* they parse them in the particular way that they do. Through student written work on algebra tasks and interviews with students in post-secondary math classes, we analyze their conceptions, particularly regarding how they parse or chunk mathematical syntax.

Literature Review

Existing research on structure sense has focused more on what students do with symbolic representations than on how they conceptualize them. Additionally, current literature is underspecified regarding the meaning of "structure" and which structures are most important. Our research seeks to clarify what structure sense is by framing it as a set of conceptions.

In this research we focus on one of the most basic components of structure sense: *syntactic meaning*, which we define as reading the intended grammatical meaning of symbolic representations. Syntactic meaning refers to the literal interpretation of symbols based on conventions (e.g., order of operations), without considering broader concept images or the ability to simplify or transform the expression. For instance, the syntactic meaning of $(2x + 1)(x - 7)$ refers to two subexpressions being multiplied, but it does not include concept images like multiplication or real-world meanings of x.

Research shows that students often assign syntactic meaning based on surface features, like spacing (Erlwanger, 1973), which Kirchner (1989) calls "visual salience," rather than on mathematically-salient features. For example, students may give precedence to operations that are visually closer together (Landy & Goldstone, 2007) or interpret spacing and grouping based on familiar visual cues from previous experiences (Nathan & Koedinger, 2000). Visual salience can overshadow mathematical salience, leading students to group elements visually rather than apply mathematical properties. Helping students distinguish between visual and mathematical salience can guide curriculum development and instructional strategies to focus on meaningful relationships rather than superficial visual cues.

Well-defined definitions allow students to interpret algebraic expressions consistently, using mathematically-salient criteria, rather than relying on surface features (Edwards & Ward, 2004). For instance, a student may have an extracted definition of syntax, where they identify subexpressions based on what "looks right" (visual salience) rather than on mathematically salient features. A mathematically well-defined, stipulated definition would use the order of operations and other syntactic conventions to determine how the representation should be "chunked" into subexpressions.

Process-Object Theories and Subexpressions

Research shows that students often struggle to recognize substrings of symbolic expressions as unified objects. This ability is crucial for algebraic manipulation, such as applying the distributive property or identifying forms like $a^2 - b^2$ (Hoch & Dreyfus, 2004). This is related to how and whether students conceptualize representations as processes vs. objects. For example, students might interpret $2(x + 1)$ as a process of adding 1 to an unknown value and then multiplying by 2—an operational view (Sfard & Linchevski, 1994); or they may be able to treat it as an object representing the theoretical result of calculation without needing to perform the operations—this is classified as a structural view. Pseudo-structural views (Sfard & Linchevski, 1994a, 1994b) occur when students treat symbols as objects that have not been reified from processes.

We extend this theory to focus specifically on what we call *subexpressions*, or substrings of symbolic representations that can be treated as unified objects without altering the syntactic meaning of the representation. For example, in the expression $3x^2$, students may conceptualize $3x$ as a unified object (subexpression) because of visual salience (3 and x are close together) or extracted experiences from instruction ("the instructor always treated $3x$ like a single thing in class"), or they may conceptualize x^2 as a unified object (subexpression) because the order of operations prioritizes exponents over multiplication. The latter is the necessary interpretation to understand the normatively-intended syntactic meaning of the expression $3x^2$. In existing structure sense literature, treating substrings as objects is often combined with other skills (see e.g., Hoch & Dreyfus, 2006), yet understanding this ability as a distinct conception is important, which is why it is the focus of this work. Identifying subexpressions is a critical precursor to

almost all other structure sense skills because it is the first step to identifying the structure of expressions and equations, which is essential to transforming them.

Method

This research focuses on the following questions: What conceptions do tertiary students have of structure sense, symbolic representations, and mathematical syntax? Specifically, what conceptions do tertiary students have of parsing and chunking mathematical syntax?

To address these questions, we use data from a broader study generating an algebra concept inventory for post-secondary students (AUTHORS, 2024). In this study, we collected thousands of responses to multiple-choice items, including 138 cognitive interviews of student thinking while completing these items. It is these cognitive interviews that we focus on here, specifically analyzing items where students were asked to attend to syntactic structure.

Our research is grounded in prescriptive conceptions (conceptions that serve as the intended goals of instruction) of symbolic representations and mathematical syntax. We generate these conceptions using literature and conceptual analysis, providing criteria for how students should "read" syntactic meanings and view mathematical syntax as reifications of normative processes. These can be seen in Figure 1, where the rows and columns represent broader categories, and the two cells at middle and bottom right represent combinations of these broader categories.

	Well-Defined Syntax Conception: Syntactic meanings must be unambiguous—the same syntax can't dictate more than one possible operational precedence at once, and different members of the mathematics community can't choose different conventions and operational precedence for the same symbolic objects.	**(Normative) Operational Precedence Conception:** Syntactic meaning is dictated by the order of operations and other mathematical conventions/norms stipulated by the mathematics community (whether or not those calculations can be carried out directly).
Structured Syntax Conception (Object): Symbolic representations can themselves be thought of as objects that can be "chunked" into unified subexpressions (or other unified sub-objects).	***Do-first Conception (operational):*** *The operational precedence norms govern the order of arithmetic (and where possible algebraic) calculations (e.g., which operation should be "done first").*	***Subexpressions Conception (structural):*** *The operational precedence norms govern how algebraic symbolic representations are structured, or "chunked" into unified subexpressions. (This may be a reification of the "do-first" conception or might be learned separately and linked to operational precedence later.)*

Figure 1. Prescriptive Conceptions of Mathematical Syntax

These categorizations of student thinking offer a way to classify students' *in-the-moment* thinking, providing insights for improving instruction and curriculum.

We used the prescriptive conceptions from Figure 1 to guide our noticing of student work, but also allowed themes to emerge more organically by employing theoretical thematic analysis

(Braun & Clarke, 2006) to constructed rich, thematic descriptions of students' structure sense conceptions. Our analysis led to the framework given in Figure 2, where we see student conceptions as a cross between operational vs. structural views and the extent to which they use non-mathematically salient vs. well-defined mathematically-salient features to determine syntactic meanings.

<table>
<tr><td>Reading syntactic meaning of symbolic representations (mathematically-salience features of syntax)</td><td>Non-mathematically-salient features, without being able to link this reasoning to stipulated criteria</td><td>Stipulated, logically-consistent and well-defined, normative criteria</td></tr>
<tr><td rowspan="2">Operational View
Conceptualizing syntax as an instruction to calculate, or as describing a process of calculation (process view)</td><td colspan="2">pseudo-process view ⟷ process view</td></tr>
<tr><td>"I have to add these two terms first because this just looks/feels right."</td><td>"I have to add these two terms first before I square them, because they are enclosed in parentheses and parentheses comes before exponents in the order of operations."</td></tr>
<tr><td rowspan="2">Structural View
Conceptualizing syntax as a structure that contains sub-structures (e.g., an expression is made up of multiple [potentially overlapping] sub-expressions) (object view)</td><td colspan="2">pseudo-object view ⟷ object view</td></tr>
<tr><td>"I know that $2x$ is a unit because I have just developed an instinct of how this works."</td><td>"I know that $2x$ is unit because multiplication comes before addition in the order of operations. Because the 2 is being multiplied by the x, $2x$ has to be thought of as a unit that could then be added to the other terms."</td></tr>
</table>

Results and Discussion

Classifying student work was our first step towards understanding student thinking. We allowed students' conceptions to emerge through the research but a priori theories also guided our noticing of students' conceptions as we attended to the following: (1) what criteria students were using to "read" the syntactic meaning of symbolic representations, and (2) whether students were able to conceptualize the syntax as dictating a process or describing an object, including how they "chunk" symbolic representations that they conceptualize as objects. We now provide examples of student work in each category (pseudo-process, process, pseudo-object, and object conceptions of syntax).

Pseudo-process Conceptions

Consider the following item: In the expression $3x^2 - 2(x^2 + 1)$, which part is being subtracted, or taken away? One student who was enrolled in a math course with elementary algebra as a prerequisite originally chose $2x^2$. During their cognitive interview, the student expressed uncertainty about the choice, so the interviewer followed up and eventually, the student says, "Honestly, the answer in my mind is not in the options." When asked to explain what is in their mind, the student and interviewer have the following exchange (Excerpt 1):

Excerpt 1:

Interviewee: It should be this one, like I would reverse this one like x square- I don't know. So, x squared minus two minus x squared plus one. So, this has to be taken away [pointing to $x^2 - 1$].

Interviewer: So, okay, so you would be subtracting the x squared plus one, kind of like what's in the parentheses?

Interviewee: Right.

Interviewer: And why would you do that?

Interviewee: I don't know.

Interviewer: It's just a feeling you have about how it should be done?

Interviewee: Yeah, but why is the parentheses still, it's not subtraction... If I'm going to multiply this one 3x- Wait, I think I did it wrong. 3x squared this, this [draws lines connecting $3x^2$ and 2 to other terms in the expression].

Interviewer: So, now you're multiplying the 3x squared by both of the things in the parentheses and also the two times both of those things in the parentheses?

Interviewee: Yes, 3x squared, it's going to be 3x four, am I correct? And then plus 3x squared and then this to the other negative 2x squared negative two.

Interviewer: Okay, so then would you be subtracting this minus 2x squared minus two?

Interviewee: Yes.

Interviewer: Okay, I got it. So, you're multiplying first and then whatever you're getting as a solution that is what you think you should take away, but that answer is not there?

Interviewee: Correct.

This student has an operational rather than structural view because they appear to struggle with identifying what is being subtracted without simplifying the expression. The student attempts to "distribute" every term that is outside the parentheses, even though the $3x^2$ is not being multiplied, which may be an extracted procedure from instructional experiences with the distribute property (e.g., AUTHOR, 2023). Further, the student appears to view $3x^2 - 2$ and $x^2 + 1$ as separate expressions to operate on, and are inconsistent with, the operations they apply. The student's attention to simplification appears to get in the way of being able to identify what is being subtracted in the expression in its current form, which may require conceptualizing $x^2 + 1$ as a unified subexpression. This student seems to be capable of thinking structurally, but they do not have a concrete justification for interpreting the syntax of the existing expression in a particular way. Thus, they may view symbolic representations as instructions to simplify rather than as an existing syntactic structure in and of itself, representing a pseudo-process view.

Process Conceptions

A different student displayed a process view for the following item: Consider $2 \cdot x^3$ in its current form (don't rewrite it or do anything to it). Which part of $2 \cdot x^3$ is being multiplied by 2?

Students were provided two options: "Only x^3" or "Only x ". This student chooses the first option because, "It's just that three is only being multiplied, not anything else. So, since it's to the third, then it would be only being multiplied by to the third because it says it on the question. And plus, it says what's being multiplied by two and it's obviously to the third." After some back-and-forth with the interviewer in which the student does not provide further detail about *how* they came to their answer, the interviewer asks, "How does the order of operations help you understand what's being cubed in the expression?" The pair then has the following exchange (Except 2):

Excerpt 2:

Interviewee: Well, since I know that there's no parentheses only thing that I will say is that there's an exponent here. So, we should focus on that first. Then we could get to the multiplication part.

Interviewer: Okay, so you said you noticed that there's no parentheses. So, where would the parentheses be if there were parentheses?

Interviewee: Now if it was a parentheses it would be around the x and then it would cube that. And once you cube that it will become x cubed.

Interviewer: Okay, if the parentheses were around let's say the $2x$ the two and the x how would that change things if it does at all?

Interviewee: Well, actually it would change both of them to my opinion, it would change both of them. Because you have to cube the two and the x.

The student uses a well-stipulated definition of the order of operations in their justification by when they describe how the order of exponents in operational precedence impacts the meaning of the syntax. However, their justification points to an operational view because they talk about processes (e.g. "multiplication", "cube") rather than describing any substrings of the syntax in a way that suggests that they view them as subexpressions. Further, since the student relies on mathematical (justification through operations) rather than visual salience, we classify this as a process view.

Pseudo-object Conceptions

We now consider an example from a student responding to the following item: In the expression $-(4x)^3 + 7$, which part is being cubed (or raised to the third power)? The item provided the following choices: a. $-4x$, b. $4x$, and c. x . The interviewee explains their choice of $-4x$ (Except 3).

Except 3:

Interviewee: So essentially the minus, the negative sign in the beginning gets distributed into the $4x$ that is within the parentheses, which is why I chose minus four x, which would get cubed.

Interviewer: Okay, so you're saying that the negative gets cubed as well as the $4x$?

Interviewee: Yes.

Interviewer: And I notice also that you included the 4 and some of the other answers don't include a 4. Why did you include the 4?

Interviewee: Because the 4 is together in the parentheses with the x.

Interviewer: With the x, what do the parentheses mean here in this expression?

Interviewee: It's saying that this whole expression, $4x$ is being cubed, not just one specific number.

Interviewer: Does the order of operations help you to understand what's being cubed here?

Interviewee: That didn't come to mind when I was doing this.

This student refers to $-4x$ as a unified object ("whole expression"), suggesting a structural conceptualization. However, what they consider the subexpression is not consistent with the standard order of operations, and when asked directly, the student admits to not considering the order of operations when doing the task. It is also notable that the alternate procedure of distributing the negative sign before cubing may also be indicative of a pseudo-process view, and thus this student may be moving between a pseudo-object and pseudo-process view or may hold both conceptions simultaneously.

Object Conceptions

For our final conception, we consider a student looking at the following item: In the expression $(x-7)^3$, which part is being cubed (or raised to the third power)? This student chose $x-7$ (over the other provided options -7 or 7).

The student explains how the third power, being "outside of the parentheses on the right side" means the third power should apply to the subexpression in the parentheses. The interviewer follows up with the question, "Do you know what the order of operations is?", leading to the following exchange (Excerpt 4):

Excerpt 4:

Interviewee: Yes, parentheses, exponents, multiplication, division, addition, and subtraction.

Interviewer: So, does the order of operations help you tell what's being cubed in that expression?

Interviewee: Correct, it does. It shows that what's in parentheses is actually being cubed. So, I would again follow what's inside the parentheses if I cannot do that equation then I would work outside of the parentheses.

This exchange reiterates the student's view of the string inside the parentheses as being a unified subexpression as well as their normative definition of the order of operations. This structural interpretation of the expression coupled with the normative definition leads to the object conception classification.

Conclusion and Implications

In this research, we not only classified conceptions, but also found interesting and common patterns of holding multiple conceptions that may have important implications for instruction and curriculum. For example, many students had a process conception of syntax in the context of numerical expressions but a pseudo-object conception in the context of algebraic expression. This may happen, for example, when students experience algebra as a sequence of transformation algorithms, without clear, consistent and ongoing connections to algebra as generalized arithmetic. We also observed students who had both process and object conceptions together, and where the object conceptions appeared to be reifications of the process conceptions. This illustrates that at least some students are able to reify processes into more structural object views in normative ways—better understanding how that happens for these students could help us to improve instruction so that all students have access to these conceptions when working with algebraic symbolism.

Overall, student responses illustrated how they often relied on surface-level cues or procedural steps when interpreting mathematical expressions, rather than understanding how symbolic representations can be conceptualized as objects that have a particular structure that is dictated by straightforward rules and conventions like the order of operations. Instruction could help students recognize and rely on mathematical salience for interpreting symbolic structures rather than visual patterns by using activities that encourage students to articulate why expressions or equations "chunk" into subexpressions in particular ways, and how this is linked to operational precedence and other notational conventions.

Students often identify subexpressions for non-normative reasons, relying on surface-level cues, and their process conceptions, such as the application of the order of operations, influences their ability to develop object conceptions, such as understanding subexpressions as unified structures. Instructors could also assist students' transition from process- to object-oriented conceptions by explicitly connecting the two, for example by consistently engaging students in tasks that illustrate how algebraic expressions are generalized representations of arithmetic.

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Why Are YOU in the RUME?

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In this study we explored the intended outcomes that participants in the SIGMAA on RUME conference had for their conference participation. We conducted in-person, semi-structured interviews at the 2024 RUME Conference and engaging in thematic analysis to arrive at three broad categories of intended outcomes. People intended to achieve specific career-oriented goals such as 'gaining a line on a CV' and research-oriented goals such as getting feedback on their work. Finally, participants expressed multiple personal outcomes driven by their sense of the RUME participants as a community that they enjoy being part of. In addition, we found aspects of the conference and community that both support and inhibited the participants in their pursuit of their intended outcomes. The analog to the perception of a strong community is that participants find it difficult to become part of the intellectual and social conversations.

Keywords: professional conferences, professional development, undergraduate mathematics education researchers

Disciplinary conferences are believed to be critical sites in the academic trajectory (Henderson, 2015), yet their roles and purposes for participants are under-researched. Generally, conferences are understood as sites of networking and as an opportunity to present work and learn new ideas (e.g., Fakunle et al., 2019; Henderson, 2015; Hill et al., 2018). There has been some research focused on people at specific career stages such as undergraduates (Hill et al., 2018) or doctoral students (Fakunle et al., 2019), or on attending specific conference types (Weber & Fukawa-Connelly, 2023). There is, however, relatively little exploration of the value of disciplinary conference attendance to the participants, how and why those values might be different for different participants, and the aspects of the conference that facilitate or inhibit the participant's ability to realize those values.

Researchers have studied the value of disciplinary conferences (Hill et al, 2018), the importance of conferences for the discipline (c.f., Carpay, 2001; Walford, 2011), and linguistic explorations of the content (e.g., Hood & Forey, 2005). Fakunle et al. (2019) surveyed and interviewed doctoral students in Schools and Colleges of Education at four institutions in the US, UK, and Australia to explore their motivations for attending academic conferences. They found that student's primary motivations were to network with other academics—with a focus on career advancement—develop ideas for new research, develop new research collaborations, explore employment opportunities, and get guidance on research tasks. The students also claimed to view conferences as learning opportunities including learning about a new field, learning who works in a field, and honing their abilities to describe their work. Kneale et al. (2016) explored what undergraduates attending STEM conferences believed they might learn from giving presentations or posters. They found that students' main learning goals related to presentation skills, such as confidence in public speaking, structuring the presentation, and writing. In addition, the students valued getting critical feedback to gain knowledge and advance their research progress.

In this study, we explore the reasons that people at various stages in their careers attend the RUME conference–what they value about attendance, how they believe they (might) benefit from attendance, and how those values have changed (if at all) throughout their careers. We also explore how and why they might realize or be inhibited from realizing their intended outcomes. The themes we examine expand on prior research by including senior researchers and focusing on a narrow research field to better explore the ideas of networking and conferences as sites for learning. Our research questions are: *What are the different reasons that RUME participants give for coming to the conference? What helps or inhibits their ability to achieve the desired goals?*

Theoretical Perspective

We draw on Lave and Wenger's (1991) construct of *legitimate peripheral participation*. Conferences are ephemeral sites of congregation, although peripatetic, with no fixed participants. Instead, different groups of people, some within the discipline and some 'interested outsiders,' briefly gather for academic and quasi-academic purposes. These purposes are influenced by social and academic issues; disciplinary and social norms; issues of power and prestige, and career stages; and types of ties to a particular community. We find the notion of *communities of practice* (Lave & Wenger, 1991) salient in conceptualizing the practice of attending conferences. From this perspective, communities include "old-timers," who embody the socio-cultural practices of the community, and newcomers, who begin participating in the practices of the community and, in so doing, begin to adopt its activities, identities, and artifacts.

A conference can be framed as a sociocultural space where various aspects support and require different types of social participation, without clearly articulated rules. Participants have varying levels of familiarity with each other, academic spaces, and the conference itself. They are also members of disparate departmental, institutional, and disciplinary communities and have different reasons for attending the conference. Thus, we draw on communities of practice and legitimate peripheral participation to conceptualize the continual process of becoming a more central participant in the RUME conference community. We explore attendees' ways of and goals for participating, as well as what aids or inhibits attendees' participation.

Positionality

All authors are former elected members of the board. Two authors have been plenary speakers. Three authors have mentored students and supported (and expected) their conference attendance. Thus, we are most likely understood to be central members of the community. Although we tried to convey to participants that we were conducting research, some told us ideas for improving the conference, indicating they believed us to have influence in the structure and organization of the conference.

Methods

We recruited in-person participants at the SIGMAA on RUME 2024 Conference through convenience sampling—typically by asking people walking through the lobby if they would talk with us. We recruited 22 participants who agreed to participate in audio-recorded interviews. Two interviews were with pairs of participants. We attempted to interview conference participants with a wide range of experience with the RUME conference, from first-time conference attendees to those who have attended for nearly two decades, and from relatively new graduate students to full professors.

We collected data using a semi-structured interview protocol. Participants were first asked to give a characterization of their career and research trajectory and to relate different reasons for

conference attendance at different times in their careers. Subsequent questions asked participants to describe their reasons for attending the conference, what initially motivated them to attend (for those who had attended multiple times), what benefits they derived from attending, and how those benefits might be derived. Finally, we asked them to contrast attendance at the SIGMAA on RUME conference with another conference to explore other possible reasons for conference attendance. Each interview was approximately 20 minutes in length.

We drew on thematic analysis (Braun & Clarke, 2006), using inductive analysis to identify latent themes and construct codes. We consulted the relevant literature, with its focus on networking and career development, to guide our identification of themes, aligned with the theoretical framing of legitimate peripheral participation. Both notions—career development and networking—are forms of participation in the academic space with networking having explicit expectations for social connection. Two authors read three transcripts initially to develop and refine initial codes. Then, all four authors met and coded three transcripts using the refined codebook. Code categories were further refined. Then pairs of authors recorded the initial three transcripts and coded all remaining transcripts. Any disagreement was resolved via conversation among all four authors.

Preliminary Results

We organize people's reasons for attending the conference into three broad categories; career outcomes, research outcomes, and personal outcomes. Most of the career outcomes such as "earning" lines for a CV, networking as a means of professional advancement, or satisfying a job expectation are well-represented in the literature. Consequently, this report focuses on the research and personal outcomes as they are less well represented in the literature.

Research Outcomes

In terms of research outcomes, we categorized participants' goals as either related to engaging in collaboration, attending research presentations, and presenting their own research. Participants described two different types of goals that they might achieve through conference attendance: developing new or sustaining ongoing collaborations, acquiring knowledge from attending presentations, and sharing their own research.

Collaborations. Several participants reported attending to engage collaboratively with colleagues on already-established research projects. Other participants described a goal of developing new collaborations. For example, Darrell (advanced graduate student) said:

> I can pick people's brains and find ways to network and collaborate with them in person that just helps. There's papers I've written just purely by running into people here. And just asking them, hey, I have this idea. What do you think about this? We just kind of spiral off.

Attending presentations. Within this category, participants identified two goals: gaining an overview of the field and learning new research topics or methods. For example, Barbara (graduate student, first time attendee), who trained as a mathematician and was engaged in the process of becoming a mathematics education researcher claimed:

> It's a super succinct way to get exposure to, you know, a lot of things that are super relevant to you, new ideas, and you don't have to go sit and read a bunch of papers, you're like, given these nice, succinct entryways into it. And we've been taking copious notes of things to go look up later.

At a smaller grain size, participants claimed that they might gain inspiration for new projects in terms of either new questions to pursue or new methods to use. For example, Joe (professor),

who had participated in the conference for over 10 years, claimed that "you hear ideas that then you think about and wrestle with and they produce new ideas for you to cool study," and Dave (mid-career associate professor who has been attending since graduate school) described a goal of learning new methods:

> I still will come to go to talks and think like, oh, wow, that's a really interesting method. Like, I didn't think about that. Yeah, right. Maybe I could employ something similar. So there is still the potential for refining my understanding of appropriate methods and theories and so on.

In each case, these are outcomes that the participants take away from the conference based on hearing talks and do not, on the face, require discussion with other participants.

Sharing research. Several participants described benefits that resulted from presenting their own research. For example, Jenny (early assistant professor who started coming to RUME as a graduate student) claimed:

> I had now built up this big network system, and I knew everybody and, and then it became more of asking richer questions about detailed questions about my research, or how to improve my research. It's, I think it's rare like here, I come to a conference and I can literally like talk to anybody in the room. And they'll just sit down with me and go over technical stuff about research or advice or, like that's common, especially with like, the bigger names.

We interpreted Jenny as saying that she was able to have conversations about her work and get specific and individualized feedback from established members of the community. Participants suggested that access to journal editors and NSF program officers for discussion about ideas and papers was particularly helpful.

Participants also described a sense of excitement about their ideas and the fact that they're doing work they think the field would find interesting. Darrell, who had attended four prior RUME conferences, described a sense of ownership of and pride in his work:

> This is the first semester that I'm gonna get to present my work. So I'm excited to actually be able to do that ... It's mine! I remember hours and hours with [my advisor] just like pounding, her pounding me on, like let's start thinking about things… And it's like, this is something that I thought of, everything that's there is something that I did, and it's my idea … I get to talk about what I like and people have to sit there and listen to me for thirty minutes, they have to they chose to be there.

Other participants described presenting their research to mark one's place within the research community. For example, Dave said:

> I think it's through the establishment of a novel research program, or focus that hopefully, it's communicated persuasively to the people who are interested in this community, I think that that can really help establish a kind of personal identity as a researcher.

Personal Outcomes

Participants highlighted three interrelated personal reasons for their conference attendance: they think of the conference as their intellectual home, they enjoy the community, it supports their mental health in important ways.

Intellectual home. Several participants described the RUME conference as their intellectual home. For example, Brooke (associate teaching professor, trained as a mathematician) stated:

> RUME is consistently the conference where I feel like I belong, because it's more I feel like I find more people who are mathematicians and math ed researchers here. I like that I can talk about advanced mathematics but also frame it theoretically in conversation about

> research. So I got just really excited about that there are a lot of people here that I have a lot of respect for as mathematicians who are also doing really good math ed research.

Similarly, Ralph (associate professor) described the conference as an academic home:

> It's my academic home. … I've always tried to straddle – well, since grad school - like trying to straddle like, Math Department life and education mindset kind of life and RUME is like a really good place to do that. I think we're all kind of—a lot of people, have experienced similar kinds of issues. And they're like—most of us, or a lot of us, are housed in math departments and a lot of us are math ed or schools of ed. But like, just like understanding that kind of balance of like, needing to always like having to kind of put in a little bit extra work to get the cred among your own peers in your in your department, stinks. And then you come here, and it's like, we all kind of like, have earned this. Like, we don't have to do that in this group as much, I don't think. So that's good.

In both cases, the participants highlighted the importance of balancing significant mathematics with quality research. Ralph went farther than Brooke and noted that he, like other participants, may not feel valued or appreciated in his home department, but found that his work is valued and appreciated within the RUME community.

Community. This notion of professional and intellectual home was coupled with claims about enjoying the community of scholars who gather for the conference. For example, Sam (professor, attending for 15 years) said:

> This conference is a delight for me. I love coming to it. I'm paying my own way this time. Like, I am just doing it because I want to be here so much. You know, I have I don't have a grant this year. And we have no travel money in our department. So yeah, that's how much I like, really, like I love the people here. There's a lot of friends that I have that I don't want to miss.

Similarly, Robert (associate professor, attending since graduate school) described the community and friends that make the conference enjoyable:

> I just enjoyed, like, the ultra-supportive and just good group of people that this conference generally brings about, there's something different about this particular conference compared to other education conferences or other even math conferences.

For both Sam and Robert, the conference community included numerous friends who they enjoy reconnecting with each year.

Mental health. Participants described coming to the conference for a variety of reasons that we described as mental health, which included being re-energized, "loving people as human beings," and being offered support for life challenges. For example, Kelly (associate professor, attending since graduate school) said, "I come here to fill my cup. And I'm never disappointed. And I want that to be the title of the paper."

Structures that support goal achievement

In addition to describing personal attributes that supported their goal achievement, such as being willing to "just go talk to people," participants described two aspects of the conference that helped them achieve their intended outcomes. First, participants mentioned the importance of conference size. For example, Sally (full professor, first RUME conference in 2009) said, "this conference matters because is small, it's intimate," and contrasted this with larger conferences (such as PME-NA or JMM) where it can be challenging to find talks of interest and to find people among the other participants. Second, participants mentioned structural elements of the conference that they believed made it easier to participate both intellectually and socially, often

claiming that social interactions would facilitate subsequent intellectual ones. For example, Irene (first attended RUME in 2016) said:

> What I really appreciate is the intentionality that the RUME—like the RUME conference has done to make it more welcoming to have spaces where it's more inclusive. I mean, having more non-drinking options is great. Having like the faith community this year, having the trans community this year.

Marge (started attending RUME in 2017 as a graduate student) highlighted some of these same elements, concluding " Like, it makes it clear that our community is diverse and welcoming of that diversity".

Darrell also noted the importance of the option for conference buddies because:

> For outsiders, it can be intimidating. But I like that idea of the big buddy system.to at least get hey you're alone. … Here's someone that would be willing to like chaperone you or lead you to people or things like that. I thought that was really cool.

Darrell claimed that the conference space could be intimidating to outsiders but that by offering people the opportunity to sign up for a buddy, it could help decrease barriers to participation.

Factors that Limit Participation

The participants identified two factors that inhibited their ability to achieve their conference goals: a perceived lack of intellectual or disciplinary capital and a perceived lack of social capital or difficulty 'breaking into' the conversations. Barbara described feeling that she lacked disciplinary knowledge and expertise, which could have inhibited her participation:

> I guess the big takeaway from my first RUME, which was the Boston one … two things I took away; first thing was like, "Oh, my God, I know nothing." And the second thing was like, but that's okay, this community will accept me and nurture me and I will eventually know more. Not everything, but I will learn I will start knowing more.

Brooke highlighted a similar challenge, framing this perceived lack of disciplinary capital in terms of impostor syndrome:

> The first time I came to RUME was in 2017. And I was a massive imposter. I spent a lot of time in my room because I'm, like, so nervous, and I didn't know how to speak the language. Now I'm a lot more comfortable with that. And I'm, I'm more aware, like I'm, I still have plenty of gaps in my, I'm not formally trained as a researcher. So I have plenty of gaps. We're more aware of that. But I'm also aware of what different skills I bring to the table by being here.

Both Barbara and Brooke were trained as mathematicians and transitioned to mathematics education research. With Barbara stating that "My PhD is in mathematics. And then as postdoc, I started my transition into room. And I'm a rumer 100% of the time" and Brooke explaining "sort of sidestepped from algebraic topology into math education in 2016. Now I'm starting to feel like I kind of know what I'm doing".

Multiple participants described difficulty "becoming part of the conversation." For example, Marianne (fifth year attending RUME) framed this challenge in terms of access:

> I guess certain people have more regular access to key players in the field…. And it's hard to know how to get it'd be nice to be able to sit down sometimes with key players and just talk a little bit like see, even if it's just like as friends just to like, not feel so intimidated. So that then that could lead to conversations

Anne (mid-career, started coming as a graduate student) also described the amount of time and effort it could take to begin engaging in conversations:

> Yeah, cause you're right. I don't just like walk up to people during those breaks…. So, like, because it's every year and you see the same people every year, like, there's someone I met, maybe two or three years ago that I barely spoke to, but then I see them again the next year, and we speak a little bit more. And I'll find them during a coffee break and be like, oh, yeah, remember, we met last year and we'll talk a little bit more and then so even if it's a long term, slow relationship.

We might understand the difficulty in breaking into conversation to be a consequence of the existence of a strong community with tight friend groups among the conference participants.

Discussion

We make two minor claims advancing the literature on professional conferences, share some speculative points, and make a call for additional research. First, the participants we interviewed provided intended outcomes from attending a professional conference that went beyond what we found in the literature. They described the excitement of presenting their own work and ways that they learned from participating in the conference—in particular, through discussions with individuals—that could help them improve their research beyond what they might learn by only attending talks. Second, participants described numerous personal outcomes from attending the conference related to the fact that they felt a real sense of both intellectual and personal connection with the conference community. In contrast, though, some participants cited feelings of deficiencies in both their intellectual and social capital, and, thus, difficulty in being part of the ongoing conversations at the conference. Multiple participants cited the existence of conference structures such as affinity groups and the "buddy system" that lowered barriers to participation and signaled that all are welcome. Finally, as a speculative point, we noticed that claims about career advancement or technical goals tended to be cited by participants who were further from the "center" of the community. Conversely, participants near the "center" cited community and friendship as significant aspects of the conference. We suggest that if one reason for attendance drops in importance, another factor must continue to motivate attendance.

There are some significant limitations to the study that necessitate further research. Our sampling methods (approaching people in the lobby during breaks and throughout the conference) resulted in potential selection bias. In particular, our participant recruitment pool did not include people who were in-person RUME conference attendees but were not in the lobby when we were recruiting participants, people who attended RUME virtually, or people who had attended RUME in the past but were not in attendance in 2024. Potential participants could have also been too busy to participate in an interview or may have not been interested in or willing to participate. Thus, we cannot make claims regarding the generalizability to all RUME attendees – past, present, in-person, or virtual – regarding their intended and realized outcomes for their personal RUME conference participation. Further, while we noted career stage and prior attendance at the conference in the results as a proxy for centrality in the community, research is needed to understand trends and nuance. For instance, do attendees in different career stages value the social or community aspects in different ways, how does attending virtually facilitate or hinder attendees in achieving their intended conference outcomes, and is it the case that people who longer attend had outcomes that were not being met?

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Computational Tools' Mediation of Argumentation in Undergraduate Probability and Statistics

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As computation becomes increasingly central to mathematics education, instructors must balance competing forces when choosing which computational tools to use in their courses. This is compounded in probability and statistics where computation is widely used. Grounded in a social constructivist perspective, we believe that tools mediate our activities and that different tools play different mediational roles. As such, this study explores how different computational tools mediate undergraduate students' mathematical activity of argumentation. Using Toulmin's argument model, this research investigates how two classes in probability and statistics using different computational tools, R or Minitab, performed on a mirrored assignment. Through analysis of students' assignments, a difference emerged across the classes use of visuals. Our findings suggest Minitab promoted more deliberate consideration and use of visuals than R, leading to a difference in arguments produced by the students.

Keywords: Computation, Toulmin's Model of Argumentation, Statistics Education

The use of technology in undergraduate mathematics settings has routinely been emphasized over the last fifty years. This emphasis has been supported by technology expanding the set of possible activities for students and enhancement of their learning experiences (e.g., Chan et al., 2023). One such framing of technology is that of *computation*. Computation positions the use of specific kinds of technological tools as a means of mediating and augmenting problem solving.

DiSessa (2018) directly emphasized the impact computation can have on mathematics by drawing a parallel between computation and symbolic algebra. Since the invention of symbolic algebra, algebra has been integrated into society and has expanded the availability of complex mathematical understanding. Similarly, the promise and ubiquity of computational tools opens new opportunities for people to engage with and experience new mathematical understanding.

Indeed, many computational tools have been used in mathematics education to improve student understanding (Chan et al., 2023). In probability and statistics, in particular, there has been a consistent push for the use of computation to better augment and improve their learning experience, but also to prepare students for their future careers (Carver et al., 2016). Instructors are left to balance different considerations when choosing which tools to use in their courses. This work explores how different computational tools used in introductory courses influence students' mathematical activity.

Theoretical Framework

Our work is grounded in two theoretical frameworks: Vygotsky's social constructivism and diSessa's computational literacy. A central tenant of social constructivism is the claim that learning does not occur in isolation and, instead, happens in relation with others (Vygotsky, 1978). Beyond relations between other individuals, learning also occurs through relations between an individual and a culture, and a primary way this happens is with cultural artifacts or tools (e.g., a computer for the culture of American society). Significantly, all actions performed

with a tool are transformed through the mediation of the tool. The higher quality the tool, the more influence it will have on our actions. According to Vygotsky (1987), this is characteristically illustrated by language, the quintessential tool of human culture.

For this study, we are interested in the broad set of cultural tools that leverage technology for problem solving, which we refer to as computational tools. Computational literacy (CL) is a framework that grounds our argument that computational tools are a cultural tool.

Computational literacy draws parallels between the use of computation and other representational forms that had, and continue to have, lasting effects on society (e.g., written language, algebraic symbols; diSessa, 2000). Framed in this way, computation shapes society and, as such, is a cultural tool. Computation can be explored through the material, cognitive, and social pillars of a literacy. Specifically, material CL encompasses the mechanisms of using computation (e.g., writing syntax); cognitive CL includes the application of computation to solve problems (e.g., using computation to visualize and solve differential equations); and social CL involves communicating through and about computation (e.g., the presentation of visualizations).

In summary, tools mediate our actions, different tools mediate differently, and computational tools are an important cultural tool. As such, the use of computation across society, and within mathematics specifically, raises several important questions about how computational tools mediate activities. Our research question guiding this project is as follows: *How do different computation tools mediate undergraduate students' mathematical activity of argumentation?*

Literature Review

Several projects have framed their inquiry of what we refer to as computation, broadly, under the exploration of technology in mathematics education (e.g., Biehler, 1997). Additionally, many projects have explicitly probed computation in mathematics education through the lens of computational thinking (e.g., Chan et al., 2023). Across both bodies of literature, there has been supporting empirical evidence for the use of technology to solve problems in mathematics.

While there is evidence supporting the use of computation at the undergraduate level—and in probability and statistics, in particular—there have also been contrasting claims about what those tools should look like. One school of thought centers on the opportunity to give students experience with technology that better prepares them for their future careers (National Science and Technology Council, 2023; Nolan & Temple Lang, 2010). For example, Nolan and Temple Lang (2010) proposed the importance of preparing introductory statistics students for the ever-evolving landscape of technology available to practicing statisticians. For these authors, this suggested that the tools being used should be versatile and have features that promote general computational skills (e.g., troubleshooting). On the other hand, many computational tools have been specifically designed for their use in introductory settings (e.g., TinkerPlots). For example, Biehler (1997) documented the design and possible uses of a tool called Modelling in connection with Exploratory Data Analysis and Stochastic Simulation (MEDASS). The goal of MEDASS was to provide a learning environment that leveraged the affordances of technology (e.g., for data analysis) in an easy-to-use environment. Ultimately, easy-to-use and versatile can be conflicting features. As such, instructors must balance the different affordances when choosing a tool for their introductory undergraduate courses (Johnson and Berenson, 2019).

To date, several projects have probed the affordances of different computational tools from either the expert or student perspective. In Abbasnasab Sardareh et al. (2021), four tools were compared (SPSS, basic R, Jamovi, and R Commander) across their usability (e.g., the presence of a graphical user interface [GUI]), technical features (e.g., variety of visualizations), and software capabilities (e.g., ability to manipulate data). Through the authors' assessment of these

software, they concluded that tools with GUIs (e.g., SPSS) are important for introductory statistics, but acknowledged "[through] real software in [students'] introduction to statistics, they will be better positioned to use that software for real-world purposes later (p. 159)." In a similar study, Johnson and Berenson (2019) identified 11 criteria of computational tools that they had 13 faculty teaching introductory statistics use to rank a set of computational tools. JMP, a menu-driven statistical software, was the highest rated tool. These studies suggest that experts believe user-friendly tools like SPSS and JMP best serve their instructional goals for teaching statistics.

There have also been a handful of studies comparing students' uses of differing tools. Myint et al. (2020) compared students' uses and peer assessment of plots done in base R to those done using ggplot2, in R. Myint and colleagues reported that the users' of ggplot2 were more persistent and more likely to complete the assignments. Furthermore, through peer assessments, the plots produced in ggplot2 were perceived as clearer, containing more labels, and aesthetically pleasing. In another comparative study, Rode and Ringel (2019) measured students' anxiety and confidence about the use of R and SPSS outputs after they were exposed to either R or SPSS in their statistics course. Rode and Ringel reported no differences across the treatments in their pre/post-test changes in anxiety or confidence. Interestingly, both treatments evidenced reduced anxiety in using R *and* SPSS. These studies suggest that a difference can emerge in statistics courses that use different computational tools, but that is not always the case.

A question arises as to how differing tools influence specific mathematical activities (e.g., the activity of argumentation). As Wilensky (1995) demonstrated, computation can influence students' argumentation. As such, in this study, we narrow our exploration to the difference in students' argumentation mediated by different computational tools.

Methods

To compare how different computational tools mediated students' mathematical argumentation, a comparative design study was conducted. A homework assignment was designed to be completed by two classes in introductory probability and statistics at a university in the Northeastern United States. The assignment was co-designed by two of the instructors of record for the respective classes. As a separate part of their course design, the instructors had chosen computational tools that best served their instructional goals. In one class, students used Minitab (a menu-driven statistical software), and in the other, students used R (a programing language designed for statistical analysis) in Google Colab.

The primary goals of the assignments were for students to learn to use their respective tools, explore the gamma and Weibull distributions, and communicate their results. The focus of this report's analysis centers on the students' exploration of the parameters of the gamma distribution. In the gamma task, students generate random data sets and use histograms to visualize the gamma distribution. The goal of this task was for the students to manipulate the parameters, explore how they affect the distribution, and describe and justify their conclusions.

The assignments were designed to mirror each other, but differences in the tools required small variations in the assignments. Specifically, in the assignments the process of using the specific tool to generate data and visuals was described. In the Minitab assignment, the sequence of menus and windows was described. For Minitab, generating a random set of data fitting the gamma distribution begins with menu navigation (*Calc > Random Data…*). In the *Random Data* window, the user must enter a sample size in *Number of rows of data to generate* and a column name in *Store in column(s)*. Next, they must select the distribution they want the data to fit in the *Distribution* drop down menu (i.e., *Gamma*). Finally, they must enter values for the parameters

in *Shape parameter* and *Scale parameter*. Once they select *OK*, a column of random data fitting the specifications of their gamma distribution will be generated under the chosen column name.

For R (in Google Colab), the analogous process can be done in a code cell using the function *rgamma()*. Thus, the R assignment began with a text block describing the *rgamma()* function and a linked resource describing the different parameters of the function (e.g., *n* for sample, *scale* for the scale parameter, *rate* for the rate parameter). Then, in a code block, scaffolded code was provided that required manipulation. This scaffolded code began with the assignment of a sample size, *n*. Students were told this through a code comment to "assign a number of samples to take." In the next line of code, a variable name is provided, "random_gamma_data," which takes the *rgamma()* function with input of *n*, and the two inputs of *shape* and *rate*. Students were prompted to set values for the latter two inputs, again, through code comments.

After following these steps in both assignments, students will have a set of random data fitting a gamma distribution either stored as a vector called "random_gamma_data" in R or as a column with a given name in Minitab. Next, both assignments described the process of turning that data into a visual in the form of a histogram. For Minitab, this again entailed instructions for the sequence of menus and windows to navigate. For R, scaffolded code, linked resource on the *hist()* function, and in code comments were used to guide students. After these instructions for using each tool, the assignments provided the same instructions, questions, and prompts.

Data and Analysis

Two data sources were used in this project. First, all students' work on the assignments were collected. In total, 68 artifacts were collected across the two classes (35 from Minitab users and 33 from R users) In addition, artifact-based interviews (Brennan & Resnick, 2012) were conducted with two students from each class. During the interviews, students were given their artifacts and asked to describe their work, thought processes, and documentation.

Analysis focused on students' work on the gamma distribution task because it was characteristic of students work throughout the assignment. Using Toulmin's model of argumentation (Toulmin, 2003), we coded for claims, grounds, and warrants. Claims, or conclusions, were centered on the effect the two parameters of the gamma distribution have on the distribution. Grounds are the ideas/data supporting the claim, and warrants provide the link between grounds and claims. The use of visuals as grounds emerged from our data. As such, artifacts were coded for the number of graphs included. This was further categorized into either no visual, one visual, or multiple visuals included. Open coding was used for interview data.

Findings

The following section is broken into two subsections. To begin, we present interview data showing that Minitab and R users' interpreted the goal of the task in the same way and engaged in a similar exploration process on the assignment. Next, we show the emergent differences within students' artifacts. This centers on an exploration of students' argumentation structure.

Students Goals and Processes

During the interviews, students were asked what they thought the goal of the gamma distribution task was. Both Minitab and R users claimed the goal was to use their respective tools to explore the distribution and its parameters. For example, one Minitab user said, "the main point of this task was … take those [distributions] and play with them, see what the different parameters do." Similarly, the other Minitab user explained, "[the goal was] predicting, trying to make a pattern … If I increase this parameter, do I expect the range to increase?" For the R

users, one stated, “the main task was creating the gamma and Weibull distributions, changing the shape value and rate value, and seeing how those affect the distribution.” The other R user said, “[the goal of the task was] to do a deep dive on this distribution, and you come up with your own ideas about what it means.” Consistently, students identified exploration as the goal of this task.

Next, during the interviews, students were probed about their exploration process and their choice to include visuals in their assignment. Both sets of students created multiple visuals that they did not include in their artifacts, but only Minitab users evidenced a consideration of which to include. For example, an R user who did not include multiple visuals in their artifact, described their process as, “just changing my histogram function over and over. Trial and error.” This user continued, “I didn't really, while I was going through, think to copy each iteration I had done or anything. So instead of leaving a trail, I was just rerunning the code every time I made the change.” This R user generated multiple visuals but explicitly did not think to leave “a trail” of them. Similarly, the other R user said “So it never came across my mind to show other graphs. I just thought, just change a value at a time, but then moved on.” While this R user included multiple visuals in their artifact, they claimed to not think about which of their visuals to include in their artifact. On other hand, a Minitab user, who included multiple graphs described a curation process in choosing visuals to include. They claimed, “I considered doing seven or five [visuals]. An odd number just to show a balance of decreased and increased parameters… Then I chose the three that made the most sense and, to me, display the most drastic changes.” Across the interviews, all students engaged in a similar exploration process of generating multiple visuals. Uniquely, Minitab users thought about which visuals to include in their artifacts.

Differences in Artifacts

A difference also emerged in the students’ artifacts. 34 (or 97%) of Minitab users included multiple visuals in their artifacts versus just 16 (or 48%) of the R users. For the remaining R users, 15 included one visual and 2 included no visuals.

The importance of the inclusion of multiple visuals was in the students’ ability to use the visuals as grounds for their claims. In the next two sections, we provide characteristic examples of the kinds of arguments contained in students’ artifacts. Namely, a difference emerged in arguments from students' artifacts that contained multiple visuals and those that did not.

Arguments using multiple visuals. When multiple visuals were included, the visuals served as grounds for students’ claims. Warrants supporting these grounds came in two forms. The first kind of warrant was a description of what the visuals illustrated. For example, one Minitab user included three graphs as *grounds* for each claim they made about the parameters. For the shape parameter, this user selected three graphs (Figure 1) and produced a warrant in their artifact:

> “In the three graphs above I varied the shape parameter to see how it affected the random data. It can be concluded that the shape parameter has an effect on the skewness of the data, as the [shape] parameter is increased the data becomes less positively skewed.”

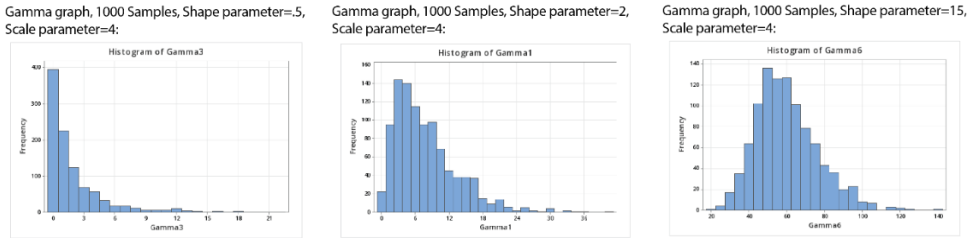


Figure 1. Three graphs generated by a Minitab user that served as grounds their claim.

This user made a *claim* that increasing the shape makes the skew "less positive." This was supported by the three graphs (Figure 1), thus the graphs served as the *grounds*. Additionally, this user provided a *warrant* connecting the graphs to their claim by connecting the visuals to the effect they were illustrating (i.e., increasing the shape parameter caused a change in skew).

Another way in which the grounds of multiple visuals were supported by warrants was through examples. In one R users' artifact, they produced multiple visuals, including two with a rate parameter of 100 (Figure 2). They wrote, "At a very high rate the points on the graph are extremely close together. For example, when using the sample size of 100 with a rate of 100, all the values became less than 1." This users' *claim* was about the relationship between large values for the rate parameter and the spread of the data. The *grounds* for this user came in the form of four different visuals (including two visuals generated by the code in Figure 2). Then, a *warrant* of a specific example was used to illustrate how the visuals (i.e., the grounds) support the claim.

```
gamma_data3 <- rgamma(100, shape = 1, rate = 100)
hist(gamma_data3, breaks = 20)

gamma_data4 <- rgamma(100, shape = 50, rate = 100)
hist(gamma_data4, breaks = 20)
```

Figure 2. Segment of code from an R user's artifact

Arguments without multiple visuals. When users did not include multiple visuals, their arguments lacked grounds for their claims. Many of these users included the last visual they produced and, as the R user from the interviews described it, "moved on." This resulted in claims that made no reference to their visuals. As such, the visual did not serve as grounds. For example, one R user included one visual in their artifact and made two claims with no grounds:

> "As the value for shape goes towards infinity, the shape of the curve appears more like a bell curve… The peak of this curve also becomes centered on the x-value of the shape. So if shape = 1000, the curve will be centered on x = 1000."

This users' first *claim*, about skew, was accurate, but they did not provide grounds because the visual they included was of a gamma distribution with a shape parameter of 1 (i.e., right skewed). Their second *claim* was that the curve is "centered" on the shape value. This student then produced an example. Ultimately, this student did not include a visual of this example. As such, the example was grounds for their claim, but the lack of a visual of this example meant their argument did not contain a warrant.

In addition to providing visuals as warrants for claims, the inclusion of multiple visuals could have served to clarify the claims of many R users. One R user wrote, "Increasing shape decreases maximum value of the distribution." One interpretation of this response is that the student *claims* shape and range are inversely related. Alternatively, this user's *claim* was related to the normalizing effect of increasing the shape parameter. In either case, the use of visuals as grounds for this claim would have supported *and* clarified their claim. Similarly, several students made claims about the skew moving away from a right skew, but what the skew was moving towards was vague. One user wrote, "The gamma data starts skewed right when the shape value is low, but as I increase it the graph starts to become skewed further left." In the same vein, another user wrote, "When I change the shape parameter k, the skew-ness of the graph changes. When I have $k < 1$, it is skewed to the right, and when $k > 1$, it begins skewing more to the left." If these students' claims are to be taken as written, then these R users are making a *claim* about a left-skewed gamma distribution. This is not possible for a gamma distribution because its skew is strictly positive (i.e., right). Through the process of producing grounds for their claims, these users may have made an important discovery or honed their claims, as well as their arguments.

Discussion

In exploring how students' argumentation was mediated by the computational tool they used, it emerged that Minitab users put more consideration into their use of visuals and used more visuals. We then showed how these visuals were used in arguments by students as grounds with warrants for their claims. When students did not have multiple visuals, their arguments contained only claims. The production of justifications for claims was an explicit goal of this task. Thus, the differences in R and Minitab users' arguments, consideration, and number of visuals included suggests the tools were mediating their mathematical activity of argumentation.

While possible, we do not believe that the assignment or students' interpretation of the assignment played a mediating role. As described in the methodology, the assignments were designed to mirror each other as closely as possible. Furthermore, through our interviews, we identified that both R and Minitab users had viewed the goal of the assignment in the same way. This suggests that the assignment was not mediating the emergent differences.

We conjecture that the difference in each tools' ease of manipulation could explain the differences in the students' artifacts. In R, students can easily modify a segment of code and immediately observe changes (e.g., the effect of increasing the shape parameter). In contrast, Minitab requires users to navigate a sequence of menus to make a similar change. As an R user described, they were "changing [their] histogram function over and over" to quickly explore the distribution. This evokes a dynamic process of students looking at a visual, changing a parameter in their code, running their new code, and watching as the graph changes. Ultimately, this process leaves no "trail," or grounds that could be used to support claims. In Minitab, on the other hand, the process is not quick, and users must navigate a series of menus and windows. To see the dynamic effect changing the syntax produced for the R users, the Minitab users had to have multiple visuals produced and change tabs to view them. As such, the Minitab users had multiple visuals simultaneously accessible, and were able to include them in their final artifacts. In fact, this led them to pick which of their graphs "display[ed] the most drastic change."

Furthermore, when users produced their artifacts, R users were working in an environment that they exported as their artifact (i.e., they submitted an exported Colab Jupyter notebook). This means they could quickly go from seeing a pattern, writing about it in a text block, and moving on with the assignment. The Minitab users, on the other hand, could not produce text blocks in Minitab. As such, they needed to move their visuals to another document along with their written claims and justifications. It is possible that this extra step for Minitab users necessitated deeper consideration and curation of visualizations being presented.

When considering what computational tool to use in introductory courses, instructors must balance the desire to use a tool that students can easily leverage, one that meaningfully impacts their mathematical activities, and a whole host of other considerations specific to a given context (Johnson and Berenson, 2019). Importantly, different computational tools can mediate students' mathematical activities in different ways. Our findings suggest that the methodical process of exploration in Minitab produced better argumentation, and R's dynamic visualization emphasized exploration over argumentation. While both are productive activities, instructors must weigh their own goals when choosing what tool best serves their students. One suggestion we have is, consider a balanced approach between multiple computational tools. Leveraging a beginner friendly tool like Minitab can ease students into the use of computation and can illicit stronger argumentation structures. Then, integrating more powerful tools like R, can give students the opportunity to explore, dynamically, concepts that are easy to visualize in R.

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Tutors' Perceived Differences in Working With Students on Online and Written Homework

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Mathematics tutoring centers are places where students learn math, but little work exists about learning from tutoring. Much of the research about tutoring is from an observer's perspective. We build on this work by presenting a study of tutors' experiences and actions when tutoring students with written and online homework, as described by tutors. We also suggest ways the Instructional Triangle could be adapted to account for the presence of a tutor as an agent in learning.

Keywords: tutoring, online homework, instructional triangle, written homework

Introduction and Background Literature

Undergraduate peer drop-in mathematics tutoring is generally defined as a collaborative educational support system in which experienced students assist their peers with course material with no specified time for a particular interaction (Burks & James, 2019; Johns, 2022; Johns & Burks, 2022; Johns et al., 2022; Mills et al., 2020). Previous research has indicated that the majority of higher education institutions in the United States provide some form of mathematics tutoring (Bressoud et al., 2015; Mills et al., 2020). Moreover, studies have shown that undergraduate peer mathematics tutoring may correlate with improved final grades, increases in students' mathematical confidence, and more ownership in the learning process (Byerly et al., 2018; Rickard & Mills, 2018).

Tutoring centers play a critical role in providing peer homework help, which is a crucial supplement to undergraduates' learning experience (Johns, 2022; Lepper & Woolverton, 2022; Topping, 1996). Undergraduate students are expected to dedicate significant time outside of the classroom to complete homework assignments and study (Dorko, 2020, 2021, 2024; Ellis et al., 2015; Krause & Putnam, 2016; Lew & Zazkis, 2019; White & Mesa, 2014). Instructors and students view homework as essential to learning (Dorko, 2019, 2021; Glass & Sue, 2008; White & Mesa, 2014). Hence, at the undergraduate level, students tend to have written and/or online mathematics homework (Kehoe, 2010; White & Mesa, 2014). However, there is a paucity of research comparing differences in online and written homework in terms of the activity of the people who might be involved in doing it – that is, students and tutors. Our study, which looks at tutors' experiences with tutoring online and written homework, contributes to a better understanding of learning from homework.

Although undergraduate mathematics peer tutoring is common, the research community is just beginning to focus on this out-of-the-classroom learning context. Some work has been done. For example, Johns and Burks (2022) discuss the general structure of a tutoring interaction as students seek assistance from a tutor, the tutor asks them what they need help with, students request help with broad content material or, more commonly, ask for assistance with a problem where they have hit a dead end or cannot find an error in their work. The tutor-tutee interaction concludes when the tutee has an answer accepted by the homework system or when the tutor and tutee agree upon an acceptable solution or summary of the problem. Johns et al. (2022) did a finer-grained analysis, looking at the observable actions tutors made during an interaction with a student. Their study identified 83 observable behaviors through the examination of 31 tutoring interaction transcripts. These behaviors were subsequently classified according to knowledge

type (i.e., Factual, Procedural, Conceptual, and Metacognitive) and further organized into specific categories (e.g., Supplied Facts, Immediately Reducing Cognitive Orientation, Heuristics, etc.).

The Johns et al. (2022) work was from an observer's perspective. Few studies have explored the knowledge and experiences afforded when carrying out the work of tutoring with different homework platforms (Johns et al., 2022). Our study, by interviewing tutors about their experiences and activities when tutoring written and online homework, aims to help fill that gap. That is, a key contribution of our work is adding to the knowledge of tutoring actions by documenting those actions from two tutors' voices.

A second contribution is the expansion of the Instructional Triangle theoretical framework (Cohen et al., 2003; Herbst & Chazan, 2012) to describe tutoring. This framework, which conceptualizes instruction as interactions between an instructor, students, and content, does not, in its current form, provide a useful way to look at tutoring. Hence, the paper focuses on the following three research questions:

1. What are some of the ways the Instructional Triangle can be adapted to account for the presence of a tutor?
2. What differences do undergraduate peer drop-in mathematics tutors experience while tutoring in the context of online homework as compared to the context of written homework when working with students?
3. What differences do undergraduate peer drop-in mathematics tutors describe in their tutoring actions when working with students in the context of online homework as compared to the context of written homework?

Introduction and Background Literature

We follow Cohen et al. (2003) in defining instruction as interactions between teachers, students, and content, situated in environments. Cohen et al. (2003) use the teacher, student, and content as vertices of the Instructional Triangle (IT). Edges of the triangle represent interactions between vertices, which need not occur face to face but are defined as encompassing the collaborative work of teachers and students. For example, an interaction may include a student engaging with a teacher-assigned written homework. Others have used the IT to study student learning in various environments, including student learning from homework (e.g., Dorko, 2020, 2021; Ellis et al., 2015).

Herbst and Chazan (2012) elaborated on the IT to frame studies of teachers' work. Their triangle, developed to focus on the teacher's perspective, does not account for the presence of a tutor in undergraduates' learning. In our study, tutors are humans who have face-to-face interactions with students around content. Tutors also have their own content knowledge and have historical and "present" interactions with instructors. As an example of the former, tutors may be tutoring for a class they previously took as a student and hence know something about the instructor's aims for a particular problem. As an example of the latter, professors may communicate with tutoring centers about homework to support more efficacious tutoring interactions. It is necessary to figure out where the tutor should be in the IT to employ the IT to study tutoring interactions. One possibility is as a resource, since the role of a tutor is that of a helper. We argue that because tutors are humans - like the student and teacher - positioning them as a resource is limited. Instead, we propose a theoretical modification to the IT framework that integrates a tetrahedral model with a tutor as one of the vertices (Figure 1). We developed this model on the hypotheses stated above: there are interactions between tutors and students, tutors and professors, and tutors and content. We propose that the incorporation of a tutor vertex within

the IT is imperative in fully elucidating the experiences and actions of undergraduate drop-in peer mathematics tutors during their interactions with students on both OHW and WHW. This model, shown in Figure 1, guided our data collection. In data analysis, we sought to verify and refine our hypothesized model.

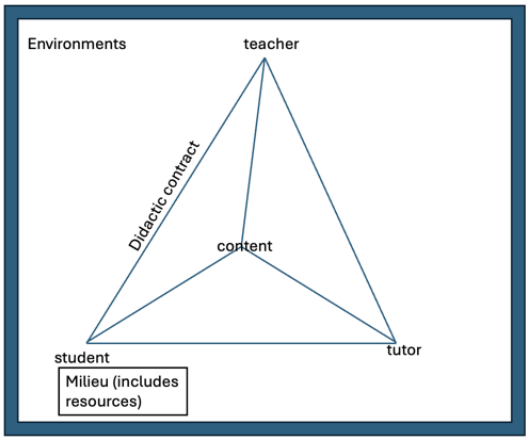


Figure 1. Our proposed modified IT with a tutor component resulting in a tetrahedral model.

Methods

The data comes from transcripts of 45-minute audio-recorded narrative interviews (Agee, 2009; Tracy, 2019) with two undergraduate drop-in peer tutors at a large public university in the United States in the spring of 2024. The first author selected two participants who worked at the university's mathematics tutoring center, possessed a minimum of two semesters of tutoring experience to ensure their familiarity with tutoring, and had experience working with students on OHW and WHW. Interview questions focused on having tutors describe their interactions with students when doing online homework and when doing written homework.

To analyze the data for RQ1, we aimed to verify and refine the six theoretical tenets we hypothesized (Figure 1; edges: tutor-student, tutor-professor, tutor-content; faces: tutor-student-content, tutor-student-teacher, tutor-teacher-content) and to identify theoretical tenets that are important but that we did not account for in our hypothesized model. We sought quotes that represented each of the six components (verify/refine our hypotheses) and instances that did not fit into those six components (identify new important parts of a model). This process allowed us to confirm, refine, and contest our expanded model. If we had at least one quote for a tenet, we kept it in the model. We had quotes supporting all six hypothesized components. With the quotes that did not fit in the model, we sought themes via grounded theory (Strauss & Corbin, 1998). That is, we looked for themes within the quotes. The emergent themes were the didactic contract and tutors' experience with the milieu. To analyze the data for RQ2 and RQ3, we used grounded theory to seek themes in tutors' experiences and actions.

In analyzing the data for RQ1, we read one transcript together, highlighted any statement that commented on an experience with a particular homework format, and then looked for themes in the statements. We then individually read the second transcript, coding it with the themes we had identified. We met to compare our coding of the second transcript, added to our list of themes, and then re-coded the first transcript together. This supported the validity of the analysis. The analysis for RQ2 and RQ3 was the same iterative, together/separate/together procedure. In the analysis for RQ1, the difference in coding was that we were looking for themes we hypothesized, with our minds open to themes we had not thought of.

RQ1 Results: Adapting the Instructional Triangle to Account for Tutoring

We found evidence of the three additional edges and three additional faces we hypothesized (Figure 1), as well as three emergent components. The emergent components were the didactic contract between the tutor and student, the didactic contract between tutor and instructor, and the tutor's knowledge of the milieu. Figure 2 shows the visual answer to RQ1; the verbal answer is that the IT can be adapted by adding a tutor vertex, the edges, and faces between the other vertices that represent the interactions with the other vertices/edges and make the model a tetrahedron; and by considering the didactic contract between the tutor and student, the didactic contract between tutor and instructor, and the tutor's knowledge of the milieu.

Due to space limitations, we provide evidence for only some of the edges and faces we hypothesized to be important, though the data contain evidence that all exist and are key components of an IT model that accounts for tutoring.

As an example of the tutor-content edge, tutors drew on their own content knowledge to make sense of questions in written homework. For example, Tutor 1 said, "written homework sometimes it's like okay, how should this be interpreted… I have to just say, 'I think this is how it should be interpreted'". We infer that the tutor drew on his knowledge of math to interpret the question. As an example of the tutor-content-instructor face, a tutor said online homework often followed class notes. We infer tutors looked at class notes to help students and see class notes as a way tutors and instructors interact because students tend to write what professors write (Lew et al., 2016). That is, the instructor communicates content to the tutors via class notes.

The first emergent component was that tutors felt there were expectations between the tutor and instructor, which we call a tutor-instructor didactic contract. For example, a tutor said tutoring interactions were "like trying to explain concepts, explain stuff, but you're not in a lecture situation… it's almost like you're trying to explain like the bare minimum to get this problem". Another tutor agreed, saying, "I catch myself… trying to explain this concept from scratch when that's not what I'm supposed to be doing… I'm not supposed to be replacing a professor." We infer the tutor expected the instructor to have explained the concept fully and the tutor expected to be able to fill in gaps or explain just enough for the student to connect back to what they had learned in lecture. These quotes are some of the evidence that tutors have expectations for instructors. They also support our hypothesized tutor-instructor edge because tutors thinking about the instructor represents a tutor-instructor interaction.

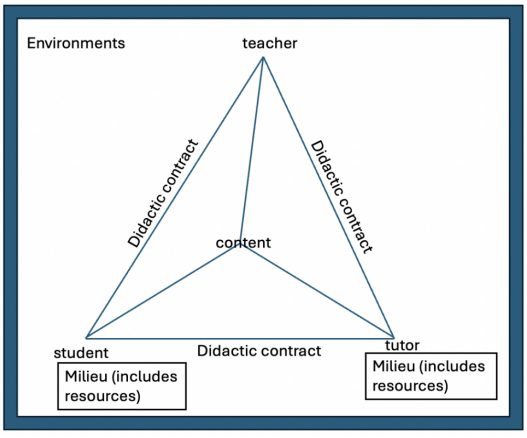


Figure 2. Our adapted tetrahedral model to include the three emergent components.

The second emergent component was a didactic contract between tutors and students. Tutor 1 said, for instance, his job "isn't to make sure the student gets homework right" but rather "learn with the student or like help, bare minimum, make sure the student is better off than when you sat down with them." This emphasizes the role of tutors as supporters of learning and growth, rather than just correctors of mistakes.

The third emergent component was the tutor's knowledge of the online homework platform (in this framework, a milieu). Tutors described how their experience with online homework affected their interactions with students. For example, Tutor 1 described, "the student is frustrated because they're like, I thought I did this right' and then I look at it and I'm like, oh, well, actually, I've seen this before, you have the right answer, there's a formatting thing that's wrong." This quote shows how the tutor knew online homework platforms often have precise expectations for answer format. A tutor also described how he felt written homework was "more detailed, it's more intricate… questions that require you to think about things in certain ways" whereas he felt the online homework was "practice, practice, practice." That is, tutors had different expectations and activities for the different milieu, which was reflected in their activity with students because of the ways they helped students navigate the two milieu. RQ2 and RQ3 lend insight into the nature of that navigation.

RQ2 Results: Differences in tutors' experiences working with students on OHW vs WH

The answer to RQ2 is the list of themes that describe the differences in tutors' experiences working with students on online homework as compared to written homework. These themes were the frequency with which they were asked questions about either, that students were more reliant on solution templates and online resources for OHW, that they thought more about the instructor while tutoring WH, that the expectations for correctness felt different, that students were more willing to write for WH, and that the differences between OHW and HW depended on course level.

Tutors differed in whether they said they answered more questions about written homework, or more questions about online homework. Tutor 2 said he fielded more written homework questions because those assignments were "usually created by the professor, so [students are] not going to Google and find that. [They] have to come in person". That is, he attributed the difference in number of questions asked to the availability of resources. In contrast, Tutor 1 answered more questions about online homework, which he thought was because students simply had more online homework tasks than written homework tasks.

Tutors said one difference in their experience with the two formats was that for lower division courses, HW and OHW were similar and tended to follow the notes, but upper division courses' WH differed from OHW because "[instructors] are able to get a lot more creative and force a lot more of like critical thinking into [it]" (Tutor 1).

Both tutors described a key part of their experience with OHW was working with students on template-based solutions, which was not the case with WH. Tutors detailed how, on OHW, students often defaulted to pattern-matching strategies from worked examples and were less inclined to engage deeply with the material. Tutor 2 said, "the most frustrating thing sometimes is the trying to pull them away from 'I saw [the solution] online, and they did it this one way'… it kind of like sucks them in… I try and be like 'yes, I understand that, but let's look at it like this'… in terms of like online versus written homework… the biggest struggle… it's just trying to get them in the realm where they're open to like different approaches to solving the problem." That is, the tutors felt students were attached to worked examples in OHW, and more open to problem-solving in WH.

Tutors described how they thought more about the instructor's goals and expectations with WH than they did with OHW. They often discerned the instructional design of the problems, in part because they often had taken a particular class with that professor. Tutor 2 commented, "different professors will look at things very differently" and that they knew professors often wanted "certain things shown in certain ways."

Part of tutors' experience with the two formats was helping students navigate their affect toward the two formats. Tutors described helping students work through frustrations with OHW platform constraints, something they did not have to do with WH. For instance, the tutors talked about students being frustrated with OHW wanting answers in particular formats, or students feeling demoralized by having limited attempts. Tutors also described students "enjoy [WH] less" than OHW because students felt WH sometimes had "confusing wording" or required more critical thinking. They also said students were more willing to write on paper on WH, while getting students to write anything down on an OHW task was challenging.

Tutors described one difference in formats was whether they knew their answers were correct, though Tutor 1 said he knew his job "isn't to make sure the student gets homework right. Your job is to learn with the student or like help, bare minimum, make sure that the student is better off than when you sat down with them… written homework… we don't get feedback on it." The tutor noted that OHW gave immediate feedback as to correctness.

Finally, tutors noted that a key difference in their experience between working with students on the two formats was students' willingness to write. Tutor 2 said, "in WH… you have to write it. It's not on a screen… it takes less effort for me to get them to be doing stuff… in WH… they are expecting to write. Online HW, it's on the computer… you do see students more willing to engage [with WH] because the more that they're writing, the more that they're thinking about it". In summary, the tutors found students were more willing to write and engage in work on WH than OHW.

RQ3 Results: Differences in tutoring actions for OHW and WH

To identify the differences in tutoring actions for OHW and written homework, we found themes in their actions for both milieu, then compared and contrasted the themes. We found in OWH, tutors worked with students to navigate using platform-provided similar examples; sought procedural and conceptual mistakes in students' work; had students explain problems; leveraged immediate feedback; directed students to resources; encouraged students to write down their work; and had students attend to particularities in the formatting of answers. We found in WH, tutors worked through problems themselves while modeling their process for students and directed students to use their course notes.

As an example, both tutors talked about how a key difference in their activity for OHW and WH was that in the former, they spent a lot of time helping students make sense of the worked solutions available in most OHW platforms (called "Help Me With This" (HMWT) in one platform) at the click of a button or directing students away from that resource. For instance, Tutor 2 tried to avoid HMWT. He said, "I always try to steer away from…usually I have them explain the problem, and then if [the students] have work, I'll start to go through that, and that's usually where I catch mistakes. Sometimes I'll catch the conceptual mistakes". Tutor 2 also describes here how when tutoring with OHW, one of his actions was to try to identify mistakes.

As another example, tutors leveraged electronic resources with OHW and directed students to those resources. For instance, Tutor 1 said he used the immediate feedback and multiple tries with online homework to try ideas, telling students, "I like the idea that you have, let's go through that, see if that answer's right". Tutor 2 said he used online resources a lot, describing

"online homework, it's easier to get them to approach a question graphically. 'Oh hey, can you open a tab of Desmos? Like that kind of stuff is a lot easier to show online than written".

As a final example of a difference tutors noticed in their activity with OHW versus written, tutors said they spent time encouraging students to write their work down in OHW. One said of WH, "this is on paper, you have to write it" and so students more naturally wrote steps in written homework, but had to be reminded the same could be useful for online homework.

Tutors described in written homework, some of their activity entailed thinking about how the professor wanted the students to approach the problem, and they often directed students to look at their class notes. Tutor 2 described that the WH were usually "tailored specifically to the notes".

In summary, the answer to RQ3 is that the key differences in tutoring actions between WH and OHW were related to the nature of the various milieu. In OHW, much of the tutors' activity was helping students to navigate the resources available online, while in WH, much of their activity was helping students to leverage their resources from class.

Discussion and Conclusions

We aimed to build the body of knowledge about tutoring by adding tutors' voices to complement literature about tutoring actions from an observer perspective. While Johns et al. (2022) conducted a fine-grained analysis of tutoring session transcripts by open coding, our study introduced the invaluable voices of the tutors themselves, thereby contributing new insights to the field. In akin with Johns et al. (2022), our research also identified specific instances where tutors employed specific actions to actively engage students and facilitate their understanding of mathematics.

We also aimed to build the body of knowledge about student learning from homework. We did this by comparing written and online homework from a tutoring perspective. We found many of the differences in tutors' actions and experiences were related to differences in milieu, in particular, students' reliance on worked solution templates in online homework. Our finding affirms those of Dorko et al. (2023) and Dorko (2021, 2020) that such templates often play a key role in students' online homework completion.

A key area for future work is to describe the specific content students learn from tutoring interactions, and how tutors contribute to that learning. We hypothesize the adaptations to the IT that we have proposed are a possible frame for such work. Another area for future work is to continue to develop this adapted model of the IT, in particular, because our adaptations were based on interviews with only two tutors.

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An Exploration of Differences Between Undergraduate Mathematics Students' Knowledge About Graduate School at R1 versus Liberal Arts Institutions

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The Carnegie Classification sorts institutions by their different styles of education. Two prominent types of institutions are high research activity universities (R1) and liberal arts focused colleges (LA). Institutional characteristics may positively (e.g. Eide et al., 1998) or negatively (e.g. Astin, 1997) affect graduate school aspirations. We analyzed responses from a national undergraduate mathematics major sample using chi-squared and Mann-Whitney U tests to identify differences between students' knowledge about graduate school and its application process by these two types of institutions. Using this same sample, we used chi-squared tests to explore the differences between departmental (professors, advisors, and mentors) support by institution type. We interpret these results with the theories of social and cultural capital and offer suggestions for future research investigations.

Keywords: Graduate School Applications, Graduate School, Liberal Arts, R1, Social Capital

Introduction

Higher education institutions are diversified to accommodate different educational goals. Some universities and colleges focus on research activity, others focus on teaching and mentoring of smaller student bodies, and others prepare students for specific skills-based professions. Studies show that these institutional characteristics affect student outcomes including whether or not students attend graduate school (Adedokun et al., 2012; Astin, 1997; Blaney & Wofford, 2021; Bruthers & Matyas, 2020; Eide et al., 1998; Ethington & Smart, 1986; Faurot et al., 2013; Hanson et al., 2016; Hathaway et al., 2002; Hearn, 1987; Kot, 2014; Mu & Fosnacht, 2019; Ostrove et al., 2011; Willis et al., 2013; Zhang, 2005).

Institutions with a similar focus often have many characteristics in common. The categorical system used in classifying institutions is the Carnegie Classification of Higher Education Institutions or Carnegie Classification for short (Indiana University Center for Postsecondary Research, n.d.). The Carnegie Classification sorts universities into doctoral universities, master's colleges and universities, baccalaureate colleges, baccalaureate/associates colleges, associates colleges, and special focus institutions. Each category contains smaller subcategories and the two main categories we will focus on are doctoral universities with high research activity (R1 institutions) and baccalaureate colleges with a liberal arts focus (LA institutions). In the most recent classification from 2021, the baccalaureate colleges with a liberal arts focus were labelled as baccalaureate colleges with an arts and sciences focus. We maintain the LA (liberal arts) label since most of those institutions still self-identify with that label.

Literature Review

Devereux (2023) states that an institution's classification ranking has a direct linear relationship with its level of sponsored research and foundation funding with higher rankings associated with larger amounts of funding demonstrating a Matthew effect, in that institutions

with resources are more adept at securing more resources (Devereux, 2023). Thus, non R1 institutions experience limitations or barriers in growing their faculty and administrative resources leading to strain on existing faculty and staff to secure externally sponsored funding (Devereux, 2023). Grunig (1997) discusses similar findings with research and reputation arguing that institutions increase their research levels to supply satisfaction to their customers so that they can receive the resources (funding) needed to continue their mission (education). Thus, the higher the institutions' classification, the more resources they obtain, and thus their customers (students) perceive higher quality service (education).

Before examining the literature on undergraduate institution effects on students' graduate educational goals, it must be noted that Etmanski (2019) suggests that the "cooling out" hypothesis (students opting out of pursuing an advanced degree) does not play out within the Ph.D. program, but instead prior to the entrance into the program. Additionally, Astin (1997) reports that attending an R1 institution negatively affects graduate school aspirations. Thus, it is imperative for all undergraduate, but especially R1, institutions to not adopt cultural characteristics that discourage students from pursuing a Ph.D.

Some studies note that the quality of the undergraduate institution affects graduate school aspirations and attendance. Eide et al. (1998) and Zhang (2005) report that graduates from high quality or elite colleges were more likely to enroll in graduate programs and for that program to be at a R1 institution. These studies viewed attending R1 institutions for a Ph.D. positively since these institutions focus on providing Ph.D. students with necessary and higher quality research experiences due to their additional resources (Devereux, 2023; Grunig, 1997). With regards to undergraduate research experiences, participating in research experiences in general have been shown to have positive effects on graduate education (Adedokun et al., 2012; Bruthers & Matyas, 2020; Hathaway et al., 2002; Willis et al., 2013). Personal connections with professors through interactions or teaching also affects graduate school aspirations. Researchers have shown that good teaching practices (e.g. non-classroom interactions, frequency of interactions, teaching clarity and organization), mentorship by faculty on undergraduate research projects, integration within the social and academic systems of the institution, and supportive departmental climate have positive effects on persistence to graduate school (Ethington & Smart, 1986; Faurot et al., 2013; Hanson et al., 2016; Ostrove et al., 2011). Researchers have also shown that personal relationships with faculty have a strong positive effect on graduate school aspiration (Blaney & Wofford, 2021; Hearn, 1987). Thus, personal relationships with professors impact graduate school aspirations.

A different aspect of relationships with professors and departments is the quality of advising the students receive. Mu and Fosnacht (2019) found that students' interactions with advisors were crucial for learning outcomes and that the quality of their peers' interactions with advisors influences students' self-perceived gains, over and above their individual interactions. With regards to centralized advising, students with centralized advising reported a better perception of advising compared to faculty advising and achieved better grade point averages (GPAs) compared to students without any advising (Kennedy-Dudley, 2007; Kot, 2014). It must be noted that these studies have primarily occurred at R1 institutions where faculty members may not have enough time or be offered incentives to invest the necessary amount of time and resources to advise and mentor students well (Faurot et al., 2013; Kot, 2014). Thus, faculty advising at LA institutions might have similar effects since the professors are more likely to be offered the necessary incentives. Therefore, advising affects graduate school aspirations.

In this paper, we address the following research questions:

1. What are the differences between R1 and LA mathematics students' interest in attending graduate school? What are the differences between R1 and LA students with respect to their knowledge about graduate school and applying to graduate school in mathematics?
2. What are the differences between R1 and LA mathematics students with respect to the quality of their relationships with their mathematics departments? Do these relationships have an impact on the answers to the previous question?

Theoretical Framework

In this paper, we apply the theories of social and cultural capital (Bills, 2000, 2003; Bourdieu, 2018; Coleman, 1988). Coleman (1988) defines social capital by its function, as a "variety of different entities all sharing two elements in common: they all consist of some aspect of social structures and they facilitate actions of actors, whether persons or corporate actors, within the social structure" (p. S98). For this paper, we consider the potential for information abiding in social relations as social capital (Coleman, 1988). Specifically, Coleman (1988) describes "social capital relations that constitute a form of social capital that provides information that facilitates actions" (p. S104). In our case, we view the relationships that undergraduate students have within their departments (peers, advisors, staff, professors, and mentors) as this form of social capital with respect to applying to graduate school in mathematics. Another form of social capital inherent to relations within departments is access to letter writers. Possessing a strong enough social relation with a person (in our case, a mathematics professor) enables the action of asking for a letter of recommendation. In this paper, we focus on student social relationships with their advisors, professors, and mentors.

Bills (2000) defines cultural capital to be "the degree and ease of familiarity that one has with the dominant 'culture' of society" (p. 90).[1] For our purposes, since most graduate schools in mathematics are located at R1 institutions, we argue that R1 students have more cultural capital than LA students regarding graduate school aspirations. Indeed, R1 students are surrounded by graduate students and graduate faculty, and even more so, graduate students at R1 institutions teach undergraduate courses. So, undergraduate students might interact with graduate students as much as the faculty. Therefore, this observation leads to the fact that R1 students have more exposure and so more opportunities to learn about graduate school compared to their LA peers.

Methods

Instrument Development

The research team created a survey partly based on another used to investigate undergraduate physics majors' interest in graduate school and how important they believed different aspects of the application process were (Chari & Potvin, 2019).[2] Nineteen survey items were adapted from that instrument. Our survey also provided an opportunity for participants to express their lack of knowledge about different parts of the application process. The final survey had 57 items in four categories: (a) knowledge about different aspects of the application process, (b) barriers for applying, (c) interest in graduate school and what students want in programs they apply to, and

[1] The published manuscript does not contain this quote, but an earlier study cites the unpublished version containing it (Bills, 2000, 2003; Pascarella et al., 2004).
[2] The full survey is available at this link: https://researchrepository.wvu.edu/faculty_publications/3291/

(d) demographic items. Most were Likert scale or multiple choice, but four were open-ended and some multiple-choice items allowed participants to type in text responses.

Data Collection

The research team sent the survey to department chairs and undergraduate program directors at all undergraduate mathematics programs at U.S. colleges and universities with at least 1000 students total ($N = 985$). We asked programs to distribute the survey to all undergraduate mathematics majors. Emails were sent Fall 2022 through Spring 2023, via Qualtrics, and follow-up emails were later sent to increase response rate. The survey was also posted on social media, listservs, and in several professional mathematics organizations' newsletters.

Data Analysis

We received 1090 responses from students at 181 colleges and universities, with 519 complete responses. Note that students could miss part of a question and still have their response marked as complete. Thus, the *N*s for different items are not always the same. Statistical tests were run in IBM SPSS.

To address our research questions, we analyzed responses to six survey items. Four were multiple choice items. The first asked participants about their interest in graduate school in mathematics. Possible responses included not interested, interested in earning a masters degree only, interested in earning a masters then a doctorate, interested in attending a post baccalaureate program before a doctorate, interested in earning a doctorate without a masters or a post baccalaureate first, and not sure. The second asked participants to rate their familiarity with the following concepts about the GRE Mathematics Subject Test; response choices included never heard of the item or if they have heard of the item before. The list of sub-items included the subject tests are only available three times a year, the subject test is not available to take from home and you must travel to a testing center to take it, and the Mathematics Subject GRE costs $150 to take. The third asked participants if their advisor (the person who helped the participant sign up for their courses) was faculty in mathematics, faculty in another discipline, staff (not faculty), or other. Then, the fourth asked if their advisor would support them in applying to graduate school. Possible responses were Yes (but I have not asked), Yes (they either suggested it and/or are actively supporting me), No (I have not asked), and No (I have asked). Another item was binary asking if students had a mathematical mentor (someone who had not necessarily been formally assigned to the participant by their institution but gives them advice on how to advance their career in mathematics). The last two were Likert scale items: *To what extent are the following factors a potential barrier to your pursuit of graduate school?* and *To what extent do you…* (followed by a set of phrases). The first item was adapted from Chari and Potvin (2019) and had 17 sub-item topics rated on a scale of 1 (not at all a barrier) to 5 (very significant barrier). The second item had 11 sub-item topics rated on a scale of 1 (not at all) to 5 (very much so). Chi-squared tests of association were employed to analyze the multiple-choice items. For the Likert scale items, a one-way analysis of variance (ANOVA) was not employed because for three out of the four sub-items, the Homogeneity of Variance assumption was violated. Thus, for ease of comparison and consistency, Mann-Whitney U tests were performed using the R1/LA variables for all sub-items.

Results

Before we present our main results, a few small reports are in order. First, a chi-squared test of association found that there was no significant difference between R1 and LA participants in wanting to attend graduate school. Second, for all the χ^2 tests of associations applied to questions regarding knowledge of graduate school, only two questions were statistically significant and they covered knowledge about the GRE Mathematics Subject Test, specifically whether participants knew: (1) Subject Tests are only available three times a year in September, October, and April [$\chi^2(1, 211) = 4.279$ ($p = 0.039$, $V = .14$)]; (2) The subject test is not available to take from home and you must travel to a testing center to take it [χ^2 $(1, 211) = 4.244$ ($p = 0.039$, $V = .14$)]. For those questions, only participants who recorded that they knew about the GRE Mathematics Subject Test saw them. Thus, there was not a significant difference between the populations with knowing about the GRE Mathematics Subject Test, but there was a significant difference in specific knowledge about the test.

Table 1. Descriptive statistics for the question, "Who is your undergraduate advisor?" using the R1/LA variable.

Group	Type	Faculty in Mathematics	Faculty in Another Discipline	Staff (not faculty)
R1	Observed	115	31	70
	Expected	139.1	32.0	50.3
LA	Observed	98	18	7
	Expected	73.9	17.0	26.7

Table 2. Descriptive statistics for the question, "Do you think your undergraduate advisor would support you in applying to graduate school in mathematics?" using the R1/LA variable.

Group	Type	Yes (not asked)	Yes (suggested or actively supporting)	No (not asked/asked)
R1	Observed	144	39	41
	Expected	138.7	57.9	27.4
LA	Observed	74	52	2
	Expected	79.3	33.1	15.6

Table 3. Descriptive statistics for the question, "Do you have someone you consider a mathematical mentor?" using the R1/LA variable.

Group	Type	Yes	No
R1	Observed	119	123
	Expected	137.6	104.4
LA	Observed	92	37
	Expected	73.4	55.6

When analyzing between the populations about departmental support, we found that LA participants were more likely than R1 participants to have an advisor that is a faculty member in mathematics [χ^2 $(2, 339) = 38.384$ ($p < 0.001$, $V = .31$)] (Table 1), feel supported in applying to graduate school by their advisor [χ^2 $(4, 372) = 45.248$ ($p < 0.001$, $V = .35$)] (Table 2), and have a

math mentor [χ^2 (1, 371) = 16.823 ($p < 0.001$, $V = .21$)] (Table 3). All these results were statistically significant differences ($p < 0.05$). The first two have a medium effect size (r between 0.3 and 0.5) and the third one has a small effect size (r between 0.1 and 0.3).

Note that only 2 (< 2%) out of 128 LA participants reported not feeling supported by their undergraduate advisor in applying to graduate school compared to 41 (18%) out of 224 R1 participants. This perceived lower lack of support, mentorship, and connection in mathematics was also reflected in Tables 4 and 5, related to Mann-Whitney U results on four Likert scale items.

Table 4. Mann-Whitney U test results for selected items for the question, "To what extent are the following factors a potential barrier to your pursuit of graduate school?" using the R1/LA variable.

Item	Group	*N*	Mean	Mean Rank	*U*	*Z*	*p*	*r*
Obtaining letters of recommendation	R1	242	2.77	208.79	10335	-5.61	<.001	0.29
	LA	130	2.01	145.00				

Table 4 contains Mann-Whitney U test results for R1/LA groups for the 372 participants who responded to the selected sub-item. The output of a Mann-Whitney U test is a Z value on a normal distribution. The Z value in Table 4 indicated that the mean of the R1 group was higher than the LA group. The result shows that there was a significant difference ($p < 0.05$) between the R1/LA groups. The R1 participants were more likely to view obtaining letters of recommendation as a barrier to their pursuit of graduate school compared to the LA participants. This result has a small effect size (r between 0.1 and 0.3).

Table 5. Mann-Whitney U test results for selected items for the question, "To what extent do you…" using the R1/LA variable.

Item	Group	*N*	Mean	Mean Rank	*U*	*Z*	*p*	*r*
Have mentors who encourage you to study mathematics	R1	243	3.42	168.41	11278	-4.63	<.001	0.24
	LA	129	4.14	220.57				
Relate to your classmates in mathematics classes	R1	243	2.88	174.65	12794.5	-2.99	0.003	0.16
	LA	129	3.29	208.82				
Feel like you belong in mathematics classes	R1	243	3.42	176.27	13187	-2.60	0.009	0.13
	LA	129	3.78	205.78				

Table 5 contains Mann-Whitney U test results for R1/LA groups for 372 participants who responded to the selected sub-items. The Z value in Table 5 indicated that the means for the LA group was higher than the means of the R1 group. The results show that there were significant differences ($p < .05$) between the R1/LA groups. The LA participants were more likely to have mentors who encourage them to study mathematics, relate to their classmates in mathematics classes, and feel like they belonged in mathematics classes compared to their R1 peers. These results had small effect sizes (r's between 0.1 and 0.3).

Discussion

Results from this exploratory study revealed that there was no statistically significant difference between our R1 and LA participants about knowledge of graduate school and the graduate application process except for specific knowledge about the GRE Mathematics Subject Test. We find these results encouraging, suggesting that in general, relationships at both institutions provide students with similar information. This is important given all the additional opportunities and resources available to R1 institutions and students (Devereux, 2023; Faurot et al., 2013; Grunig 1997; Hathaway et al., 2002). We conclude that LA students are receiving the same resources from their institutions or outside of their institutions, e.g. research experiences (Adedokun et al., 2012; Bruthers & Matyas, 2020; Faurot et al., 2013; Hathaway et al., 2002; Willis et al., 2013). Further research needs to be done to figure out which relationships are giving these students their social capital. Further, we found no statistically significant differences in aspirations to attend graduate school between these two groups.

We believe the supportiveness of the LA departments is providing their students with their capital (see results from Tables 1 through 5). In congruence with the literature, it seems that their personal relationships with their professors, advisors, and mentors are leading them to want to aspire to graduate school (Blaney & Wofford, 2021; Hanson et al., 2016; Hearn, 1987; Mu & Fosnacht, 2019). In further congruence, it appears that the supportiveness of their departments is also contributing to them wanting to aspire to graduate school (Ostrove et al., 2011). However, we note that, based on Table 1, 32.4% of the R1 participants had an advisor who was staff and only about half (53.2%) of our R1 participants had an advisor who was mathematics faculty. Based on the literature, these results are not surprising (Kennedy-Dudley, 2007; Kot, 2014), but they show that a sizeable portion of our R1 participants perceive that they are missing out on a crucial component of support from their mathematics community compared to the LA participants. We see this lack of support in Table 2 where 18% of R1 participants either assumed their advisor would not support them or that their advisor refused to support them in applying to graduate school. This leads us to conclude that our R1 participants are not forming personal connections with their professors, advisors, and mentors, or, at least in the case of mentors, they do not have one (see Tables 3 - 5). Further research must be done to investigate why this phenomenon occurs. However, these students still acquire the same knowledge as the LA students but without utilizing the social capital within relationships with their professors, advisors, and mentors. Instead, we argued that they acquire it through their cultural capital. We also hypothesize that they wield social capital but through their relationships with their peers and within math clubs. Further research must be done to confirm this argument. In conclusion, the LA participants acquire through their social capital the same advantages that the R1 participants acquire through their social and cultural capital, but further research must be conducted to decipher where R1 participants are acquiring their knowledge.

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Exploring the Concept Image and Concept Definition of Periodic Functions

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This paper presents the results of a study where a student's concept image and concept definition of periodic functions were explored. The findings suggest that a student thinks about periodic functions as visual repetition of patterns within a function's graph. He also analyzes output values, particularly at distinguishable points such as maxima, to determine whether a function is periodic. His concept image and concept definition include the idea that a function's period represents the length between any two points where the output values are the same and the pattern repeats throughout the domain.

Keywords: Periodic Function, Concept Image, Concept Definition

Introduction and Background

The periodic function is an important mathematical concept that plays a central role in understanding various topics in mathematics, physics, and engineering (Jia et al., 2011; Mays & Loverude, 2019). Some researchers have mentioned the concept of periodic functions through the lens of students' understanding of trigonometry and focused on periodicity as a property of trigonometric functions (Weber, 2005; Ortega, 2017; Brown & Rice, 2011). Dreyfus and Eisenberg (1980) noticed that although most students hold an intuitive understanding of the periodicity of a function, applying the properties of a periodic function in different contexts is often difficult for them. Although the literature on students' thinking of periodic functions is rare, I found a limited amount of work that discusses students' understanding of the concept of periodicity. The analysis of Buendía and Cordero's (2005) study revealed that predicting future movements based on the movements of the graph of a function is a characteristic that students may use to make meanings for the notion of periodicity. Shama (1998) investigated ways of understanding periodicity. The findings suggest that (1) the students consider periodicity a dynamic collection of elements (time dependency, motion, etc.), (2) the students think about periodic phenomena as a whole unified structure (the segment bounded between two points in one-dimensional space is considered to be a period as whole). Although both studies (Shama, 1998; Buendia & Cordero, 2005) primarily focused on students' perception of periodicity, they did not delve much into the role of periodicity in the concept of periodic function in mathematics. This study aims to contribute to the limited literature on the students' thinking of a periodic function.

Therefore, this paper's motivation is rooted in the fact that the mathematics and STEM education research community will benefit from learning the students' conceptual understanding of a periodic function. The results of this study provide opportunities for researchers to explore the role of a periodic function in various mathematical topics and use the ways of thinking to teach students. In this paper, I explore how an undergraduate engineering student describes his thinking of a periodic function while he works through a series of tasks.

Theoretical Perspective

The constructs *concept image* and *concept definition* were first introduced in a PME article (Vinner and Herschowitz, 1980). Then Tall & Vinner (1981) published the first journal article on these constructs and applied them in reference to limits and continuity. This research uses

concept image and *concept definition* to frame a student's thinking about periodic functions. These constructs provide researchers with viable ways to interpret the data and categorize students' mental pictures based on what they say, do, draw, point at, etc.

Tall & Vinner (1981) introduced the concept image as follows: "*We shall use the term concept image to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures" (p. 151).* The concept definition was mentioned as follows: *"We shall regard the concept definition to be a form of words used to specify that concept. It may be learnt by an individual in a rote fashion or more meaningfully learnt and related to a greater or lesser degree to the concept as a whole. It may also be a personal reconstruction by the student of a definition. It is then the form of words that the student uses for his own explanation of his (evoked) concept image. Whether the concept definition is given to him or constructed by himself, he may vary it from time to time. In this way a personal concept definition can differ from a formal concept definition, the latter being a concept definition which is accepted by the mathematical community at large" (p. 152).*

We have adopted this framework to analyze our data. We use the construct *concept image* as a whole structure that includes evoked mental snapshots of the student we interviewed while he worked through a series of tasks about periodic functions. We also use the term *concept definition* to describe the evoked personal definition he developed from his intuition after exposure to the tasks we presented. We address the following research question in this study by adapting the concept image and concept definition as our theoretical background-
What concept image and concept definition of periodic functions may a student hold while he works through a series of tasks?

Methods

Data Collection

I conducted a semi-structured clinical interview (Clement, 2000) with one participant, Fred (pseudonym). He was enrolled in an engineering major at a large southwestern university. The only requirement to participate was to have completed at least the first level of the Calculus course. Thus, Fred was familiar with the concept of functions, and upon asking, he mentioned that he associates periodic functions with trigonometry. However, periodic function concepts were never discussed in his Calculus classes. Fred participated in two interviews (each about an hour long) over Zoom. The sessions were video recorded via Zoom. The interviewer shared a screen with Fred with the tasks for each session. Blank spaces between the questions allowed him to annotate or write to capture his ideas and diagrams. He worked on a total of 12 tasks during the interviews. The tasks involved some open-ended questions such as: Have you heard of the concept of 'periodic function?' If yes, can you please say about your conception of periodic function? Some tasks presented graphical and imaginary situations to the participant, and interview questions were designed to learn what thinking process students use and how they use it in specific given scenarios.

Data Analysis

For analysis, I utilized in-vivo coding (Saldaña, 2021). A researcher uses in-vivo coding when s/he focuses on the words from data as a first step to code. I first transcribed using Adobe Premier Pro and then manually corrected auto-transcription errors. I read through the data and

named codes based on words and phrases. I labeled the data into discrete parts based on the codes I created. Next, I compared similar events in the data and collated excerpts that were labeled with a particular code. Then, I drew connections between the codes with the goal of finding how the codes can be grouped into categories. These steps of analysis aimed to open new theoretical possibilities. I used the terms concept image and concept definition constructs to frame the thinking of periodic functions students inhibited during the interviews by thinking out loud, drawing, gestures, pointing out, etc.

Results

In this section, I highlight the various ideas of periodic functions in Fred's concept image and his choice of a personal concept definition that has evolved while he worked through Tasks 1-7.

Periodicity as Repeating a Pattern

Fred was never exposed to the formal definition of a periodic function. Tasks 1 and 2 explored the student's current conception of periodic functions, followed by an example. In response, Fred described periodic functions as repeatable functions. To elaborate on Fred's description of a 'repeatable function,' he used the example of the sine function-

> "So, I mean, like, there might be some function like probably $\sin(x)$. Right. So, it would go like it would be a periodic function starting from zero that would go on to one and then come down again and then come to zero again. Right. So, if that's a period, it would be the same. The same. How do I say this? The same pathway of that function would be repeated after the period itself, like the period of the function".

I note that Fred was imagining the graph of a sine function within the interval of 0 to 2π. He then imagined the same pattern of the graph being repeated onwards. Here, Fred was thinking about the unchanged pattern of the graph within an interval as the period of the function.

Focus on Output Values

Task 3 presented a graph of a function that represents a recurrence pattern but does not return the same output value for any period. Fred acknowledged that the pattern is repeating but also mentioned that the graph does not represent a periodic function. He noticed that although the pattern is similar, the function does not have the same output value for any period. To validate his reasoning, shown in Figure 1, Fred chose segments of patterns within the graph. First, he separated intervals where the pattern repeats, and then he chose half segments within those intervals. He chose imaginary points where he assigned half segments of the pattern within the first two intervals, and then he chose imaginary corresponding output values for the input values. Then, he drew horizontal and vertical lines to show how the output value of the function does not return for a period.

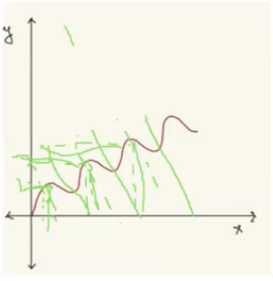


Figure 1. Fred's scratch work shows how the function does not return the same output values for any period.

Task 4 presented three apparent graphical examples of periodic functions, and one graph of a non-periodic function and the interviewer asked Fred to determine which of the following graphs represents periodic functions. This task aimed to determine whether a student demonstrates a consistent conception of periodic functions in relation to his/her previous responses and to learn what thinking process students use and how they use it to determine whether a function is periodic from graphs. Looking at the graphs, Fred quickly determined the three graphs representing periodic functions. The interviewer then asked him to explain his rationale for choosing each graph as a periodic or non-periodic function. Fred chose the word cycle to refer to one complete graph pattern within a period. To justify the first graph of the choices as a periodic function, his responses are as follows-

Fred: So suppose we are looking at this point x equals to something between -2π to $-\frac{3}{2\pi}$, okay? And if we take this one here (Fred points to another input value on the graph for which the output is the same as the point he initially chose) it would give the exact y value. It's the same y value for that point. That is why I would say this is a periodic function. And the cycle repeats itself. And it's exactly the same over here on the left side as on the right side. And it also, like the same interval, same time values. If you start from here, then after that, it would give you the same.

Interviewer: so, when you say, start from here, what do you mean? …

Fred: ….So, start over here, I mean, when the cycle starts here, (pointing to a value in the graph), we mentioned like…

Interviewer: -2π?

Fred: …Yeah. When the -2π starts and after a small interval…..probably like π over this (pointing to another point where the repetition occurs). If we consider this point, this would give the exact same y value as if we started here and took the same interval, like 0.2π. … It would give the same exact y value. So that's the reason I think it's periodic.

Interviewer: Okay…then like what is the period? How would you determine the period of this function?

Fred: Yeah. So, the same thing as I said before. So, whenever we have a function, and it starts, the cycle starts repeating itself, and we can see a trend. Right? So, that's how we divide it (he made segments of repetitive patterns in the graph). So, I can see that at x equals to 0, there, the cycle starts repeating itself again from here to here (pointing to the pattern within a period in). And it's the same exact function that starts from here again.

Figure 2 represents Fred's annotation in relation to the excerpt above.

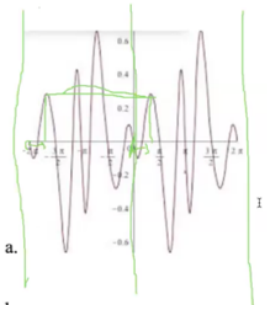


Figure 2. Fred shows repetitive patterns in the graph with the repetition of output values.

To determine whether a function is periodic, Fred initially identified a pattern of the graph that repeats, and he referred to the pattern within a period as a cycle. Then, in support of his

argument, Fred chose two points where the output values are the same for two different input values. He thinks about periodic functions as functions when the same pattern repeats after a period, and the output values repeat for different input values after each period. Figure 2 shows Fred's way of separating periods of a function and identifying the same output values for different input values as the pattern repeats. Fred drew a green horizontal line from one point to another to show an interval between two same output values. He also identified this length of the horizontal green line as the period of the function. Fred maintained similar reasoning to identify other periodic functions in task 4.

Fred's Concept Definition

Task 5 presented five statements that describe periodic functions. Fred used the ideas of repeating a pattern and finding the same output values while he comprehended those statements. I designed task 5 anticipating that students would use these statements to organize the cognitive structures they would have developed about periodic functions.

Task 6 asked Fred to reflect on the tasks he had worked through thus far and to define a periodic function. Initially, Fred mentioned that a periodic function is like a cycle that repeats itself, and for a specific interval after the cycle starts, it would give the same output value. He drew Figure 3 to elaborate on his definition of a periodic function.

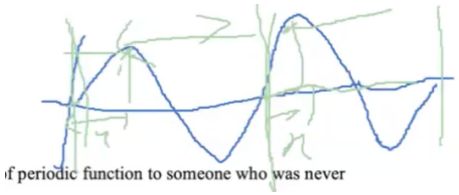


Figure 3. Fred shows a periodic function with two repeating cycles.

Here, Fred drew the graph of a function with two repeating patterns. He wrote two x's to show the starting points of each cycle. He also highlighted the length between two maximum points in the graph to show the period of the function. The interviewer asked Fred to pretend to explain the concept of periodic functions to someone who had never been exposed to it and how Fred could tell whether a function is periodic or not. Fred mentioned that he imagines a periodic function as a uniform repetition of patterns. In his words-

> "These two parts (referencing two repeating patterns in Figure 3) are uniform here. That's (uniformity) one of the criteria is that we can define it as a periodic function. And then again, we have the fact that a specific interval, for example, this (choosing $(0,0)$ point) is a starting point of this period for a specific interval. This length, (a length Fred chose named x in the graph) probably x, will give us the same y (output) values here from x (pointing at the starting point of the second repeating pattern). Here, it's going to be exactly same y values here. So, two things to keep in mind are uniformity and the cyclical nature of the function. And the other thing is that whenever we take a small interval after the periodic function starts, it would give us the same output on both intervals".

Fred summarized his concept definition again when the interviewer asked how he could tell whether a function is periodic. Fred defined periodic functions as functions that maintain a uniform repetition of patterns throughout the domain after certain intervals, and the functions have a cyclical nature. He also included the idea of the same output values in his definition, as he mentioned that for a function to be periodic, we can choose any length of intervals from the

starting points of each period, and the output values for the corresponding input values would remain the same. As we see in Figure 3, Fred noted two lengths of intervals named x's from the starting points of each segment of repeating patterns. Fred mentioned that the output values for the corresponding input values in both intervals would be the same.

The Distance Between *Turning* Points is the Period

Fred's primary criterion for determining a periodic function is to look for a pattern that repeats throughout the function's domain. However, to locate the period of the function from the graph, Fred relied on choosing a distance between two maximum points instead of choosing the distance between the starting and end points of one pattern that repeats itself. The interviewer asked his reasoning for choosing these points, and he replied that these are turning points in the graph. Fred used the term turning points to note the visually easier points (shown in Figure 3) so that he could measure the length in between without calculations.

In task 7, Fred encountered three imaginary periodic situations. The interviewer asked him to represent the context graphically and determine the function's period. The first context given was- *when the heart contracts, blood pressure in the arteries rises rapidly to a peak (systolic blood pressure) and then falls off quickly to a minimum (diastolic blood pressure). Blood pressure is a periodic function of time.* Fred chose some imaginary measurement of the blood pressure. The blood pressure would vary between 65 to 115 units of measurement, and at the initial moment, the blood pressure would be 90 units of measurement. A part of Fred's reasoning for graphing the situation and finding the period is mentioned here-

Fred: I am assuming that we're starting in the middle….90 blood pressure. I'm not sure about the unit…..so this goes up from there, the maximum which is 115. And it reaches a maximum of 115. It goes down again to the minimum. To 65. Then, when it hits the minimum, it starts going up again. And it hits 115 again. It starts coming down again, and the *cycle* keeps repeating itself. So, here we are considering that we're starting from the 90 and we're just stopping at 90 here as well. Let's assume that this is two seconds that it took for one *cycle* to be completed. So, two seconds would be the period for this function.

Interviewer: So, where does the *cycle* start, and where does it end? Tell me this, using the context.

Fred: So, for this case, …..we could just make it easier for us to say,….whenever we reached the maximum pressure in the heart, like blood pressure. The next time, we reached the maximum blood pressure again. We can say that it is one *cycle*.

Interviewer: Can you use a horizontal line to show me one *cycle*?

Fred: Yeah, this would be one cycle.

Interviewer: So, one maximum to another maximum?

Fred: Yeah.

Interviewer: Okay, so that's one cycle. And what is the period?

Fred: It would be the same. The period here is two seconds. That's what we assumed. Suppose this is 0.5, and here, it's 1, it's 1.5, this is 2, and this is 2.5. So, 2.5-0.5. That would give us 2.

I note that Fred used the words cycle and period interchangeably when he was thinking about the cycle length as the period of the function. To determine the period of the function from a graph, Fred relied on the easily distinguishable points on the graph. While working on task 6, Fred named his choice of maximum points (shown in Figure 3) as turning points when he was using a graph of a periodic function without context to determine the period of the function. His reasoning for choosing turning points to determine the period of the function was sustained in

task 7. However, this time, Fred was graphing a real-life situation, and he assigned numerical output values. Fred chose the distance between two maximum points of the function to determine the period. The contrast between tasks 6 and 7 is that in task 6, Fred chose a cycle from the starting to end point of a pattern that repeats, but in task 7, Fred mentioned the pattern between two maximum points as one cycle. When the interviewer asked Fred to determine the period of the function, he assigned numerical input values. He determined the distance between one maximum point and another as the period of the function.

Discussion

In this section, I summarize the important findings and discussion points based on the results. I highlight the key ideas of periodic functions in Fred's concept image and distinguish his personal concept definition of a periodic function from a formal definition.

Fred's concept image of periodic functions centers on the idea of a repeating, unchanged graph pattern. He refers to this as the same pathway of the function, emphasizing the consistent repetition of the graph's structure. To determine whether a function is periodic and to identify its period, Fred primarily focuses on the function's output values. He prefers choosing clearly distinguishable points on the graph, such as maximum points, as reference points. According to Fred, selecting these prominent points makes it easier to visualize the period of the function without needing to assign specific numerical values. Moreover, Fred's concept image includes a flexible understanding of the period as the distance between any two points where, for different input values, the output values return to the same value. He perceives the pattern between these two points as repeating consistently for subsequent intervals throughout the domain.

Dreyfus and Eisenberg (1980) formally defined a periodic function as $f(x)$ being periodic if there exists a positive number, p, such that $f(x+p) = f(x)$ for all x in the domain of f. In contrast, Fred's concept definition focuses on the idea of repeating patterns that remain consistent throughout the domain and the recurrence of the same output values after each period. Since Fred had not encountered the formal definition of periodic functions, it is reasonable that his concept definition did not include any mathematical notation. Instead, Fred relied on an evolving, descriptive understanding of a periodic function and how to recognize periodicity based on patterns without formal mathematical expressions.

Fred's understanding of periodic functions reveals an intuitive grasp of the concept, despite his lack of exposure to formal mathematical definitions. His focus on visual patterns and the repetition of output values allows him to construct a personal definition that aligns with certain aspects of the formal definition. By examining Fred's understanding of periodic functions, the study highlights how learners can meaningfully engage with mathematical concepts through visual and experiential reasoning, even in the absence of formal definitions. By understanding how individuals like Fred perceive mathematical concepts, educators can tailor their teaching strategies to help students connect intuitive understandings with formal mathematical language. Overall, Fred's case emphasizes the value of concept image and concept definition as a foundation for developing a deeper comprehension of mathematical ideas.

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Linear Algebra Student Thinking of Magnitude, Unit Size, and Coordinates

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This study explores how linear algebra students reason with three entities: magnitude, unit size, and coordinates in change of coordinate systems during teaching experiments. Two conceptual frameworks are employed in this study. The first framework, the coordinate system framework, was developed prior to the study and is used to frame a larger study that includes clinical interviews and teaching experiments. The second framework, the entities framework, was developed from the results of clinical interviews and is used to analyze student work from the teaching experiments. Students interacted with a GeoGebra applet that creates coordinate planes using matrices. The results present how students establish relationships among the three entities while working with the applet and how they represent their reasoning through matrix entries.

Keywords: Linear Algebra, Change of Coordinate System, Framework, Entity

Introduction and Literature

Coordinate systems are widely used in undergraduate mathematics. Students work with various coordinate systems, such as the Cartesian and Polar coordinate systems, and may switch between them in their courses. In linear algebra, students also encounter non-traditional linear coordinate systems that are similar to the Cartesian coordinate system but scaled and/or rotated. To graph a mathematical expression in a coordinate plane, students start by identifying coordinate pairs and comparing the components of each pair (a, b) with the unit length imposed by the coordinate system in each direction. (Lee, 2023) Observing this process, the author recognized the three entities that students engage with: coordinates, unit size, and magnitude.

Numerous studies have explored how students conceive of the magnitude of an object and how they interpret measures with changing units. (Steffe & Olive, 2010; Thompson et al., 2014; Vergnaud, 1988; Wildi, 1991). Attending to relationships of quantities or coordinating quantities has been a decent way to promote students' flexible reasoning with multiple coordinate systems in precalculus and calculus context (Joshua, Musgrave, Hatfield, & Thompson, 2015; Levenberg, 2015; Lee, 2017, 2020; Moore, 2015; Moore & Thompson, 2015). In linear algebra, change of coordinate systems is primarily discussed in the context of change of basis. Hillel (2000) pointed out that students tend to think of coordinate systems as the coordinate axes rather than something generated by a pair of basis vectors. Some studies concerned with students' difficulties in understanding matrix representations (Hillel, 2000; Hillel & Sierpinska,1994; Sierpinska et al., 1999). Specifically, Hillel & Sierpinska (1994) claimed that it is particularly hard for students to distinguish between the matrix representation of a linear transformation operator and the transformation itself. Wawro et al. (2013) developed a task sequence that helps students reinvent the [1]matrix diagonalization $A = PDP^{-1}$ in the context of Re-Naming points in a non-traditional coordinate system which uses y=x and y=-3x as the new axes.

Building on prior studies' insights into student conceptualizations of quantities, numbers, operations, and matrix representations, this report highlights the three entities (magnitude, unit size, and coordinates) and examines how students reason with them in coordinate planes. Given

[1] P is an invertible matrix and D is a diagonal matrix.

the increasing importance of linear algebra in STEM education (Tucker, 1993; Stewart et al., 2022), this study focuses on linear algebra students' thinking when working with non-traditional *linear* systems and matrix representations.

Conceptual Framing

Naming, Locating, Re-Naming, and Re-Locating

The coordinate system framework was developed from the author's calculus and linear algebra textbook analysis (Lee, 2023). I found that the types of examples illustrated in the textbooks handle coordinate systems differently, and that readers become engaged with different ways of thinking accordingly. The framework starts with two fundamental processes of representations: Naming and Locating. In Naming, a location in space is being measured following the convention imposed by a coordinate system and creates the measurement, a name. For example, a location in the 2D plane gets its name as (1,1) with the Cartesian coordinate system. On the other hand, in Locating, an existing name creates its location in space following the convention imposed by a coordinate system. An example of Locating is that the ordered pair (1,1) puts on a specific point in the Cartesian coordinate plane. Figure 1(left) illustrates that Naming and Locating are the reverse processes to each other. The two processes can be extended to represent an object with multiple coordinate systems: Re-Naming and Re-Locating. In Re-Naming, a location that has been measured by a coordinate system gets its new name measured in a new coordinate system that is laid atop the location. That is, the location previously paired with (1,1) is being renamed with $(\sqrt{2}, \frac{\pi}{4}) \approx (1.414, 0.785)$ in the same space using the Polar coordinate system (Figure 1, middle). On the other hand, in Re-Locating, an existing name creates two different locations in space depending on coordinate systems being used. The new location may appear different from the first, but they share the same name. For example, (1,1) corresponds to two locations: one defined by a horizontal and vertical distance of 1 each, and the other determined by a distance of 1 from the origin and an angle measure of 1 radian from the horizontal axis. (Figure1, right)

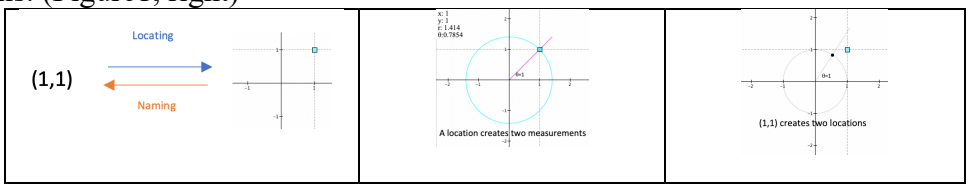


Figure 1. Naming and Locating (left), Re-Naming (middle), Re-Locating (right)

Three entities: Magnitude, Unit size, and Coordinates

The entities framework was developed from a part of a larger dissertation study (Lee, 2024). Through the process of Locating and Naming, I found that students engaged with three distinct entities in clinical interviews: magnitude, unit size, and coordinates. Magnitude refers to the size or extent of an object, characterized by its measurable attributes such as length, area, and volume. In Locating, students recognize the location of a coordinate as the distance of an object in each direction from the origin, which I specify as 'location magnitude'. Unit size refers to the measuring scale or dimension associated with each unit of measurement within a given coordinate system. When Locating, students compared a coordinate with a value 1 to determine the coordinate's location. In this case, unit size itself is a magnitude but is used as a measuring stick. Coordinates refer to number values assigned to the magnitude after measurement utilizing the corresponding unit size. The number value implies the amount of unit size. The three entities are conceptualized in a formula:

$$M(Magnitude) = U\ (Unit\ size) \cdot C\ (Coordinates).$$

The magnitude of an object is characterized by U (unit size), the measuring scale, and by C (coordinates), the amount measured. In Locating, the measured amount (C) is given, and then the location magnitude (M) is determined by multiplying the C by its measuring unit size (U). For example, in a clinical interview when locating an ordered pair (1.3, 0.5) on a plane, a student said " *the y* [0.5] *is the midpoint here* [points on the vertical axis]…*and…this is 1* [points to the unit in the horizontal axis], *then 1.3* [marks on the horizontal axis the right of 1]." The student needed the unit size of 1 to determine the magnitudes of 1.3 and 0.5 using multiplication either by scaling or through repeated addition. In Naming, the location magnitude (M) is fixed on the coordinate plane, and then its coordinate (C) is named by determining the amounts the measuring unit size (U) covers. For example, another student in the clinical interview rewrote the coordinate pair for a location $\begin{bmatrix}1.3\\2\end{bmatrix}$ as $1.3\begin{bmatrix}1\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\end{bmatrix}$, breaking it down into components along the unit vectors consistent with the conceptualized formula. This formulaic idea extends to situations where the unit size changes to another size with multiple coordinate systems, leading to two scenarios emerged with switching from one coordinate system to another: 1. Under constant magnitude M: If U_2 (unit size for the second coordinate system) is k times U_1 (unit length for the first coordinate system), then C_2 (coordinate in the second coordinate system) will be $\frac{1}{k}$ times C_1 (coordinate in the first coordinate system). (Figure 2, left) 2. Under constant coordinates C: If U_2 (unit size for the second coordinate system) is k times U_1 (unit length for the first coordinate system), then M_2 (magnitude in the second coordinate system) will be k times M_1 (Magnitude in the first coordinate system). (Figure 2, right)

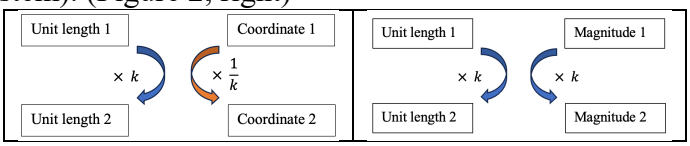


Figure 2. Re-Naming (left), Re-Locating (right)

This proposal reports on student reasoning with non-traditional linear coordinate systems in a teaching experiment to address the research question: How do students interact with various entities, such as magnitude, unit size, and coordinates when a coordinate system switches to another?

Methods

As part of a larger dissertation study, I conducted individual face-to-face teaching experiments with three students: Hann, Wilson, and Jeraldo. At the time of interviews, the participants were STEM majors who had completed Calculus 1 or 2 as prerequisites and had taken linear algebra at a large public university in the United States. Both their written work and interview conversations were recorded. The interview data were transcribed into spreadsheet and coded line by line, based on the author's entities framework. All the names used in Results are pseudonyms.

Description of the Applet (https://www.geogebra.org/classic/vz2mcxqr).

The teaching experiment utilized a GeoGebra applet created by the author. The applet was designed to engage students actively in creating a new coordinate system by modifying matrix entries. The applet features two panes: one on the left displaying the Cartesian plane in black, and one on the right showing the new coordinate plane in pink, generated by the matrix entries.

(Figure 3) Earlier part of the teaching experiments, the students interacted with the applet and learned how each entry in the matrix M (top left in Figure 3) influences the new coordinate plane on the right pane. They began to conceptualize the 2 × 2 matrix as a set of two column vectors, with each column vector representing a basis vector in the plane. For example, with $M = \begin{bmatrix} 6 & 1 \\ 2 & 2 \end{bmatrix}$, the students recognized it as $\{\begin{bmatrix} 6 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\}$ generating a basis on the plane. (Figure 3, right pane)

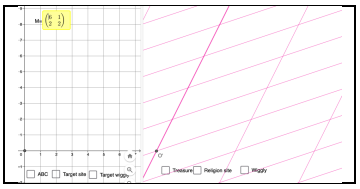


Figure 3. An example of the GeoGebra applet screen

Description of the Task

I designed the task protocols in three packets each called clinical interview, teaching experiment 1, and teaching experiment 2. This proposal includes Task 7 and Task 8 (Figure 4) in the teaching experiment 1 which aims to teach and learn matrix representations with a focus on individual entries. Previously, the students participated in the clinical interview tasks themed around Gulliver's Travels, where one of the main tasks was a treasure hunt, following the central idea of Realistic Mathematics Education (RME). During the clinical interviews, the students identified the location of the Treasure on Gulliver's coordinate system as (1.23, 2). In Tasks 1-6 of teaching experiment 1, the students engaged with matrix entry modifications using the GeoGebra applet and found that Gulliver's coordinate system can be generated using the two vectors $\begin{bmatrix} 6 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Thus, the matrix M for Gulliver's gridlines is $M = \begin{bmatrix} 6 & 1 \\ 2 & 2 \end{bmatrix}$. Task 7 (Figure 4) revisits the treasure and provides the students with a new coordinate pair (Josh's) for the same treasure, asking them to describe the new coordinate system. Task 8 then requires them to generate the coordinate gridlines using the 2 × 2 matrix. These tasks involve multi-step thinking commitments. Students determine what has changed (or not changed) in each component of the coordinate pairs, understand the implications of the change, identify the causes of the change, and modify the matrix entries accordingly to match the new coordinate pair.

Task 7: You just got a phone call from your friend, Josh, and he said that the treasure is at (1.23, 6) on his map, what does the Josh's map look like? (The treasure is at (1.23, 2) on the Gulliver map.)	Task 8: Find the entries of matrix M for Josh's map. In other words, the matrix M for Josh's map is $\begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix}$.

Figure 4. Task 7 and Task 8 (From Teaching Experiment 1)

Result

As shown in Figure 4, each task has its own goal aimed at examining student reasoning related to entities and matrix representation. In this section, we will explore the different ways that linear algebra student reasons with the three entities (magnitude, unit size, and coordinates) as they modify the matrix entries.

Hann: Reciprocal Relationship between Coordinates and Unit Size

Hann: I'm noticing that our x coordinates are the same, which means that our horizontal unit vector is the same. Um, so Josh's unit vectors… it's going to be the same as Gulliver, which means that our horizontal is 6 2 [$\begin{bmatrix}6\\2\end{bmatrix}$].

Interviewer: Why is it the same as Gulliver's horizontal?

Hann: Because we have the same x coordinate as Gulliver does. so that means I think it means it has to be the same… so now I just need to find his vertical vector and I'm noticing that it [coordinate]'s just *three times* what ours [Gulliver] is. So, for his [Josh] vertical [unit] vector, it's just going to be *one third* of ours, so it's going to be one third and two thirds, one third.

Interviewer: Why does it have to be reciprocal?

Hann: The location of the treasure [on Josh's map] is at six vertical units, whereas for us [Gulliver] it's two, so the only way that that can be true is that each of his units are three times smaller.

Hann recognized a reciprocal relationship between coordinates and unit size: enlarging the coordinate implies reducing the unit size. When the coordinate increases threefold from Gulliver to Josh, the unit size decreases to one third. His reasoning for multiplication relationship between unit size and coordinate is depicted in a diagram. (Figure 5) Then, Hann modified the second vector in the Gulliver matrix M by multiplying by 1/3.

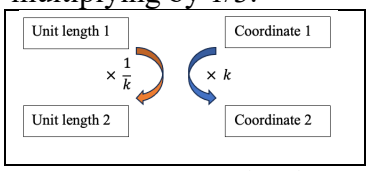


Figure 5. Hann: Reciprocity

Wilson: Proportional Thinking to Reciprocal Thinking with Coordinates and Unit Size

Wilson: His [Josh] vertical axis is three times what the value for I guess ours would be. This would probably look closer to 3 6 [$\begin{bmatrix}3\\6\end{bmatrix}$]… [after modifying the GeoGebra app] Oh, jeez.

I note that Wilson's visualization and verbal expression of Josh's vertical gridline was fine; However, when he tried representing the unit utilizing vectors, he modified from $\begin{bmatrix}1\\2\end{bmatrix}$ to $\begin{bmatrix}3\\6\end{bmatrix}$ to the GeoGebra applet. Then, he encountered a discrepancy between his expectations and what he observed in the app. It appeared that Wilson reasoned that the unit needed to be scaled by a factor of 3, as the coordinate changed three times from 2 to 6. Consequently, he modified the second column from $\begin{bmatrix}1\\2\end{bmatrix}$ to $\begin{bmatrix}3\\6\end{bmatrix}$, managing unit length and coordinate at a proportional manner (Figure 6, left).

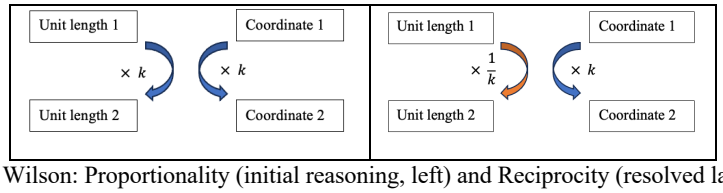


Figure 6. Wilson: Proportionality (initial reasoning, left) and Reciprocity (resolved later, right)

When Wilson discovered the discrepancy between his anticipated grid lines and the actual grid lines after modifying the entries of M on the applet, I followed up on him:

Interviewer: So, when you just hear the two coordinates, (1.23, 2) in Gulliver's while the treasure location is right here (1.23, 6) on [Josh] map, it has the same x value, right? The

second coordinate is six now. When you're picturing the grid line using these two coordinates for the same location, what kind of image you have in terms of the grid line?

Wilson: There [Josh's map] would be an extra … lines… [then modified the second column in the app to 12 6 [$\begin{bmatrix}12\\6\end{bmatrix}$].

Interviewer: Why did you make it to 12 and 6?

Wilson: So, with the horizontal point being the 1.23 being the same between both maps, that's where my thoughts are thinking. That first column isn't going to change. That second column is where the big change will be. I was initially thinking with the 6 and 3 and 12, that's something along those lines could change, could scale it in a way that it is on the six…So, I think the point I need to find is what would one that would scale it down or closer….[later] What I think needs to happen for is that it needs to keep the same slope, … but also needs to be scaled down. So maybe about one third Oh, so maybe one third, two thirds because it [coordinate] needs to be scaled two, three, four, five, six.

In this episode, Wilson initially attempted to modify the entries of the second column using larger numbers, such as $\begin{bmatrix}3\\6\end{bmatrix}$ then $\begin{bmatrix}12\\6\end{bmatrix}$. It seemed that he continued to reason proportionally between coordinates and unit size when Re-Naming the treasure location. Upon discovering its invalidity of this approach with the applet, he experimented with small numbers while maintaining the same ratio to ensure that the slope of grid remains constant. Eventually, he figured out that the entries should be 1/3 and 2/3. He realized that to scale the coordinate up from two to six, he needs to scale the unit size down by one third, discovering the reciprocal relationship in matrix representation.

Jeraldo: Reciprocal Relationship between Coordinates and Unit Size

Jeraldo: On the horizontal direction, it stays the same. They have the same exact, um, the same exact number in the first entry [column]. The only difference I would say is that it probably looks the same with the angles and everything like that. It's just a little bit smushed down. So, there would be more lines… I'm thinking of smashing this down by a factor of three, but it has to be the same exact line [same slant]. Because if I change the entries [not holding the same ratio], then that would change the vector. So, if we keep one, two [$\begin{bmatrix}1\\2\end{bmatrix}$], it looks like this [same slant], right? But if I change this entry [not holding the ratio, one two], it would be at a different angle and completely different.

Jeraldo employed reciprocal reasoning between two entities, unit size and coordinate, in Task 7. Given the Treasure coordinate pair (1.23, 2) and (1.23, 6) in the Gulliver coordinate system and in the Josh coordinate system, respectively, where the vertical coordinates are tripled from 2 to 6, Jeraldo focused on the ratio of the vector components. He emphasized that the ratio needs to remain the same for the Josh's unit vector to hold the same slant of grid lines. He divided the Gulliver's unit vector $\begin{bmatrix}1\\2\end{bmatrix}$ by the scaling factor 3 to make it smooshed down. (Figure 7, left)

[On the second day of teaching experiment when revisiting the same task]

Jeraldo: So if I can keep this same line and smush it down, then we would able to have this [coordinate of 2] to be equal at six. So, I imagine if we just scale this down by three…scale down every entry by three. So essentially, take the original unit vector [Gulliver unit vector $\begin{bmatrix}1\\2\end{bmatrix}$] and divide it by the same factor.

Interviewer: Not multiplying?

Jeraldo: I guess you could multiply it by the reciprocal of the scale factor, but in my mind it makes more sense. If he wants to switch from, then you're gonna divide it. But I'm not sure how I will find the scale or the scale factor, but I think this might be the closest I'll get from

Interviewer: Why did you pick the number 3?

Jeraldo: Just because 2 and 3 are factors of 6. I just assume since we already have 2 right there, we can use 3. (Figure 7, middle)

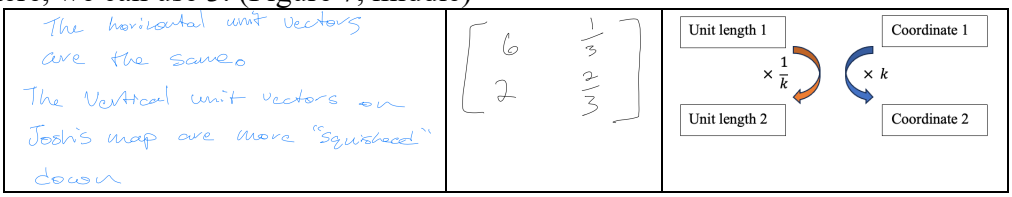


Figure 7. Jeraldo: Description for Josh's map (left), Matrix M entries (middle), Reciprocity (right)

Jeraldo recognized the reciprocal relationship: enlarging the coordinate implies reducing the unit length. His reasoning for multiplication between unit length and coordinate is depicted in a diagram (Figure 7, right).

Discussion

Reflecting on the entity framework, which initially began with the analysis of student work in clinical interviews, the formulaic idea, $M(Magnitude) = U(Unit\ size) \cdot C(Coordinates)$, extends to situations where the unit size changes to another size with multiple coordinate systems. This leads to two scenarios emerged with switching from one coordinate system to another (Figure 2).

First, under constant magnitude M: If U_2 (unit size for the second coordinate system) is k times as large as U_1 (unit length for the first coordinate system), then C_2 (coordinate in the second coordinate system) will be $\frac{1}{k}$ times C_1 (coordinate in the first coordinate system). Task 7 is designed to support students in thinking of coordinates in relation to unit size while keeping the location magnitude (the Treasure) constant across coordinate systems. According to my coordinate system framework (Lee, 2023), this task was developed under the Re-Naming perspective. As a coordinate system switches to another, the results show that students engaged with the three entities, establishing the reciprocal relationship between coordinates and unit size in Task 7.

Second, under constant coordinates C: If U_2 (unit size for the second coordinate system) is k times as large as U_1 (unit size for the first coordinate system), then M_2 (magnitude in the second coordinate system) will be k times M_1 (Magnitude in the first coordinate system). Even though Task 7 is better suited for the first scenario, where the location magnitude remains constant, students also demonstrated an understanding of the proportional relationship between unit size and magnitude. Specifically, when the unit size decreases to one third from Gulliver's unit to Josh's unit, the magnitude also decreases to one third (Figure 8), establishing a proportional relationship between unit size and magnitude: reducing the unit size leads to a reduction in magnitude.

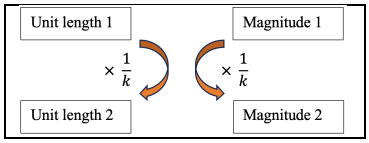


Figure 8. Proportionality between unit size and its magnitude

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Navigating the Transition from University to School: A Case Study of Incorporating Reasoning and Proving in Secondary Mathematics Teaching

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University teacher preparation programs extend significant efforts to support the development of secondary teachers' competence in teaching mathematics. However, it remains an open question whether or to what extent beginning mathematics teachers enact ambitious practices in their teaching – specifically, engaging students with reasoning and proving. We report on one teacher, Nancy, who, after two years of autonomous teaching seized the opportunity to design and enact a "proof unit," by utilizing materials from her undergraduate studies. Using the framework of practical rationality and four professional obligations, we examine Nancy's discourse over various time points on her professional journey, focusing on how she held on to the ideas from her university training, culminating in the reasoning-and-proof unit. Implications for strengthening mathematics teacher preparation beyond university programs are discussed.

Keywords: Reasoning and proof, Internship, Beginning secondary teachers

University teacher preparation programs are pivotal in strengthening knowledge and shaping the pedagogical practices of future mathematics teachers (AMTE, 2017; Wasserman et al., 2023), including their competence to integrate reasoning and proving into their teaching (e.g., Conner et al., 2014; Buchbinder & McCrone, 2020). However, transitioning from university to classroom teaching poses significant challenges for beginning teachers, who must balance conflicting commitments to both the university and their mentor teacher during student teaching (Bieda et al., 2015; Thompson et al., 2013). Stylianides et al. (2013) describe specific challenges beginning teachers face when attempting to enact reasoning and proving activities in classrooms, like teachers' limited knowledge of students' proof conceptions and pre-existing school cultures. When entering their first teaching job, teachers may be overwhelmed by the increased responsibilities and competing dilemmas (Gainsburg et al., 2012), which they need to reconcile to support students' engagement with proof (Conner et al., 2014). Stylianides et al. (2017) alert to the limited number of studies that explore the development of proof-related practices of beginning teachers longitudinally. This study attempts to contribute to this research gap.

We use a case study of Nancy (a pseudonym) to investigate beginning mathematics teachers' proof practices. As part of a larger NSF-funded study, we followed Nancy for four years, from her participation in the undergraduate capstone course *Mathematical Reasoning and Proving for Secondary Teachers (*Buchbinder & McCrone, 2023), her year-long supervised internship, and first two years of autonomous practice. In her second year, Nancy seized the opportunity to design and implement a unit centered on reasoning and proving, drawing on ideas and materials from the capstone course and the professional learning community on reasoning and proving comprised of her former classmates. Nancy's case provides a unique window into the long-term effects of university training on teaching practices. We utilize Herbst and Chazan's (2003, 2011) framework of four professional obligations (described below) to explore the following research question: *How did Nancy navigate the university-to-school transition, balance various professional obligations, and apply reasoning and proving practices in her classroom?*

Theoretical Perspectives

Reasoning and Proving in Secondary Schools

The term *reasoning and proving* denotes a collection of disciplinary practices involved in creating and validating mathematical knowledge, like identifying patterns, conjecturing, explaining, justifying, and proving (Ellis et al., 2012; Reid & Knipping, 2010). We adopt a definition of proof as "a mathematical argument for or against a mathematical claim that is both mathematically sound and conceptually accessible to the members of the local community where the argument is offered" (Stylianides et al., 2017, p. 238). That is, both formal and informal mathematically valid arguments can be accepted as proof by a particular community.

While the main purpose of reasoning and proving in classrooms is to support student mathematical learning and understanding, research consistently shows that teachers find it challenging to establish such classroom environments (Stylianides et al., 2017). One of the challenges may be rooted in teachers' limited knowledge of proof or unproductive views of proof (Ko, 2010). In response, several university-based teacher preparation programs attempted to support future teachers' competencies to enact reasoning and proving in classrooms (e.g., Buchbinder & McCrone, 2020; Conner et al., 2014). Other researchers suggest that the barriers to the integration of reasoning and proving in classrooms are rooted in sociocultural factors inherent in the institutions of schooling (Chazan et al., 2016). Teachers need to navigate a complex system of often competing expectations by various stakeholders: students, parents, and administration. In such an environment, a teacher's desire to enact reasoning and proving is weighed against a "need to cover" the curriculum (Kotelawala, 2016).

Practical Rationality and Four Professional Obligations

The theory of *practical rationality* (Herbst & Chazan, 2003, 2011) suggests that becoming a mathematics teacher involves adopting a particular framework for making decisions about classroom teaching. This framework involves balancing four types of professional obligations. The obligation to the *discipline* of mathematics involves accurately representing mathematical concepts and engaging students in meaningful mathematical practices, such as reasoning and proving. The obligation to students as *individuals* involves attending to fairness and catering to each student's cognitive and emotional needs. *Interpersonal* obligation considers the class as a whole, requiring the teacher to manage social dynamics and intergroup relations to ensure fair sharing of resources. The *institutional* obligation requires the teacher to follow school, district, and state policies related to curriculum assessment and standards, and adhere to practices and guidelines shared by members of school mathematics departments. While obligations do not prescribe actions, they collectively influence how teachers navigate their professional roles, regardless of their personal beliefs or competencies (Chazan et al., 2016).

We draw on the theory of *practical rationality* and the four professional obligations to explore how Nancy navigated different spaces—her university-based teacher preparation program, supervised internship, and autonomous teaching —and how she balanced various obligations while developing her own teaching practices around reasoning and proving.

Method

Nancy was one of eight beginning secondary mathematics teachers whom we followed for four years as a part of the larger, NSF-funded longitudinal project, and for whom we compiled the most complete data set. Nancy had a strong mathematical background and expressed productive dispositions toward reasoning and proof, making for a rich case study (Yin, 2003).

Data Collection and Analysis

As a preservice secondary mathematics teacher (PST), Nancy took a specially-designed capstone course *Mathematical Reasoning and Proof for Secondary Teachers* (Buchbinder & McCrone, 2020, 2023). The course focused on four proof themes—topics considered in the literature particularly challenging to learn and teach: direct proof and argument evaluation, conditional statements, quantification and the role of examples in proving, and indirect reasoning. This course supported PSTs' developing proof-specific mathematical knowledge for teaching by providing opportunities to crystalize their knowledge of the four proof themes; connect it to students' conceptions of proof and apply this knowledge by designing and teaching in local schools four proof-integrated lessons. Of all the data sources, for this paper, we focused on analyzing Nancy's written reflections about teaching in the course.

An internship is a one-year, post-graduate, supervised teaching experience. Nancy was present full-time in the classroom of her mentor teacher (MT), gradually expanding the scope of her classroom responsibilities. The data for this period included five lesson observations, related instructional materials, and follow-up debrief interviews with a researcher on the project.

As a novice teacher (NT), Nancy moved to another state across the country, so data were collected through online interviews (two in year 1 and three in year 2). Nancy also participated in the Professional Learning Community on Reasoning and Proving (PLC-RP), led by the first author of this paper. The PLC-RP met online, four times a year for 1.5 hours. The video recordings of individual interviews and the PLC-RP meetings comprised the main data sources for the NT period, supplemented by lesson plans and sample student work.

Nancy's transcribed interview recordings from INT and NT years and the written report from her PST period were segmented into thematic units of varied length (from a single utterance to ~10 sentences). Each segment was analyzed to identify Nancy's actions or decisions and her justifications for these decisions. The justifications were examined to discern underlying obligations according to Chazan et al.'s (2016) framework of four professional obligations. Subcategories for each obligation were developed through open coding and the constant comparative method (Corbin & Strauss, 2015) based on Nancy's specific circumstances, e.g. her interactions with her MT during the internship reflect how she upheld the institutional obligation. Some actions were justified not by obligations but by Nancy's personal beliefs and values reflecting her emerging educational approach. We illustrate these in the results section.

Results

The PST Period

As a PST, Nancy did not have to attend to institutional obligations – her sole focus was on developing her mathematical and pedagogical knowledge and experimenting with proof. Nancy shared, "before taking this [capstone course], I only associated proofs with high school geometry and didn't realize all the different proof topics that can be integrated into a [secondary] math classroom." While Nancy did her course practicum in a high-school geometry class, she felt that the course allowed her to develop a more open approach to proof, saying, "I've learned […] a number of ways to integrate these [proof] topics into math classrooms." Illustrating how she integrated the proof themes into her lessons, Nancy wrote:

> For example, with conditional statements, I had the students investigate the P & Q, the truth value, and the converse and its' truth value of conditional statements about triangles and segments in triangles since that was what they were learning at the time.

Nancy felt that direct proof and conditional statements proof themes could be "seamlessly" integrated into her lesson plans compared to quantification and role of examples in proving, and indirect reasoning, which she described as "a little forced." Still, Nancy found creative ways to integrate these proof themes into her lessons. For example, for the lesson on indirect reasoning[1], Nancy designed *The Quadrilateral Detective* game, where students used analytic geometry to determine the type of quadrilateral from the given coordinates. Students applied indirect reasoning by writing statements of the form "A quadrilateral cannot be __because__" (e.g., "cannot be a square, because not all sides are congruent") (Liu & Buchbinder, 2022).

Reflecting on the course, Nancy described that it expanded her view of proof and provided her with experiences she wanted to enact in her future classrooms. She wrote:

> My teaching philosophy is centered around exploration and reasoning, but before [the course] I wasn't really sure how to enact it in my classroom. […] This course challenged me to think about proving in new ways and helped me to see its benefit in all math classrooms [and] gave me confidence in my abilities.

This quote, while containing no action, shows Nancy's emergent commitment to disciplinary values - "exploration and reasoning." She attributed to the capstone course her increased understanding of what integrating reasoning and proving in mathematics classrooms means, and increased willingness and confidence to do so in her own classrooms.

The Internship (INT) Period

Nancy's internship was with an experienced 8th-grade MT, who shared Nancy's commitment to fostering student mathematical engagement. However, they both found it challenging to enact these values due to the institutional constraints – a rigid curriculum. Nancy appreciated that "the curriculum is good about doing reasoning and proof stuff," but it came in the form of "a booklet that tells us all the lessons that we need to do, and [the] estimated times for everything." To address this, Nancy and her MT redesigned lessons to make them more rich. One example was Nancy's *Halloween Scavenger Hunt*–a Halloween-themed activity that she designed to replace a mundane exam review packet on transformational geometry. Working in groups, students solved problems worded as riddles hidden around the room; each solution led them to a new problem location and a clue to a combination lock box containing a prize. But, except for one problem involving indirect reasoning ("Of the four given sequences of transformations, identify one that does *not* result in transforming a shape FRANK into F'R'A'N'K'"), all other problems focused on calculations since Nancy's goal was for students to "enjoy doing some practice problems."

Thus, Nancy's focus was on the obligation to individual students' engagement, while the disciplinary obligation to reasoning and proving moved to the background. When asked directly about this, Nancy explained that "eventually students are going to prove […], but right now, day to day, we're doing a lot more reasoning than we are doing proving." She also intentionally tried "to ask a lot of *why* questions because it gets them thinking about justifying and explaining."

Compared to Nancy's PST period, this might seem like lowering the bar for integrating reasoning and proving. However, the institutional context is critical to this change. While Nancy enjoyed the support of her MT to follow her aspirations, she was constrained by the institutional

[1] We refer to indirect reasoning as any reasoning of the form "it cannot be so because otherwise", whereas indirect proof is a proof by contradiction or contrapositive (Yopp, 2020).

obligation to adhere to her school's rigid curriculum and procedures. Thus, sustaining a classroom culture where students are actively engaged in "fun" activities became Nancy's primary goal, a pre-condition for enacting her "teaching philosophy centered around exploration and reasoning." This is evidenced in her repeated use of "why does it work?" questions, supporting students in "justifying and explaining" and "understanding mathematics as a whole." She also used self-designed problems to substitute or supplement the curriculum. Nancy's approach reflects the tensions between her own values, aligned with individual and disciplinary obligations, and the obligation to meet the institutional standards of her internship school.

The Novice Teaching (NT) Period

Nancy started her first year as a high school teacher in a different US state. In the spring of her first year of autonomous teaching, Nancy described tensions between her desire to support students' mathematical learning and the need to adhere to school standards and curriculum, saying, "the standards and the things that [the school] wants [teachers] to teach are taking away from some of the fun stuff." Nancy also shared her frustration with some of the pedagogical approaches of her colleagues, especially the teacher with whom she was expected to coordinate. Nancy said: "Basically what she [the teacher] does is […] walks through guided notes for 30 minutes and then gives them [students] practice problems for 30 minutes. And that kills me!"

Several teachers in Nancy's school shared her desire to "move away from the guided notes" toward a "more student-focused curriculum" but met with opposition from other teachers whom Nancy described as "old school," clinging to "old traditional ways." Despite the institutional challenges: the divisive school culture, traditional teaching practices of her direct colleague, and teacher-centered instructional resources "handed to her," Nancy strived to teach her classes in student-centered ways, aligned with her teaching philosophy, even at the cost of extra work hours. Her persistence paid off in her second year when she developed her *Proof Unit*.

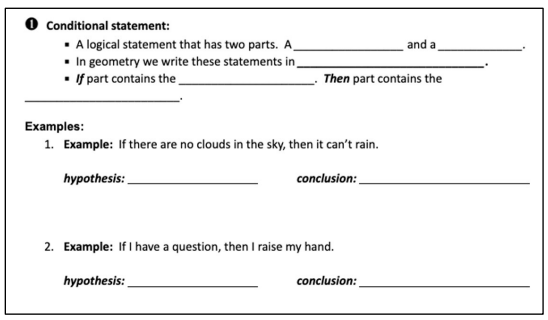
1. Introduction to Proof Concepts ❑ PowerPoint defining conjectures, indirect and inductive reasoning and counterexamples. ❑ Interactive Activities: SET Rules, Snowflake Activity ❑ Practice with patterns and counterexamples	2. Conditional Statements and Logical Puzzles ❑ PowerPoint presentation on vocabulary: conditional statements, converses, contrapositives and inverses. ❑ Practice creating conditional statements ❑ Interactive Activity: Sudoku	3. Proving the Area of a Circle ❑ Proof 1: Rearranging sectors of a circle to form a parallelogram and a rectangle (hands-on task) ❑ Proof 2: Unfolding concentric rings to form a triangle
4. Combinatorial Problems and Game Strategies ❑ Video introducing the "Game of 100" ❑ Movement activity on game strategies ❑ Introduction to the "Handshake Activity" for combinatorial problems	5. Combining Mathematical Concepts and Problem-Solving ❑ Continuation of the "Handshake Activity" ❑ Triangle Inequality Theorem Exploration ❑ Fibonacci Rectangles Exploration ❑ Sudoku puzzle for Problem-Solving	6 & 7. Summative Assessment ❑ Poster Project: students work individually to create a poster summarizing and expanding on one of the proof activities from the unit. ❑ Followed by a gallery walk.

Figure 2. Nancy's Proof Unit outline.

Nancy's Proof-Unit. The 11th-grade geometry new state-wide curriculum standards called for a unit on "reasoning and proof." However, there were no pre-existing curriculum materials. As a result, the school allowed the teachers to use curriculum materials of their choice and relaxed the usual requirement for teachers to coordinate assessments across classes. In Nancy's words, this was a "rare situation." With this relaxed institutional pressure, Nancy jumped on the opportunity to integrate her desired teaching practices, saying: "I got this! I have all these really great ideas, I have the PLC where we do proving and reasoning, like, I can come up with this really cool thing!" Undeterred by the lack of her colleagues' collaboration, Nancy decided to

"pull from the proofs […] that we've done [in the course and the PLC] to do some of the cooler parts of math." The outcome was a three-week unit (Figure 2) where each lesson involved "different activities about reasoning and proving." For the summative assessment, Nancy had each student create a poster that summarized and extended one of the unit's proof activities.

Nancy utilized many activities from the PLC-RP (e.g., proving the area of a circle) and the capstone course, like introducing proof-related (e.g., conjectures, patterns) and logic-related concepts (e.g., conditional statements, contrapositive) using both mathematical and real-world examples (Figure 3a). She started by having students explore visual and numeric patterns, introducing them to conjecture, inductive reasoning, and counterexamples (Figure 4).



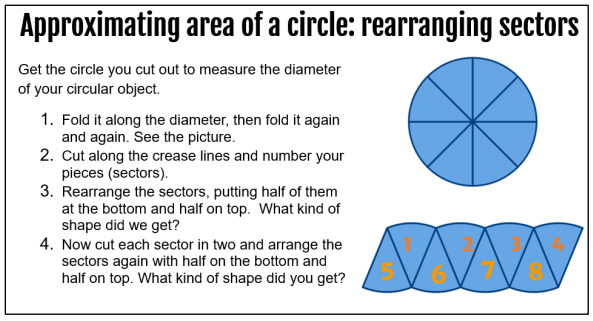


Figure 3a (on the left). A slide introducing conditional statements using non-mathematical examples. Figure 3b (on the right). A slide from the "proving the area of a circle" activity (lesson 3).

Nancy explained that the first two lessons in the unit "weren't necessarily proof, but they were just like, things that have to do with proving." The rest of the lessons involved what Nancy called "a bunch of different mini-proofs" – pattern explorations and informal arguments. For example, describing lesson 3 on proving the circle area formula (Figure 3b), Nancy said:

> [We] did the proof of the area of a circle, where the students have a circle and then they change it into a parallelogram […] That was kind of a fun day because […] I had them go through the exercise of folding [the circle] and cutting it and laying it out on their desk […] So, they actually had hands-on proof that they engaged in the entire time.

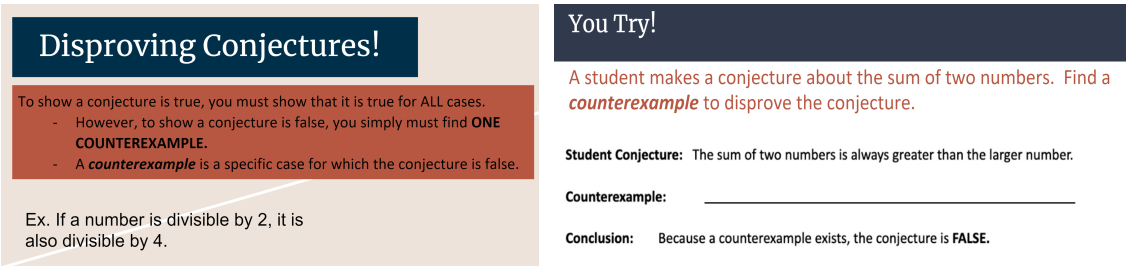


Figure 4. Explanation of the meaning of a counterexample and a task for students to do on their own.

These examples show that Nancy's focus was not on writing formal proofs, but on having them visualize, generalize patterns, formulate conjectures, and justify their thinking. This aligned with her disciplinary obligation to introduce mathematical reasoning and proving while considering students' minimal prior knowledge (individual obligation). The individual obligation is also evidenced in the focus on students' active engagement through games, hands-on tasks, and explorations. Nancy worked to make reasoning and proving enjoyable and appealing.

Takeaways and community reactions. Nancy enacted her Proof Unit with two lower-level

geometry sophomore classes. She was pleased with the outcomes, especially when some students took the initiative to explore concepts beyond class notes. Nancy's feelings were apparent in her voice and discourse: in a 49-minute interview, she used the words "cool" 30 times and "excited" 15 times. Encouraged by the positive experience, Nancy planned to revise the unit next year, to further emphasize proof, which reflects her increased attention to the disciplinary obligation.

Returning to institutional obligation, Nancy shared her unit and student outcomes with the administration and other teachers. She felt that many of her colleagues "were inspired," but some had reservations, motivating them by the need for consistency in teaching. Nancy, apparently, refused to be constrained by this. She said: "The big takeaway for me is - try new things even if other people aren't on board. Still do it, and you might get some really good outcomes from it."

Summary and Discussion

Putting Nancy's experiences in the broader context of preparing mathematics teachers to enact reasoning and proof in schools, we ask: What is Nancy a case of? Nancy is a well-prepared teacher candidate (AMTE, 2017) with robust mathematical knowledge, a strong pedagogical orientation toward active learning, and a commitment to engaging students with reasoning and proof—ideas she cultivated in the capstone course (Buchbinder & McCrone, 2023). As a PST, Nancy was shielded from the institutional obligation, enabling her to focus on mathematics, students' interactions with it (i.e. disciplinary obligation), and her own professional growth. As an INT, Nancy faced institutional constraints, including a rigid curriculum and a quick teaching pace. However, with her MT's support, Nancy was able to design and enact some reform-oriented activities to uphold her obligation to student engagement and to the discipline, which Nancy described as "interactive things and exploratory things and proof-based things," like her *Halloween Scavenger Hunt.* This illuminates the importance of placing INTs with mentors whose teaching philosophy aligns with that of university teacher preparation programs, and who can support rather than constrain INTs' professional aspirations (Gainsburg, 2012; Thompson et al., 2013), in this case – Nancy's "teaching philosophy centered on exploration and reasoning".

As an NT, Nancy faced increased institutional constraints and found it even more difficult to balance them with disciplinary and individual obligations. The need to coordinate assessments with other teachers, some of whom had vastly different pedagogical approaches from Nancy's, created tension for her. This is consistent with Wang et al., (2008) and Feiman-Nemser's (2003) findings on the influence of school culture on the nature and quality of the early years of teaching practice. Despite this, Nancy remained determined to teach in the way she saw as most advantageous to student learning. She utilized activities from the capstone course ("I did pull a couple of things from my time [in the capstone course]") to integrate "exploratory" and "proof-based activities." Nancy's nonconforming attitude and sense of agency are possibly unique to her personality. However, what is not unique is the fact that she had strong mathematical and pedagogical foundations, as well as access to instructional resources and continued professional support around reasoning and proof, aspects that have been identified in the literature as positively affecting novice teachers' practices (Wang et al., 2008; Vescio et al., 2008). Nancy's case illustrates that despite the challenges, novice teachers can draw on their professional preparation to develop rich opportunities for student learning of reasoning and proving.

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Examining Connections Between Students' Mindsets and Beliefs About Mathematics

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Students' beliefs influence the ways in which they learn mathematics. These beliefs can be categorized in different ways and related to growth or fixed mindsets towards mathematics. In our work, we seek to better understand connections between students' beliefs and mindsets about learning mathematics. We present case studies of two undergraduate students who were enrolled in a first-year college algebra course who demonstrated a shift in mathematical mindset. Our findings highlight the complexity of students' beliefs and how students can hold both growth and fixed mindsets towards mathematics. We explore how mindsets can be tied to specific categories of beliefs and how those beliefs can influence a student's overall perspective about learning mathematics. We conclude that mindset interventions and future research should consider how to leverage these mixed mindsets to help students foster more growth-oriented beliefs about mathematics.

Keywords: Beliefs, Mindset, First-Year Mathematics, College Algebra

Students' beliefs influence the ways in which they learn mathematics. These beliefs are situated at the intersection of the cognitive and affective domains and have been categorized in various ways. For example, McLeod (1992) emphasized the affective aspect of belief systems and distinguished between beliefs about mathematics, beliefs about the self, beliefs about mathematics teaching, and beliefs about the social context of mathematics education. Similarly, Kloosterman (1996) organized beliefs into two primary categories: beliefs about mathematics as a discipline and beliefs about learning mathematics which also includes beliefs about oneself as a learner of mathematics and beliefs about the role of the teacher as subcategories. Building on past research, Op't Eynde and colleagues (2002) used models of different student beliefs to develop a theoretical framework of beliefs related to mathematics learning and problem solving. More recent work relating to affect and beliefs in mathematics has focused on mindsets (Dweck, 2016; Boaler 2016) and how these can influence the ways in which students engage in learning mathematics. In our work, we seek to better understand connections between students' beliefs and mindsets. We present case studies of two undergraduate students who were enrolled in a first-year college algebra course. These cases highlight how mindsets can be tied to specific mathematical beliefs and how those beliefs can influence a student's overall perspective about learning mathematics.

Theoretical Framing

Our research draws on the concepts of mindsets and mathematical beliefs. In her seminal book, Dweck (2016) described two types of mindsets that affect the way students view their learning abilities. Students with a growth mindset believe that their mathematical abilities can be improved with practice, while students with a fixed mindset believe that they only have a certain amount of intelligence or mathematical ability (Dweck, 2016). Boaler (2016) applied Dweck's mindset theory to the field of mathematics, describing how growth and fixed mindsets can affect

students' views of themselves as mathematical learners. In her work, Boaler framed having a mathematical mindset as "knowing that math is a subject of growth and that [a student's] role is to learn and think about new ideas" (Boaler, 2019, p. 29). Students' mathematical mindsets are tied to their beliefs about themselves, the nature of mathematics, and their role in relation to it (Boaler, 2019).

In our study, we draw connections between mathematical mindsets and beliefs using Op't Eynde et al.'s (2002) beliefs framework. We analyze how students' expressed mindsets may be linked to their mathematical beliefs. Op't Eynde et al.'s (2002) framework, outlined below in Table 1, describes how students' beliefs about mathematics fall into three categories: beliefs about the discipline of mathematics and mathematics education, beliefs about themselves, and beliefs about the social context of learning. Our findings suggest that students can have a certain mindset linked to beliefs in one category, but a different mindset about beliefs in another category. In addition, we postulate how mindsets and beliefs in one category may influence those in another category. This research gives practitioners insights into undergraduate students' mathematical mindsets and how they might change over a semester. Our work also provides a basis for researchers to continue investigating connections between mathematical mindsets and beliefs.

Table 1. A Framework of Students' Mathematics-Related Beliefs (Op't Eynde et al., 2002)

Category	Belief	Example
Beliefs about mathematics education	Beliefs about mathematics as a subject	Mathematics is a collection of facts and procedures to be memorized.
	Beliefs about mathematical learning and problem solving	There is only one correct way to solve a mathematics problem.
	Beliefs about mathematics teaching in general	A good teacher will show me how to solve a problem before I am expected to do it myself.
Beliefs about the self	Self-efficacy beliefs	I am confident that I can understand the mathematics material in this course.
	Control beliefs	I will be able to learn the material if I put in enough time and effort.
	Task-value beliefs	It is important for me to learn the material in this course.
	Goal-orientation beliefs	Fully understanding the mathematics material will give me great satisfaction.
Beliefs about the social context	Beliefs about social norms in their own class	If I speak up during a class discussion, my classmates and teacher will consider my ideas and value my contribution.
	Beliefs about socio-mathematical norms in their own class	I must be able to explain my thinking to present a good mathematical solution to my classmates and teacher.

Research Question

The purpose of our research is to better understand how students' mathematical beliefs and mindsets are related. In this paper, we address the research question: *What types of mathematical*

beliefs are expressed by students who exhibit changes in mathematical mindset in a first-year mathematics course?

Methods

The research presented in this paper comes from a larger study of students' perceptions of a first-year mathematics course. Data for this study comes from two written mathography reflections that were assigned to 611 students enrolled in 15 sections of a corequisite college algebra course at a large, Midwestern, metropolitan university during Fall 2023. This course involved active learning strategies and standards-based grading; it was designed to help students foster a more positive relationship with mathematics. Mathographies were collected at the beginning and end of the semester. These reflection assignments were based on past work by Drake (2006) and had students respond to prompts such as "What do you think it takes for a student to be good at math?" Out of the 611 students, 445 students submitted a reflection at the beginning of the semester. Of these 445 reflections, 55 were selected for further analysis using results from a Qualtrics survey on mathematical mindset and self-efficacy that was also administered at the beginning of the semester.

Beginning mathographies were analyzed during the Fall 2023 semester using MAXQDA 2022 (VERBI Software, 2021). These reflections were coded using a set of primary codes including Growth Mindset, Mixed Mindset, and Fixed Mindset. We coded excerpts that included both fixed and growth mindset characteristics as a "Mixed Mindset." For example, "I believe math is a strong suit for some but not for others but if you really took the time to practice and pay attention you can be successful." At least two authors independently coded the mathographies and met to reconcile differences until agreement was reached. At the end of the semester, ending mathographies were collected from the same cohort. Of the 55 students whose beginning mathographies were coded and analyzed, 51 students submitted an ending mathography. These ending mathographies were tagged using the same codebook and coding process as the beginning mathographies, with at least two authors meeting to reconcile differences to ensure consistency.

To understand how the mathematical mindsets of students expressed at the beginning of a semester compared to mathematical mindsets shown at the end of a semester, each of the 51 student's beginning and ending mathographies were analyzed side by side, specifically examining any potential changes in mathematical mindset codes. From this secondary analysis, we selected six students who exhibited a noticeable shift in their expressed mindset for further analysis. We then analyzed each student individually and created a detailed description of their expressed mindsets and mathematical beliefs as categorized by the Op't Eynde et al.'s (2002) framework from both their beginning and ending mathographies. Our goal was to compare the different types of beliefs relating to changes in mindset from the beginning to the end of the semester. In this paper, we use a case study approach (Stake, 2003) to better understand the mathematical beliefs and mindsets of two students who exhibited a change in mindset from the beginning to the end of the semester. To protect the privacy of the students, both names are pseudonyms.

Findings

In studying the mathematical beliefs expressed by students who had a change in mathematical mindset during a first-year mathematics course, we looked closely at individual stories. We found that students hold different beliefs about math, which potentially influence

their mindsets. Some students stuck to their initial mathematical mindset that they expressed at the beginning of the course, while others seemed to exhibit a change in mathematical mindset by the end of the semester. These case studies help us understand what types of beliefs are expressed by students, and why some students might have a change in beliefs and mathematical mindset.

Case 1: Emily

Emily's case provides an example of a student who initially demonstrated a fixed mindset towards mathematics but ended the semester with more of a growth mindset. In her intro mathography, Emily expressed a fixed mindset relating to her beliefs about mathematics as a subject stating, "I think students who are good at math honestly were able to grasp the concepts at a young age like in elementary school," and "I believe math is a strong suit for some but not for others." This view of mathematics as an innate ability corresponds to having more of a fixed mindset about mathematics as a subject. However, Emily went on to add "but if you really took the time to practice and pay attention you can be successful," which reflects more of a growth mindset relating to control beliefs about the self. Thus, in her initial mathography, Emily demonstrated both a growth and a fixed mathematical mindset relating to two different categories, beliefs about mathematics education and beliefs about the self, under Op't Eynde et al.'s framework.

In addition to discussing beliefs about the self and mathematics education, Emily also discussed past experiences with mathematics teachers relating to beliefs about the social context in her initial mathography. She explained that "when asking for help [my high school teacher] would make me feel not smart with the way she responded to me." She also described negative experiences with assessment: "when taking a test and not understanding I would breakdown and cry and have to step out of the classroom. I felt like the odd one out like everyone else understood but I did not." Thus, the social context in Emily's past mathematical experiences potentially contributed to her statements reflecting a fixed mindset, shaping her belief that mathematics ability is innate, with some inherently or 'naturally' strong in it while others, like herself, struggled due to negative interactions with teachers and comparing her abilities to her classmates.

At the end of the semester, Emily's reflections had transformed. She acknowledged her capability in math, attributing her progress to consistent study and support from her instructors and colleagues in class. Emily wrote, "I learned that I am capable of doing math and doing a good job as long as I keep studying and making time for math outside of school." She highlighted the positive impact of a supportive learning environment and her beliefs about the social norms in the class, noting that her interactions with teaching assistants were judgment-free and beneficial. She reflected that, "On some problems I just did not know what to do and when I would get help, I felt no pressure whatsoever when getting it incorrect." Emily added that,

> I really enjoyed how much assistance we got during class and the involvement with other students really helped me out, because when I would find myself confused my group would always make sure to ask if we all understood, it was like a team effort.

In her ending mathography, Emily's beliefs about the socio-mathematical norms were different than the ones she expressed in her initial mathography when she felt like "the odd one out" and "not smart" because of the social context. In her initial mathography, Emily demonstrated both fixed and growth mindsets in the category of beliefs about the self. In terms of control beliefs, she expressed more of a growth mindset, using the third person language of

"...you can be successful". However, Emily demonstrated a fixed mindset pertaining to her self-efficacy belief shaped by her previous class experiences. In her ending mathography, the social context in the class seemed to influence her self-efficacy beliefs positively as she used first person language to state, "I learned that I am capable of doing math." Thus, Emily's mathematical mindset changed from a fixed to a growth mindset in relation to her self-efficacy beliefs while also internalizing a growth mindset about her control beliefs. Emily's case exemplifies the types of mathematical beliefs expressed by students who experience a change in mathematical mindset during their first-year mathematics course. This finding underscores the role of the mathematics teaching and learning environment in influencing students' mathematical beliefs and mindset development.

Case 2: Ryan

Ryan's experience in mathematics shows a transformation from a mixed mathematical mindset at the beginning of the semester to a growth-oriented mindset at the end of the semester. In his intro mathography, he reflected, "In general, my experience with math has been pretty poor. My intelligence favors the English/Reading side of education by a vast margin." Ryan's mathematics experiences prior to college seemed to have created a self-efficacy belief that he was intelligent in subjects other than mathematics, corresponding to a fixed mathematical mindset. However, in addition to stating his limitations with mathematics as a subject, he discussed how with hard work, one can be successful at learning mathematics. He wrote,

> I used to think that some are born with the ability to do math, and some aren't. Although I feel that still applies to a certain extent, it's not that black and white. It's true that for some people, it just clicks and for others it doesn't. However, I have learned that working hard and applying yourself is all it takes ... I know it's a bit cliché saying, "all it takes is just hard work."

Here, Ryan shared a belief about mathematics education learning and problem solving – that some people are "born with the ability to do math" and "for some people, it just clicks." Yet he also expressed a control belief about the self that one can be successful in mathematics through hard work. Thus, we can see in this example how someone can hold a fixed mindset relating to beliefs about mathematics education (learning and problem solving) while demonstrating a growth mindset about control beliefs relating to the self.

In his intro mathography, Ryan compared his current classroom experience to past ones. He stated, "I really like how this class emphasizes group work […] also, the attitude in the classroom is very open and non-judgmental. I feel like in my previous math experiences, as horrible as it sounds, I would feel dumb for getting help or the teacher would try not to roll their eyes when I asked a question like they were thinking, 'Again? Seriously?'" This quote shows how social contexts have the potential to shape beliefs, although those beliefs may be difficult to identify. Ryan's comparison of classroom norms may relate to a belief that the role of the teacher should be to support and help students, rather than to judge their mathematical worth. This belief may also affect mindsets and beliefs in other categories such as beliefs about the self and mathematics education. Students who have these types of negative experiences in their educational past (e.g. mathematics teachers who make them feel "dumb") are more likely to view mathematical ability as innate and hold fixed mathematical mindsets as the classroom social norms contribute to defining students who are or can be good at mathematics.

In his ending mathography, Ryan expressed more of a growth-oriented mindset stating that anyone could succeed in math with enough effort. He reflected,

> I do believe that any student can succeed in math. My 16-year-old self wouldn't believe this, but all it takes is effort. It may seem like you're predestined to suck at math, but that isn't the case. It'll just take you a little more work and effort. If you really want to succeed, you will.

This statement corresponds to a control belief about the self where Ryan used third person language to describe what it takes to be successful at mathematics. In addition, Ryan also used first-person language to discuss another control belief specifically relating to himself, which potentially was influenced by being in a positive and non-judgmental environment. He stated,

> When it comes to new aspects, I've learned about myself that relates to Math, one of them would be just show up. As long as you just go (to the class, in this instance), then 75% of the work is done.

Thus, Ryan's experience in a positive and non-judgmental classroom environment promoted a control belief in the value of participation and effort. He realized that engaging in a supportive class setting can significantly contribute to a student's success and that he could be successful at mathematics by "just showing up."

Ryan's case highlights how an encouraging learning atmosphere can help promote growth-oriented control beliefs about the self. Having held a mixed mathematical mindset at the beginning of the semester, Ryan's development towards having a growth mindset was facilitated by his beliefs about the social norms in the class and the support from his instructors and peers.

Discussion

The mathematical beliefs of students are complex and often show characteristics of both fixed and growth mindsets. This phenomenon, which we call having a "mixed mindset," is evident in the expressed beliefs of our participants. For example, Emily expressed a fixed mindset belief about mathematics as a subject, i.e. "I believe math is a strong suit for some but not for others" while maintaining a growth mindset belief about their mathematics learning and problem solving, i.e. "but if you really took the time to practice and pay attention you can be successful." This complexity exists between the general categories of the beliefs framework (beliefs about mathematics education, the self, and the social context) and also between the subcategories of the main mathematical beliefs (e.g. different beliefs about self-efficacy and control beliefs in Ryan's case). This finding suggests that if students express different mathematical mindsets and beliefs about mathematics education, the self, and the social context, then a one-size-fits-all approach to mindset interventions may not be sufficient to address the full spectrum of students' beliefs.

Our findings also show that the social context plays a crucial role in shaping students' mathematical mindset, although these influences are often more nuanced and harder to categorize as strictly fixed or growth oriented. In our analysis, it was more straightforward to classify beliefs about mathematics education and the self as either fixed or growth. However, it became more difficult to apply the same classification to beliefs about the social context because oftentimes this relied on making inferences from students' mathematical experiences. For example, Ryan's experience of his teacher making him "feel dumb" potentially relates to a belief that the teacher's role is to support and assist students, which could be classified as a growth mindset. We were able to infer this based on Ryan's other statements about the social context of his current mathematical classroom. However, without this context, a statement similar to Ryan's could be taken as a belief that a teacher's role is to pass judgment on students, which may suggest more of a fixed mindset. Thus, while it was sometimes difficult to categorize beliefs

about the social context as relating to fixed or growth mindsets, it was possible in some cases by looking at the broader set of student experiences.

Finally, beliefs about the social norms in the class can influence a student's beliefs about the self and mathematics education. A supportive and encouraging classroom atmosphere can foster a growth mindset by helping students feel capable of improvement. The social context defines expectations of what it means to be a good student, and these expectations are assimilated by students as a personal belief about the self, which influences the student's mathematical mindset. Ryan's experience illustrates this point. Similarly, Emily's story tells how positive social norms in the class can nurture growth mindsets in beliefs about mathematics education. Mathematics-related beliefs in one category can influence and even change beliefs in another. For instance, a student's belief about the self can impact their attitudes toward mathematics education and whether they hold fixed or growth mindsets along with these beliefs.

Our findings raise important questions about different types of mathematical mindset related beliefs and the role of the social context in influencing these beliefs. As students begin to adopt different mathematical beliefs through shifts in social norms in their class, they may revise their beliefs about what it takes to succeed in mathematics, hopefully pushing them towards developing more growth-oriented mindsets. Interventions should address specific areas of fixed mathematics-related beliefs while building upon existing growth-oriented beliefs. In our data, we found that students appreciated the supportive classroom environment and multiple opportunities given to understand the material and demonstrate their knowledge. Interventions instructors incorporated in the course included activities related to productive struggle and metacognition (McGuire, 2015). In addition, the course curriculum emphasized understanding mathematical structures rather than only developing procedural knowledge. These types of interventions and teaching practices can help shift student beliefs and mindsets. It is important to acknowledge that even though a student may display a fixed mindset related to a certain type of mathematical belief, they may still hold a growth mindset related to other belief categories, what we have termed in this paper as a "mixed mindset." Future research could explore how educators can leverage this concept of mixed mindsets to foster more growth-oriented beliefs in students, particularly in areas where fixed mindsets persist.

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Examining Inequitable Talk and Perceived Status During Group-Worthy Tasks in an Undergraduate Topology Course

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We report on the initial findings from a project aimed at enhancing equitable group work in undergraduate proof courses through the design and implementation of group-worthy tasks. This mixed methods study reports on three group-worthy tasks implemented in a topology course. We investigated the extent to which student interactions in these small groups reflect participatory and relational equity in relation to perceived academic status. Quantitative results indicate patterns in which the group-worthy tasks may disrupt students with high status from dominating talk turns taken, the amount of talk turns taken, and number of words spoken by the students, or neither depending on the tasks. Qualitative results explored relational inequity via silencing, ignoring, or no talk during these three tasks. Our initial findings suggest that group-worthy tasks alone may not guarantee equitable participation or relations in group interactions.

Keywords: Status, Equity, Group-Worthy Tasks, Proof

There is a growing consensus that active learning approaches can benefit students taking undergraduate mathematics courses (Freeman et al., 2014). While there is some evidence that active approaches may lead to more equitable outcomes (e.g., Laursen et al., 2014; Theobald et al., 2020), it is important to recognize that such strategies do not automatically guarantee equitable participation among students. Research in inquiry-based settings suggests that inequities can persist, including in proof-based courses where group work is a central component (Brown, 2018; Ernest et al., 2019; Johnson et al., 2020). A possible cause for these inequities is how students perceive each other's status; that is, their assumptions about who is "better" or "smarter" at math influence their participation in group work.

One factor that can amplify or attenuate inequitable participation is the nature of group work tasks themselves (Shah & Lewis, 2019). Both content and structure can create opportunities for every student to engage and contribute mathematically. For example, well-designed group-worthy tasks can help address status-related dynamics that may otherwise hinder some students' participation (Cohen et al., 1999; Esmonde, 2009). Our understanding of the interplay between group-worthy tasks, status dynamics, and equity in undergraduate courses using active learning strategies is far from complete (Adiredja & Andrews-Larson, 2017), especially in advanced mathematics courses. This setting may exasperate status issues as the formal proof becomes the focal object of study (Weber & Melhuish, 2022). Here, we share analysis of a subset of data from the STructuring Equitable Participation in Undergraduate Proof (STEP UP) project aimed at supporting instructors in designing and implementing group-worthy tasks in proof-based courses. We investigated the following research question: To what extent do students' interactions in small groups reflect participatory and relational equity in relation to perceived status during three group-worthy tasks in a collaborative topology course?

Background and Framing

In our work, *equity* is when every student in a group has the necessary resources to support their learning (Shah & Lewis, 2019). *Inequity*, in contrast, occurs when power dynamics are structured in a way that prevents students from accessing these resources. Some resources might include physical access to the shared workspace (e.g., whiteboard, worksheet), opportunities to make written or verbal contributions to the group's work, and access to another group members' ideas. Two aspects of equity related to group dynamics are participatory and relational equity. *Relational equity* (Boaler, 2008) is characterized by interactions that allow each student in a group to have the opportunity to develop an identity as a capable learner. *Participatory equity* refers to the fair distribution of both opportunities to participate and the actual participation among all students involved in a learning interaction (Shah & Lewis). No situation is entirely equitable or inequitable; instead, relational and participatory equity fluctuate in subtle ways from moment to moment and can be influenced by social factors.

Inequities in group work can emerge due to imbalanced power dynamics rooted in differences in status. A status characteristic is defined as a "social ranking that everyone agrees is preferable to have a higher rank rather than a lower one" (Cohen et al., 1999, p. 84). In the classroom, academic status characteristics, such as perceived mathematical ability, are particularly influential (Cohen & Lotan, 2014) and can be affected by broader status characteristics like gender, race, and ethnicity (Cohen et al., 1999; Ridgeway, 2018). Students with higher status often have more chances to participate and contribute, which can lead to greater learning opportunities (Cohen & Lotan).

Small group interactions impact opportunities to learn during group work because students who have more opportunities to participate often participate more, and as a result, learn more (Esmonde, 2009). Research has shown that the nature of the task influences the extent to which participation is inequitable between group members (Chizhik, 1999, 2001). One strategy to balance participation and status is to utilize group-worthy tasks, a term borrowed from Complex Instruction (Cohen et al., 1999). A task is "group-worthy" if it affords various ways to approach the task, includes mechanisms to ensure that every participant is engaged and learning, and promotes a positive sense of interdependence among group members (Cohen & Lotan, 2014). Such tasks can elicit more opportunities for each member to participate in mathematically meaningful ways, potentially disrupting imbalanced participation due to status differences.

Methods

Participants and Procedures

The data comes from an NSF-funded collaboration between mathematics educators and mathematicians to design and implement group-worthy tasks in proof contexts. The research team video and audio recorded student interactions during three group-worthy tasks in a Topology course at a Hispanic-serving institution with a high first-generation population in the southern United States. The class was taught by a topologist during the Fall 2023 semester and consisted of 17 undergraduates and 3 graduate students (see Table 1). Daily classroom activity included randomized group work, usually with students working in groups of 3 or 4 at whiteboards. The three group-worthy tasks had different content and structure from the usual group activities. The three tasks were implemented on non-consecutive class days in weeks 8, 9, and 12 of the 16-week semester. All audio was transcribed and cleaned alongside the video data.

Task 1 focused on proof comprehension of the Universal Property of Quotient Spaces Theorem, with four groups of four students studying two examples to understand the quotient

space and its induced map. Each individual student worked on one part (a-d), then collaborated on part (e) to create a composite function. Task 2 focused on constructing a proof of the proposition stating, "The diagonal is closed if and only if X is Hausdorff", with groups of four investigating two examples, answering questions for each, and recording their findings on the front whiteboard to identify patterns. Each group member was assigned a unique role[1] with specific information needed to complete the task. Task 3 focused on proof analysis of the proposition stating, "Let $f: X \to Y$ be homeomorphisms. If X is connected, then Y is connected." In this task, pairs of students compared two proofs (A and B). Each student studied one proof, then explained it to their partner to compare and identify similarities and differences.

Table 1. Participating Student Demographics, Perceived Status, and Task/Group Information.

Task #.Group Name	Name (Perceived Status)	Gender	Race/Ethnicity
Group 1.A, Group 2.C, Pair 3.B	Lily (low)	Woman	Hispanic
Group 1.A, Group 2.B, Pair 3.C	Zahir (low)	Man*	Hispanic*
Group 1.A, Group 2.A, Pair 3.A	Pink (mid)	Woman	White, Hispanic
Group 1.A, Group 2.B, Pair 3.D	Kieran (mid)	N/A	South Asian
Group 1.B, Group 2.C, Pair 3.A	Kendra (high)	Woman*	Black*
Group 1.B	Brandon (mid)	Man	White
Group 1.B, Group 2.A, Pair 3.D	Joe (mid)	Man	Spanish
Group 1.B, Group 2.C, Pair 3.E	N (mid)	Woman	Mexican
Group 2.A	Alexis (high)	Woman	White-Caucasian
Group 1.C, Group 2.A, Pair 3.B	Luke (mid)	Man	White
Group 2.B	Ariella (low)	Woman*	Hispanic*
Group 1.C, Group 2.B, Pair 3.C	Xander (high)	Man*	White*
Group 1.C, Pair 3.E	Julie (high)	Woman	White
Group 1.C	Samantha (low)	Woman*	White*

*Indicates researcher assumption; all other data is student self-identified. Note: Group 2.C was not analyzed qualitatively due to a non-consenting fourth member.

Data Analysis

This study used a convergent mixed methods design, combining quantitative and qualitative results (Creswell & Plano Clark, 2018). The goal was to gather complementary data on students' interactions in a topology course to assess participatory and relational equity in group tasks (Morse, 1991). Phase one involved coding video transcripts of group work for three group-worthy tasks, calculating turn and talk point ratios, and conducting a Spearman Rho analysis. Phase two merged these results to explain group interactions and interpret the findings.

Quantitative Analysis. The constructs captured in the quantitative analysis are: Perceived Status, Turn Ratio, and Talk Ratio. Perceived Status was determined through the results of a status survey that each participating (and present) student completed the class period following the third group-worthy task implementation. For the status survey, students (n = 10) circled peers they thought were "best at math" which captured perceived academic status via mathematical

[1] (1) Linguist assisted group with provided key definitions of Hausdorff, diagonal, and closed. (2) Zoologist assisted group's understanding of provided "strange and exotic" topologies: trivial, discrete, finite-complement, and poset topologies. (3) Product Manager assisted group with provided definition of the product topology. (4) Summoner assisted group by "enlisting additional resources" for the group.

ability. Those circled four or more times were "high" status (coded as 2), two to three times were "mid" status (coded as 1), and less than two were "low" status (coded as 0).

We adapted Reinholz and Shah's (2018) ratio approach to quantifying participatory equity to the individual level. We calculated Turn Ratios and Talk Ratios. Turn Ratios were calculated based on each student's talk turns divided by the total talk turns and then by the ideal portion (e.g., 0.25 for a group of four). A turn is defined as a continuous segment of uninterrupted speech by a student. A new turn starts when a different student speaks, or is addressed by the instructor. Talk Ratios were calculated using a word count total by summing words per talk turn, coding as 1 for < 5 words, 5 for ≤ 20 words, and 20 for >20 words (Reinholz & Shah, 2018). The overall Talk Ratio for everyone was then calculated in a similar manner to the Turn Ratio.

Turn Ratios and Talk Ratios both provide information about participatory equity. Spearman Rho was calculated for Task 1 ($n = 12$), Task 2 ($n = 11$), and Task 3 ($n = 10$) to evaluate the correlation between Turn and Talk Ratios. Given that Turn Ratios are based on talk turns and Talk Ratios on word count, they are expected to reflect each other. A Spearman Rho was chosen over Pearson correlation due to the analysis being more appropriate for a small sample size and ordinal data (Siegel & Castellan, 1988). Additionally, Spearman Rho requires, and considers, the variables to be ordinal for the objects to be ranked (Siegel & Castellan, 1988).

Qualitative Analysis. We used Stovall et al.'s (2023) concepts of Silencing, Ignoring, and No Talk to analyze relational *in*equity in group interactions. "Silencing" occurred when one or more students interrupted or talked over another so that the former student did not finish their idea. "Ignoring" occurred when one or more students did not verbally acknowledge another student's verbal contribution. "No talk" occurred when a student did not verbally contribute during the entire activity and the other students and/or instructor did not address this. We reviewed videos together to resolve disagreements and only finalized codes after reaching a consensus. After coding, we wrote memos summarizing group dynamics (including physical space and student actions), inequitable talk instances, and initial inferences. The memos also included each group member's perceived status to further link the qualitative summaries with the quantitative results. A research team member, also a student in the class, provided insider knowledge on classroom dynamics, which helped resolve coding conflicts and clarify norms.

Results

Quantitative Results

Since both Turn Ratio and Talk Ratio reflect participatory equity to some degree, we used Spearman's Rho to assess their correlation, expecting to find a relationship between the number of talk turns and the number of words spoken. As suspected, the relationship between a student's Turn Ratio and Talk Ratio was found to have a strong positive correlation that was statistically significant for both Task 1 ($\rho = .874$, $p = <.001$) and Task 2 ($\rho = .897$, $p < .001$). This means that students with a high Turn Ratio are likely to have a high Talk Ratio, meaning the more talk turns a student has the more likely they will say more words. However, unexpectedly, the relation between the two ratios was not found to be statistically significant ($\rho = .530$, $p = .115$) for Task 3. To potentially explain the relationship between the ratios, we parsed the relation via perceived academic status (see Figure 1). In Figure 1, the red circles represent a student with high status, the blue diamonds represent a student with mid status, and the green squares represent a student with low status. We note three patterns from that data when accounting for status.

First, with Task 1, the Turn Ratio and Talk Ratios are highly associated, and perceived status is consistently spread throughout. This suggests that, when regarding Task 1, status did not

influence a student's Turn or Talk Ratios. Second, with Task 2, Turn Ratios and Talk Ratios are highly related, however, those with high status are clustered at the top right meaning that those with perceived status talked more in both the amount of talk turns taken and words spoken. This relationship would not be captured by Spearman Rho. Furthermore, other than one green square on the right (low status), those with low status are clustered at the bottom left. Lastly, with Task 3, we note that most of the students had a Turn Ratio between 0.90 and 1.05, however students with high status (red circle) have higher Talk Ratios than those with low status (green square). Our initial findings suggest that the group-worthy tasks may disrupt high status students from dominating talk turns taken (only), talk turns taken and amount of talk, or neither depending on the task. Thus, while the Talk Ratios and Turn Ratios may be related in some instances, our tasks may not be disrupting status as originally conjectured. These patterns may not be generalizable and could result from various factors such as task design (e.g., Task 3 was paired), course culture, timing, or student interactions.

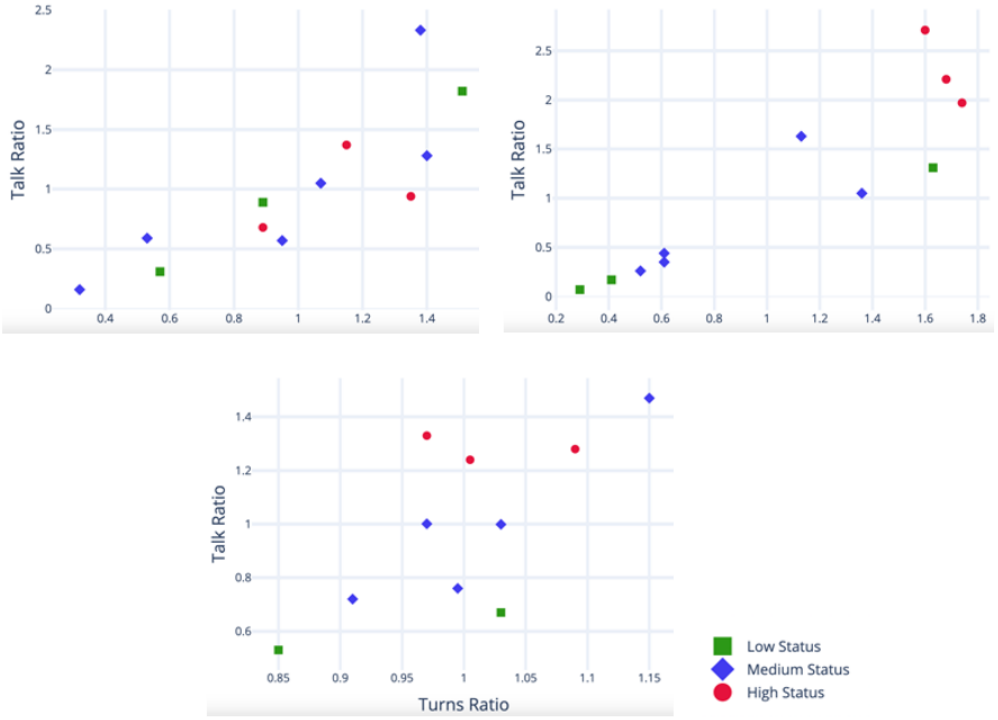


Figure 1. Correlations between Turn Ratios (horizontal axis) and Talk Ratios (vertical axis) Parsed by Perceived Status for Task 1 (Top Left), Task 2 (Top Right), and Task 3 (Bottom Middle).

Qualitative Results

The qualitative analysis occurred concurrently and focused on the relational equity element. In this section, we report an overview of these results for each task and consider how relational and participatory equity may relate.

Task 1: Proof Comprehension. Task 1 showed a more equal distribution of talk and turns during the task. There were no instances of "no talk", meaning everyone in each group verbally contributed to the group's mathematical work. However, there were some instances of ignoring and silencing. In Group 1.B, there were two cases of ignoring directed toward N (mid status) and Joe (mid), three instances of Brandon (mid) silencing N, and two instances of Brandon silencing

Kendra (high). Although Kendra spoke often, she asked many questions and listened to her peers' responses without talking over them. Kendra was the only member of this group that attempted to engage collaboratively with all members. Comparatively, Group 1.A, Kieran (mid) was ignored by everyone else in the group many times, and silenced (both physically and verbally) several times by Lily (low). Interestingly, Lily, who had low perceived status, dominated this group interaction and engaged in inequitable talk the most compared to her group mates. At one point Kieran stated, "So... this is something—" then Lily interjected, "We know this because of this," which cut Kieran off and he did not get a chance to finish his idea. In Group 1.C, inequitable talk occurred as students of equal or lower status silenced or ignored others, while a high-status student encouraged collaborative engagement. This suggests that relational inequities may account for the more evenly distributed participation ratios.

Task 2: Proof Construction. In this task, there were students who dominated the talk time and talk turns according to the quantitative analysis. The two focal groups had different patterns of interaction relationally. Group 2.A's interactions were largely respectful and inclusive with no instances of ignoring. Alexis (high status) made sure everyone contributed to the work by passing the marker and explicitly saying things like, "Okay. Who wants the marker next?" There were three instances of silencing directed towards Joe (mid) by Alexis and Luke (mid). For example, after Alexis made an observation, Joe tried to make a statement ("Yeah. I think it-") but was interrupted by Luke, and Alexis continued talking. Alexis and Luke's verbal contributions took up a majority of the overall talk time.

In contrast, "no talk" was indicated for Group 2.B because Zahir (low) contributed only one-word responses (e.g., "yeah", "oh okay") and only spoke more than one word to the instructor when he joined the group to ask how things were going. There was one instance of Kieran (mid) and Xander (high) ignoring Ariella (low), and one instance of Xander (high) silencing Ariella. For example, Ariella started to say something (although it is unclear if she was reading to herself or talking to the group), and Xander interrupted her to ask for the definition of a product topology (she was the "Product Manager"). She stopped talking, gave him the paper with the definition on it, and said, "It's right over here. I guess it's the product topology." She did not continue speaking. Kieran and Xander contributed almost all of the mathematical work, with Xander writing on the whiteboard nearly the entire time. Together, Xander and Kieran's verbal contributions took up about 84% of the total talk time. This group exemplified relational inequities in which higher status students ignored and silenced lower status students. That is, the participatory inequities may be at least partially explained by the relational inequities.

Task 3: Proof Analysis. The quantitative results pointed to this task having balanced turns in talk, but an imbalance in how much talk occurred. There were no cases of ignoring or "no talk" across all five pairs. There were few instances of silencing across each pair (except Pair 3.C), Xander (high) and Zahir (low), in which Xander repeatedly silenced Zahir. For example, while Zahir was explaining his proof, he stated "And then—" but Xander interrupted him by saying, "We know F is a homeomorphism…" barely giving Zahir time to talk before he started explaining to Zahir how to do the proof. Xander's verbal contributions took up about 67% of the total talk time, while Zahir's took up only 33%. Comparatively, there were some instances of silencing between partners in Pair 3.D, Joe (mid) and Kieran (mid), however one person was not overly silencing the other throughout the interaction, unlike with the previous pair.

Overall, the interactions across the remaining pairs were respectful and collaborative. In Pair 3.A, Pink (mid) and Kendra (high), there were no instances of silencing. The two made sense of both proofs together throughout the task, with Kendra asking a lot of questions when she was

confused and Pink helping Kendra process through the reasoning behind both proof approaches. Kendra's verbal contributions made up 62% of the talk time and Pink's took up 38%. Then, in Pair 3.B, Luke (mid) and Lily (low), there was only one instance of Lily silencing Luke. They noticed the proofs were similar, and while Luke shared what he noticed, Lily interrupted him to share her own noticing. Luke's verbal contributions made up 73% of the pair's verbal contributions, meaning Lily's verbal contributions made up only 27%. For Pair 3.E, Julie (high) and N (mid), there were very few cases when Julie silenced N. For instance, at one point N stated, "So then because of the surjection or the bijection, we know it's—" then Julie interjected, "It's a bijection, just cover both bases." N did not have a chance to finish saying what she knew. Julie's verbal contribution took up 64% of the total talk time while N's took up 36%. In four of the pairs, the Turn Ratios were overall balanced, yet the partner with higher perceived status spoke more words than the other partner. This trend did not apply to the only status-balanced pair, Joe (mid) and Kieran (mid), in which both Turn and Talk Ratios were balanced. We found that in the low-high pairing, there were substantial relational inequities that may account for the imbalanced talk, but this did not account for the imbalance in the mid-low status pair.

Discussion

This study set out to examine the extent to which group interactions reflected participatory and relational equity during three group-worthy tasks in an inquiry topology course. Task 1 seemed to disrupt inequitable status and participation patterns, despite some instances of relational inequities. In fact, the inequities appeared to disfavor those with higher status. Task 2 seemed to reproduce inequitable status and participation patterns and was the most relationally inequitable of the three tasks, with the only instance of "no talk" occurring, and higher-status students ignoring and silencing lower-status students. Task 3 seemed to balance the number of talk turns, but high-status students still dominated the conversations in words. There were some relational inequities with silencing in a low-high pairing, but this was not sufficient to account for the overall imbalance in talk across the pairs.

Reflecting on the task design, we can form further conjectures about participation differences. Task 1, which had the most balanced participation, was structured for group members to become experts on specific parts and assist others, culminating in collaborative work for the final part. This design fostered individual accountability and interdependence, making each member's contribution essential for the group's success. In comparison, Task 2 used group roles and it was observed that students largely ignored these roles. Each role card had important information on it, but students did not have to become "experts" on their role. In Group 2.B, we saw that Ariella was assigned the "Product Manager" role and was meant to be the expert on the product topology definition. However, since the definition was written on a card, Ariella simply handed it to Xander, rendering her role obsolete. It may be that students perceived these types of group roles as unnecessary to complete the task. Finally, Task 3 was a paired task with distributed expertise of two different proofs, which necessitated student interaction and contributions, leading to more balanced Turn Ratios. However, Talk Ratios showed that higher-status students still dominated the conversation.

What is clear is that group-worthy tasks alone may not guarantee more equitable group interactions. In the qualitative analysis we noticed the impact the instructor had on disrupting inequities. Instructors can engage a quiet student or highlight a contribution that may otherwise be ignored. Future work will explore the relationship between instructor interventions during group work (e.g., assigning competence; Cohen & Lotan, 2014), as well as how the culture of the classroom and the interplay of race and gender may also impact group work dynamics.

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Examining Subjective Factors Shaping Students' Combinatorial Thinking

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Research on combinatorics education indicates a crucial need in mathematics education to address students' ways of thinking at a level that benefits a deeper understanding of how students conceptualize counting problems. Attending to the ways in which students approach and interpret counting problems reveals the subjectivity in their decision makings. In this study, I report data from written works submitted by thirty five prospective high school teachers as they considered a disagreement between two possible responses to a combinatorial problem. Defining subjective combinatorics as a subjective aspect of students' interpretations of sets of outcomes facilitates describing students' counting activity. I propose subjective combinatorics as a construct to shed light on our understanding of students' counting activities, focusing on the subjectivity inherent in their combinatorial thinking.

Keywords: combinatorics, conceptual analysis, counting problems, undergraduate mathematics education, subjective combinatorics

Enumerative combinatorics, the area of mathematics focused on solving counting problems, is part of many K-12 and undergraduate mathematics curricula. This prominence is justified, as counting problems not only have practical applications in fields like computer science and probability (e.g., Abrahamson, Janusz, & Wilensky, 2006; Shaughnessy, 1977) but also challenge students to engage in critical mathematical thinking (e.g., Kapur, 1970; Martin, 2001; Tucker, 2002). In recent years, the mathematics education research community has placed greater focus on students' combinatorial reasoning. Numerous studies have examined common errors, strategies, and different ways of thinking involved in solving counting problems (e.g., Annin & Lai, 2010; Batanero, Navarro-Pelayo, & Godino, 1997; Eizenberg & Zaslavsky, 2004; English, 1991, 2005; Halani, 2012; Lockwood, 2011, 2013; Lockwood, Swinyard, & Caughman, 2015; Tillema, 2013). Despite the critical role of combinatorial reasoning, many of these studies highlight the difficulties students face in solving counting problems. Given these difficulties, it became apparent to investigate the factors that influence students' success in combinatorics. Understanding these factors requires a detailed examination of the elements involved in students' combinatorial thinking. To this end, research on combinatorics education has progressively focused on understanding students' ways of thinking about counting problems. Lockwood (2013) conceptualized students' combinatorial thinking by identifying three primary components: formulas/expressions, counting processes, and sets of outcomes. This model has significantly enhanced educators' ability to analyze and understand students' combinatorial thinking. This study aims to focus on how personal interpretations and subjective judgments impact students' combinatorial thinking. By investigating this dimension, I propose subjective combinatorics as a construct to better understand the role of subjectivity in students' combinatorial thinking.

Literature Review and Contextual Insights

This research focuses on subjectivity in students' combinatorial thinking. I am not aware of any research that attends to this explicit focus. As such, in this section I turn to prior research on the related topics; I first describe several studies on subjectivity and then focus on research on students' combinatorial thinking.

On Subjectivity

To explore how subjectivity can influence the process of solving counting problems, it is essential to start by considering the definition of counting problems themselves. A counting problem highlights restrictions in representing a set of outcomes and tasks students with determining the cardinality of the set of outcomes. Expanding upon the definition of counting problems, the highlighted restrictions can be presented either explicitly or implicitly. When restrictions are given implicitly, they can be interpreted subjectively. For example, in counting the number of outfits when there are two shirts, five pairs of pants, and three hats, the word "outfit" could be interpreted subjectively. For one student, having a hat as part of an outfit might be a necessity, while another student might find a shirt and pants sufficient. As exemplified, the word "outfit" has many distinct interpretations. Hence, students' answers will depend on how they interpret "outfit". Since a counting problem can be interpreted differently from one person to another, the set of outcomes that needs to be counted could therefore be different.

Attending to the ways in which students specify outcomes while solving counting problems highlighted the subjectivity in their combinatorial thinking. This led me to consider a new aspect of combinatorics, to which I refer as *subjective combinatorics*. Although the subjective aspect of some areas in mathematics such as probability has been considered in the mathematics education literature (Chernoff & Zazkis, 2011; Jones, Langrall, & Mooney, 2007) the subjective aspect of combinatorics has not been studied explicitly.

According to Jones et al. (2007) "*subjective probability*, describes probability as a degree of belief, based upon personal judgment and information about an outcome" (p. 913). Batanero et al. (2005) highlighted "in this subjective view, what is random for one person might be nonrandom for another…a degree of uncertainty specific for each person" (p. 24). Borovcnik et al. (1991) clarified "the basic assumption here is that subjects have their own probabilities which are derived from an implicit preference pattern between decisions" (p. 41). According to Hawkins, & Kapadia (1984) "to a greater or lesser extent the probability is an expression of personal belief or perception" (p. 349). Combinatorics, like probability, involve a degree of subjectivity influenced by the problem itself and the solver's knowledge and experience. I introduce the notion of *subjective combinatorics* and in what follows, I provide further motivation for this construct. I make a case that subjective combinatorics provides a valuable lens through which to study the learning and teaching of combinatorics. In particular, I argue that recognizing subjectivity in students' combinatorial thinking is beneficial to enhance our understanding of how students conceptualize counting problems. To this end, my research aims to develop a detailed description of subjective combinatorics and to explore its potential effect on students' combinatorial thinking. This dual focus aims to provide evidence for – and to lay groundwork for – future studies to examine subjectivity in students' combinatorial thinking. In particular, this study attempts to answer the following research questions: *How does subjectivity affect students' combinatorial thinking? How do students generate restrictions as they solve and evaluate counting problems that are susceptible to be considered subjectively?*

On Students' Combinatorial Thinking

Lockwood (2013) introduced a model of students' combinatorial thinking that included three components: formulas/numerical expressions, counting processes and sets of outcomes. Building on Lockwood's model, this study explores how subjectivity influences students' combinatorial thinking, particularly in their interpretation and generation of sets of outcomes. The concept of subjective combinatorics, introduced in the following section, serves as a guiding theoretical principle, with a central focus on the role of personal judgment in interpreting.

Theoretical Framework: Subjective Combinatorics

Drawing from the concept of subjective probability, *subjective combinatorics* views combinatorics as a matter of belief, shaped by personal judgment and information about outcomes. It depends on two things: the combinatorics problem and the knowledge and experience of the person solving the problem. Just as in subjective probability, where the likelihood of an event is assessed based on personal perspectives, subjective combinatorics acknowledges that different solvers may approach counting problems with varying assumptions, leading to different yet potentially valid solutions. This construct emphasizes the variability in how individuals conceptualize and reason through combinatorial tasks, which have been shaped by their experiences, understanding of the problem, and the context in which the problem is presented. I propose that the differences between "correct" and "incorrect" responses to counting problems, which stem from different subjective interpretations, are merely differences in the description of sets of outcomes. To determine the specific interpretation applied among the possible ones, insights can be gained by examining the comments from individuals who have solved the problem.

In daily life context problems, students might need to navigate two different interpretations simultaneously: one based on their understanding of the problem's requirements and another shaped by their personal experiences and prior knowledge. Balancing these interpretations can be challenging as they may need to reconcile what the problem explicitly asks with how they relate to it from their own experience. The notion of subjective combinatorics can shed light on understanding these diverse interpretations.

Methods

In this study, as part of a larger study, thirty-five preservice secondary mathematics teachers participated. At the time of data collection, they were enrolled in a mathematical methods course for teaching secondary mathematics within a professional development program in mathematics education. While their mathematics background and experience varied significantly, all held degrees in mathematics or science. The course is typically completed in the final semester of studies toward teaching certification. In this course, students are expected to enhance their personal mathematical knowledge and investigation skills, and to explore pedagogical approaches to secondary mathematics. The focus is on topics often overlooked in school but not requiring knowledge beyond the school-level mathematics.

Counting problems was one of the chosen topics for exploration. During class sessions, the participants were either reminded of or refreshed their knowledge of the multiplication principle and addition principle from their high school studies and discussed combination, permutation, the binomial theorem, and combinations with and without repetition. The students revisited different counting problems and discussed possible errors in counting problems. After these discussions, participants were given the following task:

You work with a class on counting problems. One of the problems asks students to solve the *Kittens problem*: There are 10 kittens up for adoption. How many different ways are there to distribute kittens to 7 people so that everyone gets at least one kitten and at most two kittens? Two students suggest two different answers.

Jamie's answer: $\binom{7}{3}$

Lucas's answer: $\binom{7}{3} \times \binom{10}{2} \times \binom{8}{2} \times \binom{6}{2} \times 4!$

1) Which answer do you agree with? Explain your reasoning. If you disagree with both, please provide your own answer and justify your reasoning. 2) What do you think was implicitly

assumed in each answer? 3) Why do you think the students made these assumptions? 4) Reflect on your engagement with the assignment: How have you dealt with it? Did you have any difficulties and/or dilemmas while working on the assignment? What have you learned through the process? The suggested length for responding to each question is about 150 words.

The *Kittens problem* was intentionally designed to allow for subjective interpretation, making it a suitable task for exploring subjective combinatorics. Some students might view the kittens as identical, while others might treat them as distinct entities. Additionally, one interpretation might require that all 10 kittens be distributed among the 7 people, whereas another might allow for scenarios in which only 9, 8, or 7 kittens are adopted. In the latter case, this would result in 2, 1, or 0 people receiving an extra kitten, respectively. Both suggested responses in the task are correct based on different interpretations. Jamie's and Lucas's responses suggest consideration of distributing 10 kittens among the 7 people, however, they do not share the same consideration for kittens. Jamie's answer assumes that the kittens are identical and focuses on choosing 3 out of the 7 people who will each receive 2 kittens. Since Jamie considers the kittens to be identical, there is no need for any additional counting processes or any additional consideration of the set of outcomes. Lucas's approach builds on Jamie's by considering the kittens as distinct entities. While Lucas also starts by choosing 3 out of 7 people to receive 2 kittens each, he incorporates additional steps to account for the distinct nature of each kitten. Lucas counts the number of ways to assign 2 distinct kittens to each of the 3 chosen people and then distributes the remaining 4 distinct kittens to the remaining 4 people. This involves considering $\binom{10}{2}$ for the first person, $\binom{8}{2}$ for the second, $\binom{6}{2}$ for the third, and 4! for the final distribution.

The participants' written works comprise the data for this study. The claims highlighting subjective interpretations served as a unit of analysis. Having repeatedly read the data, I looked for common themes by highlighting noteworthy explanations. I then categorized them based on participants' views on the uniqueness or interchangeability of kittens. This process involved examining how unconscious assumptions, real-world experiences, personal and mathematical connections, and level of detail influenced participants' interpretations.

Results

In the following analysis, participants' interpretations of the problem, whether viewing the kittens as identical or unique, are presented along with their reasoning. While the excerpts primarily focus on participants' interpretations and their reasoning, it is important to note that these interpretations directly informed their combinatorial thinking. Specifically, participants' interpretations shaped their understanding of sets of outcomes and their counting processes, as reflected in their alignment with either Jamie's or Lucas's approach.

Interpretation 1: Kittens as Identical

Unconscious assumptions. In solving counting problems, personal assumptions often play a significant role in shaping how individuals interpret and solve problems. The following excerpt illustrates this well.

> *Shawn:* I could immediately understand Jamie's answer because it matched my initial thinking. I found Lucas's answer very perplexing until I discussed it with my friend. He mentioned that the kittens could be considered unique or interchangeable. That is when I realized that I had been making the assumption that the kittens were interchangeable without even consciously making that choice. By thinking of the kittens as unique, I could then begin to understand Lucas's answer.

Shawn's explanation highlights an important aspect of subjective combinatorics: the role of unconscious assumptions in interpreting the problem. Initially, Shawn found Jamie's approach to the problem intuitive because it aligned with his own interpretation that kittens are identical. This assumption was made without explicit awareness. It was only after discussing the problem with a friend that Shawn recognized this underlying assumption and acknowledged that kittens could also be seen as unique. Shawn's experience underscores how unconscious biases towards viewing items as identical or unique can significantly influence students' combinatorial thinking.

Interpretation 2: Kittens as Unique

Real-world experiences. While all participants' responses suggest they agree people are inherently unique, some extend this perspective to kittens, recognizing their unique attributes.

> *Max*: Jamie was assuming that the kittens were not unique. The kittens, like the children, are different, so Jamie's expression does not account for all the permutations of different children holding different kittens.

Max challenged Jamie's assumption by likening kittens to children, rejecting the idea that they can be considered identical, and emphasizing that each kitten, like each child, should be considered unique. This perspective is echoed by Heather, who supports Lucas's interpretation by emphasizing that kittens have unique markings, sizes, and personalities, much like humans.

> *Heather*: I agree with Lucas. Kittens have both unique markings, sizes, and personalities therefore it is fair to treat them as unique objects. Humans are similarly typically considered unique objects. Therefore, we are ordering one set of unique objects onto another set of unique objects.

Heather extended this idea further by noting that the problem involves matching one set of unique objects (kittens) to another (humans), reinforcing the importance of recognizing these individual attributes in her combinatorial reasoning.

Matthew's explanation links real-world experiences and combinatorial thinking.

> *Matthew*: When thinking about this question, Lucas appears to have an image of 10 distinct kittens that are up for adoption. This is a fair assumption, since each kitten would naturally be different. A typical animal shelter that offers kittens for adoption would have kittens that already have their own names, and a personalized description for each kitten before they are put up for adoption. During the process of acquiring a kitten, a person also usually chooses a kitten based on a number of factors such as their personality, gender, or compatibility with other animals, thus making each choice unique.

Matthew highlighted the realistic context of an animal shelter to emphasize the assumption of unique kittens. He drew a parallel between the counting problem and the real-world scenario of adopting kittens, where each kitten typically has a distinct name and description. This real-life context supports the idea that each kitten is perceived as unique, similar to how they are treated in actual adoption settings. Matthew's explanation illustrates how real-life contexts can shape students' assumptions in combinatorial reasoning.

Dual Interpretation: Kittens Could Be Either

Personal and mathematical connections. While the earlier discussion highlights the influence of real-life contexts on interpreting the uniqueness of kittens, the following excerpt explores how personal experiences in combinatorics and emotional connections can simultaneously lead to seeing kittens as both identical and unique. This dual interpretation highlights the complex interplay between combinatorial experiences and personal experiences in shaping one's approach to combinatorial problems.

Heather: Most math questions are set up so that if there are 10 kittens all must be distributed unless otherwise stated thus people tend to compare to other previous examples and will assume all kittens must be distributed. Also when discussing the adoption of kittens most people have an emotional preference to having them all adopted out. The confusion as to whether kittens are unique or not may either stem from a comparison to other math examples where objects are frequently treated as identical unless otherwise stated or from their experience with kittens and how they picture the scenario. For example, most people who have previously adopted a kitten and picture this scenario as 7 people going to an agency meeting the kittens and then going home with one or two kittens will know that going home with kitten 1 instead of kitten 2 could feel drastically different for that person. However, if the person is picturing a scenario where an adoption agency is just going to deliver kittens they have never met it might not matter which one you get since you never met kitten and kitten 1 is still a kitten you get to play with or perhaps the person has never met a kitten before and is unaware of their unique personalities, markings and sizes.

Heather's assumption that all kittens must be distributed and might be treated as identical stems from her experience with combinatorial problems. This reflects how her past experiences in combinatorics have shaped her initial subjective interpretation of the problem. However, she also noted that real-life experiences and emotional preference with kittens can influence students' interpretations, leading them to see the kittens as either identical or unique, depending on how they picture the scenario.

Level of detail. While Heather's explanation brings attention to how mathematical experience and personal experience both contribute to a dual interpretation, the following excerpt further explores how the level of detail and personal connection can shift one's interpretation.

Shawn: If I want to categorize something as "bird", I am probably only looking to see if it has a beak and wings. If I want to categorize something as a wood duck, I now have to consider the size, the colouration of the feathers, the body shape, and the habitat. The word kitten is a high level category. It can easily lead to thinking of kittens as generic and interchangeable. Whether one particular person thinks of them as interchangeable or unique probably depends on how they feel about cats in general, and whether those ten cats are physically in front of them. I initially approached the problem seeing the kittens as interchangeable, but if I had to choose a pet from a specific group of kittens, I would have a preference for which one I wanted, meaning that I felt that the kittens were unique.

Shawn's explanation touches on how the level of specificity in categorization can influence whether one perceives objects as interchangeable or unique. This suggests that how one frames the problem—whether focusing on broad categories or specific details—can shape the subjective interpretation of the elements involved. Shawn's shifting perspective—from seeing kittens as generic to recognizing their uniqueness based on situational factors—illustrates how subjective interpretations can evolve as more context or personal connection is applied. Similarly, the following excerpt contrasts how different levels of detail in viewing the kittens can lead to different interpretations.

Dmytro: If Lucas views kittens as complex with different colours, personalities, genders, etc. then in his eyes kittens aren't just kittens. Each one is complex enough in that they should be distinguished from one another. Where Jamie may view kittens as just kittens. More on a macro level like they are all just one species.

Dmytro's explanation highlights the varying levels of complexity with which individuals might perceive and categorize objects, such as kittens, in combinatorial problems. His distinction between viewing kittens as a homogeneous group versus recognizing their unique attributes underscores how subjective judgment influences whether objects are seen as interchangeable or unique. This analysis suggests that the level of detail one applies to a problem can significantly shift the outcome of their reasoning.

Discussion and Conclusion

In this paper, I introduce *subjective combinatorics*, with the goal of providing language to describe different interpretations involved in students' combinatorial thinking. As discussed in the introduction, in recent years, the mathematics education research community has increasingly focused on addressing students' underlying conceptualizations of combinatorial ideas. This study addresses subjectivity in students' combinatorial thinking, thereby filling a gap in the mathematics education research on the area of combinatorics.

To explore subjective combinatorics, I examined how participants attended to a counting problem that allowed for distinct interpretations. The study focused on two main aspects: the restrictions participants generated while engaging with counting problems and the impact of their interpretations on their combinatorial thinking. In particular, this study addressed the following research questions: *How does subjectivity affect students' combinatorial thinking? How do students generate restrictions as they solve and evaluate counting problems that are susceptible to be considered subjectively?*

In addressing the research questions participants were presented with a counting problem which highlighted students' disagreements regarding solutions, drew participants into a situation where they faced varied interpretations. The task encouraged participants to engage with the numerical expressions/formulas component of Lockwood's model. The variation in interpretations—whether treating kittens as identical or distinct—affected how participants made sense of the suggested answers. This variation demonstrated how different conceptualizations of the problem influenced their counting processes and considerations of sets of outcomes. The results highlight how different conceptualizations of the problem influence students' combinatorial thinking. Furthermore, the results indicate that participants' interpretations of the set of outcomes were significantly influenced by unconscious assumptions, real-world experiences, personal and mathematical connections, and the level of detail considered. The findings illustrate how these factors shaped students' views on the uniqueness or interchangeability of objects in combinatorial problems. This underscores the importance of considering subjective factors when analyzing students' counting activities, as these factors can lead to diverse and sometimes conflicting interpretations of the same problem.

This study reveals the subjective nature of students' combinatorial thinking and the significant impact of personal context on their interpretations. Recognizing how these subjective factors affect students' combinatorial reasoning can offer valuable insights for educators. This understanding can lead to more effective teaching strategies that consider the diverse and personal ways students engage with counting problems. Lastly, the focus on a specific type of combinatorial problem may not capture all aspects of subjectivity in students' combinatorial thinking. Future research could benefit from exploring subjectivity across a broader range of combinatorial problems to gain deeper insights into students' combinatorial thinking. Additionally, future research might explore approaches to formulating combinatorial problems that reduce subjectivity, potentially leading to more consistent interpretations of students' combinatorial thinking.

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Utilizing Video Resources in Undergraduate Mathematics Classes: A Systematic Review Research Synthesis

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The increased use of educational technology has raised questions about technology equity in mathematics education research and pedagogy. This paper provides a subset of findings from a larger systematic review, which aims to investigate literature on equity and personal device use in undergraduate mathematics courses from 2009-2023. For the systematic review, literature was searched and screened for information on personal device access, utilization, and outcomes. This paper reports on the subset of findings specific to the utilization of video resources in mathematics classes. Through the synthesis process variable information regarding student outcomes and student perceptions of use was pulled from 82 studies. Findings suggest that the use of video resources correlates to either higher academic outcomes or no difference in academic outcomes. Additionally, student perceptions suggest that video resources can afford an equitable educational experience in a variety of ways, but elements of distraction and communication can serve as limitations.

Keywords: research synthesis, personal devices, video resources, technology

Technology use in the teaching and learning experience has become more ubiquitous since the turn of the century (Meehan & Halmun, 2016), especially in the form of utilizing personal devices, such as laptops, tablets, or smartphones (Galanek et al., 2018). Mathematics education scholars have raised questions about the implications of personal device use in terms of equitable education. To investigate the research on equity and technology in the literature Hennesey & Dunham (2008) published a research synthesis, *Equity and use of Educational Technology in Mathematics.* The results suggested individual students' access to technological devices varied by household and utilization proficiencies differed by identity traits, such as gender. Hennessy and Dunham concluded that a variety of factors are intertwined into answering questions involving equity and device use, including levels of access and how the technology is being integrated into the mathematics curriculum.

Since the publication of Hennessy and Dunham's 2008 research synthesis technology use in the classroom has undoubtably changed over time. For example, during the early 2000s home and school technology access was limited (Swain & Pearson, 2002). However, personal devices have become more commonplace in society at large over the last two decades. A new population of students, known as *digital natives*, are now navigating higher education with a variety of life experiences involving technology (Filippi & Agarwal, 2017; Galanek et al., 2018). This shift in the undergraduate population, as well as the gap in time since the last research synthesis, motivated a large systematic review project analyzing more recent literature on personal device access, utilization, and outcomes for undergraduate mathematics students. The results for this paper focus on a subset of the systematic review results, utilization of mathematics video resources on personal devices. Utilization is synthesized in terms of both student perceptions of the video resources and the relationship to student outcomes.

Framework

Fennema's (1990) equity framework, accommodated by Hennessy & Dunham (2008) for technology-specific research, was used in this review. This framework navigates literature on equity and technology in mathematics with three lenses: access, utilization, and outcomes. The access component explores mathematics students' access to personal devices and the affordances and barrier that may exist around device access. Personal device utilization encompasses the differing uses of personal devices and their affordances and limitations in mathematics. The multitude of resources available through personal devices have the potential to accommodate a variety of students' learning needs, which is necessary for equitable education (Tan et al., 2019). Student outcomes can also be used as measurable data for statistical analyses in research. This review addressed outcomes in terms of improved learning outcomes and retention rates, both of which are important for investigating equity in education. This paper reports findings using the utilization and outcome portions of the framework.

Research Question

1) In the literature since 2009, what are the results surrounding undergraduate mathematics students' video resource utilization in terms of student perceptions and outcomes?

Methodology

There are four steps in the systematic review methodology: searching, screening, extracting the data, and synthesis & analysis. In order to search and screen the literature inclusion and exclusion criteria must first be created.

Inclusion/Exclusion Criteria

Literature inclusion and exclusion criteria were established through the SPIDER framework (Cooke et al., 2012), which comprises the following elements.

Sample. The sample of interest within the literature is undergraduate students using personal devices in mathematics courses.

Phenomenon of interest. The phenomenon of interest is student access and utilization of personal devices in undergraduate mathematics courses. Personal devices in this review include tablets, computers, and smartphones. Research on calculators is beyond the scope of this review.

Design. This review incorporates qualitative, quantitative, and mixed methods studies through content analysis and Miles & Hubermann's (1994) synthesis methodologies.

Evaluation. Studies were first reviewed to establish the variables for evaluation within access, utilization, and assessment. For the systematic review the access variable was used to evaluate levels of access and barriers to access. Utilization was evaluated in terms of the type of personal device utilization, student perceptions on the device utilization, and the relationship to student outcomes in undergraduate mathematics courses. Outcomes involved both student learning outcomes and retention rates.

Research type. This review included empirical research with quantitative, qualitative, or mixed methods methodologies. Peer reviewed literature and grey literature (dissertations, conference proceedings, etc.) were included and found through the search and snowballing process.

Searching

The goal of the searching process was to find literature involving undergraduate mathematics students' personal device access, utilization, and outcomes. Literature published between 2009-

2023 was searched, to prepare an updated synthesis since Hennessy & Dunham's 2008 publication. ERIC (EBSCO Host), Education Source (EBSCO Host), and The International Journal of STEM Education and PRIMUS, through the University of New Hampshire PRIMO system, were searched to find the 27 foundational studies before snowballing, otherwise known as the start set. The searching methods included Boolean operators with synonyms found either on the ERIC thesaurus or through discussions with colleagues. The search string included four sections, (personal devices) AND (mathematics education) AND (equity) AND (undergraduate students). Each section included a list of synonyms for search terms, separated by OR to find a greater variety of literature.

Screening

Two types of screening were completed for this systematic review: title/abstract screening and full text screening. Studies found through the search terms and systems previously mentioned were screened using the SPIDER framework's inclusion/exclusion criteria. Three rounds of snowballing (forward and backwards) were completed on the 27 foundational studies to find additional literature for inclusion in the systematic review. Through the screening of 17,196 studies, a total of 357 were found to meet all inclusion criteria and were synthesized in the systematic review. The 27 foundational studies in the start set were then reviewed to establish how the literature discusses access, utilization, and outcomes to establish synthesis variables for extraction. For example, studies discussed resources utilization in terms of who assigned or chose the utilization (teacher vs. student choice) and student perceptions of the utilization. Therefore, utilization variables included sub-variables for data extraction of "assigned vs. choice" and a "student perception".

Data Extraction

After establishing the variables of interest within access, utilization, and outcomes, data was extracted from each study using data extraction forms (Briner & Denyer, 2012). Variable data and general literature data, such as country and methodology of research, were extracted to supply some background information on the included studies.

Data Synthesis and Analysis

Synthesizing literature is a process of aggregating the results of multiple studies to establish new knowledge about a phenomenon of interest in the field (Campbell et al., 2003; Denyer & Tranfield, 2009). Literature was first synthesized in this review by looking for common themes or results amongst the data extraction forms. It was then analyzed through cross case data analysis methods and content analysis.

The Miles and Huberman methods involve coding results and establish themes amongst those codes (Dixon-Woods et al., 2004; Miles & Huberman, 1994). This process was used when aggregating and synthesizing results surrounding student perceptions of video utilization. For example, some student perception codes included "Flexibility, mobility, can learn anywhere, helpful for pacing learning, can pause videos, can rewind videos, and can split into smaller learning chunks". Codes were grouped together into the theme "Flexibility and Pacing for Learning". Content analyses were also conducted to count frequences of variable information occurrences amongst the results of each study (White & Marsh, 2006).

Results

Of the 357 studies synthesized for the systematic review project 86 use qualitative methods, 144 used quantitative methods, and 127 used mixed methods. Additionally, studies were conducted in 52 different countries. Through synthesizing and analyzing the studies for this review, six major categories of utilization were found amongst the literature: online learning modality, the flipped classroom, video resources, online homework/practice problems, communication, and additional tool interventions. Please note the results reported in this paper are a subset of the larger results and focus only on the utilization of video resources.

Utilization Perceptions & Outcomes of Video Resources

This section reports results surrounding video resources, from 82 studies, in terms of both student perceptions and student outcomes. Video resources include videos made by the instructor, class recordings, and supplemental videos, such as YouTube or Kahn Academy.

Student perceptions. Student perceptions on the utilization of video resources with personal devices were investigated in 75 studies. Table 1 outlines the different student perceptions and the larger themes that they fit into (underlined). Frequency counts through a content analysis were taken in order to display how many studies reported a given perception, affordance, or limitation. While some studies reported a general liking or positive perception for video resources, counted in the positive perception frequency count, others went further detail about the elements of video resources that students found useful or limiting. A perception, affordance, or limitation was included in the table if it appeared in at least two studies.

Table 1. Student Perceptions of Video Resources

Theme Subcategory/Perception	Study Frequency
Affordances or Positive Perceptions	75
Flexibility and Pacing for Learning	25
Flexibility or Mobility	10
Helpful for Pacing Learning	10
Pausing & Rewinding Features Useful	11
Can Split into Smaller Learning Chunks	3
Modes of Representation and Engagement	8
Visual Aids in Videos	7
Multiple Modes of Representation & Engagement	3
Supplemental Learning	16
Improved Confidence on Prerequisite Knowledge/Closed Learning Gaps	6
Students Found Supplemental Videos Useful	9
Makes Math More Enjoyable	3
Limitations or Negative Perceptions	7
Need for More Communication with Instructor or Other Students	5
Distraction or Lack of Motivation Using Technology	2

Throughout the literature undergraduate students reported both positive and negative perceptions for using video resources for learning mathematics. Positive perceptions include themes of self-pacing and learning flexibility, videos providing multiple forms of representation and engagement for learning, videos affording additional supplements for learning, and videos making mathematics more enjoyable. Negative perceptions included students wanting more direct communication with peers and instructors for collaborative learning and a general lack of motivation or distraction using personal devices to watch mathematics videos.

Student learning outcomes and retention. Student learning outcomes for the systematic review were assessed in one of three ways: better outcomes with device use, no difference in outcomes, or better outcomes with control group (no device use or less frequent device use). Also note that the assessment to measure learning outcomes varied by study. For instance, some used exam scores, while others used final grades. In the review 22 studies investigated video resources and learning outcomes. Some compared video resource use to no resource use, while others looked at frequency of use. Figure 1 shows the frequency of study results in terms of the three possible outcomes.

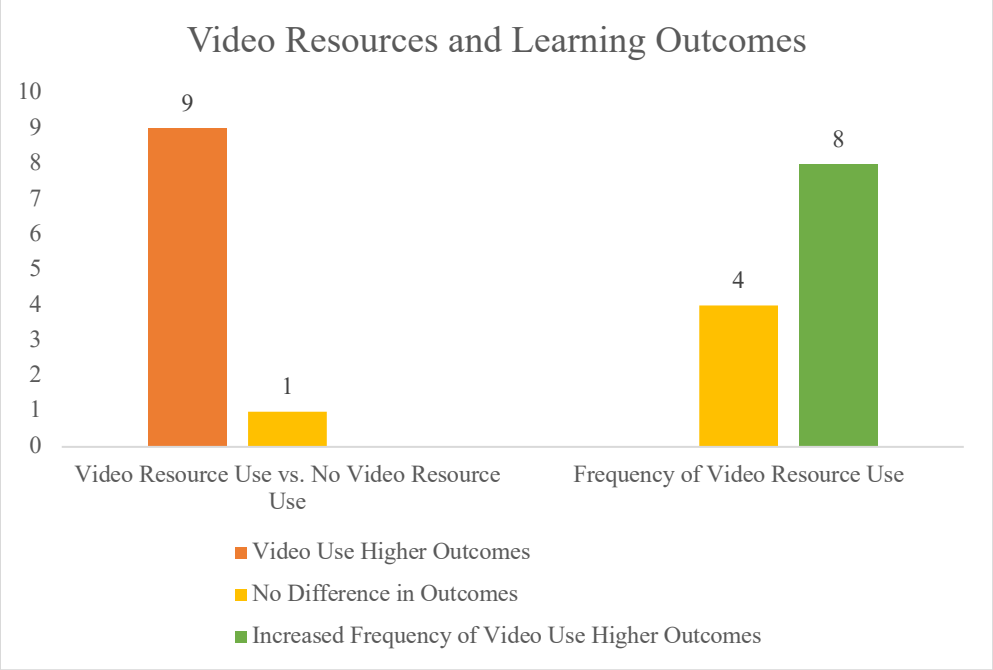


Figure 1. Video Resources and Learning Outcomes Results Amongst Studies

In the studies examined, the results suggested that groups who utilized video resources either had higher academic outcomes or no difference in academic outcomes compared to the control groups. The same was true when comparing the higher frequency use group to the lower frequency use group. Only one study reported on retention rates in relationship to video resource use and suggested that video resource use was related to higher retention rates than the control group, implying that students were more likely to drop out of the course if there were no video resources available for learning in the course.

Discussion

Systematic reviews can afford both scholars and educators a better idea of the existing knowledge within the field and assist with future research and pedagogical choices (Briner & Denyer, 2012). This section will provide an overview of the findings from this part of the larger systematic review, how they may impact the field of mathematics education, and the limitations to the study.

The literature in the systematic review was from 52 countries around the world, suggesting it is an area of interest for scholars at a global level. Additionally, all three empirical methodologies were used amongst the literature: qualitative, quantitative, and mixed methods. These multiple methodologies afforded a variety of data types for synthesis and analysis and suggest there is no one way to investigate personal device use in undergraduate mathematics courses.

Video Resource Utilization

While six categories of utilization were established in the systematic review project, this paper only focused on personal device utilization for video resources. In terms of student perceptions there were more papers reporting positive perceptions or affordances than negative perceptions or limitations. According to the literature, students found video resources useful for flexible, paced learning practices, multiple modes of representation and engagement, such as visual aids, and closing learning gaps. In turn, student participants in the literature reported feeling easily distracted while using personal devices and wished for more live communication with others while learning, which the videos could not provide. Learning outcomes were either higher with video resource interventions or there was no significant difference. These findings are significant for educators when deciding what resources to provide or encourage students to utilize in mathematics learning. Students may be more likely to utilize video resources than others because of the positive perceptions and ability to pace for student agency over learning. The research suggests video resources do not hinder learning outcomes and may even improve them. Additionally, student participants reported many video resource affordances that align with equitable educational practices. The affordance of multiple modes of representation and engagement is beneficial for diverse learning needs, which supports equitable learning practices (Tan et al., 2019). The process of closing learning gaps, reported by students in multiple studies, is equitable because it provides the ability for students to achieve a similar starting line. Students found videos to be beneficial for self-pacing and splitting into smaller learning chunks as well. These student-made learning decisions can promote learner autonomy, a skill known to be necessary for undergraduate learning and career development (Caprara et al., 2008; Zimmerman, 2008).

Given these findings amongst studies, educators may want to include video resources for their mathematics students. By providing supplemental resources that students generally enjoy utilizing, and find useful for learning, pacing, and multiple modes of representation, students may be able to learn mathematics in a more equitable, inclusive way. Furthermore, the review found either no change or improves academic outcomes with video resource use, suggesting that video utilization has the potential to improve mathematical learning. However, the synthesized findings indicate some cautionary measures should be taken, such as still maintaining educator-student communication and being aware of the potential for distraction while on a personal device. Educators may be able to accommodate students' communication needs by opening new avenues for inquiry, such as a discussion board, to assist with questions or concerns that may

arise while watching videos. Additional research can be done to further investigate how to address elements of distraction and focus with personal device use as well.

Limitations

Synthesizing data from studies with multiple methodologies can be difficult in a systematic review (Mays et al., 2001). While the techniques for data synthesis and analysis in this study afforded the ability to aggregate qualitative, quantitative, and mixed methods data, the analysis is not as robust as more rigorous statistical methods, such as meta-analysis. Hence, assumptions made from the findings should be made with caution. The methodology for searching in a systematic review is used to find the most literature possible on a given topic or phenomenon. The process of snowballing was used to find a large variety of research findings and grey literature, but note it is not possible to complete a fully exhaustive search, so other studies on this topic may still exist outside of this review.

Conclusion

This paper includes a subset of findings from a larger systematic review project on personal device use in undergraduate mathematics classes. Literature was searched and screened for evidence of findings involving the three elements of Fennema's (1990) equity framework: access, utilization, and outcomes. Through review synthesis and analysis, results regarding personal device video resource utilization student perceptions and video resource outcomes were discussed. Video resources were utilized in 82 studies, with results regarding both student perceptions and student outcomes. Students reported affordances involving equitable benefits, such as multiple modes of representation, self-pacing, and closed learning gaps. However, students also reported negative perceptions involving elements of distraction and lack of communication. Student learning outcomes were either improved with video resources or no difference was found.

This review is beneficial for both research and pedagogy. Scholars can use the information to further investigate ways to improve upon the limitations reported and find elements of video use that are most beneficial to learning. Educators can use this review to inform decisions around personal device access and utilization within the classroom. The findings suggest a general student liking and perception of usefulness, as well as unchanging or improved academic outcomes. Therefore, providing video resources for student use may be beneficial to both student enjoyment of mathematics courses, as well as mathematical understanding. However, educators should take caution with providing meaningful communication outside of video use, and the potential for distraction.

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"These Topics are Off the Table": Shifting Discourse Toward Critical Transformations in Mathematics Departments

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Many undergraduate mathematics departments are seeking to improve equity in their programs; however, they may struggle to translate these goals for equity into action. In this study, we used Cultural Historical Activity Theory (CHAT) theory to investigate the evolution of a Networked Improvement Community (NIC) focused on improving equity within their mathematics department, highlighting ways that NIC discourse evolved to support more critical, action-oriented activity. By examining field notes of meetings, documentation (e.g., agendas), interviews, and journal entries from NIC members, we identified key components of the NIC activity system. Our analysis revealed ways that NIC leaders mediated tensions between their object (goals) and implicit discourse norms governing NIC conversations. Specifically, we found that to make progress toward their goals, NIC leaders needed to establish an explicit rule to take certain topics "off the table" in NIC meetings. This rule facilitated discourse that was more action-focused than previously observed.

Keywords: critical transformations, equity, activity system, mathematics

Local data typically show mathematics departments have equity problems: inequitable student outcomes, encompassing not just grades, but course-taking patterns and trajectories, as well as mathematical efficacy and views on the value of mathematics in the world (e.g., Denaro et al., 2022). However, there is little in the way of coherent direction for department members to know what to do to address inequities in mathematics teaching and learning. Indeed, there is no singular "checklist for equity" that can magically "fix" inequities that have deep roots in our culture, educational systems, and institutional systems.

The ACT UP Math project partnered with three mathematics departments to establish networked improvement communities (NICs; Martin et al., 2020) as part of a research-practice partnership designed to help mathematics departments wanting to use data to make critical transformations to address inequities. A NIC is a community of people working toward a common improvement goal, in connection with others across a network. NICs typically are driven by improvement science, iteratively planning and testing improvement ideas through multiple improvement cycles of observation, reflection, planning, and action. NICs are an ideal vehicle for approaching critical transformation efforts; the emphasis across the network is adaptation to local contexts, not replication. For this paper, we focus on the activities of one of the mathematics departments - Tau University as they worked toward critical transformation. In our work (see Hagman et al., 2024), we define making critical transformations as challenging and critiquing the existing systems that create inequitable mathematical experiences.

In seeking to understand the progress of NICs, where "progress" is defined as moving toward the NIC's improvement aim, it can be difficult to tease out the impacts of individual beliefs, collective beliefs, perceived constraints, and systemic constraints. Cultural Historical Activity

Theory (CHAT) provides a valuable lens through which to understand the activities of a NIC (Vygotsky, 1978; Engeström, 1987), particularly to understand change (Turner et al., 2017). Núñez (2021) applied CHAT perspectives to investigate equity-oriented higher education organizational and systemic change; Núñez found the application of CHAT afforded a fine-grained analysis of group activities that delve into cultural practices and how those are sustained or changed over time. CHAT provides a way to understand the roles and values of individuals, while maintaining a focus on the activity system, and allows for interrogating power dynamics and inequities. **Our overarching research question for this study is**: How did the NIC evolve as an activity system during one improvement cycle (from Summer 2023 - Spring 2024) to better support critical transformations?

Methods

Our study methodology is critical ethnography (Palmer & Caldas, 2015). As researchers engaged in a research-practice partnership (Coburn et al., 2021), we are immersed in the data as ethnographic researchers and as participants. Since Fall 2022, we have been working closely with the NICs in our project to better understand their institutional contexts and goals related to critical transformations toward diversity, equity, and inclusion and documenting all interactions.

Data for this study come from a larger project studying three NICs engaged in critical transformation efforts. The focus of this study is on the Tau University NIC, led by three faculty who all teach courses for the mathematics department but have different roles (Natalie, a lecturer; Garrett, a professor of practice and course coordinator; Olivia, a professor and college-level administrator). Tau University is a public, doctoral degree-granting university with high research activity located in the Southern United States. For this study, we draw on the following data sources collected between May 2023 and May 2024 during one improvement cycle of the NIC: individual interviews with NIC leaders (60 min/annual), structured field notes of monthly NIC meetings and planning meetings, NIC member reflective journals (approximately three times per semester), and project documentation (e.g., meeting notes; products from project activities, such as when NIC members reflected in a Google Jamboard about definitions of equity).

The research team, who are regularly observing Tau NIC meetings and facilitating interviews, met weekly over the past year to debrief observations, discuss the goals and challenges of the NIC, and make plans to support the Tau NIC's work. As part of these discussions, we kept notes of research observations as well as themes and storylines present in the efforts of the Tau NIC. One storyline that was consistently present in the NIC data was the shift toward critical transformations in the form of discourse patterns about students at their institution. The Tau NIC leaders were introduced to an anti-deficit framing of students through a presentation from and conversations with our project team, where we drew on the equity-oriented work of Adiredja (Adiredja, 2019; Adiredja & Louie, 2020) and Gutiérrez (2009; 2013). From this point, we kept a comprehensive evidence log of data for instances where Tau NIC members defined critical transformations, described their efforts toward equitable or critical outcomes, or talked about students in their courses or at their institution. After the presentation, NIC members more clearly articulated a goal toward critical transformations and began explicitly addressing their assumptions and language about students in NIC meetings.

Our data analysis consisted of several iterations of analyses, for each data source (e.g., interviews, observation field notes, journals) and for each storyline (e.g., leader goals, discourse about students), similar to narrative coding (Saldaña, 2013). Once we determined that we were viewing the evolution of the Tau NIC as an activity system with the leaders working toward

specific goals via NIC activities, we began to align the NIC's evolution with the seven components of the Cultural Historical Activity Theory framework.

Analytic Framework: Cultural Historical Activity Theory

Cultural Historical Activity Theory (CHAT) is a theory to understand object-oriented activity undertaken by an individual. In other words, a subject's thoughts and actions are directed toward a goal or object; Vygotsky's (1978) initial theory was that the object is realized (as an outcome) through meditating tools/artifacts. Later, CHAT was extended by Engeström (1987) to emphasize the collective nature of object-oriented activity; this work added the components community, rules, and division of labor to the framework (see Figure 1). CHAT can be used to frame how we understand complex, dynamic systems, particularly those that involve tensions and evolve over time (Engeström, 2001). CHAT also examines the history of the activity system and multivoicedness of those involved, emphasizing that the subjects bring various perspectives, interests, and conventions, such as stakeholders in different roles within an educational setting (Hancock & Miller, 2018). We draw upon the second generation CHAT (Engeström, 1987) to examine the activity of the Tau NIC.

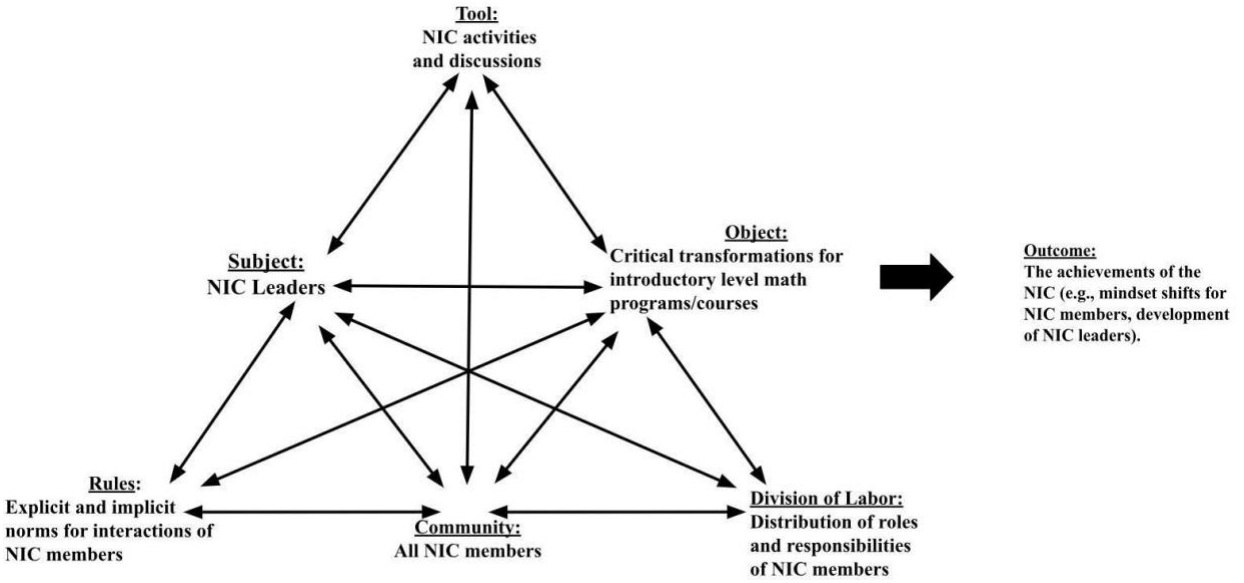


Figure 1. Second Generation CHAT Framework (Engeström, 1987) with Tau NIC activity system components

When applying the CHAT framework as an analytic tool, one of the initial steps is to operationalize the framework by determining what are the subject, object, tools, rules, community, division of labor, and outcomes in the activity system. Figure 1 briefly describes the Tau NIC features we drew upon to determine each CHAT component. The subject, community, and division of labor were operationalized based on observations of NIC planning meetings and artifacts, and our understanding of the formation of the Tau NIC. We operationalized the object, tools, and rules of this activity system based on an in-depth qualitative analysis of journal entries, leader interviews, observational field notes, and (ongoing) outcomes of the NIC within their institution. From this analysis, we identified a significant evolution in the rules of discourse governing the NIC, brought about by tensions between the object of the NIC activity system and the implicit rules governing how NIC members interacted with one another. This evolution supported the NIC's capacity for critical transformation. In the findings section, we describe this evolution in detail.

Findings

In this section, we focus on the NIC leaders' creation of an explicit rule for discourse due to tensions between the goals (object) and implicit rules for discourse established within the NIC's activity system. Based on observations of planning meeting between the three Tau NIC leaders and their journal entries, we identified three primary goals guiding the activities of the NIC:

1. Improve diversity, equity, and inclusion in mathematics at Tau University
2. Improve student experiences in mathematics at Tau University
3. Support mathematics faculty in creating equitable and inclusive experiences at Tau University

Meetings with the ACT UP Math research team in August 2023 equipped leaders with ideas for NIC activities (tools) which could support NIC members in thinking about critical transformations, including promoting an "anti-deficit" perspective of students. Natalie highlighted the importance of this critical language to her goals, "I have learned a lot about my own biases. The biggest impact on my attitude has been the phrase 'students are experts on their own experiences,' rather than just novices in the classroom" (Fall 2023 journal). Similarly, in a Fall 2023 journal entry Olivia wrote, "And a big part of [achieving those goals] is shifting the way that they look at our students, not from a deficit standpoint, but from looking at their student's potential." Thus, for at least one NIC leader, a major goal was to shift faculty mindsets in support of critical change.

Two Relevant Rules for Discourse

The NIC activity system had two rules which we identified as important in understanding the NIC leaders' decision to introduce a more explicit rule governing the NIC's discourse. One rule was that NIC members trust one another and are open to sharing about their experiences. The second rule is that NIC members are allowed to share frustration about students, including a perceived systemic lack of mathematical preparation of the students in their classes. We next describe the formation of these two rules, and the ways in which they supported or inhibited the object of the Tau NIC activity system.

Rule: NIC members trust one another and are open to sharing about their experiences. Evidence from observations indicates that most NIC members were willing to share vulnerable teaching experiences with one another. This was likely supported by the example set by NIC leaders. For example, Garrett often shared his "mistakes" and personal realizations with the group, and we observed Garrett and Natalie intentionally planning what experiences they would share in whole group meetings as a way to model vulnerability with others and ways to question one's practices. This openness often supported the NIC in thinking critically about practices and in working to understand student experiences. However, the level of trust and openness that NIC members felt for each other also supported an implicit rule which at times impeded the object of the NIC, namely a rule that NIC members could share their frustrations. Olivia commented on how the NIC has a "tendency to focus on the negative" in conversations (Tau NIC Planning Meeting - 2023.02.16). Often these frustrations turned to negative comments about students, particularly a perceived lack of preparation for precalculus courses.

Rule: NIC members are allowed to share frustration about students, including a perceived systemic lack of mathematical preparation of the students in their classes. Since the beginning of the NIC, it was apparent that NIC members felt comfortable sharing their frustrations about students, particularly frustrations with their prerequisite mathematical knowledge. We observed multiple occasions of faculty publicly discussing their frustrations during NIC meetings. In the first NIC planning meeting, Garrett expressed being nervous "that

people will just complain about students if there is extra [unplanned] time" during NIC activities (Tau NIC Planning Meeting - 2023.01.19). At first, the NIC leaders seemed to operate as if this was inevitable, and planned around it. For example, in one planning meeting Garrett described how he chose to share data and analysis from his own classroom (mediating tool) to help shift the conversation away from complaining about students:

> I know that whenever faculty get together, we just talk about whatever is bothering us. So we got into that [complaining] and I was like, 'oh I have to get us out of that conversation'. So then I started talking about my data and just sharing my analysis.

In another planning meeting, Garrett shared an additional strategy (mediating tool) for shifting conversations away from complaining: "if it comes to where people start complaining, I will try to have some questions in my pocket." (Tau NIC Planning Meeting - 2023.10.23) In a later meeting, Olivia assured Garrett that she would "jump in" if faculty did start going "off the rails" (Tau NIC Planning Meeting - 2023.11.08). Thus, it seems that the leaders desired to minimize deficit discourse about students, but it was an implicit goal that was not shared with the other NIC members (community). There was also some evidence that NIC leaders felt empathy for faculty members as they shared their frustrations. In his Spring 2024 interview, Garrett described how when "you're in your class, and these things are happening, and then you are in some sort of situation where there's other people with a shared experience, and you want to share - you vent." Garrett felt that this venting was not "necessarily deficit" oriented and stressed the importance of having discussion prompts planned ahead of time so that when instructors "start grumbling" he can move them "forward." In fact, Natalie's observation of her own "venting" and its effects prompted an introduction of a new, explicit rule in the Tau NIC activity system, which we describe below. Garrett was drawing on his experience as a course coordinator when he normalized instructor complaints about students, and also argued that part of the role of facilitating instructor meetings is having explicit strategies or tools (e.g., discussion questions) that can move a group forward. Such tools were used by NIC leaders to shift discourse, but a lack of progress toward critical transformations in the NIC eventually contributed to a desire to make more explicit rules for discourse about students within the NIC activity system.

Explicit Shifts in Discourse: Re-examining What is on the Conversation Table

During Spring 2024, the NIC leaders observed that deficit conversations about students were impeding their ability to reach their goals of critical transformation. In a journal entry, Olivia reflected on the mindset of some NIC members, "I think some of our faculty are still in a space of, it's the students and it's really their problem." She highlighted that at times students are discussed from a deficit viewpoint but continued with a hopeful observation of NIC members, "But I do think that there's some evidence that some of our faculty recognize that there are things that we can do to better support our students, but they're just not really sure how." At this point, shifting discourse about students (within the NIC and among faculty in the department) became a new goal for the NIC leaders and an explicit rule about leaving deficit-oriented conversation "off the table" was shared with the NIC. The following excerpt illustrates how the Tau leaders decided on this rule.

In a planning meeting, the three leaders discussed the structure of their final semester meeting in April 2024. Olivia suggested questions to guide NIC members to consider what they are noticing when viewing data from their classrooms: "What is the problem we're seeing, what are the opportunities we can impact? Maybe strengths? Let's get them thinking about what they've been noticing." Natalie agreed that it would be a good idea to focus on identifying

potential problems. Garrett recalled a conversation with NIC members where he tried to challenge traditional notions and goals in the department about mathematical content and procedures in these courses (e.g., focused on computational fluency). Based on these reflections, the leaders discussed how they should facilitate this conversation in the April NIC meeting:

> *Natalie*: How do we stop focusing on what we can't change, and focus on what we can control?
>
> *Olivia*: Should this conversation take place in a small group?
>
> *Natalie*: Sure
>
> *Olivia*: Then come to the large group; people can vent in the small group, then have a productive conversation in the large group. […]
>
> *Garrett*: What are some potential problems we can actually address that may come up? Obviously I'm focused on changing the computational fluency and thinking everything needs to be this hard. What other problems will come up?
>
> *Natalie:* What we always go back to, is complaining about student prerequisite knowledge. On the one hand, the prereqs don't have to be that hard, on the other hand, they're way too easy.
>
> *Garrett*: Do we want to change their minds on this?

Garrett and Natalie continued to discuss NIC members' and their own views on students' prerequisite knowledge. This exchange ended with Natalie saying "I'm just venting now." In response, Olivia joined the conversation with a suggestion of taking prerequisite knowledge "off the table" for discussion during the April NIC meeting:

> *Olivia*: Ok, these things are off the table for today's conversation. We can't say "we can't do this because of X,Y, Z."
>
> *Natalie*: I like that. We all know that's an issue, let's not talk about that.
>
> *Olivia*: Yeah, and more so, today we're identifying potential areas of places we can consider. Today is about determining the problem that needs to be addressed.
>
> *Natalie*: Then, items off the table: Prerequisite preparation has to be off the table. We all know it's an issue and we have a problem with it.
>
> *Garrett*: Maybe co-construct this list with the NIC members.
>
> *Natalie*: Initially, the other [item] is higher up things we can't change [i.e., trying to make Calculus 1 two semesters long].
>
> *Olivia*: Maybe leave [the list of items off the table] as an open thing, and if they don't say it, then we say it [i.e., we suggest adding it to the list].

In the subsequent Spring 2024 NIC meetings, the Tau leaders established their new rule and worked to navigate conversations away from deficit discourse and toward more productive avenues. Garrett explained, "I don't think it's productive to repeatedly come up with the same challenges." In her interview, Natalie reflected on the impact of their decision to take deficit discourse about students off the table:

> We should focus on what we can fix. Regardless of how you want to label the problem, taking that off the conversation table still didn't remove it completely, but it didn't take up 50% of the conversation, instead 10-15%. I hope the NIC members understand why it was taken off the table. I hope we can train our brains to stop complaining about things we can't fix.

The NIC leaders realized that if their overall goal was to work toward critical transformations in their undergraduate mathematics programs, then they needed a collective, explicitly stated rule to focus on productive areas rather than get bogged down in deficit discourses about students.

Discussion and Conclusion

In institutional or departmental change efforts, there need to be clearly established norms or "rules" for the community of change agents. In the Tau NIC, working toward critical transformations and improving student experiences meant that it was necessary to shift the discourse that was occurring in the meetings and take on an anti-deficit perspective (e.g., Adiredja, 2019). Whereas NIC members collectively viewed students' mathematical preparation as a problem in their introductory mathematics courses, it was not productive to dwell on what students were lacking. Instead, the NIC leaders guided the community to notice students' strengths and attend to challenges they have the agency to address.

The CHAT framework emphasizes the evolution of the NIC through the interaction of the CHAT components, especially the objects, tools, and rules within the community. Viewed through this lens, the object evolved over one year to include a focus on shifting discourse and perspectives about students, and NIC leaders worked to achieve this through the tools of the NIC activities and the established rule of taking specific topics and anti-deficit discourse "off the table" for conversation. CHAT highlights how this shift was a result of observed tensions within the activity system and required clarification of objects and new rules and tools to address the tensions. CHAT was a helpful framework to analyze the multi-voiced activity system, as NIC leaders operated within a community of various stakeholders situated within a large mathematics department at a diverse institution, which complexified the decisions and impacts of their work.

Our work so far has focused on the second generation CHAT framework (Engeström, 1987) to understand the evolution of the NIC as an activity system across one improvement cycle. One of the most prominent shifts in the NIC occurred due to tensions between the object and goals of NIC leaders, and the rules governing discourse within the NIC. Future research could extend by applying third generation CHAT to look at tensions among connected activity systems. Third generation CHAT is better able to consider power dynamics within and across activity systems (Engeström, 2009), along with the inherent tensions involved in negotiating human activities (Sannino et al., 2009). In our case, since ACT UP Math is a research-practice partnership, we plan to examine the research team activity system together with the Tau NIC activity system, and how those systems interact to influence the progression of the NIC activity system. Future work can also apply CHAT to additional NICs – both those within the larger ACT UP Math project and those in other settings.

Finally, one implication of the application of CHAT to understanding NIC evolution is for NIC leaders to explicitly engage NIC members in discussing their activity system as a NIC. Applying a CHAT framework to an activity system or set of activity systems can help system participants see the objects and illuminate what are often implicit rules. Transformation work is more effective when participants are working toward a common goal (Kezar, 2014; Martin et al., 2020), and having a NIC map out their activity system in a CHAT framework can be useful in encouraging conversation about common goals and rules governing collective action.

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Using Poetic Transcription to Showcase Multilingual Students' Experiences of Silence in Introductory College Mathematics Education

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In this paper, we explore the role that silence plays in the experiences of multilingual students of color within undergraduate mathematics classes. The poetic narratives presented in this paper focus on exploring the role of silence in students' experiences. While silence often reflects marginalization (e.g., being silenced), it can also function as a mechanism for navigating different spaces, and as a form of resistance. As such, we take an asset-based and anti-deficit framing to explore how silence interacts with students' cultural capitals, including, for example, navigation capital, resistance capital, aspirational capital, and language capital. We use poetic transcription to present students' stories using only their own words.

Keywords: multilingual students, undergraduate mathematics, critical methodologies, poetry

Educational institutions at all levels continue to serve more students who are multilingual learners (henceforth, this term is used to refer to students whose home languages are different from the language used for instruction). For decades, research in K-12 mathematics education has advocated for tending to the care of these students, and the importance of nourishing their home language and culture in their learning of mathematics (Civil, 2009; Gorgorió & Planas, 2001; Moshkovich, 2002; Planas & Setati, 2009). In this paper, we focus on elevating the voices of multilingual students of color by sharing their narratives about introductory college mathematics education using their own words through the process of poetic transcription.

Background and Literature

Language is inherently political and can function, in conjunction with race, to empower some students while marginalizing others (Barwell, 2015; Planas & Civil, 2013). This has been discussed significantly at the K-12 level; however, the same challenges documented in the K-12 literature persist in higher education (Rios, 2024; Hwang, 2023). For example, Rios (2024) found that multilingual students were often positioned by others in deficit-oriented ways, particularly when they are not frequent vocal participants in the classroom. Civil (2014) states that "if most of the communication is expected to be in oral form, this may silence students who are not comfortable speaking up or who have a hard time producing oral explanations" (p. 14). In response, Planas and Civil (2013), Civil (2014), and Rios (2024) challenge deficit narratives about students' participation by emphasizing ways that home languages and prior experiences serve as assets to students' mathematical learning.

Focusing on student assets is an approach research has taken to promote an anti-deficit orientation of multilingual students. This involves recognizing the valuable role that students' cultural capitals can play in their education. In addition to language resources, other cultural capitals include students' families, communities, aspirations, and knowledge about navigating and resisting existing systems that are oppressive (Peleaz et al., 2023; Yosso, 2005). Research

has shown how different forms of capital, rooted in community, are linked to student motivation in pursuing and persisting in their education (e.g., Dilka et al., 2018; Samuelson & Litzler, 2016). Moreover, in Peleaz et al. (2023), the authors discuss the tensions around celebrating 9th-grade multilingual student Layla's strengths, agency, and ability to overcome, without minimizing the structural barriers that are potentially marginalizing to students of color. They state that "while other students may navigate or resist the educational system differently, their relative empowerment does not change the overall reality of oppression" (p. 11).

The growing body of literature examining multilingual students' experience in K-20 mathematics emphasizes the importance of amplifying students' voices and documenting their lived experiences using their own words. For example, Hwang (2023) uses narrative inquiry to explore, in greater depth, the lived experiences of four international, post-secondary students studying mathematics. In her methodological approach, Hwang constructs narratives—using primarily students' own words—to capture each student's experiences studying advanced mathematics.

There is an undercurrent in this body of literature related to silence, and the lack of students' voices, in their experiences. Recognizing and responding to this juxtaposition, this paper aims to showcase the individual experiences of multilingual students of color in undergraduate mathematics classes, especially around silence, while using only their own words via poetic transcription (Glesne, 1997; Prendergast, 2009; Tremaine, 2022). Our methodology builds off of Hwang's (2023) narrative inquiry approach to present multilingual students' experience in undergraduate mathematics. The poetic narratives we present here attend to both the challenges shared related to silence and the cultural capital demonstrated through silence.

Theoretical Perspectives

Asset-based and anti-deficit lenses

Two theoretical lenses guiding this work include asset-based perspectives (emphasizing the resources students bring rather than their deficits) and anti-deficit perspectives (explicitly and intentionally challenging the deficit narratives). Historically, the language resources of multilingual learners have been viewed as obstacles to learning mathematics (Barwell et al., 2016; Setati & Adler, 2000). To shift this narrative, asset-based and anti-deficit perspectives (Adiredja, 2019; Adiredja et al., 2020; Celedón-Pattichis et al., 2018; Harper, 2010) take the firm stance that "all students are capable of reasoning mathematically and that they bring productive resources for learning" (Adiredja et al., 2020, p. 521). In the context of language, these lenses assert that all students are capable of learning and participating meaningfully in the classroom, and that students' language backgrounds enrich their ability to do so. While asset-based perspectives highlight student's strengths and resources, anti-deficit perspectives go a step further to challenge deficit narratives (Adiredja, 2019). This asset-based, anti-deficit approach informs our work by acknowledging that some multilingual students of color may be quieter in the classroom, not due to disengagement, but because they may feel uncomfortable speaking in an environment that lacks diversity and emphasizes vocal participation (Civil, 2014; Rios, 2024). An anti-deficit lens also challenges the "disengaged" narrative by emphasizing alternative ways these students actively engage in their math courses (Rios, 2022).

Cultural Capital Wealth framework

Yosso's (2005) Community Cultural Wealth (CCW) framework exemplifies the transition from a deficit to an asset-based perspective. Specifically, this framework identifies existing resources and capitals that students of color often bring to their college experiences. Although

not typically valued by the dominant culture, these forms of capital originate from students' communities and can play a crucial role in how students of color navigate educational systems and environments. In particular, Yosso's framework identifies six cultural capitals: (1) *linguistic capital,* (2) *resistance capital*, (3) *navigational capital*, (4) *aspirational capital*, (5) *familial capital*, and (6) *social capital*.

In addition to language capital, in this work we focus on resistance and navigational capital. *Resistance capital* represents the knowledge and skills that students of color develop to stand up to inequities and injustices in their daily lives. Similarly, *navigational capital* is the knowledge and skills that students of color develop through constantly navigating spaces that are not designed for them. In this paper, we examine how silence intersects with these two capitals as manifestations of the resources students draw upon when facing potentially harmful experiences.

Perspectives on silence

Finally, we draw upon theoretical perspectives on silence. This literature provides distinct forms of silence: that of being silenced, broader notions of silencing, as well as using silence for navigation, and as an act of resistance. While the first two are clear acts of marginalization, the latter two, though they may happen in conjunction with being marginalized (Peralta et al., 2013), can also be seen as components of students' cultural capital, and related to navigational capital and resistance capital respectively. Silence can be shaped by students' sociopolitical realities (Smith & King, 2018). Silence is intermeshed with the distribution of power, and social roles, among dominant and marginalized groups of people (Mills, 2006; Verma, 2017). For example, the act of silencing—that is, the deliberate or inadvertent suppression of the voices, ideas, or agency of individuals or groups—functions as a way to "exclude", "constrain", and "devalue" others (Mills, 2006, p. 1). Moreover, silence can relate to assimilation, and how certain aspects of people's identities are silenced as a result of the dominant culture, including their language and cultural values (Migliaccio, 2016; Ricketts et al., 2007). At the same time, silence is also a form of communication and can have a specific navigational or resistance function (Verma, 2017; Ellis & Wrightsman, 2024). For example, in Gilmore (1985), African American students in an inner-city school deliberately used silence as a way of shifting the burden of communication back to the teacher, thereby asserting their resistance to the teacher's authoritative role in the classroom. We draw on these perspectives to guide our investigation and discussion of how multilingual students of color experience silencing in their introductory mathematics courses, as well as how they navigate, and resist being silenced.

Research Questions

Drawing from these perspectives, the research questions we explore are: (1) How do multilingual students of color experience silencing in college precalculus and calculus classes? (2) How do these students respond to silencing in college precalculus and calculus classes?

Methods

The data utilized for this paper is part of a larger study aimed at exploring the experiences of multilingual students in precalculus and calculus courses. The study was conducted at a large Hispanic-Serving Institution in the Southwest, which also serves a large population of international students. As part of the study, 28 students were selected to be interviewed, all of whom identified as having a home language other than English. All 28 students were students of

color[1], and although their racialized experiences in mathematics were not the central focus of these interviews, it is an important context that distinguishes their experience from multilingual students from Europe, where accents are often privileged and seen as an indication of education status (Moscovitch, 2007; Rosa & Flores, 2017). Interviews were conducted over Zoom, lasted for 60-90 minutes, and were semi-structured. Interviews were recorded and were transcribed verbatim. As will be detailed below, we identified six of these students to answer our research questions. In the results section, we include a brief introduction to each student.

We use poetic transcription (Glesne, 1997; Prendergast, 2009; Richardson, 2004; Tremaine, 2022) to showcase the stories of multilingual students of color within these courses. Poetic transcription is an arts-based methodological tool that re-presents interviewees' words in the form of a poem; this allows the interviewee's own words to be used to tell their story while also bringing in space for the researcher's interpretation and analysis through choices in poetic juxtaposition. Particularly as we discuss the cultural silencing of multilingual students of color, our research methodology must be one which centers those students' voices; we see re-presenting these students' own words verbatim through intentionally constructed poetry as aligned with this aim.

We developed a process for co-creating poetic representations of students' experiences in undergraduate mathematics. First, the first author identified 10 student interviews that explicitly discussed or related to silence. Four students were initially selected, and we began by individually engaging in poetic transcription using Tremaine's (2022) methodology. After creating poems for two of these students, we paired up to co-create collaborative versions. This proved challenging due to our attachment to our individual drafts.

To address this, we adopted a new approach: two team members would each create individual drafts, while two others would merge them into a cohesive poem by selecting stanzas that told a unified story. The first author then reviewed all the poems to ensure a cohesive voice across the collection. As we refined the process, we eliminated four students' stories after team discussions about their relevance to the theme of silence.

Through this process, we established a set of guidelines for consistency, such as using students' exact phrasing without creating repeated words, removing filler words when necessary, and preserving students' sentence structures. We used italics to indicate direct quotes and combined experiences from both high school and college to create a cohesive narrative. Our approach prioritized methodological rigor while embracing the creative elements of poetry to authentically capture students' experiences with silence.

Results

Qamar: *Not Something I Wanted to Hide*

Qamar (she/her) is a Syrian refugee student who spent the pandemic living in Saudi Arabia, while her college classes were all remote. In her poem, we see her share about two class experiences: first, where a teacher refused to record the video sessions, and so Qamar had to attend that class remotely at 3am her time; second, where her teacher muted Qamar in the video session so that the class could not hear Qamar's contributions, to which Qamar responded by unplugging her audio so the teacher could see that Qamar was attending but not listening.

[1] Drawing from Yosso's (2005) work and terminology, we have chosen to use the term *students of color* to refer to students from racially minoritized communities, based on the U.S. context. We recognize that international students may not be from racially minoritized populations in their home countries and that different countries use different terminologies to describe minoritized populations.

I was like:
Hey Professor,
can you please record the Zoom thing?
I'm not sure I could make it,
it's like really late.
Actually
I will not record this class.
We're actually going to make it more harder for you.
Because you know what?
It's your struggle, you have to deal with it.
We will not record the class.

I speak a different language.
I don't think it shows.
Maybe it shows in certain words.
I'm not sure how that would impact this.
There was definitely some circumstances where
the teacher is treating someone else better.
Ask me all the questions you guys want.

I was asking a question,
and the professor literally muted me.
I'm a good student, I'll do it–
You will not do it.
She'd interrupt me.
I'm not physically muting you,
I'm talking-muting you.

(continued)
Okay.
It's stay silent situation.
It was really belittling,
Rude.
and not welcoming–
I'm not asking any more questions;
that just creates more problems.
It's better to be silent.
I'll stay silent.

I'm a very extroverted student
I usually just want to talk and share–
You're not really welcomed in this class.
Why are you even here?

You know what?
I'm never responding to you;
this will only harm me.
You muted me in your class.
I'm attending
but I'm not really listening.
I did not want to listen to you.

I would literally leave the computer audio
I don't want to hear you anymore.
Your class is not even valuable to me.
I'm in the class but I don't want to hear you.
She definitely could see that
And it was not something I wanted to hide.
I didn't even apologize.

We use italics and formatting in this poem to illustrate different voices, though all of the words were spoken by Qamar during her interview. This structure conveys a back-and-forth dialogue between Qamar (represented by italics and left justified) and her teachers (represented by italics and right justified), with reflection from Qamar in regular font. While this dialogue is a poetic transcription of Qamar's reflections from two different class experiences, the juxtaposition emphasizes Qamar's desire to be a "good student" who was working hard to attend class and engage, but feeling shut out and treated differently than other students. By the end of this poem, we see Qamar stand up to her teacher by choosing to literally silence her teacher by turning off the audio feed from class (which was conveyed back to the teacher) - an act of resistance for which Qamar "didn't even apologize."

Suveer: *The Switch Just Turns On*

Suveer (he/him) is an international student from India who majored in computer science and took Calculus II.

I am always myself.
Most of the time I use Hindi to talk to myself.
I just use American English, sometimes.
Your mind automatically switches to the Hindi mode for conversation.
We sometimes may use Hinglish.

Everything is English;
the resources are all in English.
Nothing is in Hindi.
My mind would be a bit confused initially:
Oh my God
Why are you doing this?
It is so awkward
But then it is our teacher
and you can't really disobey.

I don't talk a lot.
Every student knows English.
The switch just turns on.
But I won't say I was uncomfortable
or I felt really strange.
I just bend myself into the surrounding;
I have to adapt to the thing that is with me.
It happens automatically.

(continued)
It's not like
I'm calling upon
a second personality within me.
It's just habituated,
it's there in me.
I am being myself.
I have that background of watching American films,
superhero films, and all of that.
So I am being myself.

Being bilingual doesn't make things better in the class.
Classroom exams, homework assignments–
no, not particularly there is an advantage of being bilingual.
But when I'm talking to myself,
studying math,
then I use Hindi.
That is one essential part
I won't give that up.

Italics is also used in Suveer's poem to portray the voice in Suveer's mind that felt "confused initially" by the lack of Hindi used in his math classes. While this voice is still part of Suveer and not meant to portray a second person as in Qamar's poem, this poetic structure helps to highlight the "second personality within" Suveer that can "bend' and "adapt" to the English-only mathematics class. He relates his ability to adapt (or one may say assimilate) to the expectations and norms of his math classes to his "background of watching American films", and so he asserts that this adaptation version is still him "being [him]self." The poem ends with Suveer naming "one essential part" of himself that he "won't give up": talking to himself about math in Hindi.

Discussion

Our poetic narratives explored multilingual students of color experiencing silencing in college precalculus and calculus classes and highlight how these students respond to silencing. Several students described explicit acts of verbal silencing, carried out by others' lack of engagement with their questions and contributions. For example, Qamar details being explicitly silenced by her teacher, via muting her on Zoom, when she asks questions in class. She notices that other students frequently ask questions and do not receive the same treatment.

Beyond sharing these students' stories of silencing while using their own words, the poetic narratives reveal three different responses: (1) becoming silent, (2) intentionally using silence as a navigational tool, and (3) using cultural capital, sometimes through silence, to manage and resist marginalizing experiences in the classroom.

We see instances where students become silent in the face of explicit actions by others that are meant to silence their voices, ideas, or identities. Qamar says "It's better to be silent. I'll stay silent." Qamar refers to these as "stay silent" situations. Qamar's poem exemplifies how the silence instructors or peers might see in the classroom is not necessarily indicative of disengagement, but rather a response to marginalization within the classroom.

We also see examples that demonstrate students' agency to intentionally use silence as a navigational tool, allowing them to more comfortably maneuver within the classroom environment. For instance, Suveer describes code switching as a way he silences his home language to navigate the classroom. Suveer recognizes that the math classroom is a place for speaking English, and navigating this space requires turning on his internal English "switch", "bending" himself into the "surroundings". Although he feels like he can "be himself" and does not feel "uncomfortable" speaking English, Suveer shares that he does not "talk a lot" in class. These students' narratives highlight the way that silence can still include speaking, allowing them to still engage in some of the mathematical discussions in class while silencing aspects of their experience - their home language (Fivush, 2010).

In recognizing silence as a form of navigational capital, we are pushing back on the narrative that multilingual students are quiet because they are not interested, not engaged in learning, or not capable of communicating with sufficient fluency in English to maintain mathematical discussions. Instead, we showcase how students' "silent" responses to the silencing around them can be tactical, and that their participation should not be interpreted in a deficit way. We offer their narratives and our reframing as an explicit counter-story to the discourse of multilingual students' silence.

Finally, students' narratives exemplify a third response to silencing: using cultural capital, sometimes through silence, to manage and resist marginalizing experiences in the classroom. Qamar describes agentively using silence as a form of resistance. Recognizing that her teacher's behaviors are harmful to her, Qamar intentionally disconnects her computer audio during her online class and decides: "I'm never responding to you". Moreover, she chooses this form of silent resistance, knowing that it is a visible act that she was not trying to hide from her instructor. This skill of knowing how to resist is an asset that Qamar brings to the classroom and allowed her to exist in a harmful space in ways that mitigated this harm.

Our work shares (counter) narratives in the form of poetic re-presentations of multilingual students of colors' experiences of silence within undergraduate mathematics. These poems allow us, as researchers, to share students' stories in a way that conveys more of their human experience and preserve the emotion of these experiences, and we offer our methodology to support more such work. These poems also teach us, as college instructors, to view students' participation (or lack of participation) in our classes with a different lens, and specifically with an anti-deficit lens. We join other scholars (e.g. Boistrup & Samuelsson, 2018; Hinestroza, 2020) in calling other teachers in to recognize how silence can be used as a tactical resource in students' mathematical sense making, and to encourage instructors to recognize and disrupt the harmful ways other students or they themselves may be silencing their multilingual peers or students.

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Complexity, Signaling, and Goal-Orienting Feedback in Specialized Interactive Digital Tools

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The implications for students and teachers of educational digital tool design have thus far been under-examined. We find it important to problematize and bring attention to relationships between the explicit or hidden choices of digital tool designers and implications for the possible learning taking place with the tool. Exploring one aspect of these relationships, we examined extant digital tools within the undergraduate mathematics curricula. Grounding our analysis in Piaget's genetic epistemology, we took on the perspective of the epistemic user and identified features of extant digital tools that hold implications for the learner-user experience. With this we generate initial categories that contribute to a user-oriented phase of a framework for digital tool design and discuss implications for further research.

Keywords: digital tools, digital task design, undergraduate student learning

Digital tools can be leveraged in a wide variety of contexts ranging from asynchronous learning environments, in-class digitally supported activities used by individuals or groups, discussion-orienting didactic objects (Thompson, 2002), or tools to be used throughout a curriculum. Designers of educational digital tools engage in unique task design, at times facing different constraints than those of traditional in-class or at-home task-design. For instance, digital tools are necessarily contextualized within software constraints, and designers have varying levels of control over user-interface and visual design features that can enhance or constrain the user experience. These decisions, made implicitly or explicitly by the designer, all have the potential to orient the user towards learning goals set by the designer or to potentially obscure the underlying intended mathematical content.

Not yet directly addressed within the literature on digital tools is the relationship between designer choices and the experiences of users engaging educational tools. The broad aim of our research agenda is to highlight this relationship and infer implications for users. Pursuant of this goal, we analyzed a variety of extant digital tools (e.g. "micro-worlds", online interactive textbooks, educational games, computational tools, etc.) to infer features of the tools that might constrain or afford users' experiences for learning. We address the following research question: *What design features of specialized interactive digital tools might enhance or constrain the learning of users engaging with the tools?* We provide initial explanatory and operationalizable constructs in the form of digital tool features that demonstrate the relationship between design choices and the user experience.

Literature and Theory

This report is part of a larger project exploring educational *digital tool* design. By *digital tool*, we mean an instantiation of a digital technology created within a specific software for the purpose of student use in learning (Margolis et al., 2025). We aim to provide a framework for designers of undergraduate mathematical digital tools, enabling them to approach tool design with attention tool features that might constrain or afford users' progress towards established

learning goals. We think of this design process as unfolding in two phases: 1) a macro-orienting phase in which the designer situates possible digital tools within *design space*, and 2) a micro-orienting phase in which the designer makes tool-level design choices, continually de-centering to consider the implications of the choices for users (Margolis et al., 2025).

This conceptualization of the design process draws from Yerushalmy's (2005) notion of a *design space* for interactive digital mathematics textbooks, which she broke down according to "forms of interaction", "interdependency of diagrams and text", and "forms of represented data" (Yerushalmy, 2005). We adopt our own notion of a design space to all digital tools and think of the space as variations in the spectra of *specialized to generalized*, *interactive to inert*, and *dynamic to static* (Margolis et al., 2025). For the purpose of this report, we analyze extant tools from undergraduate mathematics curricula that are specialized (e.g. supporting a specific learning trajectory) and interactive (e.g. users manipulate controls and receive feedback). Our analysis attends to features of tool-level design choices that impact the experience of the epistemic user.

Situating Within the Literature

Thus far, the literature on digital tool design includes well-investigated but siloed regions of design space, such as micro-worlds (Kaput, 1992), didactic objects (Thompson, 2002), videos, or projects like TELMA (Bottino & Kynigos, 2009) or ReMath (Artigue & Mariotti, 2014). When not siloed, much of the literature also takes broad framings of the digital tool use that foreground the interplay between the student, teacher, tool, and situated social or pedagogical environment (Leung & Bolite-Frant, 2015). While these silos are of value, to our knowledge links have not yet been made between design decisions, the user experience, and the learning taking place.

In a recent review of the literature on digital tool design, Leung and Bolite-Frant (2015) described a still extant gap between our knowledge of "...how to use a tool and how the tool usage leads to mathematics concept development." (p. 207). They cautioned that without more careful attention to tool use and students' cognition, "…the use of tools may result in epistemological and didactical obstacles for students, depriving them from reaching the desired or intended mathematical knowledge." (p. 207). They concluded that no theory yet exists that productively encompasses "tool-based task design", which is a gap that we intend to remedy.

Other authors have problematized the interplay between pedagogical intention and design. Bottino & Kynigos (2009) consider the pedagogical intentions of digital tools as inseparable from the tools itself. Our work aligns with this general conception of digital tool design, and we intend to generate features of design informed by foregrounding the user experiences and the possible hindering or supporting of users' achievement of the designers' pedagogical goals.

Theoretical Framing

For this paper, we foreground the experience of the epistemic user of digital tools. The user engages with the digital tool to complete an activity or to learn a mathematical idea within an undergraduate mathematics setting. For space, we will consider only individual student-users, leaving other scenarios such as groups of student-users or teacher-users for future work.

With this in mind, we take a radical constructivist framing (Glasersfeld, 1995) of the user, who learns through interaction with the digital tool. The epistemic user takes actions within the digital tool, receives and interprets feedback from the digital tool, and imputes and develops mathematical meanings from their tool engagement via assimilations to and accommodations of schemes of meaning (Piaget, 1971) for which the inscriptions or visualizations within the digital tool are figurative material for their re-presentations. Within this epistemological framing of

users' experiences with these tools, in analysis we consider the epistemic student's digital and mental actions using the tool and inferences possible from assimilations, abstractions and generalizations (ibid) resulting from their interaction with the digital tool.

Similarly, while our analysis primarily considers the user perspective, we find it also important to note that extant digital tools for mathematical learning were necessarily created in the context of the (however broad or specific) pedagogical goals of the designer. We consider that when designing a digital tool to meet some learning objective, the designer is making explicit or implicit conjectures about how students' interactions are going to support particular ways of reasoning. While we can reasonably establish the pedagogical underpinnings of some digital tools designed with respect to curricula that well-define hypothetical learning trajectories and targeted ways of thinking, such as quantitative and co-variational reasoning (Thompson & Carlson, 2017), we foreground our analysis on the features of the tools made for such curricula.

Methodology

The primary goal of this initial analysis was to examine extant digital tools from the perspective of a user. Because of the magnitude of digital tools publicly available, we sought digital tools that would likely generate theoretical user-oriented insights, rather than attempting to survey or descriptively analyze the breadth of tools available. As such, we drew from our own diverse libraries of tools, tools that we were aware of from active curricula or research projects, and tools submitted via survey. All tools discussed are masked, and in our results we generically describe features of the tools and give general diagrams of the tools' layouts and controls. This masking is to analytically describe and explain features of tools without being read as passing undue judgment on any one digital tool or a collection of digital tools.

Analysis

Our analysis attends centrally to observable features of extant digital tools as they might matter for the actions, feedback, and potential meaning-making of the epistemic student. This involves critically examining tools by taking on the perspective of a user. Our approach is analogous to prior work taking radical constructivist framings of exam items from the perspective of "users" (i.e. the students taking the exams) (Tallman, Reed, Oehrtman, & Carlson, 2021; Reed, Tallman, Oehrtman, & Carlson, 2021). Analyzing exam items for the purpose of identifying features by which one might assess mathematical understanding, Tallman and colleagues considered the mental activity necessary for the epistemic test-taker to successfully answer the exam item. They identified reasonable requirements on mental activity and thusly inferring features of items that reasonably assessed particular mathematical meanings. In doing so, the authors foregrounded: 1) the problem contexts, 2) the relative complexity of the various mental actions and inferences necessary to successfully complete the prompt, 3) the mathematical modeling required to establish quantities relevant to the problem, 4) the necessity of the student to make strategic decisions at the onset of solving the problem, 5) and other necessary mental actions taken by an epistemic user (Tallman et al., 2021; Reed et al., 2021).

We take a similar approach, in that we consider the epistemic user of digital tools, analyze the tools with respect to the actions and inferences from feedback reasonably necessary for use of the digital tool to achieve an objective, or for mathematical meaning making. Such analyses, independent of knowledge of the designer intention, are important, as how users might perceive or engage with the tools might drastically differ from how designers intend tools to be used.

After engaging in individual analyses, the team met to discuss the salient features of tools for supporting or constraining meaning-making. We report on some of the features that emerged

from our analysis and give examples that problematize the need for a greater dialogue around the interplay between design choices, learning goals, and the user experience.

Results

We will elaborate the emergent features of two primary types of tools that fall within the interactive and specialized regions of design space. In this analysis, we highlight the features of *complexity, signaling,* and *goal-orienting feedback.*

Dynamic Interactive and Specialized Tools

We first discuss digital tools of interest that are also dynamic (e.g. "micro-worlds" (Kaput, 1992)). This particular design subspace tends to contain tools that display at least one or more non-symbolic representations of mathematical quantities and give the user some means of dynamically altering the state of the representations as well as any linked textual or symbolic information related to the mathematical representation. We attend to features that help delineate similar tools with respect to the student experience and discuss then ways that these features extend less dynamic, but still interactive and specialized tools.

The first collection of tools come from curricula that intentionally leverage dynamic tools to promote quantitative and covariational reasoning in pre-calculus and calculus. We examined tools promoting quantitative meanings for Riemann Sums and definite integration by appeal to applied contexts. As masked stand-ins for the common design elements of these tools, we will use hypothetical digital tools that we refer to as the *Hydrostatic Force Tools*.

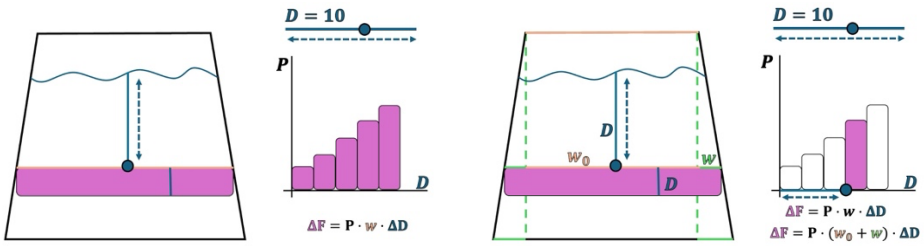


Figure 1. Variations on Hydrostatic Force Tools

These masked tools use hydrostatic force on a dam as the central context for definite integration. Figure 1 depicts variation in the typical layout of such tools and gives representations of the common design choices for control elements, feedback elements, and stylistic elements of the *Hydrostatic Force Tools*. As depicted Figure 1, common to these tools were contextual representations of the quantitative relationships relevant to defining a definite integral (seen on the left), a separate but linked representation (typically graphical) on the right, symbolic or numerical representations of the quantitative relationships at selected states, and one or more control elements (sliders) that the user could manipulate. The masked task for these tools asked students to come up with a Riemann Sum or definite integral that would calculate the hydrostatic force on the depicted dam. Some tools would be implemented as didactic objects, others as formative asynchronous quizzes, and others as supports for in-class group work.

Two important features that emerged from analysis of these tools are *complexity* and *signaling*. *Complexity* relates tool controls to the types and levels of feedback given in response to control change. *Signaling* refers to stylization of the perceptual features of the digital tool with the potential to highlight information or make connections between representations. Many Hydrostatic Force Tools employed color to signal links between multiple in-tool representations

of quantities. In Figures 1a and 1b, each representation for depth (e.g. graphical, contextual, symbolic, and slider) is colored blue. Similarly, force representations are colored pink, and width representations are colored orange (and green). Moreso in Figure 1b, contextual and graphical representations of the local force quantities are signaled when the slider varies. As the local representation of force moves throughout the dam, the rectangle at the associated depth is singularly colored pink. Conceptualized as dynamic supports for students' quantitative and covariational reasoning, signaling was a key design feature of these tools that served the purpose of not only emphasizing the key quantities in the modeling process, but also linking the representations of the quantities within the tool amidst variation of the independent quantity.

Complexity was also a distinguishing feature of these tools. Figure 2a depicts one tool in our data with higher complexity, in a way that potentially hindered present signaling. There were multiple sliders in the tool, each controlling possible variation in the setup of a Riemann Sum such as interval size, sample point location, interval location, and interval number. Many of these control elements influence each other, including the interval number and interval size being interrelated, and the sample location impacting all approximation representations at once. With this higher complexity, particularly the interrelatedness of the control variables, the user must attend to numerous feedback elements at once when altering the value of any number of sliders.

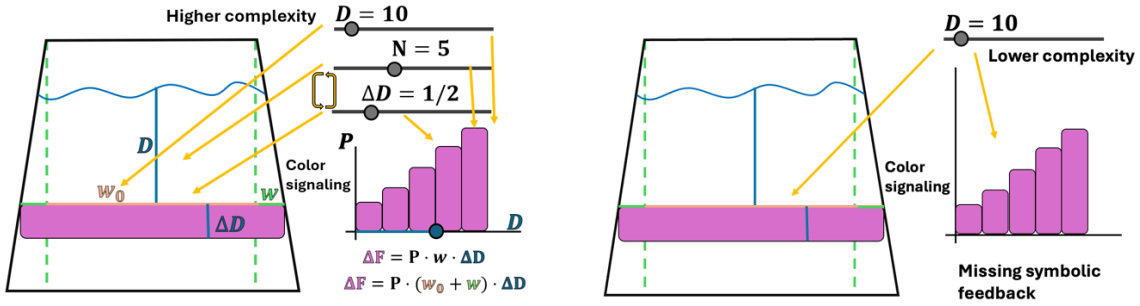


Figure 2. Tools with higher and lower complexity

This feedback variation was across representations and across other control elements. This adds mental actions necessary for a user to infer the important quantitative relationships in the accumulation process, potentially distracting from color-based signaling taking place within the tool. Contrastingly, Figure 1b has lower complexity, hides certain feedback elements, and contains similar intentional signaling. Each slider directly alters only one aspect of each representation, increasing the signaling potential to a user intentionally varying the control elements. Moreover, the symbolic feedback is hidden, reducing the amount of information that the user might observe at any one time, and increasing the weight of the color-based signaling.

River Swimmer Tools

We now broaden the design subspace by including digital tools that share features with Hydrostatic Force Tools but are more static. Such tools are largely text-based and highly scaffolded, centering on singular vector or trigonometric problems. Our masked hypothetical tool is called the *River Swimmer Tool*, involving velocity vectors and correction trajectory angles. The following directions and problem statement are placed at the top of the webpage: ***Directions*** *Use the interactive application below to visualize the problem and guide you through the process of solving. You may choose your own path to the solution.* ***Problem:*** *A swimmer wants to cross a river that is 120 meters wide. The river's current flows at a speed of 1.5 m/s parallel to the riverbanks. The swimmer can swim at a speed of 2 m/s in still water. The swimmer wishes to*

reach the point directly across the river from the starting point. What angle relative to the riverbank should the swimmer aim for in order to reach the point directly across?

This is followed only by a single statement and dropdown menu:

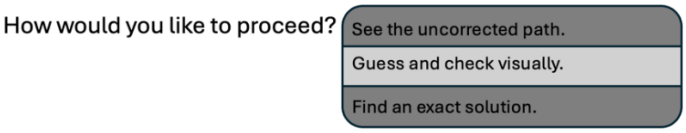


Figure 3. Branch Choices for the River Swimmer Tool

If the first option is selected, the user is presented with a simple dynamic diagram with three points: one for the starting location, another for the intended destination, and a third representing the path of the swimmer. Pressing the "GO" button in the diagram causes the swimmer point to move diagonally away from the starting point, downstream of the intended destination. Choosing the second option provides a similar interactive element.

The third option in the dropdown menu provides a sequence of steps, containing boxes in which users provides numerical data (pertaining to angles, directions, and coordinates) and the boxes give immediate feedback based on the entry. More dropdown menus then provide further branching opportunities. Each option in Figure 3 creates one of three main branches in which the user progressively acts, receives feedback, and is provided more branch options. For example, on one branch the user can choose between solving the problem via a vector table or solving via drawing a triangle, with either choice creating a new sub-branch of activities. Certain paths also contain minimalistic diagrams, such as an unlabeled triangle for the triangle solution path.

This digital tool leverages a branching mechanism, guarded by dropdown menus, as an effective means of reducing complexity. The user not only interacts with one branch at one time, but also only one activity within the branch at any one time. The more static design of the tool provides the user with clear singular feedback from any one action taken within the tool. Even more unique to the design of this tool, the hiding of further steps along any branch contrasts similarly set up tools in which information is not hidden, but the action of scrolling (rather than option selection) is the mechanism for progress through a task.

The River Swimmer Tool made less use of color-based signaling, instead signaling in ways different from the Hydrostatic Force Tools. As much of the tool was text-based, textual stylization such as italicization, bolding, and fully capitalized words were implemented to emphasize tool instructions, to call attention to situational considerations such as units, or to draw attention to optional tool features such as drop-down hint boxes.

One feature of this digital tool not present in the Hydrostatic Force Tools was explicit feedback provided in response to user actions that informed the user whether the input or action advanced them towards (or away from) completion of a goal. For instance, users received "Red X" (incorrect) or "Green Check" (correct) feedback for angle measure, magnitude, and vector coordinate inputs. Further, textual feedback given at the end of certain branches similarly oriented users to other branches that might yield more accurate results. Much feedback of this type had the function of explicitly guiding users to a final answer. Each step, including the optional step of guessing and checking, was accompanied by written and/or symbolic affirmations (e.g. "Yes! We want to use sine for the vertical component.") or formative guidance (e.g. "Consider how a negative angle would affect the path of the swimmer in this problem."). We call this feature of the digital tool *goal-orienting feedback*. Key to this feature is an explicit tool function of providing actionable feedback, indicating that the user is closer to or farther from

achieving a goal. *Goal-orienting feedback* was less present in Hydrostatic Force Tools, though it was identified in some. We interpret that rather than directly in the tool itself, any goal-orienting feedback intended for the Hydrostatic Force Tools would be given by an instructor.

Discussion and Conclusion

In this report we examined extant digital tools, attending to design features that hold implications for the user experience in the context of learning. We now discuss implications of the features for user learning and conclude with future directions for this avenue of research.

Complexity, signaling, and goal-orienting feedback all interact with each other and contribute to the barriers or supports for user learning. Use of a digital tool with high complexity of the kind seen in Figure 2a would require the user to strategically vary individual controls attending to interrelationships between the controls inferred from systematic variation, to then infer the particular quantity represented by individual controls, to then infer the quantitative operations that construct hydrostatic force. This complexity might obscure extant signaling, and in the absence of explicit goal-orienting feedback within the tool users might need to spend a substantial amount of time engaging with the tool itself before being enabled to make the necessary abstractions to the underlying mathematics. This is not to say that increasing the need for user observation and inference is a constraint for every task. In fact, there are plenty of curricular contexts in which a higher need for inference is an intentional design element.

In contrast, River Swimmer Tools were heavily scaffolded, and complexity was lessened by restricting viewable and actionable tool elements, ensuring that users were only tasked with one small component of the larger problem solving at any one time. High volume of goal-orienting feedback densely distributed across the tool provided users with more explicit indications that their actions brought them closer to or farther from the task goals. This requires less inference on part of the student, and more heavily guides users towards goal achievement, such that the task can be entirely achieved within the tool. This lessening of user inference is not suitable for all tasks, however, and in particular removes much of the need for guidance from an instructor.

As such, the three features identified from our results allow for analysis of digital tools of varying types such that one might build initial hypothetical models of interactions taken by the epistemic student, as well as the constraining or affording of learning by use of the tool. Moreover, these features also enable us to distinguish (and perhaps eventually establish) design choices for supporting students in ways unique to the location of tools within design space and unique to the designer's goals. While our features were applied across disparate tool types, we were also able to speak into the tool-specific design elements in relation to the features. In other words, reducing complexity in a static environment might be different from complexity in a dynamic environment, and tools residing in similar regions of design space might have unique design components that contribute to a feature such as complexity.

Such constructs can be incorporated into a process of design-based research on digital tools, wherein theoretical analyses of newly-built tools might provide initial models for the learning taking place by use of the tool. Such models can then be refined through observation of actual user engagement with the tool. Considering categories such as these in the process of tool design can be an important perpetual act of de-centering by designers. Further work from which more similar features might emerge will contribute to an eventual process for iterative design of digital tools that is informed by the thinking of users, thus enhancing their impact. Future work will also include interviews with designers and interview observation of students' use of these tools in clinical learning environments.

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Developing Pre-Service Elementary Teachers' Conceptions of Place Value Through Coursework

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This study investigates how instruction in the base-8 number system in a Calculus Course for pre-service elementary teachers (PSTs), affects PSTs' conceptual understanding of place value. Despite the importance of number sense and numerical operations in the K-12 curriculum, research shows that many PSTs struggle to grasp place value conceptually, often focusing on procedural rather than conceptual knowledge. This study uses a convergent mixed-methodology approach with quantitative pre- and post-tests and qualitative clinical interviews to analyze PSTs' understanding of place value after instruction in base-8. Findings suggest that instruction in base-8 improves the sophistication of PSTs' place value conceptions, though verbalizing that understanding remains a challenge for some. These results highlight the need for a focus on both conceptual understanding and consistent language use in undergraduate teacher preparation programs.

Keywords: pre-service elementary teachers, content knowledge, place value, undergraduate mathematics

The concept of number sense and operations is one that permeates all levels of K-12 mathematics curriculum in the United States (National Governors Association & Council of Chief State School Officers, 2010). As such, it is imperative for elementary school students to develop proficiency with number sense and numerical operations (Kilpatrick et al., 2001). Despite its clear importance, research suggests that undergraduate pre-service elementary teachers (PSTs) struggle to understand numerical operations and place value at a conceptual level (e.g. Southwell & Penglase, 2005; Thanheiser, 2009b).

Often PSTs are proficient in carrying out algorithms for numerical operations, but struggle to conceptually explain the algorithm or why it works (Ball, 1989; Ma, 1999; Thanheiser, 2009b). These struggles are sometimes left unresolved even by the end of PSTs' undergraduate teacher preparation program (Thanheiser, 2009b). Fortunately, some research suggests that studying alternate numeration systems can deepen PSTs' conceptual knowledge of place value (e.g. Yackel et al., 2007; McCain, 2013; Thanheiser et al., 2013a; Thanheiser & Melhuish, 2019). The purpose of this study is to analyze PSTs' conceptual understanding of place value after in-depth instruction on alternative numeration systems and basic operations in the base-8 number system.

Literature Review

PSTs often exhibit a calculational perception of mathematics (Thompson et al., 1994; McClain, 2003) and see the discipline as simply "memorization of basic facts, rules, and procedure[s]" (Simonsen & Teppo, 1999, p. 517). This perception can lead PSTs to engage with arithmetic operations on multi-digit numbers as surface-level manipulations of algorithms (McClain, 2003; Tirosh & Graeber, 1989). While the procedural treatment of standard algorithms may be a natural consequence of years of use, the tendency to focus on procedural aspects of algorithms over conceptual understanding persists when PSTs are introduced to novel alternative algorithms (Harkness & Thomas, 2008; Lo et al., 2008; Nugent, 2007). It is common for PSTs to conflate procedural accuracy and conceptual understanding of the underlying mathematics (Graeber, 1999). As such, while PSTs typically have no problems executing the

standard arithmetic algorithms, they struggle to justify why they work (Ball, 1989; Ma, 1999; Thanheiser, 2009b). Southwell & Penglase (2005) argue that some misconceptions around arithmetic algorithms originate from a lack of conceptual understanding of place value. Upon entering undergraduate teacher preparation programs, Thanheiser (2009a) found PSTs often hold incorrect conceptions of place value, which can persist through their undergraduate education (Thanheiser & Philip, 2012; Thanheiser et al., 2013b; Thanheiser, 2009b).

Research suggests that studying alternative numeration systems can help correct these misconceptions and deepen PSTs' conceptual understanding of place value. Thanheiser & Melhuish (2019) designed an instructional sequence to highlight the three key features of the base-10 number system - grouping, having a base, and place value - by studying other numeration systems without these features (e.g., tally & Egyptian). Their findings suggest that coupling this approach with the study of arithmetic algorithms using the Mayan numeration system (a foreign system possessing the three key features previously listed), left PSTs with a more robust understanding of place value in the base-10 number system (Thanheiser & Melhuish, 2019). Other research supports the idea that the study of algorithms in alternate numeration systems can help PSTs develop their conceptual understanding of arithmetic algorithms (McCain, 2003), place value (Thanheiser, 2015), number relationships, and grouping strategies (Yackel et al., 2007).

Given the promising results that engaging PSTs in instruction of alternative numeration systems has on their conceptions of place value, this study adds to that body of work by operationalizing an established framework for categorizing PSTs' conceptions of place value (Thanheiser, 2009b) to evaluate the efficacy of in-depth instruction on the base-8 number system and use of the standard arithmetic algorithms within that system to increase the sophistication of PSTs' conceptions of place value. In this instructional unit, PSTs learn how to count, represent numbers, add, subtract, and multiply in base-8 to experience and illuminate elementary students' struggles with place value when learning arithmetic algorithms for the first time. The features of the base-8 number system are identical to that of base-10 with the exception of the base, much closer than numeration systems such as the Mayan numeration system, providing familiarity to PSTs while also forcing attention to be on place value as this is the sole featural difference between the two systems. Specifically, this study aims to answer the following research questions:

1. Does studying arithmetic in base-8 increase the sophistication of PSTs' conceptions of place value with statistical significance?
2. To what extent do the qualitative findings converge or diverge with quantitative findings?

Conceptual Framework

For this study, we apply Thanheiser's (2009b) framework for the categorization of place value conceptions. Thanheiser's framework (2009b) was developed from interviews conducted with 15 undergraduate PSTs before their first undergraduate mathematics course. Participants engaged in two clinical interviews over four weeks. During these interviews, they were given problems to solve. This led to Thanheiser (2009b) developing four groups to categorize PSTs' conceptions of place value. The framework was created to "build on aspects of the framework for two-digit numbers (Fuson et al., 1997) and extend them to numbers of three or more digits on the basis of the results of [the] study" (Thanheiser, 2009b, p. 261). The framework is summarized in Table 1. It is important to note that the conceptions are listed in order from least sophisticated to most sophisticated and that the first two are incorrect conceptions, while the

final two are correct conceptions (Thanheiser, 2009b).

Table 1. Explanations of Place Value Conceptions held by PSTs

Conception Held
Concatenated digits only. PSTs holding this conception conceive of all the digits in terms of only ones (e.g., 548 would be 5 ones, 4 ones, and 8 ones).
Concatenated digits plus. PSTs with this conception conceive of at least one of the digits in the number in terms of an incorrect unit type at least some of the time. They therefore struggle when relating the values of the digits in a number to one another. A PST may correctly conceive of groups of 100 ones for a digit in the hundreds place but incorrectly conceive of one for the tens place (e.g., 389 would be seen as 300 ones, 8 ones, and 9 ones).
Groups of ones. PSTs with this conception reliably conceive of all digits in terms of groups of ones; 389 would be 300 ones, 80 ones, and 9 ones. PSTs holding the conception do not conceive of the digits in terms of reference units.
Reference units. PSTs with this conception reliably* conceive of the reference units for each digit in the number and can relate the reference units to one another; in 389, the 3 can be seen as 3 hundreds or 30 tens or 300 ones, and the 8 can be seen as 8 tens or 80 ones.
**Reliably* in these definitions means that after the PSTs were first able to draw on a conception in their explanations in a context, they continued to do so in that context.

Note. Adapted from Thanheiser (2009b, p. 263).

Methods

This study followed a convergent mixed-methodological design (Creswell & Clark, 2017). Quantitative secondary data was collected from students in the form of a base-10 pre- and post-test completed as part of the work required for their undergraduate mathematics course. Additionally, clinical interviews were used for qualitative analysis to further elicit student thinking and validate results (Creswell & Clark, 2017).

Participants

This study was conducted on elementary PSTs (n=37) at a large southeastern university in the spring semester of 2024. Participants were second-semester freshmen enrolled in a calculus course specifically designed for PSTs. The calculus curriculum for this course includes a unit on alternate numeration systems and arithmetic in base-8. These systems included the tally system, the Mayan system, the Egyptian system, and more. Students spent the next two weeks studying arithmetic in base-8. In particular, students relearned how to count, add, subtract, and multiply in base-8. The first author served as the instructor for the mathematics course. Throughout instruction, emphasis was placed on the concept of place value and what each digit represents.

Data Collection

Students were given a paper-pencil pre-test prior to any instruction. Immediately following the instructional sequence described above, students were given the same assessment as a post-

test. The assessment was adapted from the instrument used by Thanheiser (2010) to classify elementary PSTs' conceptions of place value by level of sophistication. As such, its intent was to measure the participants' demonstrated level of sophistication of place value conception. The pre- and post-tests of students who consented to participate (n=43) were used for analysis and any participants missing either a pre- or post-test (n=6) were excluded.

A subset of five students participated in a clinical interview (Hunting, 1997) to explore their conceptions of place value. However, two of the students interviewed were participants missing pre- and post-test data that were excluded, resulting in three interviews for analysis. Interviews were conducted by the second author at the end of the spring semester, approximately three months after instruction. During the interview, participants were asked to verbally explain their thinking as they solved one addition, one subtraction, one multiplication, and one division problem in base-10. Interviews were semi-structured (Maher & Sigley, 2014) and students were asked probing questions throughout the interview to elicit their thinking (Hunting, 1997).

Each interview lasted about 30 minutes and was conducted in person. Interviews were recorded via Zoom and the students' work was completed on an iPad with its screen shared to the Zoom meeting. In this way, both the participants and their work were recorded simultaneously. Transcripts of the interviews were created with screenshots of participant work for data analysis.

Data Analysis

Analysis occurred at the end of the semester after all data collection was complete, in line with a convergent mixed methodology design (Creswell & Clark, 2017). This allowed participants to remain unknown to the instructor until the conclusion of the course. Thus, the quantitative and qualitative data were analyzed concurrently as separate data sets.

Quantitative analysis. A scoring system that built upon the levels of Thanheisers' (2009b) framework (0 - *irrelevant,* 1 - *concatenated digits,* 2 - *concatenated digits plus,* 3 - *groups of ones,* 4 - *attempt at reference units,* 5 - *reference units*) was used to assign a score to each question on each pre- and post-test. Authors scored all pre- and post-tests individually with a high percent agreement (79.72%) and met to resolve disagreements. A final average pre-and post-test score was calculated for each student and used for statistical analysis. SAS Studio was used to run paired *t*-tests to determine if there were statistically significant differences between student pre- and post-test scores.

Qualitative analysis. Each interview (n=3) was transcribed and coded for instances of each type of place value conception. Since the primary goal of the study was to analyze PSTs' conceptions of place value, moments where participants were discussing place value were identified within the transcripts and assigned a conception type using Thanheiser's (2009b) levels of sophistication as an a priori framework (Miles et al., 2018). It is worth noting that Thanheiser's levels are intended to describe how PSTs *reliably* conceive of place value; however, throughout coding, the levels were used to describe *individual* instances of place value conceptions. Authors coded independently and met to establish agreement.

Data integration. After initial analysis of quantitative and qualitative data was complete, qualitative data was transformed to quantitative data (Creswell & Clark, 2017) by counting the frequency of instances in which participants displayed each type of conception. Frequencies were then used to calculate percentages to identify what portion of the time participants used each conception. For the participants with both interview and pre-post data (n=3), the post-test scores were compared with the frequency of each conception they displayed in clinical interviews. This comparison was conducted in line with convergent mixed-methodology design

principles (Creswell & Clark, 2017) to corroborate results and provide more insight into participants' conceptions of place value.

Findings

This section presents the findings from analyzing the pre- and post-tests and the interview transcripts. It is important to note that, while reliably demonstrating the same kind of conception is a cornerstone of Thanheiser's (2009b) conception categorization, in this study, every instance of place value conception used by each participant has been counted and categorized.

Quantitative Findings

To answer research question 1, participant pre-test averages (μ=2.00, σ=0.94) on the base-10 assessment were compared with post-test averages (μ=2.89, σ=0.91) in a paired t-test. Results indicate that the mean level of sophistication displayed for participants on the post-test was statistically significantly higher compared to the pre-test and had a large effect size (t=7.00, p<0.0001, d=1.15). This suggests that PSTs' level of sophistication of place value conceptions improved after instruction on arithmetic in base-8.

Qualitative Findings

The percentages of frequencies with which each interviewed participant demonstrated each conception in their interviews are displayed in Table 2 along with their base-10 post-test average and gain score. Note that the Concatenated Digits Plus code was only applied for instances where participants used both Concatenated Digits and a correct conception (either Groups-of-Ones or Reference Units) within the same sentence. As shown in the table, results of the clinical interview analysis are in line with the results of the post-test for both Student 1 and 2. Student 3's interview, on the other hand, was not necessarily consistent with their post-test score.

Table 2. Integrated Qualitative and Quantitative Results

Percentages of Each Conception, Post-Test Scores, & Gain						
Participant	Concatenated Digits	Concatenated Digits Plus	Groups-of-Ones	Reference Units	Post-Test Scores	Gain Scores
Student 1	16.13	6.45	29.03	48.39	4.6	1.6
Student 2	63.16	0.00	15.79	21.05	2.8	1.2
Student 3	46.43	7.14	28.57	17.86	4	0.6
Total	38.46	5.13	25.64	30.77		

Student 1 scored high on the post-test (μ=4.6 out of 5), demonstrated a correct conception of place value over three-quarters of the time in her clinical interview (77.42% of all instances), and showed the most sophisticated conception (Reference Units) almost half of the time (48.39%). For example, Student 1 used Reference Units language when she tackled the task of subtracting 40 from 10 (see the left-most picture in Figure 1), to describe extracting a group of one hundred from the hundreds place, augmenting the tens place to be 110 minus 40, stating, "So I take one of the 100 groups, making that 500, and then adding a one in front of the tens, since it would be 10 groups of 10." Student 1 is able to refer to numbers by their groups and change that language to

reflect the place value she was working with, a group of 100 becomes 10 groups of 10. After regrouping, when Student 1 performed the subtraction in the same figure, she switched to using a Groups-of-Ones conception. When subtracting 40 from 10, Student 1 explained her reasoning as, "Technically, it's a hundred…and then I'm putting it over here, which would make it…110 minus 40 instead of 20 minus 40." This differs from referring to the numbers as four tens or 11 groups of ten but is still correct.

The results are similarly aligned for Student 2 who scored relatively low on the base-10 post-test (μ=2.8 out of 5) and demonstrated incorrect conceptions of place value over half of the time in her clinical interview (63.16%). For example, when Student 2 subtracted 40 from 10, she used a Concatenated Digits level of conception. Explaining her steps in the middle picture of Figure 1, Student 2 said, "Then we are going to use this 10 and subtract 4, and that would be 6. Then I do 6 minus 2, just like earlier, to get 4," explaining the steps as if she were dealing with the one's place rather than the ten's or hundred's place.

For Student 3, the quantitative and qualitative results were misaligned. The post-test score was high for Student 3 (μ=4 out of 5). However, she demonstrated correct conceptions of place value less than half of the time (46.43%) and her most frequently demonstrated conception was Concatenated Digits (46.43%). To describe multiplying 700 with three in the third picture of Figure 1, Student 3 explained, "So then, when we multiply 7 times 3, you're getting what was essentially 2,100." The beginning of this statement, referring to each number as seven and three shows the use of a Concatenated Digits level of thinking, yet then recognizing that the multiplication actually yields 2,100 employs a Groups-of-Ones level of thinking. This is one example of an instance coded as a Concatenated Digits Plus conception. The contradictory use of language will be examined further in the discussion section.

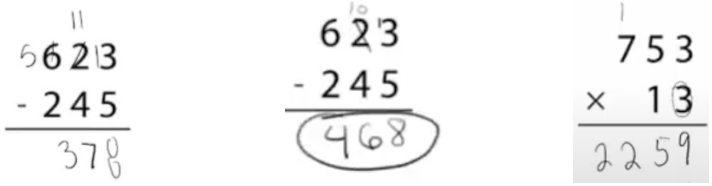


Figure 1. Student Examples of Conceptions: Student 1 on Left Using both Reference Units and Groups-of-Ones, Student 2 in Middle Using Concatenated Digits, Student 3 on Right Using Concatenated Digits Plus

Discussion

The findings suggest that the study of arithmetic algorithms in base-8 has a statistically significant impact on PSTs' level of sophistication of place value conception. However, with an average gain score of 0.89 points, the small amount of gain suggests that engaging in arithmetic in base-8 may not be sufficient to drastically change PSTs' conceptions of place value. While the pre- and post-tests offer insight into the change in the thought process of students before and after engaging in arithmetic algorithms in base-8, they also provide only a snapshot of student thinking. The interviews with a select number of participants gave greater insight into students' understanding of place value as they approached arithmetic operations. The close alignment of qualitative and quantitative findings from Students 1 and 2 provides further support for this improvement.

The misalignment of quantitative and qualitative data from Student 3 highlights an interesting additional consideration. During her interview, Student 3 switched between correct and incorrect language within single statements in a way that could be more confusing than consistently using a single conception - even an incorrect one. For example, when multiplying 700 and 3, she explained "when we multiply 7 times 3, you're getting what was essentially

2,100." For a student who has just learned 7 times 3 is 21, not 2,100, this can be more difficult than either using Groups-of-Ones language - "700 times 3, which is 2,100" - or Concatenated Digits language - "7 times 3, which is 21." It is clear from the post-test scores that Student 3 conceptually understands place value; however, she demonstrated difficulty with the consistency of language use while explaining problems in real time. This highlights the importance of having PSTs practice verbally explaining arithmetic operations using correct language. Though studying arithmetic in an alternative numeration system may deepen PSTs' understanding of place value and arithmetic algorithms, that does not guarantee they are able to verbalize that understanding to their future students.

An interesting finding that emerged from the interview data was students' response to re-learning arithmetic and place value in base-8. For example, Student 2 referenced that the instructional unit had changed her understanding of place value and that previously, for "all problems in base-10, [she'd] learned to just accept it as it is" and that carrying out algorithms in base-8 had "tricked her brain" in a way. Similarly, Student 3 reflected that "I did not have that basic understanding of place value going into this class… you know I would have never told you that when you multiply this 3 and this 1 [referring to a double-digit multiplication problem], you're multiplying 3 times 10". Additionally, Student 1 indicated that after "relearning addition, subtraction, [and] multiplication… it made me kind of understand why that stuff works more so". Student 1 also explained how she was able to easily transfer her learning in base-8 to base-10: "When I was having trouble, thinking… as soon as you hit eight, that's actually one-zero because it's one group of 8. I was like, okay, well, in tens when you hit 10, that's one group of [the base]". It is possible that the nature of base-8 and its similarity to the base-10 system made it easier for the PSTs to develop a conceptual understanding of place value and to disrupt the automatic procedural nature of their thoughts surrounding operation algorithms within base-10.

Limitations

This study was limited by a sample size that, while large enough to produce statistically meaningful quantitative results, was still small. Additionally, the involvement of the first author instructing the course naturally introduces the chance for bias in the results. The post-test was given immediately after the participants had engaged in the base-8 unit and interviews were conducted within the same semester. While the base-8 unit clearly changed the way participants described arithmetic operations, whether these changes are sustained remains to be seen. Finally, whether base-8 instruction is more beneficial than less familiar systems, such as the Mayan or Egyptian system, would require further study.

Conclusion

This study highlights the positive impact of studying arithmetic in the base-8 system on PSTs' conceptual understanding of place value. While the instructional unit successfully improved participants' level of sophistication of place value conception, the variation in individual results underscores the complexity of this learning process. Some participants exhibited strong conceptual understanding, yet struggled to consistently verbalize their thinking, emphasizing the need for more focus on clear communication of mathematical concepts. Future research should further investigate the long-term effects of learning alternative numeration systems on both conceptual comprehension and the ability to articulate mathematical reasoning.

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Students' Actions Related to Complex Exponents

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Interest has been growing in how students learn concepts in complex analysis (Troup, 2015; Hancock, 2019). A concept generalized in complex analysis is exponents and logarithms, yet little is written on how students leverage their existing knowledge to the complex case. In this paper, I present data from task-based interviews via the lens of APOS theory to present two students who leveraged their knowledge of real-valued exponential and logarithmic operations to varying degrees. I present data from a single question asking them to compare the magnitude of complex numbers and argue that their schemas for the two subjects were separate.

Keywords: Complex Analysis, Exponents, APOS Theory

Many studies concerning how students understand exponential or logarithmic operations and functions have looked at how students generalize the domains of exponents from natural numbers to rational numbers (Elstak, 2007; Williams, 2011). In the context of complex analysis, there is a notion of raising a complex number to a complex power, defined as $z^w = e^{(w \cdot Log(z))}$, where $e^{a+bi} = e^a(cos\ (b) + i \cdot sin\ (b))$, and Log(z) denotes an inverse of e^z. At the time of writing, there is yet a study on how students understand such expressions, either from the literature on learning of exponents & logarithms or learning of complex analysis.

In this study, I extend studies on exponents and complex analysis that used APOS theory (Pitta-Pantazi et al. 2007, Tabaghi, 2007). Foundational among these studies is that from Danenhower (2000) which touches on the theme of Thinking Real, Doing Complex (TℝDℂ). Extending his work, along with that from Troup (2015, 2019), Hancock (2018, 2019), and Soto (2014, 2016), I investigate the question "How do students leverage their knowledge of real-valued exponential operations to reason about the complex analog?"

Theoretical Framework and Literature Review

Next, I briefly review APOS Theory and the studies related to the learning of complex analysis, exponents, and logarithms situated within it.

APOS Theory

The four constructs of actions, processes, objects, and schemas constitute the basis of APOS theory. APOS theory, as described by Asiala et al. (1997) is concerned with the construction of schemas and mental objects, and the theory posits that objects are the result of sufficient reflection of processes or the thematization of schemas. According to Asiala et al. (1997), an action "is a transformation of objects which is perceived by the individual as being at least someone external," a process is the result of reflective abstraction on an action or the reversal of an existing process, and an object is a result of either "when an individual reflects on operations applied to a particular process, becomes aware of the process as a totality, realizes that actions or processes can act on it, and can construct such transformations" or schemas via "thematization," which is defined as a schema's inclusion in other schemas.

Dubinsky (1991) and Asiala et al. (1997) define a schema: as a "coherent collection of objects and processes," (p. 166) and argue that the main concern of math education researchers should be the construction of schemas. Although a schema is internal to the individual, Dubinsky (1991) highlights the idea of genetic decomposition as "detailed descriptions of schemas in terms

of these mental constructions" as "a way of organizing hypotheses about how learning mathematical concepts can take place" (p. 182). Although schemas, objects, and processes aren't possible to directly observe by outsiders, Dubinsky (1991) describes them as the entry point for reflective abstraction.

Exponents

Some of the earliest studies around exponents and logarithms were from Confrey (1994), Confrey & Smith (1995), and Weber (2002). The primary contribution of Confrey (1994) and Confrey & Smith (1995) is the theoretical notion of the splitting world and covariation as a means to productively understand the exponential function. Weber (2002) reports on the result of a teaching experiment primarily through the lens of APOS theory. He also uses Sfard's notion of reification and Sfard's postulation that structural understanding is "necessary to reason about mathematical concepts" (p. 1020) His test group was found to be able to "use their conceptual knowledge to reconstruct" exponent and logarithm rules. (p. 1025) Like Confrey (1994) and Confrey & Smith (1995), Weber finds that the idea of exponents as iterated multiplication is an insufficient model for generalizing to non-natural numbers.

Pitta-Pantazi et al. (2007) identify three levels of understanding exponents within APOS theory via written questionnaires and semi-structured interviews. The interviews consisted of asking students to compare the magnitude of exponential expressions. The first level they identify corresponds to "the pre-conceptual stage at which students are getting used to certain computational operations on the exponents with natural numbers," or an action understanding of exponents. The second level "corresponds to the operational stage at which students use the familiar processes on exponents with new kinds of numbers such as rational numbers," or is characterized by generalizing valid bases. In the case of rational numbers, it can also be seen as a coordination of applying exponents on the numerator and denominator. The third and final one "corresponds to the restructured stage at which students recognize exponents as a mathematical object" (p. 309). Here, "object" is used in the sense of APOS theory.

Complex Analysis

Danenhower's (2000, 2006) work is foundational to contemporary investigations into students' learning of complex analysis and is situated within a framework of radical constructivism and APOS theory. His primary hypothesis distinguishes the vector, cartesian, and polar forms of complex numbers and conjectures that students will have difficulty moving between these forms.

One part of his dissertation focused on the "multirepresentations of complex numbers" (p. 109), which is also how he thematizes his findings (p. 110). One of these themes was "Shifting Representations," which refers to how students change the representations of complex numbers to expedite (iterated) multiplication, division, or addition/subtraction. In such context, Danenhower describes four stages of understanding complex fractions in terms of APOS theory, the last one being "the ability to indicate when to shift representations" from "our conception of complex fractions" (p. 124). He also presents the following criteria for each stage of the theory.

Table 1. Danenhower's (2000, p. 125) findings on simplifying positive integer powers of complex fractions in APOS theory

Action	"Concerned with the details of the realizing algorithm for each problem. Applies algorithms routinely and apparently without thought of other

	possibilities (absence of evidence of a thought process leading to a decision to perform the calculations that were undertaken by the student)."
Process	"Shifts to the polar representation at some point, but is still primarily concerned with the details of the simplification. Possibly uses some "short cuts", e.g., uses (a + ib)(a - ib) = $a^2 + b^2$ without calculation. Generally fluent technique, i.e., is able to do the simplification correctly with few obstacles. Shows that they are doing the calculations by decision, rather than because 'it is what you do in this kind of problem'".
Object	"Uses prior knowledge freely. Indicates consideration of more than one method of solution. Uses geometry to aid calculations. Freely modifies realizing algorithm or polar method to save work."

Similar findings regarding Shifting Representations would be replicated by Panaoura et al. (2006), Karakok et al. (2014), and Soto-Johnson & Troup (2014). Panaoura et al. (2006) found that "the identification of two distinct clusters of pupils revealed that there are two fundamentally different perspectives from which complex numbers are viewed and approached: the algebraic and the geometric approach" (p. 700). Soto-Johnson & Troup (2014) found that algebraic and geometric reasoning decreased as time went on with a bias toward algebra.

Data and Methods

This paper draws on a wider data set concerning how students understand the complex exponent and logarithm for a pilot study. In this paper, I mainly look at responses to questions concerning exponents based on items from Pitta-Pantazi et al. (2007) which looked at comparing the magnitude of exponential expressions, as well as responses to earlier questions based on those from Danenhower (2000) which give further context to their responses.

Research Design

Participants and Setting: Two participants, Monk and Thomas, were recruited from an American R1 university. These students were currently enrolled in an undergraduate complex analysis course and recruited with the aid of their instructor. The study was structured to have each participant participate in two individual interviews and two paired interviews, totaling six interviews, but this paper reports only data from the individual interviews.

Instrument: Complex exponents can be understood to require a coordination of three actions to achieve an object-level understanding: complex multiplication, the complex exponential function, and the complex logarithm, and so the item I present requires coordination of these actions. This paper reports on the result of a specific question based on Pitta-Pantazi (2007): Given expressions a, b, c, compare them in terms of "<," ">" or "=." The expressions were as follows:

1. $|(2e^{i\pi/4})^3|, |(2e^{i\pi/4})^4|, |(2e^{i\pi/4})^5|$
2. $|(-1+i)^{3+i}|, |(-1+i)^{4+i}|, |(-1+i)^{5+i}|$
3. $|(-1+i)^{3+i}|, |(-1+i)^{3+2i}|, |(-1+i)^{3+3i}|$

These items came after the participants calculated the following: $\frac{-2+2i}{(1+i)^3}$, $-\frac{8}{(\sqrt{3}+i)^6}$, $(3\sqrt{2}+3\sqrt{2}i)^{\frac{1}{2}}$, $i^i, i^{2i}, 3i^{1+i}, (-(1+i)^{-5-3i})$, as it contextualizes their behavior for the three items and

gave them opportunities to interiorize their relevant actions into a process for taking complex powers. Some of these items were chosen due to their use by Danenhower (2000).

Analysis

Although there were similarities between the two participants, there were also differences in how they reasoned about these three problems to such an extent that they reached different conclusions for question 3. Thomas deduced the sequence was decreasing while Monk deduced it was increasing. To understand how Monk and Thomas compared the magnitudes of exponential expressions, it is necessary to understand how they computed exponentials.

Calculating Real Exponents

Common between both participants was their strategy in computing the factors with purely real exponents. When faced with an exponential term in the denominator, they immediately multiplied by some power of the conjugate. To simplify $\frac{-2+2i}{(1+i)^3}$, they multiplied by $\frac{(1-i)^3}{(1-i)^3}$, and similarly, for the second they multiplied by $\frac{(\sqrt{3}-i)^6}{(\sqrt{3}-i)^6}$. And then when calculating the square root, $(3\sqrt{2}+3\sqrt{2}i)^{\frac{1}{2}}$, Monk deferred to the definition of $z^w = e^{w\cdot log(z)}$, writing $e^{\frac{1}{2}log(3\sqrt{2}+3\sqrt{2}i)}$, and then eventually simplified to $e^{\frac{1}{2}ln(6)}(cos\,(\frac{\pi}{4}) + i\,sin\,(\frac{\pi}{4}))$. In contrast, Thomas drew a picture of the vector $(3\sqrt{2}+3\sqrt{2}i)$ in the complex plane and reasoned that "the square root should stay on the same line, because the angle is retained... and the modulus is greater than 1 so the length should be shortened." Then he started to reason with the Pythagorean theorem, concluding "I wouldn't know how to expand it otherwise."

According to Danenhower (2000), they operated at an action-level understanding of complex numbers, characterized by their reliance on algebraic definitions and manipulation. Their responses to the square roots are stronger evidence of this claim, where according to Pitta-Pantazi (2007), they were operating on a Level 1 understanding of fractional exponents with a complex base. Monk relied solely on an algebraic calculation, and although Thomas attempted to use geometric reasoning to calculate the square root, he was ultimately unsuccessful and incorrectly stated that taking square roots leaves angles unchanged. So there was also a lack of evidence that complex multiplication was reversed for either participant. Thomas' assumption that square roots stay on the same line is possibly an instance of the TRDC phenomenon. Thus, I conclude they have not achieved a robust process understanding of complex multiplication in general but were working with a process understanding of algebraic complex multiplication.

Calculating Complex Exponents

Monk: When tasked with calculating $3i^{1+i}$, Monk immediately deferred to the definition by writing $e((1+i)(Log(3i)))=e(Log(3i))e(iLog(3i))$.[1] When asked how he got the second term, Monk described a mental calculation of distributing the Log(3i) term and using the established property of $e^{z+w}=e^z e^w$. He continued to simplify by writing 3ie(iLog(3i)), using the inverse relationship between logarithm and exponent, and tried to "do [Log(3i)] in my head," subsequently simplifying to 3ie(i(ln(3)+iπ), and finally to 3e(-π)i(cos(ln(3))+i·s(ln(3))). In this inscription, "s(ln(3))" stands for "sin(ln(3))."

When asked for another means of calculation, he checked if the value he calculated prior would match with the identity $3i\cdot(3i)^i$ as an instance of the conjectured rule $a^{z+w}=a^z a^w$. He

[1] At this point, Monk stopped writing exponential expressions with the power above the base, hence the change in style.

simplified $3i\cdot(3i)^i$ into $3i\cdot3^i i^i$ using another conjectured rule that $(ab)^z=a^zb^z$, and used his past work to replace i^i with e(-π/2). Then, he replaced 3^i with e(iLog(3)) and had a final answer of 3e(-π/2)i(c(ln(3))+is(ln(3))). He recognized that these two expressions were not equal, due to a different factor of e(-π) and e(-π/2), but corrected it after the researcher pointed out that the argument of 3i is not π, but π/2. After this, he concluded that the rule $a^{z+w}=a^za^w$ was valid.

He immediately used this rule in calculating $(-(1+i)^{-5-3i})$, after first distributing the negative sign. By reducing the calculation to $(-1-i)^{-5}(-1-i)^{-3i}$, he looked to simplify the term $(-1-i)^{-3i}$, again by using the definition of complex exponents. He rewrote it as e(-3iLog(-1-i)) and then as $e(-3i(\ln\sqrt{2}-\frac{3\pi}{4}i))$, and then by distribution of the -3i term, arrived at a final answer of $-(1+i)^{-5}e(-\frac{9\pi}{4})e(-3i(\ln\sqrt{2}))$.

His performance demonstrated a strong action understanding of complex multiplication, but as evidenced by his calculation of the square root, he has not reversed iterated multiplication. His initial claim that the argument of $3i$ as π reveals his attempt to conceive of the logarithm as a process, and his adoption of the rule $a^{z+w}=a^za^w$ indicated that he was attempting to coordinate the processes of exponentiation with real power and with complex powers into a single process.

Thomas: When tasked with calculating $3i^{i+1}$, Thomas first wrote the definition $e^{(i+1)Log(3i)}$ and replaced Log(3i) with the definition, giving $e^{(i+1)(ln|3i|+i\cdot Arg(3i))}$, then replaced the magnitude and argument functions with their outputs, giving a final expression of $e^{(i+1)(ln(3)+i\frac{\pi}{2})}$, saying "honestly, I would leave it at just that." I asked if there was an alternate way to write it. Thomas proceeded by using the conjectured rule that $(ab)^z=a^zb^z$, transforming the calculation to $3^{i+1}\cdot i^{i+1}$ He wrote both factors in terms of the definition, similar to how he did with his first calculation. Because Arg(3)=ln(|i|)=0, he got the expression $e^{(i+1)ln(3)}e^{(i+1)(i\frac{\pi}{2})}$. Next, he distributed and used an exponent rule simultaneously to get the expression, $e^{iln(3)}e^{ln(3)}e^{-\frac{\pi}{2}}e^{i\frac{\pi}{2}}$ identifying the outermost terms as rotations and the inner terms as scalars. He then identified that $e^{i\frac{\pi}{2}}$ is a 90-degree rotation, and is therefore equal to i, and substituted, giving a final answer of $ie^{iln(3)+ln(3)-\frac{\pi}{2}}$. The final question I asked about this calculation was if it said anything about the final angle and magnitude. He "could simplify a bit more" using the inverse relationship between ln(x) and the exponential function, resulting in the expression "$3ie-\frac{\pi}{2}3^i$ but "in terms of angles I can't say much," and indicated that the magnitude could be read as a product of "all the scalars here," referring to 3i, $e-\frac{\pi}{2}$, and 3^i. There was no clear evidence if he thought of 3^i as a scalar, or just the 3.

When calculating $(-(1+i)^{-5-3i})$, Thomas again used the definition of z^w, writing $e^{(-5-3i)Log(-1-i),}$ and then the definition of the logarithm, writing $e^{(-5-3i)(ln|-1-i|+i\cdot Arg(-1-i))}$. Although he instantly calculated |-1-i| as $\sqrt{2}$, he drew out the argument to get $\frac{3\pi}{4}$, "because φ is between -π and π." He again did a multiplication and exponent rule simultaneously to get $e^{-5ln\sqrt{2}}e^{-i\frac{15\pi}{4}}e^{-3iln\sqrt{2}}e^{\frac{9\pi}{4}}$. Finally, using the inverse relationship between the exponential and logarithm, he got a final answer of $(\sqrt{2}\ \ )^5(\sqrt{2}+\sqrt{2}i)(\sqrt{2})^{3i}(\sqrt{2}+\sqrt{2}i)$, and commented that he would not know how to simplify $(\sqrt{2})^{3i}$, and that the furthest he could go in simplifying would be to identify the real and complex parts.

The simultaneity of operations indicates a process understanding of finding the magnitude and angle of a complex number, although these two processes did not appear to be coordinated

into a process for calculating logarithms. Thomas also demonstrated a process understanding of complex multiplication but continued to rely on an action understanding of taking complex powers via the constant execution of a standard algorithm.

Comparing Magnitudes of Exponential Expressions

Question 1: per the framework described by Danenhower (2000). By this, I mean that both Thomas and Monk identified that the polar form of a complex number has an independent magnitude and angle component and that for the question, they only needed to isolate the magnitude. This is the "free modification" that is characteristic of an object-level understanding. Consistent with prior data, they both treated integer exponents as iterated multiplication. I infer this because they both alluded to notions of stretching and rotating and did not use the general definition of complex exponentials in their calculations. According to Thomas "the $e^{i\theta}$, that's always just going to be 1," and according to Monk, "This (referring to terms of the form $e^{i\theta}$) is just going to be our spinny component, so it'll just go away." They reasoned that because the base had a magnitude of 2, taking higher integer powers caused the sequence to increase.

Question 2: The second question caused the participants to think aloud the most out of the three as they used the techniques they developed for Question 2 for Question 3. Thomas used his prior observed method for computing $(-1+i)^{3+i}$, concluding that "the things that affect our length" were the terms $e^{3\ln(\sqrt{2})}e^{\frac{-3\pi}{4}}$, upon which he concluded that the sequence was increasing. Upon further questioning, he went back and forth between each factor determining the magnitude or only one. Initially, he thought that the increase of the real component of the exponent affected only the first term, but likely due to both factors having a 3, he changed his process to have each 3 increment by 1, resulting in this chain of expressions: $e^{3\ln(\sqrt{2})}e^{\frac{-3\pi}{4}}$,$e^{4\ln(\sqrt{2})}e^{-\pi}$,$e^{5\ln(\sqrt{2})}e^{\frac{-5\pi}{4}}$. However, he eventually realized that the second term of each expression corresponded to the angle of the base, which stayed constant, and was able to settle on a conclusion that the sequence was increasing. Notably, despite the length of the interview and his continued practice in calculating logarithms, he was still careful to write out the definition of logarithms to calculate his final answer. From this, it is evident that his thinking at that point relied on a mix of actions and processes, as he consistently deferred to the definition of logarithms, and kept track of each quantity so he would not have to redo the other calculations.

Monk operated with a different understanding of Question 2. He explicitly used the assumption that $z^{a+b}=z^a z^b$, and likely implicitly used the assumption that $|zw|=|z||w|$. Hence his sole focus on the term $(-1+i)^3$, ignoring $(-1+i)^i$, possibly because it was a common factor in all three terms. After calculating $(-1+i)^3$ as $2(i+1)$, he reasoned that higher powers would increase the magnitude, verbalizing the magnitude of each, concluding that the sequence is increasing. Essentially, he identified the consequential calculations to attend to and reduced it to a question about iterated multiplication of real numbers.

Question 3: Thomas approached Question 3 with his usual approach of decomposing each operation into its definition. However, once he arrived at the term $e^{(3+i)(ln|-1+i|+i\frac{3\pi}{4})}$, he replaced the (3+i) term with a variable, (3+ai), "since we're changing the coefficient in front of the i." He proceeded to distribute and use a power law to get $e^{3ln\sqrt{2}} \cdot e^{i\frac{9\pi}{4}} \cdot e^{ai \cdot ln\sqrt{2}} \cdot e^{\frac{-3 \cdot a \cdot \pi}{4}}$. After simplifying each term, he identified that "the main thing here that's changing is this $\frac{3 \cdot a \cdot \pi}{4}$." Although the third term also has an a inside of it, he reasoned that "this one's also going to be an angle, and we're just dealing with the modulus here… we know that we can split the modulus up

across products, so $[e^{a \cdot i \cdot ln\sqrt{2}}]$ is equivalent to 1." He also noted that $e^{i\frac{9\pi}{4}} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$ "because of periodicity, we can subtract 2π from it." He concluded that because "our thing that's changing is $e^{\frac{-3 \cdot a \cdot \pi}{4}}$, so a larger a means our overall quantity is decreasing" and concluded that the sequence is decreasing. There is evidence here that Thomas is operating with an object-level understanding of complex exponentiation, given that he established a useful form of the numbers he was working with and used a variable to reason about the problem. He developed an algorithm for this question and realized exponentiation as a function, corresponding to object-level understanding in Danenhower's (2000) and Pitta-Pantazi et al.'s (2007) frameworks. The question possibly made him generalize his schema of functions to accommodate complex exponents, as he first led me to believe that he would calculate each term individually.

Monk's initial reaction was that "$[(-1+i)^3]$ will give us $2\sqrt{2}$ and $[i]$ is going to spin it each time." From this, I infer that he knew immediately that $|-1+i| = \sqrt{2}$, and that cubing it would give it a magnitude of $2\sqrt{2}$. I also infer that he was breaking up the calculation into $|(-1+i)^3| \cdot |(-1+i)^i|$ This is further evidenced by his next move where he looked at the second term, and wrote $2\sqrt{2}|(-1+i)^{i+i}|$. He then broke $(-1+i)^{i+i}$ into $(-1+i)^i \cdot (-1+i)^{i+i}$, after which he said "and that's just taking your vector and spinning it so the magnitude isn't going to change, so it's just going to square itself." In this utterance, he partly associates purely imaginary powers with rotations. Next, he said, "Well, this is going to be a squared magnitude," referring to his inscription of $(-1+i)^{i+i}(-1+i)^{i+i}$ "And because the magnitude of that is $-1+i$ is $\sqrt{2}$ because that's not less than 1" he concluded that the sequence was increasing. However, the magnitude of $(-1+i)^i$ is approximately 0.09, which would instead lead him to conclude the sequence was decreasing.

I conclude that Monk demonstrated a process understanding of complex multiplication and complex exponents with integer coefficients, given his instant claim that the magnitude of $(-1+i)^i$ was greater than 1. This is possibly an instance of the TℝDℂ phenomenon as described by Hancock (2018) where he described participants "extending real intuition to the complex setting erroneously," (p. 392) the intuition being that a complex number with magnitude greater than 1 raised to a power with magnitude greater than or equal to 1 will have magnitude greater than 1.

Discussion and Conclusion

In extending research on student understanding of complex analysis, I replicated the finding of Shifting Representations as found by Danenhower (2000), Panaoura et al. (2006), Karakok et al. (2014), and Soto-Johnson & Troup (2014) in how the participants tackled their calculations of complex exponentials. My participants chose the representations to use for a given question based on either its initial representation, or its representation in the definition of the complex exponential. They rarely experimented to see if alternate representations would be useful.

Because they only used familiar rules after they established their plausibility, participants seemed hesitant to leverage their knowledge of real-valued exponents and logarithms, instead reasoning from the definition. Only toward the end were instances of TℝDℂ seen explicitly. It is possible that during the experiment, the schemas related to complex and real exponents were separate, and the participants were actively generalizing both into a more general schema. Additional analysis of their performance on other items will need to be done to make definite conclusions, but this works toward a more complete understanding of the TℝDℂ phenomenon.

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Implications From Interviews of and About International Students and Researchers

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Following the work by Shultz et al. (2023), I present data related to surveys and interviews of math education researchers from underrepresented cultures in the wider literature. I present findings related to constructs they deem important in math education and have not seen widely represented in current research.

Keywords: International Students, Equity

Long ago in ancient India, there was a group of blind monks. One day, they learned that something had come to their village. Only able to touch the object, one thought it to be a thick snake, another a large fan. When the third touched it, he determined it to be a leathery barricade, and the fourth concluded it was a tree trunk. The last concluded it was a curved spear. This is The Parable of the Blind Monks and the Elephant, which illustrates the variety of conclusions that people can arrive at despite studying the same phenomenon. Even if math education research might be a well-defined area of study, different researchers may view the nature of math education, teaching, and learning quite differently. So wider adoption of differing, even contradictory viewpoints, can help paint a more holistic picture of any topic of interest, including math education.

In this paper, I present data from a preliminary investigation into the cultural differences that math education researchers from non-American backgrounds notice in math education spaces. When I use the term "math education spaces," I refer to math classrooms and math education research. This paper reports on the specific research question "What attitudes toward mathematical learning do scholars identify as important and underrepresented?" To work toward an answer to this question, I started by looking at the experience of researchers who came to the US as international students. Due to survey responses, I expanded my view to include anyone who might fall outside the dominant, White culture of the United States. The scope of the participants' experiences ranges from learning or teaching in an American math classroom to conducting research in a math education setting. In this paper, I present a wide variety of sentiments, opinions, and judgments of the current state of mathematics education research based on the lived experiences of a similarly wide variety of researchers and graduate students.

Theoretical Framework

Just as the theoretical frameworks we use are inherently influenced by their authors, sometimes quite explicitly in the case of individuals like Vygotsky and the cultural influence of Soviet Marxism (see Fu (1997) for his commitment to furthering the Marxist project) researchers also bring their own biases and influences to their work. It is not only the implicit biases and beliefs of American and European researchers that are worth examining but also of outside the Anglosphere, especially as non-Western education and math researchers and instructors continue to work in the West and international students continue to be a significant (and changing) demographic of college math classrooms. We must also ask if we allow international students to bring their frameworks to their professional work and if we allow their influence to perpetuate throughout the field.

For this paper, I draw upon Sfard & Prusak's notion of identity and draw a parallel to their belief that "the notion of identity is a perfect candidate for the role of 'the missing link' in the [researchers'] story of the complex dialectic between learning [about mathematics education] and its [American] context." They define identity as "collections of stories about persons or, more specifically, as those narratives about individuals that are reifying, endorsable, and significant." Key to this idea is the distinction between Actual and Designated Identity, where "actual" identity is characterized by how the individual understands themselves and "designated" identity is how they imagine themselves in the future. Sfard and Prusak (2005) argue that one's education is how one realizes one's Designated Identity starting from one's Actual Identity. In the context of this study, this means that one's understanding of American classrooms starting from outside the United States shapes one's designated identity as a math education researcher.

Literature Review

In their paper on the role of mathematical identity in math education research, Sfard and Prusak (2005) report on a study of Prusak (2003) comparing the learning outcomes of national Israeli students and Soviet-Israeli immigrants, both of Jewish religious backgrounds. While the outcomes of each group were comparable in terms of overall success, the methods each group of students employed were not. Sfard and Prusak argue that the identity of the students is powerful enough to explain the different study methods. The Israeli students saw mathematics as a way to further their career prospects, and the Soviet immigrant students instead saw mastery of mathematics as important to becoming "full-fledged as a human being" (p. 20). That is, they saw mathematical mastery as having inherent value.

Shultz et al. (2023) investigated how Hispanic serving institutions enact culturally relevant pedagogy (CRP) in the sense of Ladson-Billings (1995). They found two barriers to enacting CRP: a reluctance to discuss various aspects of identity (racial, gender, ethnic) and a belief that "mathematics is culture-free" (p. 503). They linked the various attitudes and instructors' reluctance toward implementing CRP to their identity. For example, one Korean instructor guessed that shallow cultural references in word problems would be more insulting than inviting based on how he imagined references to various aspects of Korean popular culture. They found that despite both barriers, instructors generally enacted aspects of CRP in their classes. To address these barriers, they call on the field to "take more risks in conversation and invite a more diverse range of people to the undergraduate mathematics community" (p. 486). As RUME researchers are part of the undergraduate mathematics community, this call applies here as well.

In summary, Sfard and Prusak (2005) linked identity to how math students understand the experience of mathematical instruction, and Shultz et al. (2023) did a similar investigation, except with instructors instead of students. In this work, we link identity to how education researchers understand the experience of mathematical instruction.

Methods

To begin to understand the cultural influences that international students of math and math education bring to their work, I sent out a survey primarily aimed at students who started their math education in the United States as immigrants or international students and recruited conference attendees as participants. Recruitment took place in person and online.

Participants filled out an initial survey consisting of demographic questions and two main questions: 1. What are some things in math education that you feel are important but are underrepresented in Western math education research? 2. Briefly describe American cultural practices or attitudes regarding teaching and learning that initially surprised you. Participants

were also asked if they would participate in a follow-up interview, of which 8 agreed. Interviews were semi-structured and lasted for 45 to 60 minutes. Interview participants were all relatively new to math education research, either as graduate students, postdocs, or established scholars who have recently started math education research. Transcripts were edited for clarity.

After cleaning up the transcripts, I engaged in a thematic analysis (Braun and Clarke, 2006) to identify common themes between the survey and interviews. The thematic analysis consisted of two rounds of coding. The first identified answers that expanded upon what was asked in the initial survey. The second round identified common themes between the questions. Although a common finding that I will discuss later is the importance of ethnic identity, I will discuss them in general terms to protect anonymity, thus cultural origin will be discussed only when necessary. It would be overly ambitious to sketch out an entire theory of attitudes toward math education or to think that a handful of interviews can provide enough material to even do so. As such, the results are meant to illustrate the diversity of opinions even within a small dataset.

The interview participants consisted of 2 Central Americans, 1 South Asian, 1 West Asian, 1 North Asian, 2 Southeast Asians, and 1 East Asian. This demographic information is based on self-reported data but generalized to grant wider anonymity. A limitation was my inability to collect interview data from large groups such as Africans, Hindus, or Chinese. Furthermore, because there was only a single researcher involved in coding, there is no measure of inter-reliability.

Results

After organizing my findings, multiple themes appeared: the cognitive aspect of mathematical learning, the social aspect of mathematical learning, and the American mathematical practices in a global context. All these reflect some facets of math education that the participants do not see reflected in their work, either as researchers, instructors, or students.

Cognitive Nature of Math Learning

One cultural difference participants noticed regarding the cognitive aspect of mathematical learning was the emphasis on calculations and memorization. One survey response described an "arbitrary distinction between procedural and conceptual mathematics," and another wrote about the importance of not only memory, but also rhythm in mathematical learning, and that "reciting math orally often uses rhythm."

Two interview participants went into greater detail about the importance of mastering calculations. Both discussed how the use of calculators was discouraged in their upbringing outside the United States. When asked how such practices affected their mathematical education, one said "I felt like if I learned by doing things by hand I understand things better." Another discussed how achieving algorithmic competency helps achieve "mastery" and importantly, move beyond memorization, "deeper understanding," and "critical thinking."

These respondents placed large importance on memorization and algorithmic competency in their personal theory of mathematical learning and see that Western research largely marginalizes it. It is unlikely they can use it in their professional theories. Although the interviewers rarely cited existing research on why strong procedural knowledge advances one's conceptual knowledge, one survey response has insight into that "mathematical ideas and justifications underpin every algorithm and procedure."

Relatedly, one participant discussed how some of her work's results described different attitudes among students regarding the usefulness of lectures. Of note was an international student who deeply enjoyed lectures. In summary, some researchers come to the United States

placing importance on what is typically deemed “rote” or “passive” methods of learning mathematics, and the research they are exposed to does not align with such attitudes.

Social Aspect of Mathematical Learning

There were more findings based on reactions to American classrooms, typically in higher education, as many of the participants first came to the United States as graduate students. However, some reactions were rooted in coming to the United States as children.

Attitudes Toward and of Teachers: Americans' broad cultural attitudes toward teachers and the teaching profession stood out to many participants. In describing the teacher education program from overseas, one participant said the “teacher education program is really small and [back home], being a teacher is really competitive. It is still crazy competitive. In the United States, we can't really imagine it.” A similar difference was backed by another participant from a completely different culture, who said that American teachers “do not get treated like professionals and often don't see themselves as professionals practicing a profession.” Other participants discussed how this broad societal attitude toward teaching manifested in students. One participant wrote about the surprise of “how little authority teachers hold in teaching and learning in America” and the independence of students, even though “the student-teacher ratio was a lot higher in my culture.” This comparative lack of authority was reinforced by another survey response which described seeing students “talking back to the teacher” as a culture shock. These culture shocks impacted participants’ learning as well as their work as instructors. One self-described Middle Eastern researcher who first came to the United States after college described the culture as “very different” and the adjustments they needed to make when teaching American students. “We were the ones that were always guilty if we didn't know anything. And here it was, it was like the exact opposite.” The participant discussed more on how in her upbringing, the onus fell on the student to motivate their own learning, and how they had to adapt to the shift of such responsibility onto the instructor.

Classroom Dynamics: When participants discussed topics related to the wider classroom dynamics, they made their ethnic, religious, or cultural background the most explicit.

Learning as a Community: One theme that appeared in one Southeast and one South Asian culture is the idea of Learning as a Community. I offer a broad, initial definition as a culture where students, particularly higher-performing ones, have an obligation to ensure their classmates understand the material. One participant described it as follows:

> “A sort of idea where it can't just be me who's learning something. Like if I'm in a classroom as a student, I can't be the only person learning because I'll feel bad. Because I want other people to also pass. The idea that somebody has to fail does not cohere with my mind. It doesn't make sense in my head like somebody has to fail. Cause it's sort of like, it takes a village to pass. And so for me, I'm definitely the type to try to study and learn and work with people, regardless of the fact that I can handle it by myself. Sometimes I take the position of a teacher.”

Moreover, this idea of Learning as a Community stood out to the participant as a noticeable absence from the American classrooms he was in, and he differentiated it from ideas such as “active learning” or “inquiry-based learning.” Another participant from a similar cultural background considers a similar phenomenon part of his initial motivation to pursue math education research. He described how, due to his relatively higher performance in his math courses even in high school, his classmates sought help from him, forming a “helper” identity and sparking an interest in math education. The first participant attributes this to how

"individualism" is characterized in their culture, where it is not the individual person that is prioritized, but the individual family unit, and how it is practiced. He said, "Constantly when I was going up… we would get told, ask your big sister, ask your big brother or like make sure they help your little brother, make sure to help your cousin."

The emphasis on relying on others and making oneself reliable to others seemed to extend to a classroom setting and impacted the participant's theory of classroom dynamics. This awareness of how one's knowledge can help their classmates was present in another interview in a completely different religious context.

> "In Islam, there's a lot of emphasis on learning about how the world works, how the universe works…the other thing is like when I'm learning something, I feel like this is something important I'm doing. It has an impact. I can help other people. If I learn something I can help other people to learn."

The participants, with their varied backgrounds, described a variety of perspectives on the nature of mathematical authority and how it is exercised in and beyond the classroom varied among participants, but a common theme was the larger responsibility of students to themselves and each other.

Closeness and Caring Relations: Another difference was how participants characterized "caring" in American instruction. Some participants elevated the caring of their own culture over the aspects of American culture they experienced. One preferred to characterize it as a distinct construct using the Spanish equivalent of "Cariño." In her Hispanic culture, she characterized "Cariño" as "this value placed on being friendly and loving and caring, like this caring warm quality." She described how it was a prominent feature of teachers from her experience teaching at an overwhelmingly Hispanic secondary school, as this extended to how the teachers treated their students "Like calling them my child… Like there's a lot more like mothering or fathering I think that goes into that role that you wouldn't see" in other cultures. Other participants partially validated this by discussing how they understood "caring" instructors from their cultural backgrounds.

One participant with an Asian background compared his experience in the United States with two Asian professors. Although both taught "traditionally" either in mindset or instructional practices, he saw different levels of caring between the two. One was "incredible in terms of he was a caring guy" and although "he would do the lecture, make stuff really hard, he really wanted you to work hard for his class and he was super." He attributed this partially to this professor's flexibility, such as working with him for longer than expected, and even at night. Compared to another professor "he wasn't as caring. There was an instance that I asked them a question after class. And he said to me 'You should just practice more. You don't understand.' And he just left." Another participant discussed availability in terms of the expectations for teachers in her home country. Although students were expected to be in school for more than 12 hours a day, so were teachers. She characterized the instructors who adhered to the same schedule as their students as "really passionate." While neither the teacher nor the student was in class every moment of the day, teachers were expected to be available similar to "office hours." This same participant also described having minimal connection with her American professors and felt a lack of caring on the side of their professors.

Situating American Culture in a Global Context

Finally, the participants' perspectives help put different aspects of American society into greater context. One of these is Americans' beliefs about people's capacity for mathematical

learning, especially in children. One participant described how the math content in the United States seemed a year behind what she was used to in her original country and how returning home left her unequipped for her school. Another described her experience as an instructor. In her home country, high school students in math regularly graduate with integral calculus, which required an adjustment to teaching what she considered high school material at a college.

Another finding is that the current diversity and equity research paradigm does not align with how non-European American researchers understand diversity. Participants do not encounter much work regarding Middle Eastern and North African, Asian, religious minority, or even White ethnic cultures. Some discussed their motivation to seek out Critical Race studies related to their cultural background in other departments, and the wider discourse on diversity seemed alienating, and overwhelmingly shaped by White researchers with "little room for opinion from those who are non-white, non-American."

The participants' opinions on diversity differed from the more dominant racial discourse. Aside from an implicit desire to see their culture represented in the wider discourse, some were also interested in more granular factors such as linguistic, ideological, religious, and ethnic diversity. One rooted their interests based on their culture where multiple indigenous languages existed in a classroom where everyone was taught their country's national language. For example, studies about linguistically diverse classrooms in America might also benefit by drawing from frameworks drafted by those with upbringings from multilingual societies. More broadly, there might be a place for frameworks that illustrate American attitudes as foundational rather than as a reaction, as the former is the perspective of these researchers. Others placed specific importance on their ethnic background on their Actual Identity, as there were instances of individuals from Southeast and East Asia identifying not as Asian, but as their specific ethnicity. This was alongside an interest in cultural differences among White Americans. Thus, the investigation of the link between identity and attitudes should not be relegated to just non-American, non-White researchers, or even restricted to a single White identity, as there is potential interest in the relationship between White ethnic identities and math education. In general, the field can benefit from a closer inspection of these ethnic identity.

There were also different attitudes regarding the inclusion of women in academic spaces. One woman described avoiding the sexism of her home country as part of her preference for working in the United States. Another described her understanding of her place in academia as rooted in her culture. "[There's] a very conservative mindset, but then there are Muslim scholars who are women. When we date back to the beginning of Islam. One of the wives of the Prophet was a scholar and she just devoted her life to education." Further investigation concluded that she was likely referring to Aisha, who became of renewed interest among Arab scholars in the 20th century (Elsadda, 2001).

Discussion and Conclusion

I end this paper with a brief discussion on counterstories. Solórzano and Yosso (2002) describe one kind which counters majoritarian stories, and another in the sense of Ikemoto (1997) which is not "created as a direct response to majoritarian stories" (p. 32). This final quote regarding the beginning of Islam could be an instance of the former. It could be a counter-story within the context of Islam, not the Christian one that dominates North American culture. Alternatively, it could be a counterstory within an Islamic context.

My findings may well be instances or parts of counter-stories that can exist outside the context of the majoritarian paradigm. Regarding authority relations, there are similarities to authority relationships as described by Amit and Fried (2005) as a development of Weber's

(1947) framework, as well as near-peer mentoring (Stringer, 2024). It would be inaccurate to describe these frameworks as wholly inappropriate for characterizing the dynamics that have been described. However, non-Western perspectives can likely enrich current authority frameworks by providing another foundation. For example, unlike the frameworks from Amit and Fried (2005) or Hackenberg (2010), it does not need to be surprising that students would see each other as an authority by supposing such dynamics from first principles, as might be the case for Confucian frameworks.

Similarly, my findings illustrate two distinct notions of "caring," one as a construct related to immediate, present behavior and another related to availability. These notions have been expanded on mostly outside of the Western European tradition. Lomei (2023) offers a more concrete definition of "Cariño" as "the manifestation of caring grounded in Latino cultural values" (p. 20). Her work is designed as "a departure from a universalistic conception of care and makes an intentional departure from Eurocentric views of what care might entail" (p. 21). and it remained until her work produced a workable framework to recognize cariño in predominantly Hispanic classrooms, and its absence can be felt in other studies. For example, Hufferd-Ackles et. al's (2004) study about the classroom culture of a predominantly Hispanic classroom arguably captures "cariño" from the teacher in a salient way, yet they could not recognize it. A Korean researcher, Ye Ryung Chun, worked to capture the second notion of caring. In her dissertation (2007), she acknowledges the concept of "chuk eun ji shim" which is instead related to the idea of "how to care in situations where we almost do not want to care" (p. 106). In her estimation, this dimension of obligation is absent from Noddings' work. Both these researchers shed light on conceptualizations of caring outside an American perspective and corroborate claims from the participants that such ideas are outside the current mainstream of math education research.

Conclusion

The diversity of opinions I found demonstrates the large diversity of perspectives among non-European American researchers. One does not necessarily need to conclude that the differences researchers notice between their culture and the dominant math education and research culture indicate definitive neglect of such concepts. To do so outright would downplay the contributions of scholars who have contributed in these ways. That said, the data demonstrates an opportunity to leverage the unique perspectives of international researchers working in America, as many of their personal theories of mathematics education were shaped outside a strong Western influence. They can help Western researchers have a more holistic perspective of math education as these researchers demonstrate the capacity to embody multiple cultural paradigms in a single researcher. More generally, future frameworks should not be responses to the dominant culture of the United States, but ones developed outside of it.

The variety and cultural salience of the responses raise questions for the field going forward such as: How can we understand and situate existing frameworks in relation to the cultural identity of their original authors? Explicit awareness of an author's cultural identity does not inhibit the adoption of their frameworks. For example, Lev Vygotsky worked within the Marxist tradition (Fu, 1997) but his theories were eventually adopted by those in a capitalist society. So should work from scholars like Lomei or Chun be restricted to only Latino or Korean contexts? Our field has always applied learning theories to cultural environments distinct from cultures where said theories were initially developed, so I propose not.

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A Gendered-Language Analysis of Abstract Algebra, Real Analysis, and Introduction to Proof Textbooks

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Mathematics education scholars have asserted that proof can be viewed as a unique genre of communication with specific norms and conventions. Further, some have argued that these norms and conventions are gendered and may dissuade women from participating in advanced mathematics. However, these assertions for the most part have not been investigated empirically. In this proposal, we systematically analyze the linguistic features of 90 proofs from 15 textbooks in advanced undergraduate mathematics. Key findings include that many feminine ways of communication rarely occur in proofs, including hedges, references to emotions, and intensive adverbs. Some masculine ways of communication, including the use of directives, are common in undergraduate proofs. Implications of this research are discussed.

Keywords: Gender; Genre; Linguistics; Proof

Mathematical proof plays a vital role in advanced mathematics. Proof is the dominant means of justification in advanced undergraduate mathematics classes and testing students' ability to prove statements is a primary means of assessing students' performance in those courses. For this reason, proof serves as a gatekeeper to studying advanced mathematics and participating in the community of professional mathematicians. Students who do not master the practice of proving will neither succeed in their advanced undergraduate mathematics courses nor be invited to study mathematics further.

There are several reasons for why reading and writing proofs is challenging for university mathematics students (see Stylianides et al., 2017, for a review of this literature). In this paper, we focus on one particular source of difficulty: to read and write proofs, undergraduate mathematics students must work with a new genre of academic writing. In recent years, several scholars have remarked that proof can be viewed as rhetorical genre (Bowers & Küchle, 2020; Lew & Mejía-Ramos, 2019; Selden & Selden, 2013). There are a number of norms and conventions in proof writing that seem unique to this genre and are not present in most other forms of discourse. For instance, mathematics educators claim that proofs usually do not use the "I" pronoun (Burton & Morgan, 2000; Dawkins & Weber, 2017). Lew and Mejía-Ramos (2019) demonstrated that students are not aware of these norms, and they speculate that this may contribute to students' difficulties in engaging with proof. We note, however, that the norms and conventions of proof described in the mathematics education literature are usually not based on empirical studies on how proof are written. Rather, authors commonly justify their claims by referencing their own experience, or by citing other scholars who have made similar assertions. One contribution of this report is to provide an empirical study exploring trends on how 90 proofs are written in undergraduate textbooks.

Scholars have further asserted that the norms and conventions of proving may lead to inequities and discourage students from underrepresented populations from participating in advanced mathematics (e.g., Brown, 2018; Tanswell & Rittberg, 2020; Minggi, et al., 2021). In

particular, some scholars claim that proof may represent a masculinized way of communicating (Keith, 1988; Mendick, 2006), since proofs are impersonal and directive. If true, this can dissuade women from participating in advanced mathematics, as there is a larger gap between the norms of proof writing and the way that women would ordinarily communicate in natural settings (c.f., Tanswell & Rittberg, 2020). However, claims that proof is indeed a masculine form of communication are in need of empirical support. In this paper, we investigate this hypothesis by analyzing 90 proofs from undergraduate textbooks.

Related Literature and Framing

Like Selden and Selden (2013) and Bowers and Küchle (2020), Dawkins and Weber (2017) assert that proof is a genre where proofs are expected to adhere to specific norms. Dawkins and Weber argue that these norms are not arbitrary. Rather, these norms are in place for epistemological reasons. These norms support axiological virtues of the discipline of mathematics and facilitate the epistemological aims of the intended audience for these proofs.

Dawkins and Weber (2017) first assert that mathematicians desire that mathematical truth be *a priori* and objective, corroborating Ernest's (1995) claim that mathematicians have absolutists views where their knowledge claims are correct, universal, and incorrigible since they flow from firm foundations via deductive logic. Two norms associated with this value are that justifications must be traced to stipulated definitions, and that justification should be deductive and not allow for rebuttals (c.f., Inglis et al., 2007). This leads proofs to be written as chains of deductions. As Pedemonte (2007) summarizes: "[W]ithin proofs, the steps are connected by a 'recycling process' (Duval 1992–1993): the conclusion of one step serves as an input condition to the next step" (p. 24).

Dawkins and Weber (2017) further argue that "mathematical knowledge and justification should be independent of (non-mathematical) contexts, including time and author" (p. 129). This points to a common refrain that proofs aim to conceal the author and render them without a human element (e.g., Burton & Morgan, 2000). Further, proofs are "addressed to a universal audience" (Pedemonte, 2007, p. 27). This leads to the norm that references to the personal experiences of the author are omitted, including the process used to generate the proof and even the pronoun "I".

A final epistemological virtue mentioned by Dawkins and Weber (2017) is that proofs should be written to enhance understanding. Proofs are written in a brief and clear style so that mathematicians can recognize the overarching structure of the proof and recognize their key ideas. To avoid getting bogged down in logical minutia, routine details from a proof are omitted, and the proofs do not contain redundancy (Selden & Selden, 2013). We note here that this sparse writing style might not actually increase comprehensibility, especially for students. Logically compact statements can be difficult to parse, and "routine" details might not be obvious for a student to supply (see Minggi et al., 2021).

Gendered Communication

Women and men tend to communicate differently. In Table 1, we present Mulac et al.'s (2001) synthesis of the documented significant differences between male and female communication. As can be seen from Table 1, the literature reveals that men are more likely to use directives and judgmental adjectives in their communication and women are more likely to express uncertainty, add hedges, or use intensive adverbs. With Table 1 in mind, our research questions are: (R1) To what extent do proofs in advanced undergraduate mathematics textbooks

contain the 18 language features described in Table 1 and (R2) Which masculine features are common in proofs in advanced undergraduate mathematics textbooks, and which feminine features are absent?

Table 1. Features of Language Found in Empirical Studies (Mulac et al., 2001) where M: male features, F: female features; X: mixed results from empirical studies.

Language Variable [1]	Definition
Questions (X)	But not including directives in question form
Tag questions (F)	An assertion that is followed by a question asking for support
Directives (M)	Telling another person what to do
Negations (F)	A statement of what something is not
Sentence initial adverbials (F)	Answers the questions: how, when, or where? regarding the main clause [why and how long]
Oppositions (F)	Retracting a statement and posing one with an opposite meaning
Judgmental adjectives (M)	Indicating personal evaluations rather than merely description
Uncertainty verbs (F)	Verb phrases indicating lack of certainty or assuredness
Progressive verbs (X)	Verbs presented in the "-ing" form
Intensive adverbs (F)	Adverbs such as "really" or "so"
Hedges (F)	Modifiers that indicate lack of confidence in, or diminished assuredness of, the statement
Justifiers (X)	A reason is given for a previous statement
References to emotion (F)	Any mention of an emotion or feeling
References to quantity (M)	References to an amount or quantity
Locatives (M)	Usually indicating the location or position of objects
Personal pronouns (X)	Words that stand for beings
"I" references (M)	First person singular pronoun in the subjective case
Fillers (X)	Words or phrases used without apparent semantic intent

1 We do not report on two categories from the framework: elliptical sentences (due to lack of consensus in coder team) and dependent clauses (due to their common appearance across all types of texts Rafajlovičová, 2013).

To avoid misinterpretation, we wish to emphasize two points. First, Table 1 represents broad trends in the differences between how men or boys and women or girls communicate, but these are only statistical trends, and do not tell us how a particular individual will communicate. There are doubtlessly specific females who will frequently use directives and specific males who will avoid them. Second, Table 1 does not specify *why* males and females communicate differently. We agree with Mulac et al. (2001), gender can be understood as a culture in which communication is learned and practiced. That is, "much of what people know about interpersonal communication is learned from same-sex peers during the ages of 5 to 15 years," and therefore particular cultural elements, including language and communication, are developed within same-sex groups. Of course, power, societal ideologies, and ambient cultural trends likely shape how children learn to communicate in their formative years. We also note that this empirical synthesis only allows for exploration related to binary gender identities.

Methods

To explore the nature of language in proof textbooks, we selected the five most used books at the undergraduate level for each of: Introduction to Proof, Abstract Algebra, and Analysis. To identify these books, we used the analysis by David & Zazkis (2020) for Introduction to Proof, Melhuish (2015) for Abstract Algebra, and use the Open Syllabus (2023) project to identify Analysis textbooks. Based on a power analysis (80%) to capture a 0.5 effect size (f) with 15 groups (reflecting 15 textbooks), a sample size of 90 was needed to detect medium differences between books. In this paper, we focus primarily on overarching trends, but this will allow for more detailed analysis in the future.

We selected six proofs per textbook. We selected these six proofs as follows. For each textbook, we set n equal to the total page count divided by 6. We went to page n, and proceeding backwards, identified the first proof. We then repeated this process for $2n$, $3n$, $4n$, $5n$, and $6n$, thus collecting six proofs that span the textbook. Each proof was then coded using the linguistic features in Table 1. This began with a common coding of eight proofs in a pilot to develop a shared understanding of codes. After initial coding, the team met to discuss questions, inconsistencies, and to come to a consensus on how each code was applied. At this point, a revised primary coding was finalized for each proof and is reported below.

Results

Many of the linguistic features we analyzed were consistent across proofs. Table 2 provides an overview of the frequencies of the different features.

Table 2. Frequencies of Features in Proofs Analysis where) where M: male features, F: female features; X: mixed results from empirical studies (Mulac et al., 2001).

Language Variable	Proofs with This Feature (n=90)	Instances of This Feature
Questions (X)	0	0
Tag questions (F)	0	0
Directives (M)	78	209
Negations (F)	26	54
Sentence initial adverbials (F)	90	626
Oppositions (F)	4	5
Judgmental adjectives (M)	23	40
Uncertainty verbs (F)	2	2
Progressive verbs (X)	38	83
Intensive adverbs (F)	10	13
Hedges (F)	3	3
Justifiers (X)	27	41
References to emotion (F)	1	1
References to quantity (M)	46	198
Locatives (M)	18	46
Personal pronouns (X)	76	323
“I” references (M)	0	0
Fillers (X)	0	0

To organize the remainder of the results discussion, we suggest three distinct types of features: necessary, arbitrary, and epistemological. We borrow the distinction between arbitrary

and necessary from Hewitt (1999) but delineate between the two slightly differently. A necessary feature is one necessitated by the mathematical content involved in the proof. We then make a distinction between two types of non-necessary features: arbitrary and epistemological. An epistemological feature is one rooted in the way knowledge is established by the mathematical community. We use arbitrary feature for one that does not link to a particular content need or means of knowledge generation. Both epistemological and arbitrary features are reflective of convention, but as noted by Kontorovich and Zazkis (2017) convention choices are meaningful in the discipline.

Necessary Content-Based Linguistic Features

Some linguistics in proof are what we refer to as *content-based features*. Many proofs had references to quantity (n=46) and locatives (n=18) , which seem unavoidable in mathematical domains that concern number and space. Arguably, negations are a content-based feature as well. The negations that we observed in the proofs for the most part concerned proofs by contradiction.

Epistemological Linguistic Features

In many cases, we could relate the presence or absence of some linguistic features to epistemological values such as those described in Dawkins and Weber (2017). We refer to these linguistic features as *epistemological features.* For instance, across all 90 proofs, there were three instances of hedges, two instances of uncertainty verbs, and no tag questions. Each of these three linguistic features was described as more common in feminine communication. The absence of these linguistic features is consistent with the view that proofs should provide certainty. There was only one reference to emotion across the entire data set, which also has been found to be a feminine way of communicating. This finding is consistent with the content of proofs being independent of the author or the reader (in particular, the emotional reactions or states of the authors or readers).

The use of the "I" pronoun is found to be a masculine way of speaking. Both Burton and Morgan (2000) and Dawkins and Weber (2017) assert that the "I" pronoun is rarely used in proofs, with Dawkins and Weber (2017) speculating that this norm is in place so that proofs are independent of their author. We found support for the norm to avoid the "I" pronoun; none of the 90 proofs used this pronoun. The "we" pronoun was used frequently, accounting for all 323 personal pronouns in the proofs. Although the use of personal pronouns was not distinctly masculine or feminine, we found that the "we" pronoun was often used to give directives (e.g., "We observe…"), and as a "royal we". So, this may still nonetheless may represent a depersonalized and masculine form of communication (see Weber, 2023, for more discussion). Interestingly, Hungerford (2012) did not use the word "we" in any of the six proofs that we analyzed, suggesting that the use of "we" may be a stylistic choice by the author.

In contrast, we were surprised to see judgmental adjectives (a masculine way of communicating) were present in 23 of the 90 proofs that we analyzed with statements such as "we merely have to show..." or "it is obvious that…" This would seem to go against upholding the value that proofs are independent of the authors who wrote them.

Arguably other features that may align with the values of the discipline and knowledge generation include the frequency of "sentence initial adverbials" (found in every proof on average 7 times) and "justifiers" (found in 27 proofs, 41 times). Justifiers provide a reason for a fact after it is presented. These were unexpectedly low. Sentence initial adverbials serve to connect a new state from the prior. The justification in proof is often found *before* a statement,

and thus not meeting the justifier definition. That is, these justifications are found in the initial sentence adverbials which often look like "Thus" or "By theorem X." We note that these features align with the deductive logic that underscores proofs. Each line follows deductively from the previous illustrating the recycling process suggested by Pedemonte (2007).

We also note progressive verbs were not found commonly amongst the proofs; another feature found more often in how women communicate. Thirty-eight of the 90 proofs contained progressive verbs. These are action verbs that end in "ing." We suggest this again aligns with epistemological features as truth is not established via the problem-solving process (Dawkins & Weber, 2017) and therefore active verbs would be uncommon in the final product.

Arbitrary Linguistic Features

Some of the linguistic trends that we observed appeared to us to be independent of the content being studied and did not appear to support an epistemological value. We refer to such features as arbitrary linguistic features. Intensive adverbs, a feminine form of communication, were infrequent in the proofs that we surveyed. We found only 13 intensive adverbs appearing in 10 proofs. We suggest that proofs tend towards brevity and words that do not serve to provide additional meaning are not commonly found. We also suspect this desire for brevity may account for the breach of the objective value because statements like determining something is "obvious" often serve the purpose of reducing the amount of information needed to be shared.

Directives, a masculine form of communication, were common in the proofs that we observed. A total of 209 directives appeared in 78 of the 90 proofs that we surveyed. We discuss directives as an arbitrary feature because they do not necessarily connect to the deductive argument development of knowledge. Sometimes they serve the role of introducing notation (e.g., "let x be a...") while other times serve to draw attention to information (e.g., "note ..."). Directives are one of the few features that have been empirically studied. Inglis and Tanswell (in press) found that imperatives where common in mathematicians' proofs on the mathematical archive, and Weber (2023) illustrated how directives (which he labeled "instructions") were frequently used in graduate set theory textbooks. Our results corroborate these findings in undergraduate mathematics textbooks.

Discussion

A number of mathematics educators have speculated on the nature of the genre of proof (e.g., Bowers & Küchle, 2020; Selden & Selden, 2013). However, empirical analysis of the linguistic features of proofs in advanced undergraduate mathematics courses is uncommon. Table 2 reveals several trends based on 90 proofs in 15 textbooks that we analyzed. In many cases, our results corroborated the claims made by other scholars, such as proofs rarely use the "I" pronoun. In other cases, we documented (perhaps not surprising) trends in proofs, such as the lack of hedges, uncertainty adverbs, and intensive adverbs. Finally, there was the curious finding that judgmental adjectives appeared in 23 proofs. This warrants further analysis.

Our findings also indicate several linguistic features of proofs that are representative of masculine ways of communicating. These include the lack of tag questions, uncertainty verbs, hedges, intensive adverbs, and references to emotion, the frequent use of directives, references to number, and locatives, and the occasional use of judgmental adjectives. We note that not every masculine linguistic feature of communication from the literature was common in the proofs that we analyzed. In particular, the pronoun "I", a masculine form of communication, was not present in any of the 90 proofs that we coded.

This raises the question, do proofs need to contain masculine linguistic features and avoid feminine linguistic features? Our current analysis is speculative. It amounts to us asking whether proofs need to have the linguistic features that they do, and can we imagine presenting proofs of the same content in different ways. We cannot see how the content-based linguistic features that proofs have can change. Much of mathematics concerns numbers, quantities, and location, and as such, proofs will often have references to quantity and locatives. It is easier to imagine altering the arbitrary linguistic features. For instance, one can add intensive adverbs to a proof without changing its content, and one can write proofs in a way that does not direct the reader to perform actions. What we coded as epistemological linguistic features present a more interesting case. It is true that avoiding reference to emotion may uphold the value that proofs are independent of author, and avoiding hedges and uncertainty may uphold the value that mathematical results can be known with certainty. However, as noted by Ernest (1995), these epistemological values themselves may be masculine. He suggests:

> There is a radical view that the kind of popular image of mathematics described above serves conservative interests in the mathematics community and in society in general. For if mathematics is viewed as difficult, cold, abstract, ultra-rational, important and largely masculine, then it offers access most easily to those who feel a sense of ownership of mathematics, of the associated values of western culture and of the educational system in general (p. 454).

He further references the work of Gillian (1982) which argues that women and men have differed values in their development where, women are more likely to have values that are, "relationships, connections, empathy, caring, feelings and intuition, and tends to holistic and human-centred" (p. 454). This is contrasted with a separated view where men are more likely to have values that emphasize, "rules, abstraction, objectification, impersonality, unfeelingness, dispassionate reason and analysis, and tends to be atomistic and thing-centred in focus" (p. 454). If this assertion is accurate, then the proof features aligning with epistemological values may need to be interrogated for the way they encode gendered and power dynamics. For example, one can imagine proof norms where the author is allowed to express an emotional state (surprise at a non-intuitive result, or frustration that a particular proof approach is difficult). Further, while proofs must employ deductive reasoning, we can imagine proofs where the author acknowledges that they are not absolutely certain that they carried out each deductive step correctly. Mathematicians offer these types of hedges in their ordinary discourse around proofs (Weber, Mejía-Ramos, & Volpe, 2022), so we can imagine permitting them to do so within the proof itself.

We have demonstrated that proofs in undergraduate mathematics frequently possess linguistic features that are associated with masculine forms of communication. Bowers and Küchle (2020) noted that genres have a power dimension and may play an exclusionary role. It is therefore reasonable to conjecture that the linguistic norms around proof writing may serve to exclude students (Keith, 1988). This suggests two lines of future research. The first is seeing how these linguistic norms encourage or discourage participation in the practice of proving, particularly in gendered ways. Would women, nonbinary individuals, (or even men) be more comfortable in writing proofs in ways that did not adhere to the masculine norms of proving? The second is more explorative and involves designing classrooms that re-imagine proof discourse.

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Exploring Undergraduate Mathematics Students' Math Identities Using Sentiment Analysis: Affordances and Precautions

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Students' identities, orientations, and experiences in mathematics courses shape their participation and longevity in the discipline. Yet, undergraduate mathematics instructors do not often integrate pedagogy or curricula that help them build knowledge of their students' math identities. We believe that in many math courses this is a problem of time and scale. We propose that language processing methods can help alleviate this tension by giving instructors a baseline understanding of their students' math identities. In this study, we applied sentiment analysis to 28 mathematics "origin stories" written by undergraduate pre-calculus students. We compared the predictions from four language models with qualitative codes applied by two mathematics education researchers. We conclude this paper by discussing the strengths and limitations of sentiment analysis in our data, and discuss implications for undergraduate mathematics researchers and practitioners who may wish to apply these methods to their work.

Keywords: math identity, sentiment analysis, mixed methods, math instruction

Research has demonstrated the importance of *math identity*, how students see themselves in relationship to mathematics, as doers of mathematics, and their ability to persist in mathematics courses (Boaler, 2002; Cribbs et al., 2016). When mathematics instructors attend to students' math identities with culturally sustaining pedagogy, scaffolding curricula, or creating supportive course infrastructure, they can help students persist by diminishing issues of power and authority historically present in mathematics. However, an important first step in teaching responsively to students' math identities is building knowledge of students. While small class sizes make this a tractable task in K-12, organizational arrangements in undergraduate education pose difficulties for many teachers who wish to know their students' math identities. Undergraduate mathematics courses often enroll hundreds of students, and many instructors serve in a lecturer role. Even by incorporating pedagogical best practices that scale well to lectures, building knowledge of students' mathematics identities remains an intractable challenge. Furthermore, from a research perspective, undergraduate students' math identities are difficult to study at scale with qualitative methods that require humans to apply and agree upon qualitative codes.

To address these challenges, we explore the potential of using *sentiment analysis*, a technique from natural language processing, using community college mathematics students' math origin stories. Though limited to affect, sentiment analysis can give teachers a first glimpse at their students' positive/negative valences with respect to mathematics. We then investigate the limitations of sentiment analysis by comparing predictions from four language models to qualitative affective codes applied by the researchers. Our goal is to explore the extent, if at all, sentiment analysis can aid mathematics instructors and researchers in building knowledge of students' math identities (particularly in large math courses). We ask:

1. How accurately can sentiment analysis predict students' affect in math origin stories?
2. What limits exist for studying students' math identities with sentiment analysis?

Theoretical Framework

Math identity refers to "the ways that students think about themselves in relation to mathematics" including "a person's beliefs, attitudes, emotions, and dispositions about mathematics and [their] resulting motivation and approach to learning and using mathematics knowledge" (Froschl & Sprung, 2016, p. 1). Building and acting upon knowledge of undergraduate students' math identities can impact students' perceptions of their ability to succeed in mathematics. Incorporating math identity requires teachers to inquire about students' math histories and interrogate students' ongoing tensions in seeing themselves as agentic doers and persistors in mathematics. Research has presented several strategies that can aid teachers build knowledge of students' math identities and histories. For instance, Nasir and colleagues advocate for making explicit the relationship between mathematics knowledge, students' cultures, and identities (2008). Rubel and Chu (2012) found that utilizing students' math histories as a starting point for enacting instruction led to increased participation and improved problem-solving skills among underrepresented minority students. However, both of these studies were conducted in K-12. It is unclear how the strategies presented in these studies would be implemented in larger undergraduate mathematics courses.

Implementing pedagogy based on students' math identities is significantly more challenging in large undergraduate mathematics courses. Hornsby and Osman (2014) highlight that the sheer size of lecture halls and high student-to-teacher ratios often impede instructors' abilities to know their students personally. This conclusion is echoed by Cooper and Robinson (2000), who note that large class sizes can hinder the development of a classroom community that is conducive for students to share their personal mathematical experiences. Furthermore, Jaworski et al. (2017) point out that the diverse backgrounds of students in large lectures make it challenging to address all individual math identities effectively. To mitigate these issues, some researchers have suggested using technology to collect data on students' math histories (e.g., Cotner et al., 2013), or using active learning pedagogies during lecture, such as group problem-solving, that could give students opportunities to share aspects of their math identity with peers (Freeman et al., 2014). However, these strategies require time investment and pedagogical knowledge, which may not always be available in large, traditional collegiate mathematics departments.

Sentiment Analysis in Mathematics Education

Although limited to affect, sentiment analysis could give mathematics instructors and researchers initial signals about students' math identities. *Sentiment analysis* is a supervised machine learning technique that has been used in social science research to study emotions and affect in large text-based data sets (Thelwall & Buckley, 2013). Within sentiment analysis, language models make inferences on input text by weighting linguistic features that are likely associated with affect (e.g., "great," "fail," or "!") towards a positive or negative prediction. An input text assigned a "positive" prediction, for example, represents the model's inference that features extracted from the input text align most strongly with patterns associated with positive sentiment, based on the data that the language model was trained upon. In the case of mathematics identity, sentiment analysis can be applied to predict the positive/negative valence in students' writing samples about math identity, which could give suggestions about how students perceive their past and present experiences in mathematics courses.

To date, sentiment analysis has been applied in mathematics education research to analyze pieces of student writing *after* mathematics courses have been completed. For instance, Crossley and colleagues (2018) used sentiment analysis on students' emails to an emotional support

"Genie" (intelligent agent) to study correlations between students' math identities and K-6 mathematics assessment scores. Other studies in higher education have used sentiment analysis to study students' dispositions in end-of-course reviews (Bringula et al., 2022; Hew et al., 2020). However, as large language models are now available to consumers, we believe that these tools may afford mathematics instructors and researchers opportunities to study students' math identities vis-à-vis writing artifacts produced *before* and throughout mathematics courses. Our goal is to study the affordances and limitations of this method with texts about math identities.

Context and Methods

The participants in this study are part of a larger research project in a community college undergraduate mathematics course sequence. The course sequence contains one semester of pre-calculus and one semester of calculus, taught by the same instructor. Students are STEM majors in a cohort pursuing 4-year degrees. Many come from diverse linguistic, cultural, and educational backgrounds and have had mixed experiences in mathematics prior to the study.

The second author requested that the instructor give students a "math origin story" assignment at the beginning of the semester. Here, we consider "math origin story" to be roughly equivalent to other math identity activities (e.g., "mathography," "math autobiography," and "math history"). 28 students submitted math origin stories. The prompt for this assignment was:

> "…Write a short paper (approximately 300 - 500 words) OR make a media recording (audio or video, 3-5 minutes) and talk about your experiences of mathematics [before this class]. Describe the ways that any important experiences you had made you feel about math and how they shaped your learning of math. You can draw from formal or informal experiences, in the home, school, or any other setting. Maybe you had conversations with a family member that was important to you? If you were fortunate, you may have had an amazing teacher or perhaps you were unfortunate enough to have had a negative learning experience. Be descriptive, this is not an academic paper, it is a chance for you to share your math history and for me to learn more about you and your relationship with math."

Data Analysis

To determine the usefulness of sentiment analysis from the perspective of college mathematics researchers and educators, we analyzed the data using mixed methods. First, both researchers individually coded students' math origin stories for positive, negative, and neutral sentiment at the level of the entire math origin story. Next, we met collectively to agree on sentiment codes and resolve outstanding disagreements (Cornish et al., 2016). Drawing from research on math identity, we placed importance on students' statements that described their relationship to mathematics, such as "I am excited by the idea of doing mathematics," or "Using mathematics, I see myself as a future engineer." We placed less emphasis, although still took into account, sentiments about students' experiences with individual mathematics subjects or teachers unless the description indicated a transformation about their relationship with mathematics (e.g., "After taking Ms. G's class, I no longer see myself as a mathematician."). Table 1 shows criteria that we used to assign positive, negative, and neutral sentiments with examples from the data.

Table 1. Qualitative sentiment codes applied by the researchers.

Sentiment	Description	Example from Data

Positive	The text indicates affirmative behavior and/or thoughts in relationship to math identity, such as self-affirmation, seeing oneself as a doer of math or persisting in math. When positive and negative sentiments exist, positive sentiment is stronger.	"...I did notice at an early age that math was something that came natural to me even though I didn't apply myself as much." (Student 6)
Neutral	The text does not indicate sentiment direction, or, is not about the subject's math identity.	"The next math class I took was algebra 2." (Student 3)
Negative	The text indicates disaffirming behavior and/or thoughts in relationship to math identity, such as poor experiences, not seeing oneself as a doer of math or not persisting in the discipline. In the case of positive and negative sentiment, the negative sentiment is stronger.	"There were very few instances in math where I felt that what I was currently learning was genuinely …something that would … help me." (Student 17)

Both researchers held at least six years of teaching and research experience in mathematics education that informed how we conceived of math identity in undergraduate mathematics. The first author taught statistics and calculus for seven years in high school, and the second author taught calculus for nine years in higher education. Both authors taught mathematics to populations of students who have indicated complex relationships with "doing math."

After we applied sentiment codes, the first author pre-processed the text in students' math origin stories to ensure students' stories were compatible with the language models (Haddi et al., 2013). All program-specific acronyms were converted to their full name to reduce the likelihood that the language model would incorrectly infer alternative linguistic meanings. Then the corpus of words was spell-checked, and some words were edited for clarity (e.g., "highschool" was converted to "high school"). Any non-math symbols were removed from the data.

The first author then compiled a corpus of data containing students' origin stories to be analyzed at the level of each story. Sentiment analysis was performed with four language models: TextBlob, a python language processing package chosen for its accessibility; BERT, selected as a former state-of-the-art model for language inference tasks; GPT-4, chosen because it is a current state-of-the-art large language model accessible to consumers for free, and GPT-4o (ver. Sept. '24), selected because it is a current state-of-the-art large language model available to consumers (via OpenAI's API) for a monthly subscription fee. We then analyzed the nature of the errors each model made with confusion matrices and examined patterns of misclassified sentiments. The first author then created a new data corpus with all math identities misclassified by the highest performing model, at the level of individual sentences. We then performed sentiment analysis again with the highest performing language model.

Results

We present the findings from our study by research question. First, we give aggregate performance by each language model; then we analyze the sentiment of misclassified examples to further investigate differences between machine and human-applied sentiments.

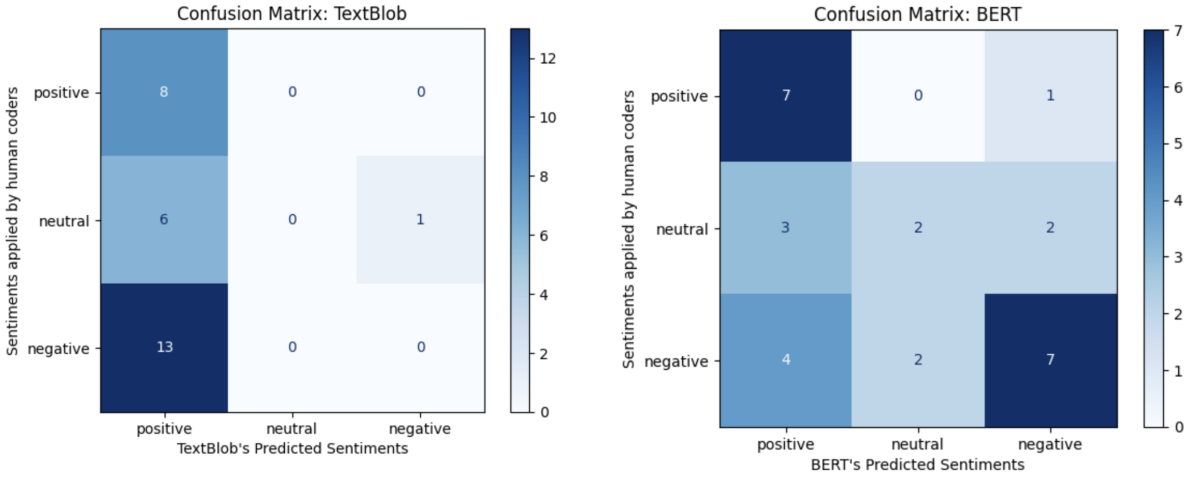
RQ1: How accurately can sentiment analysis predict students' affect in math origin stories?

Table 2 shows the performance accuracy of the four language models. While most models performed poorly (and Textblob performed worse than random guessing), GPT-4o's 82.19% accuracy is close to what is considered to be state-of-the-art performance (about 90%) among language interpretation tasks in English at the time of this writing.

Table 2. Sentiment analysis prediction accuracy.

Model	Python TextBlob	BERT	GPT-4	GPT-4o
% Accuracy	28.57%	57.14%	64.28%	82.14%

Figure 1 shows four confusion matrices that demonstrate how each model differed in sentiment predictions from the sentiment codes we applied as human researchers. Counts along the top-left, bottom-right diagonal indicate math stories where the model predicted the same sentiment as we did. All other areas of the confusion matrices indicate classification errors. For instance, TextBlob's low accuracy was almost entirely due to the fact that 27 out of 28 predictions were "positive," whereas we only labeled eight math origin stories as positive. On the other hand, GPT-4o's errors are due to predictions of neutral sentiment on positive math origin stories. We explore these misclassified results in detail in the second portion of the findings.



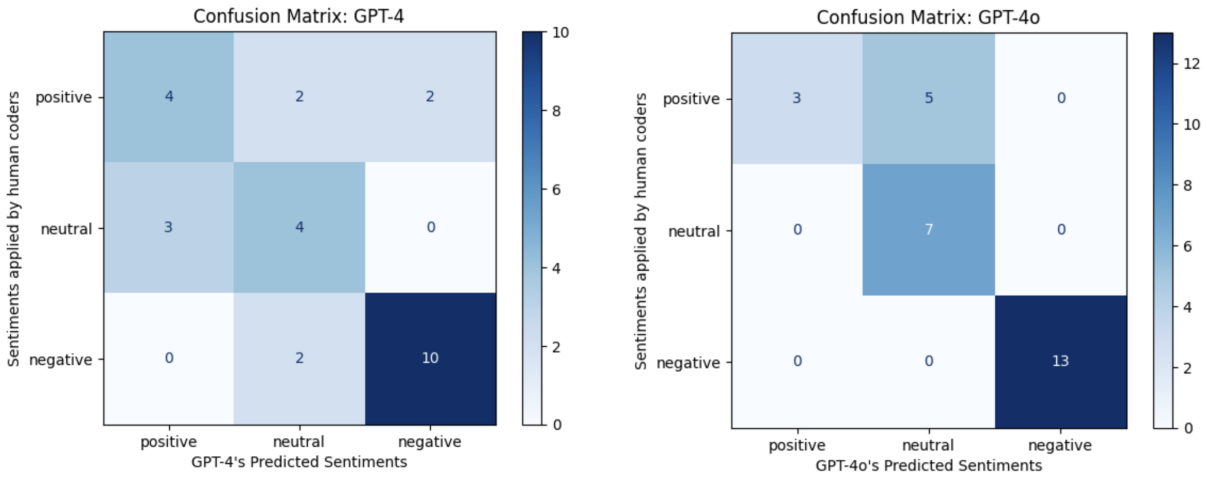


Figure 1. Confusion matrices for each language model.

RQ2: What limits exist for studying students' math identities with sentiment analysis?

We compared the sentiments of the misclassified math origin stories from GPT-4o by sentence to better understand why misclassifications occurred. Figure 2 shows the relative frequencies of positive, neutral, and negative sentiments among the 83 sentences in stories of five students whose positive sentiment was misclassified as "neutral." Results show that GPT-4o classified math origin stories as "neutral" if the ratio of positive to negative sentences was within ~20%. We examined misclassified data for common themes, and found three key distinctions.

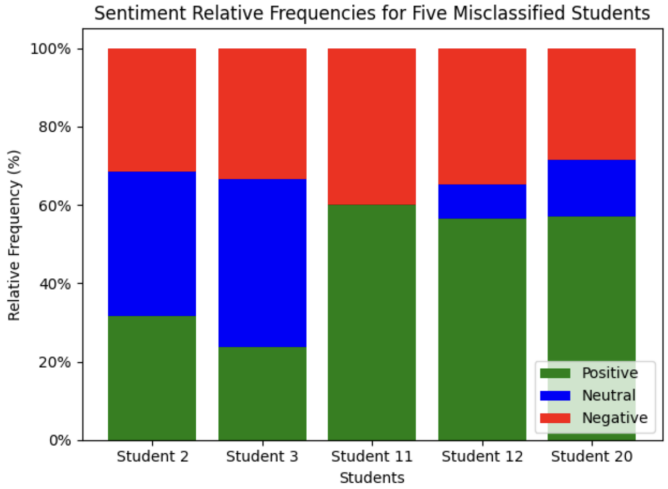


Figure 2. Relative frequencies of sentiments in five misclassified math origin stories.

Theme 1: Math identity in mixed sentiments. Sentences with higher word count frequently contained both positive and negative sentiment. In some cases, GPT-4o classified these sentiments as "neutral." For example, student 20 wrote: "As soon as the semester was over, I immediately dropped [math] class and became a teacher's assistant for my cooking teacher." We inferred that GPT-4o predicted "neutral" because "Dropped the class" is a negative sentiment and "became a teacher's assistant" is positive. However, as mathematics educators, we believed that dropping a mathematics course was more consequential for students' persistence in math,

and therefore was more closely related to the student's math identity than becoming a teaching assistant for a cooking class. We therefore rated this sentence, and others like it, as "negative."

Theme 2: Attribution of sentiments. Unlike human raters, GPT-4o did not seem to weight sentences that referred to individuals other than the student author with less importance. For instance, student 12 wrote, "The most important person to my journey understanding math is my mom. She was always very good at math and knew most things." GPT-4o rated both sentences with positive sentiment. We agreed; however, we did not believe these sentences were as closely related to the students' own math identity, as they described the mother's ability to know and do mathematics. We rated these statements as "neutral" because they described another person.

Theme 3: Prioritization of sentiments that directly described students' math identity. All sentences seemed to be weighted equally by GPT-4o toward the overall sentiment prediction of the math origin stories, regardless of the subject-matter within the sentence. It appeared that the model tried to give an "average" sentiment prediction in neutral cases. For instance, student 3's story contained five positively-classified sentiments, nine neutrally-classified sentiments, and seven negatively-classified sentiments, and was given a "neutral" label. We show a portion of student 3's story below, *italicizing* negative sentences and **boldfacing** positive sentiments:

> "…*My algebra 2 class was a joke. My teacher retired and we did not have a teacher for the rest of the year. We changed substitute teachers and I learned nothing.* The next math I took was pre-calculus. *I would always skip class so I really didn't learn anything during the last 2 years of math.* During all my high school math I have had nothing crazy or anything important enough to remember. *My math classes shaped me to know the basic understanding of math but did not prepare me to thrive.* **Even though I had poor math experience it does not scare me or deter me from becoming a computer science major and I am excited to complete calculus and move on to future maths."**

Since Student 3 described their overall relationship with math as positive in the final sentence, and indicated that they wished to study more math (indicating persistence), we labeled their math origin story as positive, despite the student's description of negative past experiences.

Discussion

Performing sentiment analysis on a class of community college mathematics undergraduates gave us insights about the affordances and limitations of the methodology for both researchers and practitioners. Although GPT-4o underestimated the amount of positive math origin stories, it accurately predicted negative stories. This implies that state-of-the-art language models could potentially assist researchers and teachers to determine how many students express negative affect toward math early in the term, affording opportunities to provide instructional interventions proactively. Of course, sentiment analysis is not instructors' only potential tool for assessing the affective dimensions (McLeod, 1992) of students' math identities. Kaspersen and colleagues (2017) constructed a validated survey to measure math identity in STEM students. We note that the underlying data sources that support surveys (student questionnaires) and sentiment analysis (math origin stories), as well as students' relationships to those data sources, are structurally different from each other. We are curious to explore the measurability affordances of surveys versus sentiment analysis in math identity research in future work.

Furthermore, sentiment analysis by sentence demonstrates that instructional interventions can go beyond mathematics performance and address common themes in students' math identities

such as challenging teachers or struggles to understand math concepts. Doing so could signal to students that their teacher is receptive to negative sentiments about mathematics, and will help students persist in mathematics regardless (Cribbs et al., 2016).

We explored sentiment analysis on students' math identities as a proof-of-concept methodology with a small sample intentionally. Our goal was to unpack the nuances of misclassified sentiments and make conjectures about how human mathematics researchers weigh linguistic features and assign labels in comparison with machines. In future work, we aim to explore this methodology at scale. We also wish to investigate how, if at all, knowing students' math sentiments at the beginning of the course impacts instructors' pedagogical decisions.

In the meantime, we recommend that instructors and researchers wishing to experiment with sentiment analysis and math identity begin their inspections of the data with neurally-labeled sentiments. Throughout this analysis, we learned that "neutral" can mask "mixed" or "nuanced" statements about math identity, which was the case for Student 3. Yet, a balance of "good" and "bad" math experiences does not imply neutrality in one's math identity. Humans apply their own weights to past experiences to determine their present relationship and future aspirations when seeing themselves as doers of mathematics. Our students deserve culturally-responsive instructors and researchers that understand the mixed nature of students' sentiment toward math.

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Enacting Care in a Community College Math Classroom: A Case Study

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Too often college mathematics classes become a gatekeeper that pushes students off of STEM pathways. This study looks at an example of a community college mathematics instructor who appears to be successful in supporting students to stay on a STEM pathway through creation of a warm, caring classroom environment. Using the lenses of a caring relation and a teacher as a 'warm demander' this case study examines what pedagogical practices he used to create and sustain this environment. We illustrate in the findings how the instructor used his emotional intelligence and skill in relating to students to sustain a caring, warm classroom. He also incorporates metamessages and 'off-topic' talk in his classroom which demonstrate care to students. This case study serves as a beginning point in thinking about the social and emotional work of teaching and we end by proposing a model for caring instruction in community college mathematics classrooms.

Keywords: Community College Math, Care, Pedagogical Practices, Equity

The study of Science, Technology, Engineering, and Mathematics (STEM) disciplines is often seen as an avenue to good jobs and a requirement for many in the modern workforce of the 21st century. Politicians and business leaders regularly lament the lack of qualified students prepared to fill these roles. Unfortunately, STEM fields also continue to be a place where historical inequity around race, gender, and socio-economic status also continues to persist.

At the undergraduate level, mathematics is often the subject that determines whether students begin or continue these majors. Students are required to pass through the calculus sequence before they can access other higher-level classes in their desired major. Studies have shown that many students who drop the subject are not less prepared or capable but cite a negative experience in their early classes (Rasmussen & Ellis, 2013). Work over the last few decades has examined how to teach these disciplines better and has shown the effectiveness of active and inquiry-based learning for improving results for all students (Freeman et al., 2014; Hassi & Laursen, 2015; Laursen et al., 2014). However, a few studies have shown that discrimination can still exist in active learning classrooms and impact the experiences of students from marginalized groups (Ernest et al., 2019; Johnson et al., 2020).

In further understanding how to address the issues of inequity, researchers have also pointed to the value of focusing on community college contexts (Sitomer et al., 2012). While community colleges educate a disproportionate share of students from backgrounds underrepresented in STEM fields, they are also understudied in comparison to 4-year university contexts (Mesa et al., 2014). In a 2019 study, Nabb and Murawska found that community college instructors identified care as an important component of their jobs. This case study provides a detailed examination of what care may look like in the context of a successful community college mathematics program. The research question of focus for this report is: What pedagogical practices characterize care in a community college math classroom?

Theoretical Framework

In defining care, we draw on the work of Nel Noddings and her definition of care as a caring relation (Noddings, 2013). Noddings' definition of care importantly distinguishes the caring

relation from a definition of caring as a virtue simply dependent on the orientation of the individual in the role of 'carer'. The caring relation only exists if the 'carer' offers care and the care is acknowledged or received by the individual to whom it is directed, the 'cared-for' (Noddings, 1999).

This relational definition of caring offers important additional considerations in the enactment of caring. If caring is defined in the relationship built between the teacher and student, the experience of the student and how they respond to their teacher's acts of care is an important factor in determining whether the teacher was in fact 'caring.' This need for a relation means that there may sometimes be structural factors at play that could prevent a recipient from receiving the care intended, such as in a large class where an instructor is not able to build relationships with all students. Beyond structural factors, the instructor's cultural knowledge or skills, such as emotional intelligence, might limit or enhance an instructor's ability to enact care that is felt by their students. The enactment of this type of relationship with students relies on a skill set that many community college and undergraduate instructors have never been trained in. Caring in this definition requires judgment, awareness, and nuance to enact in ways equitable toward all students (Noddings, 1999).

Noddings' work on care is limited in that it does not provide specific guidance on what care looks like in educational arrangements (Roberts, 2009). Multiple scholars have also noted a lack of engagement with the impact of race and racism in the contexts of care, such as educational settings, that Noddings addressed in her work (Patterson et al., 2008; Thompson, 1998). We believe it is important to address these limitations, and so make use of Noddings' framework in connection with the idea of a 'warm demander' bringing in race conscious, critical perspectives.

Kleinfeld (1975) identified two characteristics that appeared to separate effective from ineffective teachers. The first characteristic, which she deemed the most important, was teachers' ability to "create a climate of emotional warmth that dissipates students' fears in the classroom and fulfills their expectations of highly personalized relationships" (p. 318). The second was, "teacher's ability to resolve his own ambivalent feelings about the legitimacy of his educational goals and express his concern for the village students, not by passive sympathy, but by demanding a high quality of academic work" (p. 318). This framework is important to pair with Noddings caring relation because it points to the importance of maintaining high academic standards while also creating warm environments. It adds more context to the caring relation that must be felt. Care in this context is not just creating "a climate of emotional warmth" but also communicating to students that you believe they are capable of academic success and pushing them to work hard toward achieving at high levels.

This framework also helps us bridge the gap between a caring relation and pedagogical practices. To define practices we look to social practice theory and the work of Reckwitz (2002), who defined a 'practice' in the social theory sense as a "routinized type of behaviour which consists of several elements, interconnected to one [an]other" (p. 249). This definition casts practices in the realm of actions that may be driven by mental processes, beliefs and learning but is enacted day to day through the actions taken. In the case of teaching these practices are a unit of study or focus, but whose existence depends on the interplay of beliefs, learning, emotions, social context and background of the individual engaging in the 'practice.' In the case of teaching, pedagogical 'practices' can take the form of instructional lectures or actions to facilitate student engagement in the classroom. Taking a more expansive view on the work of teaching to include motivating and engaging students in the discipline of study, pedagogical practices also include efforts to build relationships with students and help them develop a sense

of belonging and identity in the field. This study with extensive time in the classroom and interviews with the instructor and students allows for detailed examination of the pedagogical practices engaged in by the instructor.

Study Context and Methods

Study Participants

Students. The classroom population reflects the demographics of the school and the broader California Bay Area where it is located. In Fall of 2021, around 40% of the students identified as Asian and 20% as Latinx, the two largest demographic groups. While the racial and ethnic makeup of the class was similar to the school demographics, it was considerably more male, over 70%, and younger, 85% of students were 19 years or younger than the school as a whole. These differences likely reflect that the focus of the program was students interested in engineering or computer science, which are fields that skew male and included the introductory math sequence, so the majority of the students were in their first year of community college and came directly from high school.

Instructor. The instructor who is the focus of study, Professor O[1], is an experienced faculty member at the college with over 20 years of experience teaching mathematics at the school. Furthermore, he has been an instructor for the program of study for all 9 years of its existence. While Professor O identifies himself demographically as a white male, an important aspect of his orientation to teaching is how he identifies with his students. Professor O is a first-generation college student and describes himself as a bad student in high school. He is also a community college graduate himself. This shared experience provides him with important insight into his students' struggles and likely supports his ability to be highly empathetic towards them.

STEM-Core Model

The class is a part of a program initially started with funding from an NSF grant, STEM-Core. The STEM-Core model creates a cohort of students, who follow an accelerated math course sequence and stay together for one year. This study was conducted in the 9th year of the program at this school and is no longer supported with NSF funding.

Site implementation

This particular program was chosen for study because students are broadly succeeding at high rates. Students are prepared to pursue a STEM major and ultimately go on to complete 4-year degrees in STEM and find jobs in the field. Institutional data shows that students have a higher pass rate in Pre-Calculus 1 and 2 and Calculus 1 than the college as a whole. Historically, 64% of students who started the program in Algebra II reached Calculus 1 within one year, compared to the statewide average of 16% of students who start in algebra reaching calculus (Mejia et al., 2016). Furthermore, when students do struggle and drop or fail classes, they re-enroll in the classes at higher rates demonstrating resilience.

Analytic Methods

The study makes use of a case study design with the instructor and a single cohort of students serving as the case under study. This design was chosen because it allows for a fine-grained and

[1] All names are pseudonyms

detailed description of the environment. The first author spent a full year in the class, attending classes and documenting observations through fieldnotes. During the spring semester videos were also recorded. In addition to class observations, interviews were conducted with students and the instructor to understand their perspectives and experiences. The complexity of the environment and constructs of interest lend to the need to consider depth over breadth in this study and makes it particularly well-suited for an ethnographic case study methodology (Merriam & Tisdell, 2015). This need for deep understanding of a complex environment leads naturally to adopting a case study design for this work (Starman, 2013).

The fieldnotes and student interviews were coded with a priori codes based upon the 'warm demander' (Kleinfeld, 1975) and caring relation frameworks and observations of the classroom (Saldaña, 2021). Adjustments were made to each codebook to allow for analysis of the interviews to focus on students' description of the instructors' actions and their experiences in the class while the codebook for the fieldnotes was focused on observable behaviors in the classroom. After initial coding, the excerpts were pulled out and a thematic analysis conducted to develop descriptive categories for the codes (Clarke & Braun, 2017).

Results

Coding of field notes and interviews showed the practices breaking down into three broad categories: active learning, warm language and meta messages, and off-topic talk. The combination of both data sources allowed for important triangulation of observable actions in the classroom, conversations with the instructor to understand his intent, and hearing from students to understand their experiences.

Active learning

The instructor extensively made use of pedagogical practices associated with active learning. The most common code that appeared in the data was 'students relating to each other'. This code was driven by the frequent turn-and-talks initiated by the instructor. A turn-and-talk occurred when Professor O paused his lesson and told the students to check-in with one another on what he just shared or to ask them to attempt the next step in the problem. The frequency of these turn-and-talks led to a very active classroom, with around 8-15 turn-and-talks per a 110-minute class period. Professor O intentionally initiated interaction among students, occasionally telling students to stop working if he felt the class had gotten too focused on working individually and say, "Everyone needs to stop writing and talk to the person next to them." (FN 27 Oct)

An outcome of his push for regular student to student engagement, is that students appeared to start engaging with one another apart from the turn-and-talks. In interviews, students shared how they had built relationships with another in the class because they were pushed to interact. One student contrasted this experience with a previous instructor who did not push students to talk in these moments. He appreciated how Professor O pushed students to talk when necessary. Since talking with one another was normal, students started to talk together before class, or during breaks. Another strategy Professor O employed was structuring his long class with "working breaks" and "group quizzes" to encourage students to engage in math problems together. In interviews students used words like "community" in describing their experience in the class.

Students using one another as a source of knowledge would also often overlap with Professor O's use of turn-and-talk. He would make requests like, "I need everyone to pick up their calculator to attempt to do this calculation in your calculator and when you get the answer share

it with the person sitting next to you." (FN 13 Oct) This push to check with each other helped demonstrate to students that in working together they could build confidence in their answers. In addition to having students check in with one another, Professor O would also regularly solicit ideas from the class and make use of their ideas. He very intentionally encouraged students to challenge each other to make sure they understood what the other person was thinking and use necessary tools when it made sense. This environment of encouraging struggle aligned well with the second most common code from the field notes: a supportive learning environment.

Warm Language and Meta Messages

Professor O also appeared to use language intentionally for initiating the turn-and-talks. He regularly used language and phrasing that is not common in college math classrooms. Phrases like, "check in with the person next to you and say, 'I care about you and I want to make sure you are doing okay' say it" (FN 27 Sept) as the initiation for a turn-and-talk when he wanted students to check in with one another about whether they were understanding the next step in solving a problem. Another prompt he would use frames the need for each student to check-in with their neighbor to ensure the other person was understanding, "check with the person next to you to make sure they understand." (FN 22 Sept) Sometimes this statement was followed up with the rejoinder, "no one gets left behind". His phrasing for turn-and-talks brought the language of care into the classroom and emphasized that students are not only responsible for their own learning, but they are part of a community where everyone is learning. When interviewed 10 of 13 students made comments that explicitly addressed their comfort in the classroom.

He also often made comments and initiated discussions to push students to engage in productive learning behaviors. In talking about watching the videos before class he said, "If you come to class confused you will leave class not confused. If you come to class not confused because you have not watched any of the videos or done any work, you will leave confused." (FN 11 Oct) With this statement he is simultaneously pushing students to watch the classroom videos ahead of time and encouraging them that if they are confused after watching the videos then that is normal and where he wants them to be. He is also reinforcing to them that they have some agency about how confused they are, based on their behavior/choices to watch the videos.

Normalizing struggle also showed up in a number of small ways through comments from Professor O and norms he set in the class. This came up in the initiation of turn-and-talks with "explain to the person next to you what I just said, 'I know you understand but they were confused.'" Everyone in the class is simultaneously being identified as the person that understands but also the neighbor that is confused. He would sometimes take a moment in class to acknowledge, "that it might be overwhelming right now but by the time you get to the exam it will be okay." (FN 27 Oct). This acknowledgement did not mean that he was going to cover less material or slow the class substantially because that would not be possible with the requirements of material to be covered, but it made space for emotion and feeling and served as an acknowledgment of what many students might be feeling. Messages like this one about studying and learning were common supplements to the mathematics content in this classroom.

Students also commented about how Professor O handled mistakes in ways that helped them see their mistakes as part of the learning process. One student Zara commented, "every time I've made a mistake, it was more just like, oh no, it's fine. This is the actual answer and this is how we're supposed to do it. Just remember that for next time. And I was like, okay. Not that big of a deal." Messages like this were interspersed with regular, micro growth mindset messages he

gave by encouraging students that their brain was growing when they made a mistake in class, and struggle is a normal part of the learning process.

Off-topic Talk

Professor O regularly took the time to address relevant issues for the students that were not directly related to mathematics, which we coded as "off-topic talk." These topics could range from broad educational and career goals and skills to mental health and university resources. He would often connect these topics to aspects of the class. On several occasions he mentioned vaccine clinics for both COVID and flu vaccines that were being hosted by the school. (FN Oct 11) When talking about the flu vaccine he commented to students about how getting the flu at the wrong time can really mess up your academic plan and encouraged students to get vaccinated.

In addition to health-related issues, he would occasionally address topics that he felt were important even if they were completely disconnected from mathematics such as Banned Book Week and the connections between book bans and racial issues in the U.S. Professor O did not generally address race explicitly in class, although he would regularly wear t-shirts with messages around racial, gender, LGBTQ and immigration equity issues. However, in this case, Professor O made use of the moment to name and address his racial identity and share support for work that would allow him to better understand other people's experiences. In interviews several students share how these comments demonstrated to them Professor O's awareness of structural inequities and made them more comfortable approaching him when they had an issue.

Professor O regularly connected what students were learning to their future careers and educational trajectories. He would often share about the importance of internships and creating resumes to apply to those internships because, "you aren't going to put a math topic on your resume, so that is why you need to make sure that you are doing clubs and activities." (FN Sept 6) He pushed students to attend information sessions about various internships they were eligible for and supported them through the application process. Professor O also used these opportunities to help offer perspective to students on what they were working towards, "no one is here because the one thing missing in your life is trig identities, but you are all here to get a job and internships will help with that." (FN Oct 27)

Professor O also took time out of class to invite two speakers from the counselor's office and host a 'field trip' to the school's Transfer Day so the students could engage with schools they would be interested in attending in the future. All of these conversations communicated to students that Professor O had a bigger picture in mind when he was teaching them math. He wanted them to learn the content in his class but ultimately to use it as a stepping stone for careers in their desired fields.

Interviews with students confirmed that they received his inclusion of these outside as helpful, caring actions. In interviews (9 of 13) shared how Professor O cared about them and would support them in various ways, beyond just learning mathematics. Several students explicitly connected his discussion of non-math or 'off-topic' content as an indicator of his caring about them.

Discussion and Limitations

These findings illustrate that while one area of practice related directly to content, active learning, the other two involved engagement and work not directly related to the curricular content of the course. From this case study, we propose a framework of caring instruction in a

community college mathematics classroom. The major components of the framework are: *relationships*, *holistic support,* and *active learning*. *Active learning* has been discussed extensively in math education literature and shown to be effective in increasing student success and learning. (Deslauriers et al., 2019; Freeman et al., 2014). However, some studies have also shown that it alone may not be able to address all the challenges and equity gaps that exist in the mathematics classroom (Johnson et al., 2020). In this case, *active learning* appears essential to enabling the other portions of the framework because without space to engage in mathematical conversations and interact with other students and the instructor, the other pieces are not possible. However, *active learning* alone does not equate to caring instruction.

The data from this study reveal that *relationships* are a fundamental driver of success in this math classroom. Students felt like they had access to a warm relationship with their instructor who knew their name and cared about them as people. Professor O worked to cultivate opportunities and encouraged students to develop *relationships* with one another. The richness of these *relationships* and normalization of mistakes and struggle in the classroom appears to have improved students' sense of their ability to belong in and succeed in mathematics as shown by their comments in interviews.

Finally, the *relationships* are also intertwined with the *holistic support* offered in this classroom through the inclusion of 'off-topic' material such as health information, school transfer fairs and various speakers sharing about resources on campus. These discussions helped show students that their instructor cared about them as people beyond just being math students in his class, which helped build trust and enable *relationships*. This broader, *holistic support* also helped to even the playing field by making explicit aspects of the education experience that are often part of the hidden curriculum and not revealed to first generation and low-income students.

While not addressed in this paper, it is important to note that in the background of these three pieces: *relationships*, *holistic support,* and *active learning*, are the challenges of implementing these ideas with students, which requires critical consciousness and emotional intelligence. It is important for instructors to be aware of systemic inequity and the realities of position, power, and privilege when they are working with diverse groups of students (Rubel, 2017). They need to understand that certain groups of students may be less comfortable in their classroom or may have a different experience because of their identity or social positioning. At the same time, instructors need to be careful not only to rely on broad generalization or stereotypes but instead know how to engage with the wide variety of students they teach and meet each student where they are. Such work requires a high degree of emotional intelligence.

This case study involves only a single instructor and class, which means it can only serve as a starting point. An important next step for this work is the observation and study of other successful instructors to see if and how these pieces show up in those classrooms as well. An active classroom appears to be a necessary foundation for enacting care, but it alone does not appear to be sufficient. We need to build capacity for the social and emotional work of teaching in college math classrooms. Instructors need support in learning how to enact care in their classrooms for all their students. Community college math instructors know care is an important part of good teaching (Nabb & Murawska, 2019) but not all know how to enact care. This work shows the need to support teachers to move beyond an orientation of care, but to caring relations that are received by students. We as a community of educators and researchers need to think about how we can create learning environments that are truly welcoming and supportive for all students.

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Floating in a Most Peculiar Way: The Role of Feedback in a Playful Calculus Task

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We created and piloted a playful calculus-based activity called Major Tom with preservice secondary mathematics teachers over two classroom sessions. Players create sequences of jetpack boosts (accelerations), controlling an astronaut's movements to arrive at a pick-up location at a particular time. The learning goal was for students to draw upon and enhance their knowledge of integration to determine velocities and displacements. We found that the imprecision of the system's feedback inhibited the students' productive use of their formal calculus knowledge, while on the other hand an instructor-led structured discussion about velocity graphs supported it. Neither of these features of the playful math environment much affected the students' playfulness, but a third feature, the opportunity to create and solve your own challenges, fostered significantly increased play and laughter. This contributes to our broader goal of how to elicit play with undergraduates while simultaneously engaging and building their formal mathematics knowledge.

Keywords: velocity, integral, calculus, play, acceleration

Mathematical play is beneficial for students. It can improve enjoyment and engagement (Plass et al., 2013), increase equitable access to mathematics (Widman et al., 2019), and help students develop creativity and a disposition for innovation (Barab et al., 2010; Gresalfi et al., 2018). It has also been shown to support mathematical learning and reasoning at the elementary and middle-school levels (e.g., Edo & Deulofeu, 2006; Jasien & Horn, 2018; Levine et al., 2012). However, there are still significant gaps in our knowledge of how play can support learning, particularly at the undergraduate level. The aim of our work is to support undergraduate students' mathematical play while at the same time meaningfully leveraging their formal mathematics knowledge. We consider this kind of "productive" playful engagement to be an interaction between the task(s), student(s), and the teacher's implementation. We use the term *playful math* to describe the elements of classroom activities and environments that facilitate mathematical play. To investigate this broader question, we studied students engaging in playful math activities in an undergraduate class. Our research questions for this study are: (1) What features of the playful math environment supported or constrained students' productive use of their formal mathematics knowledge? (2) What features of the playful math environment supported or constrained students' mathematical play?

Background and Framing

Our two-part definition of mathematical play is grounded in the traits of a person's behavior and experience that are considered foundational in the literature. Firstly, mathematical play entails *agency in exploration* and *self-selection of goals or subgoals*. A person experiences freedom and control over their activity, including how they explore and the goals they select (Holton et al., 2001; Gresalfi et al., 2018). Exploration is essential (Mason, 2019) and often accompanied by unexpected outcomes (Davis, 1996). Goals are inherent to play (Huizinga,

1955) and can emerge and change as play proceeds (Jasien & Horn, 2022, Radke & Ma, 2018). Secondly, mathematical play entails *immersion* and *enjoyment*. People who play are highly engaged (Gresalfi et al., 2018) and may exhibit a state of flow (Csíkszentmihályi, 1990). They find pleasure in their activity and experience it as being worthwhile for its own sake (Su, 2017).

This definition highlights a tension that arises when mathematical play is used to support learning in classroom environments. Students with high agency might pursue goals that have little to do with the teacher's learning goals. Pyle and Danniels (2017) view play-based learning to be situated in a continuum, where the activity ranges from student-directed to teacher-directed. In more teacher-directed play, the broad goals can be prescribed if students have significant leeway in the subgoals and actions they can take to pursue them. This type of continuum provides a way to situate playful math activities at the undergraduate level also, where a teacher may direct students to engage with formal mathematics while still playing.

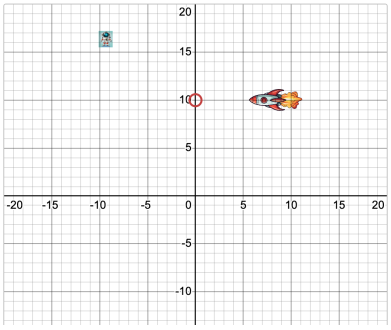
We have identified six design principles for playful math activities (see Ellis, et al., in press): (1) foster understanding of specific mathematical ideas; (2) enable free exploration within constraints; (3) engender anticipation within the task; (4) provide a method for intrinsic feedback; (5) offer meaningful challenge while still being feasible; and (6) allow the student to act as both designer and player. For this study, we considered what the students did with the intrinsic feedback they received from the task environment. A student experiences intrinsic feedback in direct correspondence to their actions, rather than as mediated by a teacher or some other extrinsic source. Students can increasingly come to anticipate the results of their actions in the environment when they are able to use their mathematical understanding to simulate and predict the experience first (Tzur, 2007). The feedback students receive can therefore inform their mental models and anticipated actions and outcomes, rather than just allowing for reactive adjustments to their actions. For example, Williams-Pierce et al. (2024) studied a case in which students programmed a robot to navigate a path and received perceptual feedback when the robot missed the mark. Because the students had developed and used a quantitative model while doing the programming, they anticipated the path and were able to interpret the feedback in light of their model, productively engaging with their prior mathematics knowledge.

Methods

We collected data with 16 secondary pre-service teachers (PSTs) at a public university in the southeast United States. The PSTs had taken at least two calculus courses and were enrolled in an algebraic reasoning content course. They formed 8 pairs, and each pair engaged in two playful activities on an iPad over four sessions. We collected audio and screen recordings of each pair's work as well as any written work not captured by the iPad. This report is based on the engagement of one PST pair, Audrey and Penny (pseudonyms), in one of these activities called *Major Tom*. *Major Tom* (Figure 1) consists of 6 tasks whose common objective is to move the astronaut ten miles "north" in space, from (0, 0) to (0, 10), in 10 minutes by adjusting "boosts" (accelerations). The PSTs entered magnitudes, angles, and durations of the boosts, hit Play, and the system showed the astronaut's motion. The system also provided quantitative feedback (time and fuel usage during the first session, position also during the second session). The activity's introduction read: "You are floating in space, at the origin. In 10 minutes, the spaceship will swing by the pick-up point ten miles 'north' of you. Can you be there at the right time? Your jetpack can give you a boost if you open its valve, in any direction you want. At each moment, the valve setting determines the acceleration you experience (in mi/min^2) at that moment. Change the table to make whatever sequence of valve settings you want."

On successive screens are follow-up tasks. Task 2 requires Tom to be stationary at the pick-

up point. Task 3 introduces a fuel gauge (a valve setting of n for m seconds uses nm kg of fuel), and requires the use of specific amounts of fuel. In Task 4, an asteroid blocks the way, so Tom cannot stay on the y-axis. In Task 5, boosts can now include variables. In Task 6 you design your own challenges, which you solve in Task 7. *Major Tom* satisfies our design principles, allowing PSTs to select and explore their own goals, use feedback to reflect on their solution strategies, and design challenges. To save space we will use a shortened notation for inputs. For instance, the sequence of two boosts in Figure 1 would be written as {1, 90°, 0, 2} and {2, 180°, 4, 5}.



Minutes: 9.192
Position: (-9.384,16.384)

Play

Valve setting	Angle	Time Begin	Time End
1	90	0	2
2	180	4	5
0	0	0	0
0	0	0	0

Figure 1. Major Tom Activity.

We intended the *Major Tom* activity to promote an understanding of the relationship between position, velocity, and acceleration in terms of derivatives and integrals. For instance, suppose a PST tried out one boost {1, 90°, 0, 2} followed by Tom coasting the rest of the way from $t = 2$ to 10. One can mathematically model Tom's velocity, whose derivative is the acceleration $a(t) = \begin{cases} 1, & 0 \le t \le 2 \\ 0, & 2 < t \le 10 \end{cases}$, to get $v(t) = \begin{cases} t, & 0 \le t \le 2 \\ 2, & 2 \le t \le 10 \end{cases}$. One could then find the total displacement by integration, either by reasoning with areas under the velocity graph or symbolically as $\int_0^{10} v(t)dt = \frac{1}{2}t^2|_0^2 + 16 = 18$ miles. These models could support PSTs to get exact answers and examine more phenomena such as minimizing fuel usage or going in non-vertical directions. The PSTs engaged with *Major Tom* in two 75-minute sessions.

This is part of a larger study, so we adopted an existing coding scheme to describe the characteristics of mathematical play (Ellis et al., in press) as well as developed emergent codes to investigate the PSTs' cognitive engagement with a particular emphasis on their use of feedback. Part of the existing scheme relevant to this report concerns the characteristics of playful engagement, in which we distinguished moment phenomena from task phenomena. Moment phenomena are discrete instances of experience or behavior, and include *competitive fun*, *feeling proud*, *enjoyment*, *taking on authority*, and *laugh* (as evidence of enjoyment). Task phenomena reflect an outcome of the PSTs' engagement with a task as a whole, and include *investment*, *immersion*, *agency* in goal exploration or accomplishment, and *perseverance*. We coded the transcripts independently, discussing and resolving any disagreements to refine and establish a common understanding of the codes during weekly meetings. Once we developed a stable coding system, Authors 2 and 4 then re-coded the data corpus with the final system.

Results

Features influencing PST's productive use of their formal mathematics knowledge (RQ 1)

In the first session, the PSTs worked in pairs on Tasks 1-7. They did not always clearly distinguish between acceleration and velocity, and many had difficulties drawing on calculus concepts to try to produce exact solutions. At the beginning of the second session, the first author

therefore led a structured discussion on constructing a velocity graph from a sequence of unidirectional accelerations (i.e., parallel to the y-axis) and discussed what the graph informs about displacement. The PSTs then revisited the tasks from the first session and solved new challenges they created. In examining what supported and constrained the PSTs' use of formal mathematics, we found two features. One is the imprecision of the program's feedback, which acted as a hindrance; the other is the structured discussion, which acted as a support.

A hindrance: Imprecise feedback from the system. The program's imprecise feedback hindered Audrey and Penny's productive use of their formal knowledge. We initially programmed *Major Tom* with a relaxed success condition: the spaceship would stop if Tom was less than 1 mile from the pick-up point (0, 10) at time $t = 10$ minutes, and would otherwise cruise past without stopping. Furthermore, the screen did not show a position readout for Tom. For the second session we added the position readout and shrunk the window of success to 0.1 mile. In this section we illustrate how this imprecise feedback in the first session diverted Audrey and Penny from drawing on the derivative/integral relationships between acceleration, velocity, and displacement to determine exact displacements for Tom.

On Task 1, Audrey and Penny started with the values seeded in the table and changed them one at a time for five tries, to understand the system. They then tried just one boost: {1, 90°, 0, 2}. When this took Tom too far, they tried {1, 90°, 0, 1}. With this, the spaceship stopped, with Tom at (0, 9.5) at time 10 (although the exact position was hard to discern). They considered this a success and moved on to Task 2, at no point using symbolic or numeric calculation. On Task 2, Audrey and Penny kept the same first line in their table, but added a second boost: {1, 90°, 0, 1}, {-1, 0°, 9, 10}. Tom veered at the end, so they corrected the angle from 0° to 90° and tried again. This time Tom stopped at (0, 9), short of the pick-up point. They kept making small adjustments, six different times, and eventually solved the problem with {1, 90°, 0, 1}, {-2, 90°, 9.5, 10}. The alert reader may calculate that this solution puts Tom at (0, 9.25) at $t = 10$, which is short of (0, 10), yet the program considered this close enough and the spaceship stopped.

Audrey and Penny continued to almost exclusively rely on this repeated guess-check-adjust method for the rest of the first session. They experienced success each time after multiple tries, never attaining an exact answer of (0, 10) at time 10. On Task 3 they used exactly 3kg of fuel with {1, 90°, 0, 1.5}, {-1, 90°, 1.5, 2}, {-2, 90°, 9.5, 10}, ending with Tom at (0, 9.75). On Task 4, with an asteroid in the way at (0, 5), they adapted their previous approaches to split the journey into two parts, and after many adjustments ended up with an approximation that got the spaceship to stop: {2, 45°, 0, 1}, {-2, 45°, 4, 5}, {2, 140°, 5, 6}, {-2, 140°, 8.8, 10}. They did not appeal to a mathematical model, never calculating anything with $\sqrt{2}$ or using a symmetric 135° angle to try for an exact solution. On Task 5, they were asked to use some variable non-constant valve settings (and still avoiding the asteroid at (0, 5)). The aim was to push the PSTs to reason with integrals to calculate displacements when the accelerations were non-constant (and thus required calculus). Penny did briefly start by writing some calculus symbolism, "ds^2/d^2t," but she quickly abandoned this approach and instead decided to adapt their solution from the previous task, inputting {4x, 45°, 0, 1}, {-4(x-4), 45°, 4, 5}, {4(x-5), 140°, 5, 6}, {-4(x-9), 140°, 9, 10}. She explained to Audrey, "We want our average acceleration for one minute to be 1, which means it should be 2x." Thus, an acceleration of 4x over the time interval $x = 0$ to 1 would average to 2, so she assumed that replacing every 2 with 4x would not affect the displacement.

Penny conflated some aspects of velocity with acceleration here. If the *velocity* were 4x, it could be averaged to 2 over this time interval without changing the total displacement, but the same is not true when the *acceleration* is 4x. The displacement s over one minute is 2/3 when

$\frac{d^2s}{dx^2} = 4x$ and 1 when $\frac{d^2s}{dx^2} = 2$. This difference, accumulated over their four small time intervals, put Tom a bit beyond (0, 10) when the spaceship came by. But because Major Tom was within 1 mile of the target, it stopped anyway. Audrey and Penny made further adjustments to get Tom visibly closer to (0, 10), but, like before, they moved to the next task without getting an exact answer. Getting an exact answer was unimportant to them, as it was not necessary to make the spaceship stop. This laxity of feedback thus allowed Audrey and Penny to avoid confronting a conceptual issue about the relationship between acceleration, velocity, and displacement.

A support: Structured discussion. During the beginning of the second session, the PSTs discussed how the area under the curve (i.e., between the graph and the time axis) represented the total distance traveled. This discussion supported Audrey and Penny's productive use of their formal mathematics knowledge. They proceeded to engage with the activity very differently when they revisited the tasks from the previous session, including the challenges they had invented. They no longer conflated acceleration and velocity. For each task they drew a velocity graph, and they consistently interpreted the accelerations as the slopes of this graph, recognizing acceleration to be the derivative of velocity. They interpreted Tom's total displacement as the definite integral of the graph. An example is in Figure 2, which shows Penny's mathematical model of her solution to Task 2, where Tom must be stationary at (0, 10) at $t = 10$. This allowed them to calculate inputs to yield an exact solution which they could then test with the system, instead of repeatedly guessing, checking, and adjusting as they did in the first session.

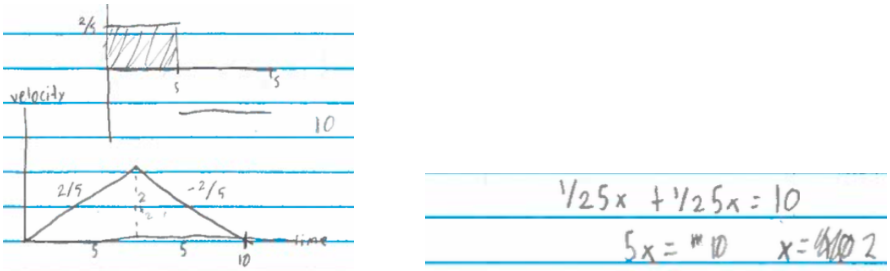


Figure 2. Penny's velocity graph model for solving Task 2, second session: {2/5, 90°, 0, 5}, {-2/5, 90°, 5, 10}.

Velocity graphing also helped the PSTs use their calculus knowledge to create new scenarios. For instance, they invented this challenge: "Major Tom needs to go to (0, 15) to pick up some space litter but still needs to get to the rocket. Figure out how he can do this and be stationary at the pick-up point at least one minute early." They used a velocity graph with negative area (Figure 3) to calculate a solution: {5/3, 90°, 0, 3}, {-5/3, 90°, 3, 6}, {-20/9, 90°, 6, 7.5}, {20/9, 90°, 7.5, 9}. Penny initially calculated a solution that she found to be incorrect when she checked it on *Major Tom.* Seeing Tom miss the mark, she said, "Oh wait, no, no, no, I see what I did. I see what I did. So, I made the area of this [the negative region] ten. It's supposed to be five." Then she said, "This is why I like having this [the program] to check. I wouldn't have realized I did it wrong." Thus, the velocity graph model not only helped them solve the problem, but also helped them interpret and resolve perturbing feedback from the system.

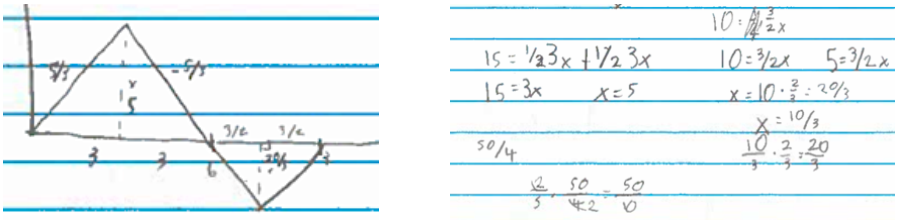


Figure 3. Penny's model for solving their invented challenge, second session.

Features Influencing Mathematical Play (RQ 2)

To investigate the features of the environment that may have influenced mathematical play, we considered the incidence of our moment and task codes. *Competitive fun* occurred when the PSTs demonstrated excitement about their challenges, *feeling proud* occurred when they showed pride in their work, *enjoyment* occurred when they demonstrated enthusiasm (further evidence of this was through the *laugh* code), and we coded *authority* when seeing either PST take on that role. We coded for *investment* when the PSTs showed that they cared about their work or the task's outcome. Evidence for *immersion* included the PSTs ignoring distractions and lacking awareness of time passing. *Agency* occurred when the PSTs demonstrated autonomy in choosing a strategy or posing their own problem, and we coded for *perseverance* when we found evidence of the PSTs voluntarily persisting through difficulty.

Audrey and Penny did demonstrate evidence of mathematical play. Across both days they engaged in a total of 14 tasks, spending on average eight and a half minutes per task (although there was great variation across the tasks). Considering the moment phenomena, competitive fun occurred 0.8 times per task on average, feeling proud 1.6 times, enjoyment 1.1 times, taking on authority 0.6 times, and laugh occurred 6.2 times. The task phenomena codes we either applied zero or one time per task. Out of the 14 tasks, we found evidence for investment during 8 of the tasks, immersion during 2 tasks, agency during 10 tasks, and perseverance during 5 tasks. In considering whether the imprecise feedback influenced Audrey and Penny's mathematical play, we noted that it was a feature of every task, and thus the presence of mathematical play phenomena suggest that it did not prevent them from playful engagement. However, some aspects of mathematical play were minimal, such as immersion, taking on authority, and experiencing a sense of competitive fun. To examine the potential effect of the structured discussion, we compared the incidence of mathematical play prior to and after the discussion. Audrey and Penny solved 8 tasks prior to the discussion and 6 tasks after. The length of time they spent per task was slightly more prior (9.2 minutes) than after (7.4 minutes). If the structured discussion affected the PSTs' playful engagement, we would expect to see a noticeable difference in the average incidence per task prior to the discussion compared to after. Except for the laugh code, we did not see any meaningful difference in the moment phenomena. Two phenomena decreased slightly (competitive fun and feeling proud), and two increased slightly (enjoyment and authority). Laughter, however, did decrease noticeably, occurring 7.4 times per task on average prior to the structured discussion and only 4.7 times per task on average after. The students made more verbal expressions of enjoyment after the discussion, but they laughed less. Considering the task phenomena, we saw a slight decrease in the incidence of investment and perseverance and a slight increase in the incidence of immersion. Agency, however, doubled, occurring for only half the tasks prior to the structured discussion and for every task after. It may be that Audrey and Penny experienced greater agency having gained an appreciation for the utility of velocity graphing, derivative, and integral to tackle the challenges.

We then returned to the data to see whether there were any features of the task / instructional environment that *did* influence the PSTs' playful engagement. We found one feature, which was the creation and solution of their own challenges. Audrey and Penny solved nine regular tasks (in which they solved a problem given to them), two "make a challenge" tasks, and three "solve your own challenge" tasks. Figure 4 shows the average number of moment and task phenomena per task for the regular tasks compared to the make-a-challenge and solve-your-challenge tasks. One caveat is that the task time differed significantly, which would affect the incidence of

moment codes (but not task codes). Audrey and Penny spent longer on average on the regular tasks compared to the make a challenge tasks (5.9 minutes versus 3.3 minutes), and they spent a whopping 19.4 minutes on average when solving their own challenges. Thus, there was more time to observe and code moment phenomena during the solve your challenge tasks. This accounts for the large discrepancy of laugh codes, for instance. However, it is interesting to observe that the PSTs laughed 7.5 times per task when making a challenge compared to 2.3 times per regular task, despite make-a-challenge tasks being shorter. Moreover, when we consider task phenomena, which are not swayed by task length, we still saw greater levels of investment, immersion, and perseverance when solving their own challenges. It appears that having the agency to design and solve problems of their choice was meaningful for Audrey and Penny, and it enabled them to take ownership in their work and invest in the outcome of their activity.

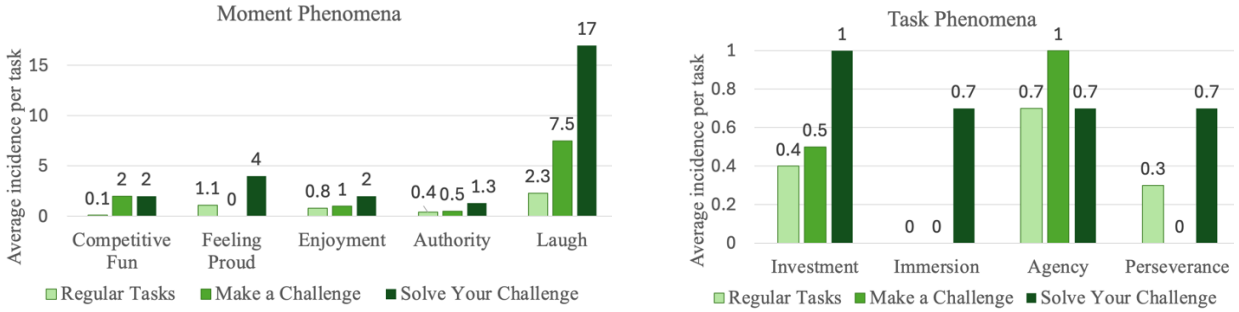


Figure 4. Comparison of mathematical play phenomena across task types.

Discussion

Our findings with Audrey and Penny playing with the *Major Tom* activity provide detail that can augment our design principles for playful math tasks. One of those principles is to provide a method for intrinsic feedback. This case indicates that it is worth considering whether this feedback is also precise and quantitative enough to be reasonably guess-and-check-proof. This can help to necessitate (or at least to not allow the students to easily avoid) the engagement of formal math knowledge. The feedback can then bear on their mental models, not just on their immediate actions. The design factor of allowing students to act as both designer and player was certainly reinforced for these students, for whom inventing and solving their own challenges was propitious for play. Our findings also reinforce the design factor of fostering understanding of specific mathematical ideas. In particular, it illustrates that there need not be a trade-off between play and engagement with formal mathematics; the teacher-directed discussion supported the latter without diminishing the former or dampening student agency.

As we reported on just one pair of students, we do not wish to make any general claims from them, but rather to note that they reveal how various features of a playful math environment can function. Since other students may be hindered and helped by different factors, based on their dispositions and knowledge, we aim to continue studying this question in the context of other playful activities in other undergraduate mathematics classrooms.

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Calculus I Students' Logical Reasoning About the Definition of Continuity of Function

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We examine undergraduate calculus students' reasoning about the continuity of a function at a point. The study was conducted in the context of an NSF-funded project that supported the redesign of Calculus I recitations at a large public university. One aspect of the reform effort involved introducing conceptually rich, collaborative activities into Calculus recitations. One such activity focused on the concept of continuity at a point, with one particular task probing students' understanding of the definition of continuity at a point using true-false statements. Data from 84 student in-class worksheets were analyzed with a focus on the type of reasoning students used to respond to these questions. The results reveal a variety of graphical, logical, example-based, and rules-based strategies.

Keywords: Continuity at a point, Limits, Calculus learning, Logical reasoning

The concepts of limits and continuity are fundamental to calculus, yet they present many challenges for students to understand practically and conceptually (e.g., Amatangelo, 2013; Bezuidenhout, 2001; Denbel, 2014; Williams, 1991). Some researchers locate these challenges in the procedures-oriented emphases that are prevalent in many calculus textbooks (Bezuidenhout, 2001), while others suggest that students' challenges may reflect the historical, mathematical, and psychological complexity inherent in these concepts (Juter, 2006; Tall & Katz, 2014). Consequently, many studies focused on revealing various types of students' conceptions and misconceptions about limits and continuity. Our study seeks to add to this literature. In particular, we examine how first-year Calculus students reason about the definition of continuity at a point as they learn about this concept in an authentic classroom setting.

The study was conducted in the context of an NSF-funded project aimed at improving the teaching of Calculus. The enacted changes included using more conceptually oriented tasks, which students solved by working in small groups during the recitations. This context constitutes an authentic learning setting where students, after attending a lecture on continuity, work together to solidify their understanding of this concept. Learning construction is expected to be characterized by students expressing incorrect or incomplete ideas, drawing on prior knowledge and naïve conceptions that may or may not apply to the mathematical situation at hand. Within this learning situation, we explore the research question: *How do students reason about the definition of continuity at a point, and what types of justifications do they use to make sense of this definition*?

Literature Review and Theoretical Background

Definition of Continuity at a Point

The formal definition of continuity at a point states that a function f is continuous at a point c, in its domain, if for every $\varepsilon > 0$ there exist a $\delta > 0$ such that if $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$. This definition is rarely introduced to first-year Calculus students. Instead, a more intuitive definition is commonly used: a function f is continuous at a point c if $\lim_{x \to c} f(x) = f(c)$. While this definition relies on notation and concepts that are more familiar to students, it has some non-obvious challenges. First, it is often written as a conditional rather than

a bi-conditional statement, which may affect how students interpret and apply the definition in practice. Second, $\lim_{x\to c} f(x) = f(c)$ implies that a function is defined at a point c, and the limit $\lim_{x\to c} f(x)$ exists. Thus, some textbooks, introduce a *criterion* for determining if a function is continuous at a point c, comprised of three rules: (1) $f(c)$ is defined, (2) $\lim_{x\to c} f(x)$ exists, and (3) $\lim_{x\to c} f(x) = f(c)$. This three-rule structure streamlines the procedure for determining if a given function is continuous at a point, but complicates the logical structure of the definition. Let V (value), L (limit), and E (equal) represent propositions corresponding to the three rules above, and C (continuity) the conclusion. Then the definition of continuity at a point has a logical form of $(V \wedge L \wedge E) \Leftrightarrow C$. The criterion for continuity at a point is the direction $(V \wedge L \wedge E) \Rightarrow C$, while the contrapositive $(\neg C \Rightarrow (\neg V) \vee (\neg L) \vee (\neg E))$ represents the property of a function being not continuous at a point. While first-year Calculus students are not expected to unpack this complex logical structure, they may be expected to use it flexibly and implicitly when solving problems. For example, students must understand that a function can be discontinuous at a point, even if it is both defined at that point and has a finite limit at that point.

Students' Reasoning about Continuity at a Point

The concepts of limit and continuity are closely related to each other and are the core of differential Calculus. However, they pose persistent difficulties to learners (Núñez, et al., 1999; Oehrtman, 2009; Tall & Vinner, 1981). Studies distinguish between how students evaluate limits and continuity at a point (often utilizing algorithmic, procedural approach) vs. how students interpret them conceptually (Amatangelo, 2013; Bezuidenhout, 2001; Denbel, 2014).

When reasoning about limits and continuity, students rarely rely on the formal definition, even if they are familiar with it, but rather on what Tall and Vinner (1981) call a personal concept image – a collection of all mental images, words, metaphors, and associations a student has developed about a particular mathematical concept. Thus, students may interpret continuity using intuitive, dynamic images of graphs that can be drawn without lifting a pen or interpret limits using the language of movement, e.g., "getting close." Such reasoning is associated with a dynamic conception of limits and continuity and can lead to either correct or incorrect conclusions (Núñez, et al., 1999; Oehrtman, 2009; Przenioslo, 2004). For example, Denbel (2014), found that pre-engineering students often viewed limits as "unreachable" or "approximations", and many mistakenly believed that if the limit exists at a point, then the function is continuous at that point. Similar misconceptions were observed among first-year Calculus students in South Africa (Bezuidenhout, 2001) and Sweden (Juter, 2006).

Amatangelo (2013) identified four common misconceptions related to limits and continuity at a point: (1) a function defined at a point implies continuity at that point, (2) the limit value is equivalent to the function value at a point, (3) the existence of a limit implies continuity, and (4) a limit at a point necessitates discontinuity. That study involved over 800 students and revealed that these misconceptions were prevalent and persistent until students took advanced calculus courses like Multivariate Calculus or Theory of Analysis.

Logical Reasoning

As mentioned above, interpreting the definition of continuity at a point requires (possibly implicit) reasoning with logical operators of conjunction ("and"), disjunctions ("or"), and negation. Most first-year calculus students have not received explicit training in mathematical logic. Thus, their conceptions of logical principles are rooted in everyday language that often

misaligns with conventional mathematics (Epp, 2003). For example, students tend to interpret disjunctions as exclusive rather than inclusive "or" (Dawkins & Cook, 2017) and confuse between the statement and its converse (Durand-Guerrier, 2003). Additionally, negations and reasoning with contrapositives have been shown to be challenging for students (Buchbinder & McCrone, 2019; Brown, 2018; Yopp, 2020). While the logical form of a statement may affect how it is interpreted, some psychologists argue that people mainly reason by analyzing the *semantic* content of the statements: by forming and mentally manipulating mental representations of that content (Johnson-Laird & Byrne, 2002; Oaksford & Chater, 2002). Bringing back the notions of concept image and dynamic reasoning, this may imply that when interpreting the definition of continuity at a point, students form mental images of continuous and discontinuous functions and mentally manipulate them to arrive at certain conclusions.

Method

This study is a part of the larger, NSF-funded project that focused on reforming the teaching and learning of STEM gateway courses, particularly courses in the Calculus sequence, at a large public university in the northeast of the USA. The reform efforts involved re-envisioning of Calculus I recitations by introducing conceptually rich activities, active-learning strategies such as collaborative problem-solving (CBMS, 2016), and introducing learning assistants (LAs) alongside graduate teaching assistants (GTAs) to support student group work. Calculus I is taught in the lecture-recitation format. Large lectures (~120 students) are taught by a faculty member, and coordinated recitation sections of about 20 students each are led by GTAs.

Data Collection

The data for this study were collected in Spring 2023 during one of the first recitations of the semester. The topic of the activity was the continuity of functions. We focused on the task, modified from Straumanis et al. (2013), that probed students' understanding of the definition of continuity at a point by asking students to respond to five true-false statements (Figure 1).

Given five statements. For each statement, decide if it is true or false. For each false statement, provide a counterexample: sketch a graph of a function showing that the statement is false. Then, use another representation: a table, a symbolic equation, and/or words to explain why the graph you sketched disproves the statement.

1. If $\lim_{x\to c} f(x) = f(c)$ then f is continuous at $x = c$.
2. If f is not continuous at $x = c$, then $f(c) \neq \lim_{x\to c} f(x)$.
3. If f is not continuous at $x = c$, then $f(c)$ is undefined.
4. If f is not continuous at $x = c$, then $\lim_{x\to c} f(x)$ does not exist.
5. If f is not continuous at $x = c$, then either $f(c)$ is undefined, or $\lim_{x\to c} f(x)$ does not exist.

Figure 1. Definition of Continuity at a Point Task

Item (1) is a true statement, which is an abbreviated version of a semiformal definition of continuity at a point. The hypothesis $\lim_{x\to c} f(x) = f(c)$ suggests that both the limit at point *c* and the value of the function at point *c* exist and are equal to each other, and therefore, the function is continuous at *c*. Item (2) is the contrapositive of item (1) and is logically equivalent to it, thus it is also true. Items (3-5) contain false statements comprised of the elements of the contrapositive; the conclusion of item (5) is a combination of items (3) and (4).

Data Analysis

To analyze the data—students' written responses to the task—we used open coding (Patton, 2002), informed by the literature described in the theoretical background section. While we noted whether students responded correctly or not to each question, our main focus was on the reasoning they used to justify their responses. The following questions guided our analysis: Did the student's explanation rely on graphs, words, or both? If graphs were used, what kinds of graphs were they? If discontinuities were mentioned (verbally or graphically), what kinds of discontinuities were they? Were their reasoning strategies rule-based, or did they reason about the semantic meaning of the statements (or both)? Were students' justifications logically sound (did they reach valid conclusions based on true premises)? If logical mistakes were made, what was the nature of these mistakes?

Students' work for each item (Figure 1) was tagged and grouped in different ways according to the answers to these questions. Next, we examined trends in data across all five sub-questions. Due to space constraints, in this paper, we report mainly on the phenomena identified in student responses to individual items, followed by a discussion of general outcomes about students' reasoning about the definition of function's continuity at a point. Before describing the results, we emphasize that our goal in this analysis was not to make any assertions about students' conceptions or cognitive processes. Rather, we seek to understand how students reason about the definition of continuity at a point and how they justify their thinking.

Results

Overall, students were relatively successful in responding to all items in Figure 1.

Item 1, which presents a definition of continuity at a point had a 100% success rate. Four out of 84 students initially selected "false" but crossed it and wrote "true," possibly after discussing it with their peers. Most students' justifications (52) mentioned the definition of continuity at a point in some way. Of those, 17 students re-formulated the definition of continuity in their own words (e.g., "the function is continuous because the limit and the function value agree at a point"). 11 students wrote that the statement is true based on the "three rules of continuity" and stated these rules. We interpret these responses as rules-based reasoning. 19 students drew graphs of continuous functions in their responses, but only one student explained how their graph relates to the statement at hand. It seems that in these cases, the graphs served as illustrations, and students may not rely on these graphs to reason about the meaning of the statement, especially if they recognized it as a definition of continuity at a point.

11 students mentioned that given the statement, there cannot be any discontinuity at $x = c$ and therefore, the function was continuous at $x = c$. This is an attempt to apply contrapositive reasoning, where the negation of the conclusion implies the negation of the condition, or "eliminating counterexamples" (Yopp, 2020). While this reasoning could be valid, most students did not specify how they knew that there were no discontinuities, but some mentioned the different types of discontinuities they learned (hole/removable, jump, infinite).

Six students exhibited a "defined = continuous" misconception, previously identified in the literature (e.g., Amatangelo, 2013). For example, one student wrote: "If we are able to plug c into the function $f(x)$ then it must be continuous because it can approach that limit."

One student claimed that statement 1 is true unless the function and the limit are both undefined and therefore "equal DNE." The student seems to treat the acronym *DNE* (does not exist) as a number, which can be equal to another DNE "number," resulting in an equality

$\lim_{x \to c} f(x) = DNE = f(c)$. The student realized that a function cannot be continuous in such a situation but thought it implies that equality is true and should be treated as an exception.

Although this was a singular occurrence, it indicates the ambiguity of notation and a potential misinterpretation of what DNE stands for (undefined for a function and does not exist for a limit) and that the acronyms cannot be treated or equated as numbers.

Item 2 is a contrapositive of item 1. Despite the high success rate (80 students correctly responded "true"), the analysis suggests that it was a much more difficult item to comprehend compared to item 1. Of the students who correctly determined the statement was true, 11 merely restated it as their explanation. 23 students mentioned at least one rule of continuity. Of those, seven claimed that "if a function is discontinuous at a point c, it means it is undefined at that point," and three students said that "it means that $\lim_{x \to c} f(x)$ does not exist." Possibly, the students were showing different ways in which a function can be discontinuous, but it can also indicate a limited concept image of discontinuity, with students overlooking other ways in which a function can be discontinuous at a point. For example, one student wrote: "It is not continuous at $f(c)$ then that means there's a jump or stop somewhere, meaning the limit can't exist."

31 students drew at least one discontinuous graph in their explanation. Of those students, 24 drew a graph of a function with a hole or a removable discontinuity, while some students drew multiple graphs with different types of discontinuities. While these were correct discontinuous graphs, it is unclear whether students considered them as illustrations of where the statement is true or if they attempted to "prove" the statement with specific examples of discontinuous functions, thus exhibiting example-based reasoning (Rissland, 1999; Watson & Mason, 2006).

Rather than examining discontinuities, 21 out of 84 students used a rules-based approach. Working from the three-rule criterion of continuity, these students explained that since the function was discontinuous at $x = c$, it must not satisfy one of the three rules of continuity at that point. While this is a potentially productive approach, some students interpreted it narrowly, identifying specific rules that would be broken.

When students mentioned all three rules of continuity at a point, they seemed to struggle with negating their conjunction. For example, one student wrote: "Because the rule of continuous function does not allow $\lim_{x \to c} f(x) \neq f(c)$ to be continuous, all rules can be proven to be untrue." The student could be wrongly assuming that all three continuity rules must be simultaneously false or merely using "all" in the sense of "any" — a common confusion (Epp, 1999).

Six students justified statement 2 by proving its converse: If $f(c) \neq \lim_{x \to c} f(x)$, then the function is not continuous. Since definitions are true, biconditional statements, the students reached the right conclusion, even by using a logically problematic approach.

Items 3 and 4 are both false. Students performed well on both, using similar reasoning and exhibiting similar challenges. Most students (57 for item 3, and 53 for item 4) drew graphs showing counterexamples to justify their reasoning. Of these students, most (37 of 57 for item 3 and 38 of 53 for item 4) drew a graph of a function with a removable discontinuity at $x = c$. For item 3, these students drew graphs of functions where $f(c)$ was defined, and for item 4 $f(c)$ was not defined. Other students (33 for item 3, and 29 for item 4) mentioned different types of discontinuities which would constitute a correct counterexample to this statement. Similarly with the graphs, most of these students (24 for item 3, and 29 for item 4) mentioned removable discontinuities to prove this statement false. Overall, very few of the common misconceptions suggested in the literature appeared in these explanations.

Item 5 was the one the students found most challenging compared to other statements. 71 out of 84 students correctly responded "false," with 14 of them initially choosing the option "true". Many students noticed that item 5 is a combination of items 3 and 4 and used that in their justifications. They did this in a variety of ways, not always correctly. Most students either explicitly referred to or re-drew the graphs they used to disprove items 3 and 4: a graph of a discontinuous function with a hole (showing $f(c)$ undefined) or jump discontinuity with $f(c)$ defined but undefined or non-existing limit. The challenge appeared in making conclusions based on these examples. 11 out of 84 students incorrectly inferred from these graphs that the statement was true, not realizing they were "proving" the statement with supportive examples.

One possible reason for these types of responses could be students' limited concept image of discontinuity, restricted to only certain types, while a removable discontinuity would constitute a valid counterexample. Indeed, the majority of correct arguments relied on graphs with removable discontinuity, like Figure 2, where the student explained that "The graphed function is not continuous because $f(c) \neq \lim_{x\to c} f(x)$, but both $f(c)$ and $\lim_{x\to c} f(x)$ exists."

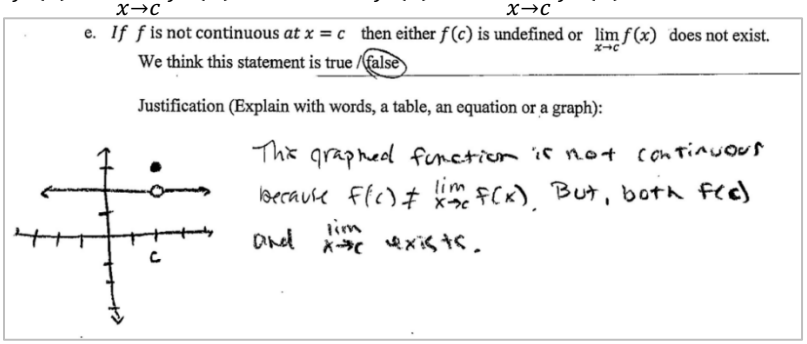


Figure 2. Correct response to item 5: a counterexample of removable discontinuity

Another explanation for incorrect conclusions about discontinuous functions whose graphs contain holes or jump discontinuities could be that the students were attempting to evaluate the converse of statement 5. That is, looking at graphs where the conclusion is false and inferring the function is not continuous. Whether this is the case or not was not always clear from written responses, but the argument in Figure 3a supports the assumption about evaluating the converse.

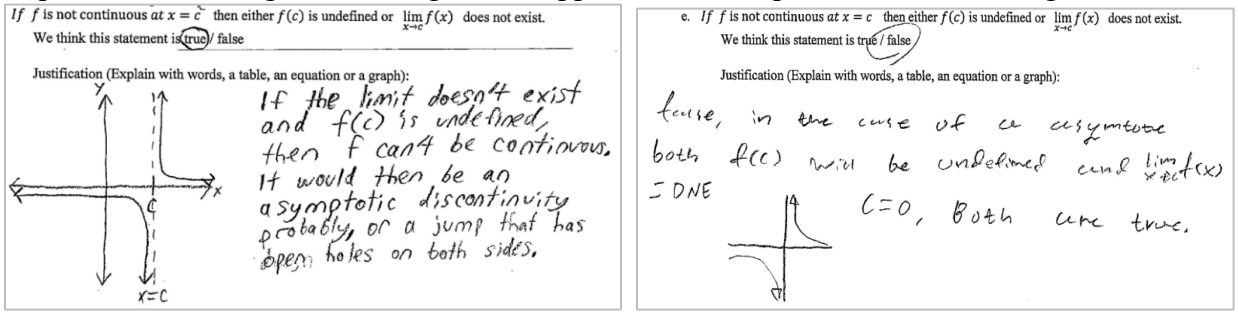


Figure 3. Different incorrect inferences from an asymptotic discontinuity. 3a (right): Evaluating the converse to "prove" the statement. 3b (left): Relying on exclusive disjunction to "disprove" the statement.

The student gave an example where both parts of the conclusion are true: "The limit doesn't exist, and $f(c)$ is undefined" but inferred that the hypothesis must be true also: "then f can't be continuous." Note that in the argument in Figure 3a, the student used "and" instead of "or." This was another recurring issue of students' difficulties with disjunction. Figure 3b shows one of the four instances of arguments where students attempted to disprove the statement by drawing a graph of a function with an asymptotic discontinuity at $x = c$ and explaining that the statement is

false because "Both $f(c)$ is undefined and $\lim_{x\to c} f\ (x) = DNE$." These students interpreted the disjunction as inclusive rather than exclusive and asserted that they found a counterexample where both parts of the statement's conclusion were true. Whereas in fact, such an example supports the statement. Challenges with negating disjunction, interpreting its meaning, or confusing it with conjunction led to 10 instances of incorrect reasoning on item 5.

Another example of this challenge appeared when students did not rely on graphs but on three continuity rules. This reasoning involved students claiming that the statement is false because: "All three rules need to be untrue in order for a function to be not continuous. This [statement] only specifies two of the three [rules]". Similar to the examples above, this argument shows a rigid application of continuity rules, but contrary to those examples, this is a clear invalid negation of the disjunction. It is also interesting to note that only 10 students mentioned $\lim_{x\to c} f\ (x) \neq f(c)$ in their answer to item 5 in some way, whereas in the previous items, students relied more heavily on the three rules of continuity.

Discussion and Scientific Significance of the Study

The results reported herein shed light on the variety of ways in which students reason about the definition of continuity at a point – an important concept of introductory Calculus. The results reveal that item 1, which presents a definition of continuity at a point, had a 100% success rate, while item 5—an incomplete contrapositive—had the lowest success rate. Some of the students' responses align with the common misconceptions identified in the literature, like conflating $f(c)$ defined or $\lim_{x\to c} f\ (x)$ with continuity (Amatangelo, 2013; Denbel; 2014; Juter, 2006). However, most of the reasoning challenges in our study were related to logical reasoning, especially the negation of the intersection of the three continuity rules ($\neg(V \wedge L \wedge E) \equiv (\neg V) \vee (\neg L) \vee (\neg E)$), interpreting the contrapositive and disjunctions (Brown, 2018; Dawkins & Cook, 2017; Yopp, 2020). Students' responses suggest that as the logical structure of the statement became more complex, it became more difficult for students to apply continuity rules in a rote way. It forced students to attend to the logical and semantic meaning of the statements, challenging their emergent understanding of the definition of continuity. Thus, students relied heavily on examples: graphs of discontinuous functions but often struggled to make inferences from these examples. Having limited example space (Watson & Mason, 2005) or a limited concept image of discontinuities (Tall & Vinner, 1981) often negatively affected students' reasoning. However, even when correct examples were used, drawing correct logical inferences from them was often challenging. This concurs with prior studies showing similar challenges in other mathematical contexts, like concluding the statement is true based on several examples or proving (or disproving) the converse (e.g., Buchbinder & Zaslavsky, 2019; Rissland, 1999).

The authentic context of the study, where students work in groups trying to make sense of the definition, distinguishes our study from previously reported in the literature, making for a unique contribution and capturing student thinking during concept construction. Focusing on written responses allowed us to analyze a rather large data set, but since our analysis is limited to written responses, we could not probe student thinking further. This seems critical since some of the students' responses were rather surprising. We intend to explore them more closely in the future.

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Students' Connections About Equivalence Across Contexts: Substitution and Known vs. Own Equivalences

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The concept of equivalence is fundamental to problem-solving in mathematics. Particularly, an oft-utilized feature of equivalence is substitution, in which the mathematics doer replaces one object with an equivalent object. Equivalence in general, and substitution in particular, have been identified as common threads that can tie together different areas of mathematics, but do students notice these connections in their own mathematics? In this paper, I report on task-based clinical interviews with students exploring the commonalities they identify among tasks implicitly involving equivalence relations. In particular, I discuss themes that students explicitly identified as relevant when answering the interview tasks: (1) substitution equivalence and (2) a distinction between students using a known equivalence for substitution or introducing their own equivalence.

Keywords: equivalence, substitution, actor-oriented, student-generated connections

Equivalence has two fundamental uses throughout mathematics: it is used to express the *sameness* of mathematical objects as well as their *interchangeability*. In fact, Liang (2021) proposed that equivalence "can be used as a thread to connect mathematical concepts and topics," but that much "remains to be done to analyze the nature of equivalence and substitution in different contexts, and in developing suitable tasks to provide students […] with opportunities to enrich their understandings of these ideas" (Liang, 2021, p. 43). Studies have extensively examined the substitutive aspect of equivalence with respect to equality, but it remains to be seen how the substitutive aspect of equivalence is conceptualized by students when working with non-equality equivalence relations, and moreover if students see these ideas as related across contexts. Thus, for this study I sought to answer the following research question: *What connections do undergraduate students identify among tasks involving equivalence relations?*

Background & Theoretical Framework

Actor-oriented Perspective

To examine whether students identified any commonalities across tasks involving equivalence in various forms, I adopted the actor-oriented perspective. This perspective was appropriate because my purpose was to "[inquire] into what types of common features students might perceive across cases" (Ellis et al., 2007, p. 227). Under this perspective, Lockwood (2011) defines a *researcher-generated connection* as a connection inferred by the researcher, i.e., "a connection that was not verbally articulated or borne out in some external activity" (p. 311) by the student, and a *student-generated connection* is one in which a student explicitly articulates or demonstrates that they view a particular connection. Student-generated connections are important to study because they are the connections that students are consciously aware off, thus students may more intentionally attend to or rely on them in their problem solving. In short, this framework provides researchers with an opportunity to examine what *students* see as relevant when identifying similarities between their current and past mathematical activity. Lockwood (2011) characterizes types of student-generated connections as:

- *elaborated* or *unelaborated* connections, where the former refers to connections in which the student explains what prompted them to draw a connection, and the latter refers to when students mention a connection but do not expound upon what led them to make this connection;
- *conventional* or *unconventional* connections, where the former "occur when students make connections to problems that may conventionally be considered isomorphic to the problem at hand" (p. 311), and the latter "occur when students relate situations that an expert or observer might not find mathematically isomorphic" (p. 312); and
- *referent type*, meaning the types of objects students draw connections to, including particular problems, problem types, and strategies or techniques for solving problems.

My work builds upon the existing work of Cook et al. (2022) and Richardson et al. (2024)—who presented *researcher-generated* views of the key commonalities in reasoning about equivalence across contexts—by examining students' views on the commonalities they attend to and identify when reasoning about equivalence across contexts.

Substitution Equivalence

Researchers have posited that a desirable conception of equality is a relational one, i.e., a recognition that equal mathematical objects have the same value (e.g., Baroody & Ginsburg, 1983; Jones & Pratt, 2012; Kieran & Martínez-Hernández, 2022). An elaborated, *full relational* conception of equality has been proposed as more completely capturing the nature of equivalence, which involves recognizing the sameness of mathematical objects and that one object can be substituted for an equivalent object (Carpenter et al., 2003; Jones et al., 2012; Jones & Pratt, 2012; Molina et al., 2009; Tirpáková et al., 2023). In particular, for the symbolic representation $x = y$, the sameness component would manifest as x "is the same as" y and the substitutive component would manifest as x "can be exchanged for" y (Jones, 2008, p. 154). I use the *sameness* and *substitutive* aspects as similarly applying to equivalence, whereby objects related by some equivalence relation have the properties that they are the same with respect to some property of interest and they can be freely substituted for one another.

Tirpáková et al. (2023) elaborated on the idea of substitution equivalence by introducing the idea of students' *own equivalences* to describe when students define an arbitrary equivalence for the purpose of using substitution in mathematics. This is contrasted with what I call *known equivalences*, which are equivalences that students do not define themselves and are instead recalled, proven to be true, or created via a sequence of appropriate transformations. For example, substitution of 0.5 with $\frac{1}{2}$ in simplification of the expression $\frac{2}{3} + 0.5 + \frac{3}{4} - \frac{4}{9}$ would be the use of a *known equivalence* because it is based on the numeric equality of the objects in question. Contrarily, using substitution of variables $u = 2x$ to rewrite the integral $\int_0^{\sqrt{\pi}} 2x \sin(x^2)\, dx$ as $\int_0^{\pi} \sin(u)\, du$ would be considered an *own equivalence* because it is an arbitrary, albeit convenient, replacement introduced by the mathematics doer and is not based on some known, given or discovered property of the given mathematical objects.

Introducing an *own equivalence*, such as in the integration example above, is important because it involves attending to the structure of the given problem without changing its content mathematically. Tirpáková et al. (2023) emphasize the importance of students' own equivalences as they can be introduced strategically to make the intended calculations more efficient or easier to perform. Moreover, Tirpáková et al. (2023) assert that "perceiving structure of the given equation allows the students to effectively connect their various knowledge of mathematics by

using appropriate substitution," and that by defining their *own equivalences*, students "can transform an unknown task into a task that they are already able to solve" (p. 486).

While the idea of substitution has been studied extensively as it relates to equality of expressions, this work has been done primarily at the level of researcher-generated connections, and it remains to be seen what findings are also applicable in more general instances of non-equality equivalence. Thus, my work addresses a gap in the literature by investigating the commonalities students identify in how they view equivalence (in particular, substitution equivalence) across contexts.

Methods

Data Collection and Task Design

I conducted task-based clinical interviews (Clement, 2000) with 16 students who were either currently enrolled in or had recently completed the last course in a three-course calculus sequence. Each student participated in an individual interview lasting between 90 and 120 minutes. Students' written work was recorded on a digital whiteboard iPad application and synced with an audio recording of the interview.

I selected this population of students because they would have had prior exposure to the mathematics required to complete each interview task, but they would not yet have been exposed to formal notions of equivalence relations. I conducted task-based clinical interviews because the purpose of a such an interview is to capture students' genuine interactions with the task setting in order to understand how they reason about a concept in the moment. My goal was to learn about students' current understandings of mathematical ideas—with the hope that the given tasks would elicit students mentioning ideas of equivalence or substitution—without deliberately influencing changes in students' thinking.

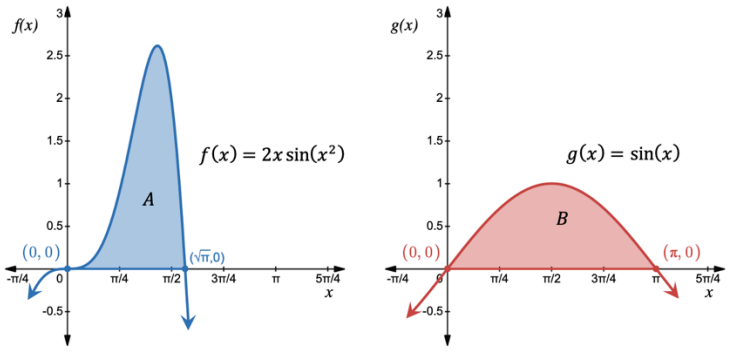
Task 1:	Simplify the following expression: $\frac{2}{3} + 0.5 + \frac{3}{4} - \frac{4}{9}$
Task 2:	Simplify the following expression: $\sin(2400°) + \cos(1110°) + \tan(585°)$
Task 3:	Show that the two shaded areas in the graphs below are equal: 
Task 4:	Solve the following equation on the set of real numbers: $\frac{x\sqrt[3]{x}-16}{\sqrt[3]{x^2}-4} - \frac{\sqrt[3]{x^2}-1}{\sqrt[3]{x}+1} = 17$

Figure 1. Tasks administered during the interviews.

The interview tasks (Figure 1) were selected because they could each be completed by using some form of substitution to make the given task simpler to execute, but the tasks involved

different contexts and slightly different notions of equivalence. For example, in Task 1, each term can be substituted with an equivalent fraction so that every term has a common denominator, and each original term is numerically equal to its substituted term (e.g., $\frac{2}{3} = \frac{24}{36}$). In Task 2, each angle can be substituted with its coterminal angle in the interval $[\mathbf{0}, \mathbf{2}\boldsymbol{\pi})$, and while these angles are not numerically equal (e.g., $\mathbf{585}° \neq \mathbf{225}°$), they are equivalent in the sense that they share the same position on the unit circle and thus would produce the same value when used as the argument for given trigonometric functions. The interview tasks also involved different notions of substitution. For example, the substitution discussed above for Task 1 would be considered using a *known equivalence* because it involves substituting one number for another which is known to be equal. Contrarily, the example of introducing $\boldsymbol{u}$-substitution to rewrite an integral (discussed in the "Substitution Equivalence" Section) for Task 3 would be considered using an *own equivalence*.

Data Analysis

The audio for each interview was transcribed verbatim and then coded using Clement's (2000) cycles of interpretive analysis. I focused on ideas that students mentioned repetitively throughout the interview and across tasks because I wanted to investigate what features of the tasks and their solutions that students deemed as relevant across a variety of contexts. In particular, I was interested in whether the students identified anything pertaining to equivalence.

I distinguished between what I deemed as common ideas that students appeared to recognize across tasks (*researcher-generated connections*) and what students *explicitly* mentioned as being common or similar across tasks (*student-generated connections*). That is, instances labeled as researcher-generated connections could have been connections that the students *implicitly* recognized but did not *explicitly* identify as being connections across tasks in their utterances or writing, i.e., there was not sufficient evidence to show that students did in fact see these as connections across tasks. Thus, connections classified as researcher-generated have the potential to be commonalities that students identify across tasks but are simply not made explicit by the students. In the results, I provide examples of explicit, student-generated connections; I used Lockwood's (2011) framework for categorizing the types of student-generated connections. While I was interested in all types of connections that appeared in the data, in the results I provide two examples of elaborated, conventional student-generated connections.

Results

I will now discuss two particular connections that students identified during their interviews. For brevity, I discuss selected excerpts from two participants: Jackson and Alex (pseudonyms). I consider these to be prototypical examples that are illustrative of the commonalities identified across contexts by multiple interview participants: *substitution equivalence* as a problem-solving strategy and the difference between *known* and *own equivalences*.

Student-generated Connection: Substitution Equivalence

Jackson identified his use of *substitution*—i.e., replacing one mathematical object with an equivalent one—as part of his strategy for answering several of the interview tasks. As evidence, Jackson continually referenced how he would "exchange" or "interchange" equivalent elements for the purposes of changing the given task into a "more convenient" format throughout multiple interview tasks. In particular, he expressed that he viewed equivalence as a necessary condition for being able to freely substitute mathematical objects.

In his work on the first task, Jackson simplified the expression by first writing each term as a fraction, then finding common denominators pairwise and simplifying left-to-right. He wrote 4/6 and 3/6 in place of 2/3 and 0.5, respectively. He then justified this step of his work by saying:

> *Jackson*: That ratio would be the same if you had 6 things and you picked 4 out of that. It's like the same ratio, but 4/6 was just more convenient to work with than 2/3 is. **And since they're the same thing, you can just interchange them.**

That is, Jackson used the idea of substitution as a solution strategy because he "interchange[d]" the fractions 2/3 and 4/6 and expressed that he knew this was allowed mathematically because "they're the same thing."

On the second task, Jackson subtracted the highest multiple of 360° that he could from each angle to convert into a coterminal angle between 0° and 360°. He then used the unit circle to read off or calculate the desired trigonometric values. When asked to justify his work, Jackson explained:

> *Jackson*: So if it's anything more than 360 you can just do, um, your starting number minus 360 because that'll take out the unnecessary rotation and then just give you the exact, like angle that you're looking for because **if it was 30 or if it was 390 those are the same thing because they both lead to the same spot.** […] And then whatever I'm finally left with, that's what gives me my angle that I can use.

He then clarified that he viewed *sameness* of angles in terms of their position on the unit circle, and expressed that this sameness allowed him to substitute one angle for another for the purposes of this task:

> *Jackson*: **For two angles to be the same on the unit circle they have to have the same end coordinates.**
>
> *Interviewer*: Okay. […] So as long as two angles […] have the same end coordinates, what happens?
>
> *Jackson*: **Then they can be exchanged** […] within trigonometric functions.

Similar to the first task, on the second task Jackson explained his process of "exchang[ing]" one object with an equivalent one that would yield the same end result as the original presentation of the task. In the first task this equivalence condition was numeric equality, whereas in the second task this equivalence condition was position on the unit circle. However, in both cases, Jackson expressed that this equivalence condition allowed him to substitute one element for an equivalent one and still preserve the solution to the original task. Moreover, Jackson was careful to clarify that his equivalence condition for Task 2 allowed him to exchange angles specifically "within trigonometric functions."

Until this point, I had viewed Jackson's use of substitution as a *researcher-generated connection*, because although Jackson was continually mentioning the same ideas of using substitution across tasks, he did not explicitly state that he did in fact view these uses of substitution as being instances of one unified idea. However, after completing all the interview tasks, Jackson summarized his work by saying, "**If two things are equal, then you can substitute them**, or like interchange them with whatever's more convenient." He expressed that in hindsight he recognized that he used substitution on every interview task:

> *Jackson*: But like, if it's the same thing, you can exchange it with something else and that makes it a lot easier. And that's kind of like one of the core [ideas] of math I guess because you use it like a lot more than you would think. Like **I used it for every problem on this without realizing it**, like, yeah.
>
> *Interviewer*: And you used what on every problem on this?

Jackson: Uh, just like **exchanging a different form of what I'm working with if it's more convenient to work with it in that form.**

Because of Jackson's repeated identification of substitution as a strategy for turning a given task into a "more convenient" form, as well as this substitution being allowed as long as the objects being substituted are "equal" within the given context, and Jackson's recognition that these were ideas he used on every task, I claim that Jackson viewed substitution equivalence as a strategy connection across the interview tasks.

Student-generated Connection: Known Equivalences and Own Equivalences

Alex elaborated on a difference between using a *known equivalence* versus defining his *own equivalence* to make substitutions that resulted in the task being easier to complete. I classified Alex as having made a *student-generated connection* because he explicitly discussed how different types of substitution manifested across tasks.

Alex used the metaphor of putting a "mask" on a complicated part of a task in order to temporarily ignore a complicated part and make the task easier to understand. He identified that this "masking" of part of a task was allowed as long as it was temporary, i.e., it is undone at some point before the task is completely answered. On Task 3, Alex applied this metaphor to the process of integration via *u*-substitution. He began the problem by writing the integral $\int_0^{\sqrt{\pi}} 2x\, sin(x^2)\, dx$ to represent the first desired area A, and then used *u*-substitution to evaluate this integral.

Alex: So for that process, that is, um, *u*-substitution, and that is a process of simplifying complex integrals to make them easier to calculate. Um, so the way that works in my brain is I think of it like **taking the difficult parts and putting a mask over them. So they're still there, but you don't have to look at them.** Um, so you can compute it without dealing with the complicated parts of those, **and then once you're done with that, you can take the mask off and put back the parts that you need to.**

Alex then clarified that that in the integral $\int_0^{\sqrt{\pi}} 2x\, sin(x^2)\, dx$, he used u as the "mask" for x^2 and then used his *own equivalence* $u = x^2$ to determine that du should be the "mask" for $2x\, dx$. When asked how the "masked" integral $\int_0^{\pi} sin(u)\, du$ compared to the original integral he wrote, Alex said that the integrands were "not *perfectly* equal because if you were to just say, um, $2x\, sin(x^2)$ is equal to $sin(u)$, that wouldn't be right." However, he then clarified that this substitution was appropriate as long as the substitution $u = x^2$ was made explicit and the final answer for the task was given in terms of the original variable *x* instead of the chosen variable *u*.

Furthermore, Alex expressed that he saw the type of substitution used on Task 3 as a different type of substitution than he had used on other Tasks. For example, when comparing Task 3 to Task 1, Alex said:

Alex: **But that's not a mask that's more switching it,** like the same way as 0.5 and ½ [from Task 1] are equal because that gets you straight to the answer. You don't have to backtrack to get back to the same […] terms that you were in. […] Um, because I can't solve it in terms of *u*. It has to be solved in terms of *x*.

In this excerpt, Alex identified that the substitution of 1/2 for 0.5 in Task 1 would "[get] you straight to the answer," whereas in Task 3, after the substitution of $u = x^2$ was made, he would "have to backtrack" or undo his substitution in order to solve the task in terms of the original variable *x*.

Alex explicitly stated that substitution was a strategy he used across multiple interview tasks, and that some type of equivalence was a necessary condition for these substitutions. In fact, Alex went so far as to say that he used some form of substitution on *every* interview task. However, Alex additionally contrasted problem types by distinguishing between the different ways that equivalence allowed him to use substitution, i.e., using an equivalence that was already known versus one that he introduced himself:

> *Alex*: Um, so while for all of these, **you do need to interchange and you do need to, um, like set them equal to something else to be able to solve the problem**, for u-substitution it's necessary to be a mask because you have to be able to take part of that mask off afterwards and be able to plug part of the original part back in. **Whereas for the other ones, you don't have to so it doesn't really need to be a mask, it just needs to be like they're the same in like equivalency, the same in nature.** Um, I think of it as a mask because it still means that there's something behind it instead of being replaced by an equivalent term.

I interpret Alex's use of the phrase "equivalent term" in the last sentence to refer specifically to *known equivalences*. Thus Alex is distinguishing between substitution with his *own equivalences* where there is "something behind" the substitution and he needs to "take […] that mask off" before finishing the task and substitution with *known equivalences* that he can use freely without needing to undo or account for these substitutions later in his problem-solving process.

Discussion

My findings address a gap in the current literature in several ways. First, my work extends the work of previous researchers by shifting the attention from the commonalities that *researchers* identify in students' utterances and work across contexts (e.g., Cook et al., 2022; Richardson et al., 2024) to focus on what *students* explicitly identify as being common across contexts. In particular, Jackson's interview provides evidence that *substitution equivalence* is a commonality that students identified across contexts, and Alex's metaphor of using a "mask" for temporary replacements provides evidence that students identified a distinction between using *known equivalences* or introducing their *own equivalences*. I reiterate that Jackson and Alex were not the only students who explicitly identified these connections, nor were the instances documented here the only connections that students made. Rather, these excerpts provide examples to illustrate that these students made elaborated, conventional connections consistent with what researchers agree is important for students to attend to when their work involves equivalence.

Future research building upon this work could examine the ways in which students' conscious attention to these connections affects their problem-solving. Work can be done exploring the affordances and constraints of students' explicit attention to substitution equivalence and known vs. own equivalences, and how students can leverage their awareness of these ideas in learning new mathematical concepts. As a potential instructional implication, introducing *own equivalences* as a problem-solving method can help student students to turn a complicated problem into one they know how to solve in contexts other than integration by u-substitution (e.g., in Task 4, temporarily replacing $\sqrt[3]{x}$ with a placeholder, or "mask" variable).

Acknowledgments

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Enacting Active Learning in Calculus: How One Graduate Teaching Assistant's Instruction Shifted with an Experienced Learning Assistant's Support

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This study explores the changes in a graduate student teaching assistant (GTA)'s teaching actions over two semesters in an active learning Calculus I course. Cole, a GTA with little teaching experience and no prior experience with active learning, was paired with an experienced undergraduate learning assistant (LA) in his second semester teaching in a purposefully designed active learning-based Calculus I course. Cole's teaching actions varied more when paired with the experienced LA, including short periods of lecture and classroom discussion with the facilitation of group work. This case study provides insight into how an experienced undergraduate LA can be an agent of change in an active learning-oriented classroom.

Keywords: Active Learning, Calculus, Teaching Practices, Graduate Teaching Assistants, Undergraduate Learning Assistants

At the undergraduate level, introductory Calculus is typically a core requirement for degree programs across science, technology, engineering, and mathematics (STEM) disciplines. However, Calculus can act as a barrier to students' continuation in such programs, with many students who fail calculus ultimately leaving STEM majors (Smith & Funk, 2021). Because of this relationship, many are concerned with the currently high rates of student drop, withdrawal, or failure from calculus courses (Bressoud & Rasmussen, 2015).

Previous research suggests that active learning is an effective means of lowering DWF rates (students who receive a D, F or withdraw from the course) in Calculus courses (Bressoud & Rasmussen, 2015; Freeman et al., 2014) and cultivating more positive attitudes toward STEM subjects (Tutal & Yazar, 2022).While active learning is conceptualized in many ways (Ting et al., 2023), we offer the following literature-based definition to guide our work: active learning is marked by a learning environment which provides meaningful opportunities for all students to learn mathematics, by incorporating engagement with cognitively demanding mathematical tasks, and instructor facilitation of collaborative learning opportunities, including the promotion of both small and whole-group discussions (CBMS, 2016).

Due to its documented success in cultivating positive content-related and affective outcomes for students, university administrators and faculty within mathematics departments are interested in transforming large-lecture introductory Calculus courses to include active learning practices. Practically, the transformation of such courses requires "buy-in" from a number of individuals, including course instructors and graduate teaching assistants (GTAs) who are responsible for the teaching of the course. For instructors, the implementation of active learning can require construction of high-level tasks to engage students, moderation of classroom discussions, and facilitation of group work, which are distinct from the skills required to teach traditional, lecture-based courses. At larger universities, many Calculus courses rely on the teaching of graduate students, either as instructor of record or as a recitation section instructor. These GTAs often have scant teaching experience, speaking generally. They also typically have little experience with active learning as a student, meaning that they have few examples to emulate when asked to teach in active ways. Consequently, recent efforts have been made to create and provide

professional development to GTAs regarding the implementation of active learning and other evidence-based strategies (Deshler et al., 2015), but still little is understood about GTAs' teaching practices and their development.

Motivated by this under-researched topic, we conducted a previous study (Sager et al., 2023), in which we examined the conceptions and teaching actions of a group of Calculus GTAs with varying levels of background in teaching and experience with active learning. We found that more experienced GTAs tended to scaffold their small-group work time with short bursts of class discussion and content review, while less experienced GTAs tended not to incorporate such scaffolding. One GTA from this study, Cole, had had little previous teaching experience and no experience with active learning, and showed little scaffolding of his group-work time during his first semester implementing active learning. However, our research team noticed significant qualitative differences in his classroom environment during his second semester as a GTA when he was paired with an undergraduate Learning Assistant (LA) who had ample experience teaching and learning in the active learning environment at the university. In this current study, we specify these noticed differences, asking: how do Cole's teaching actions and perspectives on active learning shift as he gains experience with different LAs in an active learning Calculus I course, and how are these shifts related to the actions and perspectives of the LAs? Cole's case contributes to our understanding of GTA professional development in the active learning context by showing how GTAs development and the broader classroom environment can be impacted by LAs.

Literature Review

The important role of GTAs in the introductory STEM experiences of university students has led to increased interest in professional development (PD) programs for GTAs (Braley, et al., 2023). PD programs allow GTAs to interact with student-centered and evidence-based practices, such as active learning (Archie et al., 2022, Deshler et al., 2015). Many GTAs have little to no teaching experience, and have not experienced student-centered teaching in a classroom, which leads to limited confidence in implementing these strategies in their own classrooms (Deshler et al., 2015). Additionally, while GTAs may believe students learn best through active learning or other student-centered strategies, they often do not act on this belief, instead teaching how they were taught (Patrick, et al., 2021). GTAs also need a way to gain the pedagogical content knowledge and knowledge of student thinking central to effective teaching (Kung & Speer, 2009). The research on graduate students' background and their implementation of student-centered learning strategies is sparse.

While GTAs can act as facilitators of active learning in large-lecture courses, an additional approach to facilitation is through the incorporation of undergraduate LAs into the course structure. An LA is an undergraduate student who has successfully passed the course previously and returns to aid in the teaching of that course, typically by facilitating discussion amongst small groups of students. In addition to their time in the classroom, LAs also typically participate in a pedagogy course and meet weekly with instructors to discuss course content. While much of the research around the LA model of instruction has focused on ascertaining the benefits to both students and LAs involved in this instructional model (Barrasso et al., 2021), recent efforts have centered on aspects of departmental change, since this was an original goal guiding the design of this model (Otero et al., 2010). In particular, the nature of LA-faculty partnerships in the instruction of LA-supported courses has been of interest. Faculty members working with LAs have reported improvements to their course designs, including increased emphasis on active learning classroom activities, and that LAs present them with new pedagogical perspectives (Hill

et al., 2024; McHenry et al., 2010). Others have begun to conceptualize how these faculty-LA relationships impact course design and particular instantiations of the LA instructional model (Davenport et al., 2017; Karch et al., 2024). While the literature suggests that these faculty-LA relationships influence both instructors' professional development and the design of the course, the impacts of LAs' relations to other members of the instructional team remain largely unexplored. Our study contributes to this area by exploring the relationship between a GTA and LA, detailing the impacts of this relationship on classroom instruction and this instructor's conceptions of teaching and learning.

Data and Analytic Frameworks

This case study takes place within a larger project focused on improving introductory STEM courses at a large research university in the Northeastern US. One component of the project was the introduction of active learning into the Calculus I course. The course follows a lecture-recitation structure, where the students meet with a faculty member for a lecture section (of approximately 120 students) three days a week, and with a GTA in recitation section (of approximately 20 students) the other two days. Undergraduate LAs were introduced within the recitation sections to support group learning, an integral part of the active learning incorporated into Calculus I recitations.

To promote active learning, a new recitation structure was suggested to GTAs: a short introduction, followed by small group work, and whole class discussion. GTAs were given autonomy to adapt the suggested structure to fit their classroom. At the beginning of each semester, the course team (faculty instructors, GTAs and LAs) met with project team members to discuss the active learning and recitation structure, ask questions, and give and receive advice about best practices. In addition, all first-year TAs in the mathematics and statistics department are required to take a full year course discussing pedagogy and other topics pertaining to GTAs' teaching, including active learning. Similarly, the LAs are required to take a pedagogy course designed for peer coaches. Both LAs and TAs also meet with their peers and course instructors weekly to discuss upcoming course topics, but also to share feedback and suggestions related to course activities.

This case study focuses on one GTA: Cole. Cole was a beginning Ph.D. student in statistics and served as a GTA in this active-learning recitation during the Spring 2023 and Fall 2023 semesters. During his first semester as a GTA in the active learning context, Cole taught with an LA named Ann, a second-year mechanical engineering major who was a novice LA. However, during the Fall 2023 semester, Cole worked with an LA named Fran, a junior environmental engineering major who had been a mathematics LA for the three previous semesters.

In this study, we focus on the case of Cole because the research team noticed significant qualitative differences in Cole's classroom between these two semesters, with many of these differences seeming to be initiated or introduced by the LA, Fran, in the second semester. While Cole's Spring 2023 classroom practices and conceptions of teaching were analyzed in a previous study (Sager et al., 2023), here we examine how Cole's actions and conceptions change over time, connecting these changes to his relationship with Fran. The data for this case study includes 360-degree video recordings of Cole's recitation sections across the Spring 2023 and Fall 2023 semesters and as well as two interviews with Cole (one from the end of each semester) and one end-of-semester interview with each LA. Video recordings were conducted during six recitation sections each semester consisting of class sessions in which students worked on researcher-designed, conceptually-oriented activities. The semi-structured interviews with Cole focused on his perception of, experience with, and use of active learning. The LAs' interviews

we also semi-structured, but included stimulated recall, with each LA reflecting on video clips of their interactions with small groups and discussing their general thoughts on mathematics teaching in this context.

In order to capture the actions of Cole, Ann, Fran, and the students in the classroom, the Classroom Observation Protocol for Undergraduate STEM (COPUS) (Lund et al., 2015) was used to code the recitation video recordings. COPUS is an observation framework which applies codes for instructor and student actions to every 2-minute interval of the recording. Traditionally, COPUS is used to categorize an instructor's teaching (e.g., lecture-based or active), specifically for the purposes of professional development. However, we leverage COPUS as an analytic tool to provide a snapshot of recitations across the semester. For more information about COPUS and codes in the framework, see Lund et al. (2015). Two recitation recordings were chosen for each semester – one from the first third of the semester, and one from the last third, based upon the availability within the data set. The content of the early semester activities was the same for both semesters (continuity), while the content of late semester activity was different for the two semesters (antiderivatives and logarithmic differentiation, respectively). Nevertheless, each recording took place during the implementation of researcher-designed, conceptually oriented activities. Each classroom recording was coded by two members of the research team resulting in an interrater reliability of 92% and disagreements were discussed until a consensus was reached. The codes for Cole, the LAs, (Figure 1) and students (Figure 2) were aggregated over the two recordings for each semester in order to provide a profile of the classroom over the entire semester.

The GTA and TA interviews provided context and background to the GTA and LA relationship and Cole's actions in the classroom. Interviews were coded using open and thematic coding to capture the changes in Cole's definition of active learning and the GTA and LA relationship in the classroom (Miles et al., 2018; Strauss & Corbin, 1998).

Results

In this section, we compare the classroom actions of Cole, the LAs, and the students during Cole's first and second semesters as GTA in Calculus I. While Cole's actions and perceptions during the Spring 2023 semester are more extensively detailed in Sager et al. (2023), we provide an overview here to contextualize our new analysis of his LAs' actions, their perceptions, as well as his evolving conceptions and practices in the Fall 2023 semester. We illuminate the changes that occurred when Cole worked with an experienced learning assistant.

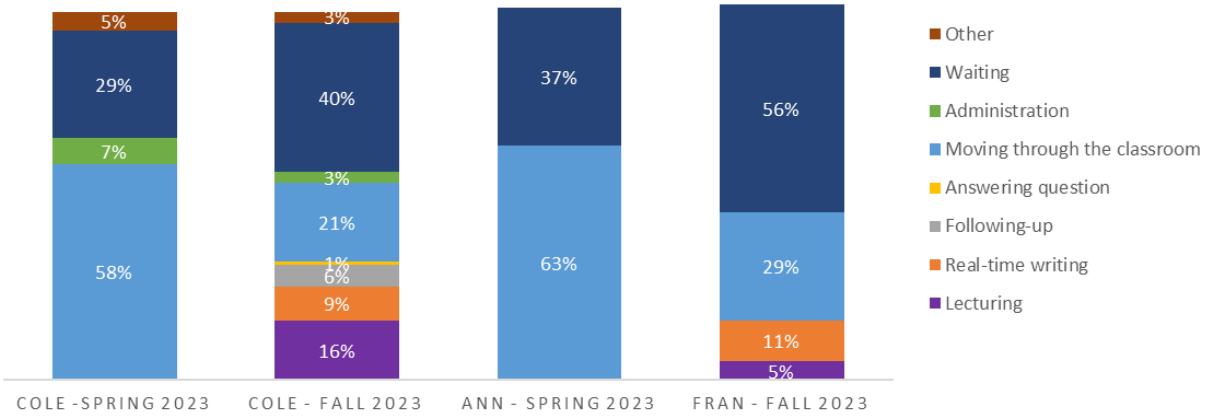


Figure 1. Instructional actions of Cole and his learning assistants.

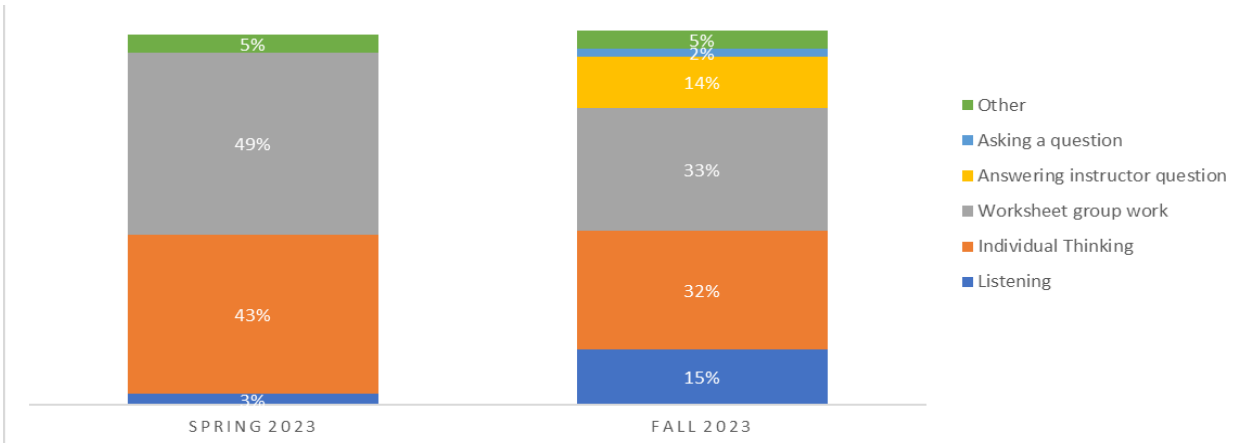


Figure 2. Classroom actions of students in Cole's classrooms.

Cole was a first-year Ph.D. student in statistics in Spring 2023, when he first acted as a teaching assistant for Calculus I. At this time, he was in his second semester of teaching as a graduate student, having taught precalculus without an LA or conceptually oriented activities in his first semester. His only teaching experience before entering the Ph.D. program was as a virtual one-on-one tutor in a program for gifted elementary students. He reported no previous experience with active learning as a teacher or a student. He "barely asked [his] teacher a question" as a student, and he would "pretty much work alone."

In contrast to his own experience as a student, Cole's working definition of active learning during the Spring 2023 semester, his first time teaching in an active learning setting, was "engagement." He believed that "most of [the] students would just copy paste notes and just leave it there. So, they learn nothing, but they think they learned a lot." Engagement and interaction were his antidote to the "copy paste" mentality of students. His actions in Spring 2023 from Figure 1 reflected his emphasis on interaction with the students -- 58% of his classroom actions were m*oving and guiding*. *Moving and guiding* was coded when the instructor moves around the classroom, interacting with students working on the activity individually or in groups. Cole's emphasis on allowing students to engage with the activity was also reflected in the student actions from Spring 2023 as well (Figure 2), as 92% of their actions were either *group work* or *individual work*.

Cole described himself as a "tutor" in his first semester as a Calculus I GTA – his first semester in a purposefully-designed course using group work and active learning. The lack of any *lecturing*, *follow-up* or *real-time writing* in his actions corroborates this description. He described that his role in the classroom was to "learn if [students] are active, thinking or learning or not ... I definitely will walk around so I'll ... check their work." In contrast, he described how students from his previous semester without active learning would "just listen to me and then copy what I wrote on a board," and characterized himself as a "lecturer." His "tutor" status was also reflected through an abundance of *waiting* codes. Because he viewed himself as a tutor, he waited for students to ask for assistance.

Ann acted similarly to Cole in the 2023 semester. In Figure 1, we see that Ann spent her time in recitation either interacting with groups (63% of her COPUS codes were *moving and guiding*) or *waiting* to interact with students (37% of her COPUS codes were *waiting*). While she spent a significant proportion of her time *waiting* to assist students, she explained in her interview that this was because, as a novice LA, it took her some time to "get more comfortable with the students" and be able to "go out of my comfort zone and initiate questions with the students". Additionally, her primary engagement in *moving and guiding* is consistent with the

responsibilities of an LA in this context as they were primarily facilitators of small group discussion. It is also consistent with her description of her role as an LA, as she stated her role is "answering questions" and acting as a "second brain" for Cole. That is, as a recent learner in the course, she provided a "second way to think about" course content and reported that Cole often sought her advice when addressing student questions, saying "when he felt like he couldn't answer it that well, he would come to me and ask, can you explain this?". From Ann's description, we see Cole's disposition toward collaboration with his LA, with Cole actively enlisting the expertise of his LA as a recent Calculus learner.

While Ann's actions of *waiting* and *moving and guiding* were consistent with and typical of the responsibilities of an LA in this environment, Fran showed more variation in her classroom actions. We see that Fran also engaged in *writing on the board* (11% of her COPUS codes) and *lecturing* students (5% of her COPUS codes). *Writing on the board* included instances when Fran would work out a solution at the chalkboard at the front of the room, walking through the steps of the problem by soliciting these steps from students around the room. She also briefly lectured about course content, explaining general concepts to students. Like Ann, these actions were consistent with her description of her classroom role, saying:

> "[Cole] and I work really well together. I feel like we pretty much have the same role. He obviously is grading the homework and stuff like that. And towards the end of class, if there's an overall concept announcement or we want to go over L'Hospital or something, he'll do that as the TA. But working in the classroom, we pretty much do the same thing"

In contrast to Ann, who viewed herself as more of an assistant to Cole, Fran viewed herself and Cole as having nearly identical roles. This alternate view was rooted in Fran's previous LA experience, since she previously worked with an experienced GTA who supported her in taking on a more active role, saying this GTA made her "feel like another TA in the class". While Fran had a higher proportion of *waiting* codes than Ann, this is attributable to both a smaller class size in Fran's semester (i.e., there were fewer students to engage with) and the increase in Cole's actions related to whole-class engagement. Overall, we see that Fran's previous experience as an LA paired with a supportive and experienced GTA, empowered her to take on a more active role in her teaching with Cole.

We see in Figure 1 that Cole's actions in the classroom changed when working with Fran. In addition to the *moving-guiding* and *waiting* codes, 16% of his COPUS codes were *lecturing*, and 9% were *writing on the board*. We notice that Fran also includes *lecturing* and *writing* with the *moving and guiding* and *waiting* expected from an LA facilitating group work. Often, Cole built upon Fran's engagement of the whole class, adding a short explanation (*lecture*) addressed to the class, or to *follow up* (6% of his COPUS codes) on a student question or problem from the activity. While Cole, like Fran, had a high proportion of *waiting*, the class was smaller and more independent in their group work, leading to less time where Cole and Fran were working with the groups.

Cole's perception of his role in the classroom also changed after working with Fran. In his interview at the end of the Fall 2023 semester, he talked about how he had "transferred [himself] from the teacher to the guide" and pointed out that his new goal was to "mix these things together." He sought to combine the "lecturer" of his first semester of teaching with the "tutor" of Spring 2023. His classroom actions indicate that he was successful in meeting this goal. His students' actions are also much more varied in Fall 2023: only 65% of their actions are either *group work* or *individual thinking* on the activity, compared to 92% in Spring 2023. In accordance with Cole's change to include more *lecturing*, 31% of their actions are *listening,*

answering instructor questions or *asking questions*. Some of the *other* codes for students include a student writing an answer on the board and explaining the problem to the class, a contrast to the *other* codes in Spring 2023, when the *other* codes were exclusively for off-task students or packing up to leave class.

When asked what advice he had for GTAs implementing active learning in the future, his response was that they should "learn from [Fran]." He mentions that she is "able to teach on the stage" and also adept at "teaching everything to the students". That is, Cole learned from Fran's example how to incorporate appropriate scaffolding through whole-class engagement into his teaching in the active setting. Cole and Fran take a similar role in the classroom, with Fran diverging from the traditional role of an LA, and Cole welcoming and assimilating her teaching actions.

Discussion

Cole's teaching actions changed dramatically in his second semester of leading an active learning recitation for Calculus I. In his first semester, he took the relatively passive role of a "tutor", spending a significant amount of time waiting for questions and interacting directly with the groups. His only interaction with the whole class was administrative in nature – giving directions or reminders. His LA, Ann, used similar actions, implementing the standard role of a learning assistant. When working with Fran the following semester, he incorporated more lecture and class discussions into his recitation. This more varied set of teaching moves led to more engagement with students, and by Cole's own definition, more active learning.

While we cannot point to any single reason for this change, Cole and Fran both describe their relationship as an important aspect of the classroom dynamic. Cole was going into his second semester in an active learning facilitation role, while Fran was in her fourth semester as a learning assistant. Fran was given the opportunity to take on more responsibility than a traditional LA, and Cole varied his teaching actions in a similar way to her. Cole cited Fran as an inspirational figure for teaching, indicating that she impacted him through their time working together.

While including LAs in a course is often viewed as a professional development experience for LAs themselves, we offer evidence that this relationship can be bidirectional, with LAs also acting as model teachers for the instructors with whom they work. Experienced LAs can be agents of change in promoting practical active learning teaching actions in the classroom. Our study has practical implications for GTA professional development programs, showing how novice GTAs can be supported by the presence of LAs in their classrooms and suggests that GTAs can benefit from real-time illustrations of pedagogy in their institutional contexts. Likewise, our results carry implications for LAs' pedagogy courses, showing that such courses should support LAs in confidently enacting innovative teaching techniques.

Overall, we emphasize that the collaboration between LAs and instructors, especially when they differ in experience, fosters a mutually beneficial professional growth process. This partnership can also enhance the implementation of active learning practices in more scaffolded ways. Our findings suggest that expanding LA programs and integrating them more closely into GTA professional development can play a critical role in advancing teaching effectiveness in higher education.

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Actively Assessing and Engaging With the Definition(s) of Active Learning

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Graduate student instructors (GSIs) in mathematics play a pivotal role in shaping undergraduate education and are the future of collegiate mathematics faculty. As part of their development, GSIs are expected to engage in teaching-focused professional development (TPD), particularly in evidence-based strategies like Active Learning (AL) methods. However, higher education is only beginning to explore how to effectively measure GSIs' growth in teaching skills through such TPD. This study examines the learning process of 47 novice GSIs from three universities, specifically focusing on their evolving understanding of AL before and after participating in TPD. By analyzing the GSIs' own definitions of AL, the research highlights changes in their knowledge and alignment with the intended TPD outcomes. The findings provide insight into the effectiveness of TPD on AL, while also offering recommendations for structuring future evaluations of TPD impact on GSI teaching knowledge and skills.

Keywords: Active Learning, Graduate Student Instructors, Teaching Assistants, Professional Development, Assessment

Mathematics graduate students are a substantial proportion of the undergraduate teaching workforce in the United States (Belnap & Allred, 2009; Speer et al. 2010) and their pedagogical knowledge and methods are greatly influenced during this time as novice college mathematics instructors (Deshler et al., 2015; Speer and Murphy, 2009). During their experiences as graduate student instructors (GSIs, graduate students who are instructors of record for a mathematics or statistics course), research has indicated that teaching-focused professional development (TPD) improves GSIs' knowledge and skills for effective teaching (Deshler et al., 2015; Yee & Rogers 2017; Hauk & Speer, 2023). Decades of work have also demonstrated that active learning (AL) approaches—using teaching strategies that actively engage all students in the classroom—are among the most effective (Freemen et al., 2014; Gobstein et al., 2016). AL has received significant discussion on its value within the classroom, with overwhelming recognition of its success *with appropriate TPD* (Reinholz, 2023). For example, Reinholz et al. (2022) reported on how and why supervision of collaborative learning is critical for AL strategies to be effective. Indeed, without addressing a classroom culture, student collaborations may include microaggressions that drastically limit the effectiveness of AL. Thus to prepare GSIs for implementing AL well, they need equitable and effective TPD (National Academy of Science, Engineering, and Medicine, 2023).

How can we know if TPD is effective? Specifically, how can we gather data and assess the impact among GSIs of AL-focused TPD opportunities? It requires data upon which to evaluate effectiveness, which, in turn, requires data collected around the objective or aim of the TPD (Yee et al., 2024; Deshler et al., 2015). In recent years, research in undergraduate mathematics

education has begun exploring ways to evaluate the effectiveness of TPD (Yee et al, 2023; Yee et al, 2024).

Purpose and Rationale

The purpose of this study was to research how collecting and analyzing data on GSI learning in an AL-focused TPD activity might be part of evaluation of TPD effectiveness. Prior research has indicated that to improve practical understanding of AL, instructors need to (1) recognize exemplars through readings about AL methods (McConnell et al., 2017), (2) visualize and imagine themselves using varying AL methods (Yee, 2021), and (3) become aware of the nuances inherent in the enactment of AL (Laursen & Rasmussen, 2019). Our approach was to focus on how GSIs' *definitions of AL* changed during their TPD experience.

Unpacking GSIs' meaning making for educational definitions is important particularly for GSIs in mathematics where deductive and axiomatic structures put significant weight on definitions. For example, a series cannot both converge and diverge because the definition of convergence and divergence is binary, founded in the logic that a mathematical statement is either true or false, hence *dependent upon the definition* (Vinner, 2002). Thus a mathematical definition's utility is uniform. However, definitions in educational research rely on human participants and evidence, as seen in the diverse use of AL across different disciplines. How a chemistry lab uses AL is distinct from how a mathematics classroom uses AL because chemistry labs inherently engage students through lab experiments, resulting in facilitating labs being the focus for AL. Our rationale for this study emphasizes GSIs' definitions of AL as dynamic and not static. To this end, our research question for this study was: *How do novice GSIs' definitions of AL change after participating in an AL-focused professional learning activity?* For this study, novice refers to GSIs who are within their first two years of teaching.

Relevant Literature for Framing the Professional Development

Dimensions and Pillars of Active Learning

Four pillars for AL support instructors in developing a definition of AL: (1) Students engaging deeply with coherent and meaningful mathematical tasks, (2). Students collaboratively processing mathematical ideas, (3) Instructors inquiring into student thinking, (4) Instructors fostering equity in their design and facilitation choices (Gobstein et al., 2016; Laursen & Rasmussen, 2019; Smith et al., 2021; Yee et al., 2024). Recent work in developing TPD for operationalizing a definition of AL has come out of the SEMINAL project (and coding of over 100 mathematics instructor definitions of AL; Williams et al., 2020). In leveraging this previous work, we sought to attend to novice GSIs with their beginning understanding of AL. This led to another dimension of interest: interconnectedness. Together with the four pillars, the dimensions reflected on and considered in our AL-focused TPD and research were: (1) Role of students, (2) Role of mathematics, (3) Role of the teacher, (4) Role of equity, (5) Interconnectedness of the other roles. The fifth dimension was added because our research team agreed that how the four roles are connected is just as important as any single role. For example, the interconnectedness dimension focused on if novice GSIs could recognize that the role of equity affects the role of the teacher and students.

Critical Reflection by Assessing Definitions

The larger scope of this work strives to understand how GSI conceptions of AL for productive use in the classroom change over time. This study focuses on definitions for multiple

reasons. We understand how educational definitions can be perceived differently from mathematicians' use of definitions (Tall & Vinner, 1981). Ultimately, learning to integrate definitions for teaching is a process. This study is not trying to argue or search for a "right" or "correct" definition of AL. Indeed, Williams et al.'s (2022) work shows the diverse ways in which experienced mathematicians view AL. The goal of this study was for GSIs to reflect on how their AL definition might change as they built experience as instructors. Consequently, the AL-focused TPD activity was designed to have GSIs critically reflect (Brookfield, 2017) on AL definitions by actively engaging in assessing their own and others' definitions. To that end, similar to other classroom practices used to help novice GSIs use AL (e.g. Observation Protocols, Yee et al., 2022), this study did not use assessment *of* AL understanding, but rather assessment *for* AL understanding.

Description of Module, Lessons, and Activities

The AL-focused TPD module included two 50-minute sessions with novice GSIs as (Lesson 1 and Lesson 2) as well as pre- and post-lessons assignment. Figure 1 describes the module.

Overarching Goal of Module*: The goal of this unit is to have novice GSIs grow their definition, understanding, and implementation of AL. *Full description of activity is at (Yee et al. 2024)*	
Pre-Module Assignment: 1. **Read** Active learning in the college classroom by Faust and Paulson, 1998. 2. In the context of Mathematics teaching and learning, please type **your own definition** of active learning in as much detail as possible.	
Lesson 1 Objectives:	**Lesson 1 Activities (50 Minutes)***
● GSIs will Identify what students and instructors can do with a selected set of active-learning strategies. ● Justify their decisions on what students and instructors can do with a selected set of active-learning strategies in their classroom.	● GSIs are given a list of 10 common GSI strategies ● GSIs are given a table with rows indicating ways students are engaging during the activity and columns about how the instructor is engaged during the activity. GSIs then work in small groups to list each AL strategy in what cells they find most relevant. ● GSIs negotiate with one another how they would enact the activity relative to what the instructor is doing and what the students are doing. ● GSIs recognize that direct instruction is just one of many teaching methods and quite limited within the scope of what students can and can't do.
Lesson 2 Objectives:	**Lesson 2 Activities (50 Minutes)**
● Notice and articulate distinctions and dimensions of definitions of Active Learning ● Revise their initial definition of Active Learning.	● GSIs review the rubric (see Figure 2) and practice evaluating three pre-generated definitions according to the rubric. ● GSIs then review their own definition ● Classroom discussion is had around any issues of confusion with the rubric and what they notice and wonder about their own pre-module AL definition

Figure 1. Active learning module structure and components

GSIs began by reading a collegiate AL article (Faust & Paulson, 1998) and creating their own initial definition of AL. Lesson 1 had them engage with 10 AL strategies by asking the GSI to envision (1) what the student is doing and (2) what the instructor is doing for each. Lesson 2 had the GSIs see a rubric for evaluating definitions of AL along the five dimensions using a scale from 1 to 5 (see Figure 2). GSIs then practiced and made sense of the rubric with three AL definition examples that were pre-generated to identify how the dimensions of AL are used. Finally, GSIs reviewed their own definition and discussed what GSIs noticed and how they might want to change their definition.

This report focuses on the Lesson 2 objectives. The activity was designed after multiple iterations of a smaller activity (Lesson Plan 1) so that results could be used to interpret and justify conclusions about GSI's change in definitions. By looking for changes in definitions (especially relating to equity), we collected data to determine if the TPD objectives had been met. Furthermore, through assessment and reflection, we could determine how to improve the TPD itself. Ultimately, the rubric is not just to rate definitions, but a means to understand the components of the definition (i.e. assessment *for* AL understanding).

Methods

To answer our research questions, this study used a sequential mixed methods approach where qualitative coding was followed by quantitative analysis. GSIs' original definitions (pre-module assignment) and the GSIs' revised definition were qualitatively coded using the rubric by the research team after the entire module TPD was completed (post-module assignment). Figure 2 offers an abridged rubric, while the full rubric can be found at https://bit.ly/RUME2025.

	(1) Missing Element (α)	(2) Basic Definition (α)	(3) Specific Example(s) (β)	(4) Sufficient Definition (γ)	(5) Rich and Nuanced Definition (γ)
Role of Students	No mention of students or what students are doing	Includes broad mentions that students are active or actively engaged	Can articulate what the student is doing during a specific example of active learning	Describes what students are doing inside and/or outside the classroom; definition may include active engagement with mathematical learning, or peer-to-peer interaction	Describes what students are doing inside and outside the classroom, including active engagement with mathematical learning, peer-to-peer interaction to communicate mathematical reasoning
Role of Mathematics	No mention of the mathematics	Mention of math class or doing math (in general)	Gives examples of specific mathematics content (e.g., chain rule)	Reflect on the mathematical content that was difficult to understand, that has become clearer conceptually than it was.	Defines how specialized mathematics knowledge for teaching is directly involved with the active learning strategy, including how mathematical reasoning is embedded to build student understanding
Role of Teachers	No mention of the teacher or what teachers are doing	Mention of the teacher in general ("the teacher plans active learning")	Gives specific examples of active learning strategies such as think-pair-share	Describes formative assessment(s) and how student feedback will be incorporated to inform teaching	Describes what the teacher is doing to orchestrate active learning, including planning considerations, instructor use of student thinking, and how the instructor engages students
Role of Equity	No mention of equity	Mention equity within the classroom around content, student, or teacher access; reference to "all students" such as active learning	Gives specific inclusive teaching strategies related to implementing active learning	Illustrate how the specific strategies will provide equitable access to the learning activity. May provide examples linking ideas to practice such as equity versus equality	Describes equity considerations and intentionality for using inclusive teaching practices, such as norm setting for engaging students in discussions; includes discussion of how active learning is not automatically equitable; may talk about other equitable or inclusive teaching strategies.
Interconnectedness	No connections among the four different roles	Defines active learning by what it is not ("not lecture"), and does not connect across dimensions	At least one connection is stated between two of the roles (teacher, students, math, equity)	Definition touches on more than one connection across more than two of the roles (teacher, student, math, and equity)	Definition touches on roles of students, teachers, and mathematics, with attention to equity, interactions, and mathematical sense making and communication; acknowledgement that the roles are interconnected

Figure 2. Abridged rubric used to assess active learning definitions by graduate student instructors

The data was then quantitatively analyzed to determine what changes, and frequency of those changes, occurred among participants' definition components and qualities. α, β, and γ are categorical codes described further in the findings.

Participants and Data Collection

47 novice GSIs from three universities in the midwestern and southeastern United States participated in the GSI TPD. All novice GSIs had just begun as full instructor-of-record in introductory mathematics and statistics courses. All three providers of the TPD were researchers in this study. Participants submitted their initial and final definitions of AL via an online form, either during a TPD activity or as homework. Participants' prior understanding of AL from the providers was limited. Students had perhaps heard the language around students should be "active," but AL was never formalized beyond personal experiences with teaching.

Data Analysis

Three members of the research team completed the qualitative coding of all 47 GSIs' pre- and post-definitions using the rubric (Figure 2). They all coded 20 GSIs' pre- and post-definitions along all five rubric dimensions, then met to compare, discuss, and agree on initial codes. They discussed how to align the rubric, then discussed and revised their individual codes

until the whole research team agreed on coding for the initial set of 20 GSIs. The remaining 27 GSIs were split evenly among this same group so that each researcher coded for 9 GSIs' pre- and post-definitions along all five rubric dimensions on their own. Inconsistencies were discussed until interrater reliability was 95%.

For example, one student wrote in their pre-definition, "My own description of it would include things such as collaborative problem solving, hands-on activities, flipped classroom, peer teaching, project based or discovery learning," which the research team agreed was a score of 1 in all categories, except Role of Students (score 5). The post definition for the same student was:

> Multiple participants are involved in AL inside a classroom: the instructor creates learning opportunities with equitable access points and then facilitates tasks so that everyone in the class is involved with the task, while simultaneously collecting feedback from the students to navigate and decide on the next learning actions and goals. All students have an opportunity to be actively involved in the learning process, while following the established classroom norm of promoting the construction of their own knowledge and ideas simultaneously respecting others' experiences and perspectives. Students engage in tasks to develop understanding and become an active participant/ a primary stakeholder in their learning and further investigate their understanding in and outside of the classroom by taking ownership of their own learning outcomes.

Researchers scored this student's post definition as still a 1 in Role of Mathematics and a 4 for Interconnectedness, but a 5 for the Role of Student, Instructor, and Equity.

Findings

Pre- and Post-Definition Descriptive Discussion

Upon completion of the initial coding, the research team decided to group some rubric score values into broader categories. The α category comprises scores of 1 (Missing Element) and 2 (Basic Definition), representing, at best, the inclusion of a rubric dimension in a GSI's definition with little specificity. The β category represents a score of 3 (Specific Examples(s)). And the γ category comprises scores of 4 (Sufficient Definition) and 5 (Rich and Nuanced Definition), representing the inclusion of a rubric dimension in a GSI's definition at a level of complexity beyond a basic statement or concrete example. These three broader categories were introduced to help identify trends via quantitative analysis, both in describing pre- and post-definitions individually and in understanding changes from pre- to post-definitions overall.

Table 1 provides a summary of how GSIs' pre- and post-definitions were coded and organized by rubric item category (ranging from less rich in the α category to more nuanced in the γ category). This provides evidence for the initial and end states of GSIs' definitions through the activity.

Table 1. Number of GSIs' pre- and post-definitions by category

	Pre-Definition (N=47)			**Post-Definition (N=47)**		
	α	β	γ	α	β	γ
Role of Students	25	9	13	6	11	30
Role of Mathematics	44	0	3	36	2	9
Role of Teachers	28	17	2	9	32	6
Role of Equity	46	1	0	38	5	4
Interconnectedness	20	20	7	0	12	35

α = rubric score of 1 or 2; β = rubric score of 3; γ = rubric score of 4 or 5

As seen in Table 1, the category with the largest number of GSIs' pre-definitions was the α category for every rubric dimension. Pre-definitions were especially highly concentrated in the α

category for both Role of Mathematics and Role of Equity, meaning that the Roles of Mathematics and Equity were either missing or included in a basic way for nearly all GSIs' pre-definitions. The other rubric dimensions (Role of Students, Role of Teachers, And Interconnectedness) were distributed more widely across all categories.

Regarding post-definitions, Table 1 shows that Role of Mathematics and Role of Equity remain relatively highly concentrated in the α category, but these are the only rubric dimensions for which the α category had the largest number of post-definitions. Researchers expected that GSIs' definitions would change after having worked with the rubric in Figure 2, but it is noteworthy that no post-definitions fell into the α category for Interconnectedness. This is likely because many of GSIs' pre-definitions completely lacked mention of certain rubric elements (such as the "Role of Mathematics" or the "Role of Equity") and therefore had fewer opportunities to make connections between roles. The rubric prompted GSIs to think about and include all roles (and naturally make more connections among the roles) in their definitions.

Changes in Definition

We observed a deepening in richness of GSIs' definitions of AL by observing their change in category from their pre-definition to their post-definition. For simplicity, a GSI's definition became *richer by category* if it was assigned rubric scores for pre- and post-definitions that represented a shift in category from α to β, from β to γ, or from α to γ. Among the 47 GSI participants, it was most common (17 of 47 participants, 36%) for a definition to become richer by category in three different rubric dimensions. Two GSIs saw their definitions become richer by category across all five rubric dimensions (15%), while seven GSIs saw no increased richness by category in any rubric dimension.

Researchers note that since GSIs' were given the rubric prior to constructing their post-definitions, one might expect to observe GSIs' definitions becoming somewhat uniformly richer by category across all rubric dimensions (after considering rubric dimensions where GSIs' pre-definitions were concentrated in richer categories, see Table 1). This was not the case at all. The "Interconnectedness" rubric dimension saw the most substantial enrichment of definitions, with 35 of 47 participants' definitions (74%) becoming richer by category, even though this rubric dimension started with the fewest pre-definitions in the α category. Conversely, the "Role of Mathematics" and "Role of Equity" rubric dimensions began with the most pre-definitions in the α category and yet saw the least amount of enrichment by category.

Role of Mathematics and Role of Equity: All but three GSIs' pre-definitions fell into the $\boldsymbol{\alpha}$ category for the "Role of Mathematics" rubric dimension, yet only nine definitions became richer by category in this dimension through the activity. Looking at the "Role of Equity" rubric dimension is even more extreme—all but one pre-definition fell into the $\boldsymbol{\alpha}$ category, yet only eight definitions became richer by category. To better understand the changes in richness for the Role of Mathematics and the Role of Equity, a more granular analysis of the participants' post-definitions was done with participants' whose pre-definition was an $\boldsymbol{\alpha}$ category. Table 2 details the exact change in rubric score (not just category) for all pre-definitions within the $\boldsymbol{\alpha}$ category.

Table 2 illustrates 8 out of 47 post-definitions (17%) received a score of 1 to represent the Role of Mathematics as being a "Missing Element." However, 23 of 35 of the GSIs' definitions (65%) that started at a pre-definition score of 1 ("Missing Element") grew to a post-definition score of 2 ("Basic Definition"). These GSIs' definitions became richer within the α category. Ultimately, pre-definitions that began in the α category lacked concrete examples of how

mathematics takes a role in the AL process (β category) or how mathematical content and reasoning shapes or is shaped by the AL process more broadly (γ category).

Table 2. Number of GSIs pre- to post-definition rubric score shifts in α category

	Pre-Definition Rubric Score (α category)	Post-Definition Rubric Score 1	2	3	4	5
Role of Mathematics	**1 (N=35)**	6	23	1	5	0
	2 (N=9)	2	4	1	2	0
Role of Equity	**1 (N=41)**	20	15	2	2	2
	2 (N=5)	1	2	2	0	0

Table 2 also states 21 out of 47 post-definitions (45%) received a score of 1 to represent the Role of Equity as being a "Missing Element." 15 of the 41 (37%) post-definitions began with a pre-definition of score of 1 shifted richness towards a "Basic Definition" (score 2) for the Role of Equity, yet 20 of the 41 (49%) had a pre- and post-definition score of 1 within the Role of Equity. Even more so than for the Role of Mathematics, GSIs' definitions infrequently incorporated specific examples or sufficiently rich/nuanced conceptualizations of the Role of Equity in AL.

Discussion & Implications

Despite limited research on GSI learning (e.g., Deshler et al. 2015; Yee et al., 2023) this study represents a transferable model for assessing the impacts of GSI TPD. Like Williams et al. (2022), we found that AL definitions rarely included any mention of equity or mathematics, much less nuanced and rich treatments of these complex topics. In answering our research question, this AL-focused TPD, overall, did lead to more nuanced and richer GSI AL definitions. Although one might expect that exposure to the rubric for AL definitions would result in uniform improvements to GSIs' definitions of AL, we did not see that. Instead, we saw that most GSIs improved in their definition with the Role of the Teacher, Role of the Student, and Interconnectedness. Although the considerations of mathematics and equity did not show overall categorical growth beyond the α category, we did see more mentions of mathematics and/or equity. The surface attention to equity was not surprising; it typically takes far more equity-focused TPD for GSIs to develop nuanced conceptualizations of equity (Reinholz, 2023; Reinholz et al., 2022). Thus, one implication for GSI TPD is that GSIs need longitudinal TPD experiences focused on developing their conceptions of equitable mathematics teaching and learning.

An additional implication for TPD providers is that the model we used for assessing the impacts of the AL TPD on GSIs' definitions is transferrable to other pedagogical topics. Thus, this study answers the call of many for more research on what makes for effective TPD (e.g., Deshler et al., 2015; Hauk & Speer, 2023; Yee & Rogers, 2017; Yee et al., 2023; Yee et al., 2024). One limitation of our study is that we just focused on GSIs' definitions of AL; future studies should look for connections between growth in GSIs' AL definitions and enactment practices of AL. We recommend some longitudinal future studies to look at more lasting impacts, and to assess if sustained TPD can further improve not only GSIs' AL definitions, but also their AL practices.

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Exploring Women-Identifying Graduate Students' Values Related to Teaching and Learning Mathematics

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Graduate students' values related to teaching and learning are critical in shaping their professional identities and future careers as faculty members. These values encapsulate instructors' convictions about what is important or worthwhile as it relates to mathematics teaching and learning. Research suggests that mathematical discourses reflect and empower dominant masculine values. Thus, we explore three women-identifying graduate students' values related to teaching and learning mathematics to highlight their voices and emphasize ways in which their values align with or challenge dominant masculine values. Utilizing interview data, we found four values across the three participants: collaboration, student-instructor perceptions, humanizing mathematics, and compassion and care for students' wellbeing. We discuss how these values interact with dominant masculine mathematical discourses and the ways in which these values may help foster equitable mathematics pedagogy.

Keywords: Graduate Teaching Assistants (GTAs), gender, values, equity

The values that women-identifying mathematics graduate students hold about teaching, student learning, and pedagogy are critical in shaping their professional identities and future careers as faculty members (e.g., Clarke et al., 2013; Austin & McDaniels, 2006). As many graduate students transition from learners to educators, their beliefs and values will not only influence their teaching practices but also contribute to the evolving landscape of mathematics education in postsecondary institutions. Given the underrepresentation of women in mathematics, especially in faculty positions (Casad et al., 2021; Kessel, 2014), it is essential to understand the perspectives and values they bring to their academic and professional lives (National Science Foundation, 2021). Research suggests that values related to teaching and learning are central to the development of a teacher identity and can profoundly influence career trajectories (Beijaard et al., 2004). For women in mathematics, these values may encompass a commitment to equity, inclusivity, and student-centered pedagogy, which are often in contrast to the more traditional, content-focused values historically dominant in the discipline (e.g., Hottinger, 2016; Leyva, 2017; McMinn et al., 2022; Mendick, 2006; Reinholz et al., 2019). Understanding these values is essential not only for supporting the professional development of women-identifying graduate students but also for illuminating how their values align or misalign with the longstanding norms and discourses of the mathematics discipline (Leyva, 2017; Mendick, 2006; Street, 2024).

The alignment or misalignment of these values with the traditional expectations of mathematics faculty can significantly impact the career pursuits and retention of women-identifying mathematicians. For instance, research has shown that when there is a mismatch between personal values and the prevailing culture of the discipline, individuals may experience role conflict, reduced job satisfaction, and a heightened likelihood of leaving the profession (Gardner & Holley, 2011). Conversely, when institutions recognize and support diverse values, it can lead to more inclusive teaching practices, innovative pedagogical approaches, and a more welcoming environment for both students and faculty (Casad et al., 2019). Therefore, examining the values of women-identifying mathematics graduate students provides critical insights into

how these emerging educators may challenge or reinforce existing pedagogical norms and helps to identify potential areas of misalignment that could be addressed to promote greater inclusivity within the discipline. In this preliminary work, we seek to illustrate values that women-identifying graduate students hold related to student learning and teaching. In particular, we asked: What do women-identifying mathematics graduate students value about mathematics teaching and learning? In what ways, if any, do their values align with or challenge dominant values in postsecondary mathematics education?

Conceptual Framing

We ground this study within the position that mathematics education is a values-laden space (Adiredja & Andrews-Larson, 2017; Ernest, 2016; Swanson, 2010). Values represent semi-stable convictions one holds about what is important or worthwhile, influenced by both individual experiences and societal factors, such as gender (Bishop et al., 2003; Seah & Andersson, 2015). Research suggests the values imbued within mathematics spaces often reflect dominant masculine values, such as individualism, rationalism, and competition (Ernest, 2003; Leyva, 2017; Mendick, 2006). Therefore, doing mathematics can produce value tension for those unable or unwilling to assimilate into the masculine culture of mathematics (Levya, 2017; Mendick, 2006). This tension can force women in mathematics to either conform or sacrifice components of their gender identity to be a mathematician (Hottinger, 2016). While a growing body of work relates to how marginalized students' values contrast with and challenge the dominant masculine values in undergraduate mathematics (e.g., Fong et al., 2019; Levya, 2021; Street, 2024), limited, if any, examine graduate students' and instructors' values related to teaching and learning mathematics.

Seah et al. (2016) operationally defined mathematics education values as tools that are "use[d] to define the relative importance of different attributes of mathematics and of its pedagogy" (p. 14) which are "inculcated through the nature of mathematics and through the individual's experience in the socio-cultural environment and in the mathematics classroom" (Seah, 2005, pp. 146-147). For example, research within K-12 mathematics suggests teachers highly value activities that focus on consolidating materials in class, problem-solving and investigative activities, and small-group collaboration (Akyildiz et al., 2021; FitzSimons et al., 2000). As one of the only, if not sole, article that focuses on graduate students' teaching values, Schwartz et al. (2022) found that graduate teaching assistants across disciplines (with 8.7% of the sample in a STEM department) most valued higher-order thinking skills in their teaching, including fostering students' written communication, critical thinking, and information literacy skills. Their results also reflect how gender may influence values of teaching and learning mathematics, finding that women GTAs in their sample valued components of ethics and diversity (e.g., adapting course content to reflect different cultures) more so than men, and that this value was associated with higher use of student-centered teaching practices. Given the highly contextualized nature of values related to teaching and learning mathematics, instructors play a major role in emphasizing the central values in the classroom. We focus the present work on women-identifying graduate students' values to highlight their perspectives in a masculine domain and connect to emerging work challenging exclusionary gendered mathematical discourses.

Methods

Context and Participants

This study was conducted at a large, research-intensive university in the West Coast of the United States. We used convenience sampling (Miles et al., 2020) to recruit graduate students in the mathematics department. Sixteen doctoral students completed a survey that elicited self-identified/self-selected demographic information and requested their participation in a 90-minute interview. A total of six students participated in the interview, but in the present work, we focus on the three women-identifying graduate students from the sample (See Table 1; all demographic information was self-selected or self-described). At this department, serving as a teaching assistant (TA) involved planning for and facilitating sections (i.e., recitation hours). All participants had experiences as TAs and as instructors of record for lower division mathematics courses. Each of the three participants researched theoretical mathematics (Kayla was interested in geometric group theory, Kendall was focused on lower dimensional topology, and Morgan was researching quantum algebra and knot theory) and all expressed intentions to continue their careers as faculty members in a mathematics department at a postsecondary institution.

Table 1. Participant information

Name	Race and/or Ethnicity	Year in Ph.D. Program	Other self-described identities
Kayla	White/Caucasian	1st	Underrepresented student, Christian
Kendall	White/Caucasian	5th	
Morgan	White/Caucasian	5th	Underrepresented student

Aligning our work with transformative feminist theories and approaches (e.g., Creswell & Plano Clark, 2011; Du Bois, 1983; Kitzinger & Wilkinson, 1997, etc.) we chose to engage with the experiences and values of women-identifying graduate students without any direct comparison to the experiences and values of men in mathematics. Centering women's experiences in mathematics and illuminating the values they hold is important in its own right. With a special focus on their gender identities, we aim to contribute to and expand the field's understanding of mathematics graduate education by describing some of the values that women-identifying graduate students hold regarding teaching, student learning, and pedagogy. However, it should be noted that our analytical focus on Kayla's, Morgan's, and Kendall's values is not intended to homogenize nor generalize to women-identifying graduate students in mathematics doctoral programs.

Data Collection and Analysis

We utilized Zoom to conduct one-hour semi-structured interviews (Rubin & Rubin, 2011) focused on students' experiences in graduate school, including the supports and barriers they encountered, their perceptions of the mathematics professoriate, and their understanding of mathematics teaching. We analyzed the interview transcripts using Saldaña's (2013) method of pre-coding, first-cycle, and second-cycle coding. We engaged in pre-coding by first independently reading through each transcript and identifying instances that participants articulated values related to mathematics teaching, student learning, etc. Our first-cycle coding process involved reading through the transcripts together and sharing and reconciling the codes

we established individually during pre-coding. Our second-cycle coding process included discussing how the codes represented themes related to participants' values related to teaching and learning mathematics. Throughout this analysis process, we wrote notes and memos (Miles et al., 2020) capturing the progression of our themes and ideas. For the present study, we highlight four themes related to participants' values about teaching and learning mathematics: *Collaboration*, *Student-Instructor Perceptions*, *Humanizing Mathematics*, and *Compassion and Care for Student Wellbeing*.

Results

Given the limited scholarship on mathematics graduate students' values about teaching and learning mathematics, we utilize a descriptive approach to illustrate these women-identifying graduate students' values. It is important to note that the following values are not mutually exclusive, and we observed participants expressing values that encompassed multiple themes.

Collaboration

Kendall, Morgan, and Kayla all valued collaboration in mathematics and reported implementing instructional practices that encouraged students to engage with one another, converse with peers, and make sense of the mathematics together. Although some of their efforts to foster collaborations were similar (e.g., group work during sections), the underpinnings of why they valued collaboration in mathematics classrooms differed. Kendall and Morgan understood collaborative sensemaking and interactions with others during class as aspects of instruction that advance students' learning of the mathematics content. They considered group work to be most beneficial because it affords opportunities for students to explain their thinking to others and teach their peers. To explain why she utilized group work in her teaching, Morgan said, "[The students] best learn by doing and teaching others…I think it's when you work together, and kind of talk about it with your peers, that's probably when you learn the most." Similarly, Kendall shared, "I do think math classes need more group work, because I think it's super important for people to explain how they think about something and compare that to how other people think about it." For Morgan and Kendall, collaboration is an aspect of mathematics instruction that supports students' learning.

Alternatively, Kayla valued collaboration in mathematics because it encouraged interpersonal relationships among the students and allowed her to foster connections with students. For example, she acknowledged while doing mathematics homework that, "I think working with other people to solve problems helps. It might not be the best to do it in a classroom setting, but I think outside a classroom works." However, Kayla did implement collaborative activities in the classroom, emphasizing how those activities helped bolster connections with her students. She reflected, "I usually would just say like, 'Alright, break up into groups of two and three. And I'm gonna just walk around, and I'm going to sit with you guys and have you talk about the problem to me… I always like trying to sit with them at the table and make it a level playing ground.'" Thus, Kayla valued group work as a means to immerse herself with students and better connect with them.

Student-Instructor Perceptions

A notable value that participants held around mathematics teaching and learning centered around how the members of the learning community viewed each other. They acknowledged the impact of students' perceptions of them, and conversely their perceptions of students, on the broader dynamics of the classroom. Morgan shared,

> Energy level is not necessary, but it helps a lot. I think for a professor who's engaging and who is willing to involve his students as much as possible, who takes the time to get to know the student, to know where they're at in the class and things like that. I think those professors are the best, and the ones who you can genuinely tell that they care. That's the truth.

She valued the student perception that professors and instructors genuinely care about their students. Additionally, participants described other pedagogical practices that would result in positive portrayals of themselves to students. For example, Kayla explained, "I think being organized is a good one...it adds that level of respectability to where if you're not organized, it just kind of gives off the wrong vibe." She made sense of how students perceived TAs and instructors and understood preparedness and organization in the classroom to be related to levels of respect. Similarly, Morgan valued providing clarity in the classroom expectations. She said, "I have a very detailed syllabus, I have a calendar up. I think students should know all the dates of the exams and everything, what's expected of them. I think that they know that in my class." Again, the notion of being clear and organized emerged and suggests that these graduate students are actively making sense of how students perceive them as TAs and instructors in the classroom.

Interestingly, we also found evidence of participants valuing students' engagement because it supported their connections with the students. Kendall shared, "I got to know [my students] really well, and they were super chill kids." She elaborated on these students, saying, "I had a really great group of students. They wanted help, they wanted to learn, they wanted to do well. They were motivated. I didn't have any students who just refused to show up to things and refused to participate." Kendall recognized and valued the students' efforts and motivation, and this made connecting with them easier and more enjoyable for her. Overall, we found that participants' value of connecting with students was deeply intertwined with their values of humanizing mathematics and exhibiting compassion. We unpack these in more detail below and recognize that participants considered these practices to collectively foster positive connections with students.

Humanizing Mathematics

Kendall, Morgan, and Kayla all discussed pedagogical components that reflect valuing humanizing mathematics, including presenting themselves as relatable to their students and presenting mathematics as a human endeavor. Kayla recollected her own experience as a student, and how she valued "professors who are more personal and kind of come in and tell you about themselves," rather than "all I know about you is you do math." In particular, she mentioned while the students were in groups, how she "like[s] trying to sit with [the students] at the table … so I come off more relatable to them." Kendall similarly valued relatability in her classroom. She noted that just being a graduate student, "especially as a TA… [the students] feel more comfortable with us." When thinking about her overall approach towards teaching, Kendall revealed she "take[s] an approach that [she] is the student, like trying to be in their shoes."

This also transitioned into how Kendall thought about the human aspects of mathematical learning, saying how "putting [herself] in [the students'] shoes, remembering how hard it was to grasp concepts can be super helpful to students." She acknowledged that mathematical learning is difficult and planned her teaching to reflect this. Similarly, Kayla recognized mathematics as a human endeavor by valuing mistakes in the learning process, saying "owning up to your mistakes [as the instructor], if you're wrong, it's fine. It's actually pretty encouraging for students to see when you do make a mistake, to be able to take accountability for it." Morgan

also discussed the value of practicing mathematics and learning through mistakes, while also emphasizing there is a point in which as the instructor, "even if you make a plus or minus mistake, and if you do that every class, then that can, even it it's not about the concepts, that can hurt the students." Thus, the participants recognized a balance, whereby mistakes are integral and inevitable in learning and teaching mathematics, but there is value in preparing for each lesson to minimize small mistakes. Morgan and Kayla also positively spoke about connecting mathematics to areas beyond the classroom. Kayla valued this as a student, saying that she really appreciated when professors "[gave] expert exposition to what they're doing" that connected the mathematical concepts to a real-world application or previously learned mathematical concepts. In her teaching, Morgan "tries to incorporate some applied things" to make the students feel more connected to mathematics.

Compassion and Care for Student Wellbeing

Each participant also emphasized valuing students' emotional wellbeing and their role as supportive and compassionate instructors. When asked about what characteristics make a successful undergraduate mathematics instructor, Kayla responded that "having compassion for students, just as people, is a pretty big one." Similarly, Kendall shared that "our role [as instructors] is to guide [the students] along in more of an emotional sense and a math sense." Morgan specified that this could look like "someone who's very available for their students, who, you know, you feel comfortable going to their office horse, and they are very supportive… and really encourage students even if they're not the best students in the class." Thus, we see how each participant valued aspects of compassion and support for students in teaching mathematics.

For Morgan and Kendall especially, this valuing of compassion frequently arose in direct challenge to traditional mathematical discourses that value innate mathematical ability over hard work and effort (Jaremus et al., 2020; Mendick, 2006). In their interviews, they recognized the prevailing mathematical discourses that paint the discipline as extremely difficult with little support and widespread stereotypes that position people as unintelligent if they perform poorly in mathematics (Leyva et al., 2021; Reinholz et al., 2019). Morgan talked about how important it was to her to be a supportive instructor to those struggling in the course, referencing professors she had who were "not supportive enough" and "who teach to the top 1% of their class." She shared how she saw how "a lot of people have that issue with math where they're like, 'Oh, I'm too dumb to be in math' and that's just not the case… like, there's too many people telling people that they're not good in math." Kendall similarly portrayed her valuing of compassion, noting how "part of [her] approach is also being forgiving in a sense." She discussed how "a lot of students get disheartened" when they fail tests, and "part of our job as an instructor… is to be there when they need a mentor and then to remind them that they can still get this stuff." Both Morgan and Kendall's interviews highlight their belief that mathematical knowledge can be learned through hard work as opposed to a solely innate skill, and they valued their role as supportive and compassionate instructors towards students in this learning.

Discussion and Conclusion

The findings above detailed four values of teaching and learning mathematics held by women-identifying graduate students. While these do not encompass all mathematics education values held by women-identifying graduate students, the ways in which similarities in values emerged across these participants suggest challenges to the sociohistorical dominant masculine values in mathematics. In particular, we see tensions within the traditional Western

feminine/masculine dichotomies of collaboration/competition, collectivism/individualism, emotional/logical, and hard work/innate ability. Mendick (2006) posits that mathematical values emerged from these dichotomies, firmly aligning mathematics with dominant masculine values. However, the four values above position mathematics along the more feminine side of these dichotomies. While mathematical environments often reflect the masculine value of competition (Hottinger, 2016; Mendick, 2006; Reinholz et al., 2019), the participants valued collaborative aspects for mathematical learning and interpersonal relationships. Collaboration alongside the values of humanizing mathematics and compassion suggest the importance of instructor-student and student-student relationships, reflecting a collectivist view of mathematics rather than the traditionally masculine individualistic view (Lane et al., 2019; Mendick et al., 2008). The valuing of compassion and concern for students' wellbeing coincide well with values related to emotional aspects of learning mathematics. While dominant masculine discourses pose mathematics as purely logical and emotionless (Mendick, 2006), the participants acknowledged the ways mathematics can be difficult and laden with emotions such as frustration or loneliness and valued their role as understanding and compassionate instructors. Lastly, the participants also valued students' effort in their courses and the ways in which, as the instructor or TA, they could support students' perseverance in mathematics, countering perceptions that mathematical ability is innate and challenging stereotypical hierarchies that position men as more innately mathematically able (Hottinger, 2016; Reinholz et al., 2019). These examples demonstrate ways in which these women-identifying graduate students' hold values that conflict and challenge traditional masculine mathematics values as mathematics instructors.

Implications and Future Work

Morgan's, Kendall's, and Kayla's values highlighted in this work resonate with and reflect existing characterizations of pedagogical practices that advance equity in undergraduate classrooms (e.g., Pai et al., 2024; Rubel, 2017). For example, their values point to the notion that mathematics teaching is an interpersonal, social activity that builds connections between people. Additionally, their values and reported practices echo our contemporary understanding of the experiences of women-identifying faculty in STEM. For example, they are seen as caregivers and feel pressure to be perceived as nice and nurturing (Guarino & Borden, 2017; Hart, 2016). Women faculty also report being more student-centered in their pedagogies than their male counterparts (McMinn et al., 2022). Future work will more closely examine how graduate students' values align with the values that underpin equity-based practices, which may potentially misalign with historically dominant values in mathematics education. We conjecture that values alignment and misalignment can influence how graduate students come to learn about and take up these practices, and this can have significant implications for how mathematics graduate programs prepare students for careers as faculty. Research has shown that integrating equity-based pedagogies into a teacher's regular practice can be a slow process (e.g., Shadle et al., 2017; Walter et al., 2021), and we suggest that examining and leveraging graduate students' values is one way we can better support their pedagogical development. Our findings revealed that these women-identifying graduate students hold values that may be conducive to developing their understanding and expertise of equity-based practices. Providing opportunities to articulate and make sense of one's values related to teaching and learning mathematics can equip graduate students, mathematics departments, and institutions with new insights and framing on how to better support their pedagogical development and transition to faculty positions.

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Not-so-Secret Watchers: What Students Can Tell Us About Teaching

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Projects to improve teaching as a means to improve learning need good tools to measure shifts in the use of effective research-based instructional strategies. We describe a survey tool to assess active and collaborative learning practices in college STEM by asking students to report what practices are used commonly in their course. Student reports correlate with their experience of the learning environment, and with their self-reported learning outcomes. Student means by section are corroborated by instructor-reported use of these practices and external observations.

Keywords: teaching, professional development, survey measures, research-based instructional strategies (RBIS)

A basic question in education research is the linkage between teaching and learning. This is surprisingly difficult to measure, however. To study how teaching activities relate to learning outcomes, we must have ways to describe teaching in diverse settings (AAAS, 2013). This need surfaces, for example, in studies of interventions to improve teaching (e.g., Manduca, et al., 2017; Archie, et al., 2022). Efforts to increase college instructors' uptake of research-based instructional strategies (RBIS) need tools to assess the initial state of participants' teaching, to measure changes over time, and to know if their change strategies are working or not.

Surveys and observations are commonly used methods for studying teaching. Instructor surveys can be efficiently administered to large numbers of instructors, but they pose challenges including sampling, internal validity, and potential biases in instructor self-report (Iverson, 2012). Social desirability bias is particularly concerning in self-reports of teaching. Williams and coauthors (2015) review properties of existing instruments for instructor self-report.

Classroom observations offer detailed, naturalistic information about teaching, but they are time-consuming and costly. Hora and Ferrare (2013) and Campbell (2017) discuss strengths and limitations of classroom observation in higher education, with particular attention to validity and reliability concerns. Weston, Hayward and Laursen (2021) discuss additional challenges to reliability when time sampling is applied to make inferences about term-long courses.

As regular occupants of classrooms, students can offer insight about the teaching taking place, especially when combined with other measurement methods. In evaluative uses, student ratings of teaching are often derided as popularity contests with known, serious biases (Kreitzer & Sweet-Cushman, 2021; Heffernan, 2022). Others find that "student ratings tend to be statistically reliable, valid, and relatively free from bias or the need for control, perhaps more so than any other data used for faculty evaluation" (Benton & Cashin, 2014, p. 315). The MET study (2012) found clear connections between student ratings of their teachers and student achievement gains: "Students know an effective classroom when they experience one." (p. 1) The key is asking "the right questions, in the right ways" (p. 1), probing aspects of teaching that students are qualified to evaluate, and doing so in ways that minimize threats to validity.

Mindful of these caveats, we developed a brief student survey to characterize the teaching in recitations of large STEM gateway courses, in the context of an institutional project to infuse active and collaborative learning into those recitations. Importantly, we did not survey students to measure course outcomes, such as attitude or learning. Rather, we capitalized on students'

everyday presence in the classroom to report on the teaching they experienced, and we used their reports to describe and measure change in teaching over time. The simple student survey focuses on the frequency of specific teaching practices and is aligned with other instruments that we have developed (Hayward, et al., 2018; Weston, et al., 2021, 2023). Alongside this measure, we also measured students' perceptions of the classroom environment and a small set of student self-reported learning gains, to assess whether these experiences were related to the teaching practices that students described. We describe results for three research questions (RQ):

1. How does student-reported use of ACL practices relate to (a) their experience of the learning environment, and (b) their self-reported learning gains?
2. How does student-reported use of ACL practices relate to instructor practices, from (a) instructor self-report and (b) observation?
3. How can these student metrics inform formative and summative evaluation?

Conceptual Framework

We follow Neumann (2014) in viewing the purpose of higher education as learning "how to live a full and democratic life," exerting essential freedoms with the abilities "to argue and write, to inquire and imagine, and to act ethically and humanely" (pp. 249-250). This view contrasts with notions of the purpose of higher education as developing a labor force to compete in the global economy. Substantive learning requires that students encounter new subject-matter ideas and connect them to their existing personal knowledge. The teacher's job, Neumann argues, is to orchestrate and bring about these encounters. Doing this well requires sophistication, and thus we recognize teaching as a complex human activity that invokes individual beliefs, values, knowledge, and behaviors, and that takes place within nested cultural and operational contexts that include the classroom, the institution, and the wider community (Schoenfeld, 1998).

Our perspective on measurement draws on Guskey's framework for evaluating teacher professional development (PD; 2000, 2002). The evaluation framework begins with two near-term outcomes, teachers' experiences of the PD and the learning they take away, and it ends with student learning, the desired ultimate outcome. Connecting near-term outcomes to downstream learning are teacher context and implementation: how supportive is teachers' context to implement what they learn in PD, and how does their classroom behavior change in ways that will lead to stronger student learning. Both these mid-stage outcomes are more complex to measure, but influence whether and how the student learning outcomes are accomplished.

Here we focus on development of a tool to measure elements of implementation of active and collaborative learning, a suite of RBIS linked to student learning (Freeman et al., 2014). While our tool yields a picture of teaching much simpler than Neumann's portrayal of complex classroom interplay, both have at heart teachers' choices about how to use class time to more or less actively orchestrate students' encounters with subject matter (Campbell et al., 2017). Effective use of active learning techniques focuses students' attention on course material and creates opportunities to connect to their own knowledge and learning needs, as students engage with each other and the instructor over the course material. We do not assume that teachers are equally skillful at implementing these approaches, nor indeed at lecturing, but we do assume that their choices are agentic and that they matter for student experience and outcomes.

Context for the Study

The study context was offered by Gateway2STEM, an NSF-supported project at George Mason University (GMU). The project sought to incorporate active and collaborative learning (ACL) into recitation sections of gateway courses in mathematics, physics and computer science

(CS). Gateway courses are foundational to a variety of educational pathways, high in enrollment, and high-risk for students (Koch, 2017). Here the gateway courses were two two-term calculus sequences, two two-term sequences in introductory physics, and introduction to programming in CS. In 2023 a new two-term intro sequence in CS was added.

Recitations were targeted as a foot in the door, seeking to give instructors concrete experiences with ACL approaches, develop their expertise and interest, and build local teaching cultures that would spur broader changes. In large courses, ACL was implemented by graduate assistants who teach recitations in math and CS; term and tenure-track faculty, who teach physics presentations and blended recitation/lecture courses in math and CS. Each department chose a guided inquiry approach, in which students worked in small groups on sequenced tasks, called worksheets (mathematics), tutorials (physics), or programming assignments (CS). Common tasks were prepared and recitation instructors were encouraged but not required to use them.

GMU is a large public institution, with about 23,000 undergraduate students. Many are non-traditional students who are working, studying part-time, resuming education after military service, or preparing for a second career. Many are in the first generation of their families to attend college. The students are ethnically diverse, like the metro area where GMU is located.

Research Methods

We surveyed students and instructors about their experiences of recitation across the whole term, and observed single class sessions for a sample of recitation sections and blended courses.

Surveys and Observations

The short survey (≤10 min) focused on student experiences. We used three blocks of items (Table 1)—two existing instruments plus our instrument, called the TAMI-SS (Toolkit for Assessing Mathematics Instruction: Student Survey) in keeping with our other TAMI instruments. The instructor survey (TAMI-IS) asked teachers to estimate their use of the same five teaching practices, by percentage of total class time, across the whole course.

Table 1: Overview of Survey Item Blocks Used in this Study

Use of ACL practices: TAMI-SS (Hayward, Weston & Laursen, 2018; Weston, Laursen & Hayward, 2023)	Learning environment: WIHIC (Dorman, 2003; Hoover & Pelaez, 2008; Skordi & Fraser, 2019)	Learning gains: SALG (Seymour et al., 2000; Laursen et al., 2011)
frequency of 5 teaching practices: lecture, small group work, individual work, student presentation, whole class discussion (0=never to 4=almost always)	frequency of experiences in 7 areas (8 items each): student cohesiveness, teacher support, involvement, inquiry/ investigation, task orientation, cooperation, equity (0=almost never to 4=almost always)	learning gains in 5 areas common to all 3 disciplines (0=no gain to 4=great gain)
S-ACL = mean for 3 practices: small group work, student presentation, whole class discussion (0-4)	WIHIC: mean (0-4)	Gains: mean (0-4)

Surveys were administered online near the end of fall term, to 112 sections in 9 courses in 2021 and to 94 sections of 11 courses in 2023, to over 3000 students each time. Student surveys were distributed by recitation instructors to protect student anonymity. We gave instructors a script to explain the survey and asked them to provide class time to respond.

In all, about 2200 students responded at least in part. Overall student response rates were 36% in 2021 and 46% in 2023, ranging by section from 96% to 0, depending on how the survey was shared with students. Because of this variation in response, we do not directly compare courses. We were able to match 25 instructor responses to student sections across both years.

We also conducted on-site observations for single class sessions of 16 recitation sections and six blended sections. Along with detailed field notes, we recorded the elapsed time for the same core teaching activities as on the surveys (lecture, small group work, etc.). Summing the time for each activity enabled comparisons with instructors' self-report and student report.

Data Analysis

We computed three composite variables. The student ACL score (S-ACL) is the mean frequency of class discussion, student group work, and student presentation, for one student. The section ACL score is the mean S-ACL score for sections with ≥10 responses. The instructor ACL score (I-ACL) is the total percentage of class time for the same three activities. For each, higher values indicate more frequent use of the ACL strategies. Internal reliability of WIHIC subscales was verified with all subscales showing a Cronbach's alpha of 0.9 or greater. Independent sample t-tests were used for statistical comparisons of courses targeted or not by the intervention. For most analyses, we combined data across both years to increase statistical power and learn about the behavior of the items. To answer RQ3, we separated data sets by year.

Results

We used descriptive statistics to report on the behavior of the survey items, and used correlations and linear mixed models to answer RQs1-2.

Descriptive Statistics

Table 2 provides the mean and standard deviation for each composite item. On average, ACL practices were reported to occur 'seldom' and the distribution was moderately platykurtic, while student-reported WIHIC items occurred 'seldom' to 'sometimes' on average, and the scale was negatively skewed. Student reported 'moderate' to 'good' gains on average, and this scale was moderately negatively skewed.

Table 2. Descriptive statistics for S-ACL, WIHIC, and Gains.

Scale	N	Mean	SE	SD	Skew	Skew SD	Kurtosis	Kurtosis SD	Alpha reliability
S-ACL	2056	2.15	0.02	1.04	-0.07	0.05	-0.68	0.11	NA
WIHIC	2236	2.70	0.01	0.70	-0.48	0.05	0.17	0.10	0.97
Gains	1973	2.82	0.02	0.92	-0.64	0.06	-0.07	0.11	0.91

Cronbach's alpha coefficient was used as a measure of the internal consistency for Gains and WIHIC scales. ACL practices are discrete, mutually exclusive practices that can be used independently, thus are not expected to show an expected relationship or response pattern connecting individual practices. Therefore, we did not measure the internal consistency of ACL practices. The internal consistency of both Gains and WIHIC scales was good, with alpha coefficients of 0.91 and 0.97 respectively.

RQ 1a: How does student-reported use of ACL practices relate to learning environment?

We tested the relationship between student ACL (S-ACL) and WIHIC scales. From correlation analysis of all student cases, we found a moderate positive correlation (0.51), indicating that students report more positive environments in courses that include more active learning. We also tested the consistency of this relationship across disciplines and found similar, low to moderate correlations across math, physics, and CS (0.40, 0.54, 0.33, respectively). Figure 1 shows both the correlation of individual scores for all student responses that include responses to every item in the scale (L), and of section means for sections with at least 10 students (R).

Lastly, we conducted a linear mixed model to test this relationship with WIHIC scale as the dependent variable, student ACL as a fixed effect, and course section as a second-order random effect. Controlling for the influence of section-level data is necessary as section responses ranged from 1 to 169, which would bias the correlations based on student reports from larger sections. For this analysis, we excluded student responses from sections with fewer than 10 responses, as larger class sizes have been shown to be more reliable (Benton & Cashin, 2014). The resulting analysis of 1640 student responses from 93 course sections revealed a moderate positive effect (0.33, t (73) = 18.64, $p < 0.001$) of ACL practices on WIHIC, when controlling for the random effect of course sections.

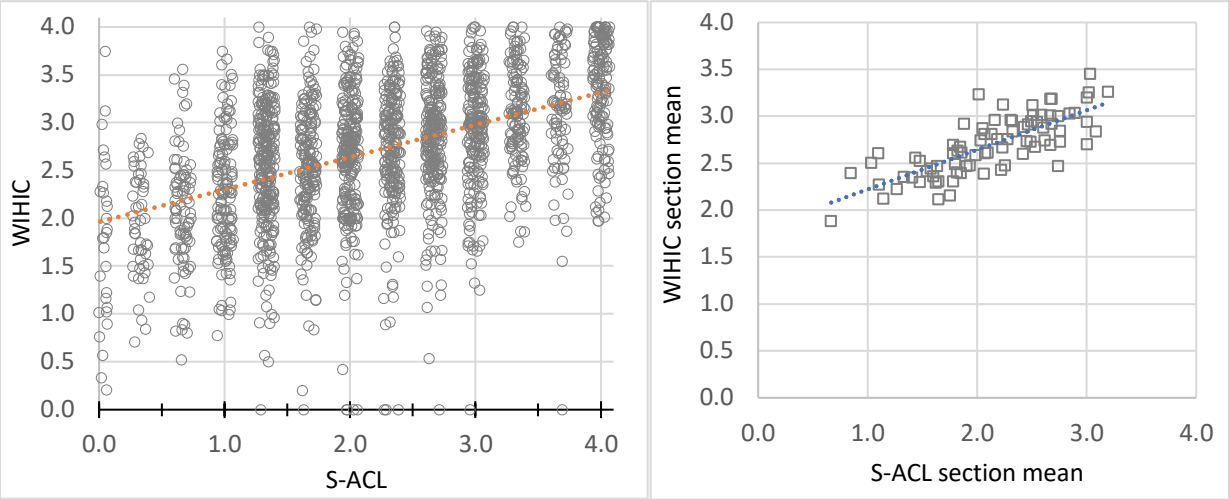


Figure 1. WIHIC score correlates positively with S-ACL, for individual students (L) and section means (N≥10) (R).

RQ 1b: How does student-reported use of ACL practices relate to students' learning gains?

We tested the relationship between student reports of the ACL practices they experienced and their perceived learning gains. A correlation analysis of all student cases revealed a statistically significant moderate positive correlation (0.25) between ACL and gains, indicating that students report greater learning gains in courses that include more active learning. We tested the consistency of this relationship across disciplines and found similar minimal correlations for math and physics, 0.25, and 0.27, respectively, but no statistically significant relationship for CS.

We then conducted a linear mixed model to test this relationship with learning gains as the dependent variable, ACL as a fixed effect, and course section as a second-order random effect. As above, we excluded student responses from sections with fewer than 10 responses. This analysis revealed a moderate positive effect (0.23, t (86) = 8.93, $p < 0.001$) of ACL practices on student reports of their learning gains when controlling for the effects of class sections.

RQ 2a-b: How does student-reported use of ACL practices relate to instructor practices?

We tested the relationship between student- and instructor-reported use of ACL practices, by instructor and course, for the subset of cases where both types of data were available. We compared the mean student reports of ACL practices for each recitation section (section S-ACL) with that section's instructor response (I-ACL). A correlation analysis of section S-ACL and I-ACL revealed a statistically significant, strong positive correlation (0.60, $p > 0.05$), indicating that students' and instructors' reports of the ACL practices in a given class are well aligned.

Similarly, we tested the relationship between instructor-reported I-ACL, and observed proportion of time spent in ACL practices from a single class observation. In general, there is good correlation between what instructors say they do, and what we saw them do (0.76, $p = 0.16$). The outliers in Figure 2 (R) are two GTAs who used the whole observed class session for a guided inquiry task, but reported using much less time for ACL on average.

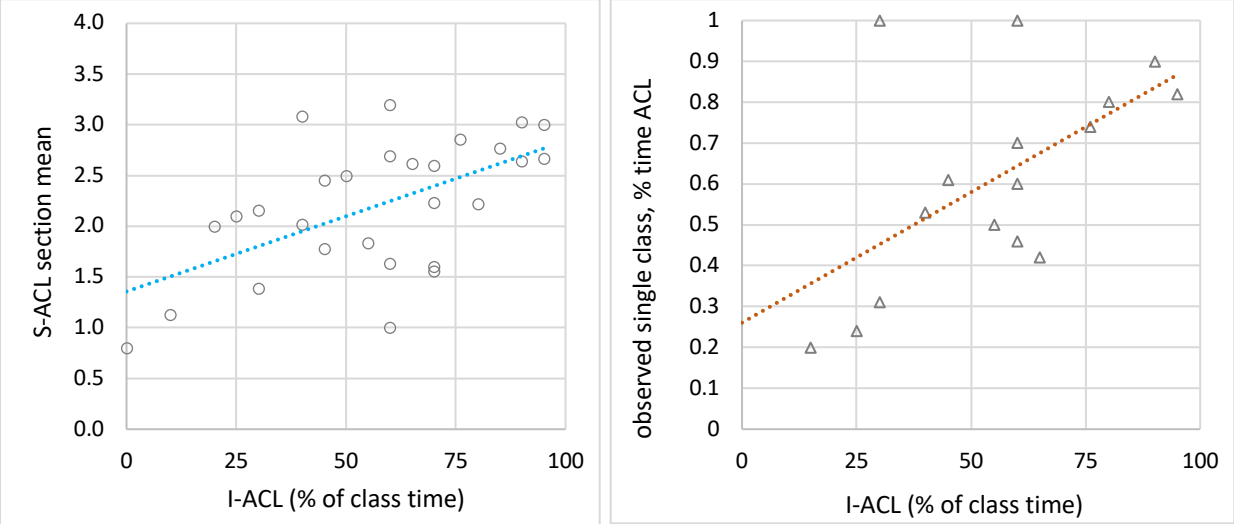


Figure 2. S-ACL section means correlate with their instructor's report, which in turn correlates with observation.

RQ 3: How can these metrics inform formative and summative program evaluation?

Figure 3 (left side) compares mean S-ACL for target courses actively involved in the project versus nontarget courses that had not yet implemented ACL in 2021. The S-ACL metric clearly distinguishes courses that had and had not implemented ACL with an effect size of 1.05.

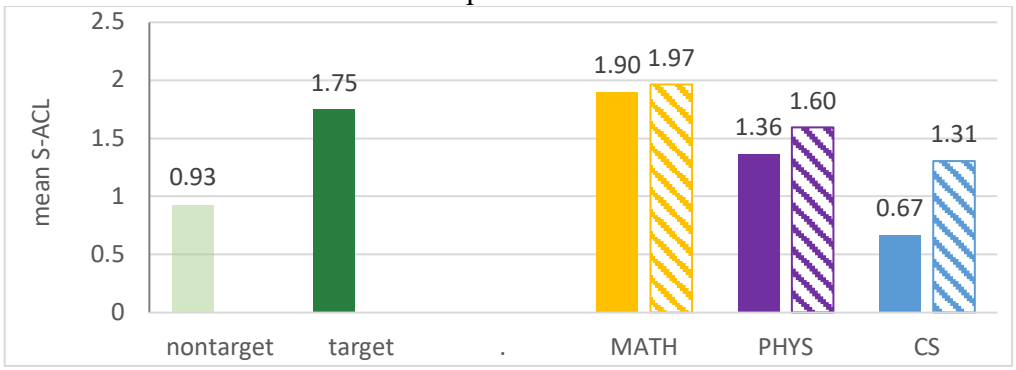


Figure 3. Project-wide and department means for S-ACL in 2021(solid), and 2023 (stripes)

Figure 3 (right side) shows department means for 2021 (solid) and 2023 (stripes). These student data corroborate the PIs' reports about implementation levels: math had already

implemented ACL in target courses in 2021 and sustained that work in 2023; physics' implementation increased from 2021 to 2023; and CS was not active in 2021, but had begun to implement ACL by 2023.

Discussion

Students responded to the items in the TAMI-SS block and were able to discriminate them on the response scale provided. Nearly all students responded; we filtered out responses missing any of the items, as the composite variable S-ACL is not meaningful without answers to each item. Over 95% of students reported coming to class most or all of the time: they are observers who have experiences to report. The S-ACL scale is normally distributed, indicating that most students reported experiencing some ACL practices, with smaller proportions reporting either high or low levels of ACL practices. Analyses are ongoing, but we have found no systematic patterns of difference in response to these items by student subgroup (Apkarian, et al., 2024).

Drawing on student observations to assess implementation of ACL strategies has several affordances. There are many observers to describe the teaching experience across the full term, not just one day. The measure is simple, behavior-focused, and not evaluative. It aligns with other measures we have developed to facilitate triangulation among measures to characterize teaching, and/or to combine with other survey or non-survey measures (exams, grades, etc.). Student-provided evidence can be a golden spike linking a specific initiative to longer-term or harder-to-measure outcomes established in the literature (Brown Urban & Trochim, 2009). However, good response rates are needed to compare courses or sections or pre/post time points. Both student and instructor response rates limited the matching of data for course sections.

The correlation of S-ACL with WIHIC suggests a positive relationship between active learning and students' classroom experience. This alone may be a benefit, if having more positive experiences encourages students to persist in their STEM studies. Further, ACL is linked to stronger learning, as measured by the smaller positive correlation of S-ACL to learning gains. However, in this setting, the impact of these benefits may be modest, given that the intervention focuses on recitations only (and a small group of blended lecture-recitation classes). It remains to be seen if the culture of ACL will "trickle up" to lecture sections as the PI team postulated.

Results from the CS courses offer an example of how these data may be useful in program evaluation and research. The increase in S-ACL (Figure 3, far right) reflects a strong implementation signal as CS joined the project work. Slightly weaker relationships of S-ACL to student experience and learning variables in CS may reflect this earlier stage of implementation. Students may not perceive benefits of ACL teaching until instructors' ACL implementations are less tentative, consistent and clearly communicated. Indeed, class observations in CS displayed less frequent cooperation among students. Moreover, in write-in comments, CS students expressed stronger preference for learning on their own, compared to math and physics students. Instructors may need to actively reshape norms about learning in CS to strengthen student buy-in. Norms can shift quickly as instructors adopt active facilitation practices that students notice—and likely share via the student grapevine (Turpen & Finkelstein, 2010).

As a metric for assessing progress on an institutional effort to increase use of active and collaborative learning, the S-ACL composite from TAMI-SS items was responsive to change over time and sensitive to differences in implementation. We do not recommend using this tool to evaluate individual instructors. Rather, it is useful to indicate progress on efforts to support instructors to shift their practices toward RBIS, at an institutional scale (Laursen, et al., 2019).

Acknowledgments

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Student Meanings for $\hbar/2$ in a Quantum Mechanics Expectation Value Problem

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In modeling physical phenomena and solving associated quantitative problems, students reason about constants in various ways. These symbols represent elements of mathematical structures with embedded algebraic operations and are imbued with contextual physical meaning. In this paper, we report on what meanings for the constant ħ/2 students conveyed in a quantum mechanics expectation value problem from a spin-½ context. Through analysis of interview data with 12 students from two different universities, our analysis documents eight different meanings for ħ/2: part of expression in problem setup, part of calculation, result (either consistent or not consistent with unit interpretation), eigenvalue, measurement, average, and bound. This analysis highlights both the cognitive complexity of reasoning with symbols and the impressive ability of students to flexibly and fluidly leverage multiple meanings for symbols at this content level.

Keywords: linear algebra, quantum mechanics, student thinking

How symbols are used in mathematics and physics can differ. For example, in contemporary mathematics the term "constant" often refers to numbers of interest such as π, $\sqrt{2}$, or the golden mean φ (Wolfram, 2024). In physics, "constant" is additionally used in ways not necessarily needed in pure mathematics, such as for fundamental physical constants that have stable values derived from experimental measurement (e.g., the gravitational constant G). As Redish (2005) states, "We have many different kinds of constants – numbers (2, e, π, …), universal dimensioned constants (e, h, k_B, …), problem parameters (m, R, …), and initial conditions" (p. 2). We note that Redish cleverly included the literal symbol "e" twice, with the first referring to the number and the second referring to elementary charge[1]; we note that e can also be considered as the unit of electric charge. The constant of interest in this study is $\hbar/2$. Planck's constant h describes the behavior of particles and waves on the atomic scale; the reduced Planck's constant[2] $\hbar = \frac{h}{2\pi}$ is the quantization of intrinsic angular momentum, or spin.

In this study, we investigate students' meanings for the constant $\hbar/2$ as they engaged with an expectation value problem from a spin-½ context. For instance, at times students were engaged with $\hbar/2$ as the measured value of spin along an axis, as a quantity with units, or as an eigenvalue of the operator associated with a spin observable. This work is part of a larger project that investigates students' understanding, symbolization, and interpretation of eigentheory and related concepts in quantum mechanics. In particular, we investigate the following research question: *what meanings do students demonstrate for* $\hbar/2$ *in a quantum mechanics expectation value problem from a spin-½ context*?

Theoretical Perspective and Literature Review

In this paper we analyze students' *meanings* for $\hbar/2$. We view an individual's mathematical meaning for a concept not to be inherent or intrinsic to a concept itself but rather be personal to the individual. Furthermore, an individual's meaning for a concept is dynamically formed through the individual's continual interaction and interpretation within their given cultural

[1] $e = 1.602176634 \times 10^{-19}$ coulombs

[2] $\hbar = 1.0546 \times 10^{-34}\ J \cdot s$ or $\hbar = 6.5821 \times 10^{-16}\ eV \cdot s$

settings (Blumer, 1969). Particular to research on students' meanings at a given time, Thompson (2013) offers that "the meanings that matter at the moment of interacting with students are the meanings that students have, for it is their current meanings that constitute the framework within which they operate" (p. 62-63). In this paper, we analyze the meanings in the moment that students are operating with for $\hbar/2$ during their work on a particular expectation value problem within an interview setting. Because we cannot access a person's meanings directly, we infer students' meanings from what they say and write during the interviews. In this way, we "delimit categories of responses according to particular mathematical meanings that we discern from them," and we categorize students' responses "based on meanings we believe might underlie the response based on the best available evidence of interviews" (Yoon & Thompson, 2020, p. 4).

The idea of particular symbols like $\hbar/2$ having a variety of meanings for students has been examined in the literature with other more prominent symbols, such as the equal sign (e.g., Carpenter et al., 2005; Falkner et al., 1999; Kieran, 1981; Knuth et al., 2005). There has also been research that considers the importance of student interpretations of the equal sign in physics contexts. For example, Alaee et al. (2022) break down the relational interpretation of the equal sign into multiple physics-specific categories depending on the relationship described, such as definitional, causality, and assignment. This illustrates a variety of student meanings for this same symbol including ones that may not appear in purely mathematics-focused research. Based on this, when considering the meaning of symbols in physics, it follows that examining and categorizing not only the field-specific meanings of standard mathematical symbols but other symbols which are entirely field-specific could provide some of this same importance.

For such an examination, we began with Arcavi's (1994, 2005) notion of symbol sense. For our purposes, we particularly focus on four of Arcavi's aspects in the specific context we are analyzing: reading symbolic expressions involving $\hbar/2$ in different ways, consideration of the meaning of $\hbar/2$, recognizing the meaning of $\hbar/2$ during intermediate (both calculational and conceptual) steps, and how it has different meanings for the students across different contexts. In Rodriguez et al.'s (2018) analysis of students' mathematical reasoning in chemical kinetics, they state, "Symbols mediate the connection between the physical world and how we think about phenomena" (p. 2115). In our study, the ways that the symbol $\hbar/2$ mediates the connection between physical interpretation and mathematical calculation are central to how students work through the expectation value problem and explain their reasoning.

Methods

The data come from individual, semi-structured interviews (Bernard, 1988) with students enrolled in a Quantum Mechanics course from two different public research-active doctoral universities. All interview participants were recruited during class, gave informed consent before being interviewed, and received compensation for their time and effort. Four interviewees (students labeled A#) were from a junior-level course at a large university in the northwestern US; the course met for seven hours a week, and interviews occurred one month into the semester. Eight interviewees (students labeled C#) were from a senior-level course at a medium-sized university in the northeastern US; the course met for three hours per week, and interviews occurred two months into the semester. Both courses used McIntyre et al. (2012) as the textbook.

This study is part of a larger research project focused on student use of and student thinking around the concepts of basis and eigentheory, as well as their use of and meanings for various notation. In this paper, we consider student responses to the following interview question: "Consider the state $|\psi\rangle = -\frac{4}{5}|+\rangle_x + i\frac{3}{5}|-\rangle_x$ in a spin-1/2 system. Calculate the expectation

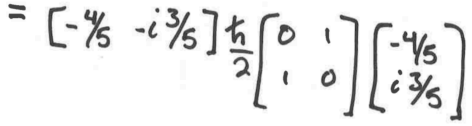
value for the measurement of S_x." Students were encouraged to think aloud as they engaged in the problem, with the interviewer asking clarifying questions along the way as needed.

The results presented in this paper started as one portion of a broader analysis. That analysis considered the correctness of students' solutions, what problem-solving approaches were used, what alternative approaches were discussed (if any), whether certain meanings and terminology were observed in the data, and whether certain concepts were discussed (e.g., basis, eigenvalue). One element we noticed in broader analysis was that some students had written their responses in forms like $\frac{7}{25}\frac{\hbar}{2}$, which prompted us to wonder about their meanings for $\hbar/2$. We then focused on how they interacted with $\hbar/2$, considering all their mentions of "h bar" and what meanings they seemed to correspond to. For our initial analysis, we focused on every instance of the students saying "h bar" in any context (or verbally referring to where they wrote it, but not if they wrote it and did not talk about it). We used open coding (Miles et al., 2013) to classify the students' statements. One of the first apparent divisions was between the use of $\hbar/2$ in a purely mechanical or calculational way, its use in a more conceptual way, and its use in a statement of a result. This led to further categorical divisions, primarily in the conceptual categories based on what was discussed by each student. The meaning categories were developed by considering each of the students in turn and either making new categories (describing what student meanings seemed to be) or grouping them into existing ones. The meaning categories were then cross-checked, combined or refined, discussed, and used to recode the data until we reached agreement.

Results

Our analysis revealed eight meanings for $\hbar/2$ in students' work with the expectation value problem: as part of expression in problem setup, as part of calculation, as part of result consistent with unit interpretation, as part of result not consistent with unit interpretation, as eigenvalue, as possible outcome of an operation or a measurement, as average measured value or weighted average, and as an aspect of positional bounds. In Table 1, we give examples of each of these meanings. In the remainder of the Results section, we share detail regarding the meanings for $\hbar/2$ as a unit with a dimensioned constant, as well as an outcome. We then conclude with an example of student reasoning that utilized many of the meanings for $\hbar/2$ in quick succession.

Table 1. Summary of the 8 meanings for $\hbar/2$ with examples.

Category Name	***Examples***
[Setup] As part of expression in problem setup	(C11) This is the h bar over two times 0 1 1 0 is the spin operator in the x direction. [The matrix used is incorrect here.] 
[**Calculation**] As part of calculation	(A8) "...so S sub x acting on plus x is equal to h bar over 2, include both vector, x, so we're just gonna use that, so then this, is equal to negative four fifths h bar over 2 plus, plus, minus i h bar over 2 three fifths minus."
[Result-Unit] As part of result with unit interpretation	(C6) "And now I like my answer. Um, then it's h bar over two 7/25, um, is the expected value of S sub x." (C5) "...you'd get zero h bar over two as your expectancy value. So, yeah."

[Result-Nonunit] As part of result not with unit interpretation	(C8) "...7, h bar out over 50 is the expectation value for spin x." (A25) "So your expectation value ends up being, uh, 7 h bar over 50, I think."
[Eigen] As eigenvalue	(C1) "And so you'd expect the average, overall the measurements to be halfway between the two eigenvalues, which are h bar over 2 and negative h bar over 2."
[Outcome] As possible outcome for an operation or measurement	(C5) "We only know this, because if I'm already in the x state, and I'm operating on x, I'm going to get an h bar over two or a minus h bar over two with these probabilities, so." (A30) "Um, so, basically when you measure psi for Sx in the Sx direction you can either get h bar over two or minus h bar over two and this is like the average of the probabilities of one or the other."
[Average] As average measured value or weighted average	(C2) If the sample was large enough for you to take, um, the average value you would get would be 7 fiftieths, 7 h bar fiftieths, would be your average measured value, [Int: Okay.] which is an impossible value for you to get with, as, or for a, which is an impossible value for you to actually measure on each individual one [...]"
[Bound] As an aspect of positional bounds	(C1) "And so the average over all measurements would be, uh, positive. It'd be something between zero and h bar over two, and that's not what I'm seeing here."

Student meanings for ħ/2 as a unit

Two of the meanings found in our overall analysis for ħ/2 are distinct but related: as part of a result consistent with a unit interpretation [Result-Unit] and as part of a result not consistent with a unit interpretation [Result-Nonunit]. The Result-Unit meaning is consistent with considering ħ/2 as a universal dimensioned constant (Redish, 2005), but also with considering ħ/2 as the unit itself. As stated by Norton (2024), "(extensive) quantities rely on a unit for measuring them. Whatever we take as a unit becomes 1, and when 1 becomes an iterable unit, every other number in the number sequence becomes some measure of 1s" (p. 466). In our context, a Result-Unit meaning of ħ/2 would be consistent with treating $\hbar/2$ as a 1 for which other numbers – in this case, the expectation value – are measures of this contextual 1.

A notable example comes from C5, who, after completing the question in Figure 1, was asked to change the state vector from Figure 1 to have the same coefficients but now in terms of the z-basis and calculate the expectation value of measurement of S_x. Immediately after correctly getting 0 as their answer, C5 stated, "So, hey I suppose, that's equal to zero h bar over two. And that's a good um, way to write that because it's well, zero with respect to what? With respect to h bar over two …" We note that C5 keeps the $\hbar/2$ in their solution even though it is not necessary in a strictly algebraic context. C5 emphasizes this explicitly by pointing out their solution can be interpreted as no $\hbar/2$'s, implying C5 can hold meaning for $\hbar/2$ as a type of object, namely a unit. By contrast, other students do not give evidence that they were treating $\hbar/2$ with a unit meaning, as seen in the two examples in Table 1.

Student meanings for $\hbar/2$ as an Outcome

In the data set, we documented instances in which students seemed to assign $\hbar/2$ a meaning of being an outcome of either a mathematical operation or the outcome of a physical

measurement. If it was unclear from the context of the utterance in the interview which of these two meanings for outcome was aligned with the student utterance, we only assigned it the category of general outcome. Overall, this was the second-most common meaning for $\hbar/2$ in our data set, behind only the Calculation meaning. Furthermore, all but two of the students made utterances that we coded as conveying an Outcome meaning.

Almost all utterances coded as Outcome were consistent with an Outcome-Measurement meaning for $\hbar/2$. For example, as C12 was beginning their solution process, they considered the measurement values associated with spin up and spin down, stating: "this measurement [points to $|+\rangle_x$] is going to be h bar over two, and this [points to $|-\rangle_x$] is minus h bar over two." A25, also while explaining how they were approaching the problem, stated "you're measuring the x component of the angular momentum and you can either get positive h bar or minus h bar over 2." In both of these examples, the students discussed the act of measuring $\hbar/2$ and $-\hbar/2$ as the result of those measurements. An example in which it was unclear if the student meant the outcome of a mathematical operation or the outcome of a measurement comes from A9: "...it's just the, like, if you did, a histogram result, right, you tabulate your results to be nine twenty-fifths of the time you got negative h bar over 2, and h bar over 2." The student uses the words "results" and what you "got," but the student does not explicitly mention either a measurement or a mathematical calculation. In the context of when the Outcome meanings occurred, namely, as students were explaining why their answer to the expectation value problem was sensible to them, the frequent Outcome-Measurement meaning is appropriate.

Only one student, C5, had meanings in the Outcome category that were consistent with an Outcome-Operation meaning. One example is included in Table 1: "We only know this, because if I'm already in the x state, and I'm operating on x, I'm going to get an h bar over two or a minus h bar over two with these probabilities, so," which was given when C5 shared their problem-solving approach to the question in Figure 1. Although the Outcome-Operation meaning is less appropriate in this context, there is precedence for it in the literature in that students may sometimes conflate mathematical operation and measurement (e.g., Gire & Manogue, 2012; Wawro et al., 2024). Gire and Manogue (2012) offer an explanation for why this could be an understandable conclusion that students might make as they are learning: "In mathematics, operators transform vectors; in physics, measurements change the state of a quantum system; but these are two different uses of the idea 'to change'" (p. 1), and that "it is common for textbooks to say that observables are represented by operators" (p. 4).

Examples of Multiple Codes Together

We conclude our Results section with a vignette highlighting how multiple meanings of $\hbar/2$ are employed in concert with each other by students to convey their problem-solving and/or explanation of an expectation value problem. While all students employed multiple meanings of $\hbar/2$ in some way, this vignette with A30 is a point where they are particularly intertwined. After reading the question, A30 writes $\langle\psi|S_x|\psi\rangle$ (see Figure 1), and says:

A30: I think I'm going to keep this in ket notation because I know what Sx does to the x, uh, kets [...] so the bra so [mumbling] sixteen over twenty, twenty five h bar over two plus nine over twenty-five which is h bar over two!
Interviewer: Okay so, what does that mean physically? What's going on?
A30: Um, so, Sx can either be, oh whoops I made a mistake [laughing] uh this should be a minus x ket and this should be a minus h bar over two which makes this minus … [corrects their answer to 7/25 $\hbar/2$]

A30: Um, so, basically when you measure psi for Sx in the Sx direction you can either get h bar over two or minus h bar over two and this is like the average of the probabilities of one or the other.

Interviewer: Okay. Okay. Do you have any way to look at that and like that seems about right? [...] I don't know if when you got h bar over two you changed your mind if something seemed wrong about it or how'd you see that mistake?

A30: Uh, well, I just I looked up here and I saw that, mm, how I put Sx on the minus x operator was h bar over two rather than minus h bar over two. I was like, oh. that's not correct.

Interviewer: Gotcha. And why - where did the minus come from?

A30: Oh, cause the eigenvalue of Sx times the minus x ket is minus h bar over two.

Interviewer: Okay. Okay, so if you look at the seven h bar over twenty five times two is there any way to look at that and think, oh that seems about right? [...]

A30: Well, I know it's - it can only be between h bar over two and minus h bar over two and so, uhh yeah if it's anything outside of that I know oh that's wrong or, I guess if you have something like this and you're measuring Sx you should not have, solidly one h bar over two or minus h bar over two.

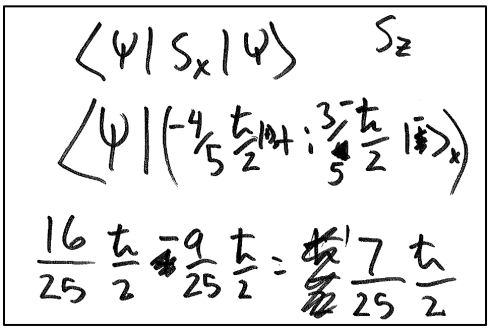


Figure 1. A30's work on the question.

In these 24 lines of transcript, A30 had 13 utterances of $\hbar/2$ or $-\hbar/2$ that show a variety of different meanings in a short span of time. First is one use of $\hbar/2$ with a straightforward Calculation meaning (line 2), where A30 substitutes in the linear superposition of psi in terms of $|+\rangle_x$ and $|-\rangle_x$. Next is A30's statement of their solution (line 3), which has a Result-Unit meaning. After explaining their error and correcting it with a Calculation meaning (line 6), we observe them interpret their solution using an Outcome - Measurement meaning twice (line 9) and an Average meaning (with "this," line 9). When asked if there was a way they knew their first answer was incorrect, they mentioned their error via a Setup meaning for $\hbar/2$ (line 15). When asked about why the negative sign was needed in that setup, A30 uses an Eigen meaning (line 18). Finally, when asked again about evaluating the plausibility of their solution, A30 uses $\hbar/2$ twice in a way consistent with a Bound meaning (line 21), and finally goes back to Outcome - Measurement (line 24). In this case, we observe A30 fluently switching between a variety of different meanings for the same symbol, including multiple procedural and conceptual ones.

Conclusion

In this paper, we report on what meanings for $\hbar/2$ students conveyed in a quantum mechanics expectation value problem from a spin-½ context. Through analysis of interview data with 12

students from two different universities, our analysis documents eight different meanings for $\hbar/2$: part of expression in problem setup, part of calculation, part of result (either consistent or not consistent with unit interpretation), eigenvalue, measurement, average, and bound. This analysis highlights both the cognitive complexity of reasoning with symbols and the impressive ability of students to flexibly and fluidly leverage multiple meanings for symbols at this content level. We illustrated this with data with from A30, an exemplar of a student quickly and seamlessly moving between several different meanings.

Additionally, the use of $\hbar/2$ in ways that algebraically and conceptually work like a unit is a notable result. Further research could point to a difference in the type of understanding held by students who use $\hbar/2$ as a unit and those who do not. We note that this split in the Result-Nonunit and Result-Unit codes has some similarities to the broader calculational/conceptual dichotomy, given the difference is whether the students' statements simply report a calculational result or suggest a related conceptual interpretation of the numerical value of the result. Accordingly, it also corresponds to Arcavi's (1994, 2005) symbol sense category of the ability to read symbolic expressions in different ways, here more focused on the consideration of meaning. Further questions could not only examine this, but also use it in contrast to other instances of units in order to gain a greater understanding of students' concepts of various units in math and physics.

Our results regarding two meanings for $\hbar/2$ related to outcome, either outcome of a measurement or of an operation, are consistent with prior research (Gire & Manogue, 2012; Pina et al., 2024; Wawro et al., 2024). We see this as further evidence that emphasizes the complexity encountered when introducing new symbols with new meanings. This variation in developing or desired meanings is also consistent with the findings of previous work related to the meaning of other symbols, such as the equal sign (e.g., Carpenter et al., 2005; Kieran, 1981; Knuth et al., 2005). In terms of symbol sense (Arcavi, 1994; 2005), developing a conceptual understanding of the mathematical equations, calculational steps, and results in terms of the related physical referents involves ability to understand the different kinds of meaning that different symbols can have across various contexts.

Limitations to our study include focusing our analysis on students' verbalized instances of $\hbar/2$. Our eight meanings for $\hbar/2$ were drawn exclusively from analysis of students engaging in one expectation value problem in a spin-1/2 setting. Future research could analyze student meanings of $\hbar/2$ from other problem contexts (other than expectation value) or settings (such as spin-3/2, which would introduce additional multiples of $\hbar/2$), which could further refine the existing meanings or introduce additional meanings.

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Mathematical Vanishing Act: Why Students are Walking Away

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This study explores the crucial issue of attrition in the undergraduate mathematics major. In this report, we thoroughly examine open-response survey data from seven former mathematics majors describing why they chose to major in mathematics as well as when and why they chose to leave. We situate these data within findings from the 2019 study Talking About Leaving Revisited. *Findings from the present study indicate students may leave mathematics because they are intrinsically drawn to a new discipline, or they may feel pushed out of mathematics due to perceived poor instruction and/or lack of outreach and support.*

Keywords: mathematics major, retention, recruitment, enrollment

Undergraduate major enrollment is a nationwide topic of discussion in university STEM (science, technology, engineering, and mathematics) departments, especially in the mathematical sciences, in which many mathematics major programs are being eliminated due to low major enrollment (e.g., D'Agostino, 2023). It is imperative to recruit students to major in the mathematical sciences for both the preparation of the STEM workforce and the vitality of university mathematics departments, yet the true issue of enrollment in the mathematics major may lie in retention; mathematics has the highest attrition rate of any undergraduate major at 52% (Leu, 2017). The question remains, why are students, who at one point chose mathematics as their major, walking away? In an effort to answer this question and provide recommendations for how mathematics departments can support retention in their majors, we launched the **Ma**thematics **J**ourneys **o**f **R**etention: **W**hy **I**ndividuals **S**witch **E**ducational Paths (MAJORWISE) Survey Study. In this report, we detail the stories of seven former mathematics majors, including why they chose mathematics as a major, when and why they chose to leave, what majors they changed into, and what supports might have retained them in the major.

Background Literature and Framing

Recruitment and retention in undergraduate STEM majors has been a topic of concern for over three decades (Seymour et al., 2019; e.g., McDade, 1988; Seymour & Hewitt, 1997). One line of research of STEM majors (broadly) has investigated trends among students leaving STEM majors, including demographic variables, college readiness, and instructional methods experienced. These quantitative studies have indicated women, students from minority racial groups, students with low mathematics standardized test scores, students who receive lower absolute (or relative) grades in university STEM courses, and students who take courses taught through strict lecture methods may be at a higher risk for leaving a STEM major (e.g., Belser et al., 2018; Chen & Soldner, 2013; Cundiff et al., 2013; Rask, 2010; Watkins & Mazur, 2013). Although quantitative data may allow researchers to find patterns and speculate why students chose to leave their major, research drawing upon qualitative data and the student perspective may help to identify the circumstances and reasons students remain in or leave STEM. A major effort to conduct large-scale qualitative research on reasons students leave undergraduate STEM

began in 1990 with the study *Talking About Leaving: Why Undergraduates Leave the Sciences* (TAL; Seymour & Hewitt, 1997; Seymour & Hunter, 2019). In this study, researchers interviewed 460 students who had either switched from or persisted in STEM majors from ten undergraduate institutions. Seymour and Hewitt (1997) classified *switchers* as those students who entered their university as STEM majors and then switched to non-STEM majors and *persisters* as seniors who were persisting to graduation in their original STEM majors. This study resulted in a framework of explanations for student academic and career decisions; this framework was characterized by a complex push-pull decision making process. *Push* factors are defined as "problems in students' precollege and college experiences that made it difficult for them to persist with their original choices of majors and career aspirations" (Seymour & Hunter, 2019, p. 7). *Pull* factors are defined as "perceived attractions or advantages that drew students to alternative majors and career possibilities" (Seymour & Hunter, 2019, p. 7). The TAL project team conducted a follow up study, *Talking About Leaving Revisited* (TALR; Seymour & Hunter, 2019) from 2013 to 2014 in which they interviewed 96 switchers and 250 persisters from six different undergraduate institutions.

TAL and TALR shared many results with respect to the reasons students chose to leave STEM. These shared top reasons included *loss of incoming interest and motivation* (push factor), *rejection of STEM careers and associated lifestyles* (push factor), *shift to a more appealing career option* (pull factor), and *poor quality of STEM teaching* (push factor). The more recent TALR study identified a few new reasons (or a dramatic increase in the prevalence of reasons identified in TAL) for switching out of STEM, such as *discovery of an aptitude for a non-STEM subject* (pull factor), *discouraged/lost confidence due to low grades in early years* (push factor), and *competitive, unsupportive STEM culture makes it hard to belong* (push factor).

The TAL and TALR studies provided a unique student-centered and qualitative perspective on attrition and issues in undergraduate STEM majors. Yet, this research neglected to investigate (a) issues specific to the mathematics major and (b) reasons students might switch from one STEM major to another. The TALR study sample was predominately life science, engineering, or physical sciences majors, with only 7.4% of participants from mathematics, even though the authorship team identified the mathematics major as having a higher-than-expected switching rate compared to other STEM majors (Seymour & Hunter, p. 76). Seymour and Hunter's assessment of mathematics students as especially high risk for leaving STEM mirrors the NCES Beginning Postsecondary Students Longitudinal Study data, which indicate that of students who declare a mathematics major only 43% (NSF/NCSES, 2012) to 48% (Leu, 2017) will complete their degree in mathematics, which is the lowest retention rate among all degree programs.

There is a need to investigate reasons students leave the mathematics major not only because it leads undergraduate majors in attrition, but also because mathematics students may encounter challenges which would not be identified as reasons for attrition in STEM more broadly. For example, students may struggle with their mathematical identities and self-efficacy as they transition from computational mathematics courses to more abstract and theoretical proof-based mathematics courses (Civian & Schley, 1996). The mathematics students learn become increasingly advanced, and the transition to abstract mathematical thinking and theoretical concepts in university mathematics can challenge students' view of the nature of mathematics and competency in the subject (DiMartino & Gregorio, 2019; Liebendörfer & Hochmuth, 2013).

A few international qualitative studies from the United Kingdom (Rodd & Bartholomew, 2006; Ward-Penny et al., 2011) and Australia (Hall et al., 2022) have investigated retention in the mathematics major specifically. These studies indicated students may become disaffected

with mathematics at the undergraduate level because they dislike lecture-based instruction and feel as though their courses place a strong emphasis on memorization. Many chose to major in mathematics because it was easy for them in secondary school, but it became a challenge at university. We seek to contribute to this extant research by focusing specifically on (a) math majors, and (b) the context of higher education in the United States. In particular, we aim to understand the journeys of students who chose to leave their mathematics majors at four-year undergraduate institutions in the U.S..

Methods

Data Collection and Recruitment. During Fall 2023, our team carried out a nationwide survey, using Qualtrics, with free-response items of students who have been enrolled in a mathematics major at four-year institutions during the years 2013-2023. In addition to general demographic information (race, gender, age, etc.), this survey solicited information such as: reasons for enrolling in a mathematics major, mathematics courses taken at the university level, type of institution attended, instructional methods experienced in university mathematics courses, and timing of and reasons for considering leaving, actually leaving, or staying in the mathematics major. We recruited survey participants through dissemination of our survey through professional networks, social media, and snowball recruitment (requesting participants share the survey with others). Participants were offered an opportunity to provide their contact information to enter a chance to win one of four $50 Amazon gift cards.

Survey Participants. The MAJORWISE online survey received 147 applicable responses from a total 176 responses, which were filtered for responses from participants meeting our sampling criteria. We removed responses when, for example, individuals did not indicate they were enrolled in a mathematics major during the years 2013-2023, the quality control question was not answered correctly, or individuals did not complete survey questions beyond providing consent to participate.

Among the 147 survey respondents, 92 (62.5%) were still pursuing a mathematics major, 48 (32.7%) had graduated with a degree in mathematics, and 7 had previously been enrolled as a mathematics major but had switched into a different degree program. We have reported on findings from the participants who are still completing a mathematics major or who graduated with a mathematics major in other publications (e.g., Heath et al., 2024), and for the purpose of this report we restricted our sample to the seven participants who left their mathematics major.

Data Analysis. Survey responses from the seven participants who left the mathematics major were analyzed by first compiling all responses from a single participant and using their responses to each question to compose an analytic memo (Saldana, 2016) summarizing the chronology of a given participant's journey through their decision to major in mathematics and eventually change that major. These analytic memos were then categorized as describing *push* journeys or *pull* journeys in leaving the mathematics major, according to the scheme presented by Seymour and Hunter (2019) in *Talking About Leaving Revisited.* In the present study, we defined a ***push journey*** as a story in which a student left mathematics primarily due to negative experiences in their undergraduate mathematics major. We defined a ***pull journey*** to be when a student left mathematics primarily because of a draw or allure to a different topic or degree program.

Findings

In what follows, we describe the journeys of the seven survey respondents who decided to leave their mathematics majors. These stories reflect a variety of reasons students may choose to change out of the mathematics major into another degree program. We report these journeys by

first presenting the participants who indicated they felt *pushed* out of their mathematics majors (Allen, Brody, Cora, Danielle, and Erika) and then presenting the participants who reported they were *pulled* into or drawn to another discipline and major (Yosef and Zanab).

Push Journeys

Participants' experiences deciding to leave their mathematics majors classified as *push* journeys were characterized by a disaffection with mathematics, their instruction, or their degree program and course requirements. The first three participants, Allen, Brody, and Cora, all expressed they left their mathematics major because they felt as though they could not succeed. The following two participants, Danielle and Erika, both expressed they left the major because they did not enjoy it.

Allen. Allen was a white, male student pursuing a B.S. in Mathematics from a large, public R1 institution in the Southeastern United States. He was enrolled as a mathematics major for three years until he decided to change his major to Business after his junior fall semester. Allen decided to major in mathematics before he began college, and he described that he chose mathematics as his major because he had wanted to be a mathematician since his childhood and an intrinsic love of the discipline. Allen wrote,

> I wanted to be a mathematician since I was a kid. I see math as a study that is trying to make sense of the universe instead of trying to make sense of man-made things. I love analyzing data and trying to find a pattern.

While enrolled as a math major, Allen took many different mathematics courses, including upper-division courses like Abstract Algebra I and Numerical Analysis. Allen attributed his experience in Numerical Analysis to his decision to change his major. He claimed the university and the mathematics department faculty had not prepared him to succeed because he was expected to know a coding language in numerical analysis, yet he did not. After his experience in Numerical Analysis, Allen changed his major to Business. Allen said if there had been different instructors teaching his course, or different requirements set up to take certain courses, he may have been supported enough to stay in the major.

Allen's story depicts a *push* journey because despite his intrinsic passion and love for mathematics as a topic, he felt there were barriers to his success embedded in his mathematics degree program, specifically with respect to how the major prepared (or did not prepare) students to take subsequent required major courses.

Brody. Brody was a white, male student pursuing a B.S. in Mathematics Education at a large, public R1 institution in the midwestern United States. Brody decided to major in mathematics before starting college, and he was enrolled as a math major for one year before changing his major to Social Studies Education. Brody said he chose to major in mathematics because he was successful in mathematics in high school, enjoyed teaching, and wanted to be like the great mathematics teachers he had in high school. Brody took College Algebra and Precalculus at the undergraduate level, but when he enrolled in Calculus I, he struggled and decided to drop the course and change his major. Brody said, "I struggled a lot in calc 1. [I] did not want to put myself through the rigor of the program and have my mental state be worsened by math classes." Brody indicated he could not think of any supports that might have helped him stay in the major.

Brody's story illustrates how a student might initially feel drawn to mathematics because of success in the subject in high school. However, Brody perceived that the challenge of Calculus and an extended study of mathematics would worsen his mental state, and thus *pushed* him out of the major.

Cora. Cora was a Black/African American, female student, pursuing a B.S. in Mathematics at a small, private HBCU in the Southeastern United States. Cora chose to major in mathematics as a first-year student, having originally declared a Computer Science major. Her decision to pursue a mathematics major was motivated by her dislike for coding in computer science and her enjoyment of learning mathematics and solving problems. Cora took Discrete Mathematics and Calculus I as an undergraduate, but she changed her major to Education during the middle of her second year at the university while taking Calculus II. Cora attributed her decision to leave mathematics to poor instruction in her mathematics courses. She wrote,

> I decided to switch when I realized that my professor and many others in the mathematics department were not efficient in teaching math courses. I did not think it was fair for me to stay in a major that did not have great professors.

When asked what supports might have helped her stay in mathematics, Cora responded that more support from her mathematics department chair and dean as well as making tutoring available for higher-level mathematics courses may have changed her journey.

Cora's story highlights how perceived poor instruction in mathematics courses and a lack of infrastructure (e.g., tutoring for upper-division courses) can *push* a student out of mathematics. Cora provided a unique perspective as a student who did not begin college as a mathematics major, and during her time as a computer science major she had potentially experienced different quality or type of instruction in another department before joining the mathematics department.

Danielle. Danielle was a white, female student pursuing a B.S. in Mathematics with a concentration in Applied Mathematics at a large, public R1 institution in the Southeastern United States. She chose to major in mathematics before starting college and she chose mathematics because she loved the subject in high school and wanted to pursue a career in applied mathematics. Danielle was enrolled as a mathematics major for three semesters; she took a variety of mathematics courses at her university: Calculus II and III, Linear Algebra, Introduction-to-Proof, Probability and Statistics, and Partial Differential Equations. Danielle's decision to leave her mathematics major was primarily motivated by frustration with her degree program and course requirements. She wrote,

> The applied mathematics concentration focused too heavily on theory, in my opinion. It seems like everyone is taking the exact same classes overall, with about 3 or 4 electives to differentiate between concentrations. The degree felt poorly designed and very rigid in what students could or could not take.

Danielle's discontentment with her degree program depicts a *push* journey because, although she still enjoyed what she considered to be "applied" mathematics, she perceived her major as requiring too many theory-based courses and the concentration as not distinct enough from the standard mathematics major at her university.

Erika. Erika was a white, female student seeking a B.S. in Mathematics Education at a large, public R1 institution in the midwestern United States. She decided to major in mathematics before beginning college because she wanted to be a mathematics teacher but changed her major from mathematics to Business at the end of her second year of her undergraduate studies. Erika decided to leave mathematics because, "Awful professors made [her] very disinterested." She noted that if the mathematics courses were structured better or if she had gone to a different institution with "better professors" she would have stayed in the mathematics major.

Erika's story highlights another example of a student leaving mathematics because they considered their course instruction inadequate, *pushing* them to a different major and department. Erika did not provide a detailed account of what made the instruction she experienced poor.

Pull Journeys

Participants' experiences deciding to leave their mathematics majors classified as *pull* journeys are marked by an allure or draw to another discipline or department. Two students, Yosef and Zanab, described their decision to leave mathematics as pull journeys.

Yosef. Yosef was a multiracial, male student seeking a B.S. in Mathematics and Statistics at a large, public R1 institution in the northwestern United States. Yosef decided to major in mathematics after transferring to his new R1 Institution. Yosef majored in mathematics for three academic years before switching to computer science. Yosef expressed lack of direction with his career, and ultimately decided to major in mathematics because he enjoyed solving problems and proving statements. Yosef took several mathematics classes before struggling in Calculus II. Yosef remarked, "After I realized that math was difficult, I fell in love with arguably something even more difficult and equally as reliant on math: computer science."

Yosef expressed he preferred the instruction in his computer science courses, which involved large projects with writing and problem solving, and his mathematics courses, in comparison, felt like "busy work." Yosef said he may have wanted to stay in math if it were taught differently. He wrote,

> I think the entire way we teach math would need to change. I think a project focused and problem-solution focused way of teaching math would've helped me. It was hard for me to think that integration techniques would help me in the real world, but knowing how to create a hash table felt more applicable.

Yosef's story depicts a *pull* journey because he was attracted to the instruction he experienced in another discipline and felt drawn to a more interesting and applicable major in computer science. Yosef indicated the instructional methods used in mathematics classes may be a deterrent to the retention of some mathematics majors.

Zanab. Zanab was a Black/African American, female student pursuing a B.S. in Mathematics at a large, public R1 institution in the northeastern United States. Zanab chose to major in mathematics because she liked the concepts and ideas of mathematics. She took many mathematics classes, including complex analysis and topology, before she switched to a computer/information sciences major during her second year of her studies. Zanab decided to leave the mathematics major due to a preference for her new major, writing, "I just preferred the new major. I wanted an intersection between compsci and psychology. I like the challenge of programming."

Zanab's story depicts a *pull* journey because she felt pulled to her true passion. Zanab said there was nothing that could have changed to make her stay.

Discussion

Across the journeys of the seven participants who left their mathematics major, there were many factors influencing their decision to leave, including: (1) challenge and struggle, (2) instruction, (3) curriculum, and (4) passion and interest. These align with and supplement the findings from *Talking About Leaving Revisited* (Seymour & Hunter, 2019) to help understand why students may leave mathematics majors.

Several of the participants mentioned challenge and struggle as a factor influencing their decision to leave mathematics. Allen, Brody, and Cora, all mentioned struggling in a mathematics course as push factors influencing their decision to leave. Researchers in TALR similarly found that students across STEM disciplines may leave STEM because they were "discouraged/lost confidence due to low grades in early years" (Seymour & Hunter, 2019, p. 92). Interestingly, although Brody and Cora did change their majors during Calculus, Allen's journey

illustrates a case in which it was not courses in early years that discouraged him. Challenge was not only associated with push journeys; both Yosef and Zanab also mentioned challenge and difficulty level in their responses. Thus, for some students, difficulty, challenge, and struggle pushed them away from mathematics, while others perceived difficulty and challenge as something pulling them to a subject, whether mathematics or their new discipline.

In addition to facing challenge in mathematics courses, students discussed instruction of their courses with respect to choosing to leave mathematics. Allen, Cora, and Erika cited poor instruction of their mathematics courses as pushing them out of mathematics. TALR also found this to be a common reason for leaving STEM majors, with 96% of switchers referencing "poor quality of STEM teaching" as a concern contributing to their switching (Seymour & Hunter, 2019, p. 92). Limited by the survey format of these data, our team could not inquire further about what "poor quality instruction" meant to Allen, Cora, or Erika. Yosef, however, described being pulled to computer science because he experienced teaching he found more engaging. He noted he enjoyed instruction based in projects, problem-solving, writing, and stated if mathematics were taught more from this perspective he may have wanted to stay. Aligning with the TALR study, Seymour and Hunter also found students may switch from STEM because they "prefer teaching approachs in non-STEM courses;" (p. 92) however, Yosef's experience of teaching he preferred was in another STEM subject.

Three students, Danielle, Allen, and Erika mentioned issues related to curriculum and program of study as pushing them from their mathematics majors. Danielle's complaint regarding her major program was that even the applied mathematics concentration required too many theoretical mathematics courses, and Allen was frustrated he was expected to know a coding language without having been required to take a programming course. The TALR study mirrors these complaints in the concern they found, "STEM curricular design problems: pace, overload, labs, alignment" (Seymour & Hunter, p. 92). It could be debated whether or not Allen and Danielle's programs of study were actually poorly designed, but it is also possible the intentions of the program design and expectations of students in those programs were not clearly communicated to the students.

Finally, passion and interest were primary reasons Yosef and Zanab found themselves pulled to other subjects. This aligns with TALR's finding that students left STEM if they discovered an "aptitude for a non-STEM subject" and "non-STEM major offers better education, holds more interest" (Seymour & Hunter, 2019, p. 92). The latter of these concerns identified in TALR also resonates with Erika's comment she "had awful professors that made [her] very disinterested," as she indicated another major would offer a better education and hold her interest.

Conclusion

The seven stories highlighted in this report provide examples of the many different factors affecting retention and attrition issues in mathematics departments across the United States. The results from this study support that some of the findings from TALR are also applicable to the discipline of mathematics, yet it is notable that three participants in this study (including both of the pull stories) switched to a different STEM discipline, rather than out of STEM. Common across both push and pull journeys is discontentment with teaching in university mathematics. As our sample of students who left their mathematics major was small and the survey format of data collection inhibited us from inquiring into further detail regarding participant comments, we intend to continue this research and gain a larger sample of students who have switched out of the mathematics major and include more in-depth qualitative data such as interviews or written narratives to gain a deeper perspective on students perceptions' of their mathematics programs.

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Real Analysis Professors' Noticing of Equity Gaps and Minoritized Students' Perspectives

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Addressing equity and opportunity gaps in undergraduate education requires change across systems, including the pedagogies and orientations of the undergraduate faculty whose courses and departments perpetuate these gaps. Hence research on how undergraduate faculty notice quantitative and qualitative evidence of inequitable experiences has the potential to advance reform efforts. We conducted an interview-based study of 5 mathematicians who were presented with data analytics indicating equity gaps in proof-based courses at their university, and statements from minoritized students in these courses about their experiences. We found that the faculty may hold split inclusive and exclusionary frames about students and mathematics; that there may be a persistence of personal experience as an interpretive lens; and that qualitative and quantitative evidence may elicit different frames of instruction and students. We discuss these findings through the lens of a noticing framework.

Keywords: proof-based course, equity, mathematicians, noticing

Background

Equity and opportunity gaps are well-documented in undergraduate mathematics education (e.g., National Academies of Sciences, Engineering, and Medicine [NASEM], 2016; Solomon & Craft, 2016; Leyva, 2016). Recent scholarship points to the need for centering minoritized students' views of undergraduate instruction (e.g., Larsen et al., 2016; Leyva et al., 2021a, Leyva et al., 2021b). Yet systemic change requires not only articulating mechanisms of marginalization, but also change in the orientation of undergraduate faculty whose courses and departments may reinforce gaps. Thus, research on how undergraduate faculty may view evidence of inequity can advance reform efforts. Such research can identify affordances and barriers to using qualitative and quantitative evidence regarding equity to shift instructor orientation.

This study contributes to this area by examining how undergraduate faculty frame and attend to data on equity gaps or data about minoritized students' perceptions. We focused our study on mathematics faculty who had taught real analysis, a course with a widespread 'weed-out' reputation (e.g., reddit r/math, 2018) and yet also recommended for inclusion into mathematics major programs (Mathematical Association of America, 2015). We act on the theory that a course with this kind of reputation and role may be one that amplifies impact on identity, particularly for minoritized students. We address the research question: *What do real analysis professors notice in data about the performance and experiences of minoritized students in real analysis courses where equity gaps exist?* By investigating this question, we aim to gain insights into how instructors may interpret and address equity issues in their classrooms.

Conceptualizing Equity

Equity can be viewed from dominant and critical perspectives, and we use both here. From a

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dominant perspective, equity can be conceptualized as "the inability to predict mathematics achievement and participation based solely on student characteristics such as race, class, ethnicity, sex, beliefs, and proficiency in the dominant language" (Gutierrez, 2002, p. 153). Equity 'gaps' can be quantified from this perspective, for instance via disparities in performance or persistence among students of different backgrounds (which may be perpetuated within an instructional and institutional environment) (e.g., National Center for Educational Statistics, 2011). For all that such "gap gazing" (Gutierrez, 2008, p. 357) may obscure asset-based views and perpetuate a static view of equity, these statistics can call attention to inequity and are often used to substantiate calls to action (e.g., NASEM, 2023).

From a critical perspective, equity concerns identity and power. This perspective "builds upon the idea that mathematics is a human practice that reflects the agendas, priorities and framings that participants bring to it" (Gutierrez, 2011, p. 21) which complicates the desire to quantify 'gaps' within. Instead, this perspective aims to document the experiences of individuals as they face situations which diminishes their identity or their power. A critical perspective is needed in cultivating asset-based – and anti-deficit – views, and to humanize the experiences of those from historically marginalized populations.

Inequitable Experiences in Departments of Mathematical Sciences

As a report of the American Mathematical Society (2021) detailed, departments of mathematical sciences have had a long history of racial discrimination, and this discrimination continues to impact equitable participation in the mathematics community today. Recent scholarship making deliberate efforts to elicit and situate the perspectives of minoritized students largely underscore three arguments. First, minoritized students continue to encounter negative experiences that impact their identities in mathematics (Apkarian et al, 2024; Leyva et al., 2022; Leyva et al., 2024). Second, researchers and practitioners can and must find more ways to recognize and showcase the creativity and cultural assets of students from historically marginalized populations (Adiredja et al., 2020; Valdovinos Gutierrez & Ko-Wong, 2024). Third, instructors can disrupt or reinforce inequities, through their actions (Ball, 2024; Cawley & Wilson, 2023; Reinholz et al., 2024).

Building on this scholarship, an emerging area of work examines what happens when departments and instructors are asked to engage with evidence toward these arguments. For instance, Bolick and Voigt (2023) used data analytics to document inequities for undergraduate mathematics leadership teams to consider in transforming mathematics at their institutions. Adirejda et al. (2024) showed preliminary evidence that professional development of undergraduate mathematics faculty such as critical conversations around race and gender and funds of knowledge interviews may lead to instructors taking a more humanizing view of instruction. These efforts show promise that programmatic consideration is possible through engagement with qualitative and quantitative data. At the same time, one area that has received less attention is instructors' framing of forms of evidence toward particular courses that they have taught and are likely to teach in the future, and over which they have relative autonomy. This focus is important to understand because it explicitly leverages instructor agency.

In courses such as real analysis, instructors often hold primary responsibility for managing and shaping the classroom experience. In this sense, instructors are on the 'front lines' of education, with the autonomy to foster an inclusive learning environment – or not. Given that instructors have significant control over classroom dynamics, meaningful change in addressing equity gaps requires a deep understanding of how instructors perceive and interact with these issues. Specifically, it is essential to explore how instructors frame, recognize and respond to

data indicating performance disparities and to accounts detailing the experiences of minoritized students–in sum, how they *notice* signals of inequity.

Theoretical Framework

We use Louie et al.'s (2021) Frame-Attend-Interpret-Respond (FAIR) framework to conceptualize *noticing.* This framework embeds the practices of attending, interpreting, and responding (e.g., van Es, 2011) within a broader action of framing. Following the work of van Es and colleagues, in teaching and reflecting upon teaching, an instructor *attends* to particular features of instruction (and does *not attend* to other features), whether consciously or subconsciously; *interprets* these features, such as in terms of student identity, potential, and competence; and *responds* to this attending and interpreting in what they say or do in the classroom or how they plan for teaching.

Louie et al. (2021) argued that *noticing* necessarily occurs within sociopolitical contexts, for classrooms are social environments where power dynamics exist (with the instructor and among students), thus learning environments are neither socially isolated nor politically neutral. As such, those who have the responsibility of noticing in learning contexts should be aware of their framings of the situation, including the local, institutional, and cultural aspects. Other framings may be related to the students, the discipline of mathematics, or interactions that occur in the learning environment, and these framings may come from a deficit or anti-deficit perspective, albeit unknowingly at times. *Frames* create "expectations for how the emerging activity should unfold" (Hand, 2012, p. 251) and may draw on unseen, dominant assumptions and cultural values that guide the actions within noticing (Louie et al., 2021); and hence *framing* shapes *attending*, which in turn shapes *interpreting* and *responding*.

Mathematics and mathematical ability may be framed in exclusionary and inclusive ways (Louie, 2017). For instance, framing mathematics as a "fixed body of knowledge to be absorbed" can serve to exclude learners, whereas framing mathematics as "about making sense of ideas and understanding connections" may promote inclusion through meaning making; and framing mathematical ability as "hierarchical" is exclusionary, whereas framing mathematical ability in terms of the potential for "everyone [to have] both intellectual strengths and areas for growth" can lead to naming and seeing competence in students (p. 496). Instructors may also frame their observations of teaching from a deficit- or asset-based view of their own teaching capabilities.

Study Context

At the University where the study took place, the mathematics department hosted a presentation of data analytics that analyzed "DFW rates" for the department's courses over a three-year period. Here, a DFW rate of a course means the percentage of students in that course who received a D or F in the course, or who withdrew from the course. The analysis revealed that underrepresented students—defined by the University as those identifying as African American, Hispanic, and/or Indigenous—had higher DFW rates in an introductory real analysis course compared to their represented peers. Specifically, at the time, the DFW rate for real analysis was 25% for represented students and 35% for underrepresented students. In this same period, the DFW rate for real analysis for continuing college-going generation students was 24%, whereas it was 27% for first generation students.

Following this presentation, two of the authors surveyed and conducted interviews of underrepresented students about their experiences in real analysis. We discuss these surveys and interviews here because the data from these protocols were written into the protocols of the study presented in this report. Given that the University is predominantly white, students who identify

as Black/African American, Hispanic/Latinx, Asian/Asian American, Native American, or Native Hawaiian/Pacific Islander are often visible minority groups on campus. In line with Shah's (2019) recommendation to avoid treating Asian identities as a monolithic "model minority," we included Asian students in our study to better understand their experiences in upper-level mathematics courses.

In all, 27 students participated; all were current or former undergraduate mathematics majors who identified as Black/African American, Hispanic/Latinx, Asian/Asian American, Native American, or Native Hawaiian/Pacific Islander and who had taken or planned to take at least two of the following junior-level, proof-based courses: introduction to proof, introduction to abstract algebra, and real analysis. Data collection consisted of three phases: informational survey, primary interview, and secondary interview. The brief informational survey included background items related to students' identity, work obligations, and school obligations (cf. Jett, 2009). This survey asked questions about the number of hours students worked while taking these math courses, if they had to commute to campus, and whether they identified as a first generation college student. Similar to Herzig (2002), we structured the primary interview as a chronological mathematical autobiography. The secondary interview allowed us to co-construct a narrative with participants based on their responses to the primary interview. This narrative structure provided a space for students to clarify and further elaborate on their stated experiences in mathematics (Leyva et al., 2021a). The secondary interview also allowed us to conduct a member-check for validity purposes.

Overall, 13 students completed the survey and both interviews, and an additional 14 students completed an extended survey where selected interview questions were written as multiple choice questions. The surveys and interviews highlighted ways that real analysis was felt as gate-keeping; that students have responsibilities outside of class that influence their participation and engagement; and how students' experiences of the class depended heavily on the instructor, including how they attended–or seemed not to attend–to students' questions and contributions. A more comprehensive description of this part of the study is found in Bennett and Lai (in press).

Methods

The participants of the present study were five instructors spanning assistant, associate, full professor levels at the University. All five taught at least one of the three courses for mathematics majors in the past three years; most had taught more than one of these courses during multiple semesters. The professors (with pseudonyms N, P, T, V, Z) completed an informational survey and participated in a hour-long, semi-structured interview. Four of the five professors identified as white (the fifth chose not to respond), and all five answered "English" for the primary language that they think about or "do" math in, although this was at times different from their "native" language. None of the five identified as first-generation college students.

In the interview, we presented professors with quantitative and qualitative data in the form of DFW rates from Fall 2018-Summer 2023 and quotes from students in the above-described interviews. In particular, we showed them four responses to the question "What do you wish your professor knew?", which students interpreted broadly, and two responses to "What does it take to succeed in real analysis?" As the professors viewed the quantitative data, we asked probing questions, such as, "What drew your attention to this part?", "What do you think is the 'problem' that is being addressed?", and "Are there things that could change this outcome?" We allowed professors to choose which student quotes to which to attend. We asked questions such as, "What drew your attention to that student's experience?", "What's your understanding of what this student is trying to convey?" "Do you think a student might feel this way in your

course? What would you do?" These questions were meant to elicit a talk-aloud of acts of noticing, in particular *attending*, *interpreting*, and *responding*.

To analyze responses, we organized professors' noticings of the quantitative and qualitative data into tables structured by the FAIR framework (Louie et al., 2021), focusing first on how they *attended* to, *interpreted*, and *responded* to the data presented. Similar to Hand (2012) and Louie and colleagues (2021), we then analyzed the noticings to look for themes to help us understand how professors *framed* students' experiences in real analysis. In articulating frames, we described how professors noticed in deficit or anti-deficit ways and the impact on frames about students, mathematics, and interactions, using Louie's (2017) exclusionary and inclusive frames as an analytic starting point.

Findings

In this section, we present three main findings from our analysis of professor noticings.

Split of Asset-based/Inclusive and Deficit-based/Exclusionary Frames

A key finding in the analysis is the complicated nature of asset-based/inclusive frames and deficit-based/exclusionary frames; how a professor can hold both kinds of frames when discussing the qualitative evidence. Professor N enjoyed teaching real analysis. He loved his real analysis course when he was an undergraduate and attributes it as being his inspiration "to be a math major and become a mathematician," affirming the view that real analysis can be a gateway. Furthermore, he shared that he tells this story with his students whenever he teaches real analysis. In this way, Professor N holds an inclusive frame of mathematics.

When asked how real analysis compared to other similarly advanced undergraduate mathematics courses, Professor N pointed to his experience teaching a linear algebra course. While he does not teach linear algebra as proof-based, he shared:

> I have them justify things, but I'm not looking for what I would consider to be a properly written form of proof. I'm just looking for something kind of a weaker level of justification... [Real analysis] has been quite a bit harder for the students. I mean, there's some that just seem to really struggle with understanding epsilon-delta proofs, and it's a real challenge to get them past the class sometimes.

We understood Professor N as having framed students as either capable or not. While the gap in DFW rates was "disturbing" to him, it was also "unsurprising" that there was a gap, as students will have varied levels of preparedness and capabilities. Professor N is an example of how two opposing frames can coexist simultaneously yet not create a disequilibrium of personal beliefs about mathematics or students when discussing the quantitative performance data.

Persistence of Personal Experience as Interpretive Lens

A second central finding in our work is how professors primarily relied on their personal experience as an interpretive lens for noticing the presented data. This manifested as professors attending to and interpreting the student quotes in light of their own teaching experiences, sometimes regarding the goals or improvements they have in mind for their classes.

In his response to "What do you wish your professors knew", a student, Vince, shared how the "lack of representation of underrepresented students in math courses could discourage a student from organizing study groups or reaching out to others for support." When prompted to discuss one of the four student quotes, four of the five professors said that Vince's response "caught my attention" or were interesting in some way. Professor N said,

> I think what Vince wrote is the one that most caught my attention just because again,

> affirming this fear I have that underrepresented students are still kind of left out. But the guilt I have about the fact that I don't really know how to solve that or at least I haven't figured out a way and I haven't really done a great job of addressing this very real issue.

Professor N connected Vince's statements to his own beliefs and feelings about teaching, particularly about being inclusive in his classroom. In addition to his feelings of "fear" or "guilt", other professors conveyed their personal emotional reactions to what students shared, feeling "crushed" or "shocked" by some of the students' experiences. Similarly, Professor T attended to what Vince was saying in light of his own musings about teaching: "I think that Vince's was the most thought provoking to me, just because I think Vince talks about some of these things that have been on my mind the last couple of semesters."

We also found it noteworthy how the professors made evaluative statements of the presented student quotes based on their own perceptions and experiences. For instance, comments such as, "I think I agree with [student]", or "I absolutely believe the things she's talking about with her experiences" emphasized the (in)validity of students' accounts. At times, the evaluative statements seemed to justify dismissing what students said, like "I think his impression…is somewhat unrealistic." However, this perception of "unrealistic" is based on the reality of the professor, not on the lived experiences of the student.

We likened this phenomenon to the notion of listening to respond rather than listening to understand. When personal experience is the listener's primary lens, they bring what others say back to their own context.

Qualitative and Quantitative Evidence May Elicit Different Frames

When reading the statements of students' experiences, the professors seemed to acknowledge that people's stories are complex and there does not exist an immediate solution to address the challenges experienced. After reading that some students expressed feeling unsupported and underrepresented in their mathematics courses, Professor V said, "I know there's no magic solution and I don't know that there's anything short term that we can do, but maybe there's things that we can put in place to support and represent the students." Similarly, Professor T underscored the importance of seeing Vince as a whole person in the classroom:

> I think what Vince is talking about is having professors that really fundamentally don't understand his experience. [...] Vince wishes that professors understood his experience, what experiences and what aspects of his life and identity are coming along with him to this classroom.

Professor T expressed that professors should get to know students on an individual level in order to understand their experiences and full selves, not just as learners of mathematics.

In contrast, four of the professors viewed the quantitative data from a classroom level, rather than an individual level. This led to ponderings about what solutions might exist "for all" students in proof-based courses. Professor Z said, "There are things that work well for everybody. I don't know things that would work well for a specific student population, but what works well for everybody is additional support inside and outside the classroom." Four of the five professors mentioned that "getting students to talk to each other" or implementing active learning strategies were a viable solution to the institutional performance gaps presented in the quantitative data. In most cases, the qualitative data helped the professors problematize and shift away from "for all" solutions for the challenges that minoritized students face, to viewing and listening to students on an individual level.

Discussion and Conclusion

We discuss our three main findings through the lens of the FAIR framework (Louie et al., 2021). In Table 1, we illustrate professors' framing of students, mathematics, and the presented student narratives. Our analysis revealed that it was important to understand professors' framing of students and mathematics *prior* to analyzing their FAIR noticings of the data. During our session, we will share similar tables for each professor analyzing their noticings of both the qualitative student narratives and quantitative student achievement data (i.e., DFW rates) and how these align with or draw on their framings of students and mathematics in the context of proof-based courses for mathematics majors. We found that frames of students, mathematics, and data differed depending on the type and presentation of the data (i.e., qualitative narratives or quantitative rates and graphs). For instance, our third finding suggested that although professors interpreted student quotes as complex puzzles that should be addressed on a personal level alongside students, they interpreted quantitative performance data needing a "for all" solution (e.g., Leyva et al., 2022). We will share how the two tables compare across professor framings and noticings of the two data types.

Table 1. Noticings of students' real analysis experiences with the FAIR framework

Students	**Framing** students as people with many responsibilities, different capabilities, and intersecting identities		
	Attending to students' experiences and their (un)success in math	**Interpreting** student challenges as complex puzzles to solve	**Responding** inclusively to the identities of individual students
Mathematics	**Framing** real analysis and proof-based courses as rigorous and necessary		
	Attending to the difficulty of the subject shared by students	**Interpreting** the difficulty of the course as unavoidable	**Responding to** quotes with worry, but not knowing a remedy
Qualitative Data (Student Narratives)	**Framing** qualitative data from students as sometimes trustworthy, valid, and compelling and sometimes unbelievable or unhelpful		
	Attending to the validity of what students say	**Interpreting** student quotes in light of their own experiences	**Responding to** quotes with an evaluative statement

Finally, the protocol we used to prompt professors to notice (i.e., to state what stood out in the student quotes, rephrase what the student is trying to convey, and describe how the account could inform their own practice) was effective in focusing the discussion on the students' experiences while rooting the conversation in practical implications for teaching. We wonder how conversations about institutional performance gaps and improving student experiences are facilitated in meetings of departmental stakeholders, and if attendees are similarly given the space and structure to attend to, interpret, and respond to presented data. We encourage those working toward equitable changes in mathematics departments to first attend to and understand what students are sharing in their accounts of classroom experiences before looking for large-scale solutions and implications for teaching.

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Students' Generalizing Activity While Using Determinant Applets

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In this article, we share results from a series of interviews with a pair of undergraduates using an applet designed to support students' understanding of determinants. We identify four generalization clusters that the participants developed, and we share our analysis of the activity types that led to these generalizations. We used Ellis et al.'s (2022) Relating, Forming, and Extending framework to characterize the participants' generalizing activity. We share results from this work and discuss the implications for applet design, instruction, and future research.

Keywords: Linear Algebra, Determinants, Generalizing, Dynamic Geometry Software, Inquiry

This paper extends prior work developing inquiry-oriented curricular materials for linear algebra instruction. We designed two web-based dynamic geometry software (DGS) applets intended to provide students with real-time geometric representations of the determinant of a matrix, visualized by the parallelogram/parallelopiped image of the standard basis vectors under a user-determined 2×2 or 3×3 matrix, respectively (Wawro et al., 2023; Mauntel et al., 2024). In the applets, users can manipulate the matrix entries and observe, in real-time, the effect these changes have on the parallelogram/parallelepiped and a dynamic "determinant" value calculated by the applet. Initial implementations of the applets showed strong potential for students' generalizing activity (Wawro et al., 2023). We aim to better understand students' in-the-moment generalizing based on their experiences with the applet and how students' use of the applet might inform design changes to the curricular materials. In service of these goals, we conducted semi-structured interviews with students as they used the applets, prompting them to discuss broader relationships they identified through their activity. We sought to identify (1) what generalizations and generalizing activities do students engage in as they use the determinants applets? and (2) how do these insights into students' generalizing activity inform our applets' design?

Literature and Theoretical Framings

We draw from several research areas to contextualize our work. We first situate our work relative to existing research on student thinking about determinants and dynamic geometry software. We then detail a framework for students' generalizing activity that we use for analysis.

Determinants and Dynamic Geometry Software

In our literature search, we found few studies that attend to student understanding of the determinant concept (e.g., Chagwiza et al., 2021; Ndlovu & Brijlall, 2016). For example, Kazunga and Bansilal (2018) focused on identifying misconceptions and errors used by in-service secondary mathematics teachers. Although most correctly solved determinants of 3×3 matrices, few were successful with 4×4 matrices because many tried to extend or generalize a 3×3 procedure. Some participants also conflated the determinant of a matrix with the matrix itself (evidenced, for instance, by using $|A^T|$ to symbolize the matrix A^T), and many did not use known properties to compute the determinant of a matrix created from one with a known

determinant (e.g., the columns were swapped). Through examining pre-service teachers' understanding of determinant using numeric, algebraic, or geometric representations, Durkaya et al. (2011) found that participants employed numeric and algebraic representations of the determinant, but not a geometric one, in answering computational questions; however, during interviews, geometric representations were helpful to improving their understanding.

Some research has focused on integrating technology to support students in developing a conceptual understanding of the determinant. Research shows that Dynamic Geometry Software (DGS) applets allow students to investigate, visualize, make predictions, calculate, simulate, and generalize certain situations, all of which are critical practices for inquiry (e.g., Gol Tabaghi & Sinclair, 2013; Gol Tabaghi, 2014; Greefrath et al., 2018; Hollenbrands, 2007; Paoletti, 2020). In linear algebra, DGS has been used to explore key topics such as linear combinations (Mauntel et al., 2021), linear transformations (Turgut, 2022), eigentheory (Gol Tabaghi & Sinclair, 2013; Gol Tabaghi, 2014), and determinants (Donevska-Todorova, 2012, 2016, 2017). Through applet and task design, Donevska-Todorova (2012) focused on conceptualizing the determinant of a 2×2 matrix as area of the parallelogram defined by the points corresponding to the matrix's row vectors, helping students discover the property $\det(A) = \det(A^{\mathrm{T}})$ and the relevance of parallelogram point orientation for the determinant's sign. The author attributes the DGS's ability to simultaneously display numeric and geometric feedback as important for connecting to students' prior geometric understanding. Donevska-Todorova (2017) further elaborates on the benefits of DGS in student learning of dot product and determinants, indicating that students took initiative in forming questions and developed a more in-depth conceptual understanding of determinant. Through interaction with the DGS, students indicated evidence of a schematic understanding of determinant by relating it to other mathematical concepts or providing different interpretations of determinants. These studies possibly indicate a potential for DGS as a viable alternative approach to support student understanding of determinant concept, providing them with visualizations of determinant.

In our own design research (Wawro, 2023), we reported on student thinking from interacting with GeoGebra applets designed to leverage students' knowledge of linear transformations in $\mathbb{R}^2$ and $\mathbb{R}^3$ to support an understanding of determinant as a measure of (signed) multiplicative change in the area or volume caused by the transformation. Consistent with the cited literature, we designed the applets so that students contextualize the DGS relative to their existing notions of linear transformations. The goal was to provide students with a foundation for the connections they make while using the applets; in particular, the applets were designed to support students' explorations of how the matrix components [M] relate to the transformed object's graph [G] and to the determinant [D]. In our analysis of students giving observations from the applets and initial explanations for them, we found that most common observation type was [MDG], meaning "[M] implies [D], which makes sense because of [G]." For example, one student wrote, "If any of the columns in the 3×3 are LD [linearly dependent] with another, the det(M) will be zero due to there being no volume, only area or a line." It was sensible that most student observations began with [M] because the most interactive aspect of the applet was adjusting the matrix component values. This potentially hindered students from conjecturing the biconditionality of the determinant generalizations. Thus, we refined the applets to include two additional features: swapping rows and columns, and dragging the graphed images of the basis vectors that define the parallelepiped or parallelogram.

Generalizations and Generalizing Activity

A core goal of this project is for linear algebra students to extend their reasoning beyond particular empirical examples and develop generalizations that can be used for a broad set of problems and further students' conceptual understanding. The process of generalizing has been highlighted in a number of mathematical areas including algebraic reasoning (Carraher et al., 2008; Ellis, 2007; Kaput, 2008), calculus (Dorko, 2023), computational thinking (Lockwood & De Chenne, 2020; van Borkulo et al., 2021), and linear algebra (Harel, 2017). There are several different theories about how students generalize material. Generalization can begin with students recognizing and continuing a pattern in early algebra (NCTM, 2000). Carraher et al. (2017) highlighted the idea of conjectural generalization, which is "treating an instance (e.g., $7 + 1$) as a case of something more general (e.g., $a + b$)" (p.264). Harel and Soto (2016) explored the idea of structural reasoning, a five-step process that ended with students exploring general structures through a process of pattern recognition and reflection.

To characterize students' generalizing activity, Ellis et al. (2022) developed the Relating-Forming-Extending (RFE) Generalization Framework, which can be used to analyze how students engage in generalizing activity:

> Relating is the identification of similarity across situations, problems, or strategies that the learner perceives as distinct contexts. Forming is the development of an initial, sometimes tentative generalization, and extending is the use of that generalization, sometimes to a broader domain. (Ellis et al., 2022, p. 362)

The relating category involves generalizing based upon a prior context, problem, or situation and can encompass the idea of transfer. The forming category is primary method for creating new generalizations and involves associating objects, finding and isolating consistent features across a number of scenarios, and anticipating or extracting regularities. The extending category encompasses continuing an existing generalization to new situations or cases, making minor or major alterations to the generalization, transforming a generalization through extension, and removing particulars. Technology can aid in the development of generalizations by providing empirical grounding for deductive reasoning and pattern recognition (Turgut, 2022).

Methods

Adopting the instructional design theory of Realistic Mathematics Education (Freudenthal, 1991), the IOLA curricular materials (Wawro et al., 2013) build from experientially real task settings to foster active student engagement in the guided reinvention of mathematics through student and instructor inquiry (Rasmussen & Kwon, 2007). Our research is guided by the Design Research Spiral (Wawro et al., 2022), in which we iteratively draft, implement, and refine task sequences, guided by various design research theories at each phase. The IOLA determinants task sequence (Mauntel et al., 2024) leverages students' knowledge of linear transformations to support an understanding of determinant as a measure of (signed) multiplicative change in the area or volume caused by the transformation. To support students' shift from situated activity to more generalized activity, we leveraged DGS GeoGebra applets that synchronously display dynamic representations of numeric, geometric, and algebraic aspects of the determinant. This makes DGS a suitable tool for supporting students' exploration of determinants, connecting to their existing geometric and algebraic knowledge. DGS applets promote mathematical curiosity (Rumack & Hunker, 2019) as students explore, observe, conjecture, and justify. This centers students' activity as the foundational mathematics to be leveraged and extended. Adopting the RME design heuristic of guided reinvention, the instructor's role includes facilitating students'

open-ended activity toward more organized and focused generalizations connecting changes in matrices to changes in the distortion of the space and how they relate to determinants.

Context

We conducted a paired teaching experiment (Steffe & Thompson, 2000) in five interviews with two undergraduates from a small, four-year university in the American Southeast: Ruby and Ahmed (pseudonyms). Authors 1, 3, and 4 identified goals for each interview, which were conducted by Author 1 and assisted by Author 4. In interviews 1 and 2, we oriented the participants to the 2×2 applet and how it depicts aspects of determinants. In interviews 3 and 4, we supported the participants in identifying mathematical relationships through their use of the applets. Toward the end of interview 4 and throughout interview 5, the team focused on eliciting as many generalizations about determinants from the participants as possible.

For this teaching experiment, we used the applets from the initial task sequence (Mauntel et al., 2024, Wawro et al., 2023) and new applets designed for these interviews, which included updated features (Figure 1). First, buttons were added to allow students to switch rows, switch columns, or take the transpose of a matrix (see (a) in Figure 1). Second, randomize and identity matrix buttons (b) were added to facilitate students generating a large assortment of matrices or testing conjectures. Finally, manipulable points were added to the images of the unit square and unit cube (c), allowing students to move the images of the basis vectors that define the parallelepiped or parallelogram (automatically changing the matrix accordingly). The modified applets are found at www.geogebra.org/m/v8hw88fs (2D) and geogebra.org/m/fppgz9q2 (3D).

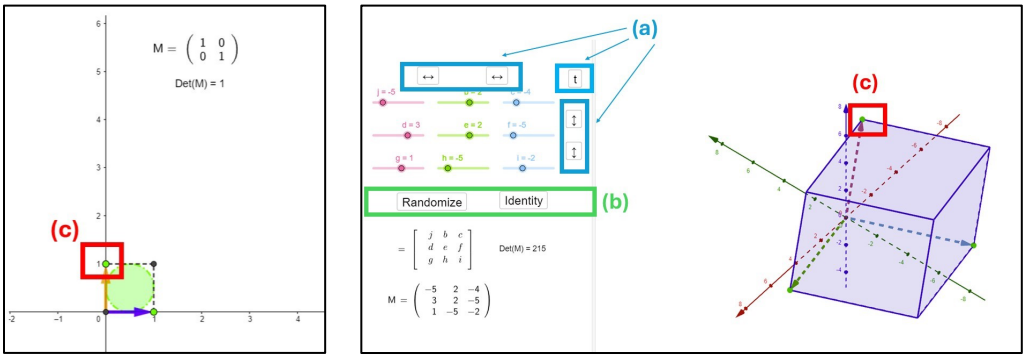


Figure 1. A screenshot highlights the new features of the updated applets.

Analysis

We individually reviewed the transcripts for interviews 2-5 and identified any statements the participants made about determinants that had a conditional or bi-conditional structure. In doing so, we centered the participants' perspectives as they generalized. We holistically categorized these student statements as generalizations through two rounds of coding: open coding and focused coding (Saldana, 2015). The first author combined the identified generalizations onto a single document, merging any instance of repeated identifications. The author team then discussed similarities and differences between the student generalizations. From this, we formed four different generalization clusters (see Table 1) based upon our perceived change to the determinant and used these to categorize participants' activity in later iterations of analysis.

We chunked the transcripts into Episodes of Mathematical Reasoning (EMR) to facilitate the coding process (Andrews-Larson et al., 2024). Each EMR was identified by what we perceived to be the major question the students were trying to answer during that portion of the interview.

The EMRs in the transcripts were primarily separated by a transition between two topics initiated either by the interviewers or the participants' goal-directed activity. For each EMR, we identified which generalization clusters were addressed by the participants' activity.

Using the RFE framework, we coded the participants' generalizing activity according the three broader categories (relating, forming, or extending) and the applicable subtypes of those categories (e.g., within extending: continuing, operating, transforming, or removing particulars). The coding process was divided into three stages. In the initial stage, our goal was to create a shared understanding of the framework's (sub)categories. We then engaged in a second coding phase, during which we noted that codes sometimes repeated across speaking turns due to the nature of the participants' dialogue. Thus, in the third stage, we grouped related chunks together with a single code. Each author coded individually on a cloud-based spreadsheet. During each coding phase, we met regularly to discuss and come to a consensus. We agreed on most codes. In other instances, one or two authors found multiple codes suitable or the interview did not provide enough information for a definitive code; we noted such cases.

Results

Generalizations

Across interviews 2 to 5, we identified four clusters of mathematical activity related to generalizations about determinants (Table 1). These clusters emerged as we sought out specific generalizations and broadened our focus to include the participants' generalizing activity while using the applets. We differentiated clusters by the described effect on the determinant. Cluster A relates to scaling the determinant by k, k^m, or k^n (e.g., if you scale a row by a number k then the determinant is scaled by k). Cluster B includes activity related to negating the determinant (e.g., switching the columns of the matrix). Cluster C mainly consists of characteristics of the matrix that make the determinant zero or nonzero (e.g., vectors are linearly dependent). Cluster D includes descriptions of the determinant remaining unchanged (e.g., swapping rows twice). Although we witnessed Ruby and Ahmed make several distinct, nuanced generalizations that consisted of multiple statements within a given generalization cluster, we do not intend these clusters to indicate that the participants consistently viewed each statement as immediately related to any other within the cluster. Instead, these generalization clusters are an organizational tool used to help identify potentially related generalizing activity across the interviews.

The first two interviews were primarily focused on supporting students' use of the applets and understanding of their connection to determinants and completing the other activities in the unit (Mauntel et al., 2024 (PRIMUS)). The fifth interview was primarily focused on soliciting as many broad, established generalizations from the participants as possible. The excerpts we discuss in this paper are from interviews 2 and 3, which consist of early, initial generalizing activity. Specifically, in this paper, we focus on the participants' generalizing activity that led to Cluster B and explore how this supported generalizing activity toward Cluster A.

Table 1. Clusters of generalizations

Cluster	Effect on Determinant	Mathematical Action/Activity/Statement (Examples)
A	The Determinant is Scaled by k, k^m, or k^n	• A vector is scaled by k • Area/volume is scaled by k. • Multiply a [row/column] by a number k • Multiply m [rows/columns] by a number k

		• Matrix A is scaled by k
B	The Determinant is Negated	• Image is flipped/reflected across an axis • Columns or rows of a matrix are switched • Vectors are swapped (geometric), either instantaneously or dynamically • Vectors cross over each other • Color of the parallelogram/parallelepiped changes • [row/column] is [multiplied by -1 / scaled by -1 / negated]
C	Determinant = 0 or Determinant ≠ 0	• Concepts related to the Invertible Matrix Theorem • Linearly (in)dependent columns/rows • Nonzero volume (3×3) or nonzero area (2×2) • Scaling a row/column by zero/having a [row/column] of zeroes • Rows/columns/vectors are multiples/linear combinations of each other • Cube turns into a plane or a line segment
D	The Determinant Doesn't Change	• Det(I) = 1 • Swapping two rows and swapping two columns • Area/volume stays the same /doesn't change • Determinant stays the same / doesn't change • Changing only an off-diagonal entry • Pushing/leaning the box/cube

The Development of Generalization Cluster B and Cluster A

Reflecting across an axis, scaling rows, and negating the determinant. In Interviews 2 and 3, we asked participants to negate the determinant of given two-by-two matrices. Near the end of Interview 2, Ruby found that the determinant could be negated by reflecting or "flipping" the image of the parallelogram across the y-axis. In interview 2, she accomplished this by negating each entry in the top row, saying, "I wanted to flip it across the y-axis," later adding, "I changed a and b [the entries in the top row] to be negative." Ruby also noticed that the color of the parallelogram changed from red to green when she negated the determinant. Around the same time, Ahmed suggested changing the signs of a and c (the entries of the first column), explaining that doing this causes the result of the 2×2 determinant formula, $ad - bc$, to change from 2 to -2 (in this case). Throughout the episode, we saw several instances in which the participants recognized connections between their actions on two different mathematical objects (here, the geometric image and the entries of the matrix) and changes in the determinant. Consistent with Ellis et al.'s (2022) RFE framework, we coded these as various types of **forming** activity, especially *associating objects*. This is based on their actions on both the geometric (G) and matrix (M) parts of the applet and their observations on how these actions affected the determinant (D). This culminated in a generalization shared by both participants and expressed by Ahmed, saying, "…if you multiply by a negative by the top half of the matrix, you'll get a y reflection. If you multiply by a negative by the bottom half [of] the matrix, you get an x reflection." Throughout these excerpts, the participants consistently scaled rows by -1, which led us to believe that Ahmed intended "multiply by a negative" to mean -1 in this generalization.

Composition of multiple reflections. In Interview 3, the participants reproduced and extended their reasoning from Interview 2, with Ahmed pointing out that the top row determines the x-coordinates for the vectors that define the parallelogram, saying, "when you multiply the top row by -1, then you get the flipped image on all flip across stuff y-axis." Ruby added, "[to]

reflect over x, you change y and to reflect over y, you change x." Ahmed and Ruby explored this connection by generating examples, which we coded as **extending** by *continuing*. Ahmed then restated their prior generalization, replacing "negative" from before with "negative number."

> *Ahmed*: …if you multiply the top row by a negative number, you will produce a matrix that is flipped across the *y*-axis, and if, subsequently, if you multiply the bottom row by a negative number, you will produce a matrix that is flipped across the *x*-axis.

In this quote, Ahmed is subtly **extending** his and Ruby's prior generalization. We note that Ahmed's statement conflates the matrix itself with the parallelogram image of the unit square under the transformation defined by the matrix. Also, as before, this quote does not mention the determinant. However, Ruby's activity before and after this quote indicates she was attending to the determinant's calculation and the color of the parallelogram produced by the applet. Indeed, when pressed by the interviewer a few minutes later, Ahmed stated, "…the determinant would be multiplied by that negative scalar." Ahmed also clarified that he is applying the negation twice, once to each row, adding, "But if you multiply that negative scalar by both axes, the *x* and *y*, then you'll end up with the same determinant just multiplied by a positive scaling." Here, Ahmed applied the generalization twice, scaling the top and bottom rows by a negative scalar. This newly formed generalization is structurally different from the former ones because it considers scaling two rows in sequence and anticipating the compounded result. We coded this instance as **extending** by *transforming* because the generalization itself changed for Ahmed.

Scaling by k and extending to k^n. Later in Interview 3, Ahmed asked to update the prior generalization, saying, "So when you're only multiplying the one row by some value, k. k times [row 1], you end up with a determinant times k." We coded this as **extending** by *transforming,* based on his explicit request to change the language of the prior generalization. Ruby and Ahmed explored this generalization using a specific matrix and scalar to test the statement, which we coded as **extending** by *continuing*. Ahmed then provided an algebraic rationale using a generic matrix with components a, b, c, and d; generic scalar k, and the 2×2 determinant formula. Like their prior generalizing activity when negating the determinant, Ahmed **extended** this generalization by applying it to each row in a 2×2 matrix, which he stated will cause the determinant to be scaled by k^2 and alluded that this should extend to larger matrices.

Discussion

The most direct result from analyzing Ruby and Ahmed's generalizing activity has been the four clusters of mathematical activity and statements, providing a foundation for understanding the generalizations that these applets can support. We have also gained insight into how students' interaction with the applet informs their generalizing activity. Specifically, we recognize that the applet first provides students with examples of matrices to explore during their **forming** activity. Students also often return to the applets during **extending** by *continuing* activity to explore more specific cases to check established but untested generalizations. This suggests potential design changes. For instance, we could incorporate features that can be toggled on once students have developed a specific generalization so that the applet might support students' **extending** activity. Though space limitations prevented us from exploring it much here, we have found some evidence that the updates to the applet supported some of the generalizing activity.

Another contribution of this work is the application of Ellis et al.'s (2022) framework in a new content area and with an additional focus on generalizing with DGS applets. Most notably, we have identified a clear instance of **extending** by *transforming*, which Ellis et al. state "are not well specified in the literature" (p. 377). This demonstrates a meaningful contribution to the field's understanding of undergraduates' generalizing activity.

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Undergraduate Student's Voices and Achievement in a Purposefully Designed Calculus Active Learning Setting

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We explore undergraduate students' perceptions of and the impact of implementing active learning in calculus recitations. Analysis of Likert-scale items indicate that speaking up in recitation helps clarify students thinking, suggesting that calculus instructors should create opportunities for students to speak up in the classroom. Qualitative analysis of two open-response questions indicates an increasing trend over time of students responding that working on problems in small groups was one of the most helpful aspects of recitation. Although students responded that dissatisfaction with group work was one of the least helpful aspects of recitation, this response was never as prominent as group work being the most helpful aspect. An independent two-sample t-test on students' course grades indicate no statistically significant difference ($p = 0.14$) between the treatment group (active learning) and control group. Results suggest that active learning with structured group discussions enhances student engagement and collaboration in learning Calculus concepts.

Keywords: Active Learning, Calculus, Collaborative Small Groups

Introduction

Undergraduate Calculus courses are foundational for all STEM disciplines. Success in the course is critical to retention since many students who struggle tend to leave STEM majors (Bressoud, et al., 2015). For students who are underrepresented in STEM, like women and people of color, having positive learning experiences in Calculus courses has been especially linked to their success and retention in STEM-related careers (Ellis, et al., 2016; Laursen et al., 2014). To address these challenges, over the last few decades, mathematics departments nationwide have invested in revamping their calculus courses (Rassmussen et al., 2019; Smith and Funk, 2021). These revamped courses include implementing a more student-centered culture that values active learning to improve student learning and engagement with mathematics (CBMS, 2016; Uhing et al., 2021).

Based on these recommendations and with support from an NSF grant (Award No. 2013427) we have substantially transformed the teaching of Calculus over the last four years. The Calculus I course at our institution is offered in the lecture-recitation format. All lectures and recitations are closely coordinated, following a common curriculum, calendar, and assessments. Starting in Spring 2022, we restructured our Calculus I course by introducing evidence-based practices. In recitations, students work in small groups collaboratively solving problems and a graduate teaching assistant (GTA) and an undergraduate learning assistant (LA) circulate the room and support student learning by responding to questions, orienting students to learning resources, and encouraging students to collaborate, share, and present their ideas. In revising the course structure and recitation materials we adopted The Conference Board of the Mathematical Sciences (CBMS) statement that defines active learning as "classroom practices that engage students in activities, such as reading, writing, discussion, or problem-solving, that promote

higher-order thinking" (CBMS, 2016, para. 1). This paper highlights the voices of the students and the potential impact on their learning throughout four semesters of implementation.

Literature Review and Theoretical Orientation

The Impact of Active Learning in Undergraduate Mathematics Classrooms

Research has consistently shown that active learning settings, broadly defined, positively impact both students' experiences (Uhing et al., 2021) and academic performance (Fuchs & Sahmbi, 2024). Freeman and colleagues (2014) specifically found that students in active learning environments outperformed their peers in traditional settings, with an average performance increase of 6%. Additionally, students in traditional, lecture-based classrooms were 1.5 times more likely to fail compared to those in active learning environments.

Delving into the reasons behind these positive outcomes, scholars report that active learning enhances comprehension and promotes higher achievement, fosters conceptual understanding (Code et al., 2014; Maciejewski, 2016; Roop et al., 2018), and reinforces mathematical competency and confidence (Kogan & Laursen, 2014; Lenz, 2015). Such findings indicate the effectiveness of active learning in mathematics classrooms and its potential to reduce failure rates while improving overall student engagement and performance.

Students Voices and Experiences with Active Learning Calculus Course Design

Research has consistently documented students' perceptions and experiences with active learning in mathematics courses, with most students indicating a high level of engagement and perceiving these courses as beneficial to their learning (Bowers et al., 2019; Reinholz, 2018; Weurlander et al., 2017). Collaborative group work has effectively promoted student engagement and success in Calculus classrooms (Hitt & Dufour, 2021; Code et al., 2014). However, some studies have reported student resistance to active learning methods (Tharayil et al., 2018), highlighting the need for continued investigation into students' perceptions of active learning environments in Calculus courses.

Even when students were neutral or did not favor the active learning classroom settings, they still found the course structure beneficial for their understanding of mathematics (Smith & Cardaciotto, 2011; Rosenthal, 1995; Weurlander et al., 2017). These findings suggest that while active learning generally enhances student outcomes, there is variability in how students perceive and respond to these methods. This emphasizes the importance of understanding students' voices and integrating them into active learning designs that are tailored to students' needs.

Active Learning

Active learning is defined as a method of teaching mathematics through student-centered pedagogies (Faust & Paulson, 1998; Weiman, 2014). It involves students engaging with the learning process in ways that require active participation and involvement (Prince, 2004), and it is a pedagogy that immerses students in the practice of doing mathematics (Fuchs & Sahmbi, 2024). In this context, active learning is about the practice of students' engagement, instructional routines, and the learning process and its outcome. This study aims to explore students' perceptions of and the impact of implementing active learning in undergraduate calculus recitations. With the following questions: (1) What are students' perceptions of an active learning environment in Calculus 1 recitations? (2) Is there a difference in students' final course grades before and after the introduction of active learning in calculus recitations?

Method

Context of the Study

This project focused on active learning in Calculus I as part of a larger research and development project to revamp introductory STEM courses at a research-intensive university in the northeastern United States. Starting in Spring 2022 a purposefully designed student-centered. active learning approach was implemented in the recitations using a particular structure (brief introduction/review by the GTA, group work on problems, student presentation of solutions), researcher-created activities, and undergraduate learning assistants working with the GTAs to facilitate discussion. Prior to Spring 2022, most of the recitation sections were instructor-centered, with whole group discussions led by the GTA; they highlighted problems from the textbook similar to the assigned homework, with the GTA answering student questions related to the concepts being covered in the lecture.

Data Collection

At the end of each semester, we collected survey data from students enrolled in the Calculus I course (a total of 694 students across four semesters, 154 students in Spring 2022, 212 in Fall 2022, 159 in Spring 2023, 169 in Fall 2023). The survey was comprised of seven Likert scale questions and several open-ended questions. Students were asked to respond on a 5-point Likert Scale (5- strongly agree to 1-strongly disagree) to each of the following statements: (1) The recitations helped me learn Calculus better (Help_learn); (2) I would like other courses to have small group work available as an option (SG_Other_Courses); (3) Ideas in this course became clearer after I heard other students in my small group explain them (S_Ideas); (4) I would recommend group work at Calculus recitations to other students (SW_Calculus); (5) When I spoke up in recitations, it helped me clarify my thinking (SR_Helped); (6) I looked forward to coming to Calculus recitation sessions (CR_Sessions); (7) I collaborate on Calculus related work with people from my group outside of class time (CC_Group).The two open-response questions analyzed for this paper are: (1) Please comment on the aspects of the recitations you think were most helpful for your learning; and (2) Please comment on the aspects of the recitations you think were least helpful for your learning.

Method of Analysis

The current analysis focuses on students' perceptions of the structure of the active learning environment in the recitation sessions and how it influenced their learning experiences both qualitatively and quantitatively. By analyzing students' final course grades, we aimed to examine the impact of the active learning approach on students' course achievement. Specifically, a two-sample t-test is applied to the final course grades to compare the traditional learning and active learning. Moreover, correlation analysis is conducted on the Likert scale survey questions to study students' perceptions about the active learning environment. In addition, we conducted a qualitative analysis to identify the aspects of the active learning environment that students found most beneficial and the factors that contributed to their engagement and motivation.

To analyze the open-ended responses, two members of the research team developed a codebook using both theory and data-driven approaches as described in DeCuir-Gunby et al. (2011). The codebook contained code labels, code definitions, and example responses to describe the individual codes. First, theory-driven (a priori) codes were developed based on Uhing et al. (2021) in their analysis of students' learning experiences with active learning mathematics in Precalculus through Calculus courses at six institutions. The three broad codes in this first stage

were helpful instructional practices, positive recitations experiences, and negative recitation experiences. During the second stage, data-driven (emergent/inductive) codes were developed to capture core issues that repeatedly showed up in student responses but did not fit under existing a priori codes. The researchers first coded the data independently, then they met to compare results; discrepancies were resolved by consensus. A summary of the final codes is presented in the results section below.

Results

Reliability Measures

We established the reliability of the perception of student active learning for the seven Likert-scale items (Altindis & Oyeniran, 2024) for both 2022 and 2023. Reliability measures (estimated reliability within the range of 0.81 to 0.87; Cronbach alpha 0.81 for both years; omega alpha 0.85 for 2022 and 0.84 for 2023) indicated that the seven items and the whole scale are equally reliable (Van den Brink & Mellenbergh, 1998). Additionally, we proved that the survey items measure a single construct: the students' perception of active learning (c.f., Altindis & Oyeniran, 2025).

Correlation Analysis of Students' Perception

With the Likert scale data from the 7 survey questions, Pairwise Spearman correlations were computed for each of the four semesters that implemented active learning activities. All the pairs of the seven survey questions produced positive correlations. Table 1 shows the Spearman correlations of those pairs with correlation value greater than 0.30 for each semester. Across all the four semesters, S_Ideas and SW_Calculus produced the largest correlation compared to all other pairs, followed by SG_Other_Courses and SW_Calculus.

Table 1 Pairwise Spearman correlations of the seven Likert scale survey questions (Likert survey, pairs > 0.30)

Scales	Spring 2022	Fall 2022	Spring 2023	Fall 2023
S_Ideas & SW_Calculus	0.671	0.679	0.724	0.624
SG_Other_Courses & SW_Calculus	0.526	0.562	0.503	0.578
Help_learn & SW_Calculus	0.460	0.539	0.335	0.539
Help_learn & CE_Sessions	0.434	0.489	0.420	0.455
SW_Calculus & SR_Helped	0.402	0.436	0.307	0.494
S_Ideas & SG_Other_Courses	0.382	0.420	0.407	0.515
S_Ideas & CE_Sessions	0.322	0.407	0.306	0.316
SR_Helped & CE_Sessions	0.316	0.442	0.431	0.467
Help_learning & S_Ideas	0.312	0.411	0.359	0.402

The correlation coefficients between the scales S_Ideas (students' perception of clarity of ideas after group discussions) and SW_Calculus (students' willingness to recommend group work in Calculus) across four semesters are as follows: 0.671, 0.679, 0.724, and 0.624. These findings suggest a robust positive relationship between how much students feel that group discussions clarify their understanding of Calculus concepts and their likelihood to recommend group work. This trend has been consistent over the four semesters, showcasing the value of group work in supporting students' learning experiences.

The correlation coefficients between the scales SG_Other_Courses (students' preference for having small group work available in other courses) and SW_Calculus (students' willingness to recommend group work in Calculus) across the four semesters are as follows: 0.526, 0.562, 0.503, and 0.578. This means that students with a positive experience with group work in Calculus recitations are moderately likely to prefer similar group work opportunities in other courses. This suggests that successful group work experiences in one course can positively influence students' attitudes towards group work in other courses.

Students' Perception of Active Learning

We calculated the mean values of survey scales for each semester to analyze the overall trends in students' perceptions of active learning, examining whether these perceptions increased or decreased over the four semesters (Table 2).

Table 2. Mean Scores and Standard Deviations of Students' Perceptions of Active Learning Across Four Semesters

	Spring 22	Fall 22	Spring 23	Fall 23
Help_learn	4.39 (0.778)	3.98 (0.920)	3.89 (1.032)	3.86 (0.999)
SG_Other_Course	4.04 (0.858)	3.88 (0.942)	3.71 (0.895)	3.93 (0.920)
S_Ideas	3.83 (0.932)	3.54 (1.036)	3.49 (1.030)	3.53 (1.023)
SW_Calculus	4.14 (0.905)	3.86 (0.974)	3.74 (1.026)	3.96 (0.937)
SR_Helped	3.95 (0.858)	3.72 (1.002)	3.82 (0.889)	3.78 (0.928)
CR_Sessions	3.37 (1.060)	3.06 (1.125)	3.02 (1.090)	3.14 (1.053)
CC_Group	2.65 (1.300)	2.63 (1.368)	2.60 (1.287)	2.72 (1.318)

Overall, responses to the 7 survey questions indicated that while students generally find value in small group work and recommend it to others, their feelings about how much it helps them learn and clarifies ideas have gradually declined. The "CR_Sessions" and "CC_Group" scores indicate that students are less enthusiastic about attending recitations and collaborating outside of class, suggesting that the structure or effectiveness of these sessions may need re-design to re-engage students. Most importantly, students' realization that speaking up in recitations helps clarify their thinking, suggests that fostering more opportunities for student participation could be beneficial.

The graphs in Figure 1 provide the results for the qualitative coding of two open-response questions (aspects that were most helpful and aspects that were least helpful) and represent the number of students who responded in each subcategory across the four semesters. It should be noted that student responses may fall into more than one category. Subcategories of responses to the question about what aspects of recitations were most helpful were further subdivided into two broad categories: (1) helpful instructional practices that include working on problems in small groups (GW), presenting solutions (PW), creating individual help space (IHS); and (2) positive experiences with group work/recitation that include working on problems (WP), TA/LA support (TAS), small class size/community building (SC). Responses to the survey question asking about aspects that were least helpful were subdivided into the following three categories: dissatisfaction with group work (DGW), concerns about recitation work/content (CRW), and class format & scheduling (CF). From the graphs we see an increasing trend over time in students responding that group work (GW) was one of the most helpful aspects of the recitations. We also note that even though students noted that dissatisfaction with group work (DGW) was a

least helpful aspect, the number was consistent across the semesters and not as prominent a response compared to group work being one of the most helpful aspects.

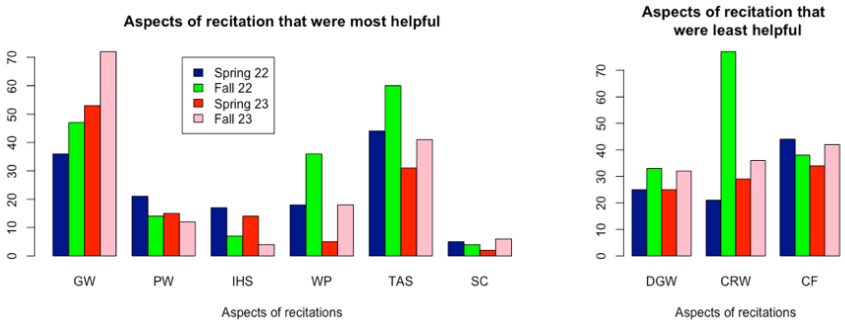


Figure 1. Distribution of codes for student responses to two open-response questions by semester

Comparing Student Achievement in Calculus

This study aims to determine if there is a significant difference in students' final course grades before and after the introduction of active learning in calculus recitations. To assess this, we collected data from students enrolled in Calculus during the following semesters: Fall 2021, where a traditional instructor-centered learning approach was employed, and Spring 2022, Fall 2022, Spring 2023, and Fall 2023, where an active learning approach was implemented.

Through preliminary examination, we found that there was a statistically significant difference (p-value < 0.001) in student achievements between the Spring semesters (n=402, mean 2.10 and sd. 1.25) and Fall semesters (n=456, mean 2.43 and sd. 1.22) where the active learning approach was implemented. Therefore, to reduce confounding effects and ensure a fair comparison, we focus on the analysis in the Fall semesters for the effect of the active learning approach with group work in recitations.

To ensure the validity of combining the data from the two Fall semesters (Fall 2022 and Fall 2023), an independent two-sample t-test showed no statistical significance (p-value = 0.84) in the difference in student achievement between the two semesters. Therefore, we combined the data from Fall 2022 and Fall 2023 to serve as the treatment group for active learning, while Fall 2021 serves as the control group.

The treatment group utilizing active learning shows an increase in average course grades (n=456, mean 2.43, sd. 1.22) compared to the control group (n=226, mean 2.31, sd. 1.43). To assess the effect of the active learning approach, a one-sided independent two-sample t-test produced a p-value of 0.14 between the control and treatment groups, indicating that the treatment's effect is statistically insignificant at the 0.05 significance level. Additional data from future semesters could further enhance the understanding of the treatment's impact. Figure 2 below shows the grade distributions for two types of learning methods.

We note that instructors in the experimental groups were not the same as in the control groups. However, the course materials, homework assignments, tests and exams, and grading policies were used uniformly by all instructors in the control and treatment groups.

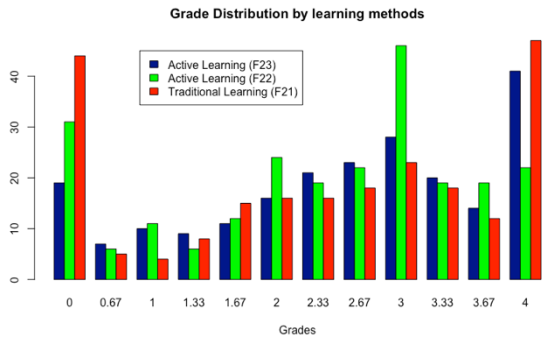


Figure 2. Grade distribution for Fall 2021 (traditional format), Fall 2022, and Fall 2023 (active learning format).

Discussion

We found a strong positive correlation between students' perceptions of the value of group discussions in clarifying their understanding of Calculus concepts and their likelihood to recommend group work to their peers, confirming previous work (Uhing et al., 2021). This indicates that as students engage more deeply in group discussions and gain clarity on calculus concepts, they recognize the benefits of collaborative learning and are more inclined to promote small group work in recitations and other courses.

These results are supported by the coding of the two open-response questions. There was an increasing trend over time of students responding that working on problems in small groups was one of the most helpful aspects of recitation. This increase over time might be a natural result of the instructional team adapting to the more student-centered process. Responses in the dissatisfaction with group category might be more related to how the groups were structured rather than group work not being helpful, e.g., the student did not like "switching groups multiple times", or how groups were assigned "random groups – I would have learned the most from my first group, but it got switched." Student concerns about recitation work/content (CRW) are consistent with results from the SEMINAL survey (Uhing et.al (2021). That is, students found some of the recitation work unrelated to the lecture or course assignments and wanted more practice, "For me recitation should be about practice problems and solving them in order to get used to them", "the recitation work felt incredibly out of sync and off topic compared with what we were learning in the main course."

An independent two-sample t-test indicated no statistically significant difference ($p = 0.14$) in students' course grades between the treatment group (active learning setting) and the control group at the 0.05 significance level. These findings do not support the claim that students participating in active learning environments achieved better final grade outcomes in Calculus compared to those in traditional lecture-based settings. Instead, we argue that we think there are other variables that affect students' achievements in calculus courses, yet our test does not measure them.

In summary, the results suggest that integrating active learning, particularly through structured group discussions, can enhance the student learning experience by promoting engagement and collaboration in understanding Calculus concepts. These findings necessitate further exploration into the long-term effects of active learning on student success and motivation in STEM courses.

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Can a Single Placement Test Effectively Place Students Into Different Math Pathways?

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Math placement policies are intended to match incoming freshmen with an appropriate math course. At our four-year institution, the same placement test is used to place students into nine courses across three pathways. In this paper we use institutional data to analyze the effectiveness of test-based placement into different courses requiring similar (and sometimes the same) cutscore. We use the classification power metric to evaluate and compare the effectiveness of three placement policies. We report that that placement tests provide minimal information about students' readiness for mathematics courses that are not algebra intensive, such as basic statistics.

Keywords: Placement, Classification Power, Math Pathways

Improving student success in entry-level college mathematics courses is a high priority for many higher education institutions. One of the approaches to improving student success involves developing and refining the process that matches incoming students with an appropriate mathematics course. In many cases, this process involves incoming students taking a placement test where the students' performance on the test, possibly in conjunction with other measures like high school GPA, determines which mathematics course(s) they are eligible to take.

Much of the research on mathematics placement has been conducted in the context of two-year institutions with a focus on the distinction between remedial and college-level coursework. Our study extends this body of research to evaluate placement at a four-year institution utilizing a test-based multi-level placement system. Placement policies at our institution are intended to simultaneously assess a student's readiness for college-level mathematics and differentiate between a student's readiness for various levels of mathematics courses. To further complicate the placement decision, there are three mathematics pathways students can choose depending on their major and career goals. Thus, the same placement test score can place students in different college-level courses. In this paper we use the classification power metric to compare the effectiveness of multi-level placement policies based on different placement-tests. Our results highlight that the placement process at our institution provides little information about students' readiness for non-algebra intensive courses.

Background

There are two main strategies to evaluate the effectiveness of a placement policy: comparative analysis of observable outcomes and classification error analysis. Researchers conducting comparative analysis of observable outcomes focus on the relationship between placement decisions and observable outcomes such as students' grades in their first math course, retention, and time to graduation (see, for example, Barnett et al., 2018, Chen, 2016, Rodriguez, 2014; and Thornton et al., 2019). Frank et al. (2022) found that the placement decision has low explanatory power for both short- and long-term outcome variables and argued that such comparative analysis of observable outcomes is not sufficient to meaningfully evaluate placement policies.

Another strategy is to treat the placement decision as a classification problem where the goal of placement is to identify, with as high accuracy as possible, the students who are ready to take

the target course and those who need to take a prerequisite (see Belfied & Crosta, 2012; Scott-Clayton, 2012). The quality of the decision can be measured with the *severe error rate* (SER). Scott-Clayton (2012) defines the SER as the sum of the percentage of students who were placed into developmental math but would have been able to pass college-level math with the grade of B or higher and the percentage of students who were placed into college-level math but earned a failing grade (F). Researchers report SERs between 16% and 28% in the context of mathematics placement at two-year institutions (Belfield & Crosta, 2012; Scott-Clayton, 2012).

Frank et al. (in press) identified three methodological challenges associated with using the SER to evaluate and compare placement decisions: (1) challenges associated with determining whether a given SER is evidence of an effective placement policy when there is no benchmark for comparison, (2) challenges associated with comparing SERs across placement policies; and (3) interpretational challenges due to possible variation in the magnitude of the SER due to chance. They propose a new metric, the *classification power* (CP), which quantifies the amount of information provided by the placement policy about students' readiness for a given course. The CP measures the reduction in the number of severe errors from implementing a given policy instead of a hypothetical placement policy that provides no information about students' readiness (imagined as a weighted coin-flip). Since the classification power compares the number of severe errors between the implemented policy and a hypothetical policy that places the same cohort of students, the classification power can be compared across courses and across placement periods.

Since the classification power measures the reduction in severe errors, the greater the classification power the more effective the policy. A policy is deemed ineffective if the 95% confidence interval for its classification power includes 0 since a classification power of 0 indicates that the implemented policy provides no improvement compared to a hypothetical policy that can be imagined as using a weighted coin-flip to place students.

To inform policy decisions, it is advantageous to consider separately the CP for both ready students (i.e., students who are able to earn a grade of B or better in a given course without taking the prerequisite) and not ready students (i.e., students would earn a failing grade in a given course if they do not first take a prerequisite). The CP for ready students (CP_R) can be interpreted as the percentage of ready students appropriately placed by the placement policy while the remaining ready students are imagined to be placed with a weighted coin-flip where the coin is weighted to match the overall placement rate. Similarly, the CP for not ready students (CP_{NR}) can be interpreted as the percentage of not ready students appropriately placed into a prerequisite while the remaining not ready students are imagined to be placed with a weighted coin-flip. Frank et al. (2024) analyze placement into Precalculus at their 4-year institution. They report lower classification powers for not ready students suggesting that placement at their institution is better at identifying ready students than not ready students.

Methodology

To evaluate and compare placement policies that place students into multiple college-level courses (e.g., Basic Statistics, College Algebra, Precalculus, etc.), we determine the CP for a single course at a time. We call this the *target course*. Each student who eventually takes the target course as part of their program of study – the *target course population* – was either placed or not placed into the target course when matriculating into our four-year institution. A student is placed into the target course when they have met the qualifying score on the placement test. Our data consists of institutional records for students in the target course population including measures of academic preparation (high school GPA, SAT/ACT) as well as demographic factors

(race, gender, age).

In order to estimate the unobserved severe errors, we follow the process described by Scott-Clayton (2012) to perform a statistical matching between students who took the target course without taking prerequisites and those who took the target course after a prerequisite. The process is fully described in (Frank et al., in press). We fit two regression models on the group of students who took the course without prerequisites. The first model estimates the probability of a student earning higher than a B in their first attempt in the target course given a vector of academic and demographic data. The model is applied to students who earned at least a B in the target course on their first attempt. Students with a predicted probability of passing over 50% are classified as underplaced; this produces an estimate of unobserved underplaced students. The second model estimates the probability a student fails the target course without prerequisites; this model is applied to students who took a prerequisite and earned at least a D on their first attempt in the target course. Students with a predicted probability of failing over 50% are classified as overplaced. These estimates, together with the students observed to be severely misplaced give an estimate for the SER since the SER is the sum of all observed and estimated severe errors.

The classification power is defined as the reduction in the number of severe errors made by the implemented policy and a hypothetical policy where the placement decision is independent of a student's readiness (Frank et al., 2024). We use ρ_R (and ρ_{NR}) to denote the SER for ready (and not ready) students for this hypothetical policy. We separately compute CP_R and CP_{NR} where:

$$CP_R = 1 - \frac{\mathbf{P}(\text{Ready and Not Placed})}{\mathbf{P}(\text{Ready})\mathbf{P}(\text{Not Placed})} = 1 - \frac{SER_R}{\rho_R} \text{ and } CP_{NR} = 1 - \frac{\mathbf{P}(\text{Not Ready and Placed})}{\mathbf{P}(\text{Not Ready})\mathbf{P}(\text{Placed})} = 1 - \frac{SER_{NR}}{\rho_{NR}}.$$

The overall CP is a weighted average of CP_R and CP_{NR} where each metric is weighted by the relative proportion of severe errors made by the hypothetical policy.

Institutional Context and Data

At our four-year institution, a student's score on a math placement test and their declared major are used to place students into one of nine math courses, eight of which offer college credit. The courses are organized into three pathways: a non-algebra pathway culminating in basic statistics, an algebra pathway culminating in Calculus for Applications (a business calc course), and an algebra intensive pathway culminating in Calculus I (this is the only pathway requiring precalculus).

Since 2014, our institution has implemented two placement tests. Between 2014 and 2018 we administered the Mathematical Association of America's Basic Algebra Test (BAT) and Calculus Readiness Test (CRT). Students wishing to take either business calculus or calculus needed a qualifying score on the BAT (17) and a qualifying score on the CRT (see Table 1). In 2018 our institution adopted the Assessment and Learning in Knowledge Spaces Placement, Preparation, and Learning Test (ALEKS PPL[1]). Both placement tests were administered to all incoming students without college math credit in an online and unproctored environment. Cutscores for a subset of courses considered in this paper are provided in Table 1. Notice that in 2016 our institution changed the cutscores for the BAT/CRT test thus we consider separately the students placed under each set of cutscores.

[1] The ALEKS placement test is an adaptive test consisting of 25 questions designed to assess students' knowledge across 314 topics (McGraw Hill, 2024).

Table 1. Cutscores for BAT/CRT placement test and ALEKS placement test.

	BAT/CRT v1 F14-S16	BAT/CRT v2 F16-S18	ALEKS F18- present
Basic Statistics	BAT 17	BAT 14	ALEKS 30
College Algebra	BAT 12	BAT 14	ALEKS 46
Precalculus	BAT 17	BAT 20	ALEKS 61
Business Calculus	CRT 11	CRT 11	ALEKS 61
Calculus I	CRT 11	CRT 15	ALEKS 76

Recognizing that there are three math pathways at our institution, the same placement test score could place students into three different courses depending on their declared major. For example, an ALEKS score of 62 would place a sociology major into basic statistics, an economics major into business calculus, and a physics major into precalculus.

Data

Our data consists of the academic records for all first-time freshmen who enrolled in Towson University, a large public university in the mid-Atlantic region, between Fall 2014 and Fall 2018. We exclude students who enrolled in/after Fall 2019 due to confounding effects of the COVID pandemic. We evaluated the CP for four courses across three pathways (college algebra, precalculus, basic statistics, and business calculus). Table 2 summarizes the demographic characteristics of these target course populations and provides the percentage of students in the target course population who were placed into the target course by the placement test. We partition the data by cohort based on the placement policy at the time of a student's enrollment.

Table 2. Summary of demographic and academic preparation for each target course.

		Sample Size n	White %	Black %	Asian %	Hispanic %	Other %	Placement Rate %
College Algebra	BAT/ CRTv1	1,076	58.27	19.42	6.41	7.90	7.99	88.85
	BAT/ CRTv2	1,566	50.13	24.58	6.90	10.09	8.30	83.21
	ALEKS	850	48.35	29.18	8.24	7.06	7.18	74.71
Precalc	BAT/ CRTv1	679	54.64	21.94	8.25	7.66	7.51	82.62
	BAT/ CRTv2	483	38.51	33.54	9.32	9.11	9.52	60.46
	ALEKS	262	38.17	37.79	8.02	8.02	8.02	59.92
Basic Stats	BAT/ CRTv1	875	59.54	18.63	7.89	6.86	7.09	71.89
	BAT/ CRTv2	939	52.82	22.15	9.16	8.09	7.77	85.20
	ALEKS	582	45.53	29.38	9.45	8.25	7.39	95.02
Business Calc	BAT/ CRTv1	635	63.62	14.33	6.93	6.61	8.50	36.85
	BAT/ CRTv2	617	56.08	21.23	7.29	9.24	6.16	45.71
	ALEKS	336	51.79	25.60	7.14	6.55	8.93	60.71

We recognize some demographic shifts in the sample over time. We do not see this as problematic since we perform the analysis on each cohort of students and the demographic characteristics are more uniform within a cohort. Additionally, we note sample sizes greater than 260, sufficiently large to perform the statistical matching described in the methodology.

Results

In this section we provide the results of the statistical matching procedure described in the methodology. Tables 3 and 4 present: the CP, the CP for ready students (CP_R), and CP for not ready students (CP_{NR}) for four courses across math pathways. We also provide 95% confidence intervals (CI) computed using bootstrapping techniques. We selected these four courses from the eight credit-bearing courses because they share similar (sometimes identical) cutscores (see Table 1). Table 3 provides the results for two courses in the algebra-intensive pathway and Table 4 provides results for two courses not in the algebra-intensive pathway.

Table 3. Comparison of classification power for courses in the algebra intensive pathway.

		BAT/CRTv1		BAT/CRTv2		ALEKS	
		Est (%)	CI (%)	Est (%)	CI (%)	Est (%)	CI (%)
College Algebra	SER	7.2	(5.8,8.8)	10.2	(8.8,11.8)	12.8	(10.6,15.3)
	CP	34.7	(22.9,45.3)	25.9	(16.9,33.7)	30.7	(20.3,40.5)
	CP_R	40.3	(26.7,53.2)	29.1	(18.2,38.8)	39.4	(26.5,52.1)
	CP_{NR}	23.3	(8.3,38.6)	19.9	(8.4,31.0)	18.8	(5.8,31.7)
Precalc	SER	7.8	(5.9,10.0)	16.1	(12.6,20.1)	15.3	(10.7,20.6)
	CP	46.1	(33.6,57.5)	30.8	(18.0,43.2)	13.9	(-5.5,33.5)
	CP_R	59.7	(44.2,74.1)	33.6	(20.0,48.5)	24.3	(3.6,45.2)
	CP_{NR}	27.4	(11.7,43.3)	24.5	(3.7,43.7)	-20.5	(-54.3,17.1)

Table 4. Comparison of classification power for courses not in the algebra intensive pathway.

		BAT/CRTv1		BAT/CRTv2		ALEKS	
		Est (%)	CI (%)	Est (%)	CI (%)	Est (%)	CI (%)
Basic Stats	SER	17.3	(14.2,21.0)	10.6	(8.7,13.3)	12.9	(10.3,16.7)
	CP	-15.7	(-27.4,-1.4)	16.7	(5.4,27.7)	3.2	(-4.7,10.7)
	CP_R	-33.4	(-50.3,-8.0)	22.6	(-0.8,49.1)	19.1	(-49.8,81.8)
	CP_{NR}	0.8	(-13.8,13.4)	14.1	(3.4,23.4)	1.8	(-3.3,7.3)
Business Calc	SER	18.0	(14.0,22.0)	18.6	(15.4,23.2)	30.4	(24.7,35.7)
	CP	21.5	(10.6,32.8)	21.3	(8.2,32.0)	-13.3	(-25.7,1.1)
	CP_R	15.5	(6.2,27.7)	19.6	(7.0,34.0)	3.2	(-13.7,24.4)
	CP_{NR}	36.3	(11.7,54.8)	23.2	(2.6,38.9)	-31.8	(-48.7,-15.5)

Effectiveness of Test-Based Placement

Across these four courses and three placement policies we report estimates for the CP between -15.7 and 46.1. As we will discuss in the next section, we take this wide range in CPs across courses as evidence that the amount of information provided by a placement test varies based on the nature of the target course and the placement test administered.

Before comparing the effectiveness of a placement test across courses, we begin with a brief discussion of the evidence of test-based placement focusing our discussion on ALEKS based placement into precalculus. For ALEKS based placement into precalculus, we report a CP of 13.9 and associated CI of (-5.5, 33.5). We interpret this CI as evidence that we can expect substantial variability in the classification power due to chance and the specific characteristics of the enrolled student population. We note that we see large confidence intervals for the SER as well. We take this as evidence that small, single percentage point, variations in any measure of the effectiveness of a placement policy may be simply attributed to chance.

We see that for placement into precalculus both BAT/CRTv2 and ALEKS made about the same proportion of severe errors (SERs around 15%). However, we see different CPs for these policies; we report a CP of 30.8 for BAT/CRTv2 and 13.9 for ALEKS. This provides evidence that two policies can make the same proportion of severe errors while providing different information about students' readiness for a course. We attribute the difference in these estimates to variation in the percentage of students ready and not ready for the course in each cohort: the SER does not take into consideration differences in the sample characteristics while the CP attends to these differences by comparing an implemented policy to a hypothetical policy that places the same population of students.

We now consider the CP for ready and not ready students separately. For ALEKS based placement into precalculus we report an estimate for CP_R of 24.3 and an estimate for CP_{NR} of -20.5. The magnitude of these results can be interpreted as the percentage of ready (and not ready) students we confidently placed correctly. That is, we estimate that ALEKS based placement correctly placed 24.3% of ready students into precalculus and *incorrectly* placed 20.5% of students not ready for precalculus into the course while placing the remaining with a coin-flip weighted to match the placement rate (here 59.92%). That is, the negative estimate (for either the CP_R or CP_{NR}) implies the implemented policy made more severe errors than using the hypothetical coin-flip policy for all students.

We emphasize that the benchmark here is a weighted coin-flip policy – an obviously suboptimal placement policy. Looking across courses most often we report CPs for ready and not ready students around 25%. That is, most often we are confidently placing 25% of ready and not ready students. We take these reported CPs as minimal evidence for the effectiveness of test-based mathematics placement in providing information about students' readiness for a target course.

Comparison of Classification Power across Courses

At our institution, the same placement test is used to place students into nine different courses. In this section we evaluate the ability of a placement test, even an adaptive test like ALEKS PPL, to provide information about students' readiness for such a variety of courses. Specifically, we compare the effectiveness of placement into College Algebra and Basic Statistics as well as Precalculus and Business Calculus since these courses share similar (sometimes identical) cutscores across placement cohorts.

Across all three placement periods, the CP for Basic Statistics is less than the CP for College Algebra. These are significant differences for both BAT/CRTv1 and ALEKS as there is no overlap in the confidence intervals. Additionally, we report negative estimates for the CP for Basic Statistics under BAT/CRTv1 and ALEKS suggesting that a hypothetical policy using a weighted coin-flip would have made fewer errors than these implemented test-based policies. We also note that Basic Statistics has the lowest estimate for CP_R across all courses. Furthermore, this estimate is negative (-33.6). Together, we interpret this as preliminary evidence that test-based placement could serve as a gatekeeper in non-algebra intensive courses as the tests are incorrectly underplacing a larger percentage of ready students than a weighted coin-flip. We do not interpret this as evidence that there should be no placement policy for such courses as we estimate around 10% of students are not ready for basic statistics and would benefit from taking a prerequisite. Our results highlight the need for better placement mechanisms.

Next, we directly compare placement into precalculus and business calculus[2]. Across all three placement periods the CP for business calculus is less than the CP for precalculus. This difference is significant for placement under BAT/CRTv1; there is no statistically significant difference under the ALEKS policy which uses the same adaptive test and cutscore for both courses. This is noteworthy since under BAT/CRTv2 students took different placement tests to place into these courses, yet these different tests did provide statistically comparable amounts of information for each course. It is also worth noting that the CRT underplaced many students ready for business calculus as evidenced by the low CP_R (15.5). This suggests that utilizing different placement tests for different target courses is not necessarily an effective way to implement a multi-level placement policy.

Impact of Cutscore on Policy Effectiveness

We notice that changing the cutscores between BAT/CRTv1 and BAT/CRTv2 had different impacts across courses. For both college algebra and precalculus, increasing the cutscore reduced the percentage of students placed into the course (see Table 2). However, this change did not improve the policy's ability to identify students not ready for the course as we see no noticeable change in CP_{NR}: the slight decrease in the reported point estimates can be attributed to variation due to chance). In contrast, decreasing the cutscore for basic statistics did improve the policy's ability to identify ready students as we see an increase in CP_R. This is a noteworthy change since the initial cutscore of 17 on BATv1 resulted in a policy less effective than a weighted coin flip, as indicated by the negative estimate of CP_R. However, after decreasing the cutscore the estimates for CP_R and overall CP increased so that we can say the policy is more effective than the hypothetical coin-flip.

These results echo Leeds and Mokher's (2020) finding that changing the cutscore can impact placement accuracy. In fact, our data suggests that increasing the cutscore does not necessarily result in improved placement accuracy but can instead serve as a gatekeeper for ready students.

Discussion

At our four-year institution, a placement test is used to place students into nine courses across three different math pathways. We questioned the effectiveness of a single test (even an adaptive test) in placing students into multiple courses. Our findings provide preliminary evidence that test-based placement serves as a gatekeeper for courses outside the algebra intensive pathway (i.e., courses not leading to Calculus I). We do not interpret this as evidence that there should be no placement policy as we estimate that there are students not ready for these courses (about 10% of students who need to take basic statistics need a prerequisite and about 20% of students who need to take business calculus need a prerequisite).

We recognize that these findings could be the result of the established cutscores. Thus, we plan to use the institutional data to construct a proposed policy that sets cutscores to maximize the CP. Evaluating the CPs associated with such a proposed policy would allow us to distinguish whether it is the placement test or the cutscore that limits the information the test score is able to provide about students' readiness for a given course.

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"Algebra shouldn't be this hard": How Developmental Math Student Perceptions of and Prior Experience With Algebra Influence Course Relearning Experiences

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We report results from a longitudinal case study of Elementary and Intermediate Algebra students' perceptions of what it means to "relearn" algebra, and the consequences of such perceptions on relevant behaviors and affective states. This data comes from a larger project investigating student experiences relearning individual topics and reflections on relearning across two semesters of developmental math courses. We found that students often conflated their perceptions of algebra as a familiar and "simplistic" subject with the notion that they understood it, leading to the idea that relearning it should be easy. This perception was held even in the face of poor course performance and was used to motivate unproductive patterns of behavior. In identifying the unique factors that informed and reinforced students' perceptions of relearning, this report provides useful insights for educators who are currently grappling with high rates of failure and attrition in such courses nationwide.

Keywords: developmental mathematics, relearning, algebra

This report describes students' perceptions of what it means to relearn algebra in traditional developmental math courses, i.e., semester-long college courses whose content mirrors that of United States middle and high school pre-algebra and algebra courses. Such courses continue to make up a significant portion of developmental math offerings (Kosiewicz, Ngo & Fong, 2016) and are subject to the greatest scrutiny in the developmental math space. In a series of prior reports on *relearning* in this space (i.e., when current learning goals are essentially the same as those of a previous learning experience for a given student), we have demonstrated the utility of focusing on *student perceptions* of their current understanding of algebra and how such understanding is impacted by learning about it again in college (Amman & Mejía-Ramos, 2021, 2022, 2023). We have argued that such perceptions may be crucial in explaining student behaviors and learning outcomes related to specific topics in those courses. Our work contributes to a long-standing call for research on developmental mathematics education to open the "black box" of teaching and learning in such courses (Grubb, 2001; Sitomer et al., 2012; Mesa, Wladis & Watkins, 2014) to better understand what factors may contribute to their historically high failure rates (Chen, 2016; Bailey, Jeong, & Cho, 2010).

The data presented in this report come from a larger longitudinal study that sought to capture how college student perceptions of relearning evolved over two semesters in a developmental mathematics program. In this report, we present results about (1) the ways students perceive their overall experience relearning in a developmental mathematics course, and (2) the relationship between a student's perception of their developmental mathematics course experience as a whole and their evolving relationship with relearning across individual topics. In addressing these issues, we also seek to contribute to educational theory regarding relearning—a phenomenon that is present in a variety of first-year college mathematics courses. Indeed, by identifying the facets of students' perceptions of relearning that may be tied to the topic of algebra itself, we become better able to understand how student experiences might vary when relearning in courses such as college-level Precalculus or Calculus.

Literature Review

Studies seeking to describe student experiences in developmental mathematics courses typically focus on one of four areas: placement into or rates of passing developmental math courses (Moran, 2008; Benken et al., 2015; Ngo, 2020; Ngo & Velasquez, 2020; Bahr, 2012; Fong, Melguizo & Prather, 2015; Xu & Dadgar, 2018); non-cognitive variables associated with placement or passing such as academic mindsets or self-efficacy (Silva & White, 2013; Ironsmith et al., 2003; Baxter, Bates & Al-Bataineh, 2016; Hall & Ponton, 2005; Zientek et al., 2013; Higbee & Thomas, 1999; Guy, Cornick & Beckford, 2015); describing instructor perceptions of student experiences (Mesa, 2012; Er, 2018; Reichwein, Schneider & Onwuegbuzie, 2014; Grubb & Gabriner, 2013); and describing student perceptions of their own learning experiences (Cox & Dougherty; 2019; Benken et al., 2015; Mesa, 2012). The current study is of the fourth type. Studies of this type are sparse, but critical to understanding and explaining students' motivation for patterns of behavior and course engagement.

Cox and Dougherty's (2019) end-of-semester interviews with 25 community college pre-algebra students indicated that 15 of the 25 students perceived the course to be useful in "refreshing" their preexisting knowledge of pre-algebra. These students identified the overall function of the course as "reviewing" or "remembering" algebra as opposed to helping them to better understand the mathematics they had seen before. In fact, even though 18 of these students indicated that they "wanted to 'really understand'" the reasoning behind their use of procedures in the course in way that they had not accomplished previously (p. 256), only three of these students indicated that they felt this had been accomplished by the end of the course. Thus, it seems that there was some distinction between what participants may have desired from a relearning experience and what they felt the course was ultimately capable of providing to them.

Benken et al., (2015) surveyed 306 first year students placed into Intermediate Algebra at California State University. A smaller sub-finding of this study concerned student perceptions of what it meant to gain a "better" understanding of mathematics in a developmental math course in which they had already seen the material before. The authors found that some, although not all, students perceived a better understanding to mean that they had recalled material they had already learned previously. Interestingly, instructors who were also surveyed in this study indicated a perception that most students approached the course as a "review" rather than an "opportunity to learn new content". Instructors perceived such an impression negatively, and assumed it stemmed from an overconfidence in skills. However, it is important to note that studies in this category involved interviewing (Cox & Dougherty, 2019) or surveying (Benken et al., 2015) students at the end of their course experience. Thus, while each of these studies gives us important insight into students' perceptions of learning in a developmental mathematics course, they are limited in their ability to explain why such perceptions are formed by students, or how they might influence student experiences learning course content. The current study sought to complement this prior work by identifying exactly such mechanisms.

Contrary to current depictions of developmental mathematics students having one uniform "type" of learning need (for example, a "refresher" of course material or an intensive review of "basic skills", Grubb & Gabriner, 2013; Cox & Dougherty; 2019), in our own previous work (Amman & Mejía-Ramos, 2021; 2022), we have found that developmental math students' learning experiences differed substantially across time and across course topics. This was because students' judgements regarding how and when to integrate their prior algebra understanding with what they were learning in their current course were dependent on their perceptions of the similarity of the topic in the developmental math context compared to their

previous algebra classes, and their confidence in their understanding of the topic at that time. Due to variations in perceptions of similarity and confidence throughout the semester, we found that existing categories from the literature were not all-encompassing descriptors or labels for a "type" of experience (e.g., a "refresher" experience) that fit each student. In subsequent work (Amman & Mejía-Ramos, 2023) we substantiated the connection between students' expectations for what relearning might look like at different points of the course and their methods of engagement with course content. The results of this previous work departed significantly from studies that advocated for the use of broad labels to describe student experiences due to the difference in relearning experiences based on topic. However, these results do not contradict the idea of students holding more holistic impressions of what it means to relearn in a course (Cox & Dougherty, 2019; Benken et al., 2015). Rather, they motivate the necessity of research that seeks to understand the relationship between students' topic- and course-level perspectives on relearning. The current study sought to do just this—to understand how students' holistic perceptions of relearning algebra were formed and how they evolved as the course progressed.

Theoretical Framework

Traditionally, conceptualizations of relearning have been oriented around the author's notion of a successful experience. For example, in her work on learning experiences in mathematics content courses for future teachers, Zazkis (2011) describes relearning as "the learning-again of concepts and ideas that are familiar or that are perceived as familiar by the learner," (p. 9). She goes on to orient this notion around a particular successful outcome. In contrast to "simply adding knowledge to what was previously incorporated into a learner's repertoire," relearning is described as *reconstruction*—a process that involves a reframing and reorganization of prior cognitive structures from K-12 learning for the purpose of teaching (p. 12). While we have found the notion of reconstruction to be useful in prior work (Amman & Mejía-Ramos, 2023), we found that orienting the conceptualization of relearning around this outcome was less useful in understanding student learning experiences in developmental mathematics courses. We were interested in characterizing what was happening to student understanding in these courses—how their cognitive framing of algebra content was impacted by their relearning experience, even if it could not be characterized as a "successful" reconstruction.

Accordingly, we define relearning in mathematics broadly as the experience (happening at a time T2) of learning about content one has already tried to learn about in a previous mathematics course (happening at an earlier time, T1). Importantly, the second learning experience cannot be simply a review of the content for the purpose of teaching something else (this would be better characterized more simply as the activation of prior knowledge in a relevant learning experience); the learning of such content must be *the target* of learning experiences happening at both T1 and T2. We do not orient our conceptualization around specific successful learning outcomes at T1 or T2. Instead, we treat the "learning" of relearning more generally to allow for a focus on describing *how* students' existing understanding of content can be (or be perceived to be) impacted by the T1 experience, without the necessity of comparing these impacts to a successful norm. Under this conceptualization, one may study relearning at multiple grain sizes: relearning a particular topic (such as relearning *solving systems of equations*) or relearning an entire course (relearning *Algebra*). Topic-level relearning was the focus of our earlier work in developmental mathematics (Amman & Mejía-Ramos, 2021, 2022, 2023) and resulted in a broader range of possible relearning outcomes than had previously been noted in the literature. This report focuses on the "course-level" relearning—students' holistic perceptions of what learning is like overall in a course. In a previous report (Amman & Mejía-Ramos, 2023) we

presented findings that described the relationship between one Intermediate Algebra student's topic-level and course-level perceptions of relearning as she continued to encounter content involving the sub-topic of factorization throughout a developmental math course. In the current study, we focus on course-level perceptions, demonstrating how such perceptions evolve over time and come to influence students' behavioral patterns and affective responses in developmental mathematics.

Methods

We conducted a multiple, longitudinal case study over a period of three semesters, following six students enrolled in either Elementary Algebra or Intermediate Algebra at a four-year university. Elementary and Intermediate Algebra are non-credit-bearing courses whose syllabi reflect those of a typical high school algebra course. Each participant spent two consecutive semesters in the study, with three participants beginning the study in Elementary Algebra in Spring 2021 and the other three beginning in Intermediate Algebra in Fall 2021. All participants were recruited at the beginning of the semester via email for a study focused on their experiences learning about algebra in their current course. Interested students completed a survey in which they were asked about their algebra course-taking history, then shown a list of topics (and related problems) that would be taught in their current algebra course. Students described how well they remembered and how confident they were in their current understanding of each topic.

Each semester, we chose participants whose survey responses varied in terms of how well they thought they remembered and understood the given topics, and their level of experience with college algebra courses. While some participants like Jina were taking an algebra course in college for the first time, all participants reported taking at least one algebra course in middle or high school prior to their enrollment in the study. Other participants had substantial course-taking experience with college algebra, such as Danasia who had failed Elementary Algebra twice before her recruitment into the study, and Becca who had passed Elementary Algebra and subsequently failed Intermediate Algebra once prior to recruitment. This abundance of prior algebra learning experiences meant that all participants were likely to characterize at least some of their upcoming learning experience as relearning. Data collection consisted of six one-hour interviews per student, surveys, grades on course exams and responses to exam questions whenever possible. Interviews 1-4 occurred in the first semester of study enrollment and focused on student experiences relearning specific topics in their algebra course as well as larger holistic reflections on the task of relearning algebra. Interviews 5 and 6 occurred the following semester and focused more on how students' perspectives on relearning evolved over time now that they were in a different course context. Details on topic-level interview questions may be found in previous reports (Amman & Mejía-Ramos, 2022; 2023). For the sake of space, we describe the components of interviews 1-4 that focused on course-level relearning for this report.

At the end of every interview, we asked students to describe, overall, how they thought the experience of relearning in their math course had been different from learning new content in mathematics for the first time. Each student's responses to this item were carried over into subsequent interviews to probe for further reflection throughout the semester. During the fourth interview, it was not uncommon for participants to be presented with three or four responses that they had previously given to this item and asked to discuss their perception of the relationships between them. This feature of the interviews served two purposes. First, it allowed us to better understand students' evolving perceptions of relearning throughout the semester and to discuss these perceptions with them in their own words. Second, it served as a member check of preliminary codes that we might use to describe course-level relearning perceptions.

Formal analysis proceeded in multiple rounds of qualitative coding. Again, for the sake of space, we only describe analysis as it is relevant to our course-level relearning data. The first round of analysis on this data proceeded once we had already built descriptions of individual cases at the topic-level. We then coded student responses to the questions at the end of each of the four interviews in the first semester of the study in which they were asked to describe their perspective on relearning algebra and how it might differ from learning something new for the first time. Subsequent rounds of analysis involved cross-case comparisons resulting in preliminary themes that either grouped various students' course-level relearning perceptions or sought to associate a course-level perception with larger patterns of behavior and affective states across the data set. This process involved engaging in explanation-building utilizing all data sources for an individual participant, then moving to the next participant to locate commonalities across experiences and produce evidence of rival explanations if that participant experienced a similar outcome despite different combinations of the study variables (Yin, 2009). The result of this stage of analysis was a set of preliminary themes that were refined over a period of several further rounds of analysis for consistency.

Results

Discussions around relearning at the course-level centered were mainly dominated by the perspective that relearning algebra should be easy due to its familiarity and perceived simplicity relative to later topics in mathematics. All six students discussed this perspective in all four of their first semester interviews and used it to justify their level of engagement with course material. However, as will be discussed in subsequent sections, students began to experience increasing moments of cognitive dissonance around this perspective when confronted with their course performance as the semester went on.

Conflating Perceptions of Algebra Familiarity and Simplicity With Understanding

All six participants persistently described the task of relearning algebra as something that should be easy because they had learned about it before. We use the word "should" to reflect the persistence with which students held this belief even in the face of substantial difficulty with the course material. This perspective influenced the behaviors and course experiences of all participants but is perhaps best exemplified by Cara and Jina's experiences throughout Elementary and Intermediate Algebra, respectively, due to their tendency towards exposition and high degree of reflection in their interviews. Additionally, Cara and Jina serve as excellent focal points for the data set given that both participants began the study with very different levels of algebra relearning experience, with Cara having taken two prior semesters of college algebra at other institutions and Jina having Intermediate Algebra as her first math course in college.

When discussing this perspective, participants often conflated the notion of algebra being a familiar subject with the perception that they either already understood it or with the expectation that they would achieve an appropriate level of understanding in the class easily. For instance, in her first interview, Jina stated, "I've already learned [algebra], so I'm not really sure what the most difficult topic would be, whether we have gone over it in the past or not. If it was something that we went over, it would be a little bit easier to understand just 'cause I have a basic knowledge of how it would go versus if it's something completely new I think I would have a bit more trouble trying to understand." In this way, Jina describes her perception of relearning as inherently easier than learning something new due to her foundation of "basic knowledge" in the subject. This perspective was often bolstered by participants' view of algebra as a more simplistic subject in mathematics. This was exemplified in Jina's case when she

described algebra as a "very straightforward" subject, one in which problems could be solved simply by "memorizing a formula." Cara echoed this sentiment in her first interview, stating that for her almost all math at this level was memorization, adding "I know once you get further into it, it's more than that and it takes a lot, but right now it's just 'remember how to do this." In this way, students expected that their experience relearning algebra would mostly be an exercise in recalling algebraic procedures. Given that the subject of algebra was highly familiar to them, this appeared to be a task that would be easy for them to complete successfully.

As the semester went on, participants continued to hold onto this perspective, even after finding it to be at odds with the actual understanding of algebra developed during their T2 learning experience. For instance, Jina reported feeling confident that she would perform well on an Exam 1 topic that would be taught in the course because she "remember[ed] that those rules were easy to remember." That is, Jina did not have any recollection of the topic itself, but rather a sense of general familiarity with it along with a perception that understanding algebra was as simplistic as memorization and recall. However, Jina often found herself frustrated when attempting to reconcile her relearning perspective with actual course experience, stating in her fourth interview, "I keep contradicting myself cause it's like I've seen these [topics], so I'm like 'I remember it being easy, so it'll be easy' but then at the same time when I'm learning it it's kind of like, 'oh this isn't what I learned'. Cara echoed similar sentiments when she indicated on her recruitment survey that she had seen systems of equations before and remembered some of what she was taught. However, after learning about it in class, she found this perception to be false. In explaining this, she reflected, "it's probably just because like at face value it's just, you know, an easy [topic], it's not that complex." In this way, Cara and Jina, like the other participants, conflated their familiarity with algebra, along with its early introduction in the K-12 curriculum, with an actual understanding of the material. As we will see in the following section, this often led to undesirable or frustrating outcomes due to its influence over students' behaviors.

Persistent Impacts on Behavior

This conflation of perceived familiarity and simplicity with understanding of algebra was critical to participants' experiences throughout the semester, given that it was often used as a motivator for patterns of in-class behavior. For example, at the end of the semester during her fourth interview, Jina noted that despite evidence from her first exam that her understanding of algebra was often not correct or unreliable, "by exam two, I still had that mindset in the back of my mind. Like 'oh algebra shouldn't be this hard.' So then it was like some things [in class] I would listen to, but some things I would just kind of be like I like, 'I already know this' like 'I don't need to be like listening' I guess." Here we see Jina explicitly conflating her perception of algebra as something that should be easy with the notion that she already understood course material. This served as a way of motivating a pattern of disengagement from the course that she mentioned for five out of seven major course topics in Intermediate Algebra.

Cara discussed a similar association between perceptions of relearning algebra as something that should be easy and patterns of disengagement from the course. For example, in her second interview, Cara emphasized that in Elementary Algebra, "we're adults and supposed to learn quickly…faster than we did when we were younger…like you already learned this but now you're just going over it and it should be like a 'oh duh' kind of moment you know?" Like Jina, Cara's perception of the task of relearning algebra as something that should happen quickly and with relatively little effort resulted in an over-reliance on memory and a lack of engagement with course material. For example, when reflecting on her lack of success with problems involving systems of equations on a course exam, Cara stated, "I think that because of how easy it was,

once I realized what you're supposed to do, I was like 'I don't need to remember this, it's gonna be fine, I could see this anywhere and be able to solve it any day of the week.' Not the case. Not the case at all." Here, Cara seemed to initially perceive the content as almost obvious, something that she could reason through without even having an explicit memory of it. In fact, Cara stopped going to class soon after the second course exam because she felt she could recall the material on her own regardless of whether she was in class or not. This decision seemed to contradict the fact that she had scored more than 10 points lower than the class average on the first two course exams, and speaks to the persistent influence of this perception of relearning algebra on student decision-making.

Conclusions

We intend to contribute to broader literature that seeks to characterize the unique challenges inherent in the teaching and learning of developmental mathematics courses that have caused them to become the target of countless reforms (e.g. Hodara, Jaggars & Karp, 2012; Yamada & Bryk, 2016). Specifically, we focus on describing how students are interpreting the task of relearning in these courses—what, exactly, they are meant to do when learning about material they have already tried to learn about in a previous course. We believe that such a focus on student perceptions is critical given previously-noted differences in the literature between instructor's perceptions of developmental math students and students' perceptions of themselves. For instance, developmental math students are often characterized by their instructors as engaging in unproductive learning behaviors such as self-handicapping strategies (Mesa, 2012) and having poor study and time management skills (Reichwein, Schneider, & Onwuegbuzie, 2014). In contrast, students in Mesa (2012) have described themselves as preferring mastery over performance goals, a lack of engagement in self-handicapping behaviors, a disposition towards hard work, and an expectation that instructors would challenge them. Likewise, students in Cox and Dougherty's (2019) study indicated a desire to form a deep understanding of the material they were relearning, and were sometimes even disappointed by the ultimate function of the course as a "review" or "refresher." In this way, it seems that developmental math students and their instructors may hold different understandings and expectations regarding the work of relearning and how they are meant to productively engage their previously-developed understanding of algebra in this new college course context.

In addressing these concerns, it seems crucial for developmental math instructors to examine the features of their instruction or of the structure of their courses that might unintentionally support students in forming perceptions of relearning algebra that ultimately do not serve them. For instance, in this report, we stress the importance of students' judgements of algebra as a "familiar" subject. Students were often unsuccessful in making accurate judgments of familiarity on their own, believing particular topics or procedures to be something they had already learned about or understood sufficiently, then finding this to be inaccurate after receiving feedback on course exams. Given this information, it would seem reasonable for instructors to build course activities that explicitly disrupt these assumptions of familiarity, either by presenting course materials in a non-standard way (see work on restructuring college algebra around the notion of structure sense; Wladis, Sencindiver & Offenholley, 2023) or by explicitly asking students to interrogate their sense of familiarity around a topic shortly after it is introduced. Likewise, instructors may want to reflect on the way that they frame the subject of algebra while teaching, noting that suggesting that students "should" already understand the material, or should come to understand it quickly (e.g. Cox, 2015), may unintentionally support students' disengagement with course content and overreliance on their own memory of their prior understanding.

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The Role of Exploring and Conjecturing When Introducing Derivatives in Calculus: To Prove or to Verify?

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In this report, I present how eight experienced college calculus instructors used exploring and conjecturing to frame their instructional tasks for introducing derivatives. During semi-structured interviews with each instructor, I prompted them to propose instructional tasks for introducing derivatives. The tasks were broken down into their smallest calculus-specific problems, which I refer to as calculus-specific task units or CTUs. Using the notion of instructional situations by Herbst (2006), I categorized the exploring/conjecturing CTUs into a single instructional situation: Exploring or conjecturing to verify statements or relationships. I further describe how the CTUs were invoked in the tasks and the kinds of mathematical works they were expected to engage students in. To conclude, I discuss why the exploring/conjecturing CTUs revealed in the study either did not, or could not, converge to proving situations.

Keywords: Calculus, exploring and conjecturing, derivatives, instructional situations, tasks

Mathematical disciplinary practices, or activities commonly undertaken by professional mathematicians—such as exploring, conjecturing, defining, and proving (Rasmussen et al., 2005)—have been promoted to enhance students' engagement with mathematics in the classroom and improve their conceptual understanding. In the context of calculus education, the reform movement of the 1980s led to emphasizing conceptual learning and active student engagement in the classroom by relying on graphical, numerical, and technological elements for exploring and illustrating ideas, while reserving algebraic manipulations more for computational purposes (Bressoud et al., 2016; Kidron, 2020). Researchers, mathematics departments, and individual calculus instructors continue to explore, study, and implement these innovations to improve student outcomes in calculus (Borji et al., 2018; Bressoud et al., 2016; Code et al., 2014; Gerami et al., 2023).

In this paper, I focus on how college calculus instructors utilize the disciplinary practices of exploring and conjecturing. I define exploring as examining mathematical objects and relationships among them, and conjecturing as a particular case of exploring, during which one produces a generic or conceptual statement after exploring relationships among some mathematical objects. To better capture the learning opportunities the instructors create in their classrooms, I turn my attention to one concept—the derivative—examining how instructors present it through their instructional tasks (i.e., problems and activities designed to "focus students' attention on a particular mathematical idea;" Stein et al., 1996, p. 460). This paper addresses the following research question: What roles do exploring and conjecturing play in calculus instructors' instructional tasks when introducing the concept of derivatives?

Theoretical Framework

By conceptualizing teaching as a social activity, Brousseau's (1997) *theory of didactical situations in mathematics* assumes that every mathematics class operates under a didactical contract, referring to "shared expectations of what it means to study mathematics in school" (Herbst et al., 2023, p. 402). Although often implicit, the contract defines the roles of the teacher

and students, governs the classroom relationships, and regulates the work of teaching and learning mathematics through social content-related norms (Brousseau, 1984).

Relying on the notion of didactical contract, Herbst (2006) introduced *instructional situations* as local or specialized didactical contracts for specific instructional exchanges within a course of study. Herbst et al. (2020) defined *task framing* from the perspective of the teacher in mathematics instruction as choosing a way of handling a problem about the knowledge at stake; once recognized by students, the task framing allows students to know what kind of mathematical work they should perform and what kinds of interactions they can have. In other words, for instruction, teachers choose how to frame a problem, so that students know the task at hand and how they are expected to handle the problem. For example, Herbst et al. (2023) provided an example of how a teacher, with respect to the same piece of knowledge—the tangent segments theorem that states "a circle is tangent to two intersecting lines if and only if the points of tangency are equidistant from the point of intersection" (Herbst et al., 2023, p. 403)—can both frame the work as a construction and as a proof. In this way, instructional situations are both each of the distinct problem framings used in a course of studies, as well as "the specific norms that regulate the teacher's and students' work on them" (Herbst et al., 2020; 2023, p. 402). Researchers have identified various customary instructional situations, or distinct types of framing tasks, mainly in high school geometry and algebra (e.g., exploring or examining a figure, shape, or solid constructing; constructing a shape, figure, diagram, or solid including physical models; geometric algebraic calculation; solving equations in one variable). This line of work highlights that teachers can select from the available set of instructional situations to frame their instructional tasks, consequently launching, organizing, and scaffolding students' work (Herbst et al., 2018; 2020).

Methods

The data for this report comes from a larger study (Gerami, 2024) in which I purposefully selected and interviewed eight experienced calculus instructors from across the United States, who reported using various patterns of inquiry in their teaching. The instructors chose their own pseudonyms: Justin, Adrian, Barry, Matthew, Gopher (he/him); Monica (she/her); Max (they/them); and Alex (any pronouns). During four semi-structured 1–2-hour long interviews with each instructor, the instructors were asked to propose eight tasks for introducing derivatives. To ground the instructors' proposed tasks in students' conceptions of derivatives, I organized the interviews using Zandieh's framework (2000, Table 1) in two ways: 1) each interview was dedicated to one representation of the derivative (graphical, verbal, physical, or symbolic), and 2) The instructors were asked to propose two tasks within each representation that would help transition students' conception from one process-object layer to the next (the first task from ratio to limit, and the second task from limit to function).

Table 1. Zandieh's (2000) adapted framework for the concepts of derivative.

		Representations			
		Graphical	Verbal	Physical	Symbolic
Process-Object Layer	Ratio	Slope of the secant line	Average rate of change	Average velocity	Difference quotient
	Limit	Slope of the tangent line	Instantaneous rate of change	Instantaneous velocity	Limit of the difference quotient
	Function	Graph of the derivative function	Rate of change of a function	Velocity as a function of time	Derivative as a function

The instructors collectively proposed 56 tasks. Because most of the instructional tasks comprised of multiple parts, I suspected that the instructors often used multiple calculus-specific instructional situations to frame the multi-part instructional tasks. Therefore, I broke down the instructional tasks into their smallest calculus-specific problems, which I call *calculus-specific task units* or *CTUs*. By this, I mean that if I further divided a CTU into smaller tasks, the results would not be recognizable as calculus tasks; instead, they would be recognized as tasks from algebra, arithmetic, or other content areas preceding calculus in a standard mathematics curriculum sequence. After finalizing the CTUs of all kinds (calculating, graphing, exploring/conjecturing, etc.), I used thematic analysis (Saldaña, 2021) to achieve two main objectives: 1) identify the calculus-specific instructional situations, as defined by Herbst and Chazan (2012), by reducing the list of CTUs; and 2) synthesize the findings. During these processes, I heavily relied on Herbst et al.'s (2010) descriptions of instructional situations developed in geometry to guide myself on what to pay attention to within each instructional situation. To identify the calculus-specific instructional situations, I asked myself *how can I identify meaningful new categories for them if I looked past the specific mathematical objects within each CTU? How are the mathematical work and the norms of such work that students are expected to do similar/different if I looked past the specific mathematical objects within each CTU?* Essentially, to define the instructional situations, I differentiated between and categorized different CTUs based on the nature of mathematical work and the norms that regulated such work by relying on my own knowledge as a calculus instructor. This process resulted in five calculating, one exploring/conjecturing, two graphing, two installing, one proving, and one solving equation instructional situations (see Gerami, 2024 for complete description). I foregrounded each instructional situation across all instructional tasks to uncover patterns and themes about situation use. In doing so, I gathered tasks with similar codes for comparison to address the following questions: *How did instructors invoke these situations? How frequently and diversely were they employed? In what sequences and patterns did situations from this category typically appear in the tasks?* In this study, I focus only on the exploring/conjecturing findings.

Findings

All instructors used exploring/conjecturing calculus-specific task units (CTUs) in their instructional tasks for introducing derivatives. Given that conjecturing is a particular case of exploring, indicating when students are expected to produce hypotheses based on their perception (or contextualized conjectures) or reasoning (or reasoned conjectures), I included them in one category. I identified 11 distinct exploring/conjecturing CTUs across all the proposed instructional tasks. These CTUs with their frequency of use are listed in Table 2 (see Gerami, 2024 for detail of how the instructors used them in their tasks). I reduced all the exploring/conjecturing CTUs to one instructional situation: *Exploring or conjecturing to verify statements or relationships*. This instructional situation requires students to confirm that a statement or relationship is true, either based on perception or by making connections between multiple mathematical representations, without deductively proving it.

Table 2. Exploring/Conjecturing CTUs identified across the tasks, their engaged layer of derivative, and their frequency of use

Layer	Exploring or conjecturing CTUs to verify statements or relationships	*n*
Ratio	**E1.** Exploring the pattern among rates of change on consecutive intervals of one unit of length	1
	E2. Conjecturing the relationship between average rate of change of a function on an interval and the slope of the secant line	1
Limit	**E3.** Conjecturing the existence and/or value of instantaneous velocity or slope of a tangent line or a best estimate for rate of change at x_0 • given a table of values for various x_i and $f(x_i)$ including x_0 and values to the right and left of x_0 • between x_1 and x_2 (to the left and right side of x_0), given a table of values for various x_i and $f(x_i)$ including x_1 and x_2 • from a pattern of slopes of secants h-units distanced to the right side of x_0, with h decreasing • from a pattern of slopes of secants to the left and right side of x_0 • from a pattern of Δx-units distanced to the right side of x_0, with Δx decreasing with formal limit definition $\lim_{\Delta x \to 0} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$ • using average rates of change from a pattern of Δx-units distanced to the right side of x_0, with Δx decreasing	8
	E4. Conjecturing a formula for instantaneous velocity at x_0 from a pattern of Δx-units distanced to the right side of x_0, with Δx decreasing with informal limit definition $\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$ as $\Delta x \to 0$ with formal limit definition $\lim_{\Delta x \to 0} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$	3
	E5. Exploring the slope of a tangent line of a cusp	1
Function	**E6.** Conjecturing how a smooth function's graph looks like if we zoom in enough	2
	E7. Exploring the graph of difference quotient $\frac{f(x+h)-f(x)}{h}$ with sliding h going to zero	1
	E8.1. Exploring the relationship between graphs (of $f(x)$ and $f'(x)$)	4
	E8.2. Conjecturing the algebraic relationship between graphs of $f(x)$ and $f'(x)$ for $f = x^n$ (e.g., f is quadratic implies that f' is linear)	
	E9.1. Exploring the physical journey of an object given the graphs of $f(x)$ and $v(x)$	2
	E9.2. Exploring the physical journey of an object given the graph of $v(x)$	
	E10. Conjecturing the shape (trend—i.e., increasing, decreasing, constant—and concavity) of a function given the sign (positive, negative, zero) and change (increasing, decreasing, constant) of its instantaneous rate of change	1
	E11.1. Exploring $f'(x)$ using patterns of $f'(x_0), f'(x_1), f'(x_2), \ldots$	4
	E11.2. Conjecturing the power rule ($f'(x)$ for $f(x) = x^n$ for all integers) using the patterns of derivatives of $f = x, f = x^2, f = x^3, \ldots$	

Note. The different terms that instructors used in their tasks to refer to the same concept are wave underlined to indicate their equivalence.

At times, the instructors used explicit expressions, such as *explore*, *hypothesize,* and *make a prediction*, in their task descriptions to evoke these CTUs (e.g., "Can you make a prediction about what will happen when you zoom in on any point on this graph?" Justin, Prompt 5's task). However, they more often directed students towards the object of inquiry by asking them: whether they could do something (to imply the possibility of performing a mathematical action, rather than assessing students' mathematical abilities), to explain their thinking, or to consider what might occur under specific conditions (e.g., "Is there a 'best' AROC [average rate of change] estimate?"; Alex, Prompt 5's task; "Can you write down the exact number of the speed at $t = 2$?", Matthew, Prompt 1's task; "Talk about the relationship between these average velocities and what [do you] think would happen if [the windowpanes became smaller?]", Monica, Prompt 1's task). The rather indirect use of language to evoke exploring/conjecturing situations suggests that instructors may have to clarify the situation in class to convey to students what they are expected to do.

The instructional situation was presented in the form of two learning opportunities in the instructional tasks:

a) Exploring graphical representations of derivatives to make sense of, or conjectures about, symbolic representations and relationships between, functions, derivative values, and derivative functions (E2, E5, E6, E7, E8.1, E8.2, E9.1, E9.2, E10), and
b) Inductively constructing symbolic representations (including definitions) to model derivative patterns (E1, E3, E4, E11.1, E11.2).

The first group—exploring graphical representations of derivatives to make sense of, or conjectures about, symbolic representations and relationships between, functions, derivative values, and derivative functions—includes CTUs in which students are expected to make connections between graphical and/or symbolic representations of objects that are related to derivatives. In these CTUs, students rely on graphs of functions, secant and tangent lines, and derivative functions to explore and conjecture how they are related to one another or to make sense of numerical or symbolic representations related to derivatives, such as the derivative value at a cusp at in E5 and the limit of the difference quotient in E7.

The second group—inductively constructing symbolic representations (including definitions) to model derivative patterns—includes CTUs in which students are expected to detect mathematical patterns about changes and/or derivatives among some mathematical objects, and to possibly come up with procedures or formulas that model the pattern. These situations, like E3, E4, and E11, can precede installing (i.e., defining) situations by allowing students to develop intuition about definitions or formulas before they are formally installed.

I identified two recurring patterns in how exploring/conjecturing CTUs were integrated into the instructional tasks: after requiring students to calculate and graph things, or before installing a new definition, formula, or procedure. In the first pattern, the instructional tasks with exploring/conjecturing CTUs required students to perform calculations and graph things via calculating and graphing CTUs. Subsequently, students were prompted to explore or make conjectures. For example, in Alex's task for Prompt 1, students were initially asked to use average velocities to estimate instantaneous velocity (C1 and C4). They were then asked to conjecture a formula for instantaneous velocity at a point by "continuing to make their estimate better." Similarly, in their tasks for Prompt 3, Max, Alex, Adrian, and Monica used combinations of graphing and calculating CTUs related to average or instantaneous rate of change at a point before E3, in which they asked students to conjecture about the existence or the value of instantaneous rate of change at a point. In another task for Prompt 5, Gopher first asked students

to estimate the rate of change at a point, and then followed the instructional task by asking students to use an applet to conjecture the slope of tangent line at a cusp (E5). As these examples demonstrate, it appears that these preceding calculating and graphing CTUs would make room for students to observe, visually inspect graphs or sequences of numbers and identify patterns. Thus, in these exploring/conjecturing CTUs, at stake are making sense of the graphs and/or calculations, connecting various concepts/representations of derivatives, or discovering how graphs or calculations would change under specific conditions. Although the role of the teacher in exploring/conjecturing situations is often to give students the object of explorations and possibly legitimize the resources and tools that students can use (Herbst et al., 2010), in these cases the work of the teacher was more directive because the tools and resources were already given in the instructional task via students working on them through explicit graphing/calculating CTUs beforehand. This limits students' free choice in selecting the representations and operations for their exploration. There were only a few instances where the exploring/conjecturing CTU did not proceed any other CTUs. Although this suggests that students were granted more autonomy, the instructional tasks already provided the related representations for their explorations (e.g., Alex's task for Prompt 6; Justin's task for Prompt 4).

In the second pattern, some exploring/conjecturing CTUs led to CTUs that stated a new definition, formula, or procedure or validated students' findings from the exploring/conjecturing CTUs for future use. For example, in their instructional tasks proposed for Prompt 5, after asking students to conjecture the existence of instantaneous velocity at t_0 given a table of values for various t_i and $f(t_i)$ with Δt decreasing (E3), Matthew used E4 to ask students to conjecture a formula for instantaneous velocity at t_0 using their explorations from E3, and Alex used I4 to formally install instantaneous rate of change. In another example, Alex and Monica installed the power rule (I7) in their instructional tasks for Prompt 8, after having students calculate derivatives of $f = x$, $f = x^2$, $f = x^3$, … (C7 and C11) and conjecture a formula for the derivative of x^n (E11). Thus, in this pattern, the exploring/conjecturing CTUs seem to create opportunities to legitimize the new knowledge gained from the CTUs. Therefore, at stake in these CTUs is introduction to or becoming familiar with new general formulas and procedures about derivatives (limit definition of instantaneous rate of change at a point, finding slope by zooming in, power rule).

Discussion

In this report, I demonstrated that although the calculus instructors in the study relied on a diverse and engaging range of exploring/conjecturing calculus-specific task units (CTUs) in their instructional tasks when introducing derivatives with inquiry, the CTUs were all to verify statements or relationships. Despite the instructors' reliance on these CTUs when framing their tasks, the instructional situation either did not, or could not, converge to proving situations. Although Herbst et al. (2010) asserted that exploring/conjecturing situations in geometry "often converges to the formulation of conjectures, or the proofs of those conjectures," it seems that calculus instructors in this study use exploring/conjecturing CTUs to have students reason abductively and inductively. It seems that in the CTUs identified in this study, deductive inferences (i.e., proving) were either: not appropriate given students' existing knowledge (e.g., proving E6 without students' knowledge of infinitesimals), not called for because these CTUs were designed to lead to installing situations (which could not be proved deductively; e.g., E3 and E7), or that even when proving conjectures were possible, the instructors chose to not

pursue them. Regarding the latter category, I suspect that six CTUs could lead to proving situations that student could work on with using their existing mathematical knowledge:

- E2: Conjecturing the relationship between average rate of change of a function on an interval and the slope of the secant line—could lead to proving that the two quantities are equal.
- E5: Exploring the slope of a tangent line of a cusp: proving that the derivative at the cusp does not exist—could lead to demonstrating that the right-hand limit is not equal to the left-hand limit.
- E8.2: Conjecturing the algebraic relationship between graphs of $f(x)$ and $f'(x)$ for $f = x^n$)—could lead to proving that $f'(x) = nx^{n-1}$ using a limit definition of the derivative, indicating that the function loses a power when differentiated.
- E10: Conjecturing the shape (trend—i.e., increasing, decreasing, constant—and concavity) of a function given the sign (positive, negative, zero) and change (increasing, decreasing, constant) of its instantaneous rate of change—could lead to proving what happens to slope of the tangent line, written using the limit definition, at a random point on any of those intervals.
- E11.1: Exploring $f'(x)$ using patterns of $f'(x_0), f'(x_1), f'(x_2), \ldots$—could be followed by using a limit definition of derivative to calculate $f'(x)$.
- E11.2: Conjecturing the power rule ($f'(x)$ for $f(x) = x^n$ for all integers) using the patterns of derivatives of $f = x, f = x^2, f = x^3, \ldots$—could lead to proving that $f'(x) = nx^{n-1}$ using a limit definition of the derivative.

Of these CTUs, however, only three—E8.2, E10, and E11.2—were about general statements. While the other CTUs could also lead to proving, they were specific to particular functions and the conclusions would not always be useful for other functions. The absence of deductive reasoning following the CTUs can be attributed to the CTUs being either about particular examples, or because when they were generic enough, the instructors found inductive reasoning more appropriate than deductive reasoning, possibly due to time-consuming or challenging algebraic manipulations.

Indeed, the instructors in this study used a high number of exploring/conjecturing CTUs, which aligns with associating inquiry in undergraduate mathematics education with authentic problem solving and the exploration of novel ideas. However, despite the emphasis on conceptual mathematical reasoning in inquiry-oriented contexts, instances where exploring/conjecturing situations could have been followed by reasonable proving situations were, in fact, not followed up with proving situations. One explanation could be that instructors perceived proving situations, especially with student-centered activity structures, as too challenging for calculus students. Paradoxically, it may be the case that more lecture-oriented instructors might use more proving situations via lecturing. It is also possible that college instructors simply do not consider calculus an appropriate subject to engage students in the work of proving, given that 'formal' proving is often introduced in linear algebra, following the calculus series. Nonetheless, my coding of proving CTUs was rather broad, encompassing deductive mathematical justifications, which researchers have shown could be done in any mathematics class (Stylianides, 2007).

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Learning Linear Algebra Through Instrumentation of Interactive Figures: Analysis of Student-Generated Mathematical Observations and Conjectures

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Linear algebra is an important topic with many in-demand applications. In this paper, we consider a set of digital interactive figures created for introductory linear algebra students. The figures were designed to scaffold student experimentation with linear algebraic objects in the process of making observations and conjectures. We analyzed a collection of student reflections written about their experience and observations from the interactive figures through the theory of instrumental genesis. Based on the results of this pilot study, we believe that these interactive figures can be effective in promoting cycles of observation and conjecture as pre-proof activity.

Keywords: Linear algebra, CAS, eigenvalues, conjecture, instrumental genesis

Literature Review

Linear algebra has been and continues to be a crucial foundational topic for many other disciplines. More recently, the modern era of data-driven and machine learning innovation has only increased its relevance. As a deeply theoretical yet also readily applied subject, the way that linear algebra is taught is of critical importance as it needs to lead students to conceptualize and yet think pragmatically.

Since the early 1990's the Linear Algebra Curriculum Study Group (LACSG), and contemporarily the LACSG 2.0 in 2022, have set forth an educational agenda for the teaching of linear algebra in the university mathematics curriculum (Carlson et al., 1993; Stewart et al., 2022). Among this guidance, of primary relevance to this project are the recommendations to implement and integrate computer algebra systems (CAS) learning support, and to further attend to the theoretically based content common to second courses in linear algebra.

In keeping with this movement, modern linear algebraic texts often encourage and integrate the use of CAS support such as MATLAB or Maple (Lay et al., 2016; Leon, 2010) Using CAS support to assist students in performing computations involving vector and matrix manipulations, solving systems of equations or computing matrix determinants, inverses, and eigenvalues and eigenvectors allows for fast and accurate calculations. The intent is to leverage CAS to reduce a student's cognitive load on routine arithmetic, thus computing should provide an arena for experimentation and tinkering that can help reinforce conceptual ideas, such as by "generating computational justifications for claims and making strategic choices to limit [complexity]" (Bagley & Rabin, 2016). This experimental mode of doing mathematics via computation also possesses affective consequences, as students may be able to shed restrictive, formulaic views of what "doing" mathematics is (Castle, 2023). Conversely, using CAS may distract students from focusing on linear algebra and shift their attention to the coding itself. As such, in implementing CAS support in the teaching of linear algebra, it is important to understand what impact it is having on student learning.

In order to connect ideas of computation and experimentation with productive modes of mathematical activity, we utilize an emergent model for the process by which mathematicians engage with mathematics, Observation-Conjecture-Proof-Theorem (Peffer, McDonald, & Stewart, 2024). Rooted in traditional Definition-Lemma-Proof-Theorem-Proof-Corollary

(DLPTPC) models, which can be opaque and intractable to introductory mathematics students (Uhlig, 2002), the Observation-Conjecture-Proof-Theorem model of doing mathematics seeks to delineate a simple path by which students can generate, critique, and validate mathematical ideas. In an introductory linear algebra course, we further wish to primarily engage students in routines of observation leading to conjecture, which can be motivated and facilitated by CAS-automated algebra and computation.

Theoretical Perspectives

In this study, we will use Trouche's (2020) model of instrumental genesis to understand how students come to understand and utilize technology (interactive figures) to perform a task. The theory of instrumental genesis describes a codependence between a subject (prototypically the student learner) and the artifact (e.g. a calculator, graphing tool, or digital intractable).

In quantifying the artifact, the affordances/potentialities and constraints of the artifact become the relevant leverages by which the subject and artifact interact. The affordances/potentialities of an artifact are "what it offers [the student], what it provides or furnishes," and the possibilities for mathematical activity that are opened to the student. In contrast, the constraints of an artifact are those limits that bound the space of action and perception available to the subject.

Thereafter, the theory of instrumental genesis describes twin processes of instrumentation and instrumentalization, which describe the mutual changes elicited in the subject by the artifact, and vice versa upon the artifact by the subject. Instrumentation is "the action by which a subject acquires an artifact, and the effect of this action on the subject, who develops, from this artifact, an instrument for performing a task. An instrument is thus made up of an artifact component and a cognitive component (knowledge necessary for/from using the artifact for performing this type of task)" (Trouche, 2020). Conversely, the process of instrumentalization describes how a subject adapts an artifact to their intentions, via stages of discovery (of functions), personalization, and customization.

Within this study, the artifact(s) in question are a collection of interactive figures developed by the second author, meant to host a space for exploratory linear algebraic arithmetic action and observation. These interactive figures primarily feature concrete, number-valued matrices and vectors, and different calculations are performed by the figures. The affordances of the figures include the mechanisms for control at the student's disposal, largely the "new matrices," "size of matrices," or scalar setter buttons. In addition, the automatic display of the example matrices generated and their relevant sums, products, scalings, and exponentiations, as well as their corresponding calculated eigenvalues constitute intended structural constraints of the figures.

As identified by the interactive figures' constraints, the interactive figures leave very little room for personalization as the source code is unavailable to students, and the discovery process is intended to be near immediate to a population of students who are familiar with mechanisms such as buttons and sliders. As such, we will not explore the instrumentalization process and instead primarily emphasize the artifact level affordances and constraints as well as the instrumentation process, by which a subject's understandings and strategies evolve as a result of working with the artifact.

Thus, we seek to utilize the framework of instrumental genesis, in conjunction with our Observation, Conjecture, Proof, Theorem conceptual framework of doing mathematics to investigate the in-situ affordances/potentialities and constraints experienced by the students, and what observable effects upon the student via the instrumentation process can be observed.

Our research questions are:

- How did the interactive figures facilitate (or not) students to participate in the Observation-Conjecture-Proof-Theorem model of doing mathematics?
- What was the process of instrumentation of interactive figures for linear algebra students, and what were the experienced affordances and constraints of the interactive figures?

Method

This case study was conducted in an honors section of a first course in linear algebra at a large public land-grant R1 university in the northwest USA. Out of 16 enrolled students, 13 participated, with most of those students interested in quantitative degrees in STEM fields and medicine.

With the intention of developing a conceptual framework by which to evaluate the interactive figures' merits, such as by collecting student perspectives on the interactive figures' instrumental affordances and constraints, a single digital worksheet activity was assembled consisting of five interactive figures related to the algebraic properties of eigenvalues (See Figures 1-4 below). As intended introductory or pre-introductory materials, the interactive figure activity was administered to students over a weekend's time to complete following a brief first lecture on eigentheory. As such, students had been shown the definitional equation of eigenvalues and eigenvectors of a matrix as well as basic necessary conditions such as non-zero eigenvectors and uniqueness of eigenvalues for an eigenvector. By the time of participation, students had not seen nor practiced how to compute eigenvalues or eigenvectors, nor had any of the eigenvalue properties displayed in the worksheet been discussed in class.

The student participants were assigned to spend approximately 15 minutes manipulating the digital figures and look for any observable patterns or trends. For course extra credit, the students then submitted one to three paragraph reflections on their observations made based on the following open-ended prompt: Write out your thoughts in a few paragraphs about whatever you found interesting or noteworthy while in the worksheet. These reflections were gathered via the course space in Canvas, which were collected and de-identified in a OneDrive file.

From these reflections, the authors used open coding (Strauss & Corbin, 1998) to establish themes and build a codebook, which include the following main codes: Descriptive language, Eigenvalue, Eigenvector, "Why" Questions, Worksheet (Figures, Features, Design), Affect, Mental Acts, Agreement/Believe as Fact, and Examples. Furthermore, we have extracted all mathematical statements included in the student reflections to analyze the stage of mathematical activity present in the Observation-Conjecture- Proof-Theorem model of "doing" mathematics.

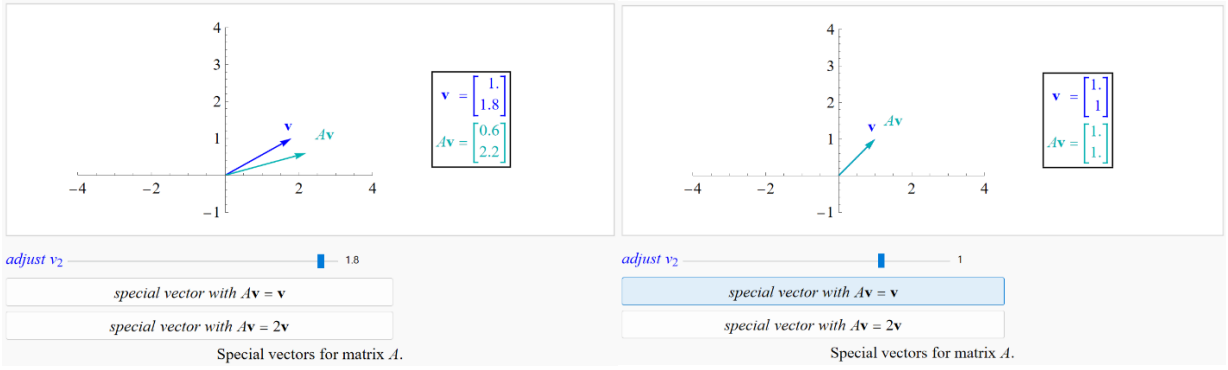


Figure 1. Geometric and algebraic connection of eigenvalues and eigenvectors.

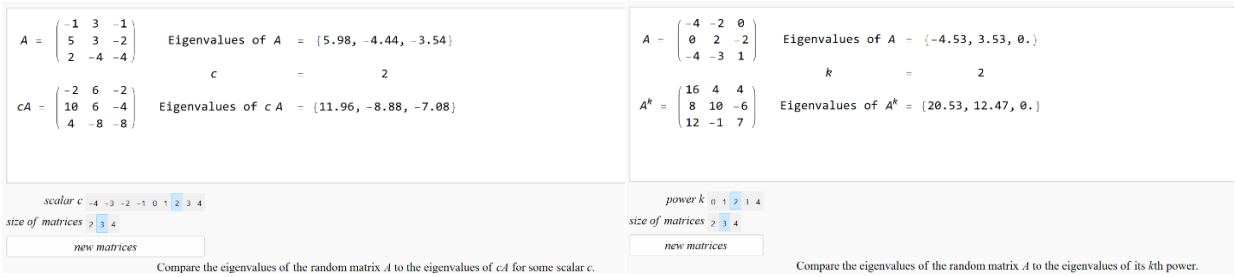


Figure 2. Scalar multiplication and matrix exponentiation on eigenvalues.

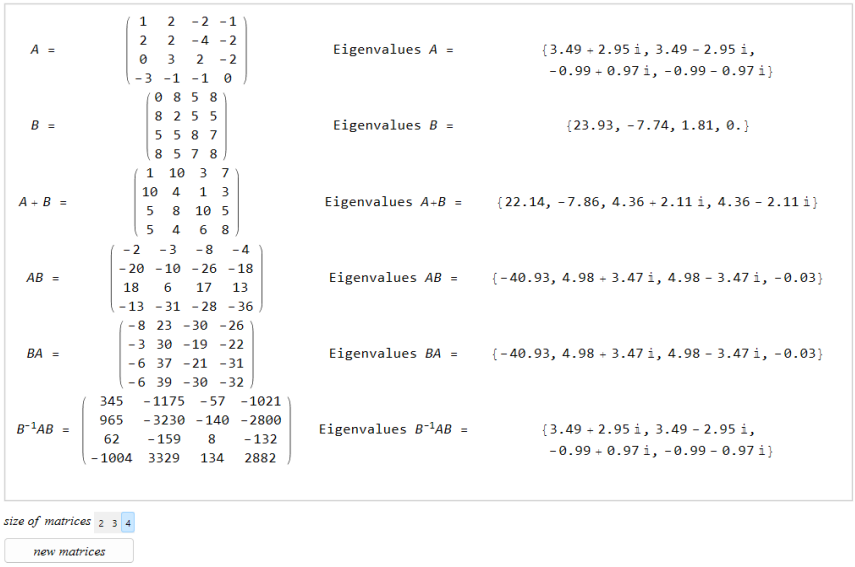


Figure 3. Eigenvalues and sums and products of matrices.

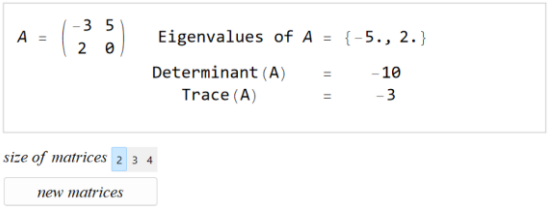


Figure 4. Relationship between eigenvalues, determinant, and trace.

Results and Discussion

Students' Construction of Mathematical Statements

Of the 13 participants in this study, 12 students attempted to discuss in some capacity the algebraic relationships given in the interactive figures. Of these students, 7 produced most or all of the desired relationships (or non-relationships, in regard to sums and products of matrices), with three others largely stating that they did observe patterns without explicating them and the final two stating that they did not understand how to interpret the figures and could not construct any of the relationships.

Scalar multiple and matrix exponentiation. Each of the 7 identified participants were able to construct valid relational statements about the eigenvalues of matrices cA or A^k from the interactive figures shown above in Figure 2, such as the excerpts below. We noticed a variety of

levels of generalization demonstrated by students, such as S1 who states the property for scalar multiples for any scalar c, while S2 only constructed the relationship for c = 2. Whether or not that student understood that the property extended to any other constant and used 2 as a prototype is unclear. Furthermore, we can observe that even though the worksheet only allowed students to select scalar values from integers between -4 and 4, students such as S1 were comfortable extending this observation to include any scalar.

> S1: "Multiplying a matrix by a scalar scales the eigenvalues by that same value"
> S2: "The eigenvalues of 2A are twice the amount as the eigenvalues of A."
> S5: "When the matrices were put to a specific power, the eigenvalues were also put to that same power."

Sum and products of matrices. Consider the more open-ended figure relating the eigenvalues of sums and products of matrices (Figure 3 above). Seven of the participants commented in some capacity that there is no identifiable pattern from the eigenvalues of A and B to the eigenvalues of the sum A+B or product AB. Nebulously, two students reflected that they could see the pattern between sums and products of matrices without explicating what pattern they were or were not seeing.

Of those students who correctly observed a lack of relationship in the matrix sum and product example, we notice a distinct difference in the style and conviction of the statements themselves. For example, S3 phrases their statement definitively whereas S5 couched their language in more uncertain terms, in particular leaving room for the possibility that they just couldn't find the intended relationship.

However, this uncertainty seemed to only be present in the case of non-patterns that the students might have expected to find, as all 7 conjecturing students presented definitive statements about the relationship between the eigenvalues of AB and BA or $B^{-1}AB$ and A, as with the final two excerpts from S4 and S6 below.

> S3: "When you add two matrices together, the eigenvalues do not correspond to the addition of the eigenvalues."
> S5: "I could not find the relationship between eigenvalues when the matrices they stemmed from were added or multiplied to each other."
> S6: "Multiplying two matrices together [does not necessarily] multiply the eigenvalues. The resulting eigenvalues always seem to be slightly greater in magnitude than the result one would anticipate."
> S4: "If a matrix A is multiplied by a matrix B, it's eigenvalues will be the same as Matrix B times Matrix A. Also $B^{-1}AB$ has the same eigenvalues as just A."

Extent of observation-conjecture-proof-theorem activity. We have seen evidence of students effectively engaging in observation and conjecture style mathematical activity through participation with the interactive figures. The below two excerpts thus demonstrate where students have stopped in the process of Observation-Conjecture- Proof-Theorem.

> S1: "As to the specific eigenvalues of matrixes, we haven't discussed how to find them, so I'm taking the worksheet's values as fact."

S3: “I also liked the brief explanations that they gave, however, I would like to have some more in-depth discussion about how to solve eigenvalues.

As expected, the students were not able to leverage the interactive figures in order to glean further conceptual meaning or computational technique about eigenvalues and eigenvectors. In this way, the domains of verification and justification as proof-type activity are not currently accessible to students following their time with the interactive figures, however by their remarks we believe that the interactive figures have helped to prime students for these further deeper conversations.

Artifacts of Instrumentation: Affordances and Constraints

The below three excerpts each contain some language relating what the student was able to “do” with the interactive figures. For S2, changing the size of the matrix being exponentiated helped to identify the relationship, at least to them. S3 remarked about the ability to change matrices, which likely came from the ability to either modify the matrix dimension or change parameters such as the scalar multiple or exponent. Finally, S9 used embodied language “scale the vector with my hand” when referring to the geometric interactive figure, indicating a level of experienced control. All of these comments indicate that the designed features of the interactive figures were largely experienced as affordances of the figures that students were able to leverage to help make sense of the encoded relationships.

S2: “For the powering up a matrix … When I made it a 3x3 matrix, it looked like the the eigenvector/the matrix of A^k is the product of A times A, and the eigenvalues of A^k = the eigenvalues of A to the power of k, but I don't know why some of the signs changed from positive to negative if it was being squared.”
S3: “I found the worksheet to be useful in that it allows you to change the matrices and see what happens to the eigenvalues.”
S9: “It was interesting to scale the vector with my hand and check the gradual change of the values on my own.”
S8: “For number three [Figure 3], I also get the products of eigenvalues and the different ways you can add or multiply them together, but like before I do not get how you created the eigenvalues in the first place.”

Conversely, the only constraint that was remarked on by students was that the operations performed to calculate eigenvalues were concealed, for example the above excerpts from S8 which communicate that the student felt limited in their ability to engage with eigenvalues and eigenvectors as they have no ability to compute and manipulate eigenvalues outside of the figures.

However, no student remarked upon other elements of the tight boundedness of the figures’ functionality such as not being able to edit matrices entry-wise, perform calculations other than those given, or change dimensions or parameters beyond the set range given. While these constraints were nonetheless present, it is unclear whether the students did feel limited by this structure and did not remark upon it, or they were sufficiently engaged by the afforded freedoms that those constraints did not impede their investigation. Further work investigating students’ thoughts while using the interactive figures could help make this distinction.

Trouche (2020) defined instrumentation as "The action by which a subject acquires an artifact, and the effect of this action on the subject, who develops, from this artifact, an instrument for performing a task." (p. 407).

From the above student excerpts, S3's statement indicates that the student felt they were in control of manipulating the matrices, even though none of the interactive figures allowed for a direct entry-wise manipulation of the matrices, while S9's comment uses embodied language of performing actions by hand. Both of these we interpret as the students actively manipulating the interactive figures in order to pursue a goal, in this case to make observations of the algebraic and geometric objects, and thus the students have, to some degree, instrumentalized the interactive figures for the purpose of engaging in Observation-Conjecture-Proof-Theorem activity.

Concluding Remarks

This study sought to investigate the types of activity in an Observation-Conjecture- Proof-Theorem perspective of doing mathematics that students engaged in while working with the interactive figures. We saw that students were able to effectively make reasonable and coherent mathematical statements relating algebraic properties of eigenvalues based upon their observations in the interactive figures, and that we believe this activity to have helped prime the students for further conceptual and computational development of eigen-theory. Additionally, the theoretical framework of instrumental genesis was useful and operationalized to examine how students experienced and engaged with the interactive figures in the action of making observations and conjectures.

In conjunction with these interactive figures focused entirely on properties of eigenvalues, other figures have been made that also display eigenvector information. Furthermore, the second author has designed several other sets of interactive figures on the topics of linear combination and span, linear transformations, matrix multiplication, and determinants, to be used across introductory and second courses in linear algebra.

Since these interactive figures are solely intended to scaffold and elicit observation and the potential for conjecturing activity, the first and second authors are in the process of integrating these interactive figures more intentionally into an introductory linear algebra curriculum by supplementing them with explicit conjecturing and proving assignments. From this pilot investigation, we believe that students are able to instrumentalize these interactive figures for the task of investigating algebraic relationships, and thus we hope that students can further utilize those observations and conjectures in the mathematical process of Observation-Conjecture-Proof-Theorem by subsequently supporting their conjectures via proof.

Furthermore, we were not able to follow up with students to make any assessment of retention or long-term effects of this interactive figure worksheet in their learning of linear algebra. Furthermore, the reflection prompt was left purposefully open-ended so that students could direct their writing towards whatever they found notable about the worksheet, meaning only 7 of the 13 participants attempted to quantify the algebraic relationships present. In future iterations of this research, we plan to pair investigations of these interactive figures in introductory linear algebra courses with subsequent proof activities that structure and draw from the observational affordances of the interactive figures. In doing so, we wish to better understand how these interactive figures can be operationalized and instrumentalized by students to accomplish a task, in this case observing and conjecturing quantifiable algebraic relationships in order to further prove them.

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Multi-Team Embodied Activity and Semiotic Games to Develop and Tune Meanings for Abstract Algebra Concepts

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As students advance in undergraduate mathematics courses, the formal (e.g., textbook definitions and proofs) and informal (e.g., metaphorical) language they encounter becomes increasingly loaded. Additional work is needed to unpack how learners can engage in semiosis "meaning making" processes beyond surface marks, symbols, and verbiage. In this paper, we conduct semiotic bundle analyses to explore the potential of a full range of embodiments in meaning making. Our results show how embodiments can be leveraged in a more advanced abstract algebra setting to mediate semiotic conflicts between students' and institutional/communal signs and expand meaning-structures. We present more specific meaning tuning and elaboration strategies that utilized embodiments and were identified during our analyses.

Keywords: Embodied Instruction, Semiotic Bundle, Semiotic Games, Abstract Algebra

Introduction and Background

Much of the existing work on semiotics in abstract algebra settings is centered on (a) spoken language with puddle diagrams that instructors' produce and use to teach abstract algebra (e.g., Rupnow, 2021; Rupnow & Randazzo, 2024a) and (b) conflicts between students' signs and institutional meanings in separate interview settings (e.g., Melhuish, Ellis, et al. 2020; Melhuish, Lew, et al., 2020; Melhuish et al., 2023). Students can be proficient with formal language and symbols without being able to leverage this language to create new signs (Duval, 2017, 2006a; Sandoval & Possani, 2016). In an abstract algebra context, students can use metaphors for operation-preserving in ways that conflict institutional meanings (Rupnow & Randazzo, 2024b), experience conflict when translating a line in a proof to diagrams (Guajardo et al., 2021), and they can tear or erase parts of Cayley graphs when trying to create structure-preserving maps (Lajos, 2021). However, the interpretations that students attach to instructional language, and how they could be tuned towards appropriate interpretations in real-time, remains unclear. Melhuish et al. (2023) noted that "semiotic conflicts and resolutions are informed by contexts beyond the scope of our task-based interview study" and suggested that "additional research into the students' broader learning ecology" is needed (p. 515-516).

In research at the intersection of semiotics and embodiment in early algebra, "Gestures are highlighted as one of the ways in which mathematical cognition is embodied and they are elevated as a potentially important semiotic device in the teaching of mathematics" (Stylianou et al., 2024, p. 259). While attention to speech and inscriptions is prominent, the inclusion of a full range of embodiments (e.g., gesture, multi-body movements, etc.) and activities in which learners are free to collectively move in the classroom, is just beginning in abstract algebra settings (Gandell, 2024; Soto et al., 2024).

Such gaps along with findings on the semiotic affordances of gestures motivated this present work. We sought to devise and pilot an in-progress embodied instructional lesson for group theory topics (e.g., group properties, homomorphisms, quotient groups) to make students' interpretations for instructional language salient and to examine how students' and their instructor might leverage embodiments (e.g., gestures, multi-party enactments, verbiage,

diagrammatic inscriptions, materials such as necklaces, markers, etc.) to develop meanings and signs for abstract algebra concepts.

In this paper, we conduct *semiotic bundle analyses* (Arzarello & Robutti, 2010) on learners' evolving sign productions, that occurred during the embodied lesson. In addition, we explicate *semiotic games* (Arzarello et al., 2009) consisting of a sequence of intervention strategies that the instructor uses to tune and enrich students' meaning structures. Thus, we address the following two research questions.

1. In what ways do learners use embodied resources to construct meanings and signs related to algebraic concepts during multi-team embodied activity?
2. How does the instructor leverage embodiments in the classroom to tune and facilitate the enrichment of students' meanings and signs in real-time?

Embodiments Serving as Semiotic "Meaning-Making" Resources

We now turn to the literature to examine gestures (Alibali et al., 2019; Stylianou et al., 2024) and how other forms of embodiments (e.g., verbiage, inscriptions, whole body performances) (Nemirovsky & Ferrara, 2009) can be used by instructors and built into the classroom ecology to support meaning-making. Drawing from Vygotsky's (1978) work, Mariotti (2018) describes semiotic mediation as a "process" that "consists in the evolution from the emergence of personal meanings related to the accomplishment of a task and in the collective development/construction of shared signs related to both the artefact's use and the mathematics to be learnt." She highlights the instructor's role in integrating a variety of different tools into classroom lessons which engender different signs and personal meanings. One can also consider embodiments (Nemirovsky & Ferrara, 2009) as semiotic mediation tools (Alibali et al., 2019; Arzarello et al., 2009; Stylianou et al., 2024). From our review of the literature, we found that semiotic mediation via embodiments can be categorized into meaning-tuning and meaning-elaboration processes.

By *tuning processes*, we mean any *mediating* action that attempts to resolve a *semiotic-conflict* defined as, "any disparity or difference of interpretation between the meanings ascribed to an expression by two subjects (persons or institutions)" (Godino et al., 2007, p. 133) or any disparity among triadic signs for the same object (Duval, 2006). Alibali and Goldin-Meadow (1993) found that children who had evoked mismatches between spoken procedures and gestures, improved their performance after instructional interventions in which their incorrect interpretations were revoiced and pushed up against the meaning of the equal sign 'is the same as'. Thus, such mismatches should be viewed as a cue to the instructor that learners are at the "threshold of learning a concept" and that a learning opportunity is immediate (Alibali & Goldin-Meadow, 1993, p. 470).

By *semiotic elaboration*, we mean any mediating action in which a set of partial yet harmonious signs from subjects (persons or institutions) are preserved and pieced together rather than replaced. Alibali et al. (2019) describes cases in which instructors' regesture a learner's gesture and use speech to attach discipline specific language. In such cases, the instructor is using the learner's gesture as a semiotic resource to carry meaning associated with the gesture and attach it to disciplinary language. In addition, to the more prominent use of a gesture, speech, and inscriptions sets, multi-party formations/movements and material interactions can also serve as resources in the process of semiosis (Kelton et al., 2018; Kelton & Ma 2020). For instance, Kelton et al. (2018) found that families ambulatory movements of walking around a double torus together prompted verbiage from the children such as "Donut!" and a spatial sense for curves (p. 550). The natural embodied impulses of a child sliding, touching, feeling, and running freely without hesitation led to sense making moments that adult family members acting according to

proper normative rules (e.g., walking slowly and stopping to read exhibit signs) did not experience.

Theoretical Framework: Action Process Communication Space (APC)

The APC Space is a theoretical framework that brings together embodied cognition, communicative discourse, and semiotic meaning-making perspectives (Arzarello & Robutti, 2008). Drawing from an embodied perspective, this framework asserts that mathematical meanings stem from sensorimotor activity which is multimodal. Over time, metaphors which represent relational linkages between an abstract mathematical concept, such as adding, and physical activity, walking forward along a path, are created as new signs for already familiar concepts (Lakoff & Núñez, 2013). The communicative perspective emphasizes that becoming an "effective" communicator through mastery of written or speech "genres" (e.g., common patterns of communication) and their "meta-discursive rules" goes hand in hand with becoming an effective thinker (Sierpinska, 2005, p. 207-210). A semiotic perspective (Duval, 2017; Peirce 1931), emphasizes that meaning-making occurs in the process of creating a sign. Meaning is developed by and revealed through interpreting agents and their interactions as they modify previous signs to create new ones. The semiotic bundle is a "living" construct and analysis tool that integrates the three aforementioned approaches to mathematics education.

Semiotic Bundle and Semiotic Games

According to Peirce's triadic conception, a sign consists of the surface *representamen*, the *object* being signified, and the *interpretant* defined to be the underlying meaning structure attached to the *representamen-object*. *Representamens* are physical in that they can be observed through human senses. For any object, multiple signs can be produced, and each sign can be associated with multiple interpretants (Kehle & Lester, 2003, Figure 5.4, p. 103; Peirce 1931). *Semiotic system/register* refers to a set of signs, a set of rules for how to produce and transform signs, and relationships between signs and interpretive meanings (Arzarello & Robutti, 2010; Duval, 2017; Ernest, 2006). Signs can be transformed in at least two ways. A *treatment* refers to how a learner manipulates signs withing a single representational register, such as symbolic manipulations of an equation. A *conversion* is a mental mapping that converts *a sign* produced in an initial register to a *sign* for the same object in another target register. Additional *interpretants* may be afforded by converting across or *coordinating* various registers simultaneously (Duval, 2006; Sandoval & Possani, 2016). Meaning-making activities can also involve metacognition in which a previous interpretant, sign, object that has been consolidated, is then taken as the object of study for which a new sign and interpretant is produced (Kehle & Lester, 2003).

A *semiotic bundle* (Arzarello et al., 2009; Arzarello & Robutti, 2010) is defined as a complex adaptive structure that consists of all triadic signs produced by individual and collective bodies of meaning-makers, transformations of signs within and across production modalities (e.g., inscription, gestures, verbiage, etc.), and their relationships both temporal (e.g., co-occurrence of gesture and verbiage) and semantic. The semiotic bundle can provide information to the instructor regarding a peak learning opportunity and offers visibility into students' multimodal resources as they become available (Arzarello et al., 2009; Arzarello & Robutti, 2010; Vygotsky, 1978). In a semiotic game, students take a move by creating signs (e.g., representamen, object, and interpretant) and instructors take a move acting as mediators between student generated signs and institutional or classroom communal signs. Mariotti (2018) describes cases in which physical artefacts are used by an instructor as *semiotic mediation tools* to bridge semantic gaps between personal and communal signs. Similarly in thinking about the role of embodiment, we

seek to understand how students' embodiments performed during multi-team enactments can be used as a mediating tool in an abstract algebra classroom.

Participants and Settings

The researcher (second author) and teacher-researcher (first author) unpacked loaded instructional language to inform the design of an embodied unit consisting of four lesson days. We reflected on past teaching experiences, training in group theory, and the literature. We asked ourselves: what semiotic ingredients or resources could be leveraged to generate and tune meanings attached to metaphorical and formal language? We began to theorize possible viable paths that learners could take to develop personal meanings and signs. During Day 1 students were asked to enact a directed Cayley graph and were asked to consider how this representation encoded algebraic structures (e.g., closure, commutativity, connectivity). Tasks for Day 2 were designed to support students in embodying properties of elements within groups and embodying homomorphisms across groups. The students worked in three teams, associated with the groups D_4, $Z_4 \times Z_2$, and with mappings/homomorphisms. The eight students in each of the D_4 and $Z_4 \times Z_2$ groups were asked to develop an enactment for binary operations within these groups. Then the students were asked to consider how their embodied representation encoded algebraic structures (e.g., identity, inverses, associativity). Meanwhile, the mapping team was asked to develop a plan for testing if a candidate mapping was a homomorphism and lead the full-class congress in analyzing their potential homomorphisms.

Methodology

We audio-video recorded a total of four classroom periods that comprised the multi-team embodied unit as well as debriefing sessions. We prepared the audio-video classroom data that included agents, verbiage, gestures, inscriptions, glances (Arzarello et al., 2009; Sabena, 2008) collective body configurations and multi-party enactments (Nemirovsky & Ferrara, 2009; Kelton et al., 2018; Kelton & Ma, 2020) other socio-material interactions (e.g., with markers, Cayley table handouts, necklaces, and technology), whole body formations/movements and eye gaze (Soto et al., 2024). Agents included individual students moving as separate entities, pairs of students, and groups of students moving together as a single entity. We conducted two semiotic analyses, *synchronic* and *diachronic* (Arzarello et al., 2009).

We began the *synchronic analysis* by looking across modes of language production invoked simultaneously (e.g., verbiage, gesture, inscription, whole body movement, etc.). A column was added to code each time-stamp as either containing or not containing a semiotic conflict. Semiotic conflicts resulting from a mismatch in triadic sign components were highlighted by both the instructor and instructor-researcher. We discussed moments coded as conflict until we came to an agreement and provided a column in the coding book to take additional notes on the mediation strategies "games" and embodiments that the instructor invoked in anticipation or in response to semiotic conflicts. Following the synchronic analysis, we considered meaning-making across time and conducted a diachronic analysis.

During *the diachronic analysis* "the evolution of signs activated by the subjects in successive moments" is examined (Arzarello et al., 2009, p. 100). We kept track of any salient semiotic elaboration strategies that occurred across time such as representational *conversions*, *coordination* (Duval, 2006; 2017) defined as the linking-up of signs across time, and *embedding* (Kehle & Lester, 2003), in which the bundle at a previous time-stamp was taken as an object of study for which new *interpretants* and *representamens* are produced. We noted the time-span stamp and a description of the elaboration strategy towards enriched interpretations and signs.

Results: Day 2 Congress and Semiotic Games that Ensued

We focus our results to whole-class discussions during Day 2 [49:36.000-1:15:51.093] for this short paper. The instructor used the term "Congress" to refer to these discussion periods because these full-class discussions allowed the group to act like a legislative body that evaluates the evidence for claims and decides collectively if they are accepted. The instructor encouraged the sets of students to share their ideas and representations with a focus on the moments of friction in their work. To make progress on research question 1, we provide evidence of students' meaning-making through embodiments. To address part of research question 2, we present selected vignettes for "semiotic games" that clarify the instructor's tuning and elaboration interventions. For each vignette, we present findings on a) how the instructor engineered or responded to semiotic conflicts, b) embodiments that agents utilized as tools to tune or elaborate meanings, and c) the effects of such interventions on the living bundle.

Vignette 1 [49:36.000-54:06.520]: Engineered conflict between personal and institutional interpretants of homomorphism definition

This vignette showcases how the instructor leveraged the mapping team's inscriptions for injective mappings from D_4 to $Z_4 \times Z_2$ and embodied enactions to engineer and bring attention to a conflict between personal and institutional interpretations for a homomorphism. Namely that, it is necessary to check that a pair of elements satisfies the inscribed property equation, but it is not sufficient. The instructor opened congress with "Alright, I think we're ready to start acting. Are we ready to start commanding teams [wiggles fingers towards mapping team]. Cierra began brewing the engineered conflict as she led her classmates to physically act out the homomorphism property for a pair that would work. Mapping first and combining first led to the same embodied result, Taylor received the marker. Following this, Cierra exclaimed "It works and why is that?!" Seth grins and Cierra remarked, "That's because rotations are commutative!" The class collectively cheered "Yaaaaaaay!"

Next, Nico led the class to test the homomorphism property for a different pair, which the mapping team had planned and knew would fail the test. The choice of two rotations again is a counter to the "commutative" conflict. Nico began by pointing at the symbolic inscription of the homomorphism property on the board to direct classmates to **combine first then map**. He stated, "okay so the first group we're going to do R-ninety acting on R-ninety" and the instructor rephrased using the language "R-ninety with R-ninety". Four different signs emerged for this **combining first** computation. First, Max verbalized "double". Second, Sean gestured [makes fist with hands, half-circular motion, bumps left fist on top of right fist, and points to classmate]. Third, Nico gestured [points to domain bubble in puddle diagram inscription]. Fourth, Jez stacked two markers and handed the stack to Jacob. This fourth sign has the potential to carry the meaning that binary operations are mappings in themselves.

After combining signs were created, additional signs emerged for **mapping** R-180 to (1,0) which consisted of pointing gestures and maker passes. First, Cierra exclaimed "Jacob" [points to inscription (1,0) in range bubble]. Second, Nico followed with "and then R-180 is mapped to (0,1). Third, Jacob and Violet leveraged the previous two markers stacked sign invoked by X and chained it with a passing sign. Jacob, R-180, walked over to Violet [raises hands] playing the role of **(1,0)** and handed her the stacked markers which completed the combining first then mapping enaction. While nearing resolution of the engineered conflict a secondary conflict that the instructor had not anticipated occurred. The next vignette highlights the role of a collective body in mediating this unanticipated conflict.

Vignette 2 [54:06.520-55:16.993]: Utilizing an explosion of embodiments to tune Julia's unanticipated conflict

Nico continued the semiotic game and directed his classmates to **map first then combine.** Nico said, "Cool. Now, we're going to **map first**". Vivian pulled the stacked markers apart clearing the previous material sign. Nick leveraged board inscriptions and stated, "we mapped R-90 [point to puddle diagram inscription on board] to (0,1)." Next, Julia R-90 attempted to make a conversion from Noah's verbiage-inscription representation to a collective body representation. Julia walked towards Noah playing the role of (0,1). She extended her arm out and handed the marker to Noah. The instructor intervened with explicit verbiage, "Julia you need both of them" referring to the markers which were needed to make the more involved conversion from Nick's verbiage "we're going to map first". Julia paused her enactions with the marker and Noah. She then turned towards the instructor and asked, "Oh because I map to myself?", clipped the two markers together and looked back to the instructor. It is at this moment that the unanticipated conflict in the form of a verbiage-classroom sign mismatch had occurred. Julia verbiage "map to myself" and the classroom sign for combining (clipping markers together) were at odds. At this point it became apparent that the meanings Julia had attached to classroom signs and instructional verbiage needed further tuning.

Immediately following this conflict, a reactive explosion of arm movements and verbiage occurred spontaneously [54:26.860-54.31.440]. The instructor remarked "No, not together you're mapping both". At the same time Cierra said "you just keep holding them separately" [grips both fists, extends arms out in front of her torso, and holds fists in air] (Fig 1, left-right). Sean straightened his bent elbows and extended his arms in parallel as if holding two markers (Fig. 1, left). The effects of collective efforts to mediate Julia's mismatch follow. Julia responded "Oh" [holds both markers side by side]. Next, she passed the left marker to Noah then passed the right marker to Noah (Fig. 1 left-middle). It is at this moment that Julia re-attempted the conversion by replacing her enacted representation with the intended target representation. Following this, June extended her pointer and middle finger in parallel. This could be evidence that June converted the previous **mapping first** signs into her fingers [54:33.300-54.34.930] (Fig. 1, right).

Figure 1. Explosion of embodiments linked temporally to mediate Julia's mismatch



Noah then enacted the classroom sign for **combine** [stacked both markers together] and passed them to Erica-**(0,0)**. Cierra exclaimed "It's not the same person! That didn't work". The instructor remarked "Didn't work? Yes, curses". The instructor explained that "no bijection between them could be a homomorphism, because then they would be isomorphic so then both would be abelian or both would be non-abelian so it's good that you found a counterexample".

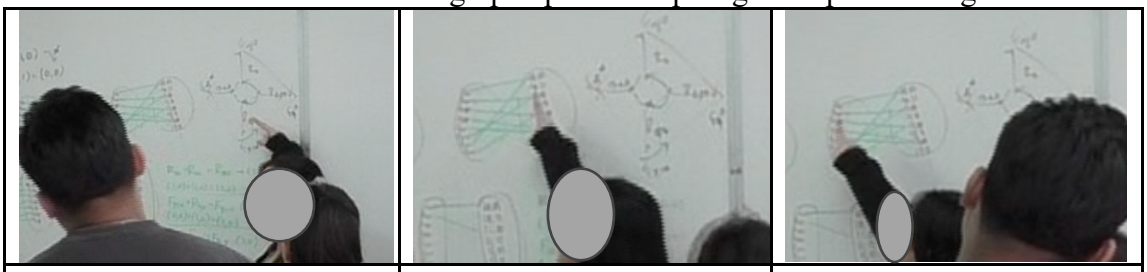
Vignette 3 [55:16.910-55:45.767]: Utilizing gesture to self-tune verbiage and convert signs

In this vignette, we first showcase how a student utilized gesture to self-tune her own verbiage. Cierra walked to the board and opened her body towards her classmates to share a more "promising" mapping. She explained, "This one we thought was a little bit more promising

[points to directed graph on the board] because we have the [pauses with gaze on board] generators make two inner circ uh [traces outer circle with right index finger into the inner circle]. Following her tracing gesture, Cierra automatically restarted her verbiage "uh an inner circle and outer circle [pinches inner circle with between thumb and index finger] with her gaze still fixed on the board. She spoke, "then some partnerships" [traces right index finger up and down over line segment connecting outer and inner circle of inscribed D4 Cayley graph]. This evidence suggests that Cierra's tracing and pinching gesture provided a mediating function. Namely, an awareness that her verbiage was incorrect. Her gesture was enough to help her self-correct and re-start her verbiage given that her gaze remained fixed on the board.

Cierra also created a compounded sign by coordinating the verbiage "partnerships", which emerged during Cayley graph enactions on Day 1, and the line segment tracing gesture over diagrammatic representations. Cierra then mapped meaning from her compounded "partnership" sign into the puddle diagram as she explained, "So we thought that the partnerships" [repeats tracing gesture] "would be mapping to the same" [traces finger across map line, points to (0,0) inscription in puddle range, and traces finger backwards] "coordinate on the other side, if that makes sense" [traces finger back to (0,0) in domain puddle] (Figure 2). Upon further enacted testing the class concluded that the "promising" mapping also failed.

Figure 2. Student coordinated directed graph "partnership" sign and puddle diagram



Repeats line-segment tracing for partnerships	Points at (0,0) in range of puddle diagram	Traces backwards to domain

Discussion

A primary contribution of this work is that it identifies the semiotic functions of an entire classroom ecology, in real-time during instruction. A secondary contribution is that this work attends to a broad scope of embodiments, in addition to those (e.g., speech and inscriptions) that are typically used to teach and conduct semiotic studies in abstract algebra settings. In this embodied lesson on homomorphisms, students and their instructor used a full range of embodiments to engage in meaning-making processes such as mediating conflicts between signs, making conversions between initial and target representations, and coordinating signs. Vignette 2 highlights how a collective body can mediate conflicts between a student and classroom sign. That is, both instructors and students together can successfully take on the role of a tuning mediator. In Vignette 2, the use of mapping leaders directing conversions between board inscription and multi-team enactions served as tool for drawing out an unknown semiotic conflict. In Vignette 3, Cierra created a compounded sign for "partnerships" which emerged during Day 1 to hypothesize and test a more "promising" mapping. Thus, a sequence of multi-team embodied activities can facilitate the compounding of signs and their respective meanings giving rise to a richer more meaning dense bundle. Thus, additional work that investigates the semiotic potential of multi-team embodied instructional units in advanced undergraduate settings is warranted.

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Meta-Analysis of How Race is Discussed in RUME Proceedings from 2019 – 2024

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In the last 20 years, mathematics education research has taken a more critical lens (Gutiérrez, 2013), in which researchers focus on salient features of students' identities (e.g., race, ethnicity, gender) to analyze the structures within mathematics and schools to motivate change to power structures and systems that oppress such groups. In recent years, there has been a notable change in the types of submissions to the RUME community. This paper details a meta-analysis of the RUME proceedings from 2019 to 2024. Using 60 keywords related to racial identity, we sorted papers based on how they discussed race and its centrality in their work. A total of 33 papers were identified as directly engaging work that focuses on race. We draw conclusions of what may be missing in current research, and the ways we may more meaningfully engage in race-conscious research in mathematics education.

Keywords: Race, Ethnicity, Meta-Analysis

In 1995, Ladson Billings and Tate introduced Critical Race Theory (CRT) into the education field, beginning a new era of research of race in education. Tate (1995) engaged CRT within mathematics education through the development of culturally relevant pedagogy, offering an alternative pedagogy that is founded on that idea mathematics should "implement a pedagogy of social change" (p. 171). Contemporary scholars have integrated CRT within mathematics education research. Martin (2009) focused on race, (re)conceptualizing the goals of mathematics education for racially marginalized groups. Gutiérrez (2009) developed an equity framework which argues that equity is achieved along a Dominant axis and Critical axis, keeping at the forefront racially minoritized students. The Dominant axis represents access and achievement within an existing system (e.g., test scores, passing rates, economic success) for racially marginalized students. The Critical axis indicates that students' racial identities are acknowledged, and that students are empowered to become agents of change in a racially unjust mathematics system. Jett (2012) and Davis (2022) have called for mathematics educators to embrace the intersectionality of CRT and mathematics disciplinary knowledge to continue to advance racial justice in mathematics education. In response to the endemic nature of racism, scholars must continue the work of pushing for social justice within mathematics education.

In recent years, we have noticed that the topic of race has become more prevalent within RUME proceedings. Due to the increased interest in issues of diversity, equity, and inclusion (DEI), and because DEI has become a ubiquitous term to capture many different student populations, we wanted to focus specifically on race to capture how our research community engages with the topic, that is: How has the RUME community engaged with race within the paper submissions for the RUME conference from 2019 – 2024? We present here a meta-analysis of the last six years of RUME proceedings related to race. This meta-analysis is guided by two tenets of CRT: 1) the centrality and intersectionality of race and racism and 2) there must be a commitment to social justice within education (Solórzano, 1998).

Identifying Papers that Discuss Race or Ethnicity

Our research team collected six years of the RUME proceedings from 2019 to 2024. These proceedings included all finalized submissions that were labeled as contributed reports (seven pages), preliminary reports (five pages), theoretical reports (seven pages), and posters (one page). Data used for this paper included only contributed, preliminary, and theoretical reports. We excluded posters because, due to limitation of length, we could not easily discern the intention of some papers in the same way as we could for longer reports.

We utilized 60 keywords to search each set of proceedings. Keywords included words that related to (a) the topic of race (e.g., multiracial, ethnicity, of color), (b) specific racial groups (e.g., Asian, Black, Latin*), (c) minority-serving institutions (e.g., HSI, HBCU), and (d) phrases related to a critical theoretical lens (e.g., critical race theory, antiracist, whiteness). After a lengthy discussion, we did not include phrases that were not specific to race or ethnicity, such as social justice, marginalized, or equity. We came to this decision as we recognized that using phrases like this groups together many identity characteristics under one umbrella and does not distinguish the focus of the research. For example, a paper can claim that it focuses on equity, yet only specifically discusses gender disparities without mentioning the race or ethnicity of the participants or impact of findings on racial groups. Similarly, papers that focus on social justice could discuss themes related to global warming or broader human rights. Essentially, a paper needed to directly discuss race in some capacity to be captured by our keyword search.

After completing the search using the list of keywords, we identified 207 papers that had at least one mention of any of the words. The highest number of words found in one paper was 26 and the least number of words present was one. Table 1 shows the distribution of papers with keywords by year. The *N* values for each year reflect the number of contributed, preliminary, or theoretical reports that were present in the set of proceedings. Notice that the number of papers containing keywords increases over the timeframe, with the 2021 proceedings containing fewer papers due to the COVID-19 pandemic. We see that about a quarter of all six sets of proceedings analyzed here contained a keyword.

Table 1: Distribution of Papers that Contained Keywords by Focus of Race in the Paper

Year	**Papers with Keywords**		**Core**		**Peripheral**		**Unrelated**	
2019 (*N*=133)	21	(15.8%)	1	(0.8%)	16	(12%)	4	(3%)
2020 (*N*=148)	30	(20.3%)	4	(2.7%)	21	(14.2%)	5	(3.4%)
2021 (*N*=49)	12	(24.5%)	1	(2%)	9	(18.4%)	2	(4.1%)
2022 (*N*=145)	40	(27.6%)	12	(8.3%)	20	(13.8%)	8	(5.5%)
2023 (*N*=151)	47	(31.1%)	7	(4.6%)	35	(23.2%)	5	(3.3%)
2024 (*N*=154)	57	(37%)	8	(5.2%)	39	(25.3%)	10	(6.5%)
Total (*N*=780)	**207**	**(26.5%)**	**33**	**(4.2%)**	**140**	**(18.0%)**	**34**	**(4.4%)**

Once we compiled the list of papers that referenced at least one keyword, we began to determine the degree to which the paper focused on race or ethnicity. We adapted three codes to capture this: *Core*, *Peripheral*, or *Unrelated* (Boeck et al., 2024). A paper was considered *Core* if it meaningfully engaged with race. This meant that the paper had race as a focal component of the theoretical framing, data analysis, or findings. A paper was not considered *Core* if the paper engaged in quantitative analysis and only used race as a control variable. If we were not sure that

the paper meaningfully engaged with race, we then turned to counting the keywords such that if the keywords appeared throughout the paper, and at least one keyword appeared at least three times in the body of the paper, then the paper would be considered *Core*. A paper was considered *Peripheral* if the keywords were used to describe only demographic information, were used only once, had limited mention, or were secondary to a larger focus of the paper. A paper was considered *Unrelated* if the keyword was only present in the references, used in a list of protected categories, or used in a citation from another scholar.

Whenever a researcher could not determine if a paper was considered *Core* or *Peripheral*, it was labeled as *Ambiguous*. The research team met to discuss these papers and determine their category. We also struggled with how to determine studies that focused on language or multi-lingual learners, and whether language not only connected to culture, but also possibly to racial identity. We drew from LatCrit (Chávez-Moreno, 2024) to support the argument that differential treatment of multilingual students is the racialization of these students.

Table 1 shows the distribution of the focus of race by year across the papers. A total of 33 papers were identified as *Core*, representing 4.2% of all proceedings (N=780), and 15.9% of reports analyzed (N=207). 140 papers were identified as *Peripheral*, representing 18.0% of all proceedings (N=780), and 67.6% of reports analyzed (N=207) papers. This group was rather large due to the wide variety of purposes for using these keywords. On one hand, some papers only had a singular keyword to describe an institution or demographic of participant. On the other hand, we had papers that had race as a secondary focus of the paper.

After identifying *Core* papers, the research team categorized the papers using the following: (a) Population of interest (e.g., students, faculty, pre-service teachers), (b) Focus (e.g., curriculum, social interactions/instruction, student supports, beliefs/attitudes), (c) Mathematics course, (d) Theoretical Framing (e.g., critical race theory, positionality), (e) Methods (e.g., qualitative, quantitative, mixed methods), (f) Dominant or Critical focus (Gutiérrez, 2009). We selected these categories to highlight the density of key components of the research studies, with the goal to inform the RUME community of areas for continued study. *Population of Interest* and *Math Course* were categories to help understand which populations are the focus of this work. The *Focus* category highlights the topic of interest for the study. For example, *Social Interactions and Instruction* include classroom instruction or group work, where *Beliefs or Attitudes* represents work that attempts to measure or reshape people's beliefs about issues around race. To understand the nature of race-conscious studies, we coded *Theoretical Frameworks* to broadly understand how researchers framed their studies. We also captured the methods that authors used to analyze studies related to race. Finally, we adapted Gutiérrez's (2009) framework of equity along *Dominant* and *Critical* axes to account for specific dimensions studied within the papers.

Findings

We discuss key findings related to the papers that contained keywords. First, we will look at overall trends of keywords over time from 2019-2024. Then, we will discuss what we found related to the Core papers related to the categories listed in the previous section. Finally, because the Peripheral paper group covered a broad spectrum of engagement with race, we discuss the different types of papers based on keyword use.

Overall Trends

As illustrated by Figure 1, the proportion of papers including at least one keyword from our list has increased each year since 2019. Specifically, 37% of the reports in the 2024 proceedings

contained at least one keyword compared to the 15.8% seen in the 2019 proceedings. However, we note that this has occurred largely through an increase in Peripheral papers using keywords. From the figure, notice that the rate at which the proportion of Peripheral papers has increased—from 12.0% to 25.3% over six years—surpasses the rate at which the proportion of Core papers has increased—from 0.8% to 5.2% over six years. This suggests that the RUME community has incorporated race more frequently into its reports, but in a way that more often accounts for demographics than providing a focused and substantive discussion of race. The 2022 proceedings had the highest proportion of Core papers seen across the six years (8.3%), about 11 times greater than the lowest proportion seen in 2019 (0.8%). The 2024 proceedings had the highest proportion of Peripheral papers (25.3%) which was about double the lowest proportion (12.03%) seen in 2019. That is, the relative increase between the highest and lowest proportions of Core papers was notably larger than that seen in the Peripheral papers.

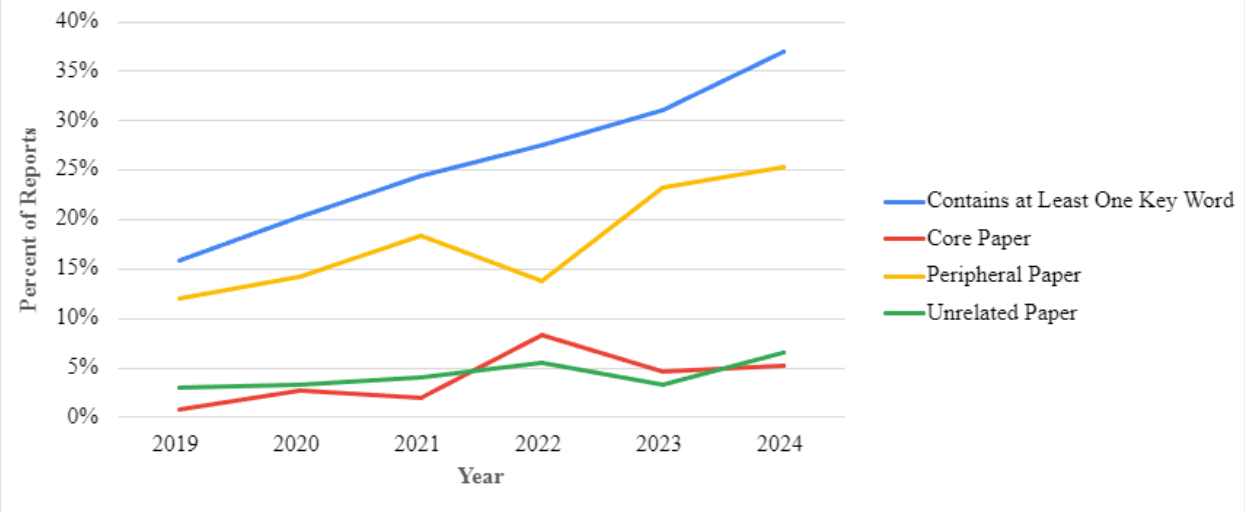


Figure 1: Line Graph of Papers Including Keywords by Focus on Race

Keywords varied by paper type, with some common keywords showing up as the most frequent among Core and Peripheral papers. Table 2 shows the top five most frequent keywords for Core and Peripheral papers. For the Core papers, seven total keywords represented the top five most frequently used; the words *race* and *racial* were present in 81.8% of papers each while *Hispanic* and *White* were present in 66.6% of papers each. You will notice that there are some common words, such as *White*, *Hispanic*, and *Black*, and that these words appeared at different frequencies. For example, in the Core papers, *Black* was the most frequently used keyword, whereas in the Peripheral papers, *White* was the most frequently used keyword. When considering the same keywords, 84.9% of Core papers used the keyword *Black* whereas only 29.8% of Peripheral papers used the same keyword. Notice that the most used keyword for the Peripheral papers is still less used than the fifth grouping of the Core papers. It is expected that the frequencies would be higher in a Core paper given our list of keywords and our selection criteria for how a paper was labeled as Core. It is interesting to note that the Core papers utilize more keywords related to the topic of race (e.g., race, racial, racialized, and of color). In fact, we found that keywords one could identify as more aligned with critical work were more highly captured in Core papers. These words include *racialized* (C: 63.6%; P: 8.51%), *racism* (C: 48.8%; P: 5.67%), *Critical Race Theory* (C: 39.4%; P: 2.84%), *racist* (C: 24.2%; P: 1.42%), and *Whiteness* (C: 21.2%; P: 2.1%), where these percentages represent relative frequency (N=33 Core, N=140 Peripheral).

Table 2: Top Five Most Frequent Words Found in Core and Peripheral Papers.

Core		**Peripheral**	
Keyword	% of Papers with Keyword	Keyword	% of Papers with Keyword
Black	84.9%	White	49.7%
Race	81.8%	Race	32.6%
Racial		Hispanic	31.9%
Of color	72.7%	Ethnic	30.5%
Hispanic	66.7%	Black	29.8%
White			
Racialized	63.6%		

Core Papers

The Core papers differed in their chosen focus. We found that 13 of the papers (39.4%) focused on the social interactions occurring in classrooms and other spaces; 13 papers (39.4%) primarily focused on the beliefs and attitudes of students and/or instructors. One paper (3.0%) focused on curriculum, two papers (6.1%) focused on student supports, and the remaining four papers looked at other issues including assessments and institutional change. Regarding population of interest, 20 Core papers (60.6%) were centered on students while 13 papers (39.4%) focused on instructors; one of these papers focused on both students and instructors. Most instructor-centered papers (64.6%) looked at beliefs/attitudes while the majority of student-centered papers (65%) investigated issues on social interactions such as instruction.

We found that 22 Core papers (66.7%) included research studies designed around a particular math course or set of courses. These 22 papers were primarily concerned with lower-division mathematics, with 13 papers (59.0%) focused on any level of Calculus and 6 papers (27.3%) focused on Precalculus. Only 3 papers (13.6%) were conducted in upper division mathematics courses, namely Abstract Algebra (2 papers) and Introduction to Proofs (1 paper).

Table 2 shows the theoretical framing found in the Core papers. 29 of the 33 Core papers (87.9%) structured their research around a theoretical framework, whereas four papers did not clearly state a theoretical framework. We found that eight papers (24.2%) utilized critical or

Table 2: Distribution of Core papers by Theoretical Framing Categories

Categories	**Example(s)**	**Frequency**
Critical and Cultural Theories	Critical Race Theory, Culturally Relevant Pedagogy, QuantCrit, LatCrit, Community Cultural Wealth	8
Identity	Emerging Mathematical and Racial Identity	6
N/A	No theoretical framing was explicitly stated	4
Other	Theory of Practical Rationality, Servingness	4
Intersectionality	Crenshaw (1991)	3
Institutional	White Institutional Space, DEI Institutional Activism	3
Sociopolitical	Sociopolitical Perspective, Social Capital Theory	3
Microaggressions	Mathematical, Racial, Gendered	2
Equity	Gutiérrez (2009) Four Dimensions of Equity	2

cultural theories (e.g., Critical Race Theory, Culturally Relevant Pedagogy, QuantCrit). The remaining papers implemented a variety of theoretical framings, which can be seen in Table 3.

Twenty-five (75.8%) of the 33 Core papers were coded as using qualitative methods, six (18.2%) used quantitative methods, and one (3.0%) was mixed methods. Twenty-three (69.7%) of the Core papers addressed race in ways characteristic of the Critical axis in Gutiérrez's (2009) framework while 10 (30.3%) papers addressed race in ways characteristic of the Dominant axis, indicating that more papers focused on issues of identity and power, rather than access and achievement. Only one quantitative paper was categorized as critical, where five studies showed characteristics of the Dominant axis. The five papers were coded as dominant because they measured levels of participation/access, graduation rates, and grades which are variables of achievement. The critical quantitative paper drew connections between intersections of students' identities (e.g., race, math) and instructional practices.

Peripheral Papers

Because the Peripheral category was so large, we noticed that there was a broad range of ways that papers focused on race. Table 3 shows the distribution of the focus of race across the Peripheral papers, listed from least to most significant. Notice that most of the Peripheral papers (58.6%) only discussed race/ethnicity in relation to the demographics of the institution, participants, or researchers—a required component of RUME proposals. We do want to note that the highest level, Secondary Focus, contains papers that were on the verge of being labeled Core papers, but because the papers had race as a secondary focus or did not clearly specify race as a focus, they were not grouped as Core.

Table 3: Distribution of Peripheral Papers by Focus of Race in the Paper (N=140)

Focus of Race	Definition	Frequency
Singular Mention	The paper had one or more keywords mentioned in one line of the paper	18 (12.9%)
Demographics	Race/ethnicity only appeared in the demographics of the institution, participants, or researchers	82 (58.6%)
Limited Mention	Mentions of race/ethnicity only appeared in the background, motivation, or future work section	25 (17.9%)
Secondary Focus	Race/ethnicity appeared throughout the paper but was not the central focus	15 (10.7%)

Discussion

The goal of this study was to provide a snapshot of the trends in the race-conscious research in the RUME community. A notable finding was the combination of population of interest and study focus. When the study investigated instructors, it was overwhelmingly to investigate their beliefs and attitudes of a racial topic. While it is important to focus on faculty beliefs and attitudes because beliefs and attitudes impact teaching practices in racialized ways (Park et al., 2024), it has also been shown that an instructors' beliefs may not immediately impact practice in the way one would expect, even if their beliefs align with best practice (Thompson, 1984). Future work could investigate instructors' teaching or assessment practices. When the studies investigated students, the focus was more on classroom instruction, group work, or other ways that we interact within learning. While these are very important areas to research, we suggest also looking into other structures in education, such as placement, course progressions, or

assessment. Other research foci in this analysis were clearly underrepresented within the Core studies. We argue that the learning process is complex and racial bias could manifest itself in localities other than social interactions and teachers' beliefs.

The student populations that were most sampled were students in lower-division courses. Most studies focused on gateway math courses, with a specific emphasis on precalculus and calculus courses. This focus on lower-division STEM mathematics courses is likely due to growing concerns about the retention of underrepresented students in STEM and introductory mathematics classes as gatekeepers in higher education (Brainard & Carlin, 1998; Bressoud, 2014; NASEM, 2016; PCAST 2012). We argue that more work should be done to include other course types. Courses like developmental mathematics, trigonometry, and college algebra also merit attention, as they have been noted as segments of the "leaky pipeline" (Larnell, 2016). Black and Brown students are placed into developmental mathematics courses at higher rates than their peers (Bahr, 2010). While developmental mathematics has been eliminated in states such as California, introductory courses have been replaced with co-requisite models that serve the same group of students (Hernandez & Cawley, 2022), and co-requisite courses merit similar attention to racial inequities. We also notice that upper-division math students are absent from race-conscious studies in the proceedings. We contend that alienation, marginalization, and attrition do not stop once racially minoritized students start taking proof-based courses.

We recognized a substantial divide between the language and terms being used within Core and Peripheral papers. In particular, the use of terms more aligned with critical work like *racist* and *whiteness* point to a substantial critique of norms and structures within mathematics education. While the overall number of studies using keywords increased over time, most of these words appeared more often in Peripheral papers. Through our investigation of Peripheral papers, the increased use of keywords seems to point to an overall increase of including participant or institutional demographic information without engaging with how the demographics inform the research topic at hand. We are unsure if this increase is due to the RUME conference call for submissions requesting that participant/institutional demographic information be included; if so, then the growing number of Peripheral papers indicates some success of this request in motivating contributors to directly name race in their work. However, our analysis here reveals the potential for the RUME community to engage in more substantive engagement with race.

We were inspired to see that a majority of Core papers engaged in research that supported identity and power (Critical axis). Unsurprisingly, we found that qualitative methods were more prominent than quantitative methods among the Critical papers in the Core category, and that quantitative papers focused mainly on issues of access and achievement (Dominant axis). While a focus on access and achievement is still important for racial equity, an overabundance of papers that measure achievement–usually a lack of achievement among racially marginalized students–can lead to a limited view of equity and a deficit view of students. Critical race scholars have argued that quantitative tools can serve social justice purposes when the questions reframe issues to highlight injustice rather than a deficit of students (Crawford et al., 2019).

In all, we found that the RUME community is making great strides to increase focus and awareness of how race exists and is experienced within post-secondary mathematics education. The findings of this meta-analysis identify where we can continue to develop our research such that we contribute more critical work that focuses on issues of race—both in ways that are obvious and others that are more historically engrained within education.

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At the Crossroads of Inquiry and Equity: Synergies and Tensions in Instructor Decision-Making

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Although some contemporary definitions of inquiry learning in college mathematics include instructor attention to equity as a core precept, evidence suggests the practice of inquiry-based instruction in proof-based undergraduate mathematics courses entails many decisions whose implications for inquiry and implications for equity and access may not be mutually reinforcing. We present the results of our analysis of over 40 design and facilitation decisions made by the second author as he taught an undergraduate-level number theory course for high school students in a summer residential mathematics program. In our analysis we identify some categories of instructional decisions that have both inquiry and equity implications. We also discuss in detail a scenario in which the instructor was able to choose a course of action that simultaneously supported inquiry and equity goals of the class, and one in which the goals of inquiry and equity appeared to be in tension.

Keywords: Inquiry, Equity, Instructional Practices, Proof-Based Mathematics

In 2019, Laursen and Rasmussen defined a set of four "pillars" for *inquiry-based mathematics education* (IBME). Of these, the fourth pillar called for instructors to "foster equity in their design and facilitation choices" (p. 138). However, Laursen and Rasmussen acknowledged that it was not yet known how the other pillars of inquiry – student engagement with rich mathematical tasks, collaborative discovery and discussion of mathematical ideas, instructor inquiry into student thinking – might best be brought into alignment with equity goals in college mathematics teaching. Indeed, several contemporaneous studies (Andrews-Larson et al., 2018; Johnson et al., 2020) have suggested, at a minimum, that an inquiry-based instructional approach is not a guarantor of equitable outcomes in conceptual learning or in public ownership of mathematical ideas. This raises a broad question: what are the mechanisms by which inquiry-based instruction supports or hinders equitable outcomes in student learning and mathematics identity development? In this report, we broach a narrower question: how does the relationship between inquiry goals and equity goals appear through the eyes of an instructor with an explicit commitment to both?

We argue that this question has significance for addressing Laursen and Rasmussen's (2019) call for equity research in IBME. Research has begun to explore theoretical perspectives that, if translated into practice and college mathematics instructor professional development, can help instructors set the stage for students' mathematical inquiry and take an asset-based approach in interpreting and responding to student thinking (Adiredja & Zandieh, 2020; Ellis, 2022; Louie et al., 2021). However, teaching in ways that center student inquiry while attending to access and opportunity for all students still presents some technical challenges for instructors. For example, stereotypes about traditionally marginalized groups may emerge in group discussion in ways in which they might not occur in whole-class discussion (Ernest et al., 2019). The skills that might help an instructor address such threats to the status and participation of minoritized students (Cohen & Lotan, 2014) are not the same as the skills required for delivering smooth, technically competent lecture-based instruction. Additionally, the process of orchestrating discussion of students' mathematical thinking, and weaving this thinking into a shared narrative that addresses the mathematical goals of a lesson, raises questions that have both inquiry and equity

implications: Whose thinking gets shared, and in what order? How to address thinking that contains reasoning errors, or that is logically sound but not communicated with the targeted level of formality or generality? How can we help students make sense of and appreciate their peers' thinking without immediately shifting to an evaluative frame of mind?

In this paper, we report on a retrospective analysis of the second author's teaching of an advanced undergraduate-level number theory course in a summer mathematics program. In this analysis we seek to address the research question: *In what ways do goals of student mathematical inquiry and equitable learning intersect in an instructor's design and facilitation decisions when teaching a proof-based undergraduate-level mathematics course?* We anticipate that findings from this analysis may inform the design of future studies of how instructors simultaneously address goals of inquiry and equity in proof-based mathematics courses and suggest possible ways for professional development to prepare instructors to manage classroom situations and decision points that have implications for both inquiry and equity.

Inquiry and Equity in The Context of Undergraduate Mathematics

To discuss synergies and tensions between inquiry-promoting teaching practices and equity-promoting teaching practices, we need to establish what we mean by both. The following sections review existing literature regarding inquiry and equity as constructs and conclude with definitions of each which we use to classify the teaching moments analyzed later in this paper.

Inquiry

There are two predominant characterizations of inquiry-centered instruction in the contemporary literature in the U.S., Inquiry Oriented Instruction (IOI) and Inquiry Based Learning (IBL). The two share many characteristics and philosophies (Kuster et al., 2018), but differ in some of their nuance and origins. Both offer valuable contributions to our conceptualization of inquiry. The tradition of IOI is deeply rooted in Realistic Mathematics Education (RME) (Freudenthal, 2005). Lockwood and Purdy elaborate on their use of RME, "Ultimately, a goal of leveraging guided reinvention (and RME more generally) is to infuse mathematics with meaning for students" (Lockwood & Purdy, 2019, p.227) illustrating that creating powerful and meaningful mathematical experiences is central to IOI.

On the other hand, IBL was borne out of work done by practitioners rather than from educational theory (Laursen & Rasmussen, 2019). Hence, IBL courses center student ideas as the main content of the course. In an IBL class, "Following a carefully designed sequence of tasks rather than a textbook, students construct, analyze, and critique mathematical arguments. Their ideas and explanations define and drive progress through the curriculum." (Laursen et al., 2014, p. 407). Both this aspect of IBL and IOI's emphasis on constructing mathematical meaning were central to both the large-scale design and day-to-day pedagogy of the number theory class.

Laursen and Rasmussen characterize IBME, a term meant to encompass both IOI and IBL, as a branch of active learning, where students do things, think about what they are doing, and talk to others about what they are thinking and doing (Laursen & Rasmussen, 2019). This take on inquiry positions students as practitioners of mathematics, something that Artigue and Blomhøj also do in their description of inquiry-based pedagogy. Artigue and Blomhøj say that inquiry-based mathematics education (IBME) "aims to offer students the opportunity to experience how mathematical knowledge can meaningfully develop." (Artigue & Blomhøj, 2013), which, like IOI, places meaningful mathematics at the center of inquiry.

Artigue and Blomhøj claim that Inquiry Based Education (IBE) historically aims to support "emancipation and democracy" (p.808) and use their analysis of IBME to set forth "ten

concerns", the last of which is "the critical and democratic dimensions of IBME" (p.809). We thus imagine that these authors might align with Laursen and Rasmussen's equity pillar for inquiry, though possibly for different reasons. Laursen and Rasmussen support their claim with seven "Equitable Teaching Practices" identified by Boaler in 2006, which they recognize as overlapping with the other three pillars. Hence, they suggest that equitable instructional practices are an essential fourth pillar of IBME. On the other hand, Artigue and Blomhøj examine the origins of inquiry, specifically the research and philosophies of Johnathan Dewey, who was a proponent of education as a tool to support democracy by producing students with both technical knowledge, but everyday skills (Dewey, 1916). To better illustrate this as an aspect of equity, we now turn to a brief review of literature in this area.

Equity

Gutiérrez (2011) characterized equity in mathematics as having two axes: the dominant and the critical. She describes dominant mathematics as, "the components students will need to be able to show mastery of in the discipline as it is currently defined and to participate economically in society." (Gutiérrez, 2011). Meanwhile, the critical axis "builds upon the idea that mathematics is a human practice that reflects the agenda, priorities and framings that participants bring to it." (p. 21). She associates the dominant axis with Access and Achievement, and the critical axis with Identity and Power, and argues that students ought to gain both dominant and critical mathematics identities. In the conceptualization of equity we adopt for the purposes of our analysis, we focus on the identity dimension of the critical axis.

Rubel (2017) identifies specific equity-directed instructional practices, two which she claims fall on the dominant axis and two on the critical. These teaching practices are: teaching for understanding, connecting mathematics content to students' experiences, fostering multidimensional participation, and connecting mathematics content to students experiences. Rubel uses these instructional practices to study teacher professional growth, and proposes that, "Indeed the practices on the dominant axis are practices that lend themselves to daily instruction, whereas the critical practices arguably could be seen as practices that teachers engage with less regularly" (p. 91) Therefore, when we later identify teaching practices and classify the extent to which they promote equitable learning experiences, we keep in mind that Access and Achievement may be more common than Identity and Power.

Our Definitions

We define inquiry-promoting teaching practices to be those which make the primary mathematical ideas of a class legitimately stem from the flow of students' thinking, thus making the content and progression of the course meaningful for the students. We define equity-promoting teaching practices to be those which give all students the opportunity to see themselves as central participants within the community and legitimate doers of mathematics.

Context And Method of Inquiry

The following analysis is based on teaching decisions made during a course on number theory taught at an informal summer mathematics program for high school students. In the program, students learn to prove foundational theorems in elementary number theory; the course is approximately equivalent to a semester-long advanced undergraduate course. The class was comprised of 36 students, all rising tenth or eleventh graders. The class met for an hour and fifteen minutes on Monday through Friday for six weeks. In addition to the class meetings, students also attended daily hour-long problem sessions with eight to twelve students facilitated

by undergraduate student mentors. The problem sessions reviewed major proofs from class and addressed students' questions about the material. The students also participated in nightly four-hour study sessions in groups of four, each led by a mentor. During these sessions, students worked on problem sets designed to foreshadow ideas in future lessons and build intuition behind major concepts in the course.

The class was taught by the second author and observed by the first. During the six weeks, both authors kept journals detailing events in class and instructional decisions made by the second author including small, in-the-moment interactions as well as larger pre-planned curricular design decisions. After the completion of the program, we used our journals to compile a list of over 40 instructional and design decisions. Separately, we utilized open coding to identify themes across the decisions then compared our codes. Ultimately, we identified six categories of decisions. We discussed possible implications of each decision for inquiry and for equity, as defined in the preceding section. We selected several decisions that we identified as having both inquiry and equity implications and performed an in-depth retrospective analysis of each. We present two of these cases below.

Analysis

In this section we provide an in-depth look at our thinking on two specific cases that illustrate distinct ways in which inquiry and equity goals can interact. We also present the six categories identified from our list of instructional decisions.

Illustrative Case 1: Building an Activity From Student Work

In the second week of the number theory course, students proved the theorem that if a divides bc and a and b are relatively prime, then a divides c. Students were asked to complete this proof in small groups during their nightly study session. The following day after class, one group presented their proof to the second author. Their approach was one that to our knowledge no students in the program had devised previously; however, there was a gap in their argument. Recognizing this gap, but seeing a way for the group to complete the proof, the second author decided that the afternoon's problem session should focus the attention of each group on completing the proof. That afternoon, mentors presented the incomplete proof and gave credit to the original authors, highlighting their creativity. The students were not told what the gap in the argument was, rather they were asked to identify a gap and attempt to fix it.

As a result of this decision, the proof became a central idea in the students' mathematical work that day. The instructor later attested to having chosen this course of action because it offered the group that authored the proof greater ownership of the class's mathematical ideas – a cornerstone of inquiry. Furthermore, the four students who originally brought forth the proof had not yet received public recognition of their ideas in class, particularly the female students who had had relatively few opportunities to share ideas publicly in class. Making the group's inventive proof an object of inquiry for the class highlighted these four students' ingenuity and the value of their ideas, consistent with our conceptualization of equity.

The instructor could have addressed these students' contribution in different ways. For example, he could have chosen to present an activity in problem session in which part of a more typical proof was given, and students were asked to critique or complete it. In this case, the students would still have had the opportunity to engage with a proof, look for gaps, and attempt to complete the argument. However, this would have focused more of the students' work on an artifact of mathematical thinking external to the class, rather than their classmates' own thinking, and would not have done as much to highlight the mathematical work of students who had not

received much prior recognition in class. In this sense the instructor had a choice available to him – using the group's proof – that advanced both his inquiry goals and his equity goals simultaneously. Thus we see this decision as an example of synergy between the two constructs, with the whole greater than the sum of the parts: greater inquiry-related goals were achieved by attending to equity, and vice versa.

Illustrative Case 2: Selecting a Class Activity on Quadratic Reciprocity

The number theory course culminated with an exploration of Gauss' law of quadratic reciprocity, which establishes a surprising relationship between the question of whether p is a perfect square modulo q and that of whether q is a perfect square modulo p, where p and q are distinct odd primes. The second author knew that the proof of this theorem is notoriously difficult for students to understand, so he wanted to provide multiple opportunities for students to engage with the ideas of the proof and try proving some of the foundational lemmas before seeing a full proof of the theorem in class.

This created a quandary for the instructor, especially given that this exploration took place during the last two days of the course: any class activity would need to run efficiently to allow sufficient time for the instructor to complete the proof of the theorem during class. He considered two possible in-class activities: either (1) have students get into groups and write up proofs of some of the lemmas that had been assigned for the previous nights' study sessions, or (2) have students do an exploratory activity in which they divide multiples of p by q, record quotients and remainders, and notice a relationship between the quotients and the remainders that serves as a foundation for one of the lemmas. Of these two options – we do not claim that there might not have been other possible options available – the instructor considered (1) more supportive of the inquiry goals of the course, because this activity would give more students the opportunity to anticipate more of the reasoning of the proof of quadratic reciprocity. On the other hand, he considered (2) more supportive of equity goals: he knew that only a few students had worked in prior study sessions on the lemmas he would assign in class, and he observed that many students were struggling with the level of abstraction of the parts of the argument the class had developed so far. In option (2) he saw an opportunity to roll some of this abstraction back and have students engage in a data collection activity (performing calculations for different pairs of primes p and q) that would generate strong evidence of a relationship that would later prove instrumental in the proof. While the reasoning available to students in this activity would not be the same quality as in option (1), the instructor knew that option (2) would be accessible to most students in class and would provide more concrete reference points for the reasoning in the proof.

We view this as an instance in which the instructor faced a decision in which he had to choose whether to prioritize maximizing opportunities for student inquiry or prioritize maximizing students' access to the proof he would ultimately present at the end of the course. This is not to say that inquiry and equity are inherently in tension with each other: this binary choice was forced by a number of factors, including the limited time remaining, missed opportunities to better engage all students in thinking about the lemmas during preceding study sessions, and the instructor's sense that completing the proof of the theorem was a non-negotiable requirement of any course of action. However, subject to local constraints, the instructor was forced to choose, in a certain sense, between inquiry goals and equity goals.

Categories of Decisions

Based on our compiled list of instructional decisions, we identified six categories of decisions that may have bearing on the inquiry culture of a class, equitable learning opportunities, or both: *defining class culture*, *exposition of content*, *facilitation structures*, *tasks and activities*, *eliciting and selecting student thinking*, and *responding to student thinking*. Given that our list is based on observations of one instructor and filtered in part by that instructor's perspective on his own practice, we do not anticipate that this list is complete. However, each instructional decision we identified fits well into at least one of these categories; some decisions fit into multiple categories. We speculate there may be categories of decisions that are overall more likely to create synergy for inquiry and equity, while other categories might generate more instances of tension, or a need to prioritize, between the two.

Discussion and Implications

We wish to acknowledge some limitations of this analysis. By taking the second author's teaching as an object of inquiry, we are able to account for insights, values, and obligations behind instructor decisions in ways that can be difficult to match in studies in which the instructors are third parties. However, the second author's perspectives on inquiry and equity are likely not perfectly representative of those of the inquiry teaching community or the wider community of undergraduate mathematics instructors. To build upon our preliminary results, a study with more instructors, in which researchers regularly check in with instructors for insight about key decisions while instructors' memories of those decisions are still fresh, would likely generate a more representative body of evidence of how instructors weigh inquiry and equity considerations in teaching.

Additionally, we acknowledge that what we identify as a tension between inquiry and equity may not be a tension if viewed from a different perspective, or with a more comprehensive set of design and facilitation options. We have both found anecdotally that as we gain more experience in inquiry-based teaching, we gain tools that help us renegotiate false choices (*e.g.*, between inquiry and content coverage) and false dichotomies. Therefore, we do not present any of the analysis in this paper as proof of any inherent incompatibility between inquiry and equity as guiding perspectives for college mathematics instruction. Rather, we present the experience of the second author as evidence that college mathematics instructors, even if experienced with both inquiry and equity, might at times perceive a tension between the two. If such a tension is perceived, then it is reasonable to infer that it could at times influence decision-making.

Based on our analysis, we see that instructional decisions arise in which the decision maker must simultaneously consider equity and inquiry related instructional goals. Our first case exhibits one way in which inquiry and equity goals synergize to create a learning experience for students in which they engage in legitimate inquiry and in which students are positioned as valued and capable practitioners of mathematics. Our second case exhibits one way in which the decision to prioritize a more equitable learning experience perhaps decreased some students' opportunity for deeper inquiry, placing the two goals in tension with each other.

The primary contribution of our analysis is considerations for future research and practice in inquiry teaching in proof-based courses. Our list of categories of decisions that impact inquiry and equity, while possibly incomplete, suggests some areas in which researchers might ask instructors using inquiry about design and facilitation decisions. If a further investigation revealed that some categories of decisions regularly generate perceived tension between inquiry goals and equity goals, professional development and professional learning communities could

equip instructors to navigate these tensions in ways that preserve the best of both constructs. For example, the [blinded] project (supported by [blinded award]) is helping college mathematics instructors address a common tension in inquiry-oriented proof-based courses: the fact that some students enter these courses with extensive background in mathematical proof, and as a result have disproportionate opportunities to drive the work of the class. One way the project is helping ease this tension is by developing group-worthy activities that offer all students high-quality, structured engagement with proof in forms other than proof construction. By pursuing such opportunities to broaden the range of activity and facilitation options available to instructors, the undergraduate mathematics education community can continue to work toward the fulfillment of the vision articulated by Laursen and Rasmussen (2019): a journey of mathematical inquiry that leaves no student behind.

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Logical Reasoning in Probability: Further Investigation Into Students' Beliefs About the Relationship Between Covariance and Independence

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In many undergraduate math courses, students are expected to interpret and use conditional statements without prior training in logical inference. In this paper we report on a portion of the third phase of a project exploring students' interpretations of a conditional statement in Probability relating covariance and independence. Specifically, we discuss the coding and analyzing of student written work in class and on assessments. We explain how this work was influenced by prior phases of the study. Quantitative results show no statistical difference between responses of students with and without prior training in logic. We do, however, see differences in students' responses overall as the semester progresses.

Keywords: Conditional Statements, Logical Reasoning, Probability, Covariance, Independence

Many theorems and definitions in undergraduate mathematics are stated as conditional statements. Consequentially, several researchers have begun to investigate how undergraduate mathematics students treat concepts from logic before they have taken a formal logic course (Case, 2015; Durst & Kaschner, 2020; Sellers et al., 2021). These studies have focused on students' meanings for conditional statements in calculus courses. In this paper, we report on an ongoing investigation into student interpretations of the following theorem from an undergraduate Probability course: "If X and Y are independent random variables, then Cov(X,Y)=0" (Casella & Berger, 2002, p. 171). Specifically, *we ask how students interpret variants of this theorem and how prior logic training impacts their interpretations.*

Literature Review

In this study, we focus on a conditional statement found in a theoretical probability course. Consequentially, we relied on literature related to both what is known about students' meanings for conditional statements as well as what is known about common hurdles that teachers might encounter when teaching an undergraduate probability course.

Students' Interpretations of Conditional Statements

A robust amount of previous literature suggests that students often interpret conditional statements as biconditionals (Hoyles & Küchemann, 2002; O'Brien et. al, 1971; Wason, 1968). Ryals et al. (2023, 2024) found that students in probability courses do often interpret the given theorem as a converse. Even if students have taken (or are currently taking) a logic course, students may not transfer principles of logic across different mathematical contexts; instead, their inferences depend on the context, particularly when it is a familiar one. (Dawkins & Cook, 2017; Durand-Guerrier, 2003; Sellers, 2020). In our study, we focus on whether students rely on rules of logic, their meanings for independence and covariance, or both in their interpretation of the theorem relating independence and covariance.

Students' Meanings for Probability Concepts

Some studies in undergraduate mathematics and statistics education investigate students' understandings of probabilistic ideas within statistics courses (Batanero, et al., 2004, 2016;

Pfannkuch & Constance, 2017). Kolmogorov's (1950) derived axiomatic theory of probability is a primary focus in undergraduate courses in probability. However, many of these studies relate to statistics education specifically and/or K-12 related domains, and little is known about how these undergraduate students think about many of the theorems like the one in our study. We do know from the mathematics and statistics education literature that many words in statistics and probability have "lexical ambiguities" (Barwell, 2005). These words may have different meanings in colloquial language than they do in mathematics and statistics classrooms (Sellers, 2020, 2024). Our study both draws from undergraduate mathematics education research in logic and what is known about lexical ambiguities to explore how undergraduate mathematics students might conceive of concepts in a theoretical probability course.

Theoretical Perspective

In this paper, we utilize a constructivist perspective (von Glasersfeld, 1995). We posit that each individual is constructing their own meanings for conditional statements as a whole, as well as constructing meanings for mathematical terms such as independence and covariance found within these definitions. We use the term meaning in the sense of Thompson et al. (2014), where a meaning is a scheme (Piaget, 1977). In light of this perspective, we focus on individual student responses related to independence and covariance first before categorizing commonalities in meanings across different students. In this phase of the study, we focus on our quantitative findings related to students' written work for the theorem. At this stage, we want to classify the characteristics of their argumentation (their justification for whether two variables are independent) in order that we might later use these characteristics as a basis for analyzing our qualitative (i.e., interview) data for students' meanings for independence and covariance.

Methods

Data Collection

Student participants were enrolled in an undergraduate Probability course for engineering and computer science majors in Spring 2024. For each participant we collected previous GPA and whether they had previously taken or were currently enrolled in Discrete Mathematics. We also gave students a survey asking about their exposure to logic instruction. We were prompted to do this by an interview participant in Phase 2; she had studied logic for LSAT preparation. The survey asked about other courses students had taken and other non-course related activities they had experienced. Through discussions with course instructors, we determined Discrete Math would be the only course used as an indicator for prior logic training, but students with select non-course experiences would be considered as having formal logic training.

The first author taught a 50-minute lecture to 7 sections of this course. This was a modification from our data collection in Phase 1, when only 4 sections were included in the study and were taught by two different instructors (Ryals et al., 2023). Moreover, in this phase of our study, all 7 sections were taught with the same lesson plan, similar to the standard treatment used in the prior phase. Student written work from this class meeting as well as responses to one midterm and one final exam problem were collected for all students present. There were 137 students who granted consent to participate, of whom 91 had received prior training in logic.

During the lecture, the instructor presented the theorem and discussed the possibility of the converse being true. Due to results obtained from interviews in Spring 2023 (Ryals et al., 2024), the instructor asked students for counterexamples of the converse. That is, she asked students how two variables could have a covariance of 0 but be dependent. In each of the 7 sections, at

least one example was presented and discussed and this was used to clarify that the theorem is not biconditional. The instructor then worked two examples that asked first whether two variables were independent and then asked for their covariance. In one example, the variables were independent and therefore the theorem could be used to conclude the covariance was 0. In the other example, the variables were dependent, so covariance had to be calculated. Students were then presented with two problems. They worked these two problems individually and their work was collected. In both problems, the students were first asked to compute the covariance of two random variables and then were asked to find whether the variables were independent. In the first, which we refer to as the "converse problem," the covariance was 0, so students were then expected to use the definition of independence. In the second, which we refer to as the "contrapositive problem," the covariance was not 0, so students had the option of using the theorem to conclude the variables were not independent. We also collected student responses to one problem on a midterm assessment and one problem from the final exam. The midterm problem was very similar to the converse problem and the final exam problem was very similar to the contrapositive problem. Responses collected are characterized in Table 1 below.

Table 1: Sample Characteristics

	No Prior Logic Training	Prior Logic Training	Overall
Average Previous GPA	3.5	3.7	3.6
Number of Post-Class Work Responses	46	91	137
Number of Midterm Responses	35	61	96
Number of Final Responses	37	62	99

Analysis

We began by coding the responses from the in-class problems. Each of the three authors individually coded each participant's response to each of the two problems, primarily using the codes established in Phase 1 of the study (Ryals et al., 2023). We met to discuss and resolve the codes we applied individually. The modification of codes was guided by analysis from the interviews completed in Phase 2 of the study and from disagreements in codes selected by the different authors. For example, on the converse problem where covariance was 0, multiple students stated "X and Y are independent because E[XY]=E[X]E[Y]." In phase 1, we had coded this response as "Assumes the converse is true." However, analysis of one interview revealed that one student believed that the expectations being equal, as in the equation above, was the definition of independence. (She was not referencing the theorem.) Therefore, we created the code, "Used expectation or variance equations." After completing the coding for the in-class problems, two authors coded each student response to the midterm problem and two authors coded each response to the final exam problem. We again met to discuss and resolve any disagreements.

Statistical analyses were conducted to investigate the relationship between prior logic training and response type as well as to examine how student responses changed over time. Multinomial logistic regression was used to study the relationship between prior logic training and response type with prior GPA and prior logic training as covariates and response

code as the dependent variable. McNemar-Bowker tests (Bowker, 1948) were conducted to study how student responses changed over time by comparing responses in the post-class problems to the corresponding problems on the midterm and final. Planned analyses included McNemar-Bowker tests, by logic training status, on the change in response from one converse problem, post-class problem 1, to the next, the midterm problem, as well as on the change in response from one contrapositive problem, post-class problem 2, to the next, the final problem.

Results

Types of Responses

We begin by presenting and summarizing the codes developed in Phase 3 which were modified from analysis in Phase 1 (Ryals et al., 2023). We will explain how these were condensed to quantitatively detect differences in how students interpret the variants of the theorem. Response rates for codes for each two types of problems are presented. Then we discuss inferential statistics assessing whether prior logic training impacted student responses and whether those responses changed over time.

Student responses closely resembled those in Phase 1 of the study. Each codes indicates how students responded to the question about independence and most often can be categorized as used the definition, used the value of covariance, used multiple arguments, or did not answer or provide justification.

Table 2: Phase 3 Codes Developed for Written Student Responses

Converse Problem	Contrapositive Problem
assumed converse was true	bypassed theorem (used definition)
claims independence can't be determined from covariance of 0	claims covariance not equal to 0 is not sufficient evidence to determine independence
used expectation or variance equations	used expectation or variance equations
no answer	no answer
incorrect or incomplete argument	incorrect or incomplete argument
no justification provided	no justification provided
used definition	used theorem
used conditional definition/argument	used theorem and also referenced definition
used informal definition argument	used definition and also referenced theorem
used or stated definition incorrectly	used informal definition/argument
wrong covariance calculation	Wrong covariance calculation
	bypassed theorem incorrectly

Prior to conducting analyses, the response codes listed in Table 2 were condensed into similar response types. For example, “used informal definition argument” and “used conditional definition/argument were combined with “used definition.” These condensed response types for each type of problem (converse, contrapositive) and their relative frequencies are listed below in Tables 3 and 4, respectively.

For the converse problems, relative frequencies stayed similar between the post-class problems and the midterm, though fewer students made the mistake of claiming that

independence could not be determined from a covariance of 0. On the contrapositive problems, however, a large portion of students who used the theorem on the post-class problem reverted to using the definition of independence on the final. Both methods can be used to correctly answer the question, though using the theorem is more efficient.

Table 3: Relative Frequencies of Condensed Codes for Converse Problems

Code Number		Post-Class #1	Midterm
1	Used the definition or meaning of independence	59.1%	58.1%
2	Claimed independence cannot be determined from a covariance of 0	10.9%	3.2%
3	Used covariance of 0 or expectation or variance equations	26.3%	32.3%
4	No complete argument given	3.6%	6.5%

Table 4

Relative Frequencies of Condensed Codes for Contrapositive Problems

Code Number		Post-Class #2	Final
1	Used the definition or meaning of independence	13.1%	41.9%
2	Used theorem	81.0%	53.8%
3	No complete argument given	5.8%	4.3%

Multinomial logistic regression analyses were conducted on the student response codes for each question controlling for prior logic training and previous GPA. The results for post-class Question 1 are found below in Table 5.

Table 5: Coefficients for Multinomial Logistic Regression for Post-Class Question 1

Code Number	Intercept	Prior Logic Training	Previous GPA
2	2.71	0.89	0.49
	p = 0.709	p = 0.838	p = 0.34
3	104.44	1.71	0.20
	p = 0.013	p = 0.254	p = 0.002
4	35.97	3.61	0.13
	p = 0.207	p = 0.292	p = 0.015

The intercept coefficients represent the odds of each student response code occurring compared to the reference code (1) when prior logic training and previous GPA are 0. As a previous GPA of 0 is not a reasonable possibility, these values are not interpreted. These results do suggest that prior logic training is not a significant indicator of any student response code at the 0.05 level. The results do, however, suggest that previous GPA is a significant predictor of student response code. The odds for student response code 3, compared to student response code 1, decrease by 80% for a one unit increase in previous GPA while the odds for student response code 4, compared to student response code 1, decrease by 87% for a one unit increase in previous GPA.

Multinomial logistic regressions were also conducted for post-class question 2 and the midterm and final questions. The results mirror those of the test for post-class question 1 with no statistically significant results at the 0.05 level for prior logic training. The smallest *p*-value found for prior logic training was 0.19. For the midterm question, which was a converse problem similar to post-class question 1, GPA was again a significant predictor of student response code. For this analysis, the odds for student response code 3, compared to student response code 1, decrease by 96% for a one unit increase in previous GPA while the odds for student response code 4, compared to student response code 1, decrease by 99% for a one unit increase in previous GPA.

To examine how student responses changed over time, McNemar-Bowker tests were conducted between post-class question 1 and the midterm question and between post-class question 2 and the final exam question. This test examines if there's a pattern in how students changed their answers from one question to the next. For example, imagine comparing two sets of answers from the same students. If about the same number of students changed their answers in each direction, the test would say there's no significant pattern. But if more students changed their answers in one direction than the other, the test would say there's a significant pattern.

Initially, tests by logic training status were planned, however, due to a smaller than expected number of students without prior logic training, conducting these tests by logic training status was not possible due to many missing response combinations. As a result, the McNemar-Bowker tests were conducted without regard to logic training status. For the comparison of post-class question 1 student response codes and the midterm question student response codes, the global test of symmetry has a *p*-value of 1, indicating that there is no pattern of movement from one student response code to another response code. For the comparison of post-class question 2 student response codes and the final question student response codes, the global test of symmetry has a *p*-value of 0.002, which is significant at the 0.05 level. This indicates a pattern of movement from one response code to another. To discover which pair or pairs of response codes have this pattern of moment, additional tests were conducted between each pair of response codes. The comparison of response codes 1 and 2 has a *p*-value of 0.007, when adjusted for the multiple comparisons. This indicates that there is a significant pattern of movement from the student response code of "Used theorem" on post-class question 2 to "Used the definition or meaning of independence" on the final. The comparison of response codes 1 and 3 has an adjusted *p*-value of 0.201 and the comparison of response codes 2 and 3 has an adjusted *p*-value of 0.617, indicating that there is no significant pattern of movement between codes 1 and 3 or codes 2 and 3.

Conclusion & Discussion

In this study we report on the third phase of a study investigating how students interpret variants of a theorem in probability relating covariance and independence of random variables. Using results from prior phases, we modified and applied an existing coding scheme and classified student written responses to two types of problems for 137 participants.

Upwards of 25% of students conclude that variables must be dependent when covariance is 0, immediately after instruction as well as weeks later on the final exam. This is despite an intended improvement to instruction in Phase 3, specifically presenting and illustrating a counterexample to the converse. These results support prior work showing students' tendencies to interpret conditional statements as biconditional and their approaches are not necessarily influenced by counterexamples (Sierpińska, 1987).

A large majority of students used the contrapositive of the theorem correctly in class when presented with two variables whose covariance is not 0. However, fewer students used the theorem on a similar problem on the final exam and this difference was significantly significant. We note that there was also an increase in the number of students correctly using the definition of independence on the converse problem on the midterm, in comparison to in class. Therefore, overall, it seems students become more comfortable using the definition of independence over time for both types of problems.

Consistent with results in Phase 1 (Ryals et al., 2023) and prior studies outside Probability (Attridge et. al, 2016; Cheng et. al, 1986; Inglis & Simpson, 2008), the differences in categories of responses for students with logic training and those without are not statistically significant. This is despite the fact that we more carefully considered how to classify a student's logic background in Phase 3.

We know from the quantitative results from this study that students' logic training may not be the only important factor to consider when discussing students' interpretations of this theorem. While logic training did not have statistically significant differences, GPA was a predictor for student success with this particular theorem. This has significant teaching implications because one might expect that some students with an overall lower GPA might have more difficulty interpreting the theorem as a conditional, one-directional statement. The quantitative data also led us to believe that there may be lexical ambiguities with i*ndependence*, *correlation*, and *covariance*, which might also be important factors to consider as we enter the next phase of this study. We further suspect that lexical ambiguities might be of concern because as we coded this data, we noticed that multiple students claimed that since Cov(X,Y)=0, X and Y are *uncorrelated*, and therefore are independent. We hypothesize that students' experience with colloquial uses of the words uncorrelated and independent have led them to view these words as synonyms. However, this needs to be further explored with student interviews that we recently conducted with this same pool of students for our next phase of the study. Future interviews may also be helpful in determining why some students provided multiple justifications, and why some used expectation or variance equations in lieu of covariance values.

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Direct and Indirect Influence of Active Learning on Women's Sense of Belonging in Calculus

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Fewer women than men choose to major in STEM, and women leave STEM majors at a higher rate than men – especially after taking Calculus. Prior research identifies low sense of belonging (i.e., feeling like an accepted member of an academic community) as a key reason why women decide to leave STEM majors. Scholars have identified factors that contribute to one's sense of belonging, including students' perceived competence and social connectedness. This study explores ways in which engaging in active learning opportunities might support women's perceived competence and social connectedness, and in turn, their sense of belonging. Women's survey responses suggest that active learning supported their sense of belonging indirectly via their perceived competence and social connectedness, as well as directly.

Keywords: active learning, calculus, gender, sense of belonging

Women continue to be underrepresented in STEM fields (Bressoud, 2023; Seymour & Hunter, 2019), and this gender gap is due in part to women's decisions to (not) major in STEM (Carmichael, 2017; Chamberlain, 2017). Fewer women than men choose to major in STEM early in their college education (Eagan et al., 2016), and women leave STEM majors at a higher rate than men (Chen, 2013; Seymour & Hunter, 2019). Prior work has identified Calculus as a common exit point for women (Ellis et al., 2016; Rasmussen et al., 2019; Seymour & Hunter, 2019). Two key reasons women leave STEM majors are that they feel a low sense of belonging to the field and they perceive the instruction in their introductory STEM courses as "poor quality" (Ellis et al., 2013; Rasmussen & Ellis, 2013; Seymour & Hunter, 2019; Shapiro & Sax, 2011). This report aims to explore women's perspectives on the connections between their sense of belonging and the learning opportunities they experienced in their year-long active learning Calculus course.

Literature Review & Theoretical Foundations

Broadly, feeling a sense of belonging involves feeling connected to or accepted within a particular environment (Rosenberg & McCullough, 1981), rather than isolated or lonely (Hagerty, 2002; Solomon & Croft, 2016). Good et al. (2012) studied students' sense of belonging to an academic domain, which they define as "one's personal belief that one is an accepted member of an academic community whose presence and contributions are valued" (p. 701). This definition of sense of belonging (SB) is most relevant to the current study.

Researchers have found that students with a stronger SB in STEM are more likely to persist in STEM (Seymour & Hunter, 2019; Shapiro & Sax, 2011). However, research shows that women are less likely than men to develop a strong SB in STEM (Good et al., 2012; Rainey et al., 2018; Seymour & Hunter, 2019; Shapiro & Sax, 2011). Good et al. (2012) found that women's SB significantly predicted their intent to pursue math in the future, and that the presence of negative stereotypes about women's mathematical abilities hindered women's ability to develop a SB.

Components of and Contributors to Sense of Belonging

Good et al. (2012) developed the Mathematical Sense of Belonging scale, which measures students' SB specifically in the domain of mathematics. In their study, a factor analysis revealed five distinct components that comprise students' SB: *acceptance, affect, desire to fade, trust*, and *membership. Acceptance* refers to whether students feel included, valued, and respected in the community. *Affect* refers to whether students feel comfortable and calm or nervous and tense within the community. *Desire to fade* is the extent to which students feel like they want to be active participants or fade into the background and not be noticed. *Trust* refers to trusting the instructor to help them learn and believe in their potential, as well as trusting that they do not have to prove their worth. Finally, *membership* refers more generally to whether they feel like members of the community and like they belong.

Researchers have identified several factors that contribute to one's SB (Lahdenperä & Nieminen, 2020; Rainey et al., 2018; Rainey et al., 2019). For this study, I focus on students' social connectedness and their perceived competence.

Social connectedness refers to the extent to which one feels like they have made connections or relationships with the people in their class. Interacting with classmates and/or the instructor allows students to develop social connections (Hoffman et al., 2003). While both SB and social connectedness involve feelings of trust and acceptance among others, SB involves feeling that one is accepted into a larger community, whereas social connectedness can be felt with just a single person. Researchers have found that social connectedness is an important contributor to students' SB (Hoffman et al., 2003; Johnson, 2012; Lewis & Hodges, 2015; Rainey et al., 2018; Thacker et al., 2022; Tinto, 1975). For example, Thacker et al. (2022) found that opportunities to interact impacted students' social connectedness, which in turn influenced their feelings of belonging. And Rainey et al. (2018) found that women in particular were most likely to attribute their SB to feeling socially connected. Further, STEM leavers attributed their low SB to lacking social connections.

One's SB to a domain can also be influenced by their perceived competence in that domain. Perceived competence refers to students' perceptions of their performance and understanding of course-related material (Rainey et al., 2018), and is not necessarily correlated to their performance in that course (Cribbs et al., 2015; Carlone & Johnson, 2007). For example, one might feel competent but perform poorly on an exam due to other factors such as testing anxiety. Research indicates that students' perceived competence can impact their SB (Lewis & Hodges, 2015; Rainey et al., 2018). In Rainey et al.'s (2018) study, students reported feeling like they belonged in their major because they felt like they understood the major-related content. Further, many STEM leavers explained that their low SB stemmed from their low perceived competence. Lewis & Hodges (2015) found that ability uncertainty, or a lack of perceived competence, was associated with low SB. They suggested that perceived competence is tied to social comparison with peers. For example, if students notice that their peers have a better grasp on the material than they do, then they might feel like they are not "good enough" or do not belong in that class.

The Role of Active Learning

One aspect of students' experiences that might influence their SB is the instructor's pedagogical approach. In their follow-up study on why undergraduate students leave STEM, Seymour & Hunter (2019) report that students are less tolerant of "poor quality" instruction than they were 20 years prior. By "poor" they mean instruction that is boring and lacking engagement. More recently, there have been efforts toward supporting student engagement, performance, and retention by incorporating opportunities for students to engage in *active*

learning (Freeman et al., 2014; PCAST, 2012; Smith et al., 2020). While there are many definitions of active learning in the literature, there seems to be consensus on the students' role in the learning process – talking about or doing mathematics. One way we can characterize types of active learning opportunities is in terms of the extent to which the opportunity encourages interaction with others (Griffin, 2023), as shown in Figure 1. In this continuum, learning opportunities like individually working on problems and responding to clicker polls would be located near the individual end of the continuum, while opportunities like group work and whole-class discussion would be located near the interactive end.

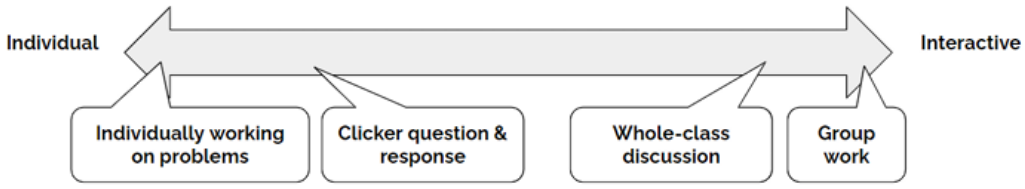


Figure 1. Active learning continuum.

I hypothesize that engaging students in active learning opportunities, especially more interactive learning opportunities, has the potential to strengthen women's SB in Calculus. First, the very nature of interactive learning opportunities enables students to form social connections with their classmates and instructor (Hoffman et al., 2003; MacMahon et al., 2020). Second, research indicates that active learning can support student achievement outcomes like performance, self-efficacy, engagement, and persistence (Freeman et al., 2014; Kogan & Laursen, 2014; Seymour & Hunter, 2019). Participating in group work or whole-class discussions presents opportunities for students to explain their thinking, which can reinforce and enhance their understanding of content, and also to form connections with their classmates. Therefore, active learning opportunities, particularly interactive learning opportunities offer promise as a way for instructors to support women's SB indirectly via their social connectedness and perceived competence.

Research Questions & Methods

The purpose of this report is to explore women's perspectives on the impact particular active learning opportunities had on their SB via their social connectedness and perceived competence. Specifically, I investigate: For women enrolled in a year-long active learning Calculus course, in what ways do they describe the impact of particular learning opportunities on their SB?

I employed a transformative (feminist) mixed methods research design (Creswell & Plano Clark, 2011) to conduct this study. I focus on women's experiences without comparing them to men's experiences because women's experiences are worthy of study in their own right (Du Bois, 1983; Kitzinger & Wilkinson, 1997), and I wanted to avoid comparisons that implicitly position the masculine perspective as "norm" and the feminine experience as "less than" (Gilligan, 1982). I administered surveys to collect data on women's perceptions of the impact of four main types of learning opportunities they experienced on their SB. The qualitative strand of this study aims to elevate women's perspectives, providing context and explanations for quantitative data.

Setting and Participants

This study was conducted at a midsized, mid-Atlantic R1 public university during the 2022-2023 academic year. The university offers a two-semester Integrated Calculus course as an

alternative to the university's standard one-semester Calculus course for STEM majors. The course and sample have been described in detail in prior work (Griffin, 2023; Griffin & Berk, 2023). Students enrolled in the course are typically first-year students who have declared or are considering a STEM major (i.e., as classified by the NSF ADVANCE Program; NSF, n.d.).

The course was designed to incorporate frequent opportunities for students to engage in active learning. During the time of the study, two sections of the course were offered, taught by two different instructors, both committed to active learning. There were 63 and 64 students enrolled in each section. The course is highly coordinated in that instructors teach from collaboratively developed lesson plans, which include recommended active learning opportunities. Specifically, instructors follow an instructional cycle of brief lecture, followed by students working on problems in groups or individually, followed by whole-class discussion. While the course was designed to incorporate opportunities for active learning, students do not know that upon enrolling in the course.

Students enrolled in the course were invited to complete a survey via email. The survey was distributed at Weeks 7, 14, and 21 of the year-long course (i.e., middle and end of fall and middle of spring). Response rates (and total number of enrolled students, N) were as follows: 90% (N=118) at Week 7, 85% (N=118) at Week 14, and 94% (N=108) at Week 21. For analysis, I include students who were first-year students and self-identified as women, which resulted in N=46 at Week 7, N=44 at Week 14, and N=39 at Week 21.

Data Collection & Analysis

To investigate my research question, I used Qualtrics, a web-based survey tool, to design and distribute an electronic survey. I administered the survey at Weeks 7, 14, and 21 of the semester. Within the survey was an item asking participants to select one of the four main learning opportunities they experienced during class (i.e., traditional lecture, interactive lecture, group work, and individual work) and describe how or why that activity has or has not impacted their SB in math. Note that the language "interactive lecture" was used in place of "whole-class discussions" because cognitive interviews with students enrolled in the course suggested that students were unclear on what was meant by "whole-class discussions" and would instead call that "interactive lecture."

At each time point, I calculated the frequencies and percentages of women's choices of activity. I analyzed their open-ended descriptions using theory-guided content analysis, anticipating the use of *social connectedness* and *perceived competence* as codes to identify times when active learning opportunities impacted women's SB indirectly via their social connectedness or perceived competence, though I also allowed for other themes to emerge. For example, I anticipated that the components of SB (i.e., *acceptance, affect, desire to fade, trust, membership*; Good et al., 2012) might emerge from women's responses, indicating more of a direct impact on their SB, so these were considered in the coding process as well. I first clustered responses by learning opportunity so that I could explore all the ways that, say, group work impacted women's SB throughout the year. Responses were not compared across time points; rather, I wanted to capture the scope of the ways in which women viewed the active learning opportunities as impactful toward their SB throughout the year.

Results

At Weeks 7, 14, and 21, participants were asked to select a learning opportunity from a list of four (i.e., traditional lecture, interactive lecture, group work, and individual work) and describe its impact on their SB. In their responses to this item, participants tended to do one of two things.

They either (1) described how the activity impacted their SB indirectly via their social connectedness or perceived competence, or (2) described how the activity they chose impacted their SB directly by invoking one of the components of SB (i.e., *acceptance, affect, desire to fade, trust, membership*; Good et al., 2012). Table 1 presents the percentages of women who chose each activity to describe its impact on their SB. Women were most likely to choose activities that fall more toward the interactive end of the active learning continuum (Figure 1). I present their responses by activity, beginning with the indirect impact it had on their SB via their social connectedness and/or perceived competence, followed by the direct impact it had on components of their SB. Given that so many women chose to discuss the impact of group work and interactive lecture on their SB, I will focus on these two activities after providing a brief synopsis of their perspectives on the impact of traditional lecture and individual work on their SB.

Table 1. Percentages of Women Who Chose Each Activity to Describe its Impact on their Sense of Belonging.

Activity	Week 7	Week 14	Week 21
Group Work	46%	52%	69%
Interactive Lecture	46%	32%	21%
Individual Work	4%	5%	5%
Traditional Work	4%	11%	5%

Note. At Week 7, N=46. At Week 14, N=44. At Week 21, N=39.

Not many women chose to discuss the impact of traditional lecture or individual work on their SB, though those who did commented on ways these activities positively impacted their SB. Those who chose traditional lecture (N=7), either discussed ways that traditional lecture impacted their SB indirectly via their perceived competence (e.g., "Traditional lecture affects how well I know the material and am able to apply it. If I feel I do not know the material well, I feel less of a sense of belonging."), or their response was vague. For those who chose individual work (N=6), responses reflected an indirect impact on their SB via their perceived competence. They either reported that they preferred to work alone or that they appreciated the opportunity to test their understanding.

Interactive Lecture

Across time, N=27 women chose to describe the impact of interactive lecture on their SB. Women's responses suggested both an indirect impact of interactive lecture on their SB via their perceived competence, as well as a direct impact on two components of SB – their feelings of acceptance and desire to fade.

Indirect impact via perceived competence. Some women (N=12, 44%) described interactive lecture as impacting their perceived competence, which in turn impacted their SB. In many cases, women explained that interactive lecture helped them understand the material. For example, one woman described, "When we have interactive lecture, we are able to discuss out all of the problems and share our ideas which helps me learn best." Others mentioned interactive lecture provided an opportunity for social comparison, which allowed them to compare their own level of understanding with that of their classmates. For example, one woman reported, "I would say interactive lecture is the most helpful because I get to hear what other people say and if they are struggling on the same things as me."

Direct impact on sense of belonging. Other women described interactive lecture as impacting one of the components of SB directly. Some women described ways it impacted their

feelings of *acceptance* (N=8, 30%). Some women described times they felt included and like they matter during interactive lecture. For example, one woman reported, "It allows my teacher to talk with us and engage us, so I feel like we are all included and listened to." Another said, "[Interactive lecture] makes me and other students feel like they matter and have a say in the class because it's encouraged for them to participate."

Many women (N=16, 59%) described how feeling involved, in contrast to fading into the background, contributed to their SB. These comments align with the *desire to fade* component of SB, though they reflect a positive view toward engagement. For example, one woman reported, "[Interactive lecture] helps with engagement and allows me to make sure I am learning which increases my sense of belonging." Others (N=2) reported that interactive lecture helped them pay attention during class, which made them feel engaged.

Group Work

Across time, N=39 women chose to describe the impact of group work on their SB. Women's responses suggested both an indirect impact of group work on their SB via their social connectedness and their perceived competence, as well as a direct impact on three components of SB – their feelings of acceptance, their desire to fade, and their affect (i.e., feeling comfortable versus anxious).

Indirect impact via social connectedness. When describing how group work impacted their SB, some women discussed how group work impacted their feelings of social connectedness, one of the contributors to SB. At times, women described group work as a means for them to feel socially connected to their classmates (N=23, 59%), which in turn supported their SB. In many cases, these were generic responses about being able to form relationships, make friends, and get to know their classmates. For example, one woman noted, "I get to know other classmates and create connections in the classroom. In a more specific example, one woman wrote:

> [Group work] has had the most positive impact on my sense of belonging in math because going over how we get an answer with my groupmates makes me feel like we're all in the same boat and I'm able to get to know my group a little better.

Indirect impact via perceived competence. Other women (N=17, 43%) reported ways that group work supported their perceived competence, which in turn supported their SB. Some simply reported that helping each other during group work supported their understanding of the material. Others were more specific and described how hearing multiple perspectives during group work supported their learning. One woman wrote, "It is helpful to get the perspectives of other students in your group and talk the problems through with them." Another wrote:

> I think talking the problems out helps me to understand the concept rather than copy answers form the professor or be confused on my own. I can also see new ways of explaining the concepts that are tricky by talking with peers.

Hearing others' perspectives also allowed students to see how well they understood the material compared to their classmates. For example, one woman wrote, "Group work is a chance for everyone to get together and share their ideas. This will allow one to get a sense of their and everyone else's understandings." Gauging others' levels of understanding made some participants feel less alone when they were struggling to understand something. One participant shared that group work allowed her to "realize that other people may also be struggling with some concepts," suggesting that she felt on-level with her classmates, which supported her SB.

Direct impact on sense of belonging. When describing the impact of group work on their SB, some women (N=9, 23%) described the ways group work impacted their feelings of *acceptance*. Some mentioned feeling valued and included. For example, one woman noted, "I

have a higher sense of belonging when working with others because it feels like I'm needed and not just there to be there." Others reported feeling like they were not alone. One said, "Group work helps with your sense of belonging because at times it's hard if you feel like you are the only one struggling. However, if you know that others are struggling too… it's a lot easier to feel like you belong,"

Other women (N=19, 49%) described group work impacting their engagement, which aligns with the *desire to fade* component of SB. Women mostly reported that engaging by getting to work together and help each other during group work supported their SB. Many women made generic statements connecting their engagement in group work with feeling like they belong. In more specific responses, women associated their level of engagement with their *affect* (e.g., feeling comfortable, calm, nervous, or anxious; N=6, 15%). In fact, all but one of the responses about affect also referenced engagement. One woman explained

> Group work makes me feel comfortable participating because I get anxious about answering questions in front of the whole class. I can share ideas with people I've grown to be comfortable and friendly with without having a fear of rejection.

Others echoed this idea that it was easier to engage when they felt comfortable. Another woman noted, "It is less intimidating to ask questions because asking in front of the whole class can be uncomfortable." Thus, participating in group work supported women's feelings of acceptance, as well as their engagement and their affect, all of which are components of their SB.

Discussion

The goal of this report was to explore women's perspectives about the impact of specific learning opportunities on their SB. The findings presented here build upon findings from Griffin (2023) by examining the impact of four learning opportunities rather than only two and providing women with a platform to describe ways their SB were impacted by these learning opportunities. When asked to select a learning opportunity and describe its impact on their SB, women were overwhelmingly likely to select group work and interactive lecture. This finding supports my hypothesis that *inter*active learning opportunities may offer more promise than solo-based active learning opportunities in supporting women's sense of belonging. In their descriptions, as hypothesized, women noted ways in which group work and interactive lecture supported their SB indirectly via their social connectedness and perceived competence. These activities supported women's ability to understand the material, as well as their ability to form relationships with their classmates, which in turn, supported their SB. In addition, women noted ways in which their SB was impacted directly as well, by invoking particular components of SB, namely, *acceptance*, *desire to fade*, and *affect,* in their responses. For example, both group work and interactive lecture supported women in feeling engaged and like their voices were valued. This is not to say that instructors should *only* employ group work and interactive lecture as women did report positive impacts of both traditional lecture and individual work on their SB. Specifically, they noted that these learning opportunities supported their perceived competence, which in turn supported their SB. However, this report provides support for employing at least some interactive learning opportunities as a way to support women's SB, which could in turn support their retention in STEM. Additionally, while other reports (Johnson et al., 2020; Reinholz et al., 2022) have noted negative impacts of active learning on women, interestingly the qualitative data in this study showed no indications of negative impacts of group work and interactive lecture on women's SB. These differences in findings could be due to several factors such as the sample, the instructors, or the enactment of the learning opportunities. More work is needed to explore aspects of enactment that are particularly important for women's SB.

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"It all started when...": Insights From Students' Math Life Stories

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This mixed-methods study evaluates the mathematical autobiographies (MABs) of STEM students to evaluate their relationship with mathematics and influential actors in their math life stories. 135 MABs were collected from students enrolled in Calculus 1, Abstract Algebra, and Topology, taught by the same instructor. Analysis revealed that students mostly had consistently positive experiences with mathematics (PME). Hypothesis testing showed that students enrolled in upper division courses and math majors had a higher likelihood of describing PMEs and that Asian students described PMEs at a lesser rate than their peers. Instructors and family members were most frequently named as influential actors, with K-12 instructors appearing more often than college instructors. These results indicate that STEM students have had primarily positive mathematical experiences, which aligns with their desire to pursue STEM degrees.

Keywords: mathematics autobiography, students' experiences in math, influential actors

Undergraduate mathematics courses have historically been considered gatekeepers to higher education (Van Dyken, 2016). As a result, researchers have focused on the equitable implementation of instructional activities that engage students in active learning and promote higher order thinking (Freeman et al., 2014; Saxe & Braddy, 2015). Instructional improvements are amiss without also attending to students' experiences in mathematics, as mathematics perception and self-efficacy have been reported as determinants of achievement (Shone et al., 2023). This study is framed by Voigt's (2020) conception of mathematics identity, which are the "dispositions and deeply-held beliefs that individuals develop…about their ability to participate and perform effectively in mathematical contexts" (p. 32). He includes three ways that a student's math identity can be shaped: 1) their perceptions of mathematical participation and ability, 2) their connection to mathematical communities, and 3) the influence of identity resources. A student's mathematics identity is the product of formative educational experiences (Silva & Roddick, 2001), and understanding students' lived experiences is an important platform on which to continue to build positive student identity (Gutiérrez, 2009). One way to understand a student's lived experience is through mathematical autobiographies (MABs) to identify and examine critical points in a student's life story with math.

Student MABs (Cook, 1995) have been essential to gain insight into students' success and struggle in math classes (Moldavan & Mullis, 1998; Williams, 2017). The benefits of MABs are twofold: math educators gain a deeper understanding of students' backgrounds and students engage in an important reflective practice. Prior work with MABs has supported the development of stronger classroom communities (Westrich, 2016) while students have also reported experiencing increased self-awareness regarding their mathematical abilities and a deeper understanding of their own math learning journey (Hauk, 2005).

The current study is focused on overall student experience with mathematics, while looking at differences by student type and influential people in their stories. Our first research question seeks to understand overall math experiences: *How do mathematics students describe their*

relationship with mathematics? We also wondered if MABs from students with different demographics (e.g., race, gender, course enrolled) differ from their peers, leading us to our second research question: *How do mathematics students' relationship with mathematics vary based on their demographics?* One aspect of identity is connection to mathematical communities (Voigt, 2020), yet little work has examined the influential actors present during these formative experiences. Our third research question asks: *How do mathematics students acknowledge influential actors in their math life stories?*

Methods

This study took place at a large public university on the West Coast of the U.S., designated as a Hispanic-Serving Institution and an Asian American, Native American, and Pacific Islander-Serving Institution. Data for this paper were a convenience sample (Creswell & Guetterman, 2019) collected between Spring 2019 and Spring 2022, totaling 133 participants enrolled in calculus 1, abstract algebra or topology taught by the same instructor. Table 1 shows the institutional student demographics. Student majors were mathematics (65%), engineering (17%), science (14%), and non-STEM (5%).

Table 1: Student Demographics (N=133)

Race	**Total**	**%**	**Gender**	**Total**	**%**	**Course**	**Total**	**%**
Latine	61	45.9	Female	56	42.1	Calculus	33	24.8
Black	9	6.8	Male	75	56.4	Abstract Algebra	86	64.6
Asian	25	18.8	Non-Binary	2	1.5	Topology	16	12
White	22	16.5						
Mixed Race	5	3.8						
Unknown	11	8.3						

Throughout the course, students were asked to submit reflection assignments, one of which students were asked to write their math autobiography. Sample prompts included to reflect on their earliest math memory, past math experiences, how they see themselves as math learners, or to reflect on meaningful mentors that impacted them. The prompts were meant to inspire, and students were encouraged to write their narratives however they wanted. 135 math autobiographies form the data set for this study. Note that the total number of MABs is larger than the total number of students as two students had enrolled in two different courses in the data collection time frame and submitted unique assignments.

The data were first analyzed using deductive coding (Creswell & Poth, 2018). Researchers coded for *overall experience with math*, *focal positive/negative moments,* and *influential actors*. Overall experience with math was categorized into five distinct codes, adapted from John et al. (2020): (a) consistently positive, (b) consistently negative, (c) consistently neutral, (d) redemption, and (e) contamination. We marked an MAB as *consistently positive*, *consistently negative*, or *consistently neutral* if their life story with math started and ended as both positive, negative, or neutral (accordingly). Even though an MAB could be coded as consistent, we also recognized that students' stories could contain ups and downs. We define a *speed bump* as a difficult or challenging moment within a student's story labeled as consistently positive/neutral. Similarly, we define a *booster* as a positive or supportive moment within a student's story labeled as consistently negative/neutral. An MAB was marked as *redemption* if their story started out negative and they had an experience that transformed their relationship with math to end

positively while an MAB was marked as *contamination* if their story started out positive and they had an experience that transformed their relationship with math to end negatively. While John et al. (2020) did not provide guidance on what was considered positive or negative in the five codes, as a team, we defined p*ositive, negative,* and *neutral* by the emotions students used to describe their experiences (e.g., "I was really curious and excited by math") as well as their self-evaluation (e.g., "I am really not great at math").

Within each MAB we also coded for any *focal positive* or *negative moment* a student wrote about that stood out as a singularly poignant experience. Within these focal moments, we coded any *influential actors* mentioned (e.g., a specific person or institution), coding whether that actor had a *positive*, *negative*, or *neutral* impact on the student. People were categorized as best as possible, and if they were a professor or mentor, we captured the course or institution type when specified (e.g., 8th grade teacher, community college Calculus 1 professor). Usually, if a student mentioned a person, they would describe a story related to that person (e.g., "Cooking with my grandma helped me count").

The entire research team deductively coded 31 MABs together, discussing coding applications and reaching agreements whenever there was disagreement. Then, three researchers coded the remaining 105 MABs, with double coding of 12 MABs (~11%), producing a reliability of 83%. In instances where there was disagreement or uncertainty, the fourth researcher reviewed and determined the appropriate coding.

Once the data were deductively coded, we utilized mixed methods to analyze the data. Thematic analysis (Braun & Clarke, 2012) was used to summarize themes within *overall experience with math* and *specific actor mentione*d. We also performed one-tailed hypothesis testing for difference of proportions for students' *overall experience with math*. We wanted to know if the difference in proportions of positive math experiences for two similarly related identities was statistically significant. To simplify the calculations, we defined two groups: *positive math experience* (PME) and *negative math experience* (non-PME). Any MAB coded as Consistently Positive or Redemption was coded as a PME, as both experiences result in current positive experience with math. Conversely, any MAB coded as Consistently Neutral, Consistently Negative, or Contamination was defined as a non-PME. We compared specific demographics (course enrolled, gender, race, and major) to determine which identities were reporting PMEs at a higher/lower rate. Our goal was to determine if there was a statistically significant difference in the proportion of PMEs for one identity compared to PMEs of all other respondents without that identity.

Findings

Table 2 shows the breakdown of overall experience with math and key events from students' math life stories. We can see that most students have a consistently positive experience with math. However, of the 87 students coded as consistently positive, we see that 86.2% were coded as having had speed bumps along their journeys. This is important to observe, as it shows the resilience of students who face adversity in their stories yet continue to push through with a positive relationship with mathematics. We also notice that students who are consistently negative or neutral also experience positive focal moments, even though their overall experience with math is negative or neutral. Most all students' MABs that were coded as consistently negative were also coded as containing a booster during their math life story (87.5%).

We wanted to compare proportions of positive math experiences (PME) by different student demographics. First, we wanted to compare lower division students and upper division students. Then we wanted to compare by gender. Because we only had two non-binary students, we

grouped non-binary and female students together as they are the non-majority gender identities in comparison to male students. We also wanted to test by race doing so for Latine, White and Asian students. We were unable to test for Black students, as our population had less than 10 Black students. Finally, we compared students by major.

Table 2: Distribution of Autobiographies as Positive Math Experience and Non-Positive Math Experience by Event

Experience with Math	**Speed Bumps and Boosters**		**Positive Focal Moments**		**Negative Focal Moments**	
Positive Math Exp (PME)						
Consistently Positive ($N = 87$)	75	(86.2%)	79	(90.8%)	59	(67.8%)
Redemption ($N = 21$)	-	-	19	(90.5%)	18	(85.7%)
Non-Positive Math Exp (Non-PME)						
Consistently Negative ($N = 8$)	7	(87.5%)	7	(87.5%)	7	(87.5%)
Consistently Neutral ($N = 15$)	11	(73.3%)	12	(80.0%)	10	(66.7%)
Contamination ($N = 4$)	-	-	4	(100.0%)	4	(100.0%)
Total ($N = 135$)	**93**	**(68.9%)**	**121**	**(89.6%)**	**98**	**(72.6%)**

Table 3 shows the eight different hypothesis tests we ran for difference of proportions. When comparing the rates of PMEs for students enrolled in different math classes, we found that students enrolled in more advanced mathematics courses were more likely to have a positive math experience ($p<0.01$). Similarly, students pursuing a degree in math or engineering were more likely to report having a PME more often than other majors ($p<0.01$). We observed that female and non-binary students had a larger proportion of PMEs but found no statistical evidence to support the idea that either gender experienced PMEs at a greater rate. Next when looking at racial identities, we were unable to find strong evidence to support the idea that Latine and White students experienced PMEs at a different rate than when compared to other races. However, with significant evidence to support, we saw that Asian students experienced PMEs at a lesser rate when compared to other races ($p<0.05$).

Table 3: One-sided Hypothesis Test for the Difference of Proportions for Positive Math Experiences

Demographic	**Test Performed**	**Test Statistic**
Class Enrolled	(Abstract Algebra & Topology) vs. (Calculus)	4.71***
Gender Identity	(Female & NB) vs. (Male)	0.87
Racial Identity	Latine vs. Other Races	1.47*
	Other Races vs. Asian	2.22**
	White vs. Other Races	1.49*
Major	Math vs. Engineering	2.11**
	Math vs. Non-Mathematics	3.99***
	(Math & Eng) vs. (Non-Math & Non-Eng)	4.05***

*p<0.1, ** p<0.05, *** p<0.01

Because so many students were coded as having consistently positive MABs (N=87), we wanted to categorize the ways students were describing their positive experience with math. Seven major themes emerged from the analysis of student responses. Table 4 shows each theme, a description of the theme, and their frequencies. MABs could be coded with multiple themes.

Easy and Fun (43.6%) and *Appreciation of Math* (34.4%) were the most common themes. For example, a female Latina math major reflected on a previous experience in middle school,

> I was doing simple algebra but I think this specific day was when I had an epiphany and really was so happy about doing math and in my 7th grade mind I was thinking wow I really love this is so easy and fun to do I can do 20 of these problems.

Many students' positive relationships with mathematics were also affected by good performance, community and support, and its perceived utility.

Table 4: Themes found in Consistently Positive Math Autobiographies N=87

Theme	Description	Count
Easy and Fun	Referenced an overall enjoyment of mathematics	38
Appreciation of Math	Found material "beautiful" or "fascinating"	30
Good Performance	Performed well on exams, assignments, or overall course grade	28
Community and Support	Received support and encouragement from teachers, peers, etc.	25
Utility	Found math useful and practical.	10
Personal Growth and Intellect	Found that mathematics helped them grow intellectually.	9
Appreciation of Difficulty and Challenge	Enjoyed solving challenging problems.	6

Of the 87 consistently positive autobiographies, 75 (86.2%) of the MABs were coded as having speed bumps. Table 5 shows the seven themes that emerged from the analysis of the speed bumps including their frequency. *Difficulty with Material* (50.6%) and *Poor Teacher Experiences* (29.3%) were the most common themes; there were times when both overlapped, meaning that sometimes if students had negative experiences with teachers and instructors, they also expressed that they found the material difficult or got low scores on exams. A Latino male math major recounted his experience entering middle school: "I struggled a lot when it came to math. Things that were considered to be easy were very complicated and I lost a lot of my confidence." A Black female engineering major in calculus wrote about when she took precalculus with the hardest teacher at her school:

> I did struggle a bit in the class and did not have the motivation to ask my teacher for help because she seemed to not want to help me when I approached her the first few times. I failed her class because one, I did not seek for help else[where] and because I lost motivation and felt as if my teacher saw me as an individual who has no skills in math and could not be successful in that field.

It was common to see that students struggled because they were unable to receive support from their teachers, or that students' confidence decreased because a teacher made them feel less than capable of doing mathematics.

Table 5: Themes found in Speed Bumps from Consistently Positive Math Autobiographies (N=75)

Theme	Description	Count

Difficulty with Material	Struggled in the course and got low scores	38
Poor Teacher Experiences	Had a negative interaction with teachers	22
Poor Study Habits	Had poor study habits that affected performance	15
Personal Reasons	Faced personal life issues or complications	12
Failing Classes	Failed their courses	10
Uncertainty of Major/ Career Choices	Uncertain about their career choices, causing stress	5
Virtual Classes	Struggled with online classes resulting from COVID	4

122 MABs (85.9%) mentioned an influential actor. We grouped these mentions into five distinct categories: (a) K-12 instructor, (b) college instructor, (c) family, (d) other support, and (e) peers. The *college instructor* category accounts for all mentions of a post-secondary instructor, regardless of the institution type. The *other support* category captured infrequently mentioned actors (e.g., mentors, college advisors, tutoring centers, clubs). Table 6 shows the distribution of mentions by positive, negative, and neutral. These counts exceed our total number of MABs because a student may have mentioned more than one actor in more than one context (e.g., a positive, negative, or neutral experience). We found that the mentions of influential actors were overwhelmingly positive (70.3%) and primarily discussed K-12 or college instructors.

Table 6: Distribution of Mentions of Actors by Positive, Negative, and Neutral

Actors	**Positive**		**Negative**		**Neutral**	
K-12 Instructor ($N = 97$)	57	(58.8%)	32	(33.0%)	8	(8.3%)
College Instructor ($N = 65$)	48	(73.8%)	14	(21.5%)	3	(4.6%)
Family ($N = 48$)	36	(75.0%)	4	(8.3%)	8	(16.7%)
Other Support ($N = 36$)	31	(86.1%)	0	(0.0%)	5	(13.9%)
Peers ($N = 13$)	10	(76.9%)	3	(23.1%)	0	(0.0%)
Total ($N = 259$)	**182**	**(70.3%)**	**53**	**(20.5%)**	**24**	**(9.3%)**

Table 7 shows the distribution of positive and negative mentions of K-12 teachers, college instructors, and family members across students experiencing PMEs and non-PMEs. We found that the frequency of positive narratives of K-12 instructors was similar among students who experienced PMEs (42.6%) and non-PMEs (40.7%). The relative frequency of positive narratives of college instructors was higher for students who experienced PMEs (38.9%) than those who experienced non-PMEs (22.0%). The relative frequency of negative narratives also differed by instructor type. We found that students experiencing PMEs and non-PMEs spoke about negative experiences with college instructors at similar rates, however more students with PMEs talked about negative experiences with K-12 teachers than non-PMEs. Interestingly, students with non-PMEs did not have more negative experiences as compared to students with PMEs. We found that students experiencing non-PMEs only mentioned one positive experience with family (3.7%), while students experiencing PMEs had 35 positive mentions (32.4%).

Table 7: Distribution of Instructor and Family Mentions by Positive and Non-Positive Math Experience in MABs

Actor	**Positive Math Experience ($N = 108$)**		**Non-Positive Math Experience ($N = 27$)**	
	Positive	**Negative**	**Positive**	**Negative**

K-12 Instructor	46	(42.6%)	27	(25.0%)	11	(40.7%)	5	(18.5%)
College Instructor	42	(38.9%)	11	(10.2%)	6	(22.2%)	3	(11.1%)
Family	35	(32.4%)	4	(3.7%)	1	(3.7%)	0	(0.0%)

Discussion

Our study showed that a majority of students had consistently positive experiences in their math autobiographies. Our analysis showed that this was more likely to take place with math and engineering majors and for students enrolled in upper division math courses. While seemingly sensical, it is important that our analysis verified this, as in other papers, the same students have shared traumatic experiences during their academic journeys (Cawley & Wilson, 2024). We acknowledge that due to the convenience sample, participants in this study were students who already "made it" to courses like calculus and upper division math. This is known as survivorship bias (Elston, 2021), which can overshadow the narratives of students who may have been filtered out of such courses. Future work will investigate MABs from other courses that support non-STEM majors. Interestingly, we also saw that Asian students described positive math experiences at a lesser rate than their peers, which we would like to investigate further.

Students in this study with consistently positive autobiographies often shared that they enjoyed and appreciated the subject or performed well in it. In contrast, most of the speed bumps were caused by moments in their stories where they faced difficulty with material, self-expressed poor teacher experience, and poor study habits. It is important to share with our students that even those who have a positive relationship with math face challenges, highlighting the notion of productive struggle (Baker et al., 2020) and a growth mindset (Boaler et al., 2021).

Students overwhelmingly acknowledged instructors as positive influential actors in their math life stories. Prior research suggests that students' relationship with mathematics is a direct product of the learning environment created by their instructor (Moldavan & Mullis, 1998). Our study found that MABs labeled as an overall positive mathematics experience cited their college instructors at a higher rate than the MABs labeled as non-PME. The trends in our data show that STEM-focused students' early experiences with math may not appear to shape their overall relationship with mathematics, as a high proportion of positive K-12 instructor experiences was present for both students with positive relationships with math and those with negative relationships. While these rates were similar, we did notice that students with negative relationships with math simply did not write as often about specific people or institutions in their narratives. This may indicate that those with more positive experiences have stronger memories tied to those people who impacted them. This may be evidenced by the result that students with PMEs wrote more about their interactions with family members – both positive and negative experiences. Studies have shown that family members can help shape students' math identities in meaningful ways (Cunningham, 2021). We argue that more work should be done to evaluate the out-of-classroom experiences that impact students' relationship with math.

Overall, our data demonstrate snapshots of positive and negative math identities as supported by the nature of the math experiences evidenced in the students' math autobiographies.

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Rethinking Single Variable Calculus for Life Sciences for Advancing Equity

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Calculus with Applications in the Life Sciences (Biocalculus) is an impactful and required course for all STEM majors at our institution, a large public university. A disproportionately high percentage of Latinx and students from low-income households (EOP) earn failing grades in this course. This study examines student learning outcomes and experiences based on project assessment (N=152) and student survey (N=130) to answer the following questions: (a) Did changing a course structure based on working within a teaching team change students' agency? (b) Did minoritized students achieve similar levels of conceptual understanding as their counterparts in the course, when the curriculum was designed with project-based learning situated within the teaching team? We found that the inclusive course structure provided minoritized students with a supportive environment and positively impacted their learning. We believe our specific implementation can be helpful to many large public universities with diverse student populations and relatively limited resources.

Keywords: Biocalculus, Teaching Team, Equity, Project-based Learning

Given the rise in the number of Hispanic Serving Institutions (HSIs) nationally, studying the impact of how we improve teaching calculus courses is of high priority. Equitable outcomes can be obtained by increasing the number of successful STEM majors among Latinx students and students from low-income households (broadly referred to as minoritized students). This course redesign project was implemented at a large research university, an HSI. As part of redesigning of the Biocalculus series, faculty worked closely with graduate student researchers (GSRs) in the summers of 2021-2024. We benefited from using their expertise to design eight authentic applied projects in the areas spanning from chemical reaction kinetics (for chemistry) to the Alee effect model (ecology and evolutionary biology) to delve deep into studying modeling (logistic and exponential models) and analyzing observations with conclusions. The GSRs also developed authentic section worksheets, aligned with the lectures and the projects' mathematical learning goals. The creativity of this approach accessed the elements of equity-focused design that helped undergraduate students showcase their strength in diverse forms of assessment where learners' agency was high because of the immediate cross-disciplinary applications (O'Brien & Reale, 2021) and alignment of students' motivations to learn the mathematics behind the science (Lee et al., 2014). It also impacted the graduate students' training as future faculty/team leaders, where they received close mentorship on aligning learning outcomes of concept(s) with authentic assessment and teaching material for better transparency and access within a learning community.

Theoretical Frameworks

Our course redesign project relied on two theoretical frameworks, Significant Learning by Fink (2013) and Backward Design (Wiggins & McTighe, 1998), both focused on achieving better transparency and equitable outcomes.

Fink's Significant Learning framework purposely engages students in six domains that are more likely to produce learning that endures well beyond the end of a course (Figure 1). The

domains relevant for our project are Foundational Knowledge - representing information and ideas; Application - developing skills, critical and creative thinking; Integration - connecting ideas (and people) using skills and content; Human Dimension - addressing learning about oneself and others; Caring - developing new feelings, interests, and values; and Learning How to Learn - promoting self-direction and metacognition. The Significant Learning framework was explicitly chosen to establish enduring learning in the calculus series as a motivator for long-lasting knowledge in the downstream science classes using the student's agency to learn the material as a dominant motivator.

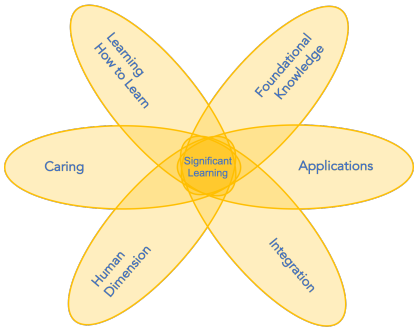


Figure 1. Significant Learning (adapted from Fink, 2013)

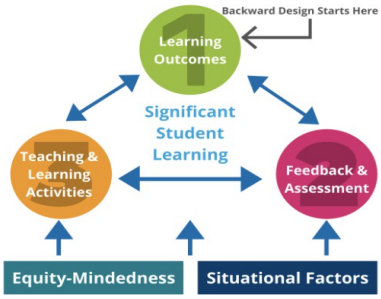


Figure 2. Backward Design (adapted from Wiggins & McTighe, 1998)

Wiggins & McTighe's Backward Design Model for equity-mindedness encompasses teaching with transparency and goal orientation (Figure 2). Since enrollment for these courses is very large (~200-500 students), establishing an alignment with teaching goals, diverse assessment types, and teaching and learning practices is imperative to achieve equitable outcomes (Garcia, 2019; Yosso, 2005). The extensive teaching team, consisting of the instructor of record, graduate teaching assistants, undergraduate tutors, and supplemental instructional team, can share a guideline for how each week progresses. Consistent instruction and messaging become more seamless once we approach the task as a team (Kerr & Samuels, 2019). Creating a sense of belonging for the students is inherent to the value held by the community, which is collaborative and reflective of the instructor's values, with the hope of changing the perception of the courses from a weeder course to a course that launches STEM students into their majors (Bhattacharya et al., 2020).

Course structure in Biocalculus for Equity

The core value of the course design was based on collaborative norms and practices.The design elements promoted a teaching team approach that integrates learning in a discussion section with large lecture experience. The course curricula aimed to develop a coherent conceptual understanding of calculus that is authentically contextualized in sciences and used metacognitive activities using reflections and other similar tools for frequent formative assessments (Cizek, 2010; Ruiz-Primo, et al., 2010) that help students become better STEM learners (so that students cultivate a sense of belonging and STEM identity). The course redesign structure is described next.

Standardized delivery of the course content (ADA compliant) includes articulating learning goals and aligning them with diverse assessments across various topics to provide all students with similar experiences in learning calculus using Canvas and Gradescope as the primary technology tools. The course learning cycle is structured by giving primary material using videos

or otherwise (Modules) to create collaborative problem-solving opportunities during lectures to develop an active learning setting. After the lecture, students worked with their TAs and tutors on worksheets relevant to the mathematical concept (calculus content) contextualized in real-world applications (precursor to the project questions). An online question bank was created using open-stack resources organized by topics (and learning outcomes), using the weekly learning outcomes of the course to better align with the lecture (and also cut costs drastically for students) and provide essential metacognitive questions/activities (reflection questions) to help students become self-regulated learners (Ambrose et al., 2010). We provided students with various formative assessment tools to develop themselves as STEM learners and to promote equitable outcomes for different types of learners.

Above all, before the beginning of the academic year, every teaching team member (graduate TAs, undergraduate tutors, supplemental instructors, and instructors of record) received professional development to work in this new framework of an inclusive teaching model, prioritizing equitable outcomes. The professional development space opened a purposeful dialogue (Alexander, 2008) to help create alignment across all teaching and learning platforms, where teaching team members with different positions and knowledge of the curriculum helped make a scaffolded learning space. The teaching team met weekly to plan and address all emerging issues to address every student's success. We hypothesized that a team-based course structure that helps create a community allowed minoritized students to ask questions. The culture of problem-solving that is dialogic is the key to promoting a sense of belonging in STEM. Achieving equitable outcomes in the Biocalculus series courses will help retain a diverse student body in STEM majors, ultimately one of our institution's (and the nation's) primary goals for this decade. An institutional culture of commitment grounded in inclusivity rather than selectivity (obstruction) promotes success among URM STEM students (Chang et al., 2014).

Methods

This study included students in the Biocalculus course taught in Spring 2024, the third quarter of the academic year. A relatively large class (N=152) was diverse in academic and demographic backgrounds. Specifically, just over a third (36%) of the class were students from low-income family backgrounds (referred as EOP students). Hispanic/Latinx students constituted a third of the class, as did White, non-Hispanic students. Of note, Hispanic/Latinx students were significantly more likely to be from low-income households than their White, non-Hispanic peers: 64% of Hispanic/Latinx and 11% of White, non-Hispanic students in this course were EOP students (X^2 (df=1, N = 110) = 32.9, p < .001).

Most students (84%) were in their first year of college, and 60% intended to major in Biology-related fields. There were statistically significant differences in students' academic achievement in the first two quarters (before taking this course): Hispanic/Latinx and EOP students had significantly lower cumulative GPAs than White, non-Hispanic students and non-EOP students, respectively (Table 1).

Table 1. Cumulative GPA at the enrolled institution by socioeconomic background (EOP) and race/ethnicity

	EOP (n=58)	non-EOP (n=105)	Hispanic/ Latinx (n=56)	White, non-Hispanic (n=54)
Mean	2.88	3.09	2.79	3.26
Std. Dev.	0.55	0.66	0.55	0.62
t-test	t(161)=-2.07, p<.05		t(108)=-4.19, p<.001	

To address the first research question, *Did changing a course structure based on working within a teaching team change students' agency?,* we used a student survey developed by the University of California Santa Cruz Assessment for Learning Excellence and Equity Center (ALEEC). The survey measures students' learning experiences in a collaborative course setting, including series of questions about student attendance and engagement during the section, reflections on the helpfulness of the section, and a sense of belonging to a community of peers. These questions were developed based on multiple studies of course- and program-learning outcomes and student focus groups. Distributed by ALEEC as a confidential, online survey at the end of the course, the survey yielded an 80% response rate and a representative sample. Survey data was merged with the demographic information, deidentified, and analyzed to examine group differences based on demographic backgrounds. We strive to provide actionable results and conduct equity analyses by focusing on specific, conceptually important points of difference (Sher & Sung, 2020). For example, students' responses about the helpfulness of attending section were collected on a 5-point scale and then dichotomized to focus on the distinction between students who indicated a moderate to high level of helpfulness (*somewhat helpful, helpful or very helpful*) vs. a low level (*a little* or *not at all helpful*). This approach allows us to report group-specific results in percentages and compare groups using Chi-square analyses. In addition, we developed an overall measure or construct for a sense of belonging to a community of peers and used t-test for examining group differences.

A rubric-based assessment was conducted to address the second research question, Did minoritized students achieve similar levels of conceptual understanding as their counterparts in the course when the curriculum was designed with project-based learning situated within the teaching team? After students uploaded their projects into Canvas, TAs used a criteria-based rubric developed by the faculty to evaluate students' skills as part of grading in Gradescope. Each specific skill was measured with four defined levels of proficiency, ranging from full proficiency (fully and correctly answered), general proficiency (minor mistakes), partial proficiency (major mistakes or omissions), to limited/no proficiency (completely incorrect or not attempted). Similarly to our work with the survey data, the rubric assessment data was extracted and merged with the demographic information, deidentified, and analyzed to measure skill-specific levels of achievement and examine significant group differences using Chi-square analyses (Sher & Sung, 2020).

Results

The survey allowed us to examine several measures of students' agency and belonging to a community. Students' attendance was very high for both lectures and discussion section. The majority (74-75%) attended most (at least 70%) lectures and section meetings. This high attendance reflects the team-based course structure and is atypical in such large courses. Most importantly, there were no differences in attendance between minoritized students and their counterparts.

Because the course redesign gave special attention to improving students' experience in the discussion section, the survey asked students about section's helpfulness for their learning. For most students, experience in section was moderately to very helpful. Notably, 80% of Hispanic/ Latinx students reported that a TA-led section helped them complete homework assignments, compared to 70% of White, non-Hispanic students (this difference did not reach statistical significance). Nearly three-fourths of Hispanic/Latinx students reported that their participation in section helped them prepare for exams; in this aspect, the activities in section were somewhat more helpful for Hispanic/Latinx students than their White, non-Hispanic peers (X^2 (df=1, N =

86) = 3.35, p = .067). Another important finding is that nearly 70% of Hispanic/Latinx students found their experiences in section helpful for feeling more comfortable in this class, and they were similar in this regard to their classmates. See Figure 3.

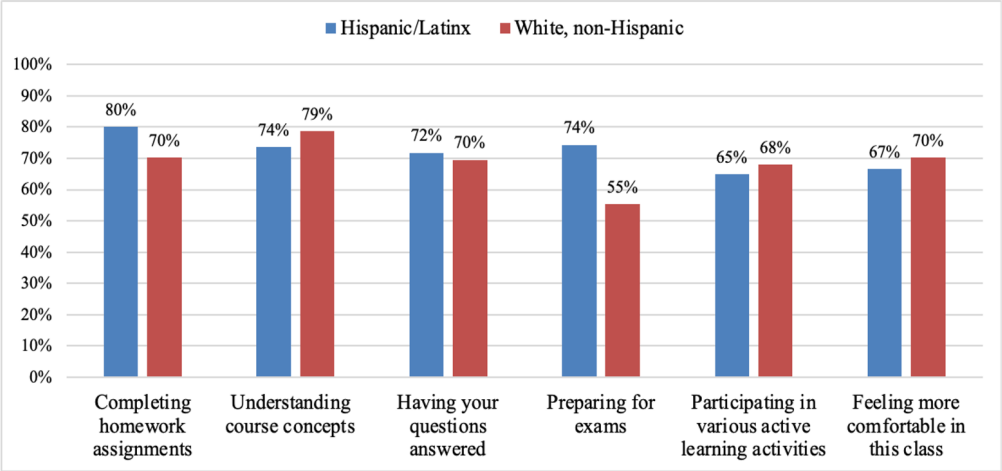


Figure 3. Equity analyses of section's impact on learning by race/ethnicity (percent of somewhat helpful to very helpful)

In addition, 65-71% of students from low-income backgrounds reported positive experiences in the section that helped them complete homework, understand course concepts, and have their questions answered. Of note, just over half (57%) of EOP students thought that their participation in the section helped them prepare for exams and for active learning activities in class (as a group, they were not significantly different from their peers). Also, just over half of EOP students indicated that section was at least somewhat helpful for feeling more comfortable in the class, which was significantly lower than 73% of non-EOP students (X^2 (df=1, N = 128) = 4.22, p < .05). See Figure 4.

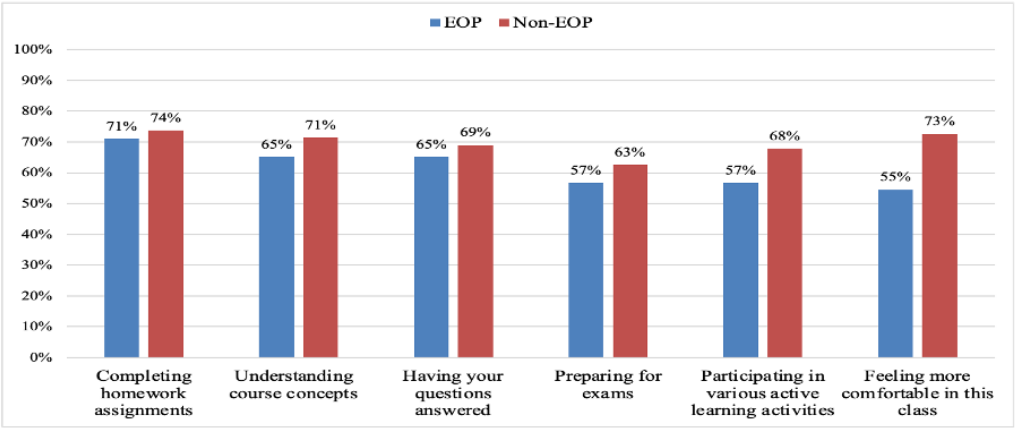


Figure 4. Equity analyses of section's impact on learning by socioeconomic background (percent of somewhat helpful to very helpful)

Such a multifaceted positive impact of the discussion section on student learning was likely possible thanks to the positive and inclusive environment created by trained graduate student TAs. When asked directly about feeling welcome and supported in section, most (89-90%)

students, including minoritized students, somewhat to strongly agreed that they felt welcome and supported, without any statistically significant differences between groups.

Another series of questions focused on students' relationships with other students in their class, designed to measure their feeling of belonging to the community of peers. In this class, most students reported high levels of connectedness to peers: the vast majority (80-86%) said that they *probably* or *definitely* had a student they could talk to, reach out for help, and share if they struggled academically. Most (87%) felt that students in this class could ask questions or give wrong answers without feeling judged by other students. As many as 70% of students reported having *probably* or *definitely* made meaningful connections with other students in this class. See Figure 5.

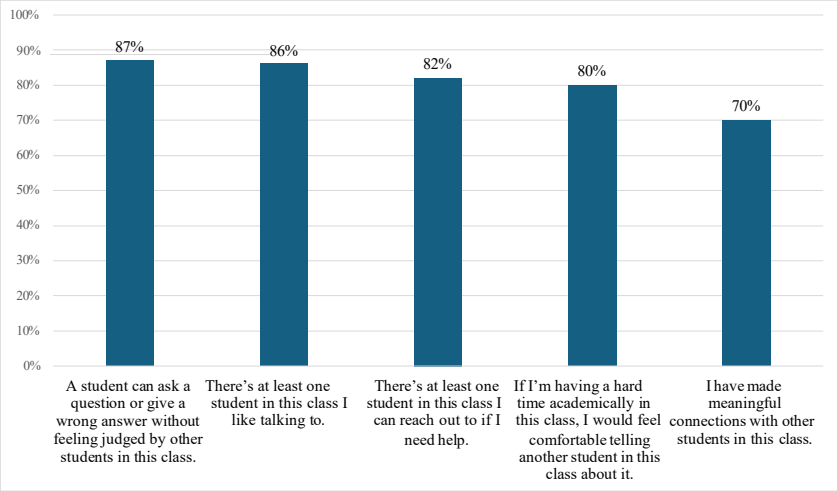


Figure 5. Students reporting belonging to the community of peers (percent of probably yes to definitely yes, based on the entire class)

We found that minoritized students experienced similarly strong belonging within the community of peers using two approaches to compare groups: (1) an overall measure or a construct based on the average across the five items on a Likert-like scale[1] (Cronbach's alpha=0.91), and (2) item-specific responses, analyzed with a focus on the more definitive responses (*yes* or *definitely yes* vs. other responses). Table 4 shows the overall measure of belonging for minoritized groups of students and their peers based on socioeconomic status and race/ethnicity.

Table 4. Equity analyses of the overall belonging to the community of peers by socio-economic background (EOP) and race/ethnicity

	EOP (n=43)	non-EOP (n=77)	Hispanic/ Latinx (n=38)	White, non-Hispanic (n=45)
Mean	4.33	4.45	4.37	4.63
Std. Dev.	1.20	1.12	1.08	0.99
t-test	$t(118)=-0.58$, $p=.56$		$t(81)=-1.15$, $p=.25$	

[1] Survey question: "Based on your interactions with other students in this course, how true are the following statements?" Responses are on a 6-point scale: 1=Definitely not, 2=No, 3=Probably not, 4=Probably yes, 5=Yes, 6=Definitely yes.

One of the major learning assessments was a project designed for students to demonstrate their conceptual understanding of rate of change in the biological context. We evaluated students' proficiencies in four related skills/concepts using a criteria-based rubric with four levels of proficiency, ranging from *full* to *limited or no* proficiency. Our faculty consider that expected proficiency in a specific skill is achieved by the whole class or specific group when close to 75% or higher percentage of students demonstrate *general* or *full* proficiency. In this class, most students (91-98%) demonstrated *general* or *full* proficiency in each of the measured skills (Table 5). Equity analyses showed that both minoritized students and their peers demonstrated high levels of proficiencies in each skill. Moreover, about 70% of minoritized students demonstrated *full* proficiency in estimating the instantaneous rate of change using the dataset (Skill 3), and in explaining the differences in the estimations and interpreting observed change using calculus in context (Skill 4).

Table 5. Results of project-based assessment of students' conceptual understanding of rate of change (N=152).

Skills and concepts:	Full proficiency	General proficiency	Partial proficiency	Limited or no proficiency
1. Interpret the average rate of change using an experimental dataset	82%	9%	1%	8%
2. Interpret reaction rate in the context of the problem	84%	12%	3%	1%
3. Estimate the instantaneous rate of change using the dataset	69%	29%	1%	1%
4. Explain the difference in the estimations and interpret observed change using calculus context	77%	18%	1%	4%

Discussion

Our study evaluated the impact of a redesigned course on student agency and conceptual understanding in the first course of the Calculus series for Life Sciences. We used direct assessment of project-based learning and a student survey to measure how minoritized students responded to a teaching team structure and frequent formative assessments aligned with learning goals. We found that all students, including Hispanic/Latinx and EOP students, demonstrated high levels of engagement and learning in an inclusive community of peers supported by a teaching team. While many minoritized students found discussion sections taught by trained graduate student TAs helpful for various aspects of their learning and social experiences, some students from low-income backgrounds still need more support to feel comfortable in a calculus class. Ultimately, our work hopes to add value to the national efforts to achieve equity in student math preparation for pursuing STEM fields by reimagining the role of the teaching team within a learning community, fore-fronted with equity-minded assessment structure using projects. We believe our specific implementation can be helpful to many large public universities (such as ours) with diverse student populations and relatively limited resources.

Acknowledgments

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Undergraduate Students' Models of Subtraction and its Significance for Calculus

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The operation of subtraction, though emphasized at the elementary level, has a central role in expressing key ideas in Calculus. In graphical representations of Calculus concepts, a magnitude interpretation of a difference, that b–a gives the distance between a and b, often supports students in making sense of the representation. A magnitude interpretation uses a determine the difference model of subtraction, in which subtraction find how much a differs from b. Using exploratory teaching interviews, we investigated the models of subtraction that nine undergraduate students used on tasks designed to support them in using a magnitude interpretation. We found four distinct models of subtraction that students employed in these tasks. We describe these models and associated observable behavior. We discuss implications of these findings for the teaching and learning of Calculus and future research directions.

Keywords: subtraction, graphical representations, Calculus, difference expressions, number lines

One way to conceive of the subject of Calculus is as the study of change. As such, the operation of subtraction, which can be used to express the change in values of a variable, figures prominently in Calculus. Difference expressions form the backbone of expressions in the limit definition of the derivative ($f(x+h)–f(x)$), the Mean Value Theorem ($f(b)–f(a)$), and the Fundamental Theorem of Calculus ($F(b)–F(a)$), among others. The operation of subtraction may be conceptualized in numerous ways, including: the takeaway model, the determine the difference (comparison) model, the part-whole model, and adding up. In particular, the determine the difference model, in which subtraction is used to find how much one amount differs from another, plays a central role in Calculus, to find the change in two values of a given variable. Researchers have highlighted the role of the determine the difference model in Calculus, and some have also argued that introducing Calculus concepts, such as displacement and net change, may motivate and necessitate this model of subtraction, especially involving negative values (Stroup, 2005). When considering graphical representations in Calculus, changes in quantities or values of a variable are often represented by horizontal and vertical segments. When students can associate a difference with the distance between two positions, they can express the length of the segment as a difference and connect such expressions with segments in a graph, what Parr (2023) refers to as a magnitude interpretation of a difference.

As part of a larger study, we conducted exploratory teaching interviews (Steffe & Thompson, 2000) with nine undergraduate students using an activity designed to support students in expressing distances between two functions in the Cartesian plane using differences (Parr & Lippe, 2024). Expressing such distances are necessary for setting up integrals to evaluate areas and volumes in Integral Calculus. In this study, we ask: *What models of subtraction did undergraduate students use on an activity designed to promote a magnitude interpretation of difference expressions?*

Empirical & Theoretical Background

We begin by describing two models of subtraction commonly introduced at the elementary level, the takeaway model and the determine the difference model (van den Heuvel-Panhuizen &Treffers, 2009; Selter et al., 2012), also called the comparison model (Usiskin, 2008). The takeaway model of subtraction involves conceiving of subtraction as beginning with one amount of a quantity, *b*, taking away some amount of that quantity, *a*, which results in a final amount of the quantity. In this model, the result of subtraction finds this remaining amount. The determine the difference model involves conceiving of two amounts, *a* and *b* simultaneously, and then comparing how much larger *b* is than *a*, to find how much *b* differs from *a*. Research has shown that students struggle with such comparison operations compared to other types of subtraction problems (e.g., Stern, 1993). Further, students often default to using a takeaway model for subtraction, even when a determine the difference model provided more computational efficiency. (e.g., Baroody, 1999; Stern 1993).

To consider how a determine the difference model for subtraction may be used to represent distances graphically, we use Parr and Lippe's (2024) description of the components of a magnitude interpretation. In this description, a magnitude interpretation (Parr, 2023), involves combining the determine the difference model with a conception of distance as a quantity in space. The magnitude interpretation combines these such that a difference expresses the distance between two positions in one-dimension. To illustrate why the difference $x_2–x_1$ gives the distance between these two positions, Figure 1 shows x_1 and x_2 themselves represented as magnitudes, or distances from zero. The expression $x_2–x_1$ may then be considered to be composed of magnitudes (Parr et al., 2021; Parr & Lippe, 2024).

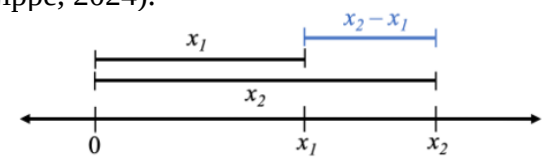


Figure 1. Composed magnitude interpretation of the expression $x_2 – x_1$ (Parr & Lippe, 2024, p. 507)

In the activity that students worked through in this study, students were encouraged to use a magnitude interpretation of differences in several ways, including drawing segments from zero to represent a given value, as in Figure 1. The magnitude interpretation of a difference expression, which aligns with a determine the difference model of subtraction stands in contrast to a cardinal interpretation (Parr, 2023) of a difference, which aligns with a takeaway model. When students use a cardinal interpretation, they often count units, such as tick marks, to show what a difference represents graphically. Even at the undergraduate level, students may favor a cardinal interpretation (Parr, 2023), perhaps because a takeaway model of subtraction is favored in their experience in elementary mathematics and daily life (Selter et al., 2012). In a previous study, we found that less than 40% of undergraduate Calculus students could correctly express a horizontal segment between a function and the vertical line x=2 using a difference expression in terms of x (Parr & Lippe, 2024). These findings motivated the development of the activity with a goal to better understand how to support students in using differences to express distances in graphs of functions in the Cartesian plane for their work in Calculus.

Methods

We conducted exploratory teaching interviews (Steffe & Thompson, 2000) with nine undergraduate math or science students at a small private college in 2023. In the interviews, students were asked to work through tasks in the Interpreting Graphs for Calculus Activity (Parr

& Lippe, 2024). The first task asked students to measure with a ruler, various lengths on a number line a certain distance to the left or right of zero, to support students in conceptualizing distances as measurable quantities within number lines. The second task, shown in Figure 2, asked students to draw a segment with a length of 5 on the *x*-axis starting at zero, a segment with a length of 3, and then a segment to represent the difference.

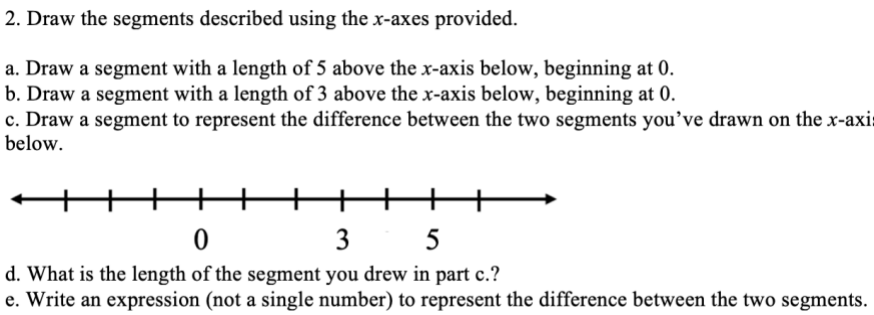


Figure 2. Task 2a-e in the Interpreting Graphs for Calculus Activity promoting a magnitude interpretation

The subsequent tasks 3-7 asked students to represent difference expressions using segments on horizontal and vertical number lines, or to write a difference expression for a segment shown. These difference expressions began with two positive integers, then included negative integers, non-integer values, and finally variables. A member of the research team served as the interviewer, with the first author serving as a witness if not the interviewer themselves. In each interview, the interviewer asked the student to further explain their thinking and may have posed additional questions to probe and test the boundaries of the students' current understanding of the task (Steffe & Thompson, 2000). The tasks were designed to promote a magnitude interpretation of a difference expression, but the interviewers refrained from expressly telling students to use this type of interpretation, unless it seemed like the student was unable to overcome a perceived obstacle without such intervention. We video recorded the interviews, as well as the iPad screen of students' work on the activity to capture their written work in real-time. The data for the study included students' words, gestures, diagrams, and written work on the activity.

We analyzed the data, with students' reasoning in the moment as our unit of analysis. That is, we considered students' models of subtraction as subject to change within their work on a given task and did not limit our description of students' work on a task to a single model. At the beginning of the data analysis process, we were attuned to the two distinct models of subtraction described above: a takeaway model and a determine the difference model. The research team used a grounded approach (Corbin & Strauss, 2014) to analyze the interviews, beginning with describing their model of how a student was reasoning in each task of the interview. At this stage, some details in students' activity became apparent, including activity using segments in distinct ways, such as moving segments and comparing segments visually. Through open and axial coding (Corbin & Strauss, 2014) we identified four distinct models of subtraction.

Results

In this study, we report four distinct ways in which the students we interviewed modeled subtraction on tasks designed to promote a magnitude interpretation of difference expressions on horizontal or vertical number lines (axes). We refer to these four models as: (a) magnitude, (b) takeaway values, (c) a takeaway segments, and (d) translation. We describe each model and the observable behaviors we used to categorize the students' activity as the given model.

Magnitude Model

We observed several students model differences as composed magnitudes, which aligned with the interpretation we sought to promote in the activity and teaching interview. Due to space, we only describe the model and observable behavior in Table 1 at the end of the section.

Takeaway Values Model

Several students modeled differences as taking away values on the number line. In these instances, students began at a position *b* on the number line, went back (or down) *a* units, and arrived at position *b*–*a*. This takeaway values model aligns with conceiving of subtraction as takeaway, more broadly, and with a cardinal interpretation of differences. We found that students used this model even after using what appeared to be a magnitude model on prior tasks.

We provide the example of the case of Ellie. Ellie had used a magnitude interpretation on previous tasks in the activity. However, on Task 5a, Ellie used what we infer as a takeaway values model. Ellie was asked to draw a segment representing *x*–2 on a horizontal number line. On this number line, *x* was represented as a point near 4.5 and labeled "*x*" (see Figure 3). These episodes show her figuring out how she would represent the result as a line. She knows the result of subtraction is a single number, so she decides that the final line she draws will be from *x* to the final product of subtraction, which in this case would be about 2.5. Here, she is using her line to represent the numerical value 2 being taken away from the starting point *x*.

Ellie: (*pauses, draws a dot near 2.5 in blue*) And by segment you really want the line?
Int: Yes.
Ellie: Okay. (*Pauses, draws line from dot at 2.5 to dot at x*)
Int: Can I ask why you were hesitant?
Ellie: Because to me *x*–2 is just a point. Like I would just take this *x*, which is…like, three, four, four and a half, and then subtract two and I would go down to two and a half. So like to me I wouldn't draw like a line segment to represent that value….I want my leftmost endpoint to represent what *x*–2 would actually be in terms of numbers…
Int: Okay so your segment is…representing, what, the answer to *x* minus 2?
Ellie: Kind of? Because I'm like literally going 2 down from *x* is what I feel like my segment is representing. Because I don't know really how else to express a single point, or at least what I'm interpreting as a single point, as a segment.

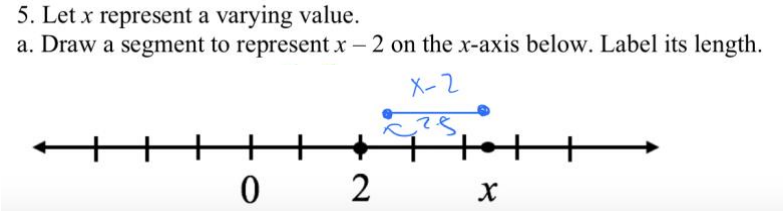


Figure 3. Ellie's work on task 5a, in which she modeled x–2 as the result of beginning at x and moving left 2 units.

In this episode, Ellie was reluctant to represent the difference as a segment, when in her mind it was "a single point." She does decide to draw a segment when she clarified the directions of the task and explained that she was going "down to two and a half." Notably, she used 4.5 as an approximation for the value of *x*, which was intended to be a generic varying value as stated in the task. Ellie's final line was a representation of the numerical value removed from *x*. By using 2 as the length of her final segment instead of one of the segment's endpoints, she suggests that she sees it as a value taken away from *x*. Therefore, we infer that she is using a takeaway values model here. For the students using a takeaway values model who were asked to draw segments,

the segments were a record of the movement from one value to another, rather than a distance whose value was the resulting difference.

Takeaway Segments Model

Distinct from the takeaway values model described above, in which students represented taking away values as counting back some amount, we found some students use a takeaway model of subtraction, but represent this through operating on the segments themselves. That is, they modeled subtraction as taking away part of a segment from another segment, leaving a segment as a result. We provide an instance with a student, Chloe. The interviewer asks Chole to explain her thinking for task 6a. She begins to explain this using the number line provided in the task, but then, she switches to thinking about how this could relate to an example about eating a chocolate bar.

Int: How would you explain for [part] a why the segment represents 3.5 minus 2?

Chloe: If you have a bigger number and you take away a smaller number, the smaller number is going to be canceled out completely, but a little bit of the big number will be left. You view almost… 3.5 as a whole, and you're taking a part out of it, and the part would be 2. So, then the remainder left would be just this tiny segment between 3.5 and 2.

Int: Is there a way you could represent what you were envisioning with a whole and a part using segments on the graph?

Chloe: Yeah. So, if I have 3.5 (*draws a rectangle next to the number line from 0 to 3.5*), and this would be 2 right here (*draws a line inside the rectangle in line with the 2 on the number line*). Okay. I'm gonna pretend that it's an object, like a candy bar. And let's say, if I only eat 2 segments out of the 3.5 candy bar, all of this will be erased right here (*scratches out the part of the rectangle under the line at 2*), and then I'm left with a segment, this red segment (*colors the remaining part of the rectangle red*)…since there's a larger number subtracted by a smaller number, I'm kind of, I'm viewing it as the larger number will stay more intact, the little number will get eaten up completely.

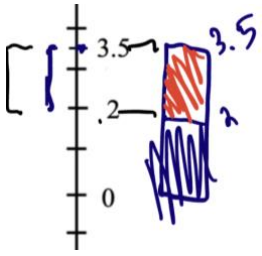


Figure 4. Chloe's work on task 6a

In this episode, we infer that Chloe was using a takeaway segments model of subtraction. Chole viewed the segment between 3.5 and 2 as what is remaining when you take away the segment from 0 to 2 from the segment from 0 to 3.5. She referred to 3.5 as "the whole" and the 2 as the "part" that is taken out. Chloe further emphasized that she was operating on these segments when she introduced the idea of a physical object, a candy bar. Although Chloe still appeared to envision a process of taking away, we note that the activity associated with this model is distinct from the takeaway values model, in which a student may begin at 3.5 and count back 2 units. We note that this model broke down for Chloe when she tries to apply it in task 6b, which asks her to represent 6 – (–2). She could not figure out how to apply the takeaway segments model and remove a negative segment from a positive segment because there was no overlap between the two segments.

Translation Model

Finally, we observed students using what we refer to as a translation model of subtraction. In these instances, students found a numerical result, and then represented that result as a length from zero. We give the example of Yasmine on task 4c. Yasmine drew a segment labeled "a" for –1, a segment labeled "b" for –5 (we note that she drew this to the right of zero rather than to the left) and then drew a segment from 0 to 4 which she labeled c, in response to task c, which asked her to draw a segment to represent –1–(–5). In this way, Yasmine translated the distance between –5 and –1 and shifted this to the right, to another segment with a length of four. Although Yasmine did not attend to the distance between –5 and –1, we refer to this as a translation model, as the drawn segment has the length of the difference, yet the endpoints of the segment have shifted. When asked to explain her reasoning, she shared that she knew the numerical result of the difference was four and that she was "more focused on getting the length right" rather than thinking about the relationships between segments a, b, and c on the *x*-axis.

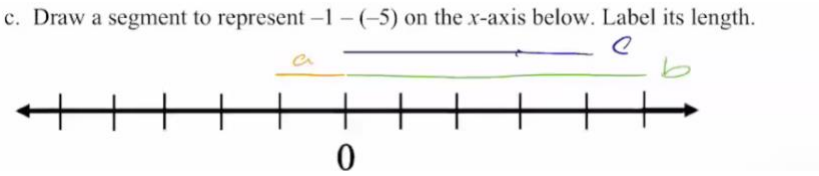


Figure 5. Yasmine's work on task 4c, an example of a translation model of subtraction

In Table 1, we summarize our description of the four models we identified and the observable evidence we used to help classify student's reasoning as such.

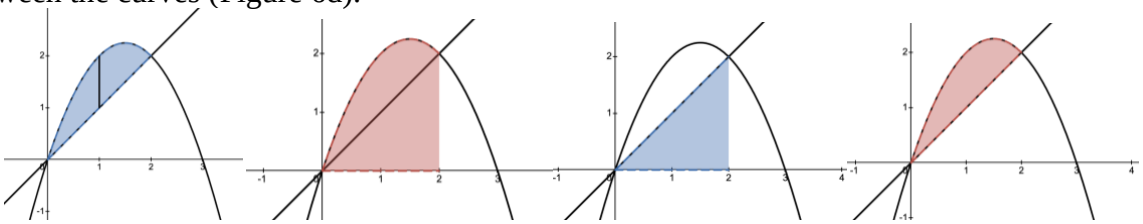
Table 1. Four Models of Subtraction

	Models *b*–*a* as	Observable Evidence
Magnitude	The distance between positions *a* and *b*	• Language: "length," "distance," "from *a* to *b*" • Draws a segment from *a* to *b* and labels it *b*–*a* • Attends to the endpoints when referencing segment
Takeaway Values	The resulting position starting at position *b* and moving left (down) *a* units	• Language: "takeaway *a*," "go *a* back" or "down from *b*" • Starts at position *b*, counts *a* units back, labels final position as *b*–*a* • Segments, if drawn, record movement from one position to another (length of segment may be associated with *a*). • Counts discrete units, uses counting language
Takeaway Segments	The resulting segment from removing a segment of length *a* from a segment of length *b*	• Language: "the segment *b*–*a*," "remove," "piece" •Considers segment from 0 to *b*, removes segment from 0 to *a*, remaining segment is result • Emphasizes remaining segment as result, may not attend to length of segment, or endpoints of the segment
Translation	The distance between *b*–*a* and 0	• Language: "length" • Finds result, draws segment of that length from 0 to *b*–*a* • Does not require end points of the segment to be *a* and *b*.

Discussion

In our study, we identified four distinct ways in which undergraduate students modeled a difference expression of the form $b–a$ in a single-dimension. These four distinct models emerged during tasks designed to promote a magnitude interpretation of $b–a$ as the distance between positions a and b. We noticed trends in the ways in which students used their models to explain their reasoning, and the extent to which students could see past their models. We theorize there may be a connection between the distinctions in figurative and operative imagery (Moore, 2016; Thompson et al., 2024) and the ways in which students were constrained or empowered by their models for modeling a range of differences, including those with negative integers, non-integer values, and variables. For instance, the takeaway segments model appeared to be tied closely to the models themselves, with students operating on an object within the model. For Chloe, these models were not able to be adapted to account for a difference between a positive and negative value. We plan further research in this direction, as well as understanding when and how students adapted their models throughout the activity.

Although three other models of subtraction emerged in a context where we were attempting to promote a magnitude model, we see ways in which some of these models can be leveraged in the teaching and learning of Calculus. We conclude by commenting on the role of various models of difference expressions in one example from Integral Calculus, area between curves. The area between graphs of functions f and g in the Cartesian plane can be found using integrals of the difference of two functions. The justification for why the difference of the functions finds the area between the curves can be conceived in several distinct ways. For example, consider the functions $f(x) = x$ and $g(x) = –x(x–3)$. Locally, the difference of the two outputs of the functions for a value of x, $f(x) – g(x)$, gives the height of a generic rectangle in the area between f and g (Figure 6a). Another way to conceive the area between the curves is as the global difference between the area under f (Figure 6b) and the area under g (Figure 6c) resulting in the area between the curves (Figure 6d).



(a) (b) (c) (d)

Figure 6. Pointwise difference gives height of any rectangle between f and g (a) vs. Global difference of area between functions gives difference between area under individual functions (b), (c), (d).

We note similarities between this global difference of the area under two functions and the takeaway segments model we observed in our study. The global difference of the area under two functions, and the area between two curves considered pointwise may each be used to conceive of the area between functions. If a student can readily perceive of these illustrations as related to the integral expression, these visuals together can serve as a proof of the difference property of integrals, that the integral of the difference of two functions is equal to the difference of the integral of each function. Still, a third way to conceive of why an integral of a difference of two functions gives the area between the curves is to conceive of a difference of two functions as translating the area to be between one singular function and the horizontal axis. In our example, the equivalent area would be under the function $h(x)=(3x–x^2)–x$. We anticipate connections between the translation model we observed in this study and this third way of conceiving of how integrals of differences of functions find areas between curves.

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Undergraduate Students' Conceptualization of Repeating Decimals

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The mathematics education community has examined how students conceptualize repeating decimals. Yet, there is potential for further investigation of how undergraduate students conceptualize the addition of repeating decimals. This paper explores pre-service teachers' conceptualization of repeating decimals. Each participant answered three open-ended questions, and our data consisted of these written responses. Our analysis highlighted that the school conventions hindered many participants from seeing how to add repeating decimals without truncating them. Initially, most participants viewed repeating decimals as processes that approximate rational numbers. Many participants came close to considering repeating decimals as mental objects rather than processes. Our data indicates that asking participants how they would check if they had correctly added repeating decimals enabled them to overcome epistemological obstacles that hindered them from seeing repeating decimals as numbers.

Keywords: APOS theory, Geometric series, Infinite processes, Infinite series, Repeating decimals

Infinite processes are vital in mathematics education, especially in undergraduate mathematics courses. Common topics that involve the notion of infinity are sets, sequences, and limits. Most students first encounter infinite series in calculus courses but rarely explicitly in school mathematics. Hence, mathematics education researchers know much more about how students think about sets, sequences, and limits (Dubinsky et al., 2005a; Dubinsky et al., 2005b) but have yet to study how students conceptualize series extensively. According to Kung and Speer (2013), the mathematics education community still needs to amass "an extensive research base that provides insights into student understanding of this topic [of infinite series]" (p. 420) because we know very little about "how students think about this topic" (p. 420).

Repeating decimals exhibit a duality between potential and actual infinity that may appear paradoxical. On the one hand, a repeating decimal exemplifies potential infinity when one views it as a continuous process of forming digits to express a rational number through long division. On the other hand, a repeating decimal is an instance of actual infinity because it represents a fixed number. This paradoxical duality between potential and actual infinity baffles many students. Some authors (see, for example, Dubinsky et al. (2013), Edwards and Ward (2004), Sierpińska (1987), Sinclair et al. (2006), Weller et al. (2009, 2011), and Yopp et al. (2011)) have investigated how students think about repeating decimals and their relationship to infinity and limits. However, researchers have yet to examine how students conceptualize arithmetic with repeating decimals. Weller et al. (2009, 2011), Yopp et al. (2011), and Dubinsky et al. (2013) have found that many pre-service teachers struggled with repeating decimals and did not believe that $0.\overline{9} = 1$. Sinclair et al. (2006) developed an instructional treatment that "draws attention away from the procedure of conversion" to allow learners to view representations of fractions in decimal form "as objects rather than processes" (p. 184). They hypothesize that pre-service teachers view the standard conversion of decimal numbers into fractions as a process "that allows them to turn decimal numbers into fractions rather than as a general relationship between fractions and repeating decimals" (p. 189). Sierpińska (1987) has also discovered that 17-year-old humanities students found the equality $0.\overline{9} = 1$ perplexing. One of her students said the left-

hand side would not equal one "no matter how many nines you put" and that it could not equal one unless "it goes to the very infinity" (Sierpińska, 1987, p. 379). Even some undergraduate analysis students who were aware of the definition of the repeating decimal "disregarded the definition" and argued that the left-hand side "was not equal to one" (Edwards & Ward, 2004, p. 417). Given these findings, our study investigates how pre-service teachers conceptualize adding two repeating decimals. In particular, we examine how adding two repeating decimals affects their understanding of the equality between repeating decimals and their rational counterparts. To frame this investigation, we turn to APOS theory because it offers one possible explanation for difficulties with repeating decimals, and we use it to analyze data from post-instructional assessments and describe the mental constructions involved in an individual's conceptualization of the addition of two repeating decimals. Our study aims to answer the following research questions: *What do pre-service teachers attend to when adding two repeating decimals? How effectively does adding repeating decimals help pre-service teachers regard repeating decimals as objects rather than processes?*

Literature Review and Contextual Insights

Repeating decimals are geometric series in disguise, and some authors have investigated how students think about infinite series and their relationship to conceptions of infinity and limits. We situate our work within this literature, focusing on endless addition, series convergence, and other obstacles in understanding series.

On Endless Addition

Sierpińska (1987) found that students generally viewed infinity as an unending and uncompleted process while studying how students face epistemological obstacles involving limits. She explored erroneous uses of the geometric series and prompted students to look for errors in her reasoning. Likewise, Martínez-Planell et al. (2012) found that students view series as an endless addition process. They observed that students continued thinking of series as infinite summations after training. Their study shows that the rigorous definition of a series may only sometimes replace the conception of series as infinite additions in most students' minds. Barahmand (2021) found that students resisted changing their intuitive use of equality and inequality signs despite facing series-related cognitive conflicts.

On Series Convergence

Alcock and Simpson (2004) suggest that some students go through a process similar to proofs and refutations because they conceive of a range of objects that satisfy the definition of convergence and debate which properties hold for all of them. They found that visual learners overestimated their understanding of the convergence of sequences and series, and they could not produce a written argument that a mathematician would consider appropriate and rate highly. However, Lindaman and Gay (2012) found that visualizing series convergence enables students to develop a longer-lasting mental image of convergence when they combine visualization with different calculus reform strategies. Fischbein et al. (1981) and Sierpińska (1987) found that some students developed implicit conceptions of convergence, but the students argued that the partial sums would never reach the limit but only get closer to it. Sierpińska (1987) also found that students could not conceive of infinite series equaling the limit of its partial sums, but some students began developing implicit conceptions of convergence. Likewise, Oehrtman et al. (2014) found that the guided reinvention approach allows students to construct a coherent definition of a sequence's limit and define the pointwise convergence of an infinite series.

On Other Obstacles

Mamolo and Zazkis (2008) and Wijeratne and Zazkis (2016) found that students perceived infinity as an ongoing process even when they indirectly dealt with convergent geometric series. Martínez-Planell et al. (2012) also show that students conceive infinite series as an unending process of addition or as limits of partial sums, but some simultaneously hold both conceptions. Martin (2013) found that experts favour the formal definition of series convergence, while students rarely do. Alcock and Simpson (2005) found that some non-visual learners can manipulate symbols correctly but are prone to errors when they restate definitions, so they shallowly consider the meaning of their manipulations. However, Movshovitz-Hadar and Hadass (1991) and Sierpińska (1987) observed that students found that algebraicly manipulating geometric series could be perplexing and paradoxical. Although Barahmand (2021) found that students resisted changing their intuitive use of equality when dealing with infinite series, Oehrtman et al. (2014) found that guided reinvention helps students construct coherent definitions of pointwise convergence of series despite notational obstacles.

Theoretical Framework

Mathematics education researchers (see, for example, Alcock and Simpson (2004, 2005), Barahmand (2021), Fischbein et al. (1981), Lindaman and Gay (2012), Mamolo and Zazkis (2008), Martin (2013), Martínez-Planell et al. (2012), Movshovitz-Hadar and Hadass (1991), Oehrtman et al. (2014), Sierpińska (1987), and Wijeratne and Zazkis (2016)) have used various theoretical frameworks and methodologies to investigate students' conceptualization of the convergence of series and their limits, but APOS theory (Dubinsky et al., 2005a, 2005b), with the Totality stage, guided our investigation.

Dubinsky et al. (2013) explain that one possible way to describe a participant's ability "to see or to imagine all of the 9s [or other repeating digits] present at once" (p. 240) without seeing repeating decimals as objects is the existence of a new stage that they call totality, between the Process and Object stages. It refers to a student's ability to view all the repeating digits as present at once and simultaneously view this process as ongoing and completed. The fraction 1/3 illustrates this more explicitly because one divided by three is 1/3, whereas using long division yields the decimal representation of 1/3. Thus, this process is simultaneously ongoing and completed. In addition, participants' evidence of both progress toward and away from a stage suggested that there might be intermediate levels in the development from one stage to another. They note that their data indicates that seeing all of the nines present at once does not give this process the status of an object. According to the genetic decomposition related to repeating decimals Dubinsky et al. (2013) provide, repeating decimals become objects when participants can think of repeating decimals as numbers, perform arithmetic operations on them, substitute them for their rational counterparts, or conclude that they approach and reach their rational representations. Dubinsky et al. (2013) also revise the genetic decomposition related to repeating decimals by including three intermediate stages from the Process stage to the Totality stage: Start toward Totality (ST), Progress toward Totality (PT), and Emerging Totality (ET). They also propose three intermediate stages from the Process stage to the Object stage: Start toward Object (SO), Progress toward Object (PO), and Emerging Object (EO). Dubinsky et al. (2013) discuss the possibility of a mental mechanism existing "by which an individual moves from thinking of a process as a sequence of continuing steps to being able to imagine these steps all at once" (p. 251). APOS theory with the Totality stage postulates that the primary mental mechanisms for building action, process, object, and schema mental structures are interiorization for progress

from action to process, de-temporalization from process to totality, and encapsulation for progress from totality to object (Dubinsky et al., 2013).

Method

Thirty pre-service secondary mathematics teachers from a mathematical methods course for teaching secondary mathematics participated in this study. At the time of data collection, our participants were in a professional development program in mathematics education. Their mathematics backgrounds and experiences varied significantly, but all held degrees in mathematics or science.

While exploring the topic of rational and irrational numbers, the participants used the definitions of rational and irrational numbers to relate them to terminating decimals, repeating decimals, and the decimal expansions of irrational numbers. The participants revisited various examples of repeating and non-repeating decimals and discussed converting a fraction to a decimal number. One student recalled the algorithm for converting a repeating decimal to a fraction by setting it equal to a variable, multiplying it by an appropriate power of ten, subtracting the former quantity from the latter, and solving for the variable. This suggestion led to a discussion about the meaning of digits and their positions, mainly focusing on repeating decimals. The students engaged in a lively debate, with differing opinions, while the instructors chose not to intervene to allow the discussion to unfold organically. Afterward, students worked in groups of two or three to discuss and respond to the following three questions:

Group Task: How would you add two repeating decimals? How would you check if your answer is correct? How would you respond to someone who says the sum of two repeating decimals is always a repeating decimal?

After that, we gave the participants a similar individual task.

Individual Task: A) How would you add two repeating decimals? How would you check if your answer is correct? How would you respond to someone who says the sum of two repeating decimals is always a repeating decimal? B) Now, add the following repeating decimals (without technology) before revising your initial responses. Please show all your work for each example: 0.252525... + 0.373737..., 0.434343... + 0.787878..., 0.858585... + 0.273273273..., 0.656565... + 0.587587587... C) After working on the above examples, please share your revised response to the three questions you answered earlier: How would you add two repeating decimals? How would you check if your answer is correct? How would you respond to someone who says the sum of two repeating decimals is always a repeating decimal?

The data for this study consisted of the participants' written group and individual responses. We categorized them by the participants' views on repeating decimals, focusing on their understanding of the underlying concepts. We analyzed their responses according to APOS's framework, focusing on intermediate stages from Process to Totality and Totality to Object.

Results

Some participants found handling infinitely many digits, converting repeating decimals to fractions, and doing column addition with repeating decimals challenging, and these particular challenges enabled us to categorize student responses.

Participants Face Challenges Handling Infinitely Many Digits

Many participants faced obstacles while checking if they had correctly added two repeating decimals in groups and their individual tasks. Many of our students viewed infinity as an unending and uncompleted process. Below, we discuss how this perception influenced their

approaches. We hoped our students would see that decimal expansions contradict the notion of potential infinity because they are approximations that consecutively grow in time while representing a completed process. Still, only some of our students observed that. We will now explain how this perception influenced their approaches.

Initial responses. Many participants argued that when adding repeating decimals, using more repeating digits would allow them to get closer to the answer. They viewed the addition as an infinite process that continuously improves the final answer as the number of digits increases.

Starting points. In class, our participants could not immediately add repeating decimals as they would terminating decimals. Hence, they used truncated versions of repeating decimals or rational equivalents because they viewed both as numbers. Since repeating decimals did not have last digits, our participants initially resorted to operating on exact rational counterparts or rational approximations instead because they had last digits. Only one participant added two repeating decimals using the left-to-right addition by claiming that "3.777... + 4.666... = 7.000... + 1.300... + 0.130... + 0.013... + ... = 8 + 0.4 + 0.04 + 0.04 + ... = 8.444... = 8 4/9."

Participants Truncate Repeating Decimals to Verify the Conversion to Fractions

We found that many participants initially viewed the sum as an endless addition process, and they could not apply mental actions to the sum (or imagine applying them). Our data indicated that all the participants could easily convert repeating decimals into fractions using algebra. However, when we asked them how they would check if their final results were correct, most did not argue for the validity of algebraic conversion. Instead, they resorted to truncating repeating decimals, approximating the sum, and using calculators. Only a few students argued that the algebraic method was valid and that they only had to check if they added the corresponding fractions correctly. We now showcase some responses exemplifying the intermediate stages between Process and Totality.

Totality stage. Some participants built on their initial responses and displayed the PT conception as they were trying to imagine all the digits present at once by physically seeing many digits. One group first "added 3.99999 to 2.22222 and got 6.22221." Then they tried "adding 3.9999999999 to 2.2222222222 and got 6.2222222221." They concluded by saying, "There is a pattern here; the sum of our repeating decimals is also repeating; the more you extend them to add, the longer the decimal of the sum will be." Another group suggested adding two repeating decimals "using regular addition," but "the answer was inaccurate." They noted that adding more digits gave "a more accurate answer" and allowed them to "see what decimals would repeat." They concluded that the more digits we use, "the closer we get to the true answer." Another group suggested using a calculator and "increasing the number of repeating digits." They started with "0.333 + 0.666 = 0.999" and then suggested using "0.3333333333 + 0.6666666666 = 0.999999999" because they wanted "to show that the answer gets closer and closer to 1 (like a limit)." Another group said we could check our work by "doing the long addition with several decimal places for the best estimation." Another group wrote that "1/3 + 1/11 = 11/33 + 3/33 = 14/33 = 0.424242…" and added that "0.33333 + 0.09090 = 0.42423" confirms their answer. Another group said, "We can check the answer by adding the decimals to several decimal points." For example, "3.333333333… + 0.7142857… = 4.0476190… = 85/21." The group added that "this shows that our approximation using fractions works except when the fraction would produce a whole number" and that "this stresses that decimals are only approximations for fractions." Another group said, "Using the rounded version of the original numbers truncated to an appropriate number of decimal places, we could find an approximate solution to compare with our exact answer." Another group said that "using the rounded version

of the original numbers" allows us to use "an approximate solution to compare with our exact answer." On their individual task, most participants abandoned the notion that adding two repeating decimals is a continuous process that improves the accuracy of the final result when more and more repeating digits are involved. We categorized some responses as the ST conception because these participants suggested truncating to estimate the sum without saying that using more digits improves the accuracy of the result. One of these participants suggested checking the addition "by manually typing out several decimal places." Another proposed to "truncate to an appropriate amount of decimals" and "add them up to check." Another suggested estimating "the approximate result of adding two decimals by manually adding the decimals up digit by digit to a certain amount." Another participant proposed "manually doing the long addition with several decimal places." Another suggested "ensuring that the sum makes sense based on an estimate."

Object conception. During the group exploration, our participants generally believed that repeating decimals were not complete numbers. They often asserted that they were approximations or approached rational numbers. We now showcase some responses exemplifying the intermediate stages between Totality and Object. Only two groups exemplified the Object conception by explicitly stating that repeating decimals are numbers. One group said that "recurring decimals are rational numbers" and "operations like adding apply and are valid when trying to add the repeating decimals together." Some individual responses exemplified the transition toward the Object conception. One participant demonstrated the EO conception by suggesting adding repeating decimals "as if they were not infinite, as the addition causes a repetitive pattern." This participant recognizes that one may add two repeating decimals, but only as if they were terminating decimals, so the participant has yet to reach the Object stage. Another proposed that "it is best to round each decimal to a certain large number of digits (12, for example) and try to add them to look for a recursion pattern." This response is similar to the previous one, but because rounding aims to identify the repetend in the final answer, this participant operates on repeating decimals as objects. Ten participants mentioned performing addition or "long addition," but we categorize these responses as the PO conception due to the lack of elaboration.

Participants Encounter Cognitive Obstacles Doing Column Addition

Our participants initially viewed the sum of repeating decimals as a process and the sum of their rational counterparts as objects. To them, the addition of repeating decimals is a dynamic process, while the sum of two corresponding fractions is a static object. Such a conceptualization involves viewing one side as an approximation to the other rather than strict equality. The standard right-to-left addition convention has prevented most of our participants from reaching the totality stage immediately. In contrast, some participants reached the Totality stage using pattern recognition but did not reach the Object conception. We now exemplify both cases.

School conventions (Totality stage). On the individual task, only one participant added using left-to-right addition, but everyone else relied on the standard right-to-left addition. This participant said, "Since repeating decimals have no last digit, we must start adding from the left to do column addition until we recognize the pattern in the new number." The same participant wrote that "$3.777... + 4.666... = 7.000... + 1.300... + 0.130... + 0.013... + ... = 8 + 0.4 + 0.04 + 0.04 + ... = 8.444... = 8\ 4/9$." This participant imagined and acted on all the digits to add from the left, reaching the Totality stage and immediately progressing toward the Object conception because the participant identified the sum as "the new number."

Pattern recognition (Object conception). Some exemplified the EO conception in their individual responses. One participant suggested "writing out the first several digits (enough to have the recurring part repeat at least three times) of each decimal and then adding them." The same participant said, "After seeing some repetition in my answer, I would assume that is the part that repeats and write the answer as a repeating decimal." This elaboration demonstrates the EO conception. The participant does not refer to the sum as a number, so we cannot conclude that this participant has reached the Object stage, but this conception is emerging. Another participant proposed to "ignore the final digit and determine the answer from the repeating portion since as the number of decimal places approaches infinity, the error tends to 0 and becomes negligible." The participant exemplifies the EO conception because the participant truncates to "determine the answer from the repeating portion" and recognizes the possibility of the sum being a number "as the number of decimal places approaches infinity." During the group exploration, only one group suggested checking "if adding the digits involves carrying ones." If it does not, "add some of the digits and look for a pattern." For example, "0.121212… + 0.656565… = 0.777… = 7/9." They added that if it does, add some digits and "look for a pattern but ignore the last digit." Since it is unclear whether this group recognizes the sum as a number or procedurally converts a repeating decimal to a fraction, we also classify this response as EO conception.

Conclusion and Discussion

Our paper aimed to investigate what pre-service teachers attend to when adding two repeating decimals and how effectively adding them helps them regard repeating decimals as objects rather than processes. Our participants demonstrated proficiency in converting repeating decimals to rational numbers and adding them that way. However, when the group and individual tasks asked them how they would verify their calculations, the majority predominantly truncated or approximated repeating decimals.

Unsurprisingly, the written group responses indicate that our participants initially viewed repeating decimals as processes that only approximated the corresponding rational numbers. The absence of the last digit and the reliance on the right-to-left addition posed the major obstacle.

Using the left-to-right addition, recognizing the sum's repetend, and stating that the final answer is a number rather than an approximation was a way of conceptualizing repeating decimals as numbers. More research could uncover the role of left-to-right addition. Most surprisingly, our data indicates that almost all the students have reached the Totality stage. A few reached the Object stage, and many came close to viewing repeating decimals and their sums as objects. Asking participants to think about how they would check that they had added two repeating decimals correctly inadvertently helped them reach the Totality stage and even the Object stage, and herein lies our main contribution to the research on repeating decimals. The data suggest that the unusual task we designed helped most participants reach the totality stage. Moreover, our study indicates the existence of the mental mechanism of de-temporalization for transitioning from process to totality, as almost all of our participants have reached it. Prompting participants for ways to check if they had correctly added repeating decimals led them to think about how to perform column addition with repeating decimals. Our participants seem to have reached the Totality stage because they eventually realized that they had to see and act on all the digits as they would when doing column addition with terminating decimals. Future research would help the mathematics education community ascertain if this is the determining factor.

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Navigating Belonging, Competence, and Career Aspirations in Mathematics: Experiences of First-Generation, Low SES Women

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This study investigates the academic journeys of first-generation, low socioeconomic status (SES) women in mathematics, focusing on their sense of belonging, perceptions of competence, and career aspirations. Using narrative inquiry, the research draws on interviews and reflections to illuminate how isolation, identification as "rule followers," and a lack of mentorship shape their experiences. The study adopts Wenger's (1998) mode of belonging framework—engagement, alignment, and imagination—to analyze how these students interact with their mathematics community and envision their roles within it. Findings highlight the urgent need for mentorship and career guidance tailored to underrepresented students in STEM. This research provides insight into institutional strategies for supporting the success of first-generation, low SES women in mathematics.

Keywords: belonging, career guidance and mentorship, competence, first-generation

The underrepresentation of women and racially/ethnically minoritized students in Science, Technology, Engineering, and Mathematics (STEM), particularly in mathematics, is a persistent issue in higher education (Beasley & Fisher, 2012; Master & Meltzoff, 2020). For first-generation, economically disadvantaged students, these challenges are further compounded by a lack of institutional knowledge, resources, and mentorship (Kim & Sax, 2018; Mishra, 2020). These students often navigate the academic landscape without the familial or social networks that their more privileged peers rely on, facing significant barriers in accessing the support necessary for academic success and career development (Choy, 2001; Mishra, 2020). As a result, many of these students experience feelings of isolation and uncertainty, particularly within the competitive and exclusive environments of STEM fields (Good et al., 2012).

Mentorship plays a crucial role in fostering student success, providing both academic and emotional support, yet first-generation, low-income students often find themselves without this vital resource (Piatek-Jimenez, 2008). The absence of consistent mentorship can limit students' career aspirations, leading them to pursue more practical, though uninspired, options that do not fully align with their academic strengths or passions (Piatek-Jimenez, 2015). Furthermore, implicit biases and stereotypes can exacerbate these issues, reinforcing feelings of non-belonging and limiting opportunities for underrepresented students in STEM (McGee & Martin, 2011; Good et al., 2012; Rainey et al., 2018).

Through Wenger's (1998) framework of modes of belonging—engagement, alignment, and imagination—this study explores how gender, race, first-generation status, and socioeconomic disadvantage intersect to shape the academic experiences of women in mathematics. Thus, the study aims to answer the following research question: How do women of low socioeconomic status experience their mathematics major, particularly in relation to their sense of belonging within their mathematics community? The study highlights the role of mentorship, the challenges of belonging, and the importance of career guidance in supporting the academic and professional development of these students.

Theoretical Framework

The theoretical framework guiding this study is based on Wenger's (1998) concept of "modes of belonging," which include engagement, alignment, and imagination. These modes provide a lens for understanding how individuals negotiate their identities and find belonging within a community of practice, such as an academic or professional field. Each mode offers a distinct yet interrelated perspective on the processes through which individuals experience inclusion or exclusion, and how these experiences impact their participation and development of mathematical competence.

Engagement refers to the active participation in a community through the ongoing negotiation of meaning, where individuals interact with others and the environment to co-construct knowledge and develop their sense of belonging. Engagement is key to identity formation as it integrates doing, thinking, feeling, and belonging (Wenger, 1998). However, engagement is not always guaranteed; individuals may face barriers to full participation due to social dynamics, unequal power relations, or institutional structures. When engagement is limited, it can lead to feelings of marginalization and prevent individuals from fully developing their competence within the community.

Alignment involves coordinating one's actions and efforts with the broader structures, goals, and expectations of the community. It requires individuals to adapt to the norms, values, and practices that define what is considered competent or valuable within the community. Alignment, however, can also be disempowering if it involves blind adherence to prescriptive norms without room for individual agency. For individuals who have historically been excluded from dominant communities, alignment may be particularly challenging, as their identities and experiences may not fit neatly within the established culture of the institution. This tension can result in a disconnection between personal goals and the broader expectations of the community.

Imagination is the ability to envision future possibilities and create new ways of being within the community. It allows individuals to transcend immediate circumstances, think creatively, and imagine their potential trajectory within the community. Imagination plays a crucial role in helping individuals see themselves as capable of growth and achievement. However, when access to role models, mentors, or opportunities for exploration is limited, imagination can be stifled, restricting an individual's ability to envision a broad range of future possibilities. This limitation can hinder one's long-term engagement and commitment to the field.

Together, these three modes of belonging—engagement, alignment, and imagination—offer a comprehensive framework for examining how individuals navigate their participation in academic or professional communities. They highlight the dynamic and often complex nature of belonging, showing how barriers in any one mode can impact an individual's overall sense of identity, competence, and future aspirations within a community of practice.

Method

Background and Participants

This study employs a narrative inquiry approach to explore the academic journeys of two first-generation, low socioeconomic status women pursuing mathematics degrees at a research-intensive large university (Clandinin, 2006). Nia, a biracial (Black and White) student, and Bethany, a White student, were selected due to their shared characteristics as first-generation students and double majors in Pure Mathematics and Secondary Mathematics Teaching (SMT). Nia had switched majors twice—first in Chemistry and then Psychology—before finding satisfaction in mathematics. Bethany, on the other hand, identified more strongly with the SMT

community and managed her studies alongside work to maintain financial independence. Both participants faced challenges related to belonging, competence, and mentorship, which shaped their academic paths and career aspirations.

Data Collection

Data were collected during the spring 2022 semester through semi-structured, in-depth interviews and weekly video reflections. Three interviews were conducted with each participant via Zoom, focusing on their sense of belonging, competence, and their future in mathematics. In addition, weekly video reflections collected throughout the semester, which allowed the participants to share real-time updates on their academic and social experiences. Throughout the process, field notes were taken to document key observations and reflections, helping the researcher triangulating the data and capturing a fuller picture of the participants' experiences.

Data Analysis

The narrative inquiry approach was used to examine the participants' experiences in depth, with Wenger's (1998) mode of belonging framework serving as a lens for data analysis. In this study, narrative is defined as a way of understanding and representing the participants' lived experiences as storied phenomena that unfold over time and within specific social and cultural contexts. Drawing on Clandinin and Connelly's (2000) perspective, narratives are conceptualized as both the means of inquiry and the objects of study, reflecting how individuals make sense of their identities and experiences through the stories they tell. By centering on the interviews and weekly video reflections as the primary sources of narrative data, the study captures authentic evidence of the participants' everyday experiences—experiences that are often overlooked or devalued by institutions, instructors, and peers within the mathematics community. These narratives provide a lens into how the participants navigate the intersecting challenges of belonging, competence, and career aspirations, revealing the subtle and overt ways in which systemic barriers shape their journeys.

Open, axial, and selective coding processes were followed to systematically analyze these narratives while remaining grounded in Wenger's (1998) framework. First, open coding was employed to identify initial categories from the interview and reflection transcripts, focusing on broader patterns related to the modes of belonging, conceptualized through *engagement*, *alignment*, and *imagination* components. Next, axial coding was used to connect related themes, integrating these patterns with Wenger's (1998) framework to explore how the participants' sense of belonging within the mathematics community was shaped and negotiated. Finally, selective coding was applied to distill the core themes reflecting the participants' evolving modes of belonging, specifically in relation to participatory actions, mathematical competence, career aspirations, and the influence of mentorship (or the lack thereof).

Wenger's (1998) framework provided a theoretical anchor, enabling the analysis to explore the dynamic interplay between the participants' identities as first-generation, low SES women and their experiences within the mathematics community. This iterative approach ensured that the findings honored the richness of the participants' narratives while being theoretically informed by the mode of belonging framework, ultimately illuminating how their identities and experiences were shaped by their interactions within their academic and social contexts.

Findings

The findings from the narratives of Bethany and Nia reveal key insights into how first-generation, low SES women navigate the mathematics community. This section explores three

central themes that emerged from their stories: (a) their sense of belonging and feelings of isolation in academic spaces, (b) their perceptions of competence and identity as learners of mathematics, and (c) the role of mentorship and guidance in shaping their career aspirations. While both women share common struggles tied to their socioeconomic and gender identities, Nia's biracial identity adds an additional layer of complexity to her experiences, creating both shared and distinct challenges in how they perceive and engage with their academic environment. Nia and Bethany's experiences reveal the interconnectedness of belonging, competence, and mentorship in their mathematics journeys, each influencing the others in profound ways. Through Wenger's (1998) modes of belonging—engagement, alignment, and imagination—both participants grappled with feelings of isolation and uncertainty about their place in the mathematics community, compounded by perceptions of competence and the absence of mentorship.

Belonging and Isolation in the Mathematics Community

Both Nia and Bethany struggled with a sense of belonging within their academic community, particularly in group settings where they often felt marginalized by more confident or prepared peers. Bethany's feelings of exclusion were exacerbated by her perception that others were more capable, leading her to disengage from group work. She reflected, "I just assume people like Troy[1] know more than me, so they must be doing it right, even if it doesn't make sense to me" (Bethany, Reflection 2). This assumption hindered her ability to fully engage, aligning herself only passively within the group, which deepened her isolation.

Similarly, Nia's experience of exclusion was tied to her peers' faster pace and greater mathematical background. She remarked, "Sometimes, especially in group work, I'm just the person watching others solve problems. It's hard when others move faster, and I can't keep up" (Nia, Reflection 4). The sense of being sidelined diminished her engagement and made her feel like an outsider in a space where she struggled to align her academic identity with the expectations of the mathematics community. For both participants, their first-generation status may have contributed to their challenges, as they lacked access to social and familial networks that could have supported them in navigating college.

Perceptions of Competence: Rule-Following vs. Rule-Making

Both Bethany and Nia saw themselves as "rule followers," a mindset that limited their perception of competence and ownership over mathematical knowledge. This tendency to follow prescribed methods without feeling a deep connection to the material undercuts their confidence. Bethany reflected on this in saying, "I'm just a rule follower, really. I do what the professors tell me to do, and that's how I know I've done it right" (Bethany, Reflection 7). For her, mathematical success was defined by adherence to the rules set by her instructors, rather than by personal exploration or critical thinking.

Nia echoed this sentiment, expressing frustration that her learning experience was mechanical and devoid of intellectual autonomy. "I don't feel like I have ownership over the math I'm doing. It's just a bunch of methods I follow, and it feels like I'm not really the one making decisions" (Nia, Interview 1). This rule-following approach led both women to feel disconnected from the deeper, creative aspects of mathematics, reinforcing their position on the

[1] Troy was one of Bethany's peers who often dismissed her contributions during group work, reinforcing her feelings of inadequacy and furthering her sense of isolation. Bethany frequently referred to him in her reflections, highlighting the impact of his actions on her confidence and participation.

periphery of the mathematics community. The shift from rule-following to rule-making—an essential transition in higher-level mathematics—remained out of reach for them, thus further impacting their engagement and alignment.

Mentorship and Career Guidance

The absence of mentorship played a crucial role in shaping both participants' academic trajectories and their ability to imagine broader career possibilities. Without a mentor to help her explore the landscape of opportunities, Bethany felt confined to practical career choices, such as teaching. While she saw it as a viable and straightforward path, she expressed uncertainty about whether it was the most fulfilling option available to her. "I really wish I had someone to guide me more. I don't think I've explored all the possibilities with a math degree, but there's no one really pointing me in a different direction" (Bethany, Interview 2). This lack of guidance diminished her alignment with the larger goals of the mathematics community and limited her ability to imagine herself in a more diverse array of mathematical roles.

For Nia, the mentorship she received in high school did not continue into her university life, leaving her to independently navigate her academic and career goals. She reflected: "My high school teacher pushed me toward math, but in college, it feels more like I'm on my own" (Nia, Interview 1). This lack of consistent support left her uncertain about pursuing more ambitious career paths like data analytics, reinforcing feelings of isolation and non-belonging.

Nia also revealed that her academic advising lacked direction, particularly in relation to STEM fields. She explained: "When I was looking for different careers, there wasn't much suggestion for the STEM field... I wasn't looking at STEM. I was looking at everything but STEM" (Nia, Interview 2). This gap in guidance may reflect systemic implicit biases that can affect the opportunities and support available to students from underrepresented groups, including Black women, in STEM fields, potentially reinforcing stereotypes about who is deemed capable of pursuing scientific careers. Without tailored mentorship to counter these biases and explore broader career options beyond teaching, Nia may have experienced a sense of disconnection from the academic community. This might hinder her confidence in pursuing other fields, such as data analytics, which she saw as a more challenging and fulfilling path, but one she did not feel equipped to follow due to the lack of encouragement and support.

Discussion

The experiences of Nia and Bethany reveal the profound impact of gaps in engagement, alignment, and imagination on their academic trajectories, career choices, and perceptions of competence. These themes highlight how the absence of meaningful support structures can hinder the success and persistence of women and racially/ethnically minoritized students in mathematics.

Belonging and isolation emerged as critical issues, directly linked to the students' engagement with their mathematics community. Both participants felt sidelined in group settings, a dynamic that undermined their engagement and reinforced their sense of isolation (Wenger, 1998). For Bethany, group work often left her feeling less competent compared to her peers: "I just assume people like Troy know more than me" (Bethany, Reflection 2), highlighting how perceived disparities in competence contribute to disengagement (Good et al., 2012). Nia faced similar challenges, remarking that she often felt like "the person watching others solve problems" in group settings, further underscoring the exclusion she felt within the mathematics community (Nia, Reflection 4). These feelings of isolation were magnified by their first-

generation status, as both lacked the social capital necessary to navigate academic spaces more effectively (Choy, 2001; Mishra, 2020).

Rule-following vs. rule-making also played a significant role in the participants' academic journeys related to their perceived competence. Both Nia and Bethany saw themselves as "rule followers," a perspective that limited their sense of ownership over mathematical knowledge and stifled their ability to engage in creative problem-solving (Di Martino & Zan, 2010). Bethany reflected on this in saying, "I'm just a rule follower" (Bethany, Reflection 7), while Nia described her frustration at being unable to make decisions independently in her mathematical work (Nia, Interview 1). This internalization of prescriptive learning models prevented them from fully aligning with the broader goals of their academic community, diminishing their confidence and reinforcing their feelings of being outsiders (Wenger, 1998).

The absence of mentorship and career guidance was particularly detrimental to the students' ability to imagine broader career possibilities in mathematics. Bethany expressed regret over not having someone to guide her career exploration, stating that she wished she had more opportunities to learn about the "possibilities with a math degree" (Bethany, Interview 2). Nia echoed this sentiment, noting that while her high school teacher encouraged her toward mathematics, she felt isolated and unsupported at the university level (Nia, Interview 1). McGee and Martin (2011) argued that marginalized students may navigate academic spaces while managing or rejecting imposed stereotypes, which could explain why Nia did not initially seek support. This lack of mentorship, coupled with implicit biases that may have shaped the career advice they received, further limited their ability to align their academic goals with ambitious career paths (Beasley & Fisher, 2012; Cribbs et al., 2021; Piatek-Jimenez, 2008, 2015).

These findings suggest that the combined effects of disengagement, misalignment, and limited imagination create a cycle of self-doubt and marginalization for first-generation, low SES women in mathematics. Addressing these issues requires a comprehensive support system that fosters engagement through meaningful participation, aligns personal goals with community expectations, and encourages the imagination of broader career possibilities through targeted mentorship that is free of stereotypes and biases (Wenger, 1998; McGee & Martin, 2011; Piatek-Jimenez, 2008b).

Concluding Remarks

An exploration of Nia and Bethany's experiences revealed critical gaps in institutional support for first-generation, low SES women in mathematics. These students struggled to fully integrate into their mathematics communities, often feeling isolated and unable to align their personal goals with the academic expectations placed upon them. Their perception of themselves as rule-followers, rather than independent thinkers, compounded their lack of confidence and diminished their potential for success. Furthermore, the absence of mentorship played a crucial role in shaping their career aspirations, which were more influenced by financial necessity than by a true exploration of the field's possibilities. Importantly, this study also highlights the need to further investigate how racial identity intersects with first-generation and low SES status to impact students' experiences in STEM, particularly regarding access to support and mentorship. This study emphasizes the urgent need for targeted institutional interventions, including mentorship, career counseling, and exploratory learning opportunities, that foster a sense of belonging and competence among underrepresented students in STEM. Addressing these barriers is essential for enhancing their academic outcomes, and future research should explore how mentorship and peer networks can be effectively leveraged to support their success.

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At-Success Students' Affordance and Barriers to Help-Seeking in an Active Learning College Algebra Course

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Despite reforms, some College Algebra students are still "at-success": they can be identified early in the semester as having a higher likelihood of not passing or completing the course (Wakefield et al., 2018). This interview study of 3 at-success College Algebra students explores the hypothesis that there are pedagogical and environmental factors to whether at-success students engage in help-seeking behaviors that could increase their chance at success. I examine: What affordances and barriers to help-seeking do at-success students describe when talking about performance in a college algebra class? Karabenick and Berger's (2013) process of help-seeking was used as a framework for the themes found using interpretative phenomenological analysis. I propose an adaptation and extension of Karabenick and Berger's framework as a result of this study.

Keywords: help-seeking, college algebra, active learning, mathematical microaggressions

Undergraduate drop, fail, and withdrawal (DFW) rates in college algebra can be as high as 50% and even over 50% (Gordon, 2008; Herriot & Dunbar, 2009). Efforts to improve student outcomes have largely focused on active learning (Freeman et al., 2014; Theobald et al., 2020), including professional development for the Graduate Student Instructors (GSIs; Deshler et al., 2015; Rasmussen et al., 2019) who may teach courses with high DFW rates, and course coordination (Rasmussen & Ellis, 2015; Rasmussen et al., 2021). Despite these reforms at the University where this study was conducted, we can still identify college algebra mathematics students early in the semester with a higher likelihood than their peers of not passing (Wakefield et al., 2018).

This study explores the hypothesis that help-seeking (e.g., Karabenick & Berger, 2013) can mitigate the likelihood of not passing the course. It aimed to examine students' perceptions of in-classroom and out-of-classroom learning supports, systems, and experiences for at-success students. In particular, it explored the phenomenon of help-seeking in an active learning college algebra course with a population of students who were at a higher risk of dropping, failing, or withdrawing from the course.

The research question addressed is: *What affordances and barriers to help-seeking do at-success students describe when talking about performance in a college algebra class?*

Study Context: College Algebra and the "MRA"

At the research site, students are asked to review and master a set of prerequisite skills needed for college algebra and demonstrate mastery by scoring 80% on a "Mastery Review Activity" (MRA; for the validity of cutoff score as predictive of course performance, see Wakefield et al., 2018). Students are allowed to attempt this activity every day over the first two weeks of the semester.

Literature Review and Theoretical Perspective

In this section, I will present a small argument for using the term "at-success" to replace the usual term "at-risk" for the population of students I am exploring. The rest of the literature review will examine what is currently known about help-seeking and the gaps this study hopes to

address.

The Term "At-Success" in Place of "At-Risk"

For higher education professionals and staff to provide services to students who have a higher likelihood of not meeting their educational objectives, the term "at-risk" has often applied to these students (Dix et al., 2020; Horton, 2015; Quinnan, 1997). When higher education professionals who are providing these services were asked to give an operational definition of "at-risk" the answers were diverse (Dix et al., 2020). The various operational definitions for at-risk students have included both academic and non-academic characteristics: academic past academic performance, social background, and both internal and environmental characteristics (Bulger & Watson, 2006; Castro, 2014; Connolly et al., 2017; Dix et al., 2020; Horton, 2015).

The label "at-risk" was initially used within K-12 educational settings (Quinnan, 1997). Unfortunately, the term "at-risk" tends to operational definitions that blame the individual and do not address needed system changes (Quinnan, 1997). Deficit language and thinking displaces all the blame for school failure on the individual. This is victim blaming. Valencia (2010) describes victim blaming as one of the six characteristics of deficit thinking in an educational setting. This type of thinking "absolves institutional structures and inequitable schooling arrangements that exclude students from optimal learning" (Valencia, 2010, p. 18).

For this study, I do not want to view students as being deficient. By examining the phenomena of help-seeking, we can learn what fosters a supportive educational environment and empowers students to navigate barriers they may encounter and move them from being "at-risk" to being "at-success". This study aims to identify possible factors that can be changed in both course structure and institutional practices to ensure students are "at-success". The operational definition of "at-success" for this study is given to students who either failed the MRA at the beginning of the course, repeating college algebra, or did not place into college algebra or higher at enrollment into college.

Help-seeking as an Analytic Framework

Help-seeking is considered a form of agency over one's learning in a particular environment (Nelson-Le Gall, 1981; Nelson-Le Gall, 1985; Karabenick & Berger, 2013). Karabenick and Berger (2013) proposed a model of help-seeking that reflected "stages" (p. 239) where decisions were made that impacted whether a learner pursued help and what type of help they pursued. Each of these "stages" occurs in a "phase" (Zimmerman, 2008, as quoted in Karabenick & Berger, p. 254). These authors emphasized that a particular learner may not experience all stages in a particular environment, and the stages do not occur in any universal order. For most individuals, a combination of these stages may occur at any point in the decision process of whether to seek help (Karabenick & Berger, 2013).

As a starting analytic framework, I used the stages and phases proposed by Karabenick and Berger (2013). I used this framework to describe and characterize help-seeking processes college students use to engage in various assignments and activities in college algebra. Since this phenomenological study uses student interview transcripts as the data source, this framework was only used after interviews with students were first analyzed using Interpretative Phenomenological Analysis (IPA) to find various themes (Smith et al., 2009). The framework was used to explain students' affordances and barriers to help-seeking; I leave open the possibility of confirming, disconfirming, or revising this model concerning at-success college algebra students' help-seeking process.

Table 1. Stages of Help-Seeking Process and Three-Phase Model of Self-Regulated Learning

	Stages of Help-Seeking Process	Phase
1	Determine whether there is a problem	Forethought
2	Determine whether help is needed/wanted	
3	Decide whether to seek help (goal)	
4	Decide the type of help	
5	Decide who to ask	
6	Solicit help	Performance
7	Obtain help	
8a	Process the help received- judge or evaluate	Self-Reflection
8b	Process the help received - react to it	

Note. From "Help seeking as a self-regulated strategy" by S.A. Karabenick, and J. L. Berger, 2013, In H. Bembenutty, T. J. Cleary, & A. Kitsantas (Eds.), *Applications of self-regulated learning across diverse disciplines: A tribute to Barry J. Zimmerman,* pp. 237–261.

Methods

Phenomenology

This study was a phenomenological study of help-seeking processes of at-success college students within the context of an active learning college algebra course. To better understand the students' process of seeking help, it is necessary to hear from the students themselves. Since the goal of this study was to give voice to the participants, understand a phenomenon, and describe the essence of engagement with class activities, this study was well suited for a qualitative design (Creswell & Poth, 2018; Merriam & Tisdell, 2016). Creswell & Poth (2018) describe two types of phenomenology studies, hermeneutic phenomenology and transcendental phenomenology. The sub-approach chosen was hermeneutic phenomenology, which aims to interpret the meaning of a lived experience.

Participants and Data Collection

To interview the targeted audience of students who did not complete the MRA activity or repeating college algebra, or did not originally test into college algebra, about their experiences with help-seeking in completing the MRA and other course activities, students were recruited through two different methods. In the Fall of 2023 and Spring of 2024, students who did not complete the MRA activity at the beginning of the semester were sent an email inviting them to participate in an interview about their experiences in college algebra and taking the MRA. Also during both the Fall of 2023 and Spring of 2024 college algebra students were invited to participate in a survey about their experiences in college algebra. In the survey, students were asked if they would be willing to participate in a follow-up interview about their experiences in college algebra. Students who indicated that they were willing to be interviewed and were from the targeted audience were sent an email to participate in an interview about taking the MRA and their experiences in college algebra. To aid in recruiting, students were paid $20 for participating in a 30-45-minute semi-structured interview. When using IPA, it is recommended that the number of participants be between three and six due to the intensity of the interviews' analysis. Three participants were recruited.

Table 2. Description of Interview Participants

Name	Gender	Race	Year	Pass MRA	Previous college math course
Amy	Female	White	Sophomore	No	Intermediate Algebra
Jacob	Male	White	Freshman	No	College Algebra
Carmen	Female	White/Mixed	Sophomore	Yes	Intermediate Algebra

Note: Pseudonyms were used to protect the privacy and confidentiality of research participants.

Participants were interviewed individually. A semi-structured interview protocol was used to guide the interviews (Merriam & Tisdell, 2016). The interviews were done over Zoom and recorded and transcribed. Before the recording began, the participants removed their names to protect their anonymity. The transcript was then checked for accuracy. After each interview and cleaning of transcripts, memoing was done to record initial impressions from each interview. The transcripts were then used for analysis using IPA.

Data Analysis

I analyzed interviews in multiple phases. After several readings of an interview, I engaged in three rounds of "noting" (Smith et al., 2009, p. 83). First, I analyzed the transcript for descriptive comments focused on the content of what the participant said. Second, I made linguistic comments focused on exploring the language and specific words used by the participant. Third, I made conceptual, interpretive comments, drawing from my "own experiential and/or professional knowledge" (Smith et al., 2009, p. 89).

After completing all the initial notations and coding, I organized data from each participant's transcript into a table. Next, I compared all participant tables to identify common patterns. From this, I constructed a merged table of cross-cutting themes. Table 3 includes an excerpt for one cross-cutting theme across all the cases.

After this was done, a summary of recurrent themes was made. These themes and others were then analyzed with respect to the phases of help-seeking (Karabenick & Berger, 2013) as each participant relayed their individual stories.

Results

Several recurrent themes emerged. These themes related to at least one participant's process of help-seeking. I now discuss the themes.

Table 3. Table of Master Theme MRA Purpose as an Assessment or Placement Test

Assessment:	
Jacob:	*Isn't that like a course readiness assessment*
Carmen:	*I personally think it [MRA] was just busy work, but I could see how it was meant to gauge where we were.*
Amy:	*I felt like it [MRA] was just to give my instructor a baseline of where everyone was at.*
Placement Exam:	

Carmen:	*The reasoning they gave us was that it was to make sure that we're actually supposed to be in college algebra. And if we're struggling with anything on the MRA to the point were it showed that we weren't ready for college algebra. They were gonna move us down a class. Move you down to 100A. Which I thought was kind of odd because of the placement exam that we have to take to come into college placed us in college algebra.*
Jacob:	*Don't you have to like test into every math class you take? So I think it's [MRA] kind of redundant. Cause like you already have to like test into the class.*

Determine Whether There is a Problem.

All three participants did not see the MRA as an activity designed for their benefit. They each framed the activity as an assessment or placement exam. See quotes in Table 3. Amy attempted to pass the MRA. She quickly became overwhelmed with the pace of the course and felt she needed to concentrate on completing her homework. The MRA was not perceived as being equal to her homework even though it had more point value. This perception of the MRA being less important than her homework was conveyed when she responded to a question about any resources she used to prepare for retaking the MRA. Amy said, "I don't think so, because I had already been into the semester. I remember at the very beginning of the semester, I was struggling a lot with math. So I was more focused on that than the MRA." Since the MRA was not perceived for her benefit, she did not prioritize learning the material on the MRA.

Determine Whether Help is Needed/Wanted

Carmen originally did not test in remedial intermediate algebra. Her advisor contacted the mathematics department to arrange for her to retake the placement exam after she reviewed the prerequisite material independently. She then had a sufficient score to enroll in intermediate algebra. She shared that she quickly learned in intermediate algebra that material she thought she knew well from high school, but she had learned incorrectly.

> *Carmen*: I was taught the rules of exponents incorrectly in like high school. Or something like that. So I got to [intermediate algebra] and we were going over the rules of exponents. And I raised my hand. I was called on. I was like 'Are you sure that's how it is? Cause I was taught this in high school, and this isn't how they taught it to me. What's going on here?' And my instructor was like, 'Oh, yeah, no they taught it to you wrong.

She also shared a story in intermediate algebra when she learned she had previously learned the order of operations incorrectly. Both of these stories had the potential of Carmen feeling threatened about asking for help, but it produced the opposite effect. She was willing to ask questions in class. She would ask questions of both her instructor and peers. She knew from positive engagement with instructors that she may have previously mislearned a topic, and immediate help was needed to correct any potential misunderstandings.

Decide Whom to Ask

The role of identity. An important component of active learning is students engaging with their peers. During this engagement, there is both help-giving and help-seeking. Each of the participants' mathematical identities with their peers impacted their decision on whether to ask

their peers for help. Jacob had previously taken college algebra. He felt confident that he knew the material better than his peers.

> *Jacob:* I know how to do most things because I've already taken this class. But if I didn't know what I was doing, I feel like I'd be really lost. Cause he [his instructor] only does a few problems, and he just kinda lets us on our own. And if no one at your table knew how to do it, you [would] kinda just be completely lost.

He also felt that his role was to help his peers. Jacob stated, "When we finish [individually] the workbook, he [instructor] wants us to [make sure] everyone else in our group [knows] how to do it."

For Amy, she did not pass the MRA. Not passing the MRA impacted her perception of her mathematical abilities in relation to her peers' abilities. Amy stated, "It made me feel so stupid. Because almost, I mean, from what I know, the people I sit with, like most people, passed it … I should be on the same level." Comments made in class by her instructor and peers also made her feel unconfident enough to let others know she needed help. Amy stated, "When everyone is going around and saying: 'This is so easy, you should know this. This is stuff you've already learned.' It kind of creates an environment that is, you know, questionable." She also stated, "My instructor had said, like, 'this is stuff you had learned [it] in the past, people have taken it, this is so easy.' And then [my] instructor [said] 'It shouldn't be too hard. Like you've already learned this." Amy experienced mathematical microaggressions (Su, 2015). These contributed to the identity she formed within the classroom context.

As stated earlier, Carmen learned in intermediate algebra that she had mislearned several mathematical procedures, but she saw her abilities as being the same as her peers'. She stated, "I'm assuming all of my classmates passed it. … [I] didn't have any problems with it." She was comfortable asking both her instructor and her peers at her table.

> *Carmen*: [I] also talk with like [my table] group. I mainly get to like a problem. I'd be like, I don't know what this problem's asking. And then my classmates would be like, 'Oh, it's either asking this.' And I'd be like, 'Oh, great! Let's you know, do it together.' And we like do it.

She was hesitant about asking questions of her instructor in front of the whole class.

> *Carmen:* I'm not a student that likes [to] ask tons of questions with the whole class. Because I don't know what pertains to just me or what pertains to the whole class. … I don't wanna raise my had and like stop the whole class.

But during independent group work time, she stated, "I'll ask questions from my instructors."

I will discuss the results related to the steps of the help-seeking process (Karabenick & Berger, 2013): deciding whether to seek help, deciding on the type of help (goal), soliciting help, obtaining help, and processing the help received in the presentation, but not here due to space constraints.

Discussion and Implications

Lack of perception of need was a barrier in the process of help-seeking. All the participants did not see the MRA activity as an activity that was for their benefit. They all framed the activity as either a placement exam or an assessment for their instructor to know the class's current ability. Amy struggled with the MRA and tried to take it twice. She did worse on the second attempt. She did not see the purpose of the MRA and saw it as an assignment not related to her coursework. To have students fully engage with assignments, students need to understand their relevance to their success in learning the course material.

Identity was both an affordance and a barrier in the process of help-seeking (Bergey, 2023). It produced a tension that had the potential of keeping a student from seeking help from their peers. For Amy, it was very clear how her identity kept her from asking questions of both her peers and instructor during class time. For Carmen, she perceived her performance as being parallel with her peers. As a result, she was willing to ask both her peers and instructor for help during class time.

In an active learning course, identity played a significant role in help-seeking during class group work. This result suggests that a student's identity may play a significant role in the help-seeking process within the classroom environment (Gardee & Brodie, 2023). Active learning requires students to engage with each other. For "at-success" students, the role of identity has a greater potential to inhibit help-seeking. Instructors must be aware of the environment that they create in the classroom and how that can help all students be successful.

The data from this study suggest that the process of help-seeking (Karabenick & Berger, 2013) should be expanded to include the role of identity. Students may or may not be cognizant of an evaluation they make about how they may be perceived by their peers before engaging in help-seeking (Bergey, 2023). The classroom norms of discourse also played a role in help-seeking. Amy experienced mathematical microaggressions, which shut down her help-seeking within the active learning environment. She lost all the advantages of active learning.

Further research should observe help-seeking within the group dynamics in an active learning classroom. There is a need to understand what type of norms aid the group dynamics to encourage help-seeking among peers. Group work has been seen as a way to afford students opportunities to seek help. Group cohesion also impacts help-seeking in K-12 settings (Kempler & Linnebink, 2006). This may also be true with college algebra students.

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Adopting a Holistic Conception of Student Success in Undergraduate Mathematics: Investigating Student Experiences With a Calculus Support Course

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This study investigates student experiences in undergraduate mathematics education, highlighting the importance of addressing inequities related to race and gender in student success rates. Focusing on support courses for Precalculus and Calculus I, where students participate in regular Supplemental Instruction sessions and an online course led by the university Math Equity Coordinator, I examine how these interventions influence student success beyond traditional measures such as grades and persistence to include the impact on students' confidence, sense of belonging and the perception of instructional practices. Results from this mixed-methods analysis show that students enrolled in the support course report higher levels of confidence and a stronger sense of belonging in mathematics. Interviews further emphasize the critical role of instructional practices have in fostering these positive outcomes. These findings suggest that to investigate the success of at-risk students effectively, researchers ought to broaden the scope of success indicators to include identity-related aspects of student experience.

Keywords: identity, confidence, belonging, calculus, student success

Introductory undergraduate mathematics, particularly Precalculus and Calculus courses, have long functioned as an access-controlling mechanism for STEM fields, often filtering students out and limiting progression through their majors (PCAST, 2012; Weston et al., 2019). National data highlights troubling trends, such as Koch and Drake's (2018) finding that 34.3% of students at 36 higher education institutions received a D, F, Withdraw, or Incomplete (DFWI) in Calculus. The barriers are even higher for Black or African American students and Hispanic or Latinx students, who experience DFWI rates of 47.8% and 47.9%, respectively. Moreover, women are 1.5 times more likely to exit STEM pathways compared to men (Ellis et al., 2016). Addressing these persistent disparities requires mathematics educators to invest in listening to student experiences and enacting meaningful, equity-driven reforms in instructional practices.

In response to these challenges, this study focuses on a large Southwestern University (SWU) that has faced similar racial and gender disparities in Calculus success rates. Over a two-year period, the institution has employed a holistic approach aimed at closing equity gaps in its Precalculus through Calculus courses. A central component of this effort is the introduction of support courses for Precalculus and Calculus I. These courses offer students an opt-in 1-unit program that incorporates regular attendance at Supplemental Instruction (SI) sessions and engagement with an online Canvas hub, overseen by the university's Math Equity Initiative coordinator. These courses represent a hybrid model of student support, occupying a middle ground between fully integrated corequisite courses and traditional SI programs.

This paper builds upon the findings presented in last year's preliminary report, moving beyond initial observations to provide a more comprehensive analysis of the impact of these support structures. Using a mixed-methods framework that integrates survey data and semi-structured interviews conducted in the Spring 2023, Fall 2023, and Spring 2024 semesters, I examine how students enrolled in Precalculus and Calculus I perceive the inclusivity of instructional practices and the effect of these support courses on their confidence and sense of belonging in mathematics. Our findings underscore that, while academic outcomes such as

grades and course completion are important, student success must also be understood through broader lenses—particularly the cultivation of confidence and a sense of belonging within the mathematical community. These results advance the argument that holistic support, including mentorship and equitable instructional practices, plays a critical role in addressing long-standing inequities in undergraduate mathematics education.

Relevant Literature and Theoretical Perspective

Investigating the experiences of students in mathematics courses requires careful attention to the unique and evolving identities they bring to the classroom. Student identities are not static but are continually reshaped by interactions with their peers, instructors, and the broader learning environment (Collinson, 2006). Within a mathematics context, these identities take on additional dimensions. Martin (2012) describes mathematical identity as the ways in which students see themselves as "doers" and "learners" of mathematics. This concept extends beyond mere participation, encompassing students' perceptions of their interest, enjoyment, and confidence in the subject. In this research, mathematical identity is explored as a multidimensional construct that intertwines with students' broader self-concepts and is closely tied to their sense of belonging within mathematics. By focusing on identity and belonging, the results in this paper illuminate some of the complex factors contributing to the persistence challenges observed in STEM fields, where nearly half of all students in the U.S. who begin a STEM degree do not complete it (National Center for Education Statistics, 2018).

Confidence plays a crucial role in shaping students' experiences and success in mathematics. Research shows that students' confidence in their mathematical abilities significantly impacts their willingness to engage in mathematical tasks and persevere through challenges (Nunez-Pena, Suarez-Pellicioni, & Bono, 2013). In the context of undergraduate mathematics, studies have indicated that students with higher levels of self-confidence are more likely to persist in STEM-related courses (Ellis, Fosdick, & Rasmussen, 2016). Conversely, low confidence has been shown to correlate with higher dropout rates and disengagement, particularly among women and underrepresented minorities (Boaler, 2016). As students navigate the often-intimidating terrain of calculus and precalculus, building and maintaining their confidence is essential not only for their academic performance but also for their long-term persistence in STEM fields. Therefore, fostering environments that actively promote confidence in mathematics can play a key role in reducing attrition and supporting a more equitable learning experience.

Closely tied to confidence is the sense of belonging, which is increasingly recognized as a critical factor in students' success in mathematics. A robust sense of belonging has been associated with improved academic outcomes, increased motivation, and greater persistence in mathematics courses (Good, Rattan, & Dweck, 2012). Particularly in the context of underrepresented students, feelings of exclusion or marginalization can undermine academic performance and lead to disengagement from the subject (Walton & Cohen, 2011). In undergraduate mathematics, where large lecture formats and competitive environments are often the norm, fostering a sense of belonging can help mitigate the alienation that many students feel, especially in fields traditionally dominated by white male students. As such, promoting inclusive instructional practices and creating supportive peer interactions in settings like Supplemental Instruction have the potential to significantly enhance students' sense of belonging and, in turn, improve their mathematical engagement and persistence.

These considerations necessitate a shift in how student success is conceptualized in mathematics education, moving beyond traditional metrics like grades and course persistence. A critically informed theoretical perspective, such as Critical Race Feminism (Crenshaw, 1989),

calls for an evolution in research approaches that view aspects of the student experience, such as student success, more holistically. Rather than focusing solely on academic outcomes, this perspective emphasizes the importance of examining how systemic inequities shape students' experiences in mathematics classrooms by listening to and believing their lived experience. By integrating insights from identity, confidence, and belonging, mathematics education research can advocate for changes that recognize the socio-cultural dynamics of learning and strive to dismantle structures that perpetuate exclusion and marginalization. A critically informed approach to systemic change in mathematics education thus positions identity and belonging as central components of student success, urging departments to reframe their strategies for fostering equitable, inclusive learning environments.

Methods

SWU is a large, public 4-year HSI research university. Administration of a survey resulted in 807 responses across two semesters of students enrolled in Precalculus and Calculus. The focus of this study are 60 total students (gender and race/ethnicity shown in Table 1) who opted-in to enroll in the newly developed Precalculus or Calculus I SI-based support course that also responded to a survey sent at the end of spring 2023 and fall 2023 semesters. While the survey collected demographic data beyond two genders, there are only two present in this subset of the sample population. Of these 60 students, 41 were enrolled in Precalculus and 19 were enrolled in Calculus and their respective support courses. The remaining data of 747 students were not enrolled in the support course with 367 enrolled in Precalculus and 380 enrolled in Calculus.

Table 1. Students self-reported identity. This table shows a count of each gender present by race/ethnicity.

Support Course Student Demographics									
	Alaskan Native or Native American	Black or African American	Hispanic/ Latinx	Middle Eastern or North African	East Asian	White	Native Hawaiian or Pacific Islander	South Asian	Southeast Asian
Man	1	1	14	1	3	11	1	0	6
Woman	0	2	5	0	1	13	0	1	4

Ten of these sixty students participated in a one-hour long, semi-structured individual interview that focused on a variety of topics related to their experiences in their current mathematics courses as well as their own histories of experiences with mathematics in general. Topics that were discussed included how their identity has impacted their learning of mathematics, how their confidence in their mathematics ability has changed, and the extent to which they have utilized university-offered support structures such as SI.

In order to develop a broader sense of experiences across all precalculus and calculus courses, the administered survey was an amalgamation of items built from subsets of two existing surveys. One of these surveys is the student postsecondary instructional practice survey (SPIPS-M), which specifically focuses on students' interpretations of instructional practices, changes in attitudes towards learning and doing mathematics, and their perceptions of the climate in the classroom (Apkarian et al., 2019). Supplemental to the focus of this survey, the second

survey developed by Brown and colleagues at Penn State University called the Inclusive Instructor Behaviors (IIB) survey was included to leverage student experience to determine various aspects of the perceived inclusivity of teaching behaviors and sense of belonging in mathematics.

Qualitative analysis of the interviews as well as free-response items from the survey was performed using MaxQDA, through multiple cycles of open coding, axial coding, and constant comparison. Quantitative analysis of the survey was performed using SPSS and R, utilizing methods appropriate for the large amount of ordinal data from the dataset and the large population size difference across support course enrollment. These methods include two-sample proportions tests, chi-squared tests, Wilcoxon Signed-Rank tests, and Mann-Whitney U tests.

Results

The results of this analysis are a subset of a larger study, my dissertation that is in publication. This paper contains selected results with an attempt to deliver important aspects of student experiences while still capturing the larger scope of the dissertation. I first present select quotes from student interviews specifically focusing on instructional practices that improved confidence and mediated aspects of their identity they found to be a barrier to their success in undergraduate mathematics. Following these qualitative results, I present select quantitative results that focus on shifts in confidence towards mathematics and sense of belonging compared across students who enrolled in the support course and those who did not.

Qualitative Results

Interviewed students shared their experiences with their perception of instructional practices that they perceived to have helped them due to their self-reported intersectional identities of women in STEM, English Language Learners (ELL), or first-generation college students. While the scope of these quotes reaches themes beyond their interactions with solely the support course and the SI leader, the comfort students found in engagements with all of their instructors (Lecturers / Professors, TA's, and SI leaders) indicates a broader effect of availability, approachability, and words of encouragement on students' confidence and sense of belonging.

Amy, a first-generation Asian American woman in STEM, shared about her experience with her TA for the recitation course for Calculus I. When prompted about instructional practices that assisted the barriers they found to succeeding in undergraduate mathematics, Amy said,

> One of the things that really helped me was my TA. He was always there during office hours, and he explained things in a way that made sense to me. He never made me feel stupid for asking questions, and he encouraged me to keep trying even when I was struggling. It was a huge boost to my confidence.

The encouraging interactions and support from Amy's TA indicate the significant role of personalized interactions in fostering a positive learning environment. She also spoke about impactful instructional practices of the professor for the lecture section:

> In one of my classes, the professor always made sure everyone had a chance to speak. He would go around the room and ask each person for their input. It made me feel valued and like my opinion mattered. It was a stark contrast to some of my other experiences, and it really changed how I viewed myself in the classroom.

Facilitating an inclusive classroom environment where the instructor actively seeks input from all students helped Amy combat feelings of marginalization. For her, this practice validated her contributions and fostered a sense of belonging, positively influencing her identity by changing her self-perception and her place in the math classroom.

Belinda, a first-generation Latina in STEM, spoke about group work and the positive experiences she had from attending SI due to her enrollment in the support course. When asked about instructional practices that mediated aspects of her identity and alleviated barriers she faced, she said,

> College group work is still a challenge, but having more support from SI sessions helps. It makes collaboration more effective. In SI, we often work in small groups to solve problems together, and the leader encourages us to explain our thinking, which helps me practice my English and math skills at the same time … the supportive environment in SI sessions makes me feel more comfortable asking questions and participating… and this has made a huge difference in my overall confidence and willingness to engage in class.

Belinda's experiences underscore the profound impact that specific instructional practices within SI sessions can have on a student's identity and confidence in math. The step-by-step problem breakdowns, small group dynamics, encouragement of peer explanations, and supportive, interactive environment provided in SI sessions have transformed Belinda's relationship with math in the classroom. These practices have helped her overcome language barriers, reduce feelings of isolation, and build her confidence and competence in both math and English.

Collectively, the instructional practices identified to help mediate identity-related barriers to engagement and increased self-confidence in women and first-generation students enrolled in the support course should appear familiar to an educator focused on equitable experiences for their students, as noted by multiple pedagogical frameworks in the field of mathematics education. For example, inquiry-based mathematics education promotes instruction that emphasize a focus on equity and engagement in one of its four pillars of implementation.

Quantitative Results

This section focuses on comparing aspects of student success across support course enrollment. First, I present the results from typical metrics such as grades and math course persistence. Then, I present results from a broader conceptualization of student success that includes shifts in student confidence as well as sense of belonging.

One metric that universities and math departments typically investigate is the DFW rate, which is the percentage of students who received a D, F, or WU in the course. For students enrolled in the support course and not enrolled in the support course, the DFW rates are 17.24% and 19.36%, respectively. A two-sample proportions Z-test was conducted to determine if there is a significant difference between these two DFW rates and the results indicate there is no statistically significant difference between them ($Z = -.401$, $p = 0.688$). A chi-square test of independence was conducted to statistically compare the distributions of all grades across both groups. The results revealed a significant difference in the grade distributions between the two groups, $\chi^2\ (5) = 14.102$, $p = 0.015$. This indicates that the distribution of grades for students enrolled in the support course differs significantly from those not enrolled, suggesting an association between support course enrollment and student grades. Further investigation revealed that this difference favored students in the support course slightly, with students in the support course more likely to receive a B and less likely to receive a D compared to their peers.

Next, I present three categories of persistence in students' math course plan: students who plan to take the next logical course, students who plan to take the same course, and students who do not plan to enroll in another math course. I conducted a series of chi-square tests of independence to compare the persistence rates between students enrolled in the support course and those not enrolled, across the three categories of interest. The results showed no statistically significant differences in the rates of students planning to enroll in the next logical course (χ^2 =

0.026, $p = 0.871$), retake the same course ($\chi^2 = 2.010$, $p = 0.156$), or not enroll in another math course ($\chi^2 = 0.029$, $p = 0.864$). Although some of the observed differences in percentages were notable, with 55.93% of support course students and 39.10% of non-support course students planning to enroll in the next logical course, the chi-squared tests indicate that these differences could be due to chance. Similarly, the rates for retaking the same course (13.56% vs. 2.06%) and not enrolling in another math course (1.69% vs. 1.51%) did not show significant differences, suggesting that the variations in observed counts are not statistically significant.

Moving beyond grades and course persistence, students were also asked survey questions regarding their shifting confidence in mathematics from the beginning of the semester versus near the end of the semester, as well as a group of items that targeted sense of belonging in mathematics spaces.

First, I ran a Wilcoxon Signed-Rank test to compare the paired ordinal data to determine if there is a significant difference in the median of the differences in confidence reported by students. This test considers the magnitude and direction of the changes in confidence levels for each individual student, providing an overall look at how positive and negative shifts in each group are distributed. This test revealed that while the shifts in confidence levels overall for students not enrolled in the support course were statistically significant ($W = 36103.0$, $p < 0.0001$), the difference in confidence levels overall for students enrolled in the support course did not show statistical significance ($W = 362.5$, $p = 0.166$). However, there is more detail I can extract from the data. I calculated confidence shift categories to indicate how many students were in each category of a positive shift in confidence, a maintained confidence, and a negative shift in confidence.

A chi-squared test was performed across the three confidence shift categories and revealed that there is a statistically significant association between support course enrollment and the confidence shift categories ($\chi^2(2) = 11.779$, $p = 0.00277$). To investigate the detail of these differences, a series of two-sample proportions tests that compared the proportion of students in each confidence shift category across support course enrollment showed that there was no significant difference in negative shifts ($Z = -1.054$, $p = 0.292$), a significant difference in maintained confidence ($Z = -2.127$, $p = 0.033$), and a highly significant difference in positive shifts ($Z = 3.404$, $p < 0.001$).

The negative Z-score from the difference in the number of students who maintained confidence indicates that a larger proportion of students not enrolled in the support course showed maintained confidence. The positive z-statistic from the difference of the proportion of students who showed a positive shift in confidence indicates that a larger proportion of students enrolled in the support course showed a positive shift in confidence. That is, not only is there no statistical evidence that students in the support course lost confidence more than students not enrolled, but that students in the support course reported they gained confidence significantly more than students not enrolled in the support course.

I performed one final analysis to determine the effect size of these changes with 95% confidence intervals, to add a finer detail to the results from the two-sample proportions tests. For effect sizes in differences in proportion, Cohen's h was calculated for each test result. The effect size of the differences from each category across support course enrollment are a negative shift in confidence with $h = -0.147$, 95% CI [-0.263, -0.031], a maintained confidence with $h = -0.299$, 95% CI [-0.418, -0.180], and positive shift in confidence with $h = 0.434$, (95% CI [0.302, 0.565]. Interpreting these effect sizes based on the standard scale, the negative shift difference of $h = -0.147$ is approaching a small but non-trivial effect but likely not practically significant, the

maintained confidence difference of $h = -0.299$ falls between a small and medium effect size, and the positive confidence difference of $h = 0.434$ is approaching a medium effect size suggesting a moderate and practically significant effect. Each of these effect sizes are in the direction that favors students enrolled in the support course. Since none of the confidence intervals contain zero, these effect sizes are statistically significant.

To investigate sense of belonging, factor scores based on the PCA loadings for the sense of belong items group were computed. These factor scores represent each student's overall sense of belonging captured by combined items that target sense of belonging. The median factor score for students enrolled in the support course is 0.377 (IQR = 2.456), while the median factor score for students no enrolled was -0.430 (IQR = 2.467).

To investigate the nature of the differences between these two groups, I used a Mann-Whitney U test to compare the combined sense of belonging items across support course enrollment. The results indicated a significant difference in factor scores between the two groups ($U = 36933.0$, $p = 0.011$), suggesting that students in the support course reported a higher sense of belonging compared to their peers. The effect size of this difference is computed to be 0.599 (95% CI: [0.524, 0.674]), which indicates a moderate effect size in favor of the students enrolled in the support course. Thus, students in the support course reported a higher sense of belonging in mathematical spaces.

Discussion

While students enrolled in the support course did not show many notable significant differences in grades and mathematics course persistence, they did report higher levels of confidence and sense of belonging in mathematics. One interpretation of the effect of the support course is that students who opted-in to a course for more support may have had a difficult time succeeding without it. That is, the experiences from being enrolled in the support course leveled the playing field to the point where there are only marginal statistical differences between their performance and success in terms of grades and course persistence.

With grades and course persistence relatively equal, with some results leaning towards a benefit for support course students, expanding the conception of student success to include affective measures such as confidence and sense of belonging illuminates the quantitative difference of experiences in mathematics between these groups. With a significantly greater positive shift in confidence and greater self-reported sense of belonging, students in the support course have "succeeded" beyond the scope of this one course. While grades and course persistence may often live in a static conception of one's mind as they navigate the myriad obstacles faced throughout the college mathematics experience, growing confidence and sense of belonging intertwine with the dynamics of identity and sense of self that endure beyond the semester or academic year context. Furthermore, the conversations I had with students about their evolving identities in mathematics were inspirational and highlight the impact humanistic and holistic experiences have on students' mathematical success.

Relating to the broader field of systemic change research, I contend researchers ought to broaden the conception of student success and listen to students' lived experience. I look forward to explaining any details or information omitted from this report and speaking with colleagues about how a support course may fit into their context.

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Reasoning About Linear Combinations and Span in a Digital Game

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In this paper we analyze the reasoning of five students who were playing a 3-dimensional digital video game called Vector Unknown: Echelon Seas. The intention of Stage 4 of the video game is to give students experience with Numeric and Geometric representations of linear combinations of vectors in the context of a pirate throwing a grappling hook to reach an anchor. We illustrate three types of Numeric reasoning, and four types of Geometric reasoning students used to solve these puzzles. For each reasoning example, we discuss issues related to which vectors the player should choose so that the anchor is within the span of the vectors, and which scalars the player should choose so that the grappling hook ends precisely at the anchor.

Keywords: linear algebra, digital games, linear combinations, span

Introduction

The purpose of this paper is to characterize the reasoning of students playing a digital video game, Vector Unknown: Echelon Seas (VUES), created to engage students with *Numeric* and *Geometric* instantiations of linear combinations of one or two vectors in R^3. The larger goal of this analysis is (1) to add to the body of knowledge of the reasoning that is involved in understanding linear combinations of vectors and (2) to provide foundational information for future creators of digital environments and tasks for the teaching and learning of linear algebra concepts involving linear combinations of vectors.

Literature Review

Literature on student understanding of linear combinations of vectors and related concepts such as span and linear independence has taken several different forms. Some papers focus on applied examples created by the research team to promote student learning (e.g., Trigueros & Possani, 2013; Cáracamo et al., 2017). Other papers consider student generated applied settings or everyday examples for basis and hence how linear combinations of vectors serve to generate or describe other vectors in the space (e.g., Adiredja et al., 2020; Zandieh et al., 2019). Other settings have been created by instructors to have students engage with linear combinations of vectors as a way to travel through space (e.g., Wawro et al., 2012; Trigueros, 2019) or to explore locations in space using Dynamic Geometry Systems (e.g., Dogan, 2019; Turgut et al., 2022). More recently there are a few examples of digital games created to engage students in learning aspects of linear combinations of vectors (e.g., Nishizawa et al., 2013; Mauntel et al., 2021).

This paper adds to this body of literature by characterizing specific types of reasoning that are involved in students' game play and connecting those types of reasoning explicitly to aspects understanding linear combinations – even when the students did not themselves make these connections. In this way, we are suggesting ways that a game such as this may be leveraged by an instructor to help students build a more nuanced, experiential understanding of linear combinations and related concepts such as the span of a set of vectors.

To this end, our Research Question for this paper is: What are the ways that students reason with linear combinations of vectors in the context of playing the 3-dimensional (3D) grappling hook puzzles of Vector Unknown: Echelon Seas (VUES)?

Methods and Theoretical Framework

Context of the Game

Our work follows the philosophy of Freudenthal (1983) that mathematics is a human activity. In this way we are interested in how students engage in tasks and the ways that their reasoning emerges through this engagement. The tasks for this study are solving the puzzles in the digital game VUES that involve a pirate using a grappling hook in a 3D environment. The goal of each of these puzzles is for the pirate to send the grappling hook from his location to the anchor. This corresponds to creating a linear combination that equals the numeric vector goal in the upper right corner of the screen. Figure 1 shows a screenshot of the VUES 3D grappling hook environment. For this paper, we have added to Figure 1 black boxes outlined in purple with the names that we will use to refer to various game elements.

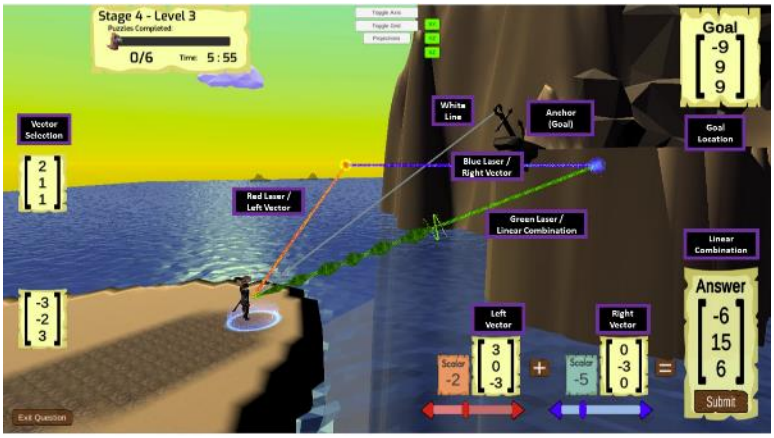


Figure 1: Illustration of Level 3 of Stage 4 (S4L3) of Vector Unknown: Echelon Seas.

As seen in Figure 1, the game provides both *Numeric* and *Geometric* information about the linear combinations of vectors. In addition, there are two main choices that the player can make in solving the Puzzle. The player can (1) select one or two vectors to move into the Left Vector and/or Right Vector slots and can (2) move the sliders underneath the Left or Right Vectors in order to change the scalar in front of the vector. Making these changes will automatically update the Answer vector and cause the Red, Blue, and/or Green Lasers to appear. We refer to these choices as *Which Vectors* and *Which Scalars*, respectively.

Data Collection and Data Analysis

To obtain information towards answering our research question, we conducted interviews with five students as they engaged in playing VUES. The fourth author interviewed each of the five students individually for at least one hour. The five students each chose their own pseudonym: Anh, Athena, Kyah, Patchy, TB. The intention of the interviews was to allow students to explore Stage 4 of VUES in which the pirate character's goal is to extend a grappling hook to connect to an anchor. For this paper we focus on the 3D level (Stage 4 Level 3, S4L3).

The video data for each student playing S4L3 was analyzed by at least two members of the author team and then discussed across team members. The S4L3 data consisted of three Puzzles

completed by Kyah and six Puzzles completed by each of the other students. An initial analysis of the data focused on *Numeric* and *Geometric* reasoning and the gameplay strategies highlighted in Mauntel et al. (2021). These strategies overlap issues of *Which Vectors* and *Which Scalars*. As pointed out by Bernier and Zandieh (2024), some of the Mauntel et al (2021) strategies focus more on the choice of vector and others focus more on the choice of scalar. For this paper we emphasize issues of *Which Vectors* and *Which Scalars* and extend these types of analyses to a 3D setting. This extension also allows us to draw out additional connections possible between digital gameplay and reasoning with linear combinations of vectors.

Conceptual Framework

Mathematically, we can interpret *Which Vectors* as referring to which vectors should be selected so that the Goal will be in the span of those two vectors, and once selected, *Which Scalars* refers to which scalars should be used as the weight of each vector in the linear combination to create the Goal vector. In this way, the questions of *Which Vectors* and *Which Scalars* point to (1) game mechanics (allowable moves a player can make in the game), (2) statements about mathematical objects and relationships (at least to an expert) and (3) in-the-moment choices a player can make in terms of their focus or attention. Our conceptual framework interprets student reasoning with linear combinations of vectors through the choices students make in the digital game (*Which Vectors* and *Which Scalars*) and their focus of attention (*Numeric* or *Geometric*) in anticipating or interpreting the results of these choices.

Results

In analyzing our data, we found multiple ways that students reasoned about linear combinations in the digital game setting. Here we highlight three *Numeric* examples and four *Geometric* examples that illustrate different ways students navigated deciding *Which Vectors* and *Which Scalars* would solve the puzzle. The examples are chosen to illuminate variety in student reasoning as well as mathematical connections across these differences.

***Numeric* Reasoning: Two 0 Entries**

We saw multiple instances of students trying to initially align one or two entries in the Answer vector (result of the linear combination) with the matching entries in the Goal vector. In this type of reasoning a student focused on initially aligning two entries and then adjusting to align the third entry as well. The more straightforward version of this reasoning occurred when the Vector Selection included a vector with two 0 entries. Each of the four students who had a puzzle including a vector with two 0 entries were able to solve such a puzzle numerically.

Anh prioritized *Numeric* aspects of the game, including leveraging 0 entries, when solving most of the puzzles of Level 2, and this continued in Puzzle 1 of Level 3 (L3P1; see Figure 2). In terms of *Which Vector*, Anh noticed that three of his vector cards had zero-valued entries saying, "Yeah, I'm thinking which one I can use because we have a lot of zeros here." To try and match the Goal vector, Anh selected <-3, -2, 3> and increased its scalar to 3 (*Which Scalar*) to match the x and z entries of the Goal vector (Figure 2(a)). After aligning these two entries, Anh used the <0, 3, 0> vector to change the y value of the Answer vector to match the y value of the Goal vector, as shown in Figure 2(b). Note that although Anh was focused on the *Numeric* features of the game, the *Geometric* effect of his gameplay was to position the tip of the Red Laser at the same height (z-value) as the Goal (Anchor) and then use the Blue laser parallel to the y-direction to reach the Goal (See Figure 2).



Figure 2: (a) Anh's use of the <-3, 2, 3> vector to match the x and z entries of the Goal. (b) Anh's use of the <0, 3, 0> vector to match the y entry of the Goal.

Numeric Reasoning: No 0 Entries

Athena's first two Puzzles of Level 3 each had a vector with two 0 entries, and she was able to solve them by reasoning in a *Numeric* way similar to Anh. However, in L3P3, Athena did not have any vectors with 0 entries. In terms of *Which Vector*, Athena initially used a single vector with a scalar of -4 (*Which Scalar*) to match two of the entries of the Goal vector (see Figure 3(a)). Eventually, she chose a second vector with $x = 2$ and $y = 3$ (*Which Vector*) which allowed her to match the x and y entries of the Goal vector, i.e., $x = -8$ and $y = -12$ (See Figure 3(b)).



Figure 3: (a) Athena's attempt to match the x and y entries with one vector. (b) Athena's initial Answer vector before she begins to incrementally decrease z by 1.

With support from the interviewer, Athena alternated between increasing the left scalar and decreasing the right scalar to decrease the z value of the Answer vector. Athena continued to alter the scalars in this manner until she finally matched the z entry of the Goal vector, decreasing the z value from 2 all the way to -10, as shown in Figure 4(a-c).

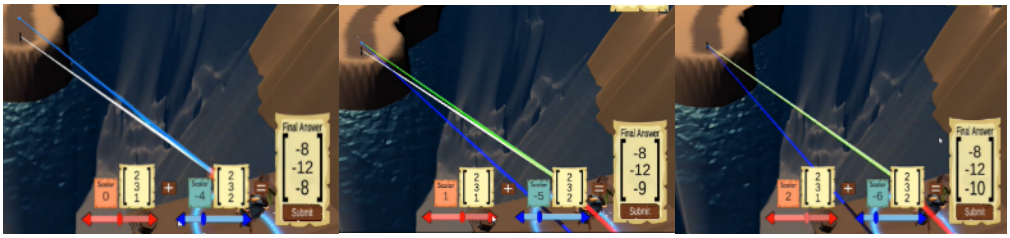


Figure 4: Examples of Athena's incrementing of the Left and Right scalars to decrease the **z***-entry.*

Athena noticed that in changing the scalars this way, "these two [x and y entries] don't change every time, but this one does." Note that in this example, Athena chose the vectors <2,3,1> and <2,3,2>, so her strategy of adding 1 to the left scalar and subtracting 1 from the right scalar is equivalent to adding (1)*<2,3,1> + (-1)<2,3,2> = <0,0,-1> to the resultant vector. This allowed her to keep the first two entries of her resultant vector -8 and -12 while she adjusted the

third entry from 2 to the *Numeric* Goal of -10. Although neither she nor the interviewer mentioned it, the *Geometric* effect of adding <0,0,-1> repeatedly was to move the endpoint of the resultant vector down vertically until it reaches the anchor (see Figure 4).

Geometric Reasoning: Green Laser onto White Line

Patchy was one of the few students who used primarily *Geometric* reasoning about the linear combination in the first two puzzles of L3. This was because a Microsoft Teams window covered Patchy's Goal vector for most of his interview. At the start of his L3 gameplay, Patchy spent about nine minutes making sense of the lasers and vector cards. Once he noticed that his Answer vector (see Figure 5(a)) resulted in the Green Laser landing on the White Line, he said "we're onto something" multiple times. Because the Green Laser landed halfway along on the White Line Patchy said, "it was like halfway there, so if like double the scaling, that should do it." Patchy increased the Left scalar to 4 and the Right scalar to 2, reaching the Anchor. Patchy answered the questions *Which Vector* and *Which Scalar* almost simultaneously. Once the Green Laser landed on the White Line, he knew he could reach the Anchor by doubling the length of the Green Laser and scaling the Answer vector <-4, 5, 3> by a factor of 2, (see Figure 5(b)).

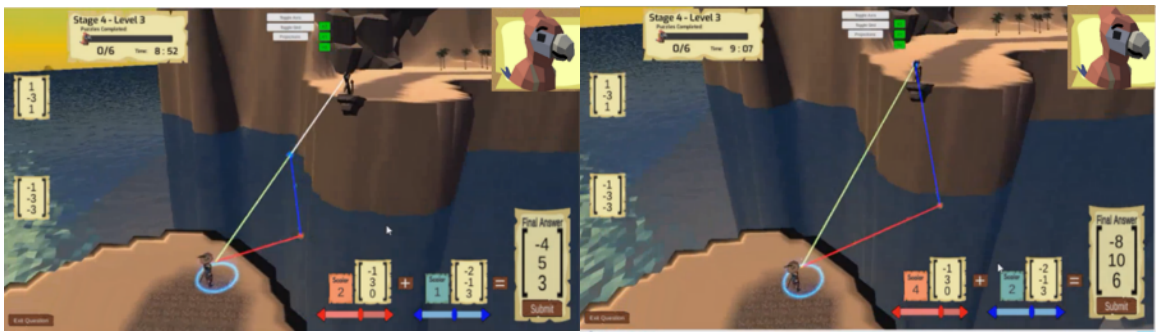


Figure 5: (a) Patchy noticed the Green Laser on the White Line, halfway to the Anchor. (b) Patchy doubled the value of each scalar to double the length of the Green Laser and reach the anchor.

Numeric Reasoning: Scalar Multiples

Previously Patchy used *Geometric* reasoning about the linear combinations because an application was covering the *Numeric* Goal vector on his computer screen. After recognizing that he could now see the *Numeric* Goal vector, he said "Oh! I haven't been seeing the Goal! This changes everything!" Patchy then said, "You can now just do it by addition." He then selected the <-1,1,-3> and <-1,-3,1> cards, and noticed that the *Numeric* Goal vector was a scalar multiple of the Answer vector, e.g., <-2, -2, -2> (see Figure 6(a)). Patchy said he wanted to "scale it up," and increased the left and right scalars each by 1 several times until he matched the Goal vector <-10, -10, -10>. Visually, Patchy was continually moving the Green Laser off and back onto the White Line each time his Answer vector resulted in a scalar multiple of <-2, -2, -2>. However, in this puzzle, Patchy was focused on making the Answer vector a scalar multiple of <-2, -2, -2> (see Figure 6(b)). Patchy's *Numeric* reasoning here can be seen as related to his earlier *Geometric* reasoning where he focused on scaling the length of the Green Laser.

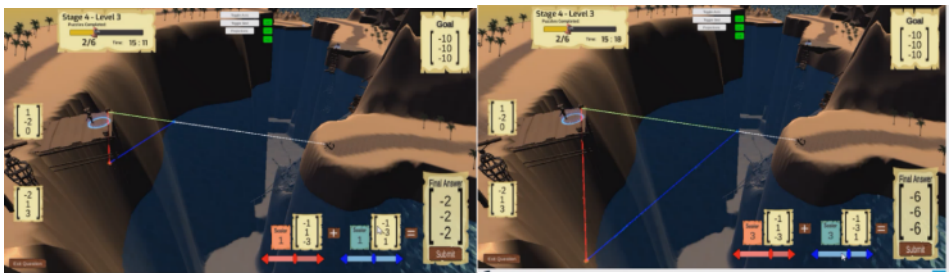


Figure 6: (a) Patchy's Answer vector <-2, -2, -2>. (b) Patchy's scalar multiple of the <-2, -2, -2> vector.

Geometric Reasoning: White Line is Not in the Span

In L3P1, Kyah moved the Green Laser onto the White Line for a fixed camera view as shown in Figure 7(a). Notice that even though she moved the Green Laser onto the White Line, her Answer vector does not match the Goal vector. After moving the camera to the view pictured in Figure 7(b), Kyah noticed that the Green Laser was not, in fact, on the White Line.

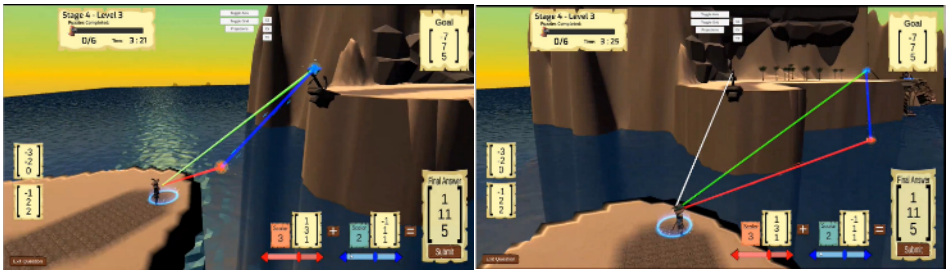


Figure 7: (a) Kyah's sees Green is on White. (b) Kyah realizes Green is not on White.

Kyah continued to leverage various camera views to determine whether the Green Laser was actually on the White Line, which helped her answer the *Which Vectors* question. From an expert point of view, this *Geometric* reasoning could be used to determine if the Anchor lies in the span of the selected vectors.

Geometric Reasoning: White Line *is* in the Span

In L3P4, TB began to answer the question *Which Vectors* using different camera views to consider the relationship between the Lasers and the White Line. As shown in Figure 8(a), TB used a top-down camera view to demonstrate how the vectors <-3, 3, -1> and <-2, 2, 2> overlapped the White Line. From an expert point of view, this *Geometric* reasoning can be seen as demonstrating that the Anchor location, as well as the White Line, lie within the span of the selected vectors. In Figure 8(a) the Green Laser appears to be on the White Line but it is actually above it. TB changed the camera view to the perspective pictured in Figure 8(b) and decreased the right scalar to move the Green Laser through the Anchor.

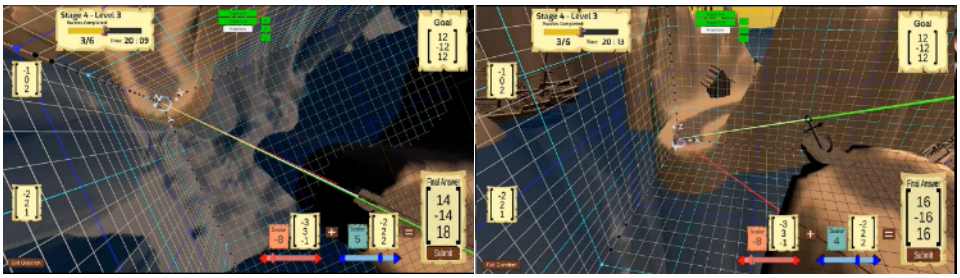


Figure 8: (a) TB's top-down view of Green above White. (b) TB's side camera view of Green through the Anchor.

Geometric Reasoning: Sweeping the Scalars

Earlier in his L3P4 gameplay, TB became convinced that the Lasers could be used to determine if a specific pair of vectors could be used to reach the Anchor. TB began L3P4 by continually increasing and decreasing the Left and Right scalars to determine possible places the Lasers could reach. He eventually turned the camera to the view pictured in Figure 9, noting that the lasers were at "a fixed angle," which would never move towards the anchor location. The narrow angle of the Lasers TB used to geometrically reason about the possible locations that could be reach by his selected pair of vectors allowed TB to consider a wedge of the plane spanned by his selected vectors. Though TB did not mention this plane, as experts, we recognize this type of *Geometric* reasoning about the linear combinations as nascent reasoning related to demonstrating that the Anchor is not in the span of TB's selected vectors, and therefore cannot be reached no matter how TB sweeps the scalars. This type of *Geometric* reasoning about the Lasers allowed TB to answer the question *Which Vectors* in the sense he recognized he needed to swap vector cards. TB summarized this by saying, "So using the line I can definitely see this combination of cards is never going to reach the right answer.

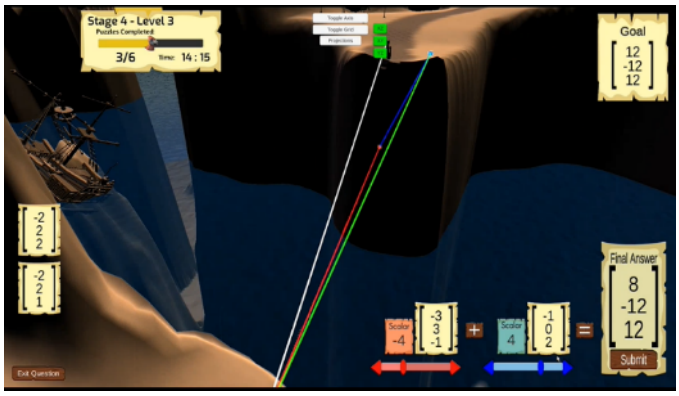


Figure 9: TB's camera view of the three Lasers to demonstrate why his selected vectors had the wrong angle and could therefore not reach the Anchor.

Discussion and Conclusion

In this paper we presented several ways (three *Numeric* and four *Geometric*) that students reasoned about linear combinations of vectors while playing S4L3 of VUES. These ways of reasoning about linear combinations were compatible with reasoning about span (*Which Vectors*) and reasoning about individual linear combinations for a specific resultant vector (*Which Scalars*) across both *Numeric* and *Geometric* conceptualizations of linear combinations.

We see this analysis as highlighting reasoning with linear combinations that adds to the body of literature in this area. In terms of *Numeric* reasoning, we see students working to coordinate three different but interconnected calculations (sum of products) to make all three numeric values align. Students must also consider issues related to orthogonal projection to consider situations in which two coordinates align and the third entry needs to be brought into alignment.

In terms of *Geometric* reasoning, we highlight the visual and experiential nature of interacting with a character that is moving in a digital 3D space and using changes in camera angles to gain deeper insight into the relationship between vectors and linear combinations in this space. We see students having a sense for what portion of R^3 is spanned by a set of linearly independent vectors and how one vector may lie within or outside of that. We also see students using both *Numeric* and *Geometric* reasoning that illustrate that if a vector v is in the span of a set of vectors (i.e., in a subspace) that a multiple of v will be in the span (subspace) as well.

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Computer-Verified Linear Algebra Proofs With Programming in MATLAB

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Research on teaching and learning proofs in linear algebra is scarce. This paper used the instrumentation genesis approach to examine students' proofs and MATLAB codes. The students were studying a proof-based second course in linear algebra. The instructor included application-focused labs and a final project as part of this study. One of the labs required a proof validation using MATLAB. This lab, a proof question from the midterm exam, and an exit survey were analyzed in this paper.

Keywords: linear algebra, proof, programming, MATLAB

Literature Review

Proof is a critical component of mathematics. In light of the current technological advances, has the way mathematicians do proof changed? According to Hanna, Reid, and de Villiers (2019, p. 3):

> Proving is sometimes thought to be the aspect of mathematical activity most resistant to the influence of technological change. While computational methods are well known to have a huge importance in applied mathematics, there is a perception that mathematicians seeking to derive new results are unaffected by the digital era. The reality is quite different. Digital technologies are influencing the way mathematicians work together and the way they go about proving.

In an article by Bayer et al. (2024), ten authors from various stages of academic careers individually expressed their views on proof within eight segments of the article. The article began by stating that:

> There is a striking difference between how a proof is defined in theory and how it is used in practice. This puts the unique status of mathematics as exact science into peril. Now may be the time to reconcile theory and practice, i.e., precision and intuition, through the advent of computer proof assistants. This used to be a topic for experts in specialized communities. However mathematical proofs have become increasingly sophisticated, stretching the boundaries of what is humanly comprehensible, so that leading mathematicians have asked for formal verification of their proofs. At the same time, major theorems in mathematics have recently been computer-verified by people outside of these communities, even by beginning students (p. 79).

The article "investigates the different definitions of a proof, the gap between them, and possibilities to build bridges. It is written as a polemic or a collage by different members of the communities in mathematics and computer science at different stages of their careers, challenging well-known preconceptions and exploring new perspectives" (p. 79). As Tall (2023) mentioned, different communities have their own needs and view things (in this case, proof) differently.

Shifting our attention to pedagogy and classroom practices, Melhuish et al. (2022) synthesized 104 papers from various countries and methods to describe what is known about

proof-based learning and what is still missing. Their exploration uncovered two schools of thought for teaching proofs: lecture-based proofs and student-centered learning. While student-based learning is promising, a significant gap remains in "understanding of the theoretical mechanisms which specific instructor moves may encourage or scaffold student activity is only beginning to emerge" (Melhuish et al., 2022, p. 17). Acknowledging the wealth of studies in proof in mathematics education literature, we focus our attention on linear algebra proofs in this paper.

Linear algebra is a key topic for many mathematics majors and other fields. In the early 1990s, Steven Leon and his colleagues initiated a national project (ATLAST) to teach linear algebra with MATLAB (Leon, Herman, & Faulkenberry, 2002). Despite these national initiatives, research studies on linear algebra and using MATLAB in classrooms are lacking. The Linear Algebra Curriculum Study Group (LACSG) 2.0 recommends that mathematics departments offer a variety of second courses, including numerical linear algebra, and include wider topics (Stewart et al., 2022). The recommendations also paid significant attention to the use of technology. In a survey paper by Stewart, Andrews-Larson, and Zandieh (2019), the authors found that research on topics in second courses of linear algebra, which contain more abstract content and proof, is rare. In response to this gap, in a study by Meyer, Stewart, and Madden (2024), they employed Schoenfeld's (2010) theory of Resources, Orientations, and Goals (ROGs) to examine an instructor's (and co-author's) textbook in a second course in linear algebra. The instructor created tailored resources with many hints and pedagogical attributes to encourage students to construct their own proofs to understand and learn the materials and invited students to explore, conjecture, and prove.

Our current paper examines students' instrumentation genesis using MATLAB to verify proofs, a proof question from the midterm exam, and an exit survey.

Theoretical Framework

Our theoretical framework for this study is based on the instrumentation genesis (Vérillon & Rabardel, 1995). An illustration of the instrumental genesis is shown in Figure 1. In the instrumentation approach, an instrument starts as an artifact before the subject interacts with it and only becomes an instrument "when the subject has been able to appropriate it for himself and has integrated it." (Vérillon & Rabardel 1995, p. 84). This development into an instrument is called "instrumental genesis" (Sinclair et al., 2023, p. 398), and it incorporates more than just the subject and the artifact itself. An important aspect of an instrument includes the actions taken and the goals of the subject that led to those actions (Trouche, 2018). It is significant to note that the artifact does have an impact on the subject's experience, as the subject's actions are formed largely in response to the design and limitations of the artifact and their own comprehension. The subject must impart their own knowledge of both the goal and the artifact into the development of the instrument, an aspect of the process that shapes the resulting instrument (Balacheff, 1994). In a sense, "the artifact does not intervene in a neutral way in the teaching-learning situation. It carries its own techniques, instrumental knowledge, signs, meanings, and a technological transposition of mathematical knowledge also takes place (Sinclair et al., 2023, p. 399). According to Trouche (2018, p. 407):

> This frame leads us to clarify our initial definition: instrumentation is the action by which a subject acquires an artifact, and the effect of this action on the subject, who develops, from this artifact, an instrument for performing a task. An instrument is thus made up of

an artifact component and a cognitive component (knowledge necessary for/ from using the artifact for performing this type of task).

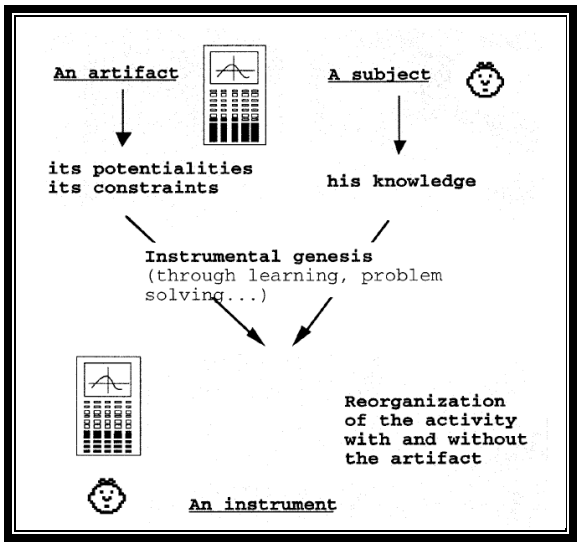


Figure 1: Instrumental genesis, from an artifact to an instrument (Guin and Trouche, 1999, p. 202).

As described by Sinclair et al. (2023), the perspective of instrumentation has been employed widely in the literature, especially with graphics calculators. Notably, Gueudet et al. (2022) studied one student's ability to use programming to explore various mathematical conjectures by examining the evolving schemes surrounding his work. Their framework sought to orient the relationship between tools and students as having an impact in both directions, with the student adapting the programming process to his needs and the student adapting to the operations of the programming language.

Our research question to guide the study was the following: What was the nature of the instrumentation of programming with MATLAB in the production of linear algebra proof?

Method

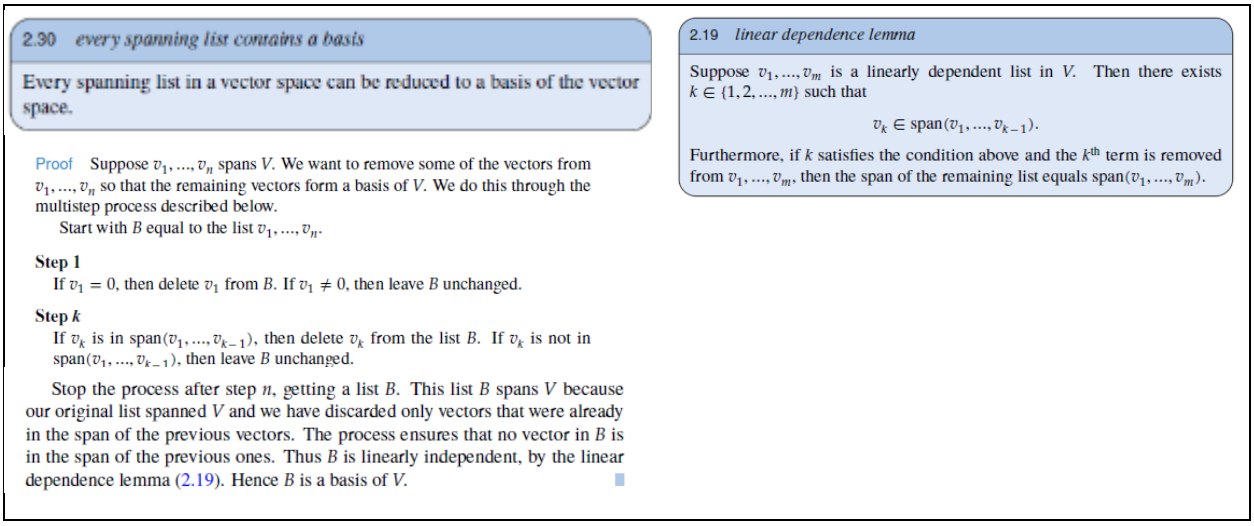
This is a case study of a proof-based second course in linear algebra. There were 22 mathematics and engineering students enrolled in this course, and 15 students agreed to participate in this study. The first-named author was teaching this second course, which was highly theoretical and proof-based, and selected the textbook *Linear Algebra Done Right* by Axler (2024) for this course. The course was modified to incorporate six labs (one on proof and five on applications) and a final project using MATLAB to help students explore numerical linear algebra and its applications via programming. The instructor (first author) met weekly with a specialist and a support person from MathWorks to create the labs and provided help with MATLAB for students. The course covered the following topics: Vector spaces and their properties (including special Vector Spaces such as Isomorphic Vector Spaces and Invertibility), subspaces, span and linear independence, bases, dimension, linear maps, polynomials, eigenvalues, eigenvectors, and invariant subspaces, inner product spaces/ operators on inner product spaces (The Spectral Theorem, Self-Adjoint, and Normal Operators, etc.), and trace and determinant. The course was taught as a mixture of lectures and various group activities, such as evaluating proofs for clarity and elegance using MATLAB in class to validate results left by the

textbook as an exercise. The class participation also included unpacking proofs in their own words and sometimes coming up with different proofs and presenting them to the class.

The data for this study was collected from students' class assessments (final exam, midterm exam, and homework), the labs, the final project, and an exit survey. This paper will only show data analysis from two students' (Student 1 and Student 11) lab 2 and a midterm test question related to the proof of Theorem 2.30. Lab 2 asked students to validate the proof of Theorems 2.19 and 2.30 from the textbook (see Figure 2 for the statement of the theorem and proof).

The exit survey questions had two parts (Proof and MATLAB). A sample of proof survey questions were: What is the purpose of the proofs in linear algebra? Describe the nature of the proofs in linear algebra. How can we best teach linear algebra proofs to enhance your learning experiences? What were the best teaching techniques presented in class that helped you learn proofs? A sample of MATLAB survey questions were: Were you familiar with MATLAB before this course? Have you done any programming before this course? Did you learn coding in this course? Did your knowledge of linear algebra increase as a result of coding?

We have included the results of whole class responses to survey questions related to MATLAB, as well as the results for Students 1 and 11 on proof questions.



2.30 *every spanning list contains a basis*

Every spanning list in a vector space can be reduced to a basis of the vector space.

Proof Suppose $v_1, \ldots, v_n$ spans V. We want to remove some of the vectors from $v_1, \ldots, v_n$ so that the remaining vectors form a basis of V. We do this through the multistep process described below.

Start with B equal to the list $v_1, \ldots, v_n$.

Step 1

If $v_1 = 0$, then delete v_1 from B. If $v_1 \neq 0$, then leave B unchanged.

Step k

If v_k is in $\text{span}(v_1, \ldots, v_{k-1})$, then delete v_k from the list B. If v_k is not in $\text{span}(v_1, \ldots, v_{k-1})$, then leave B unchanged.

Stop the process after step n, getting a list B. This list B spans V because our original list spanned V and we have discarded only vectors that were already in the span of the previous vectors. The process ensures that no vector in B is in the span of the previous ones. Thus B is linearly independent, by the linear dependence lemma (2.19). Hence B is a basis of V. ∎

2.19 *linear dependence lemma*

Suppose $v_1, \ldots, v_m$ is a linearly dependent list in V. Then there exists $k \in \{1, 2, \ldots, m\}$ such that

$$v_k \in \text{span}(v_1, \ldots, v_{k-1}).$$

Furthermore, if k satisfies the condition above and the k^{th} term is removed from $v_1, \ldots, v_m$, then the span of the remaining list equals $\text{span}(v_1, \ldots, v_m)$.

Figure 2: Axler (2024, pp. 33 & 40).

Results

Case of Student 1

The portion of code not included involves the student creating random vectors using linear combinations of basic unit vectors already in a list by multiplying them with random scalars. In this code (see Figure 3), the student starts by clearing all zero vectors from the list. Next, it is worthwhile to note the last part of this code, as the student creates a function that tests if an additional vector is in the span of a given list by comparing the rank of the list with and without the vector. Student 1 used this function in the next loop to see if the ith vector is in the span of the previous vectors. If it is, then delete that vector from the list. Thus, the resulting list is a linearly independent list of vectors and forms a basis. The only shortcoming of the code is that it does not necessarily check what the spanning space is. However, this could be considered a demonstration of Student 1's understanding that for any vector space, the number of vectors in a linearly independent list gives you the dimension.

The design of Student 1's code is also significant, as the concept of checking the *i*th vector against all previous being sufficient does directly come from the dependence lemma. In this way, Student 1 seems to have made a significant connection between their use of MATLAB as an instrument and the proofs being computer-verified.

Student 1's proof of Theorem 2.30 (see Figure 3) is similar to Axler's (2024) proof (see Figure 2). They both used the linear dependence lemma to iteratively delete linearly dependent vectors from the list until they reached a list satisfying the definition of a basis. However, Student 1 also incorporates the theorem that a linearly independent list of vectors has a length less than or equal to the length of any spanning set of vectors for that space, though this statement is not necessarily being used in the proof itself. This directly connects to their code's test of comparing the rank of a list with and without a vector appended, suggesting their instrumentation of MATLAB has impacted their understanding of the proof.

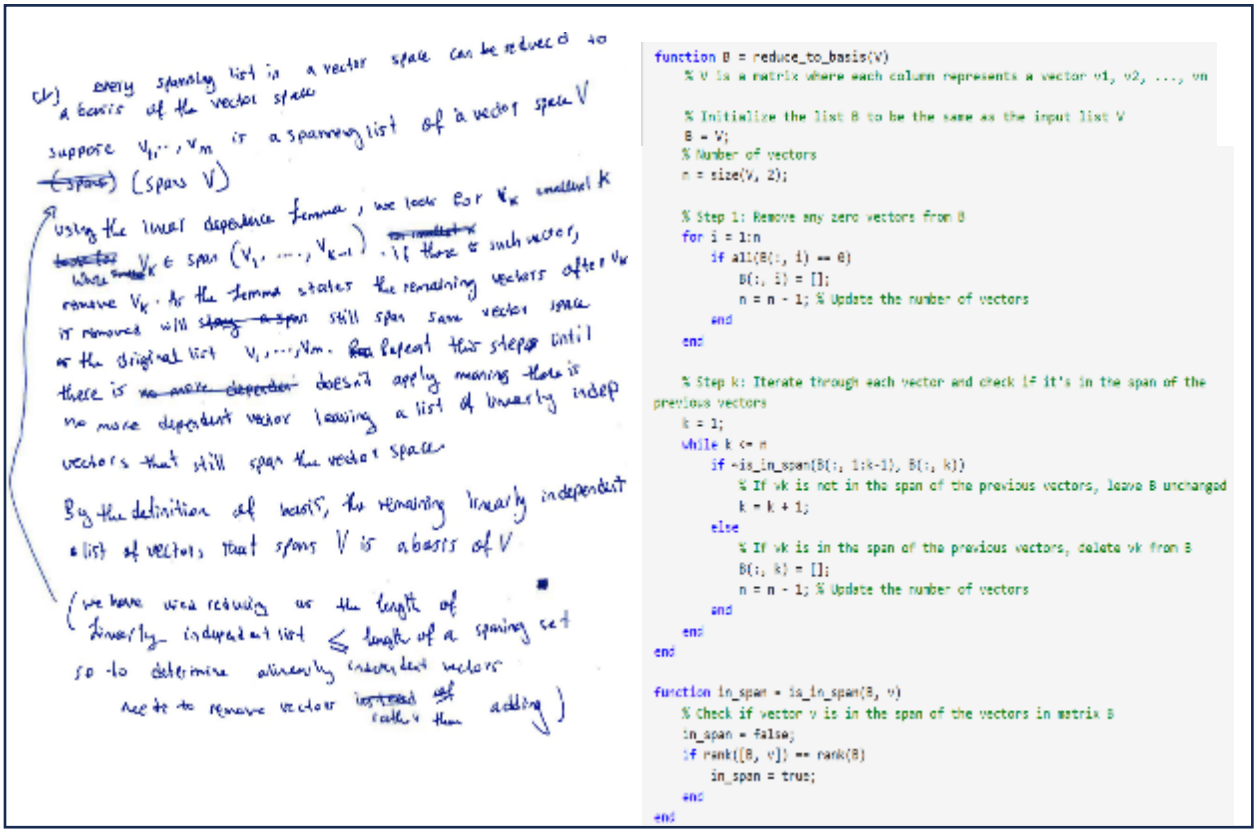


Figure 3: Student 1's proof and MATLAB code for Theorem 2.30.

Case of Student 11

While not included in this block of code (see Figure 4), Student 11 finished this exercise by verifying if the list was a basis by checking the rank against the length of the list. However, there is concern regarding the student's ability to construct vectors that meaningfully demonstrate the theorem. While generating a list of vectors, the student forced it to be linearly independent. Thus, there is not a general list of vectors being tested; there is only a list that is already going to be a basis. However, it does seem that the student has a well-developed understanding of what qualities of a list are necessary to be a basis for the space, as they verified that the length matches the rank of the list. That being said, it is not a proper test of the theorem, as they only check

whether their list has a property already built into it. It is worth pointing out that the construction of a list of vectors worth testing is an entirely distinct exercise from testing the qualities of a given list, so the student did seem to adapt the work that they had practiced and learned to MATLAB even though they were not able to properly develop a new skill in creating a sufficient list of vectors.

Student 11's proof produced in the midterm exam (see Figure 4) started with a proof of the linear dependence lemma (Theorem 2.19), which was not required as long as the result of the theorem was mentioned. In the next steps, the conditions were not set up correctly, and it is not clear what step is getting repeated and for what reason. The proof ends with the definition of linear independence without mentioning the concept of a basis.

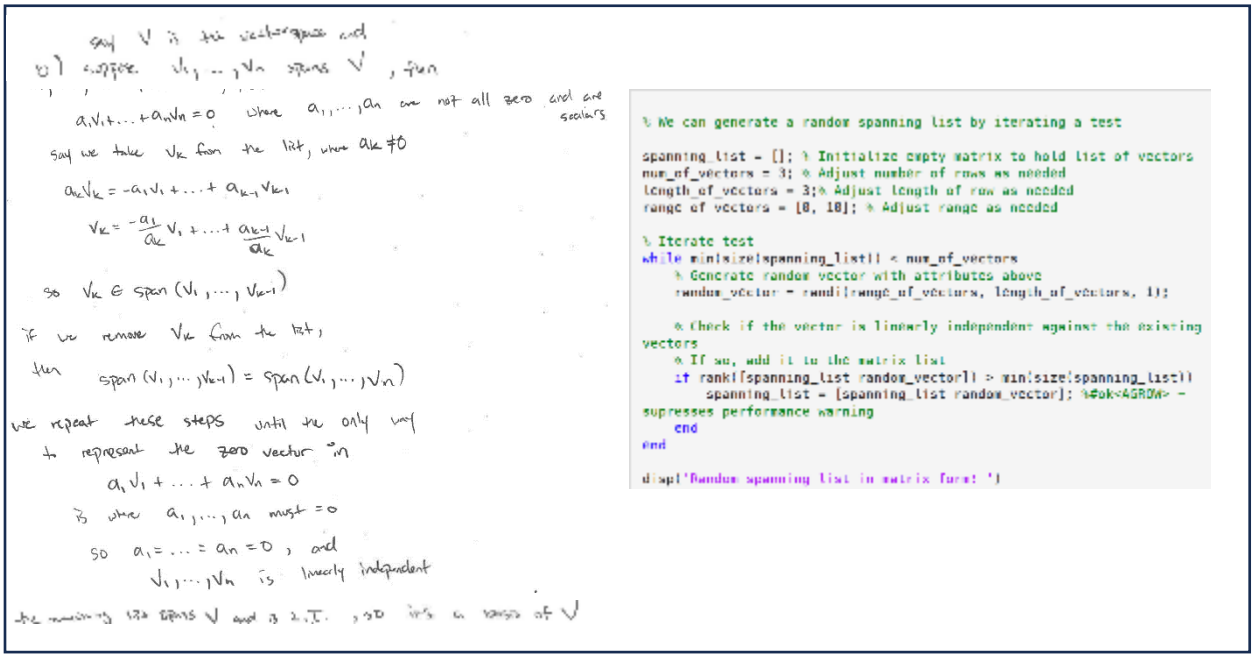


Figure 4: Student 11's proof and MATLAB code for Theorem 2.30.

Analysis of the Exit Survey

The results of the first portion of the exit survey are related to the purpose and nature of linear algebra proofs. Students 1 and 11 both referenced that proofs offer some form of verification. However, Student 11 also claimed that proofs help to understand by providing alternate ways of perceiving the concepts and equipping learners with "a skillset for applied work". Both students mentioned that proofs connect and incorporate different concepts.

Regarding teaching and how students' learning could be enhanced, Student 1 claimed that they best learned proofs by comparing different proofs. Student 11, on the other hand, suggested that they learned best by studying proofs and looking at examples of the relevant mathematics being worked out beside it.

For the best teaching techniques presented in class that helped to learn proofs, Student 1 directly referenced the labs, saying, "…the labs … showed how many of the concepts were applied. Though I enjoy theory more, I cannot deny that application further strengthens my understanding." Meanwhile, Student 11 claimed they learned best from the verbal explanations of the proofs in class.

The second section of the survey was designed to assess the students' perceptions of MATLAB and its use in the course. The results showed that four out of thirteen students were familiar with MATLAB before this course, including Students 1 and 11. Additionally, twelve students reported that they had learned coding in the course. In regards to whether or not coding helped their understanding of linear algebra, some responses were similar to one student's claim that it enabled them to "see what the theory was", as multiple students referenced some form of visualization even with labs that did not offer anything apart from symbolic results. In contrast, some students suggested it helped them learn how to apply the theorems in linear algebra. Additionally, there were a number of students claiming it helped them understand the theorems, such as Student 2's response, "Labs 2 and 6, and the final project were very important to my understanding of 2.19 [the linear dependence lemma], Gram-Schmidt, SVD, and other concepts."

Discussion and Conclusions

The results of Student 1's performance (MATLAB code and the proof of theorem) revealed that although the proof was available at the time of writing the MATLAB code, it required specific skills and expertise. For example, how to make functions, use the commands available, generate random vectors, modify lists of vectors, validate their own codes, and follow the proofs to reach a desired goal with the artifact. While the algorithmic structure of the proof made it somewhat ideal for programming, the translation of the proof into MATLAB was not trivial. As Sinclair et al. (2023) pointed out, the artifact carries its own techniques. Based on the differences between Axler's (2024) proof and Student 1's validation of it, the intervention of the artifact seems to have impacted the transposition of their proof understanding into the MATLAB environment. In the limited data analyzed here, there is an argument for some potential relationship between the quality of a student's code and their proof-writing abilities. Notably, the entire process in this particular lab was based on an abstract theorem, did not include any numerical computations, and naturally added to the complexity of the process for students. While we can conclude the results from the instrumentation process, a closer examination of students' instrumental genesis would be necessary to better understand the possible role of MATLAB in developing their understanding of the proof concept.

The results of the exit questionnaire revealed that students had a positive perception of the role MATLAB played in their learning and exposure to new experiences in an abstract course.

The authors are in the process of analyzing the remaining labs, which are primarily related to applications. Future research projects will include evaluating students' abilities to validate other linear algebra proofs with programming.

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Elaboration on Lists and Listing Processes in Combinatorics

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Lists are a foundational way to reason about and solve counting problems, yet it is unclear to what extent their role should continue for advanced learners and what that role should be. I draw on case study data of two undergraduate students who used lists repeatedly as they worked through a sequence of combinatorial tasks. I demonstrate that the students' use of lists refined over time, and their representations developed from outcome-oriented to process-oriented, thus allowing them to draw on their knowledge of lists for increasingly sophisticated problems. Further, the forms of their solutions were impacted by their use of lists. I argue that these data shed light on broader questions of the role of representations in solutions to combinatorics problems as students develop combinatorial facility.

Keywords: combinatorics, combinatorial reasoning, lists of outcomes, semiotics

Introduction and Relevant Literature

Combinatorics is the mathematics of counting and enumeration. Finding the solution to counting problems often involves reasoning about and quantifying structure in the set of outcomes being counted (Lockwood, 2014). At the same time, there are few institutional means to represent those sets of outcomes in ways that are conducive to connecting structure to expressions of cardinality (Wasserman, 2019), despite the demonstrated positive impact of using representations as tools for combinatorial reasoning (Maher & Yankelewitz, 2010; Maher et al., 2010). Listing outcomes and reasoning about how to list the outcomes has been shown to enrich student combinatorial facility across grade level (De Chenne & Lockwood, 2022; English, 1993; Halani, 2012; Lockwood & De Chenne, 2020, 2021; Lockwood & Gibson, 2016; Montenegro et al., 2021; Tillema, 2018). One drawback of listing is that it can be intractable for larger problems. Lockwood and De Chenne (2020, 2021) address this issue by having students write computer programs to list the outcomes, and then reason about their algorithm and the produced list of outcomes to find a solution to the counting problem. However, De Chenne and Lockwood (2022) showed that even when students articulated a process to list the set of outcomes in a computer program, they were sometimes unable to articulate the same structure in the produced list of outcomes. These discrepancies in student reasoning call into question the comparative roles of lists themselves (inscribed representations) and reasoning about how to list when finding solutions to counting problems.

This paper elaborates on lists and listing processes in combinatorics, and their connections to how students find solutions and what form these solutions take. I draw on case study data of a pair of undergraduate students as they worked on a sequence of counting problems intended to elicit the use of inscribed representations. I focus on these two students because they regularly engaged in listing processes across six interviews, and importantly their use of lists was often challenged by balancing tractability with providing key combinatorial insight. Thus, these data demonstrate affordances of lists and listing process while also providing insight into how they can be applied in increasingly challenging and sophisticated scenarios. I seek to answer the following research question: *what roles do lists and listing processes have as students find solutions to combinatorics problems of increasing complexity?*

Theoretical Framework

I draw from Lockwood's (2013) model of combinatorial reasoning to describe broadly how students reason about and find solutions to counting problems. I then provide a theoretical characterization of lists and listing processes in combinatorics.

Model for combinatorial reasoning, and solutions to counting problems

Lockwood's (2013) model frames combinatorial reasoning in terms of sets of outcomes, formulas and expressions, and counting processes. Sets of outcomes are the collections of objects described in counting problems, the cardinalities of which are typically the solutions. Formulas and expressions are the mathematical expressions that yield some numerical value, most frequently used to express the cardinality of the set of outcomes. Counting processes are the real or imagined processes one engages with to enumerate, or count, the outcomes. While creating this model, Lockwood observed that some students approach counting as an activity of identifying the correct problem type (e.g., arrangement with repetition or selection without repetition) for the outcomes at hand, and applying a corresponding formula/expression. When one student was unsure which problem type was correct, she would "go off [her] gut" (Lockwood, 2014). Such an approach to counting poses a challenge for continued development as it minimizes understanding of why the formula applies.

Lockwood (2014) described a set-oriented perspective on counting as one where reasoning about the set of outcomes is an intrinsic component of solving counting problems. Entailed in this perspective is that sets of outcomes can be considered to contain underlying structure, and by leveraging and quantifying this structure the counter can identify a corresponding mathematical expression that provides the cardinality of the set of outcomes. Such a solution approach may employ a combinatorial formula such as $\frac{n!}{k!(n-k)!}$, but with the understanding of how and why that formula counts the set of outcomes. Wasserman (2019) reinforces this perspective, and he additionally argues that counting experts also have a complementary perspective where a mathematical expression represents a specific counting process. That is, the form of the solution represents what outcomes were counted and how they were counted. Drawing on this perspective, I characterize finding solutions to counting problems as determining a mathematical expression with two roles: yielding an integer value of the cardinality of the set of outcomes, and representing the structure of the outcomes used to determine this value. Hence, when examining the role of lists and listing processes in student solutions I will be examining the form of the solution as a way of understanding how students leveraged structure to find a mathematical expression of cardinality.

Lists and listing processes

Drawing from De Chenne and Lockwood (2022), I define a *listing process* to be a counting process in which one imagines how a set of outcomes can be enumerated, producing an ordered sequence. A *list* of outcomes is the artifact produced by enacting a listing process. Listing processes need not be able to produce a complete and totally ordered sequence of the outcomes, and I use the term *resolution* to describe the extent of the order in the resulting list. A listing process that demonstrates complete resolution is one where every outcome is created in a pre-defined order, whereas low resolution articulates a broad structure in a set but not substructures. For example, a low-resolution listing process might state how a set can be partitioned, but not how the outcomes in each part are ordered. I make no attempt to quantify resolution here, and I only use resolution as a means of comparing differences across the course of multiple interviews.

Methods

Participants and Data Collection

The data in this paper come from a larger study conducted at a large R1 university in the Northwest United States that investigated how undergraduate students use representations in their solutions to combinatorics problems. Hallie and Leah (pseudonyms) were two participants who worked as pair on a sequence of combinatorial tasks. The students completed six task-based interviews (9 hours total) over the course of one semester, and the data reported in this paper span the first five interviews. I chose to pair Hallie and Leah based on their similar educational background and their responses to the combinatorics problems presented in selection interviews. Both students were biological data science majors (Hallie was a senior and Leah was a sophomore), and neither student successfully used combinatorial formulas to solve the problems. Hallie had previously taken a discrete mathematics course with a section on combinatorics, but her responses in the selection interview indicated that she did not use formulas from that course to answer combinatorial tasks. Leah was currently enrolled a discrete mathematics course, but at the time of the interviews she had not completed the section on combinatorics.

I chose to present these data as a case study because Hallie and Leah repeatedly used lists and listing processes as foundational aspects of their solution. That is, lists and listing processes were fundamental to how they found solutions and were not used merely to communicate a solution found through other means. Additionally, over the course of the interviews Hallie and Leah adapted their lists and listing processes to increasingly complex problems with growing cardinalities. In doing so, they leveraged lists and listing processes with other structural information such as symmetry and recursion, which provides insight into how lists might be used as one tool used in conjunction with others. These data thus provide insight into how lists might be used specifically, and how external representations might be used more generally in combinatorics.

Tasks and Data Analysis

The task sequence presented to students in the interviews consisted of combinatorics problems that could be solved using arrangements with and without repetition, and selection without repetition. These problem types are consistent with other combinatorics education literature (see Lockwood & De Chenne (2020) for a broader discussion of problem types). However, I selected tasks to elicit the use of representations, and thus while the problems could be framed as arrangements and selections the literal outcomes described did not always match this framing. For example, the outcomes in the Three Cards Problem (described in the Results section) can be framed as a selection without repetition problem even if the literal outcomes described are ordered arrangements. Parts of tasks also explicitly asked students to describe how they would represent the outcomes.

The interviews were conducted in a private office and video and audio recorded. I provided students with writing materials, and their work was collected and scanned after each interview. From these scans I created a catalogue of representations, and in conjunction with the recordings of the interviews I used thematic analysis (Braun & Clarke, 2022) to identify what types of representations students used, categories of representations, and the roles of the representations in solutions. I focus here on Hallie's and Leah's use of lists and listing processes; however these two students used other representations across the six interviews.

Results

I describe three themes that emerged in Hallie and Leah's use of lists and listing processes. First is the balance between the resolution of a counting process and the affordance of a list for providing a means of reflecting on a counting process. The second the application of lists to solve part but not all of a problem by allowing for flexibility in what is being listed. The third is moving from representations of outcomes in lists to representations of listing processes more directly, and how those listing processes are reflected in the solution.

Resolution and Reflection

Hallie's and Leah's work on early tasks demonstrates how low-resolution listing processes could be used because the act of listing allowed them to reflect on and create lists in the moment. The first task Hallie and Leah completed was the Coin Flip Problem, which asked how they would record flipping a coin three times, how many ways there are to do so, and how their answers to these problems would change if the number of coins increased. For three coins, both Hallie and Leah created complete lists of the outcomes, although their listing strategies differed from each other. These lists are shown in Figure 1. Hallie's listing process entailed using two columns, where the second column was the mirror image of the first. This approach leveraged the symmetry of the outcomes; however, she did not have a pre-determined strategy for which outcomes would occur in which column. Rather, her approach was to write an outcome in the first column, write the mirror outcome in the second column, and then in the first column write another outcome that had not already appeared in the first or second column. Hallie concluded listing when she could not find an outcome that had not appeared in either column. She described this approach as "I kind of start where I separate at both extremes [HHH and TTT], and then get kinda to the middle."

Leah's approach was to structure the outcomes by the number of heads that occurred; that is, she listed all outcomes where three heads occurred, then all outcomes where two heads occurred, etc. While this provides a broad structure to the list of outcomes, the occurrence of each individual outcome within that structure was not predetermined, and she concluded each part of the list when she could not think of any unlisted outcomes. Hence, Hallie's and Leah's listing processes demonstrate a low-resolution strategy. Despite this, neither Hallie nor Leah appeared to have difficulty solving the problem. The low resolution of the listing processes was navigated by periods of time to reflect on the list—both Hallie and Leah were able to read which outcomes had been listed and create in-the-moment strategies.

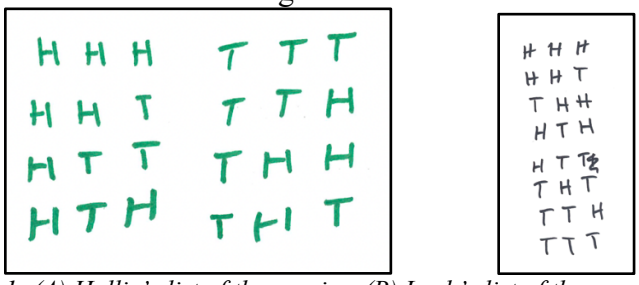


Figure 1: (A) Hallie's list of three coins, (B) Leah's list of three coins

Hallie and Leah also used lists of outcomes to determine the number of ways to flip four and five coins. Leah persevered in her original strategy of ordering the coins by the number of heads, and her list for four coins is shown in Figure 2 (A). While the broader strategy for ordering the outcomes is clear, her strategy within each of the parts appears to have been determined in the moment. She describes how she listed all outcomes with three heads as "I put all the three heads

at the start, and then I put them all at the end, and then I put one on the left, and then two on the right, two on the left and one on the right, and that got rid of... this one [crossing out HHHT] was just a repeat of what I did up here." This description shows that she produced these outcomes in an organized way (rotation around a central axis), but notably she chose a different strategy to arrange the outcomes with three tails, and this strategy was not extended for outcomes with two heads and two tails. I contend that her approach to listing the outcomes showed increasing resolution, but each part was structured using in-the-moment techniques. However, the list was a sufficient tool to enact this approach as it allowed her to keep track of her reasoning and reflect on the current state of the list to determine which outcomes had not been counted. I asked Hallie and Leah to find the number of outcomes in each part of their list, and they found that the number of outcomes was $1 + 4 + 6 + 4 + 1 = 16$.

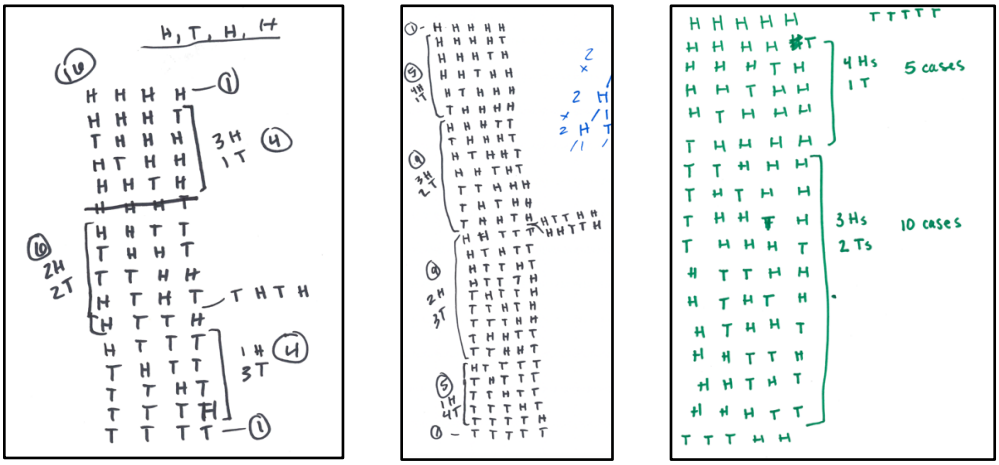


Figure 2. (A) Leah's list for 4 coins (B) Leah's list for 5 coins (C) Hallie's list for 5 coins

Both Hallie and Leah demonstrated increased resolution for five coins, as shown in Figures 2 (B) and (C). Leah's list shows repeated reasoning about ordering for outcomes with four heads and one tail, and outcomes with one head and four tails; however, her counting process for three heads and two tails, and two heads and three tails, demonstrates that the order was still determined in the moment (only after comparison with Hallie's list did she find that she missed two outcomes). Hallie relied on the symmetry of the outcomes in her list, and further she appears to use complete resolution in her listing process. Although the broad structure of her list was identical to Leah's, her organization within each part was consistent throughout. After writing half of the list, she concluded and noted "I kind of stopped writing, because of the symmetry thing we talked about." That is, Hallie reflected on her observations from prior lists and observed that by creating only half the list the remaining outcomes could be produced in the same manner. Hence, her solution reflects the mathematical expression $2(1 + 5 + 10)$. These examples demonstrate how Hallie and Leah used lists as tools for reflection, allowing them to reason about the structure of the list at hand and connecting that structure with cardinality even when using low-resolution listing processes.

Partial Lists for Part of the Problem

In the third interview Hallie and Leah worked on the License Plate Problem, which asked *A license plate consists of an arrangement of six letters and/or numbers. How many license plates consist of three numbers followed by three letters?* For this problem, Hallie and Leah were able to determine that the solution was $10^3 \cdot 26^3$ without using external representations. A subsequent part of the problem asked *How many license plates have four letters and two*

numbers? The solution to this problem is $\binom{6}{2} \cdot 10^2 \cdot 26^4$, which can be found by selecting the two positions in which the numbers occur, and then arranging the numbers and letters. At this point in the interviews Hallie and Leah appeared adept at solving problems of arrangement with repetition, but this new problem also involved selecting which of the six characters would be letters or numbers as one step in the problem. Hallie and Leah recognized that the solution was $10^2 \cdot 26^4$ multiplied by the number of ways to select two of the six characters, but they were not sure how to determine that number. Hence, they were able to solve one step of the problem but not the other. Their solution to the problem involved listing some of the ways to arrange two numbers and four letters, as shown in Figure 3. This use of listing is both partial in that it omits some ways to arrange numbers and letters, but also each inscription in the list denotes a number of outcomes rather than individual outcomes. Hence, the list was used as a tool for part of the problem, allowing them to physically outsource their reasoning for the parts of the problem they could not perform mentally. Additionally, this example demonstrates how the students began to move away from lists as outcome-oriented tools to process-oriented tools as each 'outcome' in the list stood for a collection of outcomes rather than an individual outcome. Hence, the list itself reflects one stage in the counting process.

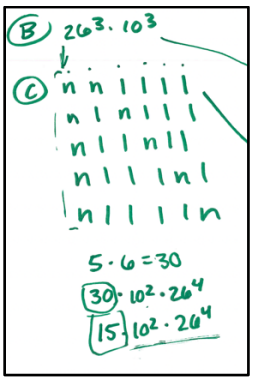


Figure 3: Partial listing of arrangements of two numbers and four letters

Representations of Listing Processes and Connections to Solutions

During the fifth interview the students solved the Three Card Problem, which states *You and your two friends, Ben and Charlene, are playing a card game that only uses spade cards (i.e., your deck consists of the Ace through King of spades). You each draw a card, and whoever has the highest card wins. How many ways are there to play this game so that you come in first, Charlene comes in second, and Ben comes in last?* The solution to this problem is $\binom{13}{3}$, which can be found by determining which three cards will be given to the players, and then giving those cards to the players in a way that preserves win order. Early in the solution Hallie stated "I could just write the cards themselves, since I know who's coming in first, who's coming in second, and who's coming in third," which indicates that she understood that finding the number of ways to select three cards would solve the problem. However, the students had not found a formula for counting selections.

Hallie and Leah employed a listing process that fixed the winning card, iterated through the second card, and then iterated through each losing card for each second card, as shown in Figure 4 (A). Hallie then created a representation of this listing process in Figure 4 (B), which demonstrates all instances where the winning card is the King. The listing process represented in these figures demonstrate complete resolution, showing how Hallie and Leah's application of listing processes was refined over the sequence of interviews. After representing this partial

listing process and observing there are $11 + 10 + \cdots + 2 + 1$ outcomes where the winning card is the King, the students then continued the process to find that there were $(11 + 10 + \cdots + 1) + (10 + \cdots + 1) + \cdots + (2 + 1) + (1)$ total outcomes, which reflects the listing process they used. To simplify this expression, Leah observed that the 11 appears in 1 of the sums, the 10 appears in 2 of the sums, and this pattern continued, as demonstrated in Figure 4 (C). I contend that he representations shown in Figure 4 (A) and (B) represent a listing process rather than simply a list of outcomes, as demonstrated by the use of arrows to signify the continuation of a process and brackets accompanied with numbers to signify the numerical value of terminating that process. Hence, it appears that the students simplified their use of notation to accommodate sets with larger cardinality by moving from notations of individual outcomes to notations for processes that generate those outcomes. In addition, their solution to the problem, $11 + 2(10) + 3(9) + \cdots + 10(1)$ reflects their use of the representations, leading to a form that differs substantially from a canonical solution of $\binom{13}{3} = \frac{13!}{3!10!}$.

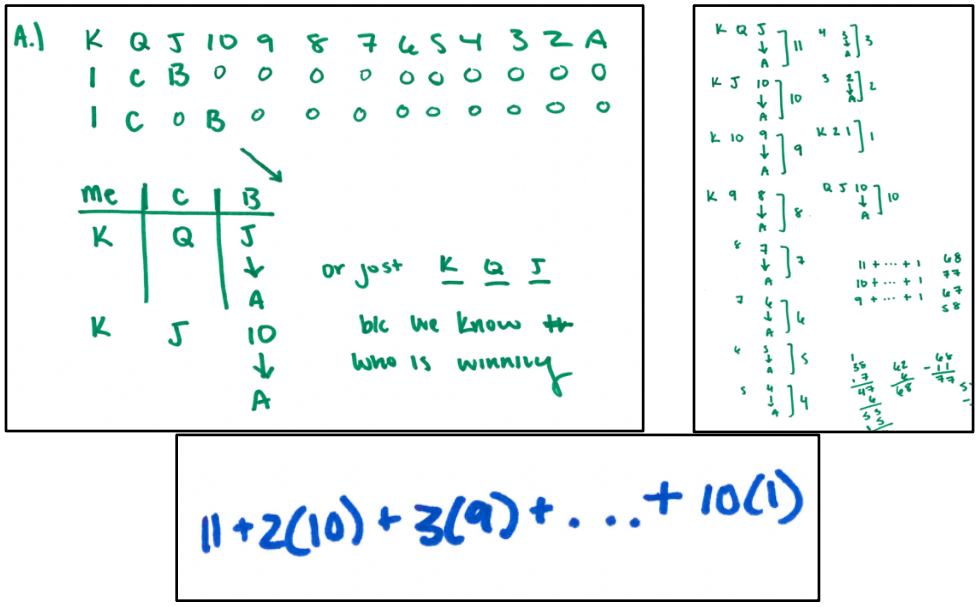


Figure 4: (A) Hallie's articulation of a listing process (B) Hallie's carrying out (C) Leah's solution

Discussion

These data demonstrate how lists of outcomes and listing processes can play a role in student understanding of counting by influencing how students approach a solution, and the form of the solution they find. This was true even for counting problems where the cardinality of the set of outcomes made a complete list intractable to produce by hand. Rather than abandoning lists as a viable strategy, Hallie and Leah adapted their listing techniques by leveraging structural aspects of the set of outcomes (such as symmetry), relying on lists to answer only parts of problems, and representing listing processes rather than lists of outcomes to articulate strategy without needing to carry out that strategy in its entirety. These data inform broader questions of how students create, use, and adapt representations as they solve counting problems because they show how students can become adept at leveraging a specific type of solution approach and become successful in applying that approach to increasingly sophisticated problems.

Acknowledgments

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Exploring Factors Impacting Students' Engagement in a College Precalculus Course

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This study aimed to gain insight into students' experiences of engagement in undergraduate Precalculus. A series of interviews were conducted with a group of 12 college students to solicit their lived experiences of engagement, using methods of hermeneutic phenomenology. The analysis uncovers that some of the key factors influencing engagement include the physical layout of the classroom, the nature of student-instructor and student-peer relationships, students' comfort level with the course material, and their interpretation of grades as formative feedback. Insights gained from these findings can provide us with important information about how students engage in college Precalculus and suggest ways to improve the teaching of the course.

Keywords: engagement, precalculus, student experiences, phenomenology.

Researchers and educators would agree that engagement is an integral part of the learning process, so engaging students in mathematics is viewed as an important goal at both the undergraduate and K-12 levels of education (Middleton et al., 2017). Efforts aiming to increase college students' success in mathematics courses emphasize the importance of student-centered instruction, providing supports for learners, and implementing pedagogies that promote students' active participation in the learning process (e.g., Apkarian et al., 2021; Rasmussen et al., 2019).

As we push for certain curricula and instructional approaches, it is important to consider not just the dehumanized numbers quantifying success and failure, but also the lived experiences of students who are on the receiving end of these reforms. Some questions we ought to consider include: Would students themselves say that they feel engaged in these classrooms? What does engagement look like for them? The study presented here aim to promote this conversation and shed light on factors that students experience as impactful on their engagement. The data and findings are part of a larger phenomenological study on students' experiences of engagement in college Precalculus.

Literature Review and Theoretical Framework

Precalculus is one of the more common gatekeepers to entering STEM majors in universities and colleges, while also ironically playing the role of a terminal mathematic course for certain majors in some universities (Bailey et al., 2008; Ganga & Mazzariello, 2018; Roksa et al., 2009; Seymour & Hewitt, 1997; Seymour & Hunter, 2019). In the last 14 years, groups of researchers in the US RUME community have conducted a number of large-scale national studies of the college-level Precalculus-Calculus sequence (Apkarian & Kirin, 2017; Bressoud et al., 2015; Rasmussen et al., 2019). The findings from these studies produced a plethora of insights into multiple aspects of teaching and learning in these mathematics courses. However, there is still a growing need for research on affective aspects (which includes engagement) of college-level mathematics instruction (Melhuish et al., 2022), and a need for more research that solicits students' perspectives on their own learning (e.g., Leyva et al., 2021).

Mathematics Engagement

Despite its importance, there is no consensus on the conceptualization of student engagement in broader education research (e.g., Appleton et al., 2008; Wong & Liem, 2022). A commonly

adopted framing in K-12 mathematics education comes from Fredricks et al. (2004) that positions engagement as a dynamic, malleable, multi-dimensional meta-construct (Middleton et al., 2017). In their Compendium chapter Middleton et al. (2017) note the complexity of the phenomenon, which makes engagement rather difficult to research, especially when it happens in the moment.

When it comes to the multiple dimensions of engagement, the three most common ones are: affect, behavior, and cognition. Affective or emotional engagement is oriented towards students' emotional reactions to aspects of schooling (e.g., interest, feeling of belonging, attitudes, relationships, etc.). Behavioral engagement is oriented towards participation in schooling (e.g., positive conduct, effort and involvement in learning, participation in school-related activities, etc.). Cognitive engagement, which is oriented towards students' investment of effort in learning (e.g., investment in learning, self-regulation, setting and working towards learning goals, etc.). When considering students in the mathematics classroom, it is easier for instructors and researchers to observe behavioral engagement and much harder to glean students' affective and cognitive engagement.

Hermeneutic Phenomenology

The method used in the proposed study is hermeneutic phenomenology, which focuses on the human experience of a phenomenon, as well as the meaning of that experience (Friesen et al., 2012). The process of interpreting meaning is valuable in education research as it allows for contextualizing both the participants' experiences within the broader socio-cultural environment where learning happens, and the individual learner's personal disposition and interaction with education. This study builds on the interpretive approach of van Manen (2014) with the influence of the reflective lifeworld approach of Dahlberg, Drew, and Nystrom (2008).

Since student engagement is a complex and dynamic phenomenon, it makes sense that by exploring students' rich descriptions of their lived experience, we can get to the dynamic essence of engagement, as opposed to capturing a static image at some point in time. In order to conduct a phenomenological study, the methods of data collection need to emphasize soliciting students' descriptions of lived experiences of the phenomenon as they happened (van Manen, 2014).

Methods

This qualitative study was conducted at a large public university in the southeast United States during the Spring semester of 2024. Data collection involved a series of four audio- and video-recorded interviews with each participant that were spread out over the course of the semester. I also conducted non-recorded classroom observations of the sections attended by the participants with field notes. To support the interview process, the participants were asked to produce engagement artifacts and an engagement graph, which allowed them to respond to the interview questions in a more nuanced way producing rich qualitative data (Alibek, 2024).

Engagement Artifact

When asking participants to describe their experiences of engagement, interpreting data can be difficult since each participant have their own interpretation of engagement. Thus, it was critical to prompt the participants to explicate their conceptualization of the phenomenon. However, due to the complexity of the term, it would have been unfair to ask participants to produce a definition on the spot. Instead during each interview they were prompted to reflect and expand on ways they believe a college student can engage in their Precalculus course with additional prompts during each subsequent interview.

During Interview 1 participants were asked to jot down on paper their response to the following prompt: What are some ways you think a college student can be engaged in their Precalculus class? The produced artifact was brought back for Interview 2 for the participants to review and edit if they wished to do so. Additionally, during the second meeting they were asked to note the specific ways they have engaged in their Precalculus course themselves within the last week. Since engagement can manifest in similar ways across different disciplines, while simultaneously being discipline-specific, it was important to contrast the participants' experiences of engagement in Precalculus with their experiences in a different university course. This was brought in during Interview 3, when participants were asked to produce an engagement artifact for another class that they are enrolled in during that semester and then to compare it to the Precalculus engagement artifact. Finally, during Interview 4 participants were asked to do a final review and edit of their engagement artifact, as well as select one item that they felt was most representative of their engagement in the Precalculus course.

Graphing Mathematics Engagement

In addition to the engagement artifact the participants were asked to sketch how their self-reported engagement has shifted throughout the semester. Graphing as an elicitation tool is not a novel approach in neither education research more broadly (Bravington, 2023) nor mathematics education (Olarte, in press; Riske et al., 2021; Satyam et al., 2022). In fact, graphing is especially useful when dealing with constructs like engagement that involve affect, are malleable, and can significantly shift and change over time.

Participants were given a blank first quadrant of the coordinate plane and during each interview participants graphed their engagement up until the moment of the interview: first few weeks during the first interview, the weeks between the first and second interview during our second meeting, etc. Thus, each graphed segment covered around 3-4 weeks of the semester. Moreover, the graphs were all done on the same sheet of paper, so participants could review their sketches from previous interviews and build on them. Participants were provided alternative color markers to show how they would revise previously sketched graphs or letting them sketch the new segment in a different color if they desired to do so (e.g., Fig. 1 and 2).

It is important to reiterate here that the graphs were not analyzed by themselves: their main purpose was to elicit rich qualitative data and thick lived experience descriptions. Participants' graphs were not compared to each other as part of the data analysis. Instead graphs together with artifacts informed the interpretation of interview transcripts.

Participants and Context

At the beginning of the semester 12 participants enrolled in the research study, with only 9 participants completing all four interviews. The other 3 participants withdrew from the course at various points in the semester and thus couldn't complete all planned interviews. The data set includes a total of 42 participant interviews.

10 out of 12 participants in the study were enrolled in the flipped classroom sections, while the other 2 were enrolled in lecture-based sections. The flipped classroom sections of Precalculus at this university were coordinated for the most part. All students had to watch a set of videos, as well as solve and submit a short assignment prior to class. Once they were in the classroom, they worked in small groups to solve the worksheet problems from a printed packet, which they bought from the university. The in-class worksheets were never turned in for grades or feedback. Additionally, students had online ALEKS homework, weekly graded quizzes, three tests taken

during class time (corresponding to the three main content units of the course), as well as a cumulative final exam at the end of the semester.

While the overall findings are based on data from all 12 participants, the comparative examples used in this proposal focus on two participants from the study: Ashley and Beth, who despite having many similarities, had vastly different experiences with engagement. They were both freshmen at the university, who have taken a year-long Precalculus course during their junior year in high school. They were both enrolled in a flipped classroom section of Precalculus, taught by different instructors. Neither participant saw herself as a math person, but both were motivated and hard-working students, who aspired to get the best grade possible. However, while Ashley ended the semester feeling positive about her involvement in the course, Beth, on the other hand, voiced strong negative sentiments towards both the course and her engagement in it.

Data Analysis

This proposal focuses on the results of an intermediate analysis question within a larger study on student experiences of mathematics engagement in college Precalculus. The research question addressed in this presentation is: *What are some factors impacting students' engagement in college Precalculus?* The process of data analysis involved rounds of phenomenological bridling (Dahlberg et al., 2008), followed by reviewing participant interviews for lived experience descriptions of engagement (van Manen, 2014), and then writings narratives of experiences. The lived experience notes, phenomenological narratives, and interview transcripts were then analyzed across participants.

As a result of the analysis, certain factors consistently appeared across narratives of engagement experiences. Two participants, Ashley and Beth, were chosen to frame the findings due to their contrasting depictions of similar factors, informed by versus coding exercises (Palacios-Díaz & Moreno-Londoño, 2023). It's noteworthy that these factors resulted from a holistic analysis of the entire data set, not just Ashley and Beth's interviews.

Findings

When contrasting Ashley and Beth's experiences in the course, on the surface it seems like their stories of engagement couldn't be more different from each other. Ashley sustained a relatively high level of engagement throughout the semester and felt overall good about the course and her own learning at the end (Fig. 1). Beth on the other hand shared many frustrations about the course and how it is taught, she would regularly say that she doesn't really feel engaged in the class (which was reflected in her graph), and by the end of the semester she admitted that she can't wait for it to be over (Fig. 2). Despite these surface differences, data analysis has shown that there are certain aspects of the Precalculus course, instruction, and classroom environment that students see as impactful for their engagement and learning.

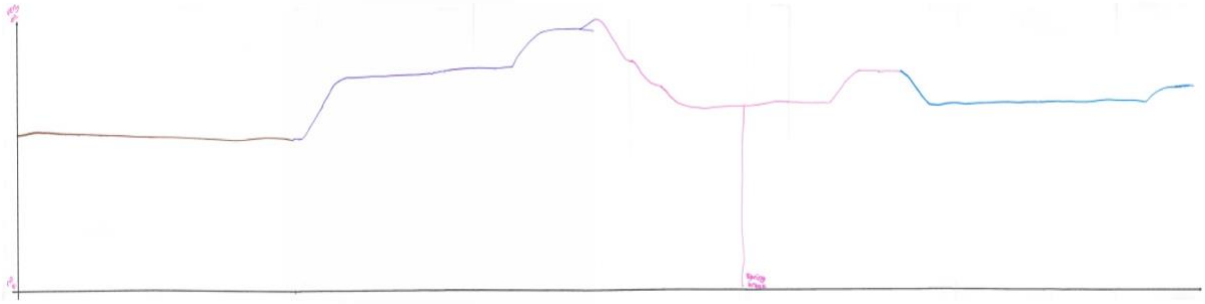


Figure 1. Ashley's final engagement graph.

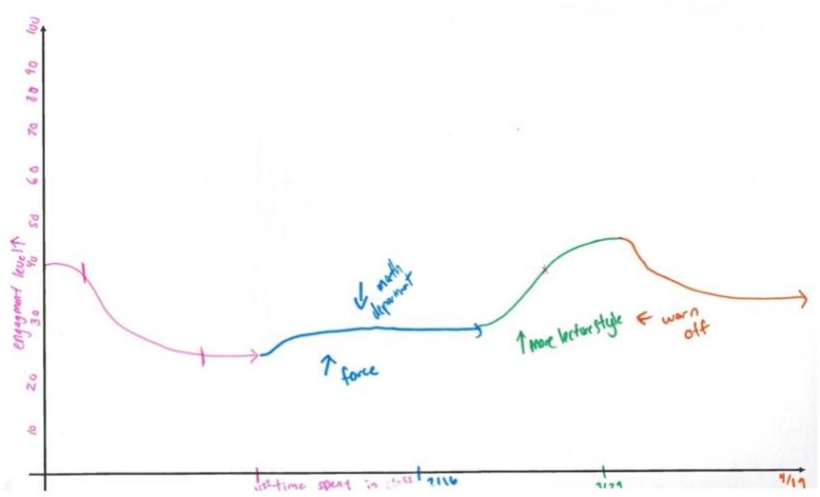


Figure 2. Beth's final engagement graph.

Physical Layout of the Classroom

Participants brought up the physical setup of their classrooms when discussing engagement, especially the way tables are set up for groupwork. When working on her engagement artifact, Beth said that one of the ways students can engage in a Precalculus course is by sitting at round tables, or at least facing each other (Fig. 3, left). However, she later noted that this is something she experienced in her high school Precalculus class, but not in her college course. Beth noted that her college Precalculus classroom had tables that were set up for a lecture-style class, where she had to move to work in a group with other students (Fig. 3, right).



Figure 3. Beth's first engagement artifact (left) and sketch of her classroom (right).

Ashley also noted that sitting together with other people and facing each other helps her feel engaged during class. She contrasted it with her experience in another lecture-based Precalculus section that she attended once during the add/drop week. She said that sitting at her desk, not looking at other people caused her to feel disengaged, which was one of the reasons she switched sections. Similarly to Beth, in her artifact Ashley noted the physical classroom setup as one of the ways to engage (Fig. 4, right) compared to the disengaging lecture setup on the left (Fig. 4).

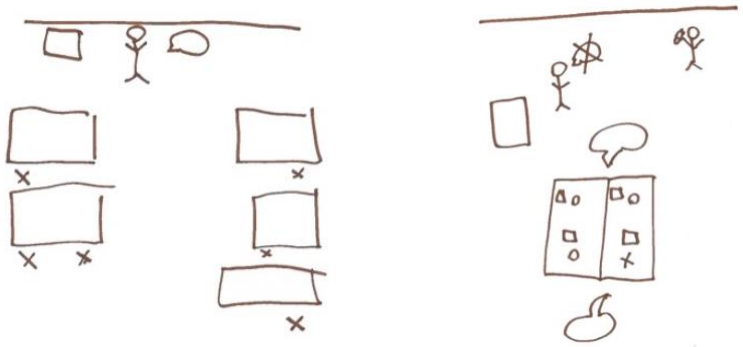


Figure 4. Ashley's first engagement artifact for the lecture class (left) and flipped class (right).

Relationality of engagement

When it comes to learning environments, it's not just the physical setup of the class that impacted participants' engagement. The students in this study noted the great role that the relationships play in the way they experience engagement.

First, classmate relationships greatly influence engagement, particularly in a flipped classroom. Initially, Ashley felt less competent in mathematics than her peers, leading to self-consciousness. However, she grew comfortable with her group and with asking questions during class, attributing her active engagement to the supportive learning environment. Conversely, Beth felt her group members knew less than her, leading to frustration as she felt her learning was hindered, describing her participation as forced and inauthentic. She would feel frustrated that she was held back by her group instead of trying to get through the in-class worksheet.

Second, the relationship with the instructor also affected their engagement. Ashley had a positive perspective of her instructor, appreciating their approachability and interactive teaching style, which significantly contributed to her active engagement. Ashley valued how her instructor structured their class time and the interactions between students. Meanwhile, Beth had less positive feedback, expressing frustration with the instructor's tendency to go on tangents and over-explain to the point of unhelpfulness, as well as perceived inefficient use of class time, leading to suboptimal engagement, despite acknowledging the instructor's care for students.

Comfort with the Mathematics Content

Something that was relevant for many participants in the study was how engagement shifted depending on the students' comfort with the topic discussed in the course. For context, the first part of the class focuses on functions in general (equations, graphs, polynomials), the second on exponential and logarithmic functions, and the third on trigonometry (angle measures, arcs, and trigonometric functions). Both Ashley and Beth shared that they struggled much more in the first part of the class but felt more comfortable with some of the latter topics. While the reasons behind their comfort levels were different, both noted that engagement would decrease a bit in level or intensity when they felt more comfortable with the content and increased when they struggled with the content.

Although Beth and Ashley experienced a similar relationship between Precalculus topics and shifts in their engagement levels, there was a clear distinction between how Beth and Ashley felt about their own conceptual understanding of the material. On one hand Ashley was thrilled that unlike her time in high school, she was finally getting the "why" behind mathematics in Precalculus at the university. She gave an example of finally understanding the unit circle and even why it was called the unit circle. Meaningful engagement in the course was paying off for

Ashley. On the other hand, Beth shared that her engagement in this class was more about studying and practicing procedures. She shared that despite her comfort with the content (especially trigonometry), she believed a lot of it was still from her time in high school. In contrast to Ashley's unit circle, Beth shared that she could solve problems involving co-terminal angles, but she still didn't know what they were.

Grades as a Factor Informing Engagement

While it is common for educators to talk about the role of formative assessment in giving feedback about student learning and their own teaching, it was interesting to see how the study participants talked about adapting their engagement using grades as feedback. This point is related to the previous idea of comfort with the content: for students their grades are an indicator of sorts whether they grasp the content or not. This informed their next steps: lower grades signaled to them that they need to do more, pay more attention; higher grades signaled that they have some level of understanding and thus can shift their energy towards engaging in other things. This was also highly related to the students' busy college schedules, when they are juggling multiple demands on their time and energy towards engagement.

One instance when grades informed students' engagement manifested for Ashley and Beth during the time leading up to Test 1. Ashley was not doing well on quizzes in the class so far. In fact, her first quiz she got a 6.5 out of 12 and the instructor told her class that anyone who got a 6 or below on the quiz should consider withdrawing from Precalculus or enrolling in a support class. Ashley's ongoing struggles with quizzes impacted how she felt about her understanding of the material. She even shared that she thought she knew the content well enough before the first quiz, but the grade proved otherwise. These struggles led Ashley to increased engagement and a lot of preparation for the first test of the semester, resulting in her getting a 90.5 out of 100 on it. One must note that the highest point in the entire engagement graph for Ashley in Figure 1 was around Test 1. Beth had quite the opposite situation compared to Ashley. She felt confident in her knowledge of the material, averaging 95% on her quizzes. She wasn't as engaged in the course leading up to Test 1, on which Beth earned a 77 out of 100.

Discussion

Many of the factors identified as impacting students' engagement in this study are in line with extant literature on mathematics engagement. In particular, the value of student-teacher interactions has been identified as having a great impact on students' engagement (e.g., Barnes, 2019; Durksen et al., 2017; Lin et al., 2018). Beth's negative experiences of engagement with her instructor's in the moment feedback have also been noted in research on flipped mathematics classrooms (Cevikbas & Kaiser, 2022). Previous research has also noted the role of the physicality of the learning environment (e.g., Imms & Byers, 2016; Waldock et al., 2017).

Simultaneously some of the findings have the potential to expand our understanding of how students experience engagement as it relates to their comfort with course content. This is especially relevant for early-college mathematics classes like Precalculus, where the material is not new for students and their engagement might shift differently compared to courses where material is mostly new (Amman, 2022). Understanding how college students use grades as formative feedback for regulating their own engagement can prompt us as a research and teaching community to think more critically about how assessment tools are used in courses like Precalculus. Moreover, this can promote highly needed conversations about further soliciting college students' descriptive experiences of mathematics gatekeeper courses as it relates to their motivation, engagement, and self-regulation.

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Mathematicians Reject, Teachers Embrace: Different Perspectives on the Role and Significance of Disciplinary Obligations in Secondary Mathematics Teaching

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A key objective of academic mathematics courses in teacher education is to enhance teachers' awareness of disciplinary values and principles, ensuring alignment between mathematics education and the discipline. However, research indicates that this goal is often not met. A possible explanation is that mathematicians and teachers have different perspectives regarding the discipline's role in mathematics education. This study examines these perspectives in three extreme cases, where mathematicians and secondary mathematics teachers disagreed on the legitimacy of secondary mathematics activities from a disciplinary standpoint. Analysis of professional obligations that underlie the mathematicians' and teachers' reasoning reveals distinct interpretations of the obligation to the discipline and of its implications and significance in consideration of other obligations. These findings emphasize that teacher education should go beyond fostering awareness of disciplinary values and principles, and support teachers in learning how they can apply this understanding effectively in school to guide and inform their teaching.

Keywords: Teacher education, Mathematicians, Professional Obligations, Decision-Making

Introduction

One of the key roles of academic mathematics (AM) in the education and professional development of secondary mathematics (SM) teachers is to help teachers orient their teaching so that it aligns with and represents the values and principles of the discipline (Ball & Bass, 2009; CBMS, 2012; Cooper & Pinto, 2017; Hoffman & Even, 2023; Shulman, 1986). A common approach in AM courses for teachers is that mathematicians provide teachers with AM learning experiences, with the expectation that this experience will trickle down into their classroom instruction (e.g., Wu, 2011). However, current research and policy documents do not specify how this orientation is supposed to happen in practice, nor how AM learning experiences become a resource for teachers' instructional decisions. There is empirical evidence that both teachers and mathematicians do not always have a clear vision of how AM studied by teachers is brought into classroom teaching (e.g., Lai, 2019; Wasserman, 2018; Zazkis & Leikin, 2010).

Therefore, in recent years, there has been a growing interest in the subtle process by which aspects of AM may become a resource for SM teachers' instructional decision making (e.g., Biza et al., 2022; Herbst & Chazan, 2020; Pinto & Cooper, 2022, 2023; Zazkis, 2020;). One approach for delineating these processes is by exposing how the obligation to the discipline may orient teachers' decision making (Herbst & Chazan, 2020). Specifically, Herbst and Chazan (2020) propose that mathematicians will share their mathematical knowledge and practices with teachers to develop their sense of the discipline and inform their teaching. However, Herbst and Chazan also emphasize the crucial role of SM teachers in revising the initial ideas offered by mathematicians in consideration of the realities of SM education as manifested in other professional obligations. In this study, we adopt and adapt this approach by engaging mathematicians and experienced SM teachers with authentic SM instructional situations in ways that invite both sides to interpret and negotiate the role of obligation to discipline with respect to specific and concrete teaching decisions.

This paper is focused on three extreme cases where mathematicians deemed mathematical activities as inappropriate for secondary education since they breach values and principles of the discipline. The SM teachers, while acknowledging the breach, still considered using these activities in their classes. In these contexts, we investigate differences between mathematicians' and teachers' perspective regarding how obligation to the discipline should orient SM teaching. Our investigation is guided by the following research question: *How do mathematicians and SM teachers perceive the role and significance of the obligation to the discipline in relation to concrete instructional situations in SM education?*

Theoretical Framework

Teachers and mathematicians may be considered as different communities of practice. According to Akkerman and Bakker (2011), encounters between such communities, in which issues related to the practices of each community are discussed, may reveal a boundary – 'a sociocultural difference leading to discontinuity in action or interaction' (p. 133). For example, differences in professional obligations expressed by members of these communities reflect such discontinuities. The cross-community dialogue in these meetings provides a fertile ground for initially identifying these differences, and negotiating the role and significance assigned to them.

Herbst and Chazan (2020) identified four central professional obligations that guide mathematics teachers in making instructional decisions: the obligation to the discipline, the student as an individual, the class as a group, and the school. In articulating these obligations, they point out that mathematics teaching is expected to be influenced by the discipline. At the same time, they recognize that instruction takes place within a specific context, involving the student, the classroom, and the institution. These obligations reflect the tensions underlying teachers' decision-making. In extreme cases, tensions sometimes turn into an outright contradiction between obligations. Such contradictions reveal different perspectives on the role and significance of the obligation to the discipline in consideration of the context, and how this obligation may orient the teaching.

Methods

This research is part of a long-term research project – M-Cubed (Mathematicians, Mathematics teachers, Mathematics), conducted in Israel between 2019-2024 (Goor et al., 2022; Pinto & cooper, 2022, 2023). In this project, small groups of mathematicians and experienced secondary mathematics teachers watch videotaped lessons together, collaboratively exploring the mathematical and pedagogical issues they identify therein. The design and facilitation of M-Cubed is informed by the literature on boundary-encounters and boundary-crossing (Akkerman & Bakker, 2011). The videotaped lessons are carefully selected by the mathematics education researchers who lead the meetings, aiming to reveal professional dilemmas arising from the exploration of practices derived from AM, as well as the pedagogical vision of both SM teachers and mathematicians. Thus, this research setting serves as a laboratory for generating and analyzing the implications of ideas and perspectives from academic mathematics in the context of real-world instructional situations. Data for this study include video recordings of approximately 30 meetings, each with a duration of about three hours. A summary was prepared for each meeting, and selected excerpts were transcribed according to the research goals.

This study focuses on extreme cases in which mathematicians strongly opposed the teaching activities presented in videotaped SM lessons, deeming them mathematically flawed and inappropriate for secondary education. In contrast, teachers did not accept the mathematicians' critiques at face value and advocated for using these activities despite their apparent

misalignment with the discipline. These cases were selected to expose differing perspectives on the role and significance of disciplinary values and principles in the context of authentic SM teaching decisions. The contrasting views between the mathematicians and the teachers prompted both groups to articulate their tacit perspectives, facilitating the identification and characterization of various professional obligations that guide and constraint teaching decisions. To this end, we first analyzed the reasoning of both groups regarding the legitimacy of the activities, identifying how they referenced broader principles or values that extend beyond the specific cases yet impact decisions about using such activities in SM lessons. We characterized these principles and values according to Herbst and Chazan's (2020) four professional obligations, guided by descriptions and examples of these obligations in the literature. We then compared the obligations to the discipline expressed by mathematicians and teachers, and how these shaped their reasoning and conclusions. Additionally, we explored how they positioned the obligation to the discipline with respect to other professional obligations. In this paper we present an analysis of three extreme cases that illustrate how mathematicians and teachers perceive differently the role of disciplinary values in SM teaching, and how obligation to discipline may intersect with other professional obligations.

Findings

In each of the three cases we will describe, we first present the mathematical and pedagogical context under discussion. We then characterize the mathematician's obligation to the discipline as reflected in their critique of the activity. Finally, we describe the teachers' voices that emerged throughout the discussion, emphasizing the different perspectives of the teachers on the roles and significance of the obligation to the discipline in consideration of other professional obligations.

Case 1: Mathematical Statements Teachers Make Need to Be True.

This case revolves around a task from an 11th grade advance-track class that the teacher, Sergei, introduced as an example that underlines the need for intuitions in mathematics. The task is to determine whether a plane 'like' the plane illustrated in Figure 1a exit, given that EK∦AB and FG∦AB. In the lesson, Sergei argued that there cannot be a plane that contains the points E, F, G and K, as such a plane would necessarily also contain the segment AB and therefore will necessarily contain the face ABD, which 'clearly' cannot be.

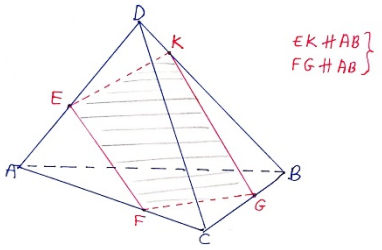


Figure 1a. Reconstruction of Sergei's board

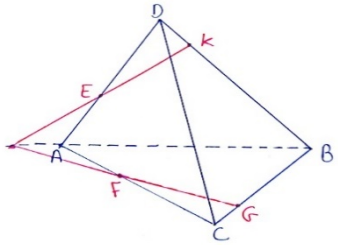


Figure 1b. Reconstruction of David's board

One of the Mathematicians, David, immediately recognized that in contrast to Sergei's argument, given a tetrahedron with vertices A, B, C and D, there always exists a plane that intersects BD, AD, BC and AC at points K, E, G and F respectively such that EK∦AB and FG∦AB (see Figure 1b). He further noted that this plane would not look 'like' the plane in the

drawing on the board, but that this drawing cannot be taken as data. He contended that Sergei made a very critical error and framed the lesson as a "mathematical catastrophe":

> [Sergei] gave the [students] a statement that is not well defined, with conditions whose sole purpose are to confuse the students because they are not relevant. False data, a false statement, a false proof, what could be worse in a mathematics lesson?

David's criticism of the task is clearly in disciplinary values, as he makes no reference to the individual student, to the class, or to the school. Notably, David recognized potential pedagogical value in the task. When considering teaching alternatives, he suggested that after convincing students that the proof is correct, a teacher may lead students to explore and where the argument fails, which in his words would afford students an "amazing lesson." However, David insisted that students cannot be left believing a false mathematical claim that a teacher made.

Some of the teachers disagreed with this position in ways that suggested that the teachers were oriented by additional obligations that were not outweighed by their obligation to the discipline. For example, Vera, noted that she loved what Sergei did in the videotaped lesson, and would use this task in the same way. She noted that as a teacher, the issue that David highlighted does not bother her as much:

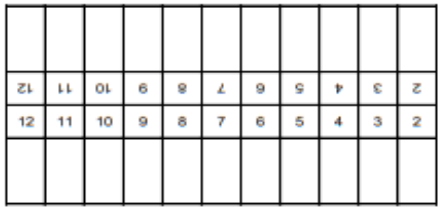
> OK, I'm convinced [there exists a plane]. But, if from now on I will be afraid to bring [to class] something beautiful because maybe somewhere, mathematically, I'm wrong, then I will never bring anything. Because there is a difference […] between a teacher that wants to show the beauty, that wants to light up [students' faces], that wants to get them to think, and between the mathematician that must have everything precise, precise, precise.

In contrast to David's criticism of the task, Vera centralizes the students when considering the use of the task in class. She considers mathematical errors an acceptable price she is willing to pay to be able to share with her students' beautiful pieces of mathematics. She considers the difference between her positions and David's as representing a distinction between how teachers' and mathematicians' disciplinary obligation to precision.

Case 2: The Mathematics Discussed in Lesson Needs to Be Well-Defined

This case revolves around the probability game introduced by a teacher, Ari, in his 8th-grade advanced class. In the game, called 'Rive-Crossing Game' (RCG), students are given a board as described in Figure 2a, and 18 office clips, which they place on numbered squares ranging from 2 to 12. Players take turns rolling two dice, and if the sum of the dice matches a number on which a clip is placed, the player can move the clip across the "river". The objective is to be the first to transfer all clips. During the observed lesson, Ari guided the students in constructing a graph that represents the optimal clip distribution on the board (see Figure 2b).

a



b

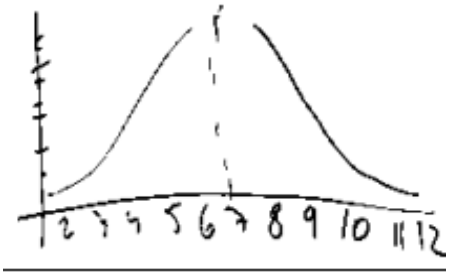


Figure 2. (a)The RCG's board and (b) A snapshot of board with the graph Alex built with his students

One of the mathematicians, John, identified the game as complex to solve, in terms of the mathematics underlying the game. He suggested the solution was likely not optimal: "I don't understand, what is the optimal strategy… I think that without a computer I don't stand a chance of calculating an optimal strategy." And then he explained: "… It's possible that there is a strategy with a slightly better expectation and slightly worse variance and another strategy that is the other way around. It's a very sensitive game."

In light of this perspective, during the continued discussion, John and another mathematician, Bill, explained they couldn't bring a mathematical problem to the classroom without being able to provide intuition for it, due to the complex mathematics behind it.

> Bill: Because I don't understand it myself. It's really hard for me to bring in something, where even on an intuitive level, I'm not entirely sure what's happening. Okay? Because I agree with John, that when the numbers are very large, it's better to approach this curve… As long as every throw moves a clip, there's no problem. The problem starts when we begin hitting zero, right? And that's something that will happen in smaller trials. It's a problem when I can't provide intuition myself.

In this case, as well, the mathematicians' critique of the task did not address the students, the class, or the school context. Again, it was grounded in disciplinary values. Specifically, the obligation to fully understand the mathematics brought to the classroom and the ability to model the problem, and well define it, to analyze it. During the discussion, John noted that without fulfilling this obligation, the RCG game loses relevance to probability learning, saying: "It is strategy thinking, game thinking. Is it related to probability? So-so".

Some of the teachers disagreed with the mathematicians' view on the activity's legitimacy. They didn't dismiss the claim that the task compromised the obligation to the discipline but gave it less weight compared to other interpretations of the disciplinary obligation the task invited, as well as other professional obligations. For instance, one of the teachers, Leah, argued that the stage of choosing how to distribute clips at the start of the game fosters initial intuition, mathematical thinking, and discussion. Unlike the mathematicians, she placed less importance on justifying optimal strategies through detailed mathematical analysis, due to time constraints that prevent students from reaching such advanced questions. Additionally, she highlighted the game's value in terms of her obligations to her students and the class, given the enjoyment it provided. She weighed her arguments against the 'damage' pointe out by the mathematician, comparing it to the 'benefits' she pointed out, as she said:

> … I don't understand what the big problem is. I see a lot of benefits. ...What's so scary … it's exactly what he was trying to say—frequency versus probability… as they said here, there's not enough time to play. They won't see the optimal strategy

And she added:

> I just want to defend the game as well. ... With all the disadvantages you listed. first of all, it's really fun. It's much less fun to flip a coin 100 times. Sorry, but that's super boring… It [RCG] really sparks a lot of interest. Kids talk about it. …usually, it works out for the ones who don't ignore the smaller tiles, who actually spread things out like a bell curve... And it's also really fun when you think you've figured out the optimal strategy, but you don't win... I think John said… opening with games. And especially with a game like this, it totally fits that purpose. In my opinion, it's a really good game.

Case 3: Motivation to learn a new mathematical idea needs to be authentic.

This case revolves around an activity from an 'introduction to function analysis' lesson in a 10th-grade high-track class. As part of the task, a student volunteer leaves the classroom while

the teacher momentarily displays on the board a graph of an unfamiliar function, such as the graph of the function $f(x) = x^2(x + 3)$. Upon the student's return, the class was asked to verbally explain how to draw the graph, without using gestures or images. One of the mathematicians, John, was highly critical of the task from the start, questioning whether it is even mathematical. He explained: "You just start with a figure and end with a figure, so [it's] more a discussion in visual communication, not in mathematics". He elaborated on what he would expect from a task designed to motivate function analysis, clarifying what he meant by "not mathematics":

> *John*: It all depends on the selection of a function, right? If you choose a function that looks like the back of a crocodile, they [students] will say it looks like the back of a crocodile, and there would be no mathematical at all
>
> *Facilitator*: …But if you wanted to use mathematical concepts to describe a crocodile's back… you could say the function has many local maxima, here it's convex…
>
> *John*: It wouldn't be natural. Anyone who would work this way would be using mathematics not for its purpose, in a way, right? We want mathematics to be used in places where it is natural… and so the function needs to invite a mathematical description.

John went as far as describing the lesson as 'terrible' and some of the other mathematicians supported his stance. In their critique, they generally did not consider the specific teaching context—the particular classroom, the specific students, or the institution. Instead, their concern indicated a strong obligation to the authentic way of things to be done in the discipline – arising naturally from a genuine need for mathematics, such as modeling in physics, rather than creating an artificial use of mathematics.

The mathematicians' utter dismissal of the task led one of the teachers, Emily, to press John to explain why he found the lesson so "terrible". Additionally, she elaborated the goals of the whole lesson and particularly the value of the activity from her perspective. In her explanation, Emily described a teaching process intended to create a need for a shared language in order to analyze functions, as another teacher explained it before John's critique:

> The goals of this lesson were different, in my view… they've spent a whole year learning about straight lines, and another whole year on parabolas. so, what now? Are we going to spend a whole year on cubic functions? … The idea is that we need tools to help us investigate functions in general... and regarding the second task, drawing something you haven't seen before. That's really... the motivation behind it is to develop a common language. Using it is part of the motivation.

Emily further elaborated: "All the words we use now, such as increasing, decreasing, positive, negative […] part of important concepts, [they] still do not have this language."

Emily and other teachers envisioned a process that achieves a pedagogical goal—creating a need for new mathematical ideas among the students. Her entire description was based on mathematics, but it is possible that her interpretation of mathematical motivation was more focused on school mathematics, whereas the mathematicians perceived it in its broader sense. Additionally, during the discussion, teachers also mentioned the students' enjoyment of the activity, which provided additional value to the activity.

Discussion

This study examined how mathematicians and SM teachers relate to breaches of disciplinary values and principles of the discipline in three extreme cases where mathematicians and teachers disagreed about whether mathematical activities are suitable for SM education. Analysis of the professional obligations that emerged in the mathematicians' and teachers' discussions of these

activities highlights differing perspectives regarding the role and significance of obligation to the discipline in SM teaching. Our findings illustrate why awareness of disciplinary values and principles that teachers are expected to develop during their academic studies does not always manifest in their teaching the way mathematicians might expect. While this phenomenon has already been discussed in the literature (Herbst and Chazan, 2020), the cases in our study stand out since they illustrate vastly different perspective on obligation to the discipline even in cases that from a disciplinary perspective could be considered as 'black and white', and even when mathematicians have the opportunity to explain their perspective and persuade the teachers.

In all three cases, mathematicians identified the activities as breaching their obligation to the discipline so fundamentally that they rejected their suitability for SM education. The teachers, however, did not dismiss these concerns; instead, they sought to refine and reconsider the issues the mathematicians had raised, weighing against other professional obligations to determine their significance from their perspective. Consequently, teachers expressed alternative interpretations of the obligation to the discipline, often taking precedence over the mathematicians' view.

Our findings underline that the role of disciplinary values and principles in knowledge for SM teaching extends well beyond mere awareness. As Herbst and Chazan (2020) have observed, teachers need to interpret and weigh the significance of these values and principles in specific situations while considering other professional obligations. Thus, our study further underlines the necessity that mathematicians and teacher educators attend to teachers' disciplinary obligations not only in theoretical settings that are detached from the realities of SM education, such as academic mathematics courses, but also within the authentic context of classroom teaching.

Moreover, our study indicates that teachers should not be passive recipients but rather active participants in the articulation of the role and significance of the discipline in SM education. Furthermore, our findings suggest that there may be merit in mathematicians and teachers collaborating rather than working separately, as proposed for example, by Herbst and Chazan (2020). In our study, the mathematicians and teachers challenged each other's perspectives and pushed them to elaborate and refine them, and eventually negotiate not only whether the activities could be used but also how the activities could be used more responsibly from a disciplinary perspective while maintaining the pedagogical productivity the teachers recognized (cf. Pinto & Cooper, 2023). Such negotiations may pave the way for the development of a clearer and shared vision about the role and significance of the discipline in SM education.

Academic mathematics studies for teachers are intended to be a vital component of their professional preparation, equipping them with tools to teach school mathematics faithfully to the discipline. However, our findings indicate that presenting disciplinary obligations in a dichotomous way, without considering the authentic teaching context, may discourage teachers from fully integrating these obligations into their teaching. If our community aims to represent the discipline productively in teaching, the challenge lies in fostering a dialogue that accounts for both disciplinary and contextual obligations. Yet, this dialogue is often missing in current teachers' education frameworks. Further research is needed to investigate how such dialogue can be integrated into teacher preparation programs to better align disciplinary perspectives and professional obligations.

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Collaborative Preferences for Learning Mathematics: A Scale Validation Study

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Collaboration within mathematics has been established as being effective in providing students with crucial opportunities to develop critical thinking, effective communication, and teamwork skills. By engaging in group problem-solving and shared learning experiences, students may gain deeper insights into mathematical concepts and learn to approach challenges from multiple perspectives. However, there remains a need for a reliable instrument to capture students' preferences for collaboration. This study aims to develop and validate the Collaborative Preferences for Learning Mathematics (CPLM) scale to measure student preferences for collaboration. Exploratory factor analysis revealed a single-factor structure, and a confirmatory factor analysis conducted with a separate sample (N = 243) demonstrated a good model fit. Further testing established the scale's strong invariance, confirming its ability to reliably measure collaborative preferences over time. The CPLM offers a valid and reliable way to capture student preferences regarding collaborative learning in mathematics.

Keywords: Collaborative learning, Group learning, Team-based learning, Peer instruction, Collaborative Preferences for Learning Mathematics (CPLM) scale

Undergraduate students are offered various opportunities to engage with mathematics, including a diverse range of pedagogical approaches, assessments, and resources. It is important to acknowledge that the backgrounds, experiences, and values students bring influence their preferences for how they engage with mathematics, particularly with their peers. Understanding and accounting for these preferences enables educators to tailor their teaching strategies, create more effective group dynamics, and foster inclusive environments that accommodate diverse learner needs. However, the extent to which students prefer collaboration or individual work, especially in the context of mathematics, remains underexplored, and there is a notable lack of systematic, quantitative tools to capture these preferences on a large scale. To address this gap, this study sought to develop and validate the Collaborative Preferences for Learning Mathematics (CPLM) scale, providing a robust measure to better understand these preferences. In the following sections, We review the literature on student perceptions of collaborative practices in mathematics, outline the study's context, present the results, and discuss the implications.

Literature Review

Collaboration is widely established in the literature as an efficacious approach to learning. Grounded in social learning theories such as Bandura's social-cognitive theory (1997), collaborative settings allow students to learn vicariously, increasing self-efficacy. Collaborative learning positively affects achievement and attitudes towards learning across various disciplines (Springer et al., 1999; Tullis & Goldstone, 2020; Xu et al., 2001). Specifically in mathematics, it has been argued that collaboration positively shapes students' learning experiences and could be used as an approach to foster motivation, confidence, and achievement for some students (Cuneo, 2008; Duah et al., 2014). However, achievement is not directly predicted by the time spent studying or the proportion spent studying collaboratively (Alcock et al., 2020).

Despite these general benefits, not all students have the same inclination to work with others (Johnson & Johnson, 1995); collaborative and individual learning preferences vary significantly. For instance, gender and age have been shown to impact this, with girls showing stronger preferences for cooperative learning than boys, who tend to favour competitive and individualised learning approaches. Additionally, preferences for cooperative and competitive learning have been reported to increase with school age (Owens, 1985). Personality traits may also be a predicting factor in students' willingness to learn collaboratively. One study reported that extroversion was significantly negatively correlated with a preference for independent study, and neuroticism, a measure of emotional stability, was negatively correlated with preferences for social modes of learning (Chamorro-Premuzic et al., 2007). Furnham et al. (2008) reported that group work was preferred by younger extroverted students with low conscientiousness and low general knowledge. Furthermore, these preferences and attitudes may also vary across disciplines (Gottschall & García-Bayonas, 2008), where disciplinary norms play a significant role. In mathematics, an emphasis on individual problem-solving, commonly used teacher-centered instruction (Dietrich & Evans, 2022; Melhuish et al., 2022), and assessments that focus on procedural knowledge may limit opportunities for collaboration and reinforce a perception that mathematics is best learned independently rather than collaboratively.

Student Perspectives of Collaborative Learning

Beyond individual factors, there is considerable nuance in students' experiences that are shaped by the social context in which they operate. In broader educational contexts, students' collaboration preferences often reflect their priorities and inclinations. Some students may choose to study alone to minimize distractions, while others may prefer group study to expose themselves to diverse perspectives. Many of the key difficulties of group work identified in the broader education literature tend to highlight the importance of equal contribution from group members (Feichtner & Davis, 1984; Salomon & Globerson, 1989; Tucker & Abbasi, 2016). This issue can be especially problematic when group work is assessed and has also been reported in the mathematics education literature.

Within the context of undergraduate mathematics, one study that interviewed three cohorts of students across different universities revealed that while students generally had positive attitudes towards group work, many still expressed a preference for learning individually in mathematics (MacBean et al., 2004). Within their study, group work, which was largely experienced in informal settings outside of class, provided a sense of camaraderie between students. These casual meetings offered the time for students to discuss their collective difficulties. Additionally, the student-driven nature of these informal interactions allows students to have the freedom to use the time as they desire with peers with whom they get along. Group work done in a formal capacity, such as during lectures or for assessments, can provide similar benefits but also expose numerous drawbacks. One such drawback is the occurrence of social loafing, which refers to "the reduction in motivation and effort when individuals work collectively compared with when they work individually or coactively" (Karau & Williams, 1993, p. 681), is more salient in larger groups than in smaller groups (North et al., 2000). Working with unfamiliar group members was another issue mentioned by students who must work to establish new relationships whilst navigating mathematical challenges (MacBean et al., 2004).

D'Souza and Wood (2003) corroborated the findings of MacBean et al. (2004): students perceived group work as a valuable experience with potential cognitive benefits and saw it as an opportunity to meet new people and increase self-esteem. However, they also identified significant challenges, including the presence of less outspoken and unprepared students who

were reluctant to participate, as well as individuals who did not contribute equally. The latter aspect might be advantageous for disengaged or weaker students who benefit from others' efforts. This observation prompts a nuanced discussion about the balance between supporting weaker group members while managing unequal workloads and maintaining fairness within collaborative environments.

Further research exploring student attitudes toward group work revealed generally positive views, with most students believing that collaboration enhances the effectiveness of mathematics teaching (Sofroniou & Poutos, 2016). However, like the previous studies, Sofroniou and Poutos (2016) reported concerns about unproductive group dynamics that were salient among students. Additionally, some students even recognized that group work "can slow down the lesson" (p. 7).

While collaborating in mathematics can be difficult for students, little is known why this might be the case. One reason could be that the strong tendencies of students to compare themselves with peers may hinder their ability to collaborate effectively in mathematics (Campbell & Yeo, 2023). When reflecting on their actions during collaborative activities, students demonstrate an acute awareness of various elements of their interactions – such as how often they speak and the overall group rapport – and the underlying factors that influence these dynamics, including confidence and personality (Campbell & Yeo, 2023).

Research Aims

The aim of this study was two-fold: (1) to develop and validate the CPLM scale, an instrument intended to measure general student preferences towards learning mathematics, and (2) to establish measurement invariance of the CPLM.

Rationale

Given the challenges and potential benefits of collaborative learning in mathematics, it is essential to better understand and gauge students' preferences for learning in this context. While collaborative learning has been explored in other subjects, there remains a need for research specifically focused on mathematics education. Much empirical and theoretical research highlights the various ways that working with others in educational settings can impact student learning; researchers should maintain a keen interest in students' preferences in these educational settings.

Firstly, this exploration of student voice indicates what students, a significant stakeholder in education, are experiencing and valuing in their learning. Kouros et al. (2006) describe students' attitudes towards learning as a reflection of the quality of their experiences. Understanding their preferences for individual versus collaborative learning in mathematics can provide valuable insights into how instructional approaches can be tailored to meet their needs better and support these experiences. Secondly, in a rapidly evolving world where social interaction preferences are continually shifting, having accurate methods for evaluating students' desires and needs can be invaluable for our understanding of the nuanced views from the students' perspectives, which has the potential to enhance overall student engagement and satisfaction. Lastly, even though student views represent just one overlooked dimension of mathematical learning, it can help identify whether the practices and policies that are implemented are aligned with their wants.

This research does not aim to advocate for or against any specific instructional method or any particular attitude towards students' collaborative preferences. Instead, it offers a way to get a snapshot of undergraduate students' preferences in collaborative settings within a mathematics context. While much existing research has focused on samples from different educational levels, our study seeks to explore straightforward and accessible methods to gauge student preferences

in learning. Specifically, this work captures undergraduate student experiences in a more generalized and context-independent manner, providing a broad overview of current student preferences. Developing a validated scale to measure student preferences for collaborative learning in mathematics can provide educators with insights into how instructional strategies might be adjusted to better align with student needs and promote engagement in this traditionally individualistic discipline.

Methodology

Collaborative Preferences for Learning Mathematics (CPLM)

This short scale contains five items that prompt students to consider their preferences for collaboration in mathematics. The items gauge different contexts and settings where these preferences may vary, including when learning for high- and low-stakes assessments and in different stages of their mathematical competency (being exposed to new concepts). All items were responded to on a slider scale from 0 to 100. The prompt and items in the scale are shown below.

Prompt: "Consider yourself learning mathematics. By moving the slider, state the extent to which you prefer to do it from 0 = Individually to 100 = Collaboratively for each of the items:"

1. What is the most effective way for you to learn mathematics? (CPLM_1)
2. What is the best way for you to make sense of mathematics? (CPLM_2)
3. What is the most effective way for you to study for high-stakes maths assessments (e.g., exams)? (CPLM_3)
4. In what social setting do you prefer to be exposed to novel concepts? (CPLM_4)
5. In what social setting do you prefer to engage in problem-solving in low-stakes assessment (e.g., homework, practice exercises)? (CPLM_5)

Analysis

Exploratory Factor Analysis (EFA). To validate the CPLM, we conducted an EFA to confirm the underlying structure of the five items (observed variables) and the latent variable (CPLM). Exploratory factor analysis was performed using principal axis factoring as the extraction method, which is more appropriate for non-normal data (Costello & Osborne, 2005). We assessed the appropriateness of the data for factor analysis using the Kaiser-Meyer-Olkin (KMO) measure of sampling adequacy and Bartlett's test of sphericity. A KMO value above the .50 threshold and the rejection of the null hypothesis for Bartlett's test were considered indicators that the data was appropriate for this analysis (Dziuban & Shirkey, 1974). Cronbach's alpha values greater than .70 were considered satisfactory (Taber, 2018). The EFA was conducted on SPSS (Version 29).

Confirmatory Factor Analysis (CFA). We conducted a CFA to validate and confirm the factor structure identified in the EFA. A 'good' model fit was defined by $\chi^2/df < 3.0$, TLI > .95, CFI > .95, RMSEA < .06, and SRMR < .05. An 'acceptable' fit was indicated by $\chi^2/df < 5.0$, TLI > .90, CFI > .90, RMSEA < .08, and SRMR < .08 (Hu & Bentler, 1999). We assessed convergent validity by evaluating whether the Average Variance Extracted (AVE) exceeded .50 and Composite Reliability (CR) was greater than .70. AMOS (Version 29) was used for the CFA and invariance testing.

Multi-Group Confirmatory Factor Analysis (MGCFA). We conducted an MGCFA to test the measurement invariance of the CPLM over three different time points. We compared changes in model fit indices to evaluate the invariance, focusing on the Comparative Fit Index (CFI). A

change in CFI of less than 0.01 between models was considered indicative of good measurement invariance (Cheung & Rensvold, 2002).

Data

In this study, we utilized self-report data collected as part of a larger project. Participants rated their preferences on a 0 to 100 scale using five slider-based, close-ended items. The questionnaires were distributed via Qualtrics, an online survey platform. Two data samples were collected. The first sample consisted of undergraduate students recruited through Prolific, an online participant recruitment platform. These participants were current undergraduates from the UK or USA, studying STEM or commerce-related majors. After data cleaning, 273 valid responses were used for EFA. The second dataset was collected during the second semester of 2023 from a second-year undergraduate service mathematics course at a large university in New Zealand. This sample involved repeated measures from 224 students across three time points. Data from the first time point (N = 243) was used for CFA, and the repeated measures across the three time points were used for the multi-group confirmatory factor analysis (MGCFA) to test for measurement invariance over time.

Results

Exploratory Factor Analysis. The inter-item correlations of the CPLM ranged from .209 to .619. Bartlett's Test of Sphericity was significant, $\chi^2(10) = 322.69$, $p < .001$, and the KMO measure of sampling adequacy was .759, both indicating that the data were suitable for factor analysis. The results of the EFA yielded a single factor with an eigenvalue of 2.005, accounting for 40.09% of the variance. All five items loaded adequately to strongly to the factor (.476 to .863) (Costello & Osborne, 2005) and had communalities ranging from .227 to .744 (see Table 1). The five items in the factor show an acceptable degree of internal consistency, with a Cronbach's alpha value of .744.

Table 1. Factor loadings and communalities for each item from the EFA.

CPLM		
Item	Factor Loading	Communalities
CPLM_1	.863	.744
CPLM_2	.734	.539
CPLM_3	.507	.257
CPLM_4	.488	.238
CPLM_5	.476	.227

Confirmatory Factor Analysis. The results of the CFA showed that the one-factor model demonstrated a good fit to the data: $\chi^2/df = 1.601$, TLI = .988, CFI = .994, RMSEA = .050, and SRMR = .0230 (see Figure 1). The factor loadings ranged from .59 to .87. The Cronbach's alpha for the factor was .848, indicating good internal consistency among the items. Additionally, the Average Variance Extracted (AVE) was 0.54 and the Composite Reliability (CR) was 0.85, indicating that convergent validity was achieved. This evidence provided a strong basis for proceeding with the MGCFA to test for measurement invariance.

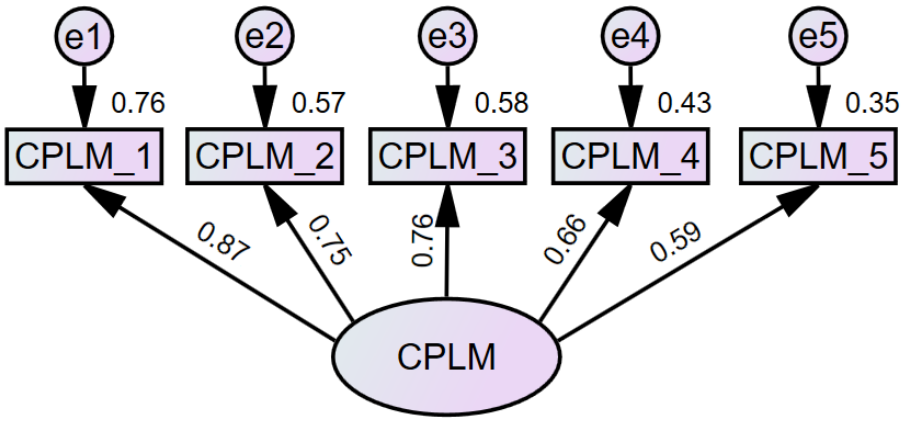


Figure 1. One-factor model for the CPLM.

Invariance Testing. The changes in CFI across the models remained within 0.01 of each model, except for the residual model, which had a change in CFI of .043 (see Table 2). From this, we can conclude that the model demonstrates strong invariance (configural, metric, scalar) but lacks strict invariance (configural, metric, scalar, and residual). Establishing strong invariance is generally regarded as adequate for determining model equivalence, allowing for valid comparisons of latent factor means across time (Meredith, 1993). Strict invariance, while rarely achieved in practice due to its stringent requirement of identical reliability across groups, is typically not considered necessary unless the research question specifically involves comparing observed variances or covariances across groups.

Table 2. Measurement invariance of the CPLM over a semester (N = 224).

Model	CFI	ΔCFI	χ^2/df	RMSEA
Configural (Unconstrained)	.993	-	1.804	0.035
Metric	.990	.003	1.803	0.035
Scalar	.991	.001	1.671	0.032
Residual	.948	.043	3.742	0.064

Discussion

This study aimed to develop and validate an instrument to measure students' general preferences for collaboration within mathematics. The results suggested that the CPLM scale is reliable and valid for measuring these preferences. EFA identified a single-factor structure, while CFA indicated the model demonstrated a good fit. Furthermore, MGCFA provided evidence of

strong invariance, confirming that the scale reliably measures the same construct across different time points.

One limitation of the CPLM is that, while it was validated in an undergraduate context, its application in other settings will necessitate further investigation and validation. Future research could expand the scale to include a broader range of collaborative experiences across different contexts, thereby enhancing its comprehensiveness and applicability.

Additionally, incorporating qualitative data, such as interviews or observations, could provide a richer and more nuanced understanding of students' collaborative preferences and the underlying factors that shape them. These insights could inform the refinement of existing tools and frameworks, ensuring they better capture the complexities of collaboration in educational settings. Such approaches would also create opportunities to explore how these preferences impact group dynamics, engagement, and overall learning experiences.

Despite the limitations, we believe this instrument offers a practical and efficient way to assess collaborative learning preferences on a large scale. Its ability to quickly gauge these preferences provides educators with valuable insight into their students' preferences towards collaboration. Such insight would enable instructors to anticipate and plan for diverse preferences within their classrooms, allowing them to better structure group activities and allocate resources. The CPLM scale is not intended to advocate for pedagogical practices that exclusively align with student preferences. Therefore, educators should use CPLM results as one of many tools to inform their teaching strategies rather than as prescriptive guidelines.

From a research perspective, the CPLM scale offers a quantitative tool to assess these preferences systematically, allowing researchers to incorporate student learning preferences into studies, which could serve as a useful variable to account for alongside other factors in future research. This integration can deepen our understanding of how collaborative dynamics influence learning outcomes and student engagement. Longitudinal studies, in particular, could shed light on how these preferences evolve over time or in response to targeted instructional interventions. In summary, the CPLM scale contributes to advancing educational research by offering a reliable tool for understanding collaborative learning preferences in mathematics.

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Go-to Strategies Used by Undergraduate Developmental Mathematics Students to Support Their Fraction Understandings

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Difficulties with fractions are well-documented for undergraduate developmental mathematics students. As such, fractions are a known gatekeeping topic for this demographic. However, research on fraction understandings for this population is scarce. In this paper, I synthesize relevant literature regarding undergraduate developmental fraction understandings and related K–12 fraction literature. I then report findings from task-based clinical interviews to share examples of various go-to strategies these participants utilized as they worked through fraction tasks. I close with a discussion relating these findings to extant literature and propose future research directions.

Keywords: Fraction Understandings, Student Thinking, Undergraduate Developmental Mathematics

Many students experience difficulties when working with fractions, a gatekeeping topic for college-level mathematics courses (Mesa et al, 2014; Ngo, 2019). In this paper, I share examples of go-to fraction strategies that Nine, Vivian, and Sofia demonstrated as they worked with fractions in task-based clinical interviews (CI) (Ginsberg, 1997). These participants were undergraduate developmental mathematics students at a large research institution in the southern United States when they participated in their respective interviews during a spring semester between 2022 and 2024. Nine viewed fractions as division problems. Vivian considered fractions to see if they were in "the majority." Sofia simplified fractions and used known percentages as benchmarks when simplification did not help.

Literature Review and Theoretical Background

Research on the fraction understandings of undergraduate developmental mathematics students is scarce (Alexander, 2013; Mesa et al., 2014). Studies involving the mathematical thinking of undergraduate developmental mathematics students build on K–12 research since undergraduate developmental mathematics students study K–12 topics while being college students (Alexander, 2013). Fraction knowledge has been shown to function as a gatekeeper for algebra readiness, though further need for research has been suggested (Booth & Newton, 2012; Ngo, 2019; Siegler et al., 2012).

In this report, I draw on the fraction literature using three of Lamon's (2020) five fraction understandings as I examine what is productive in my participant's fraction thinking. Lamon's (2020) framework includes five fraction understandings: part-whole, operator, quotient, measure, and ratio. The part-whole understanding of fractions relates the number of equal parts of a whole to the whole (Lamon, 2020). The operator understanding views fractions as a set of instructions to be carried out (Lamon, 2020). The measure understanding is a geometric meaning for fractions and involves measures, or lengths, of intervals (Lamon, 2020). For the purposes of this study, I define fractions as numbers written in the form of a/b, where a and b are not necessarily integers, such that b is not equal to zero (Lamon, 2020; Empson & Levi, 2011). By strategy, I mean the approach the participant approached the problem when working on a task. The research

question for this study is: What go-to fraction strategies do undergraduate developmental mathematics students tend to rely on as they encounter fraction tasks?

Methods

In this paper, I present data from a larger dissertation study. The overarching goal of the dissertation study is to get a picture of the way the participant is currently understanding fractions using frameworks from Lamon (2020) and Steffe & Olive (2010) and then to examine ways to create an environment to foster the strengthening of the measure understanding of fractions (Lamon, 2020). The data for this paper comes from initial task-based clinical interviews (CI) (Ginsberg, 1997). A CI is a semi-structured interview that focuses on describing the participant's current thinking and reasoning (Ginsberg, 1997; Steffe & Thompson, 2000). The CIs were conducted as the first phase in the dissertation study, in anticipation of the participant potentially continuing into a teaching experiment (Steffe & Thompson, 2000). The clinical interview data comes from video recordings, the participant's written work, and any field notes taken (Cobb & Steffe, 1983; Steffe & Thompson, 2000; Steffe & Ulrich, 2020).

Participants, Data Collection, and Analysis

Nine, Vivian, and Sofia (all pseudonyms), were recruited from randomly selected undergraduate developmental mathematics classes at a large research university in the southern United States. Each of them participated in a CI that included 12 fraction tasks, divided among two 45-minute sessions. Data was collected during the spring semesters of 2022, 2023, and 2024. I served as the interviewer, and my witnesses came from a pool of graduate students and faculty who had experience and knowledge about the CI methodology. Witnesses served based on availability. A primary camera captured the participant's written work, while a secondary camera recorded the interaction between the participant and the interviewer. Both ongoing analysis and retrospective analysis were utilized (Steffe & Thompson, 2000). For ongoing analysis, I wrote notes following each CI session to review what had taken place. During retrospective analysis, I watched the CI, wrote a thick description of each session, then analyzed the data using grounded theory techniques in the form of open and axial coding (Corbin & Strauss, 2015) to identify the strategies, or approaches, that the participants demonstrated in their sessions.

Fraction Tasks Overview

The following list contains a subset of the tasks that were used in the CI. The interviews began with two open-ended questions that asked what $\frac{2}{3}$ and $\frac{7}{3}$, respectively, meant to the participant. These questions allowed the participants to share their fraction thinking before encountering tasks that targeted specific fraction understandings. The participants were also asked to read these questions out loud, as they may be read in multiple ways.

1. *What does 2/3 mean to you?*
2. *What does 7/3 mean to you?*
3. *This figure* (a picture of a hexagon) *represents a whole. Show me* (blank). This blank can be filled in with $\frac{15}{6}$ and/or $\frac{4}{6}$, depending on the difficulty participants experienced with the previous tasks.
4. Number line task
    a. Give the participant four index cards marked with $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{8}$, and $\frac{7}{6}$ respectively. *Without using decimals, put these fractions in order from least to greatest.*

b. Give the participant a piece of paper with a line across it lengthwise. *If this is a number line, where would you place the fractions as accurately as you can?*

5. *Jade shared 5 candies among 3 friends. What amount of candy did each friend get?*
   Note: Alternative questions included sharing 12 candies among 5 friends and 2 candies among 3 friends.

Results

Nine, Vivian, and Sofia demonstrated varied go-to strategies as they worked on the fraction tasks in the CI. Nine viewed fractions as division problems. Vivian considered fractions to see if they were in "the majority." Sofia simplified fractions and used known percentages as benchmarks when simplification did not help. In the following I will explain each go-to strategy with a few illustrative examples.

Nine

Nine's go-to strategy was to primarily think of these tasks as division problems. Below, I will give examples from three tasks that illustrate this line of fraction thinking. When asked to explain what two thirds means to her in Task 1, Nine shared, "When I think two thirds, I just think two over three or two divided by three." Nine gave two examples for scenarios a teacher might give for two thirds. The teacher might give information about having pizza or brownies and the need to divide the food so they can be shared fairly. This lent to Nine viewing fractions as instructions for dividing the numerator by the denominator. She gave a similar response for seven thirds in Task 2, saying that seven thirds was "seven divided by three." This time she added that her "first instinct was generally to turn [seven thirds] into a mixed fraction." Then Nine talked through her process of using division to turn seven thirds into two and one third. Mixed numbers were not the only results of Nine's viewing fractions as division problems. When Nine was figuring out how she would share twelve candy bars among five friends in Task 5, Nine's immediate reaction was to divide 12 by five, this time using long division to find the decimal equivalent for 12 divided by five, shown in Figure 1. While we know that this problem refers to the fraction $\frac{12}{5}$, there is no evidence that Nine viewed this particular task as a fraction.

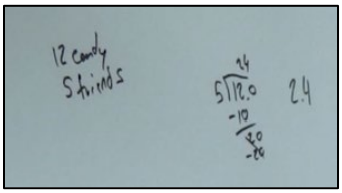


Figure 1. Nine finds how much candy each person gets when sharing 12 candies among five friends.

Vivian

Vivian's go-to strategy was to check to see how a fraction compared to "the majority," which is greater than one half. Below, I will give examples of two tasks that illustrate this comparison. In response to Task 1, two thirds meant to Vivian that "the majority of something is taken up or something has happened to it." She gave an example of three buildings in a location where two of the three buildings have people in them, shown in Figure 2. When interpreting her drawing, Vivian stated, "So, the majority of the three buildings, two thirds of the three buildings, are occupied." A second way that Vivian could describe what two thirds meant to her was to measure two thirds of a cup. As with the buildings, Vivian referred to the two thirds of a cup as "the majority of the cup."

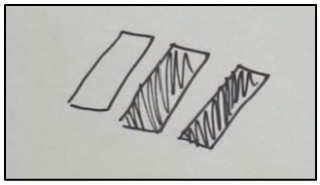


Figure 2. Vivian shared an example of what $\frac{2}{3}$ means to her.

During Task 4, Vivian decided to place the $\frac{2}{3}$ card to the right of the $\frac{1}{2}$ card, stating "Two thirds is the majority. So, that $\left(\frac{2}{3}\right)$ would probably be bigger." The $\frac{7}{6}$ card posed an issue for Vivian. To Vivian, fractions exceeding 1 did not exist. Specifically, with $\frac{7}{3}$ in Task 2, and again with $\frac{7}{6}$ in this task, she stated "seven sixths doesn't exist." Despite this issue, Vivian continued to try to analyze $\frac{7}{6}$ in this task, stating "that's (pointing to the $\frac{7}{6}$ card) more than the majority, that's more than the whole." This helped her decide to place the $\frac{7}{6}$ card to the right of the $\frac{2}{3}$ card, as two thirds is less than a whole. Vivian later described seven sixths as overflowing when she placed the fraction to the right of one on a number line. The $\frac{3}{8}$ card was placed to the left of the $\frac{1}{2}$ card because "three eighths is less than the majority." When asked about $\frac{3}{8}$ being less than the majority, Vivian elaborated that "it's less than half, so it's the smallest." This statement provides evidence that Vivian views one half as the dividing line for what is in the majority. The only time when Vivian used a benchmark other than one half was when she used one whole with seven sixths.

Sofia

Sofia's go-to strategy was to simplify a fraction and then to compare the fraction against a known percentage. Below, I will present three examples of Sofia simplifying fractions and one example of using known percentages.

Simplifying fractions. When asked what $\frac{7}{3}$ meant to her in Task 2, Sofia said that her "first thing [reaction] was to simplify" the fraction, which in this case involved working to turn $\frac{7}{3}$ into a mixed number. After determining that there were two wholes, with three in each whole, and one left over, shown in Figure 3, Sofia took out a hexagon and seven triangle pattern blocks. She arranged six triangles on the hexagon, remarking that she had one more left because she had used six of the seven triangles. She returned to her original work and interpreted her "two whole and one left" to mean that seven thirds is two and one third.

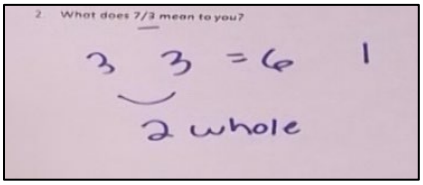


Figure 3. Sofia interprets 7/3 as two whole and one left.

Next, Sofia was presented with Task 3, where she read $\frac{15}{6}$ as "fifteen over six." Sofia's first action with this task was to simplify $\frac{15}{6}$ to $\frac{5}{2}$. Given her answer of $\frac{5}{2}$ in Figure 4a, I interpreted

Sofia's written justification to mean that she was dividing both the numerator and denominator by three in her simplification process. She reinterpreted five halves as "two wholes and one left over." Sofia initially interpreted this simplification as $2\frac{1}{6}$, but quickly changed her answer to $2\frac{1}{2}$. When she expressed that she was unsure how to continue, I asked Sofia to explain more about the "one left over." Sofia explained that since there is a two in the denominator of $\frac{5}{2}$, the one left over would indicate one half, since that is the simplified part of $\frac{15}{6}$. She still seemed to be unsure of her simplification. So, I asked Sofia to draw or represent what her answer might look like. Sofia worked to find two pattern blocks that work together to add up to the whole given in the prompt, which was the hexagon. After a bit of experimentation, Sofia chose to represent her answer with two trapezoid blocks, which she arranged on top of a hexagon block. Sofia clarified that the two trapezoids represented the two wholes as she pointed to the simplified fraction. Then she added a triangle block beside the stack, shown in Figure 4b, to represent the remainder of one sixth from the original task prompt. I interpreted Sofia's actions to mean that she was using the pattern blocks to check her fraction simplification process.

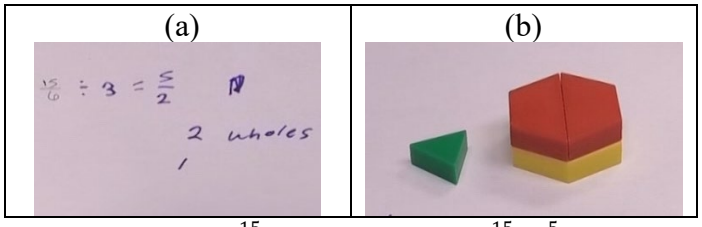


Figure 4. (a) Sofia simplifies $\frac{15}{6}$. (b) Sofia represents $\frac{15}{6} = \frac{5}{2}$ with pattern blocks.

However, Sofia seemed uneasy with her answer to Task 3, expressing that she "is so bad at fractions." To foster more confidence in Sofia, I gave her another copy of the Task 3 prompt and asked her to show me four sixths. Sofia took out a hexagon pattern block and five triangle blocks. She arranged four of the triangle pattern blocks on the hexagon block, shown in Figure 5a. Next, she took out two rhombus blocks and placed them on top of the triangle blocks, shown in Figure 5b. Sofia said that "if this (pointing to the triangles) represents four sixths, the two rhombi represent two thirds." Sofia could have left her answer in the form of the task prompt, as shown in Figure 5a. But to her, the answer was incomplete. This time, Sofia used the pattern blocks as a means of not only representing two thirds, but in finding the simplified fraction.

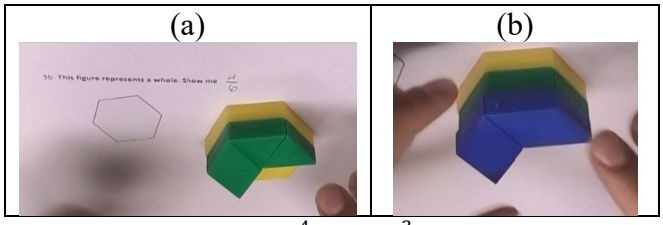


Figure 5. Sofia shows (a) $\frac{4}{6}$ and (b) $\frac{2}{3}$ using pattern blocks.

Comparing fractions to known percentages. During the number line task (Task 4), Sofia initially moved the $\frac{2}{3}$ card to the left of the $\frac{1}{2}$ card. Next, she stated that she needed to see what three and eight had in common to make sense of the fractions. I interpreted this to mean that Sofia was deciding where to place the three eighths card, and she was attempting to simplify the

fraction. After being offered scratch paper, Sofia began comparing fractions to known percentages. She wrote the fraction $\frac{2}{3}$ and set it equivalent to 66%, shown in Figure 6. I took this to mean that in that moment, Sofia's known percentage for two thirds was 66%. Next, she set $\frac{1}{2}$ equal to 50%, shown in Figure 6. Sofia is unsure about her placement of the $\frac{7}{6}$ card, so she decided to work from a percentage that she knew. She began by saying that one fifth of 100 was 20%, writing $\frac{1}{5}$ 100 = 20%, as shown in Figure 6. From there, she reasoned that $\frac{1}{6}$ as a percent was less than $\frac{1}{5}$ as a percent, adding "I'm not sure by how much, but I'm sure that it's less than 20." At this point, Sofia moved the $\frac{7}{6}$ card to the right of the $\frac{2}{3}$ card but before the $\frac{3}{8}$ card, as shown in Figure 7. She said that she was unsure she placed the $\frac{7}{6}$ card far enough to the right, but she tabled her placement of the $\frac{7}{6}$ card and moved on to considering the $\frac{3}{8}$ card. Sofia went back to her work for $\frac{7}{6}$, and noted that if she went to one eighth (writing $\frac{1}{8}$ 100 $= \downarrow$, shown in Figure 6), "then that would be even lower than from where we started from at one fifth." Sofia also noted that one fifth was "her only reference point, since she [knew] that clearly." Using this comparison to her known percentage of one fifth being 20%, one sixth being less than 20%, and one eighth being less still, she moved the $\frac{3}{8}$ card to the left of the line of fractions, indicating that three eighths was less than one half. After further contemplation, Sofia decided to leave the $\frac{7}{6}$ card at the right end of the line of cards since it was the only number that is a whole with a remainder, as shown in Figure 6 with her $1\frac{1}{6} = \frac{7}{6}$ notation.

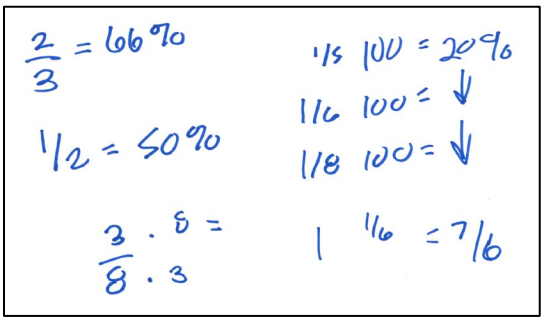


Figure 6. Sofia found known percentage equivalencies for $\frac{2}{3}$ and $\frac{1}{2}$, then reasoned to find how to order $\frac{3}{8}$.

From Figure 7, we see that the fractions have now been ordered as $\frac{3}{8}$, $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{7}{6}$. For the second part of the task, Sofia created a number line that included the fractions she had ordered. First, she marked the line with zero on the left and 100 near the right end, before adding 50 approximately halfway between them. Then she further segmented this space in increments of 10 (i.e., 10, 20, 30, 40, 50, 60, 70, 80, 90). She wrote the fractions on sticky flags and used her calibration and previously worked percent relations to place the sticky flags where she thought they should go, as seen in Figure 8. An orange flag representing $\frac{3}{8}$ was placed near the 10. A pink flag representing $\frac{1}{2}$ was placed above the 50. A green flag was used to represent $\frac{2}{3}$ and placed just to the left of the 70. At this point, Sofia added 120 at the end of the printed line then 110 halfway between the 100 and 120. A blue flag went above the 110 to represent $\frac{7}{6}$.

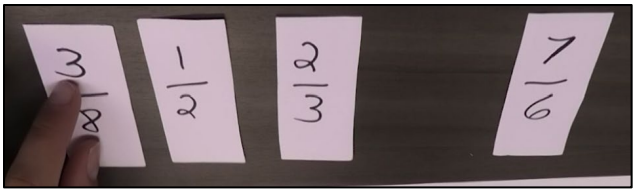


Figure 7. Sofia's final order of the fraction cards for $\frac{3}{8}$, $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{7}{6}$.

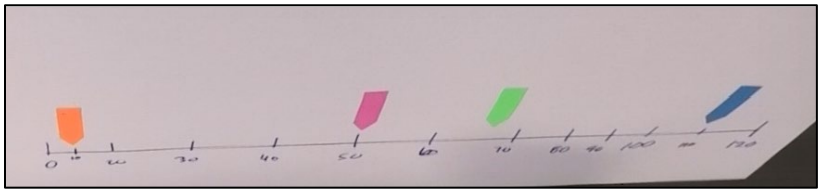


Figure 8. Sofia creates a number line with her fractions. The orange flag represents $\frac{3}{8}$. The pink flag represents $\frac{1}{2}$. The green flag represents $\frac{2}{3}$. The blue flag represents $\frac{7}{6}$.

Discussion

Nine, Vivian, and Sofia exhibited different go-to fraction strategies with both the open-ended task for what two thirds and seven thirds mean to them. These strategies also appeared when the solution path for fraction tasks were not straightforward for these participants. Their go-to strategies could be mapped onto fraction understandings and schemes from the literature. Nine tended to go to her operator understanding of fractions (Lamon, 2020), in which she viewed the fraction as a set of instructions for a division problem of numerator divided by denominator. This was carried out both to find mixed number and decimal solutions. Vivian used a benchmarking strategy. Most often, her benchmark was one half as she contemplated whether fractions were in the majority. She used a benchmark of one, or the whole, when she said that $\frac{7}{6}$ was overflowing. Sofia may have found mixed numbers, but she was doing so for a different reason than Nine did. Where Nine tended to go straight for division, Sofia wanted to find fractions she viewed as being easier to work with. Thus, Sofia's go-to strategy used commensurate fractions (Hackenberg et al., 2016; Steffe & Olive, 2010) as she sought to simplify the fractions presented to her, generally before working further with them. She also used a benchmarking strategy as she compared fractions with "known" percentages in the number line problem.

K–12 mathematics research has worked hard to provide an environment that is conducive to preparing students for college mathematics. And yet, there are students who are testing below college readiness for mathematics and being placed into developmental courses. While the K–12 curriculum often focuses on the part-whole understanding of fractions (Lamon, 2020), the go-to strategies in this paper are not within that silo. This paper provides a few examples of fraction thinking of undergraduate developmental mathematics students; it is not seeking to make a generalization of the go-to fraction understandings that are used by undergraduate developmental mathematics students. Rather, the purpose of this paper is to provide examples of what an instructor may witness as they work with students in this demographic. Sharing this information may serve to inform those working with developmental mathematics students by providing examples of the fraction understandings that developmental mathematics students hold. Future research may yield other go-to strategies used by undergraduate developmental mathematics students.

Acknowledgments

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Engendering Students' Experience of Intellectual Need: Is Task Design Enough?

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The research reported in this paper examines undergraduate students' mathematical thinking while engaged with an intellectual need-provoking (IN-P) task. Our central goal is to infer design characteristics and strategies for the pedagogical implementation of IN-P tasks that orient students to confront the key idea that the task was designed to necessitate. The results of our analysis demonstrate that provoking a cognitive state of intellectual need is nuanced and complex and extends beyond thoughtful task design.

Keywords: Intellectual need, Task design, Perturbation, Student thinking

Engaging with genuine mathematical problems crucial for students' conceptual learning of mathematics. However, much of students' interactions with undergraduate mathematics is "problem-free" in the sense that the mathematical "problems" they encounter are usually solvable by applying skills and knowledge they have already constructed (Fuller, Rabin, & Harel, 2011). These routine procedural exercises do not require students to make strategic decisions or deeply engage with mathematical concepts. While completing such exercises, students often lack a clear mental image of the problem or even an awareness that they are addressing an intellectual challenge. Schoenfeld (e.g., 2016) similarly noted that college mathematics instructors frequently engage students in routine exercises rather than complex problems. In contrast, "problem-laden" activities arise from and are maintained by students' interpretation of a mathematical task that leads them to (1) recognize that their existing knowledge is inadequate to solve the problem, and (2) develop an understanding of what new knowledge they need to construct reason productively about the problem. Well-designed tasks can promote students' awareness of the limitations of their current conceptions and support the development of new mathematical insights.

The positive impact of "desirable difficulties" and "failure-driven scaffolding" on students' mathematics learning is well documented in the literature (e.g., Dewey, 1938; Kapur, 2015; Sinha et al., 2021). For instance, Brousseau (1997) highlighted that engaging with authentic problems can challenge and evolve students' initial ideas. Chevellard's Anthropological Theory of the Didactic suggests that human activities aimed at resolving difficulties are driven by a desire to address problems within institutional settings. Piaget's genetic epistemology also posits that cognitive growth occurs when individuals experience disequilibrium, such as when they encounter unexpected situations. Empirical studies have supported these theoretical claims, demonstrating that students' experience of cognitive incongruities and their engagement in genuine problem-solving prior to participating in formal instruction supports their conceptual learning and development of mathematical reasoning skills.

Consistent with this theoretical tradition of impasse-driven catalysts for cognitive development, Harel and Tall (1991) introduced the concept of *intellectual need*, a notion later formalized by Harel (1998) as the *necessity principle:* "For students to learn what we intend to teach them, they must have a need for it, where 'need' refers to intellectual need" (p. 501). The principle of intellectual need has been extensively applied in instructional and curriculum design efforts, with empirical research supporting the proposition that students' experience of intellectual need is crucial for their learning. Despite this consensus, however, there remains a

lack of practical guidance in the literature on how to assess the extent to which students experience intellectual need and what affordances this psychological state provides for students' mathematical thinking and learning.

The research reported in this paper examines students' mathematical thinking while engaged with an intellectual need-provoking (IN-P) task. We seek to understand the "problem" that students attempted to solve and to document the precise impasses they experienced. Our central goal is to infer design characteristics and strategies for the pedagogical implementation of IN-P tasks that orient students to confront the key idea that the task was designed to necessitate. The results of our analysis demonstrate that provoking a cognitive state of intellectual need is nuanced and complex and extends beyond thoughtful task design.

Theoretical Background

The concept of intellectual need is central to the *DNR-based instruction* framework in mathematics (Harel, 2008), which outlines conditions to provoke students' experience of intellectual need and ensure they internalize and retain what they learn. Harel's *DNR* framework emphasizes that knowledge construction involves experiencing cognitive conflicts or perturbations. Consistent with Piagetian theory, intellectual need arises when students experience cognitive disequilibrium, leading to curiosity that drives them to consider important mathematical ideas and relationships. Unlike mere confusion, intellectual need involves perceiving a situation as problematic and actively working to resolve it.

Intellectual need differs from psychological need and affective need. Affective need relates to a student's initial motivation to solve a problem, driven by personal interests, social obligations, or societal goals. This motivation, along with epistemic emotions like surprise and curiosity, influences students' engagement with mathematical tasks. While affective need and epistemic emotions help initiate and maintain mathematical activity, intellectual need specifically enhances the quality and depth of that activity by driving students to develop powerful mathematical meanings. Unlike other descriptions that view intellectual need primarily as a means to stimulate motivation and engagement, our perspective emphasizes the role of intellectual need in improving the nature of students' mathematical reasoning and learning.

Methods

We engaged mathematics majors and prospective secondary teachers at two public universities in the southern United States in an IN-P task that we modified from Thompson, Carlson, and Silverman (2007) to necessitate a quantitative conception of angle measure[1, 2]. Two students from one university (Group A) and three from another (Group B) collaboratively engaged in the task during class meetings of a mathematics content course for pre-service secondary teachers. The second and third authors were instructors of the same course at two different institutions and coordinated their curriculum to align. We video recorded students' activity and discussion while engaged with the task and collected and scanned and all written products of the students' work. The instructor of Group B did not allow calculators while the students were working on the IN-P task whereas the instructor of Group A did. The students in Group A collaboratively worked on the IN-P task for one 75-minute class; the students in Group B were engaged in the task for one full 50-minute class period and the first 31 minutes of the

[1] We describe what we mean by a "quantitative conception of angle measure" in the section titled, "Conceptual Analysis of the Intellectual Need-Provoking Task."

[2] See Weinberg et al. (2024) for a discussion of the theoretical principles that guided our development of IN-P tasks.

following class meeting. All students had volunteered to participate in the study. The cameras were focused on one primary pair from each class, (Anne and Tessa from Group A and Jane and Eve from Group B). Any other student interacting with the primary pair from either class will simply be referred to as "Student."

The research team collaboratively engaged in grounded theory procedures to analyze the data. We began by developing a provisional set of open codes. Each of the three authors then conducted several iterations of independent analysis of the video data, between which we negotiated our individual interpretations to inform revisions and extensions to our open coding framework. Once we established a stable set of descriptive open codes, each author reanalyzed the videos and the research team again met several times to negotiate differences in our coding and to collaboratively construct an aggregable set of analytical notes that described our justification for applying each code and documented the significance of each coded instance on the students' mathematical thinking and activity. Our subsequent elaboration and refinement of these notes, which involved frequent reengagement with the primary data, resulted in the findings we present below.

Conceptual Analysis of the Intellectual Need-Provoking Task

Students often conceive of angle measure as a quantification of an angle's "openness." Several scholars (e.g., Moore 2010, 2014; Moore et al, 2016; Tallman & Frank, 2020; Thompson, Carlson, & Silverman, 2007) have documented the affordances of understanding an angle's measure as the length of a subtended arc, measured in a unit that varies proportionally with the circumference of the circle centered at the angle's vertex containing the subtended arc. The ratio of the magnitude of the subtended arc and the magnitude of the unit is invariant for an angle with a fixed amount of "openness."

To support students' identification of subtended arc length as the attribute of a geometric object for which angle measures are quantifications, we modified a task from Thompson, Carlson, and Silverman (2007). Our IN-P task simply prompted students to sketch a graph of the function $f(\theta) = \sin(2\cos(\theta))$, with θ measured in radians. Our goal was to problematize students' understanding of the quantity (subtended arc length) represented by the argument of a trigonometric function. Students who conceptualize "$\sin(x)$" as the ratio of the side length of a right triangle opposite the interior angle labeled "x" and the length of the triangle's hypotenuse will likely struggle to interpret this the composite trigonometric function $f(\theta) = \sin(2\cos(\theta))$ in a way that enables them to sketch an accurate graph. As an IN-P task and introduction to a conceptual approach to angle measure and trigonometric functions, we did not expect students to interpret the argument $2\cos(\theta)$ as "twice the horizontal distance of the terminal point to the right of the vertical diameter of a circle centered at the vertex of an angle in standard position." But we did anticipate that students might confront the issue of how the quantity represented by "$2\cos(\theta)$" might be the kind of "thing" (i.e., measurable attribute of an object) that makes sense as the argument of the sine function. We expected that reasoning productively about the task would require understanding the input variable as a representation of the varying values of an angle's measure (as a subtended arc length measured in units of the arc's radius), the argument of f as the length of an arc subtended by a second angle, and "$\sin(2\cos(\theta))$" as the vertical distance of this second angle's terminal point above the horizontal diameter of the circle centered at the vertex of the angle, which is in standard position. See Figure 1 for an illustration of these quantities.

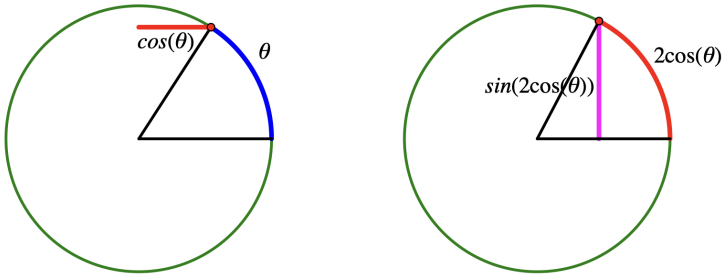


Figure 1. Quantitative interpretation of $f(\theta) = \sin(2\cos(\theta))$.

Results

We present the results of our analysis of two groups of students (Group A and Group B), respectively participating in the study from two different universities. We contrast these students' interpretations of the IN-P task, their goal orientations, and the impasses they encountered to evaluate the extent to which the students' engagement with the task necessitated the meanings we intended to promote.

Group A

The students' initial discussion focused on the graphical transformation contributed by the constant in the expression $\sin(2\cos(\theta))$. They considered whether this constant "stretched" or "compressed" the graph, or if the "2" affects the amplitude (Anne suggested that it might "make sine taller"). They then considered the graphical behavior of $y = \sin(x)$ and $y = 2\cos(x)$ separately and attempted to use sketches of these graphs to reason about the graph of their composition. Anne and Tessa quickly lost confidence in their initial strategy, and transitioned to computing values of $2\cos(\theta)$ for integer multiples of $\pi/4$ in an effort to plot points on a graph. After a long pause following the students' realization that they could not mentally compute $\sin(2\cos(\theta))$ for $\theta = 0$ and $\theta = \pi/4$, the instructor prompted the students to "try to articulate what about this task is difficult and what you are currently struggling with?" The students acknowledged that they were having difficulty with "having cosine inside of sine." Tessa suggested, "cosine and sine are like, they're obviously not inverses of each other, but they're shifted functions. … Then you also have a factor of two, so you're changing that relationship by two. There's just multiple things we have to juggle." Tessa considered how the graph of sine being a shift of $\pi/2$ units to the right of the graph of cosine might influence the graph of f. Anne followed by explaining that while their knowledge of the coordinates of terminal points corresponding to convenient angle measures on the "unit circle" was appropriate for computing values of $2\cos(\theta)$, there were several values of $\sin(\cos(\theta))$ that they were not able to compute using coordinates on the unit circle.

The instructor attempted to direct the students' activity away from their effort to plot points and instead to infer the behavior of the graph that connects these points. He suggested, "Is there a way to think about how output values are related to input values as θ varies?" Tessa was confident that the graph of f "is going to oscillate because of what we know based off sine and cosine. Um, the bounds at which it is oscillating between … that's the part that we would have to compute more." Tessa expressed the need to compute enough values of $f(\theta)$ for select values of θ to infer the graphical behavior of f between these input values. She expected that these computations might reveal "the bounds at which it is oscillating between." Anne expressed that she thought about the problem similarly to how Tessa described her reasoning.

In response to being asked to propose a question they would want answered to help them progress towards a solution, Tessa suggested, "Confirmation on basic properties and rules for layering [i.e., composing] those things." Tessa was oriented to assume that there was some algebraic property or rule for composing trigonometric functions presented in a prior course that she was currently unable to recall. In contrast, Anne expressed that she wanted an answer to the question, "When you take a radian measure and put it through a trig function, will you get a radian measure out still? That lets me reason if what I'm putting into sine makes sense."

Anne's awareness of the need to associate a quantity (i.e., measurable attribute of an object) to the expression "2cos(θ)" so that she might be able to reason about the sine value of this argument penetrates a key issue in developing a productive meaning for angle measure: How can the quantities for which sin(x) and cos(x) are respective measures be the same kind of quantity represented by the input x? Anne pursued this reflection and proposed the additional query, "The thing is when you're graphing these, I realized that the 1 and –1 that you're oscillating between, is that a radian measure? The y-value that we get out after putting in an x-value?" After considering her question, Anne concluded, "Whatever we're putting into sine doesn't make sense to put into sine." Tessa agreed and attempted to articulate a justification for Anna's conclusion:

> We were talking about when you do cos(0) and you get 1 as the output, that 1 corresponds to your length of your radius of your circle, like when you're thinking unit circle, and so you wouldn't plug a radius unit back into sine. You need to plug in radians, so you need to convert that.

Tessa's agreement with and elaboration of Anna's comment demonstrates their initial lack of association between quantities represented by sin(θ) and cos(θ) being measured in units of the "length of the *radius* of your circle" and the quantity represented by an angle's measure in *radians*. Still, that the students interpreted the task in a way that guided them to consider the quantities represented by the inputs and outputs of trigonometric functions has likely implications for their orientations towards and expectations for learning about angle measure in subsequent lessons.

Group B

After being given the prompt to sketch the graph of $f(\theta) = \sin(2\cos(\theta))$, both Jane and Eve began to evaluate the function at the angle measures for which they knew the value of cos(θ) (see Figure 2). Both Jane and Eve's scheme for graphing functions included plotting points based on evaluating functions for certain input values. This was demonstrated by both Jane and Eve as they would later add axes and plot points based on the values they computed. Their graphing of trigonometric functions additionally included selecting integer multiples of $\pi/2$ to produce readily known values for the argument of f.

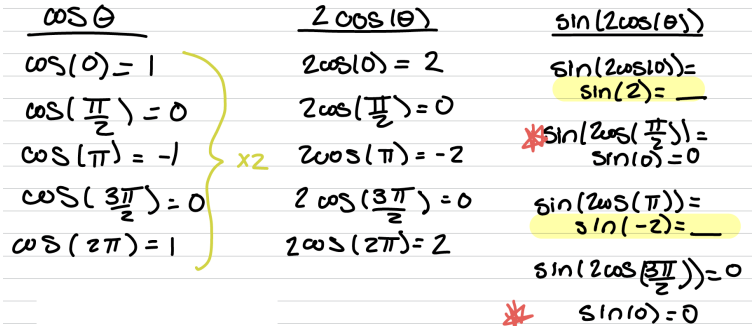


Figure 2. Eve's notes for evaluating cos(θ) *for convenient angle measures and how known values from* cos(θ) *can be used to evaluate* sin(2cos(θ)).

After about five minutes of working, Eve observed that she could evaluate $f(\theta) = \sin(2\cos(\theta))$ for $\theta = \pi/2$ and $\theta = 3\pi/2$ (see the equations marked with asterisks in Figure 2), but was unable to evaluate $f(\theta) = \sin(2\cos(\theta))$ for the input values 0, π, and 2π because she was unable to determine exact values for sin(2) and sin(–2). Since the function f is a composition of two trigonometric functions, values of $2\cos(\theta)$ are often inconvenient as the argument into $\sin(\theta)$ if a student is looking for known values from "the unit circle."

For the equations corresponding to sin(2) and sin(–2), Eve then added the blank line to the right of the equality to indicate not knowing these values, later coming back to highlight these equations to reemphasize how angles convenient for evaluating $\cos(\theta)$ did not reliably produce known values for $\sin(2\cos(\theta))$. When comparing their work, Jane described her approach and admitted that she too was unable to evaluate f for particular values of θ. "And I'm still running into that problem, where I don't know the sine of these values [pointing to her table of values that contained a column for $\sin(2\cos(\theta))$ that was empty except for when $\theta = 0$]." Jane and Eve problematized their situation as follows:

Jane: Yeah. Well not, like, sin(2).
Eve: Like, sin(2), I don't know that. [...]
Jane: [Looking at Eve's notes] So basically, all we're getting is the pi-over-twos [likely in reference to $\pi/2$ and $3\pi/2$].
Eve: Basically. But I don't know what it does in between.

At this moment, Jane and Eve had encountered an impasse that their activated cognitive structures were insufficient to resolve. This impasse included their recognized inability to produce values for the output of $y = f(\theta)$ for no more than a few select values of the input. Because of this, they acknowledged not knowing the behavior of the function between these select few values of θ. The students then proceeded to locate additional points and infer the graphical shape between the points based on contextual clues. For example, Jane indicated searching for the y-intercept. Both students confused locating the y-intercept with solving $\sin(2\cos(\theta)) = 0$ for θ to eventually get $\theta = \pi/2$. Eve noted how multiple values for the input produced identical output values, such as $\theta = \pi$ and $\theta = -\pi$, both of which correspond to an output value equivalent to sin(–2). Eve then leveraged her knowledge of the basic shapes of the graphs of $y = \sin(\theta)$ and $y = \cos(\theta)$ to produce the graph of $y = f(\theta)$ displayed in Figure 3.

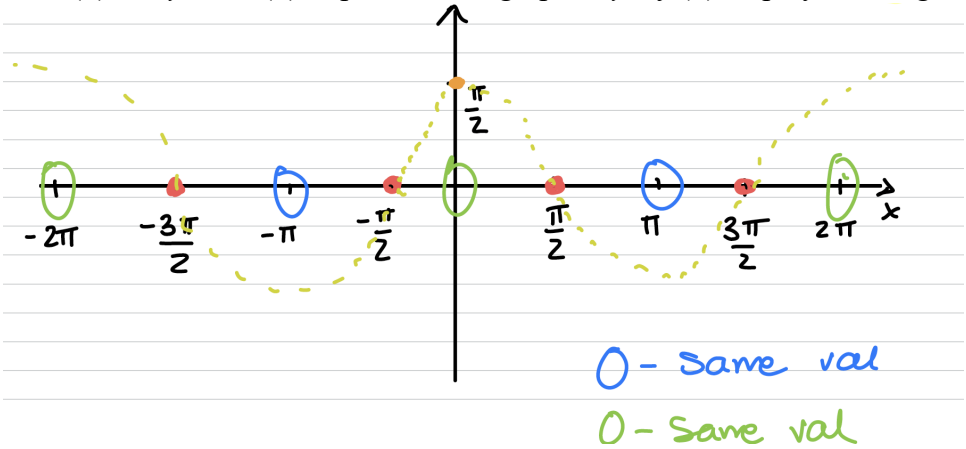


Figure 3. Eve's initial graph for $f(\theta) = \sin(2\cos(\theta))$ where she notes where particular values of input would correspond to the same value of output.

Jane momentarily proposed an alternative graph that did not include negative output values while Eve indicated thinking about "regular sine and cosine graph." They concluded that the graph of $y = f(\theta)$ must "go under" the horizontal axis because "both" graphs for $y = \sin(\theta)$ and $y = \cos(\theta)$ "go under" the horizontal axis. Other attributes of trigonometric functions were also mentioned by the pair, such as period and amplitude. Following this, Eve declared that "We have a sketch." She went on to acknowledge that she still wanted to know the value for sin(2) and indicated that if she knew this value, then she would know if her graph was accurate.

Notably absent from Eve and Jane's expressed reasoning to this moment were indications of them coordinating the quantities represented by the input, argument, and output of *f*. Although the students labeled locations on the unit circle and evaluated functions for particular values of θ, they did not indicate, for example, conceiving of $\cos(\theta)$ as a measurement of horizontal displacement of the terminal point from the vertical diameter of a circle centered at the angle's vertex. Nor did they indicate that via the function composition, the argument of the sine function $2\cos(\theta)$ could be conceived of as the directed length of a subtended arc that is twice the length of the aforementioned horizontal displacement. And, most related to the students' experience of intellectual need, neither Eve nor Jane appeared to confront the issue of how the output of one trigonometric function could be the kind of thing that can serve as the input into another trigonometric function. Although these students had encountered an impasse and eventually produced a graph for which they were confident represented the covariational relationship between θ and $\sin(2\cos(\theta))$, they had not yet confronted the central ideas rooted in quantitative reasoning about angle measure that we had designed the task to necessitate.

Discussion

Our study reveals significant variations in how individual students interpreted and engaged with the intellectual need-provoking (IN-P) task. These differences stem from students' diverse assumptions about the prior knowledge and mathematical meanings that must be applied to successfully complete the task. Our findings demonstrate that tasks alone have limited effectiveness in inducing a cognitive state of intellectual need. Instead, achieving the desired cognitive engagement requires careful pedagogical strategies and a deep understanding of the students' pre-existing conceptions and expectations. This study highlights that fostering intellectual need is more intricate and nuanced process than previously suggested in the literature.

The findings from the study suggest several important implications for educational practice. First, simply implementing intellectual need-provoking (IN-P) tasks is not sufficient. Educators need to carefully design tasks to ensure they appropriately connect with students' prior knowledge. Instructors also need to consider the pedagogical actions that might challenge students' thinking about the core mathematical ideas reflected in the IN-P task. This may involve providing in-the-moment scaffolding of tasks to orient students' attention to the central mathematical issue(s) that the task was designed to necessitate, and to do so in ways that are sensitive to the strategies a student has already employed, the operational goal the student is working to accomplish, and the mathematical facts, skills, and meanings the student brings to the task. Effective teaching requires flexibility and responsiveness. Educators should be prepared to modify their pedagogical strategies based on their ongoing assessment of students' engagement and comprehension. This may include adjusting the level of difficulty of tasks, providing additional explanations, or incorporating different teaching methods to better support students' conceptual learning of mathematics.

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The Politics of Aesthetics of Mathematical Coherence: STEM Students' Aesthetic Interpretations of Cross-Curricular Stories

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As part of a larger participatory, arts-based study, six STEM students engaged in sustained arts creation and reflection across their curricular experiences as a form of holistic, multimodal, and multisensory aesthetic critique of their cross-curricular mathematical experiences. In this contributed report, I detail instances where students' perspectives on curricular coherence (i.e., their aesthetics of coherence) deviated from privileged forms of disciplinary (logico-rational) coherence in mathematics education. In doing so, I highlight the exclusionary politics of aesthetics that result from a narrow adherence to one perspective of curricular coherence and the epistemic harm this can inflict on students in relation to how they position themselves as learners and doers of mathematics. These examples suggest a need for additional criticality concerning our often-unquestioned definitions of curricular coherence and acknowledgement that mathematical coherence seeking is governed by both aesthetic and logical forces.

Keywords: Curricular coherence, Student perspectives, Aesthetics, Arts-Based research

A common educational assumption is that curricular coherence is a pre-requisite for a "good" curriculum. Buchmann and Floden (1992), for instance, touted coherence as one of the "guardian angels of [curriculum] reform" (p. 4). In mathematics education, this perspective has persisted both nationally and internationally. For example, the recent ICMI study of mathematics curriculum reforms around the world puts coherence on a pedestal as one of the organizing themes for reform, with the final report stating matter-of-factly that "Coherence is likely to be a universal aspiration" (Morony, 2023, p. 148).

While curricular coherence is often unquestioningly accepted as desirable for student learning, some researchers have urged caution, arguing that "curricular coherence" is loosely defined with no widespread agreement over its meaning (Muller, 2022; Thompson, 2008). Canrinus (2024), for example, likened curricular coherence to a "disco ball" (p. 258) with several distinct forms and conceptualizations. Buchmann and Floden (1992) went further, referring to curricular coherence as a "rebel angel" escaping singular definition given that "coherence allows for many kinds of connectedness, encompassing logic but also associations of ideas and feeling, intimations and resemblance, conflicts and tension, previsagements and imaginative leaps" (p. 4). Indeed, the polysemous nature of "coherence" is reflected in its etymological origin, the Latin word *coharere*, meaning "to stick together" (Kolln, 1999). In other words, a "coherent whole" is one in which its parts are perceived as sticking together—or being related—in a satisfactory way. The upshot is that judgements of coherence involve a *holistic interpretation* of an object (curricular, artistic, or otherwise) formed based on one's aesthetic sensibilities and preferences for structural logics (Aschenbrenner, 1985; Repp, 2017).

For this reason, research on curricular coherence and designing coherent curricula in practice requires the consideration of multiple stakeholder interpretations of coherence (Bateman et al., 2009; Modeste et al., 2023). The perspectives of education researchers, disciplinary experts, and to a lesser degree teachers and administers have been investigated; however, outside of mathematics teacher education (Nguyen & Munter, 2024), *students'* perspectives on curricular coherence have only been peripherally considered. One result is that *disciplinary* forms of

curricular coherence have become the de facto status quo for curricular coherence in mathematics education (Cuoco & McCallum, 2018; Morony, 2023). From these perspectives, a coherent curriculum is one that consists of "a sequence of topics and performances that are logical and reflect, where appropriate, the sequential or hierarchical nature of the disciplinary content from which the subject matter derives", which requires that the particulars should evolve to "deeper structures inherent to the discipline" (Schmidt et al., 2002, p. 9).

These interpretations of curricular coherence reify a particular choice of mathematical aesthetics. As famed mathematician Henri Poincaré remarked:

> Mathematics is the art of giving the same name to different things . . . when language is well chosen, we are astonished to learn that all the proofs made for a certain object apply immediately to many new objects; there is nothing to change, not even the words, since the names have become the same. (1908/2012, p. 375)

Benis-Sinaceur (2014) called this form of generalization *idealization*, which involves "leaving aside or discarding all other aspects" thereby "extracting a *form* from sundry situations" (p. 94, emphasis in original). Disciplinary forms of coherence mirror this so-called art of idealization, privileging an aesthetic of "FUGS" (that primarily spotlights practices of Formalizing, Unifying, Generalizing, and Simplifying, see Hausberger, 2017, p. 418). Greeno (1992) has called this a "thinking-with-basic" philosophy in the sense that all concepts are seen as building hierarchically from the "basics". But what is "ideal" for some is not "ideal" for all. Not even mathematicians universally endorse such a limited form of aesthetics (Hausberger, 2017)!

Indeed, Sikorski & Hammer (2017) argued that there is a danger in a strict reliance on premeditated, expert interpretations of curricular coherence, suggesting that we must also consider students' idiosyncratic, in-the-moment coherence seeking in its many forms. Premeditated coherence, they argued, often undermines and undervalues the active role students play while seeking coherence: "sequences that attempt to guide students through a series of investigations to arrive at the canonically accepted (and 'coherent') explanation may do so at the risk of students' own epistemic agency in deciding what coherence to seek next" (p. 935).

Most importantly, there are grave ethical implications of privileging a singular, narrow view of curricular coherence. Buchmann and Floden (1991) observed that curricular coherence often has evaluative overtones which associate "coherence" with other aesthetic notions often viewed as universally positive, like "harmony" and "wholeness". They concluded that any perspective on curricular coherence required critical examination before being taken for granted as an inherent good. In other words, interpretations about what is "coherent" are fundamentally value-laden carrying with them (implicit) assumptions about the nature of learning and what "make sense" (Shemmer, 2012). As such, Richmond et al. (2019) argue that we must always question, "Coherence for whom? According to whom? To what end(s)?" (p. 188). Without such criticality, well-intentioned efforts to design curricula that is coherent from one perspective may inadvertently convey harmful messages about students who interpret these curricula as *in*coherent—i.e., that these students are *themselves* "incoherent," "bad," "not smart," "do not make sense", etc. (see e.g., Appelbaum, 2010).

In this participatory, arts-based study, I investigated the ethical implications when students' perspectives of curricular coherence and their aesthetics of coherence clashed with entrenched forms of disciplinary coherence. Six student-artists participated in sustained individual and collective arts creation and reflection across their curricular experiences with the goal of fostering "conspiratorial conversations" (Barone, 2000, 2008) on the topic of curricular coherence. An aim of these critical conversations was to destabilize privileged politics of

aesthetics underlying disciplinary coherence by drawing attention to a multiplicity of students' aesthetic sensibilities of coherence through the generation of aesthetically and emotionally impactful artistic educational critique (Barone & Eisner, 2012). In this contributed report, I focus on two research questions: (RQ1) What interpretations of coherence did these STEM students express that deviated from disciplinary forms of curricular coherence? (RQ2) What ethical implications did students' alternative aesthetics of coherence have on how they positioned themselves as learners and doers of mathematics?

Curriculum as a Storied Artform

I conceptualize curriculum as a form of art (Eisner, 2004) in the sense its content and *aesthetic form* cannot be disentangled, meaning that aesthetics cannot be ignored in curricular analysis. Specifically, I adopt the perspective that the mathematics curriculum can be metaphorically conceptualized as a *storied* artform (Dietiker, 2015b). As mathematics curricula enculturate students into the discipline of mathematics (Brown, 2022; Plut & Pesic, 2003), the stories they tell—and the *ways* these stories are told—should not be an afterthought. Indeed, the holistic form and structure of a story interpreted as a kind of narrative conveys disciplinary-cultural aesthetic, ethical, and onto-epistemological values about learning, living, and being (Stephens & McCallum, 1998). Following Dietiker's (2013, 2015a) curriculum-as-story metaphor, I take a reader-oriented perspective on narrative interpretation based on literary theory (Bal, 2017). From this point of view, meaning is not contained in the text itself but instead developed through a transactional process as the reader engages hermeneutically with the text (Rosenblatt, 1988; Weinberg & Wiesner, 2011). Through this active reading, *the plot*—a reader's aesthetic reactions to the unfolding of the story—is constructed based on their holistic interpretation of all story elements (e.g., characters, settings, themes) coordinated with their reactions to how the story events unfold. Importantly, the plot is constructed through both *logical* and *aesthetic* sense-making as a reader engages with (curricular) stories (Richman et al., 2019). From this reader-oriented point of view, curricular coherence is *not* an objective property of the curriculum itself but rather an idiosyncratic interpretation of (in)coherence a student makes while actively engaging with curricular stories.

The upshot of this theoretical framing of curriculum is that students' aesthetic sensibilities influence their curricular engagement and therefore learning of mathematics (Wong, 2007) as they construct subjective personal meanings and conclusions (Sinclair, 2005) concerning the (in)coherence of said curricular stories. In other words, one's personal aesthetics are fundamentally intertwined with how they make sense of the world and learn mathematics. By "aesthetics" I do not just mean judgements of beauty but rather "knowledge acquired through sensory perception" (Marini, 2021, p. 40), in line with its original etymological meaning where epistemology, aesthetics, emotion, and sense-making are inseparable (e.g., Dietiker et al., 2023). From this perspective, aesthetic experience (Dewey, 1934/2005) is an idiosyncratic response a person has to a particular object or event (e.g., a cross-curricular story), which could be positive (joy, wonderment, curiosity), negative (disgust, boredom, disappointment), neutral, or an eclectic mix of all these. Such aesthetic responses have been shown to play several roles in both professional mathematicians' and students' mathematical activity (Burton, 2004; Sinclair, 2004). For example, wonderment, surprise, and curiosity have been repeatedly linked to sustained engagement with mathematical inquiry (Dietiker & Richman, 2021; Gadanidis et al., 2016). Aesthetic considerations have also been shown to influence the nature of one's mathematical inquiry, including the questions one deems as (un)interesting and (not) "worthy" of further consideration (Fiori & Selling, 2016; Jasien & Horn, 2022).

Personal aesthetics are not developed within a vacuum but in relation to the *intersubjective* aesthetic sensibilities of the socio-cultural contexts individuals inhabit (Chilton et al., 2015; Conrad & Beck, 2015). Rancière (2000/2004) argued that which aesthetics become privileged within a community of practice (e.g., the discipline of mathematics) is political insofar as shared aesthetic practices result in distributions of what is considered sensible, often in ways that are covert. Consequently, aesthetic sensibilities (e.g., privileged views of curricular coherence) that are dominant within a socio-cultural context (e.g., the mathematics classroom) have a powerful influence on not only an individual's personal aesthetics but also how one's personal aesthetics are perceived (by the individual *and* others from within the socio-cultural context) (de Freitas & Sinclair, 2014). In this way, intersubjective aesthetics becomes a yardstick against which all other aesthetic sensibilities are measured and judged: privileged socio-cultural aesthetic sensibilities become a way of policing who/what is considered "aesthetic" and who/what is not.

Students' Perspectives on Curriculum in RUME

Research in undergraduate mathematics education displays many of the same general trends I detailed earlier concerning (a) a mostly unquestioned privileging of disciplinary forms of curricular coherence (e.g., Tague & Czocher, 2016) and (b) a dearth of curriculum research that centers students' holistic judgements about the holistic curriculum. There are a small number of studies that attend to the idiosyncratic ways students' engage with curricular stories at the grain size of a single course or lecture (Andrà, 2013; Krupnik et al., 2018; Weinberg et al., 2014) or, more broadly, to students' interests and what they value *across* their mathematics courses (Asada et al., in press; Leyva et al., 2022), but much less attention has been paid to students' general (aesthetic) perceptions of the undergraduate mathematics curriculum. One exception is a small handful of studies investigating mathematics *graduates'* retroactive reflections on whether their curricular experiences prepared them for their careers (King et al., 2017; Wood, 2010).

Participants and Methodology

This contributed report is based on a larger arts-based study (Tyburski, 2024) where I recruited six undergraduate STEM students from various disciplines who had just successfully completed multivariable calculus. Students were recruited from different STEM disciplines to spotlight a variety of (disciplinary) aesthetic sensibilities and promote the sharing of a multiplicity of perspectives of curricular coherence across their mathematics courses.

Student-artists came together for a series of four approximately one-hour sessions of individual and collective arts creation and reflection on their curricular experiences, with an emphasis on their interpretations of the (in)coherence of cross-curricular mathematical stories. I started by asking about their definitions of (in)coherence, beginning in familiar, everyday contexts ("a story, a TV show, or movie, a song, a conversation, etc.") and ending with the cross-curricular mathematical context. Next, students created artwork to reflect on the coherences or incoherences they had noticed or felt across their undergraduate mathematics courses. Following an arts-based research approach (Leavy, 2018), participants created and then discussed art in the form of their choosing to allow for a holistic, multimodal, and multisensory investigation of the learned mathematics curriculum (Bagnoli, 2009; Papoi, 2017). This centered opportunities for reflection on the roles that *aesthetic* as well as logical forces played in making curricular stories connect and cohere (Clark & Rossiter, 2008; Sinclair, 2009) across their everyday mathematical experiences. It also enabled student-artists to depict both positive and negative emotions related to these experiences. To allow for cross-pollination of perspectives (Leavy, 2020) on curricular coherence across artists and their artwork, private exhibitions of their art were used to elicit

richer group conversation and subsequent mathematical storytelling about their interpretations and feelings concerning the (in)coherence of their cross-curricular experiences. Interactions were spread over a period spanning three months to allow sustained space for artists to consider alternative perspectives and (co-)develop more nuanced critiques of their curricular experiences.

In line with most arts-based research, analysis went beyond reducing artwork, curricular critique, and complex discussion into a list of collectively agreed upon themes (i.e., a *similarity* analysis) (MacLure, 2006, 2013). As McNiff (2018) put it, "Of course, themes exist in life and art, but there are many other things too—characteristics, features, aspects, principles, ideas, patterns, structures, designs, compositions, similarities and differences" (pp. 29–30). In particular, analysis was conducted in two flanks: (a) a *participatory analyses* conducted by student artists across both group discussions based on each other's artwork and (2) a *collage analysis* (Butler-Kisber, 2008; Scotti & Chilton, 2018) that I used to juxtapose students' metaphors for curricular coherence with those found in the literature. Specifically, by cutting out ephemera from students' curricular critiques (i.e., visuals from photocopies of their art and transcripts from our individual and group conversations), I was able to *live and breathe* students' perspectives on coherence to further empathize with and understand the nuances of these perspectives (Margolin, 2014). I purposefully omit an image of the collage because my aim is to spotlight *students'* interpretations of curricular coherence (rather than my own). However, further methodological details and the collage can be found in Tyburski (2024).

Learning From Student Artwork

Student-artists regularly acknowledged how judgements of coherence are idiosyncratic and subjective. Fred, for example, explained how his experiences across mathematics courses felt "flowy" and coherent, but then quickly added the caveat that "Some people kind of struggle with math and, yeah, we all have our own strengths and weaknesses. I'm not really a good essay writer, so that is probably coherent to someone and that's not coherent to me". As VKN put it, "a lot of people don't experience math the same way I do . . . there are different lenses and different levels of coherence". Comments such as these foreshadowed *several* interpretations of curricular coherence that deviated from disciplinary, logico-rational forms of coherence.

Among these, I focus herein on different *emotional* forms of coherence students proposed. For example, wuyen shared that emotional investment played a leading role in interpreting whether a story (mathematical or otherwise) was coherent to her. For a story to make sense to her, it had to make her feel: "I think emotional [*sic*] is a big component for me to make things make sense because I'm a very emotionally driven person, more than logic for sure" In particular, she found Taylor Swift's discography to be coherent because of the emotional stories told in each of her songs and albums. She also expressed a similar view of mathematical coherence when I introduced the curriculum-as-story metaphor late in our first group discussion:

> The different applications of math as a story was [*sic*] very interesting because to me during our first conversation, we were talking about how stories make sense to me if it was emotionally touching in some way to me. So, applying that to the bigger idea now of the whole math experience being a story, I feel like emotions still should play a part.
>
> Well, not *should*. For me, it *does* play a part in the math journey to make it make sense.

wuyen expressed a clear aesthetic preference for stories that involved emotion and made her feel. For her, *these* stories were coherent. Meanwhile, Fred likened his love for math to his love of Disney animated movies, concluding that "I'd say math is kind of similar, *if* you care about math. I kind of care a little too much about math. Other people don't give a shit about it". In both

these cases, wuyen and Fred alluded to the inseparable relationship between aesthetic sensibilities, emotion, and preferences for particular kinds of (curricular) stories.

Emotions were not just peripheral for these student-artists: they played a leading role in their holistic aesthetic judgements of the (in)coherence of cross-curricular mathematical stories. VKN, for instance, regularly brought up a negative experience she had with unit vectors in multivariable calculus across our interactions: "I passed multivariable calculus. Got a 4.0 even. I 'understood' it. And yet, I am sitting here trying to remember what the hell unit vectors are". When I asked VKN follow-up questions, I was surprised to learn that (at least from my perspective) she understood the normative definition and uses of a unit vector, at least the recurring ones privileged most by the traditional multivariable calculus (MVC) curriculum. Rather, she was hung up primarily on how a particular unit vector was used in just *one* formula that *did not* recur throughout the MVC curriculum. From the perspective of a disciplinary expert, I perceived this formula to be largely insignificant to the overarching story of MVC, yet this local experience of extreme *emotional* incoherence colored her entire perception of the MVC curriculum as *in*coherent. This was not an isolated case: Joy similarly reflected about how she perceived her cross-curricular mathematical experiences to be largely *incoherent* for the years it took her to feel comfortable with logarithms. Just like when reading any literary story, *emotional* factors were the deciding factor between whether a cross-curricular mathematical story was enjoyable and coherent or confusing and incoherent for multiple student-artists.



Figure 1. Joy's The Smoke Trail

Joy's *The Smoke Trail* (Figure 1) is one of most evocative pieces in terms of conveying the central role that emotion played in some students' judgements about mathematical (in)coherence. In her art, Joy reflected on how she frequently *saw* the seemingly coherent "fire" of the mathematics curriculum but would often struggle to reach this fire in practice (i.e., to interpret her mathematics experiences as fully coherent), even when she had a smoke trail leading her to it. The deceptiveness of the smoke trail and how it "would come into [Joy's] lungs and choke [her] out as opposed to gently guiding [her] along" (as expressed in her artist statement) were part of why her entire painting takes place next to a frozen lake. As she shared, "I started with the color blue because that's how [mathematics] made me feel. . . . isolated. Sad." The frozen-over blue lake that dominates the foreground of Joy's art emphasizes that judgements of mathematical coherence for her are governed by a looming sense of sadness, as well as other emotionally painful memories that she likened to being choked, drowned, or burned alive. For Joy, mathematical coherence seeking was often panic-inducing and marked by negative emotion.

Discussion and Conclusion

In light of these student perspectives on curricular (in)coherence that deviate from logico-rational forms of curricular coherence, the epistemic harm (Tanswell & Rittberg, 2020) of privileging just one strict view of coherence becomes apparent. For starters, adherence to just one curricular aesthetics of coherence fails to capture the aesthetic and affective idiosyncrasies students attended to as they engaged in their coherence seeking processes. Worse, a devotion to just one set of aesthetic values can enforce an exclusionary politics of aesthetics that conveys messages about who/what is (in)coherent (Appelbaum, 2010; Buchmann & Floden, 1991). This phenomenon was on full display across Joy, wuyen's, and VKN's curricular reflections and critiques, where they often positioned themselves as struggling or incapable as they strove to interpret the cross-curricular mathematical stories they had engaged with as coherent. Recall, for example, Joy's hauntingly evocative art in which she recounted how seeking disciplinary coherence across her mathematics courses has led to her repeatedly feel as if she was being choked or even burned alive by the *supposedly* ideal "fire" of (disciplinary) coherence.

All three of these participants were highly successful mathematics students in the grand scheme of the K-16 mathematics curriculum that had successfully completed most of the undergraduate calculus sequence and all preceding pre-requisite courses. Still, I listened repeatedly as they expressed perpetual frustration about "not seeing" the (disciplinary) curricular coherence across mathematical stories expressed in other students' artwork. Though each of these students eloquently expressed highly idiosyncratic perspectives on curricular (in)coherence that deviated from disciplinary coherence, they appeared to have internalized that disciplinary forms of coherence were "more desirable" and "more correct" than their own. It was not until the final group discussion that each of these students started to challenge these deficit views they held about their own aesthetic judgements. This appeared to be a positive and perhaps even empowering outcome for these participants. But what about the other students navigating these exclusionary politics of mathematical aesthetics without sufficient space for critical reflection?

While the culture of exclusion (Louie, 2017) in mathematics classrooms is not something that will be solved overnight with just one change, there are some concrete steps that can be taken to mitigate these harms in future curricular coherence research. We can remain open to *several* perspectives on curricular (in)coherence, rather than strictly espousing just one perspective. This can be done by explicitly acknowledging the fundamentally *aesthetic* nature of making judgements about curricular coherence, clarifying that coherence seeking is *not* an objective evaluation but an idiosyncratic judgement based on one's personal aesthetic sensibilities. In other words, not everyone will find the same curricular story to be "coherent". While the definition of "coherence seeking" offered by Sikorski and Hammer (2017) does much to shift power into the hands of students to make their own judgements about what is coherent, it still positions this activity as a mostly *cognitive* act. Almost nothing is said about the *emotional* and *aesthetic* factors that influence coherence seeking (as demonstrated across students' perspectives in this study). Future research, therefore, ought to more explicitly embrace (as well as further investigate) how affective forces and other value judgements (including *aesthetics*) influence and *drive* coherence seeking. As past research in mathematics education repeatedly demonstrates, such aesthetic forces play a highly non-trivial but often overlooked role in students' mathematical activities (e.g., Fiori & Selling, 2016; Jasien & Horn, 2022). It is high time that we fully acknowledge how these personal subjectivities impact judgements of mathematical coherence and, in doing so, begin to rid our discipline of the dangerous myth that mathematics is objective, pure, or immune to exclusionary politics of aesthetics.

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Covariation-in-Pieces: Covariation as a Coordination Class

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Much of the work in covariation, to date, has focused on "large-scale" themes, such as broad reasoning levels and behaviors. We propose that covariation research could be greatly enhanced by attending to the "fine-grained" cognition happening in students' covariational reasoning. We propose that covariation can be considered as a coordination class, which models knowledge as a complex cognitive structure wherein an emergent system of knowledge resources are activated in response to context and cueing. We explain what this perspective on covariation would look like, and we provide an illustrative vignette to explore the power of this perspective.

Keywords: covariation, covariational reasoning, knowledge in pieces, coordination class

Covariation and covariational reasoning have become major themes in secondary and tertiary mathematics education research (e.g., Carlson et al., 2002; Castillo-Garsow et al., 2013; Johnson, 2022; Moore et al., 2019; Thompson & Carlson, 2017). Consequently, deeply unpacking covariation as a construct, and the reasoning done in covariational situations, is critical for building this area of work. To situate this paper, we note that most of the work done to date in covariation has focused on a relatively large cognitive scale, which tends to operate under the assumption of the stability of knowledge – e.g., broad developmental levels that encompass mental actions and behaviors (Carlson et al., 2002; Thompson & Carlson, 2017). We argue these mental actions and behaviors may involve the coordination of less stable knowledge-in-use that is manifold in structure and context-dependent in emergence. Our previous work has started to add a "fine-grained" perspective on covariation (Rodriguez, Bain, & Towns, 2019), and we have highlighted the value of applying a fine-grained lens to consider students' graphical reasoning, including students associating intuitive ideas with graphical patterns (Rodriguez & Jones, 2024). These fine-grained ideas operate at a scale of analysis that affords consideration of the flexibility of a student's knowledge, as well as their knowledge structure's dependence on context and time.

In this paper, we systematize this emerging work by discussing covariational reasoning through a fine-grained constructivist lens, highlighting the affordances provided by situating covariation within coordination class theory (and knowledge-in-pieces broadly). We claim that the importance of doing so is that it provides more accurate modeling of student thinking and it better accounts for conceptual change in developing expertise in covariational reasoning.

Brief Literature Review on Covariation

Covariation consists of two quantities or variables whose values are linked, such that changes in one correspond to (possibly) changes in the other in a smooth continuous manner (Saldanha & Thompson, 1998; Thompson & Carlson, 2017). For simplicity we use the term "variable" in this paper to encompass both real-world quantities and mathematical variables. Covariational reasoning consists of mental actions related to situations involving covariation, which have generally been described through broad "behaviors" and developmental "levels." For example, Carlson et al. (2002) described covariational reasoning through coordinating variables' values, coordinating directions of change, coordinating amounts of change, and coordinating changes in the rate of change. Thompson and Carlson (2017) gave an alternate set of broad levels, including

asynchronous tracking of variables' changes, coordinating variables' values, envisioning changes in chunks, and envisioning changes smoothly (see also Castillo-Garsow, 2012).

Covariational reasoning has been a powerful framework for building coherent mathematics instruction. It has been used to reimagine key ideas such as function (Ellis et al., 2016; Oehrtman et al., 2008; Paoletti & Moore, 2018), graphs (Carlson et al., 2002; Moore & Thompson, 2016; Rodriguez, Bain, Towns, et al., 2019), coordinate systems (Moore et al., 2013), and calculus concepts (Jones, 2017, 2019; Kertil & Gülbağcı Dede, 2022; Thompson, 1994). Covariational reasoning also coheres well with quantities-based thinking and reasoning at the intersection of math and science, or with other disciplines. It has been used to describe reasoning in chemical kinetics (Rodriguez, Bain, Towns, et al., 2019), in scientific representations (Panorkou & Germia, 2021), and in modeling quantities (Panorkou & Germia, 2021).

What might be the next step in building covariation as a construct in educational research? A major theme in the literature cited here is that covariation is generally considered through "large-grained" cognitive structure, such as a student's ability to coordinate directions of changes, or a student's ability to imagine smooth variation. Accounts of "fine-grained" cognitive structure in this area are only beginning to be produced (Rodriguez, Bain, & Towns, 2019; Rodriguez, Bain, Towns, et al., 2019). The next important step is to systematize this emerging "fine-grained" work to better understand the nature of actual covariational reasoning used by students in practice.

Knowledge-in-Pieces and Coordination Class Theory

Overview

Knowledge-in-pieces (KiP) (diSessa, 1988, 2004, 2018; diSessa & Sherin, 1998) is a paradigm centered on fine-grained knowledge and conceptual change. KiP is based on a manifold view of cognitive structure, where knowledge is modeled as an emergent system of elements that are activated in concert, in response to context and perceptual cueing. Knowledge elements can be quite small, such as intuitive extractions from experience (e.g., *closer means more intense*), or a simple idea picked up from a class (e.g., *x means a variable*). These individual knowledge elements become *resources* that a person can activate in a situation (Hammer, 2000). Thus, we call them *knowledge elements* and *resources* interchangeably.

KiP and Graphing

Because much work in covariation has been done in the context of graphing (e.g., Carlson et al., 2002; Castillo-Garsow, 2012; Moore et al., 2019; Moore & Thompson, 2015), we provide here some work done on graphing using a KiP lens. Recently, Rodriguez and colleagues (Rodriguez, Bain, & Towns, 2019; Rodriguez & Jones, 2024; Rodriguez et al., 2020) have interpreted students' graphical work through a fine-grained examination of knowledge resources students might flexibly use at different times. This work resulted in a large collection of "graphical forms" students might employ such as *curved means changing rate*, *big dot as focal point, discontinuity means sudden*, and so forth (see Rodriguez & Jones, 2024 for more). As students create a graph, they might infer graphical patterns based on what they observe in the context. For example, if one is creating a graph to model the height of grass mowed once a week, they could notice the sudden change in grass height when cut. This could lead to the activation of *discontinuity means sudden* and they might draw a jump discontinuity in the graph. However, in another situation, there may be no sudden events, so no inference is made to *discontinuity means sudden*. Thus, not observing a resource in context does not imply the students does not posses it.

In addition to graphical forms, there are also more general knowledge elements regarding axes, scaling, beliefs, conventions, and quantities (Jones & Rodriguez, 2023). These can include

resources like *spacing means duration* on an axis, the belief that *the presence of π implies trig functions*, or the quantitative resource *time as a natural independent variable*. When students are asked to create graphs, they extract information from the context that traces inference to some of these resources, which are activated and in turn may trace inferences to other resources. This activated collection of resources are used in making moves toward constructing a graph. Graphical resources play an important role as elements nested within larger knowledge systems such as a coordination class, and have strong connections to covariational reasoning, as well.

Coordination Class Theory

Coordination class theory builds on knowledge-in-pieces to provide a model to describe a "concept" within a systems view of knowledge (diSessa & Sherin, 1998). Importantly, not every concept can be modeled as a coordination class, with a coordination class being a specific type that serves the function of allowing an individual to recognize or "see" the concept across contexts. To illustrate, consider the idea of "proportional reasoning." The actual cognition pertaining to proportional reasoning is complex, with basic ideas of difference, products, scaling, balancing, ratios, rates, fractions, inverse relationships, and so on, networked together in some way (Jacobson & Izsák, 2014; Sherin, 2001). Here, a sophisticated coordination class would allow an individual to recognize and apply these ideas across a variety of situations where proportional reasoning is relevant, such as in constant difference, constant growth, or exponential growth contexts. Due to context, it is not necessary to employ the entire coordination class, and only a subset of resources might be activated in a situation. For example, a constant difference situation might elicit a distinct set of knowledge resources related to proportions than an exponential growth situation. The functioning of the concept in one specific situation (subset of resources used) is called a *concept projection* (diSessa, 2004; diSessa & Wagner, 2005).

The defining structural features of a coordination class that allow an individual to see a concept involve two major components: *extractions* (previously called *readouts*) and *inferential nets* (diSessa, 2018; diSessa & Sherin, 1998; diSessa & Wagner, 2005). Extractions are the features attended to – what are seen, noticed, or observed in the context that are relevant to the target information. In the proportion example, features that may cue a student to recognize proportionality in a situation might be specific symbol templates (A/B = C/D), the presence of rates (X lbs/min), percentages (X %), and so on. Inferential nets are the set of possible inferences that lead toward the target idea. These can connect an extraction to a knowledge element, or may connect that knowledge element to yet additional knowledge elements.

Covariation as a Coordination Class

We now turn to the main purpose of this paper, which is to propose covariation as a coordination class, and covariational reasoning as the employment of a covariation concept projection into some context. In the current literature, "covariation" levels tend to be given as hierarchical progressions toward the ideal of covariation. While this lays a useful and important foundation, we posit that we can zoom in further into the concept of "covariation" by modeling it as a coordination class. Here, it would consist of a large network of resources that function differently in different situations, such as when constructing a graph, or identifying quantitative relationships in a physics equation, or imagining different types of mathematical functions. Different collections of resources pertaining to covariation might be activated in certain of these situations, but not in others, meaning each situation represents a *concept projection* of covariation into that context. For each situation, one would employ extractions to "read" the situation and begin making inferences across the knowledge elements, to determine the desired

information. For example, in a graph-interpretation context, extractions might consist of observing features in the graph's shape, noticing axis scales or labels, or locating straight or curved sections of the graph. In a context considering a scientific quantitative relationships, extractions might consist of how those real-world quantities behave, or how their magnitudes change. In a math-function context, extractions might consist of noticing what type of function is present, or what symbols portray the input or output. Inferential nets connect these extractions to knowledge elements that might be relevant for reasoning covariationally. These might include resources like *straight means constant rate*, or *up-down as inverse relationship*, or *sine as periodic*. Inferences can further be made from these resources to yet other resources, in a chain that activates a certain subset of knowledge elements in that context.

To dig deeper, let's consider a common example of a covariational situation: the well-known bottle task (Carlson et al., 2002; Johnson, 2015a, 2015b). In this task, a bottle is filled with water and students are asked to draw a graph of volume as a function of height. Analyzing this task has previously consisted of finding evidence for certain levels of covariational reasoning through students' behaviors (Carlson et al., 2002), or describing general natures of the coordination (Johnson, 2015a, 2015b). Certainly these findings have been very helpful in understanding students' thinking. To now reframe this situation using coordination class theory, we first note that students' covariational reasoning for this task reflects a single concept projection into the bottle context. Thus, we should expect that subsets of their resources might be activated, while other resources remain dormant. In this concept projection, the extractions would be those features students see or notice about the bottle that would be relevant (to them) in identifying the covariational relationship and capturing it graphically. These might include noticing a constant rate of water pouring in, the width of the bottle, or the emptiness of the bottle of at the beginning.

An important implication of thinking of covariation as a coordination class is that these extractions then influence what the student might do next, as inferences are made to other knowledge elements. Their extractions drive inferences for which resources get activated. It is entirely possible that seeing the constant rate of pouring might activate a resource that *straight means constant rate*, leading them to initially draw a straight line segment. In coordination class theory, this action does not reflect a misconception (Smith et al., 1994). This resource is perfectly legitimate in certain contexts; it is simply misapplied here, because that volume per time rate is not what the volume-height graph is meant to reflect.

Additionally, what has been termed covariational reasoning "levels" (Carlson et al., 2002; Thompson & Carlson, 2017) may be reinterpreted as possible extraction strategies students use in their concept projection. For example, gross coordination (i.e., general increase/decrease behavior in the variables), can be a strategy for what to focus on, guiding what students may "see" in the context. Zooming in further, these extractions shape what resources get activated, and the collection of activated resources drives the actions taken. In the bottle context, a student might observe that as water comes in, height and volume both go up, leading to the activation in the inferential net of *up-right means increasing*. Consequently, the students may make a general up-right initial sketch. Increasingly sophisticated covariation levels may represent increasingly sophisticated methods for reading situations for relevant information. But if we view covariational reasoning simply as a set of levels, the in-the-moment cognition induced from that extraction strategy and the subsequent inferences might be overlooked.

Illustrative Vignette and Discussion

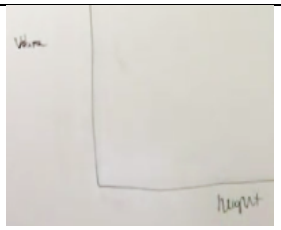
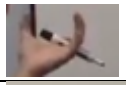
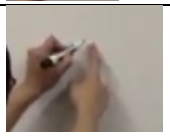
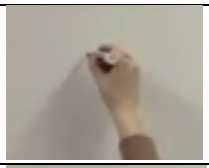
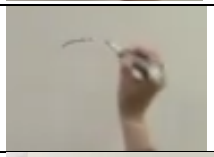
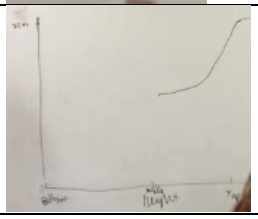
To show covariation-as-a-coordination-class in action, we provide here an illustrative example of students working in the well-known bottle context and use it to suggest the power

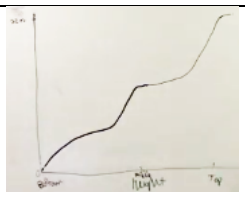
one can get from the coordination class perspective. Figure 1 shows an excerpt from two students working with a version of the bottle task, and includes their speech and inscriptions. We also document the knowledge resources activated on the right of the figure (see Jones & Rodriguez, 2023; Rodriguez & Jones, 2024 for more details on these specific resources).

Figure 1. Eric/Ellie's work in the bottle task, with identified extractions (E), resources (R), and other inferences (I)



Task: This bottle is being filled up with soda. Graph the volume of the soda in the bottle as a function of the height of the soda.

Name	**Utterance**	**Drawing**	**E=extraction R=activated resource I=other inference**
Eric:	I'm going to assume it fills up, like the volume.		E: Volume fills up
Ellie:	Let's start at zero. So, which axis should be volume?		E: Bottle starts empty R: Origin as natural start R: Axes need labels
Eric:	Probably this one [points to *y*-axis], 'cause the height [points to *x*-axis] is the constant increase. … I think it's a more, I'm guessing this is probably a curved function [i.e. graph], because the bottle's curved… It'd probably fill up the volume if you're going up height.		E: Height goes up R: Independent as 'marching on' R: Other on vertical axis E: Bottle's curvy shape R: Graphs preserve visual features I: graph is curvy
Ellie:	Like that, a little bit, maybe	Gestures 'up-right' arc	R: Up-right as increase
Eric:	… It would take up longer to fill up the areas where the bottle is wider.	Gestures fingers apart 	E: Bottle shape R: More takes longer
Eric:	…[later]… Like in the middle we should probably have a point [touches middle of graph space], kind of like right here [draws a dot in the middle].		E: Halfway point as significant R: Dot as focal point
Ellie:	Yeah. I feel like the closer we get to the top [draws a dot at the top-right],		E: Top of bottle I: scenario ends at top R: Dot as focal point
Ellie:	[continued] the more it'll kind of level off a little bit [draws slight curve going into a straight line segment].		R: Plateau as leveling off
Eric:	'Cause it's more constant [i.e. straight line segment]. This is like the middle of the bottle, so it's like a steady followed by a rapid increase. [Ellie agrees: "yeah"].		R: Straight means constant rate E: Middle is curvy I: Growing rate of increase R: Connection as transition R: Steepness as rate

Eric:	And then I think it's the same thing here [points to left of graph portion]. It's kind of followed by a sharp kind of, just kind of like gradual [finishes drawing curve].		R: Connection as transition R: Steepness as rate

In a "holistic cognition" analysis, we might identify that the students can perform basic coordination (e.g., labelling axes) and gross coordination (e.g., up-right graph), and can coordinate values (e.g., identifying height-volume pairs for the middle and top) and coordinate rates (e.g., less steep to more steep) (Figure 2, left). However, stating these does not help distinguish this pair of students from another pair who might use similar levels, but in a very different way. For example, in constructing their graph, Ellie and Eric did not move in an orderly "left-to-right" fashion, which is not accounted for in major covariational reasoning frameworks. They chose to start at the middle and end, but what accounts for those actions? Similarly, a holistic view might identify a difficulty in constructing the middle of the graph, and might try to identify the level at which the students struggle, such as the "changing rates" level from Carlson et al. (2002). But Ellie and Eric successfully manage changing rates in other places, so this description does not seem to provide much detail on the actual cognition being used.

By contrast, what might a coordination class interpretation entail? As mentioned before, it provides the ability to "zoom in" on each of these covariational reasoning levels to see how those extractions lead to other inferences, including graphical forms, graphical resources, or other knowledge elements (Figure 2, right). In Figure 3, we provide exactly this type of "zoom-in" in the form of a resource graph that models at a much more detailed level what these students' cognitive work consisted of. Notice that the covariation "levels" functioned more as broad strategies for extracting information from the task, leading to resources activation. This perspective allows us to see how their reasoning at each level operated in-the-moment. It could allow us to distinguish Ellie and Eric's work from other students' work using similar levels.

As they extracted information, they connected it to relevant resources and used those to build certain pieces of the graph, one at a time (Figure 3). They started by constructing axes they felt matched the task, based on resources related to the origin, labels, and what counts as an "independent" variable versus the "other" variable. After identifying the general "up-right" nature of the graph, they identified key locations in the graph, such as the mid-point and end-point, based on graphical forms related to dots, plateaus, and straight segments. They progressed to generating curves for the later and earlier portions of the graph, based on graphical forms related to slope, curves, and connecting points (Figure 3). When these two sections did not match up, they simply invoked a belief about the necessary "curviness" of the graph, and a graphical form *curve means changing rates* to try connect them. Thus, the incorrect middle portion seems less about abilities at the large-scale covariation levels, and more about which resources they invoked at that particular time to address a perceived discrepancy.

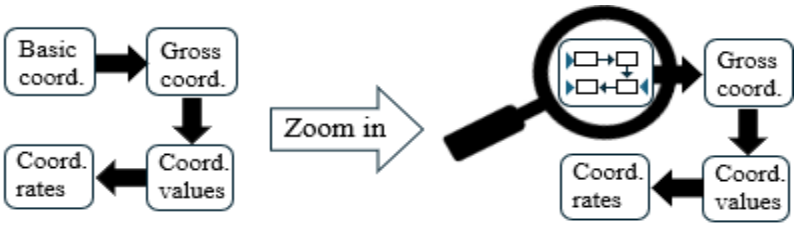


Figure 2. Coordination class theory helps with more explicit "zooming" into covariational levels (see also Figure 3)

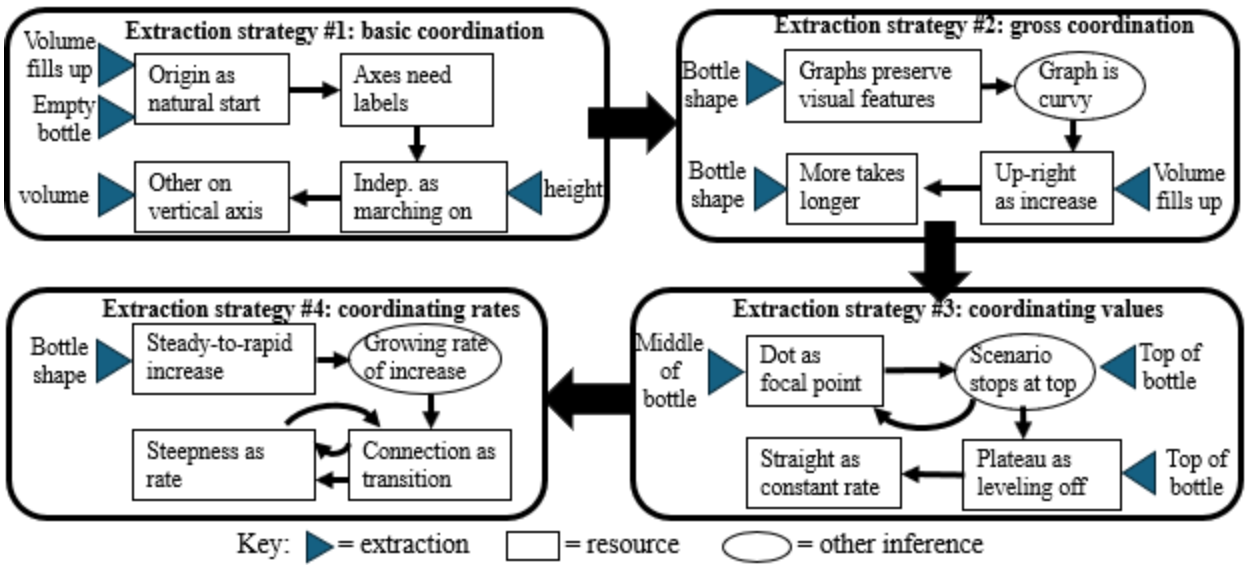


Figure 3. Resource graph as a model of the students' cognition with much greater detail

Further, consider how they determined to draw curves for a certain portions. Sometimes it was driven by an inference from the increasing rate ("it's a steady followed by a rapid increase"). Other times it was driven by the visual appearance of the curvy bottle ("probably a curved function, because the bottle's curved"). Positioning their reasoning at a certain level does not account for this subtle shift in justification. A coordination class lens helps us move away from viewing such ideas as unitary objects, toward seeing more fluid reasons and justifications.

Also, it is possible some might want to deem it "wrong" that the students appeared to use the bottle's curvy appearance at one point to infer that the graph was also curvy. But, again, the coordination class perspective helps us realize that that idea might sometimes be a legitimate inference in certain contexts, but perhaps not for others. In fact, the incorrect part in the middle is likely due to an overuse of this resource, in that they extracted the curviness of the bottle, combined it with the fact that changing rates are reflected by curves, and produced a tentative sketch of a graph they felt depicted that. This lens provides better instructional fodder, as well, because an instructor can try to highlight what resource the students drew on in that particular instance and to help the students question whether it was the right one to use. In other words, the difficulty might not be due to some larger underlying issue with one of the covariational reasoning levels. Rather, it could be as simple as the activation of the wrong resource at the wrong time. Helping the students identify their knowledge usage could help them shift from that resource to other resources, such that they could correct their incorrect portion of the graph. This constitutes, in part, the gradual shift from more novice activity to more expert activity.

Conclusion

In summary, we believe we get significant power through a coordination class interpretation in modeling these students' cognition related to covariational reasoning and to account for individual actions taken. It also provides a better narrative that follows what the students actually do. We also have better recourse for assisting the students in developing expertise, by shifting their resource usage and helping reorganize their knowledge elements. This power is important to researchers, as it improves our ability to faithfully render actual student thinking in-the-moment in real time. This power is also useful to instructors, in our ability to track what our students are thinking and doing in any given moment, in order to better scaffold and direct it.

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Using Intertextuality to Study (Student) Conceptions of Proofiness

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This work provides a theoretical grounding for proof education studies that utilize proof comparison tasks. We leverage the notion of intertextuality, borrowed from genre theories, to discuss mathematics education theory and methods. Intertextuality involves the implicit or explicit comparison of two or more texts with regard to a given genre. We discuss how this comparison can serve explorations of (student) conceptions of the genre of proof and argue that intertextual comparisons lead to novel insights into (student) conceptions of the genre. Specifically, we illustrate how someone's comparison of two (or more) justifications relative to the genre of proof can provide insights into how they view (a) proof as a genre, (b) which attributes of a justification are relevant to that justification's (potential) status as a (good) proof, (c) the relative importance of those attributes, and (d) the reasons for that relative importance.

Keywords: Proofiness, Justification, Intertextuality, Genre

Küchle and Dawkins (2024) argued that proof is a genre; consequently, genre theories might be helpful for mathematics education researchers as lenses for studying proof. They discussed the three major genre theory approaches, focusing on what these approaches have in common in their conceptualization of genre. We focus on one of those common threads, the notion of intertextuality. This construct has two competing definitions (Duff, 2002). The definition Küchle and Dawkins (2024) discussed involves texts referring to other texts. For instance, when the proof of a theorem references another proof/theorem or when a scientific paper references other scientific papers. Instead, we reference the other definition of *intertextuality*. That definition refers to examining genre via studying the (implicit or explicit) comparison of two or more texts. This definition has a rich history in studies of literary genres (Duff, 2002). When applied to proofs, the definition implies that conceptions of proof as a genre are explorable via comparison of two or more justifications with respect to their status in the genre of proof. This definition of intertextuality harnesses two key insights. Firstly, genre is a rhetorical category that expresses a relationship between texts. So, intertextual comparison is a relevant tool for studying it. Secondly, disagreements regarding which texts belong to which genre rarely resolve, so it is commonly more fruitful to further such debates by comparing two or more texts with respect to a genre.

For illustration, we discuss an example from film genre. Consider the 1988 action movie "Die Hard." Since the movie takes place on Christmas day, some have argued that this film belongs to the Christmas Movie genre. That interpretation of genre led to "Die Hard" being voted #7 in USA Today's list of 20 best Christmas movies (Truitt, 2023). However, most participating in the debate about its Christmas movie status would agree that, for instance, the movie "Scrooged" is comparatively more representative of the genre of Christmas Movie. Thus, two people who disagree on the genre status of Die Hard might still find fruitful common ground via comparison. Additionally, those people's comparison of the two films might lead to a discussion of which motifs and themes contribute to the Christmas movie genre, why they are relevant, and their relative importance. Thus, these comparisons can lead to more precise images of individual conceptions of the Christmas movie genre.

Some proof education studies have utilized proof comparisons (Brown, 2018; Mejía Ramos et al., 2021; Vroom & Alzaga Elizondo, 2024). However, these tend to focus on some properties a proof might have, such as explanatory value (Mejía Ramos et al., 2021) or its convincingness (Brown, 2018), rather than juxtaposing justifications with respect to the genre of proof as a whole. This work advocates for expanding this approach by considering the comparison of justifications with respect to the genre of proof. We argue that such comparisons can be particularly revealing. Specifically, intertextual comparison of justifications in relation to the genre of proof can reveal not just what students attend to but also the reasons for their attention. This attention need not be limited to the attributes that delineate the difference between proof and non-proof; it can also include attributes that contribute to the relative status of valid proofs, such as elegance or cleverness. Additionally, in cases where students attend to multiple attributes of a justification, comparison can also reveal the relative importance of the attributes they attend to, and the reasons for that relative importance. The phrase, "intertextual comparison of justifications in relation to perceived ideals and norms of the genre of proof," is quite onerous. So, it is helpful to create a construct to describe it.

Proofiness

Seife (2010) defines proofiness as manipulations of mathematical norms that "make falsehoods look like numerical fact" (p. 18). We borrow this construct's name (though not its definition) from Seife (2010). We instead use *proofiness* to refer to a person's intertextual conception of justifications in relation to perceived ideals and norms of the genre of proof. Our definition is naturalistic since it is consistent with student use of the word "proofy" in mathematics settings. For example, when participants in our studies were presented with multiple justifications simultaneously and asked to what extent they should be considered proofs, they responded with phrases like, "This one feels proofier to me" and "This one is not proofy." In doing so, they commented on proofiness via intertextual comparison (Authors, submitted).

Proofiness is intrinsically comparative. So, we discuss two kinds of intertextual comparisons. The first is to examine a person's conception of proof via their comparison of two (or more) (researcher-generated) justifications in terms of which is proofier. The second is to encourage participants to edit justifications to make them proofier. This editing allows them to (implicitly or explicitly) compare their edited version of a justification to the original and elucidate how those edits contributed to proofiness.

One crucial way the approach we describe differs from some past proof education studies is that we are treating the comparison of two valid and complete proofs with respect to the genre of proof as legitimate. In other words, one proof may be proofier than another, even when both are valid and complete. A consequence of this is that we explore not just the boundary between proof and non-proof but also what makes a justification a good/desirable proof. This approach is reminiscent of Paul Erdös' discussion of 'The Book,' where God keeps the perfect proofs for mathematical theorems (Aigner & Ziegler, 2004). That is, there is some (possibly unreachable) perfect proof for a theorem, and other proofs of that theorem are, using our terminology, less proofy.

The Utility of Proofiness

The remainder of this document discusses the utility of proofiness for exploring conceptions of proof. We begin by engaging the reader in an intertextual comparison of proofs. Later, we use excerpts from student interviews to illustrate insights about their conceptions of proof gained from this approach.

Your Conceptions of Proofiness

Imagine two people, Paul and Valentin, debating the proofiness of several proofs. Of the three proofs below, Paul states he believes that proof A is proofier than proof B. Valentin disagrees, claiming the opposite. Consider your stance on the matter. The three proofs are below:

Claim: $\sqrt{2}$ is irrational.

Proof A: Toward a contradiction, assume that $\sqrt{2}$ is rational. The definition of rational implies that there exists $a, b \in Z$ with $b \neq 0$ such that $\frac{a}{b} = \sqrt{2}$. Furthermore, we may assume that a and b share no common factors. Since $\frac{a}{b} = \sqrt{2}$ we know that $a^2 = 2b^2$, which implies that a^2 is even. Since a^2 is even, a is also even. This implies there exists a $k \in Z$, with $2k = a$. Substituting into the equation and simplifying, we get $2k^2 = b^2$. We then use a similar argument to conclude that b is even. Since we have shown that a and b are both even, they are divisible by 2. So, we have contradicted our assumption that they share no common factors. This means that $\sqrt{2}$ must be irrational.

Proof B: Suppose that $\sqrt{2}$ is rational. The definition of rational then implies that there exists $a, b \in Z$ with $b \neq 0$ such that $\frac{a}{b} = \sqrt{2}$. This means that $a^2 = 2b^2$ (*). The prime factorization of a squared integer has an even number of primes in its prime decomposition. Consequently, in equation (*), the left side has an even number of prime factors, and the right has an odd number of prime factors. This contradicts the fundamental theorem of arithmetic since prime factorizations are unique. So $\sqrt{2}$ is irrational.

Proof C: Let p be prime. Assume $\sqrt{p}$ is rational. By definition, there exists $a, b \in Z$ with $b \neq 0$ such that $\frac{a}{b} = \sqrt{p}$. This implies that $a^2 = pb^2$. Since the left side of the equation has an even number of prime factors, and the right has an odd number of prime factors, we have a contradiction. So, we conclude $\sqrt{p}$ is irrational. This implies that $\sqrt{2}$ must be irrational.

Consider the various stances that might emerge concerning Paul and Valentin's debate. Some might consider proof A proofier than proof B because of its similarity to the proof in their preferred textbook. Someone may believe proof B is proofier than proof A because its argument is more compact or makes fewer assumptions. Some might view proof C as proofier than proof A and proof B because it is more general. Alternatively, proof C leaves many warrants implicit compared to proofs A and B. So, some might base their judgment on that observation. Finally, a significant possibility is that one might avoid the debate entirely by asserting that if some justification is a proof, then it is no more (or less) a proof than any other. In that case, one considers relative proofiness as illegitimate. Regardless, the answers one gives and the reasons for them provide a lens into the values and norms they hold concerning proof. Both those who (1) believe the question is not legitimate, and (2) those who sighted some differences between the proofs to conclude some proof was proofier have sensible responses. The point we wish to highlight here is that it is reasonable for a person who answers (1) and a person who answers (2) to not attend to some of the differences in their respective conceptualizations of proof when discussing the subject (Author(s), submitted). That creates a potential mismatch, which can be especially problematic when, for instance, the former is a mathematics education researcher

interested in student conceptions of proof and the latter is a student interviewed about their conceptions.

Student Conceptions of Proofiness via Comparison

We argue that student explorations of proofiness can provide helpful insights. Below is an excerpt from an interview with a student, Redmond, who was near to completing an introduction to proof course (Author(s), year). Additional information regarding the tasks he worked on are in Author(s) (Submitted), and the two justifications he examined can be found below in Figure 1.

Claim 1: For integers $a, b,$ and c, if a divides b and b divides c, then a divides c.

Justification 1: Since a divides b, there exists an integer k, such that $ak = b$. Similarly, since b divides c, there exists some integer l, such that $bl = c$. Substituting for b in the equations yields:

$$(ak)l = c$$
$$a(kl) = c$$

Since k and l are integers, so is kl. Therefore, a divides c.

Claim 2: 5 divides 10, and 10 divides 30, so 5 divides 30.

Justification 2: Since 5 divides 10, there exists an integer—in particular, 2—such that $5 \times 2 = 10$. Similarly, since 10 divides 30, there is some integer—namely, 3—such that $10 \times 3 = 30$. Substituting for 10 in the equation yields:

$$(5 \times 2) \times 3 = 30$$
$$5 \times (2 \times 3) = 30$$
$$5 \times 6 = 30$$

Note that 6 is an integer. Therefore, 5 divides 30.

Figure 1: Justification Comparison Task

From our perspective, justification 1 is a valid, complete proof of the divisibility property of integers, and justification 2 is a corresponding *generic proof* (Leron & Zaslavsky, 2014). In other words, Justification 2 goes through the same steps as Justification 1 but does so using specific integers. Raymond was asked to read both justifications and compare them in terms of how proof-like (proofy) they were—that is, compare their relative proofiness. The excerpt begins after Raymond reads Justification 1:

Redmond: I like it because that is precisely how we would have to do it if that was on a test.
Interviewer: Tell me a little bit more about that. Why does it remind you of that?
Redmond: Because, my professor, she's very, very picky about the words that you use: let, assume, suppose, thus, therefore. This looks like a 6 out of 6 points, for sure. Yeah, without a doubt.
[Reads Justification 2]
Redmond: This would be the same exact proof as Justification 1, but they have integers. So, I would consider this a 5 out of 6. The reason for it is just... There are certain things like terminologies. For instance, this part here, "in particular 2." My professor would have

liked, "Since 5 divides 10, there is an integer 2", or "Let integer 2", you know, or "there's just an integer, such as 2", just sort of things like that.

Redmond indicates that Justification 1 is proofier than Justification 2 by assigning them grades 6/6 and 5/6, respectively. He believes this is reflective of how his professor would have graded these. Redmond notices the generic proof approach to justification 2 when he states, "This would be the same exact proof as Justification 1, but they have integers." However, that is not what he attends to when articulating the reasons for the two justifications' differing proofiness. Instead, the reasons for this difference in proofiness are the 'terminologies' used in Justification 2, specifically the phrase 'in particular 2.' From our perspective, those terminology differences are necessary when moving between proof and generic proof. However, Redmond's conception differs from ours. Redmond indicates that if Justification 2 had instead used the kinds of wording that his instructor expects it would have been proofier.

We can learn about Redmond's conceptions of proof from his excerpt. For instance, he believes that using the language normatively found in proofs is relevant to proofiness. Additionally, given comparable language usage, he believes generic proofs are just as proofy as non-generic proofs. In other words, moving between generic vs. non-generic proofs has no effect on proofiness from his perspective, but minor deviations from the word choice he expects do. Had we presented justifications 1 and 2 as separate tasks, without the juxtaposition of the two, we would have just learned that he regarded both as proofs and learned less about his conception of proofiness. That discussion of proofiness allowed us to glean his views on the relative importance of genericness vs. language to his assessments.

Student Conceptions of Proofiness via Editing

Next, consider the following excerpt from Claudette. When the excerpt begins, she has already participated in comparing justifications 1 and 2 (Figure 1). The excerpt is from the subsequent stage of the interview, editing. The transcript below begins after the interviewer encouraged her to revisit Justification 1 and edit it to make it more proof-like from her perspective. Her inscriptions are in Figure 2.

Claudette: ‘Since a divides b’. And then this [ak=b for some integer k] is just the definition of divides. Which in the words of [name of ITP instructor], 'sometimes you need to say that there's a definition. But also, it should always be assumed.' So here it looks there's more assuming that you know the definition of divides. In our class, most of the time you write it out… [Edits by adding ', by definition of divides.']

Interviewer: If you add that, does that make it feel more proof-like?

Claudette: I would say that it's very proofy, to begin with, but this gives more clarity. It's the idea, like if you are working with an 8th grader, whose never seen the definition of divides, then it's going to be like okay, by definition of divides, and if they have notes, they can look at the definition of divides and be like, Oh, if a divides b, ak=b for some integer k… definitely having the name of the definition so you can look it up makes it easier.

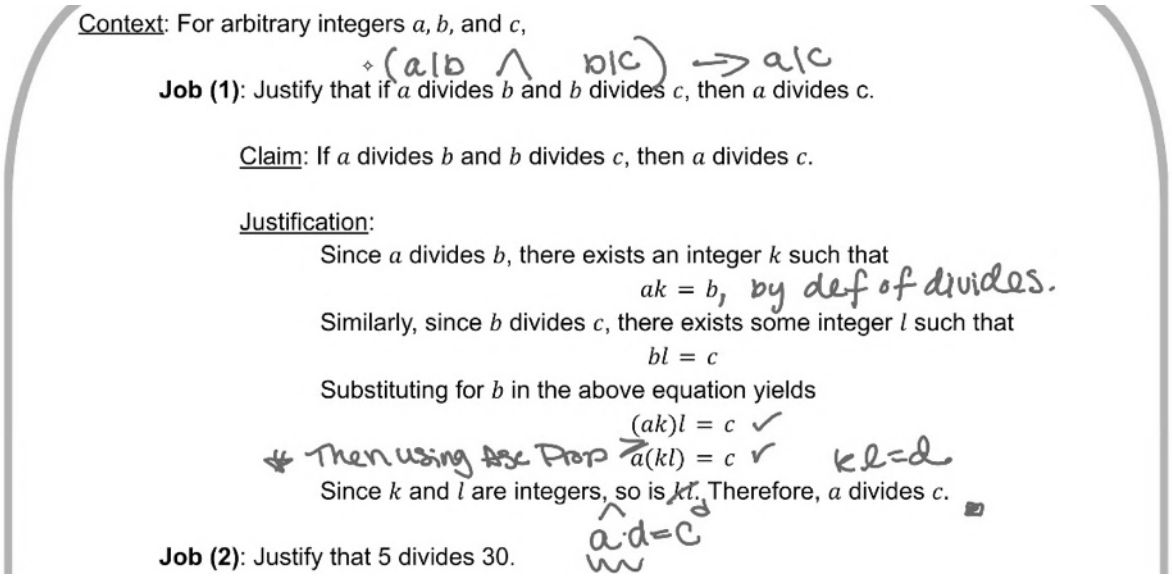


Figure 2: Claudette's inscriptions on Justification 1

Notice that Claudette indicates that her adding phrases and adjusting language is done to improve 'clarity' and 'make it easier.' So, in making these edits, she makes the justification more perspicuous from her perspective. She knows that the definition of divides implicitly warrants $ak = b$. So, that part of the justification is *transparent* to her. However, by adding 'by definition of divides' (Figure 2) she is 'making it easier' for someone else to make the same link. Additionally, she states that her intended audience is 8th graders. So, she is conscious of a potential audience for whom the justification should be clear. It is worth mentioning that proofiness facilitated our observation. By treating valid proofs as capable of being edited to be proofier, we were able to encourage Claudette to undertake such edits even though she viewed Justification 1 as initially 'very proofy.' This helped us glean what she viewed as the purpose of such edits: perspicuousness. Czocher and Weber (2020) discuss perspicuousness as one of the five properties mathematicians consider when determining whether a justification should be considered a proof. Claudette's excerpt shows that some students have similar considerations when editing proofs.

Both Claudette's and Raymond's excerpts reference their instructor. These references help illustrate that, especially in early proof courses, an instructor's discussions of proofs and how they should be communicated influence the conceptions of the genre students form. Specifically, their instructor spent significant time and effort emphasizing the importance of clarity and adherence to mathematical terminology. These themes were thus evident in their discussion of the proofiness of the justifications.

Discussion and Conclusion

Commonly, the themes studied concerning proofs are themes known to contribute to expert conceptions of proof. For instance, Czocher & Weber's (2020) expert cluster model of proof includes convincing, a priori, and sanctioned as properties mathematicians attend to when evaluating whether a justification is a proof. These three properties roughly correspond to the themes that Weber et al. (2022) argue that mathematics education researchers have historically prioritized (when studying students): 1) the psychological function of proof (Brown, 2018; Harel

& Sowder, 1998), 2) its logical or structural features (Hub & Dawkins, 2018), or 3) its social or institutional layers (e.g., Devlin, 1993). This correspondence between mathematicians' conceptions of proof and proof education researchers' priorities means that much of the current proof education research implicitly centers mathematicians' practice. The approach we described focuses on conceptions of proof as a genre. Thus, our approach leaves students open to articulate themes they believe contribute to the proofiness of a justification which may differ from themes identified by mathematicians.

Treating proof as a genre within the context of proof education points to the likely possibility that proof students and mathematicians view the genre very differently from each other. For instance some themes and motifs students view as central to the genre may be treated as surface by mathematicians (Selden & Selden, 2003) and vice versa (Chazan, 1993). Additionally, students are convinced by proofs in ways that differ from mathematicians (Healy & Hoyles, 2000; Weber, 2008). Thus, there are also differences in the role personal conviction plays in mathematicians vs. students' conceptions of the genre of proof. In this document, we have argued that intertextuality borrowed from genre theories provides one potential path for exploring student's conceptions of proofiness. This path is a potentially fruitful direction for proof education research, which does not center mathematicians' practice.

We describe two potential avenues for intertextual comparisons: comparison of researcher-generated justifications and student editing. For both, we demonstrated with brief excerpts of data that these intertextual comparisons of justifications revealed students' conceptions. The broader vision this work advocates involves using these tools to understand the themes that students view as relevant to the genre of proof, especially when those themes differ from those prioritized by mathematicians. Studying how students view the genre of proof and why can help the field better understand how students' experiences shape their conceptions of proof. Their conceptions of proofiness connect to how they approach proof construction and reading of proof. So, a better understanding of these conceptions, how and why they emerge, and their influence on reading and production of proofs can further proof education research and the design of proof-centered curricula.

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New Methods for Analyzing Students' Reasoning About Definite Integrals in Introductory Physics

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Over the past decade, research on students' understanding of integrals has grown significantly. While previous studies have noted students' difficulties in coordinating the "product layer" with definite integrals in applied contexts, they often overlook the diverse knowledge resources students use in their moment-to-moment reasoning. Additionally, effective methods for analyzing this complexity have been lacking. This paper introduces new methods for examining students' reasoning about the "product layer", $f(x) \cdot dx$, and definite integrals, combining a modified dynamic conceptual blending diagram and an expanded Visualization/Analysis/Physics (VAP)-model. We demonstrate these methods with an example of an integration task, illustrating their application and impact on integration research (see Sus & Izsák, 2024b for further examples). This contribution advances the literature on integration research by providing a more nuanced analysis of students' reasoning.

Keywords: Integration, Product Layer, Knowledge-in-Pieces, Expanded VAP-Model, Dynamic Conceptual Blending

Introduction and Literature Review

Integration is fundamental in modeling many applied phenomena and is a core topic in calculus. It is a complex concept, structured around the coordination of multiple quantities through various mathematical operations. A significant body of literature has explored students' understanding of integrals in applied contexts (e.g., Meredith & Marrongelle, 2008; Cui et al., 2006). Researchers have identified challenges students face when modeling quantities like distance and work using rectangular areas (e.g., Thompson et al., 2013; Nguyen & Rebello, 2011; Christensen & Thompson, 2010).

Students often interpret the definite integral in one of three ways: as the "area under the curve," as a "derivative/antiderivative" relationship, or as "adding up pieces through multiplicatively-based summation" (e.g., Jones, 2015a, 2015b; Ely & Jones, 2023; Jones & Ely, 2023). Research suggests that the third is especially useful when applying integrals to physics, engineering, and other fields. However, there is ongoing debate about the range of situations where the multiplicatively-based summation and Riemann sum interpretations are applicable. Meredith and Marrongelle (2008) argued that using the product structure $f(x) \cdot \Delta x$ in a Riemann sum is sufficient for modeling accumulation only when the integrand represents a rate of change or density. However, many contexts do not naturally conform to this quantitative structure.

Particular challenges arise when reasoning about the "product layer" in definite integrals, as highlighted by several studies (e.g., Ely, 2017; Jones, 2013; Sealey, 2014). For example, Sealey (2014) developed a framework for understanding Riemann sums and definite integrals, identifying the "product layer" as a particularly complex and challenging concept for students.

This paper builds on Oehrtman and Simmons' (2023) discussion of Emergent Quantitative Models for definite integrals, which includes three primary models: basic, local, and global. The researchers provided examples showing how student reasoning can involve varied interactions among these models, such as choosing which factor in a product to treat as infinitesimal and how to approach partitioning. However, they stopped short of exploring knowledge resources which

students applied while coordinating multiplication with definite integrals through the construction of all three models. This omission is significant in being able to analyze the complexity of students' reasoning about multiplication and definite integrals in applied situations.

In other studies (Sus & Izsák, 2024a; Izsák & Sus, 2024), researchers have examined how students coordinate multiplication with definite integrals, finding that students draw on a variety of mathematical and physics knowledge resources, their combinations, as well as related drawings. These resources fluctuate depending on the problem context, which can lead to shifts in student reasoning. Such instances reveal the complexity of students' reasoning about the "product layer," highlighting the need for a closer examination of students' moment-to-moment reasoning and the knowledge resources they employ.

In this paper, we propose new methods for analyzing the complexity of students' moment-to-moment reasoning about the "product layer" and definite integrals with a focus on the knowledge resources they use. This serves as a significant contribution to the existing literature on integration.

Theoretical Perspective

In this section, we outline the theoretical perspective we use to analyze students' complex reasoning about mathematical concepts, specifically multiplicative structures in applied integration contexts. This perspective includes three components: the knowledge-in-pieces epistemology, the dynamic conceptual blending framework (e.g., Van den Eynde et al., 2020), and visual conceptual blending (Cunha et al., 2020).

The Knowledge-in-Pieces Epistemological Perspective

The knowledge-in-pieces epistemological perspective was first developed in science education research on conceptual change (e.g., diSessa, 1993, 2006). It has since been applied to various topics in mathematics, including whole-number multiplication (Izsák, 2005), functions (e.g., Izsák, 2004; Moschkovich, 1998), the law of large numbers (Wagner, 2006), and integrals (Jones, 2013). From this perspective reasoning is supported by diverse, fine-grained knowledge resources and more novice knowledge evolves into more expert knowledge through processes such as the construction of knowledge resources that are sensitive to context for activation, refinement of contexts in which resources are applied, and reorganization that can involve forming new connections among some recourses and loosing connections among others. In the present study, we are particularly interested in knowledge resources students evidenced when coordinating multiplication with definite integrals in the introductory physics context.

Conceptual Blending

Original version of the conceptual blending perspective. Conceptual blending was originally introduced by Fauconnier and Turner (1998) to model how people create new meaning in linguistic contexts by selectively combining information from different domains of previous experience. Figure 1 shows a schematic representation of the conceptual blending diagram. In its basic form, the framework consists of four connected mental spaces: two partially matched input spaces, a generic space, and the blended space. Input spaces are small self-contained regions of conceptual ideas (knowledge resources). The generic space provides the underlying structure to the input spaces, identifying commonalities in content and structure. The blended space is constructed through the combination of selected elements from the input spaces. Fauconnier and Turner distinguished three approaches to form a blended space: composition (blending can

compose elements from the input spaces to provide relations that do not exist in the separate input spaces), completion (adding new elements based on background models that are brought into the blend unconsciously), and elaboration (treating the blend as a simulation and "running" it imaginatively, which creates new insights). Also, the blend has emergent dynamics: It can be "modified" while its connections to the other spaces remain in place.

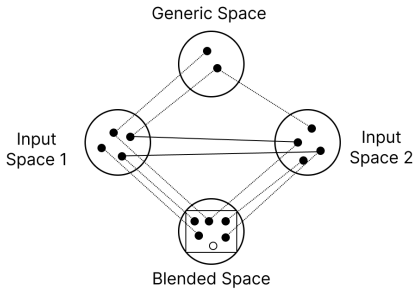


Figure 1. Schematic representation of a blending diagram (Fauconnier & Turner, 1998, p.143).

For example, Fauconnier and Turner (2002) illustrated the idea of a blend using the construct of a "computer virus" which blends meaning from the two domains of "computers" (files, software) and "viruses" (harmful, discrete, replicating) to create a new term that describes a harmful, replicating piece of software (Fauconnier & Turner, 2002, pp. 274–275).

Adapted conceptual blending perspective: Students' use of mathematics in physics. Prior research studies (e.g., Bing & Redish, 2007; Hu & Rebello, 2013; Odden, 2021; Wittmann, 2010) have investigated conceptual blending to describe students' use of mathematics in physics. For instance, Bing and Redish were the first to introduce the language of conceptual blending as a way to analyze problem solving in physics at the introductory level. Schermerhorn and Thompson (2014) used the symbolic forms framework (Sherin, 2001) and incorporated it into the conceptual blending framework to describe students' construction of differential length vectors with extensions to equation construction and interpretation generally.

Hu and Rebello (2013) adapted the conceptual blending framework from cognitive science to make sense of ways in which students combined their knowledge from calculus and physics to set up integrals. They found that many students were not able to blend their mathematics and physics knowledge in a productive way even though they had the required mathematics knowledge. An example of their blending diagram consists of three input spaces, symbolic ($\rho(x), R = \frac{\rho L}{A}$), math notion space (function, integral as a sum, '$d[\,]$' as a variable of integration), physics space (resistivity, resistance, length), and the blended space (resistivity varies along length, $\frac{\rho(x)L}{A}$ substitute resistivity, $\int_0^L \frac{\rho(x)L}{A} dx$ sum up over).

Van den Eynde et al. (2020) utilized the conceptual blending framework to study student thinking in integrating mathematics and physics within the context of the heat equation. They introduced dynamic blending, enhancing earlier diagrams by incorporating time stamps and both implicit and explicit connections to better capture the dynamics of reasoning.

Odden (2021) investigated how undergraduate physics students create conceptual connections to overcome challenges with complex and counterintuitive ideas in introductory physics. The study demonstrated how students construct and use conceptual blends to understand difficult physics concepts.

In this paper, we take a dynamic conceptual blending perspective (Van den Eynde et al., 2020) which incorporates the dynamic aspect of students' reasoning and explicitly visualizes how students switch between symbolic, mathematics, physics, and blended elements throughout their reasoning process.

Visual conceptual blending. Visual conceptual blended elements emerge from a process of visual blending grounded in conceptual understanding. Visual blending involves merging two or more visual representations (such as images) to create new ones. When visual and conceptual blending are combined, they result in what is known as visual conceptual blending (Cunha et al., 2018). This blending process plays a crucial role in students' moment-to-moment reasoning, supporting the creation of new visual elements that reflect their dynamic thought processes.

New Methods

In this paper, we use a modified version of the dynamic conceptual blending diagram from Van den Eynde et al. (2020) to explore students' reasoning about multiplicative structures and their application to integration tasks in physics. We introduce a new step for identifying activated and deactivated elements and connections and apply a different coding scheme to elements in the transcript. Our modified dynamic conceptual blending diagram (MDCBD) is illustrated in Fig. 2.

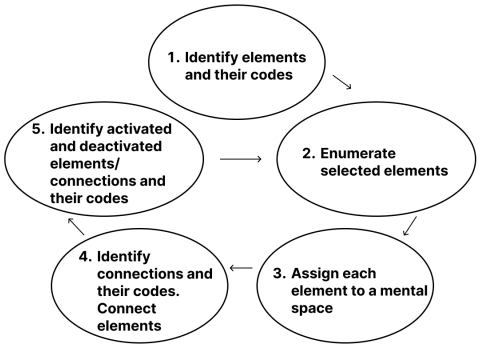


Figure 2. Schematic representation of MDCBD.

Step 1. We identify elements from verbatim transcript, which represent pieces of knowledge or resources. These elements are defined as new steps, ideas, concepts, or phrases in a student's reasoning. Unlike Van den Eynde et al. (2020), we recorded all elements from the transcript and categorized them into two groups: more confident and more tentative elements. An element is considered more confident if a student expresses confidence in introducing it, using phrases such as "I remember," "I am sure," or "I recall." Conversely, elements are classified as more tentative if the student shows hesitation, using phrases like "I guess" or "I am not sure." More confident elements are coded with solid dots, while more tentative elements are marked with empty dots. Elements introduced in the problem statement are also included in the MDCBD and can be either more confident or more tentative. Notably, we do not categorize elements as correct or incorrect within the MDCBD.

Step 2. We enumerate only those elements that capture students' reasoning about the multiplicative structure and its coordination with the concept of definite integrals. The elements are listed in the order they appeared in the transcript, allowing us to better understand the dynamics of students' reasoning throughout the interview.

Step 3. We assign each enumerated element to one of four mental spaces in the MDCBD: three input spaces (symbolic, physics, and mathematics) and one blended space. The *symbolic*

space includes mathematical and physical symbols and notations. The *physics space* encompasses elements related to physics knowledge (e.g., concepts, statements, and drawings about physics laws) without introducing new mathematical input. The *mathematics space* includes elements related to mathematical knowledge (e.g., drawings, functions, equations, derivatives, and integrals) without new physical input. The *blended space* consists of elements that integrate ideas from the symbolic, mathematics, and physics spaces, including equations and drawings. We denote enumerated elements assigned to each mental space by S1, M1, B1, P1, etc. In our study, we use the generic space to represent the overall structure based on inferred reasoning, as interpreted by the researchers. Since our analysis focuses on students' statements and reasoning, we exclude the generic space from our diagram.

Step 4. We trace students' dynamic reasoning about multiplicative structures by connecting the selected elements across all spaces and within each space. These connections are categorized into two groups: more confident and more tentative. These connections are defined similarly to the more confident and more tentative elements described in Step 1. To represent these connections visually, we use solid green lines for more confident connections and dotted green lines for more tentative connections.

Step 5. In contrast to the dynamic blending framework described by Van den Eynde et al. (2020), our approach accounts for dynamic interactions among elements and their modifications. Modifications refer to students' actions of activating or deactivating elements and connections, whether more confident or more tentative. Activation occurs when a student recognizes the relevance of an element to the current phenomena. Activated elements are represented by blue squared boxes, while deactivated elements are represented by red squared boxes. At this stage, we identify elements modified by students and repeated steps 2 to 4 for the updated set of elements. Consequently, a connection identified in Step 4 may initially be active and later become inactive. An activated connection indicates that a student has recognized the relevance of elements to the current phenomena. Activated explicit connections are represented by solid blue lines, while deactivated explicit connections are shown with solid red lines. Activated and deactivated implicit connections are denoted by dotted blue and red lines, respectively.

Figure 3a provides a schematic representation of the modified dynamic conceptual blending diagram, including all the coding conventions discussed. By elaborating on the four mental spaces, the connections among elements within those spaces, and their dynamic changes, we described students' moment-to-moment reasoning about the "product layer" when blending ideas from the symbolic, mathematical, and physical input spaces.

The Expanded VAP-Model

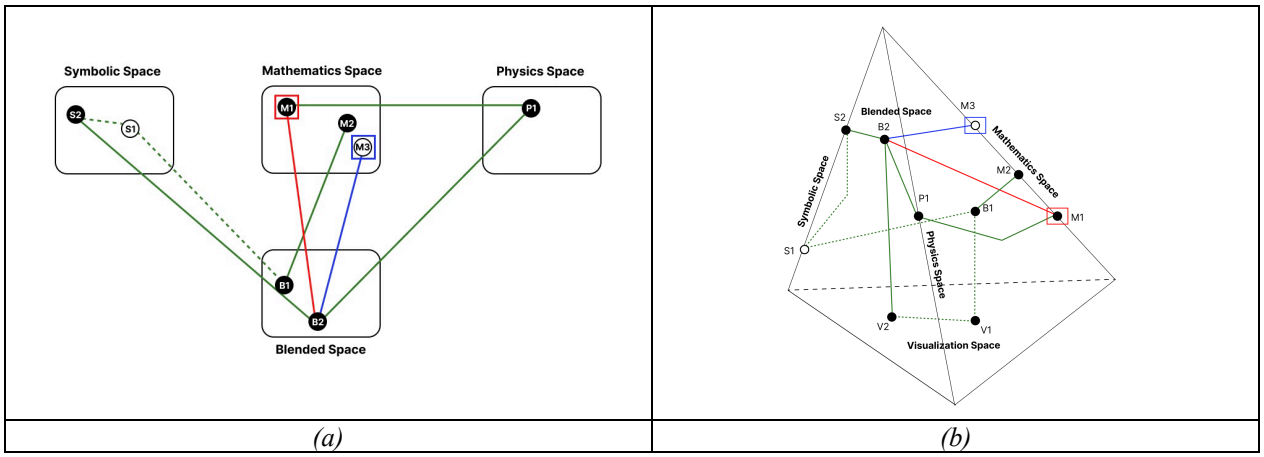
Zimmerman (1991) wrote that "visual thinking is so fundamental to the understanding of calculus that it is difficult to imagine a successful calculus course which does not emphasize the visual elements of the subject" (p.136). Other researchers contend that in order for students to construct a conceptual understanding of mathematical concepts, both visual and analytic reasoning must be present and integrated (Aspinwall & Shaw, 2002; Zazkis et al., 1996). There is also a literature dedicated to graph interpretation and construction outside of mathematics. For instance, Beichner (1994) developed a test to determine students' interpretations of kinematics graphs. Therefore, we consider visual representations (drawings) to be an important part of a student's moment-to-moment reasoning when connecting mathematics (calculus) to physics.

Our perspective extends the VAP-model which Zazkis (2013) developed to analyze students' visual (V), analytical (A), and physical (P) modes of reasoning, and the transitions between reasoning with graphs, functions, and physical concepts. The VAP-model diagram has a

tetrahedron shape where its edges represent three modes of reasoning. The transition between the modes is a disorderly, unpredictable path that moves upward with time and has a spiral form. The upward direction indicates that visualization and analysis become closer to each other as one moves to more advance levels.

In our extended model, we incorporate five reasoning spaces: symbolic (S), mathematics (M), physics (P), blended (B), and visualization (V). The spaces S, M, and P are represented by the vertical edges of a tetrahedron, with the blended space occupying the interior. The altitude of each element within the tetrahedron reflects its activation time, moving from bottom to top. The coding of elements and their connections in spaces S, M, P, and B follows the same scheme as the modified dynamic conceptual blending diagram (MDCBD) described earlier.

Under visualization space V we consider a visual blended space complemented by a conceptual layer developed through elaboration (e.g., Cunha et al., 2020). It is represented by elements located at the base of the tetrahedron, arranged chronologically (V1, V2, etc.). These elements, drawings, are connected to the blended elements in space B, with coding consistent across all spaces. Figure 3b provides a schematic representation of the expanded VAP-model diagram.



(a) *(b)*

Figure 3. The MDCBD and the expanded VAP-model diagram.

Water Pressure Task: An Illustrative Example

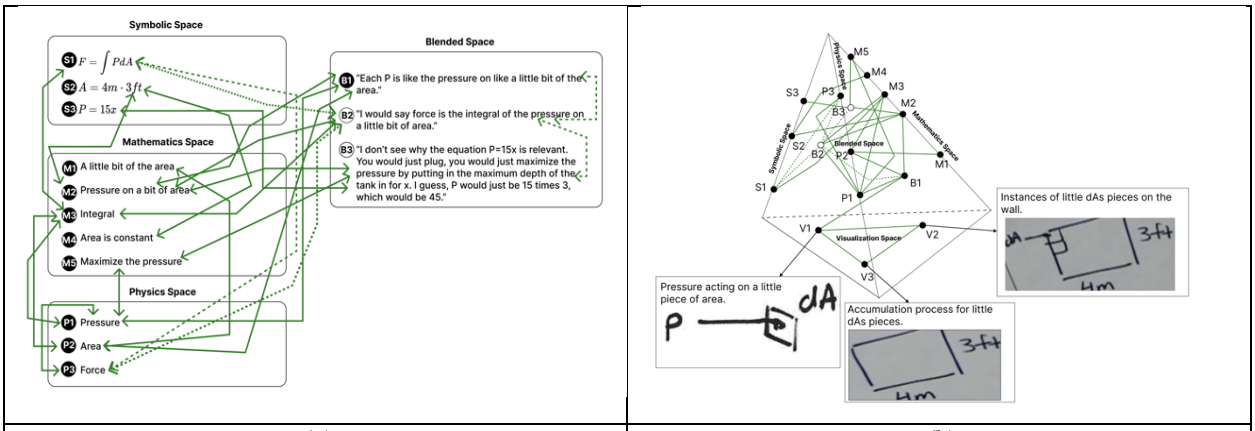
In this section, we provide an illustrative example of how the above methods can be applied to analyze student's moment-to-moment dynamic reasoning about the "product layer" and the definite integral in an applied context.

The example comes from a college student, Matthew, enrolled in a second semester calculus-based physics course. Matthew was working on a Water Pressure Task (adapted from Sealey, 2014) during a one-to-one interview session. The task asked him to estimate the total force from the water pressure on a $4m \cdot 3ft$ vertical side wall of a tank, where the pressure was given by the function $P = 15x$ and x was the depth of the water. Here is a sample of interview data.

Matthew began by writing down the given data which led him to think that "Each P is like the pressure on like a little bit of the area." He claimed that he did not know the actual relation between the quantities in the given problem situation: "I don't know the actual relation, so I'm just guessing." Then he stated: "I would say force is the integral of the pressure on a little bit of area [$F = \int PdA$]. And umm the area is, well the area is constant essentially makes sense [$A = 4m \cdot 3ft$]." Matthew expressed some confusion about the depth of the water to be given as x,

where the tank is full of water. As a result, he stated that he did not see why the equation $P = 15x$ is relevant because "you would just plug, you would just maximize the pressure by putting in the maximum depth of the tank in for x." He continued: "So, I guess P would be 15 times 3, which would be 45." Later, Matthew added the graphical representation showing how the pressure is acting on a little bit of area (see drawings in Figure 4b).

We now illustrate (see Fig. 4) how this piece of data can be analyzed using MDCBD and the expanded VAP-model diagram given in Figure 3.



(a) *(b)*

Figure 4. The MDCBD and the expanded VAP-model diagram.

Discussion and Conclusion

Modeling Matthew's responses with the modified dynamic conceptual blending diagram (Figure 4a) and the expanded VAP-model diagram (Figure 4b) allowed us to conduct a fine-grained analysis of his reasoning in the given problem situation. We can see that Matthew applied knowledge resources coming from three input spaces to generate blending elements such as B1, B2, and B3. For instance, combing elements S1, M1, M2, M3, P1, P2 and P3, the student generated a blended element B2, which stood for the force definition via the integral representation and represented the "product layer" structured as PdA. Later, the student added a visual representation of the pressure acting on a little bit of area, which was captured by the element V1 of the visualization space and corresponded to the expression PdA. He demonstrated instances of little bits of area by V2 and blending it with the element V1 produced the accumulation process given by V3.

Overall, creating these diagrams enables researchers to move beyond analyzing word choices, gestures, or inscriptions, allowing for a deeper exploration of the meanings that students construct. For instance, the diagrams in Figure 4 helped us capture the complexity of Matthew's dynamic reasoning about the definite integral by identifying a set of knowledge resources and their interactions. Analyzing these diagrams alongside the data can help researchers refine their theories of students' mathematical understanding in applied contexts and significantly contribute to the existing literature on integration.

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Supporting Learning Through Interpreting Others' Solutions From a Radical Constructivist Perspective: A Theoretical Report

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Although much research has made the case for the value of students' making sense of others' solutions, explanatory mechanisms for how such learning occurs are lacking. In this theoretical report, we consider how students' making sense of others' mathematical solutions may support learning from a radical constructivist perspective. We elaborate on radical constructivist constructs–social goals, cognitive perturbations, and reflective abstraction–and use these constructs to model how engagement with others' mathematical solutions may engender learning. We illustrate our model with a task we designed to promote students' meanings for spatial coordinate systems. We conclude with implications for research and teaching.

Keywords: Learning Theory; Cognition; Problem-based Learning

Students' making sense of classmates' solutions to mathematical problems (e.g., Webb et al., 2014) and with worked examples (e.g., Barbieri et al., 2023) has been positively associated with mathematics achievement. Although researchers have suggested reasons why such student activity may translate to achievement gains (Brown et al., 1992; Webb et al., 2023), they have not provided explanatory mechanisms for how such learning occurs for an individual. As a theory of learning focused on ways individuals develop knowledge, radical constructivism can provide such explanations. Broadly, radical constructivism explains how individuals build knowledge through persistent cognitive adaptation in concert with their lived experiences, and it defines learning in terms of shifts in an individual's mental structures (von Glasersfeld, 1995). In this theoretical report, we consider how students' working to make sense of others' mathematical solutions may support learning from a radical constructivist perspective (Table 1 provides colloquial descriptions of key terms).

Table 1. Colloquial descriptions of terms to orient the reader

Term	Brief colloquial description of constructs to orient the reader
Perturbation	Anything that creates a disturbance in a student's equilibrated (or settled) state. A perturbation is neutralized when the disturbance no longer exists.
Experiential Provocation	A perturbation that a student conceives as external in origin (i.e., from their interaction with their environment). Examples include other people, tasks, etc.
Social Goal	A goal a student establishes and assumes is shared with others.
Scheme	Goal-directed regularities in a student's functioning consisting of: (1) a perceived situation, (2) activity triggered by the situation, and (3) an anticipated result.
Cognitive Perturbation	A perturbation in which a student experiences discrepancies in their use of a scheme.
Affective experience	Experience in which a person's feelings, attitudes, or moods are activated.
Reflective Abstraction	A mechanism to explain an individual's modification of their schemes toward greater cohesion and generality.

Learning in Radical Constructivism and Connections to Social Interactions

In this section, we present and coordinate constructs to yield explanatory mechanisms through which a student may learn from others' solutions. First, we conceptualize that a student prompted to interpret a solution can experience a disturbance to their settled cognitive state (i.e., a perturbation). Second, as the student works to understand the solution, they can enact schemes relevant to their understanding of, and goals for, interpreting the solution. If the student experiences a cognitive perturbation, they may modify or reorganize their schemes to neutralize the disturbance. Such reorganizations can result in learning at a higher cognitive level (i.e., reflective abstraction). Below, we offer more elaboration on this process.

Schemes, Goals, and Learning

As students engage with others' solutions, we posit they would draw upon *schemes*. Drawing on von Glasersfeld (1995), Hackenberg (2014) defined schemes as "goal-directed regularities in a person's functioning that consist of three parts: a situation, an activity triggered by how the person perceives the situation, and a result of the activity that a person assimilates to her or his expectations" (p. 87). A student's schemes are activated through interactive experiences and involve their anticipation of the outcome of the interaction; thus, a student's goals are central to the schemes they enact. Any interaction prompting the student to draw on one or more schemes is a disturbance to the student's settled (equilibrated) state.

A student uses their schemes as part of their goal-directed activity. Moreover, in classrooms where sharing solutions is prioritized, the student may conceive they are working toward a *social goal* with others. Steffe and Thompson (2000) describe that an observer can identify a student working toward a social goal when the observer can infer the student (1) understands others in the group have intentions and (2) pursues a goal they assume is shared with the others. For example, a student may assume their classmates are working toward articulating solutions that are understandable to others and adopt this to inform their own goal-directed activity. We note even if other students are working toward a different goal (e.g., only intending to obtain a correct solution), a student's perception of a social goal can drive their goal-directed activity.

Experiential Provocations and Perturbations

If a student works to interpret classmates' solutions, they have experienced a disturbance to their equilibrium; von Glasersfeld (1980) broadly defines a *perturbation* as any input that creates a disturbance in a student's equilibrium. In this report, we use Steffe and Thompson's (2000) distinction between two kinds of disturbances (experiential provocation versus perturbation) based on the student's perception of the source of the disturbance (external versus internal). We refer to *experiential provocations* as disturbances that the student conceives to originate from their interaction with their environment (Steffe & Olive, 2002; Steffe & Thompson, 2000; Steffe & Tzur, 1994). We note the environment may include tasks, tools, teachers, other students, and, relevant to this report, others' solutions. Prompts that require a student to interpret a classmate's solutions can serve as experiential provocations from the student's perspective; in some cases, such an experiential provocation may lead to the student establishing a social goal with their classmates such as solving a particular problem or understanding a novel solution.

Experiential provocations are perturbations, but not all perturbations are *cognitive perturbations*. Students might be able to neutralize some perturbations using their current schemes and without experiencing any discrepancies as they activate and anticipate the results; such perturbations are *not* cognitive perturbations. Neutralizing other perturbations may involve a student experiencing discrepancies in their use of a scheme (Steffe & Olive, 2009; von

Glasersfeld, 1995). Such perturbations *are* cognitive perturbations. Cognitive perturbations can lead to a student reorganizing or modifying their existing schemes to achieve an equilibrated state (Steffe, 1991a, 1991b; Tillema & Gatza, 2024; von Glasersfeld, 1995). We conceptualize learning occurs through adjusting schemes in response to cognitive perturbations.

To illustrate, a student engaging in the goal-directed activity of interpreting a classmate's solution encounters a perturbation in the form of an experiential provocation. If they can interpret the classmate's solution without modification to their schemes (such as when the classmate's solution is nearly identical to their own), then the experiential provocation did not create a cognitive perturbation. However, if the student has difficulty making sense of a solution, then the experiential provocation could result in a cognitive perturbation. Such a cognitive perturbation could require the student to modify or reorganize their schemes to achieve an equilibrated state. We elaborate on this example and provide more details in the next section.

Types of Cognitive Perturbations and Affective Experiences

When a student experiences a cognitive perturbation, they may experience a minor or major cognitive perturbation. Many researchers have equated perturbations with major cognitive conflict or experiencing a 'problem' (e.g., Booker, 1996; Lerman, 1996; Simon et al., 2010). Although a major conflict is one type of cognitive perturbation, students can also experience minor cognitive perturbations without (consciously) experiencing a problem (Steffe, 2011; Steffe & Olive, 2002, 2009). If an individual neutralizes a perturbation, an observer's hypotheses about the steps taken can serve to differentiate major and minor cognitive perturbations. Major cognitive perturbations require more significant accommodations to a student's schemes to resolve, whereas minor cognitive perturbations may be easily resolvable with small adjustments to a student's schemes. Students may also experience non-neutralizable cognitive perturbations when they do not yet have schemes that allow them to neutralize the perturbation.

To exemplify this distinction, we again turn to a student work. A student might experience a minor cognitive perturbation when a solution has some feature or way of reasoning that is novel, and the student is able to neutralize the perturbation with minor modifications to their current schemes. If an observer infers the student undertakes a major modification or reorganization of their schemes, then the observer could characterize the perturbation as major. Finally, the student may experience a non-neutralizable cognitive perturbation if the student's current schemes do not support them in satisfactorily interpreting (from the student's perspective) the solution.

Moreover, as the above descriptions suggest, the perturbations students encounter, cognitive or otherwise, may bring forth different affective responses (Steffe & Wiegel, 1996; von Glasersfeld, 1995). For example, Steffe (2011) described, "A perturbation may be a surprise (e.g., unanticipated success, elation, or a sensation of pain)" (p. 258) and von Glasersfeld (1995) noted a perturbation may include "disappointment" (p. 65). A classmate's solution that a student finds hard to follow may lead them to experience frustration (i.e., a negative affective experience). In contrast, a solution they recognize as different from their own may result in a positive affective experience (e.g., pleasant surprise or delight). Thus, a student's affective experiences are important considerations in the design of learning environments that may engender reorganizations in students' schemes, thereby leading to learning.

Reorganization of Schemes and Reflective Abstraction

To further describe the reorganization of schemes that may occur after a perturbation, we use Piaget's (2001) notion of abstraction. Abstraction is a mechanism explaining an individual's modification of their schemes toward greater cohesion and generality. In this report, we use the

concept of *reflective abstraction*. In broad strokes, reflective abstraction entails two processes: a projection of actions or schemes to a higher level of thought and a reorganization that occurs at this higher level (Ellis et al., 2024; Piaget, 2001; Steffe, 2024; Tallman & O'Bryan, 2024; Tallman & Uscanga, 2020; von Glasersfeld, 1995). The reorganization can involve the creation of a coherent relationship or network of relationships between existing schemes as well as with new schemes (Piaget, 2001; Tallman & Uscanga, 2020). Such a reorganization involves taking prior meanings as input for further operating and thus can be considered a "higher" level. We note the cognizing subject need not be consciously aware of any reorganization.

Connecting to a students' affective experiences, a student might experience what an observer would call reflective abstraction as the student considers aspects of a solution they do and do not appreciate. This consideration could result in the student reorganizing their schemes to incorporate aspects of solutions they appreciated and avoid aspects they did not perceive as valuable (e.g., adopting steps or language from a particular solution they find useful and avoiding confusing lines of reasoning). In this way, affective experiences can work in concert with cognitive perturbations, leading to students' reorganizing their meanings at a higher level.

Other researchers have argued for the importance of supporting reflective abstraction and have provided suggestions for doing so. First, providing students repeated opportunities to develop schemes relevant to particular meanings can support their connecting their schemes and reasoning across similar (and different) contexts (Tallman & O'Bryan, 2024; Thompson, 2013). Second, providing occasions for students to compare activity across tasks can support reflective abstraction (Ellis et al., 2024; Piaget, 1976, 2001; Tallman & O'Bryan, 2024). We use these two suggestions to try to create opportunities for reflective abstraction as we describe below.

Exemplifying the Constructs: X-marks the Spot

We designed the *X-Marks the Spot Task* leveraging the above constructs to support students' work with spatial coordinate systems (described below). We included occasions for students to engage with their classmates' mathematical ideas with the intention they would set social goals. We conjectured multiple rounds of describing (to classmates) and interpreting descriptions (written by classmates) of locations in space could occasion experiential provocations for students. We hypothesized these experiential provocations could lead to major or minor cognitive perturbations along with associated affective experiences. Further, we provided deliberate opportunities for students to reflect on location descriptions at a higher level of thought. We intended these experiences to support students in engaging in reflective abstraction as they reorganize their meanings for organizing space.

Before describing the task, we first provide important background information about spatial coordinate systems. We then describe the task with the above constructs in mind. We provide examples from when we used the described tasks with elementary pre-service teachers (EPSTs) in a classroom setting to help contextualize how we view the constructs relating to the task.

Task Background: Spatial Coordinate Systems and Conventions

Researchers in mathematics education and spatial cognition have explored ways individuals conceive of and describe two-dimensional space (Lee, 2017; Piaget & Inhelder, 1967; Taylor & Tversky, 1996). In this report, we focus on a task designed to support students' developing meanings for spatial coordinate systems (Lee, 2017; Lee & Hardison, 2016; Lee et al., 2020; Paoletti et al., 2022). A spatial coordinate system is a coordinate system (CS) that entails either mentally overlaying a CS onto some perceived space or overlaying a space onto an already established CS. In either case, objects within the space can be located via coordinates. Radar on a

ship and GPS are different examples of spatial CSs (i.e., polar and Cartesian CSs, respectively).

We note there are countless ways an individual can organize space or construct a spatial CS. Cartesian and polar coordinates are two conventional spatial CSs. Conventions, including conventional coordinate systems, involve choices often developed or adopted for the purposes of efficiency and communication (Moore et al., 2019; Zazkis, 2008). Given the communicative value of such conventions, we conjectured we could support students in developing a social goal by providing them repeated opportunities to describe locations in space, with the anticipation of classmates' interpreting it, and to interpret classmates' descriptions of locations. This goal could lead to activity that supported students in reorganizing their schemes for organizing space towards more clear and efficient strategies. This conjecture is supported by Lee (2017), who found the goal of relaying instructions to others when locating an object in space supported ninth-grade students in developing viable spatial CSs much like conventional ones.

The Design of the *X Marks the Spot Task*

In the *X Marks the Spot Task*, we provide students with the map, buttons, and prompt shown in Figure 1a/b. Students can try each button and observe different overlays. Several overlays can be used to generate conventional coordinate systems: Star and Circles when coordinated can create a polar-like coordinate system; Vertical and Horizontal together can create a Cartesian-like coordinate system. Other overlays can be used to create unconventional yet viable ways to organize space (e.g., Figure 1b shows the map with the Waves and Vertical overlays). We conjecture the introduction of the task itself could serve as an experiential provocation for each student, and a student may experience minor cognitive perturbations as they develop associations between the action of clicking each button and the resulting change to the map.

After exploring the overlays, we ask students to mark an X. To support the creation of a social goal, we prompt students to use the overlays to provide a written description for their X's location that classmates will use. Thus, a student may assume their classmates have a shared goal of marking the same location. To achieve this social goal, a student may try to write a description clear enough for their classmates to mark a precise location.

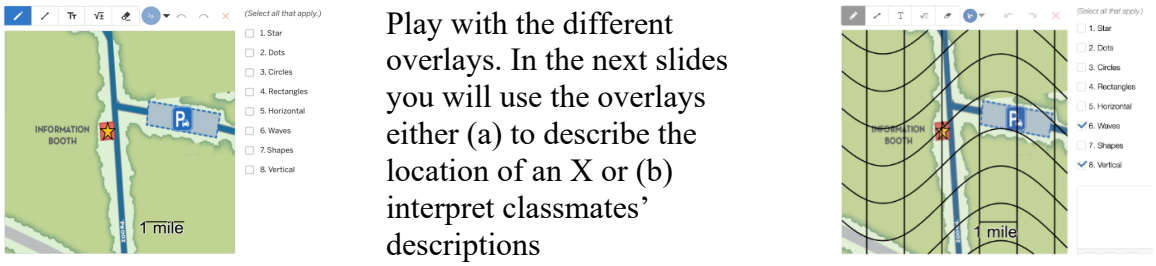


Figure 1. (a) The initial map, (b) initial prompt and (b) the map with Wave and Vertical overlays.

In classroom settings, we have EPSTs write their initial descriptions in groups. We share each group's description and the other EPSTs mark an X based on that description. After this, we engage in a whole class discussion using each group's description, original X, and an overlay of all the classmates' marked Xs (e.g., Figure 2). By comparing the locations of all the marked Xs, the EPSTs can determine whether they have achieved the *social goal* of marking their Xs in the same location. After several rounds of this activity, we provide numerous hypothetical classmate descriptions (Table 2) and ask students to mark an X from the description and provide feedback on the description to the hypothetical classmate. These hypothetical descriptions are intentionally

designed to (a) move from vague to more specific while (b) leading to more viable and/or conventional spatial CSs (i.e., polar and Cartesian).

Table 2. X Marks the Spot Task hypothetical descriptions and reflection opportunity

#	Description	Intention
1	Click the Star (1) and Circles (3) options. The X is on the 2nd circle on the line above the line going to the left from the star.	Polar-like CS that is intentionally vague
2	Click the Star (1) and Circles (3) options. Imagine the star is like a clock with the line going straight up being 12 o'clock and the line going straight down being 6 o'clock. The X is on the 4 o'clock line halfway between the 2nd and 3rd circle.	Polar-like CS that should provide enough information to locate an exact location that lies on a reference object in the CS.
3	Click the Star (1) and Circles (3) options. Imagine the star is like a clock with the line going straight up being 12 o'clock and the line going straight down being 6 o'clock. The X is 1.25 miles from the star halfway between 10 and 11 o'clock.	Polar-like CS that should provide enough information to locate an exact location that does not lie on a reference object in the CS.
[Reflection prompt for #1-3] Below are all the descriptions that used the Star (1) and Circles (3). Talk with your neighbor about which part of these descriptions was most helpful		
#4-6 provide similar phrasing with Cartesian-like CSs. There is also an opportunity for reflection on #4-6.		

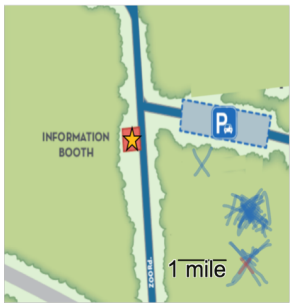
The act of interpreting others' responses occasion *experiential provocations* which could result in a student experiencing a *cognitive perturbation*. We conjecture students would be able to interpret many descriptions using their current schemes (i.e., *without cognitive perturbation*). We also conjecture interpreting vague descriptions (e.g., Description 1 in Table 2) or observing discrepancies in the location of marked Xs for the same description (e.g., Figure 2c) could create *cognitive perturbations*. For example, consider one group's description for the marked X (Figure 2a/b). During a class conversation, the teacher presented an overlay of their classmates' marked Xs for their description (Figure 2c). Observing the variance of the marked Xs, the EPSTs had opportunities to consider ways each student interpreted "two boxes up from the bottom" (i.e., does this include the bottommost box?). In this case, an EPST may have experienced a *cognitive perturbation* as they attempted to understand their classmates' descriptions or interpretations.

"With the horizontal and vertical overlays selected, the x is in lower right hand side below the parking garage. It is one box over from the right and two boxes up from the bottom."

(a)



(b)



(c)

Figure 2. (a) A description EPSTs gave for the X marked in (b), and (c) the Xs marked by the rest of the students

We anticipate the descriptions could lead to a student having a positive affective experience in concert with a cognitive perturbation. A student might have positive affective experiences as they interpret descriptions they feel are interesting, clear, or lead to precise locations (e.g., #2-3 in Table 2). For example, when tasked to provide feedback on Description 3, one EPST stated, "No advice, very precise. I like that clock analogy." We conjecture the EPST might have experienced a minor cognitive perturbation along with a positive affective experience as they made sense of the clock analogy for the first time ("very precise. I like…").

Further, we conjecture tasking EPSTs to provide feedback to the author of each description can engender reflective abstraction as students reorganize their meanings for organizing space. For instance, we observed some EPSTs adopted phrases (and associated reasoning) that included elements from the classmate description involving the clock analogy when engaging the final prompt. This prompt asked them to describe the location of a novel X (Figure 3a). Using the visual in Figure 3b, one EPST described: "Using the idea of a clock with the top line being 12, go in between 4 and 5 o clock in the third outermost circle. The x is in the middle of this box." We note how the clock language this student incorporated aligns closely with a polar-like CS used in Descriptions #2-3 (Table 2). Thus, we conjecture reflecting across descriptions supported the student in reorganizing their meanings for organizing space to incorporate elements they found useful, lending themselves toward constructing a viable and conventional spatial CS.



(a) (b)

Figure 3. (a) the last X described by EPSTs and (b) an EPST's use of polar-like coordinates

Contribution, Implications, and Areas for Future Research

In this report, we elaborated on social goals, experiential provocations, cognitive perturbations, and reflective abstraction. Importantly, we characterize how affective experiences are intertwined with cognitive perturbations in ways that can engender learning. We are unaware of examples in which radical constructivist researchers detail ways positive or negative affective experiences interact with students' learning. Given the emphasis on collaborative group work, such learning is likely to occur in classrooms where students interpret others' responses positively and reorganize their own meanings as a result of these interpretations.

We conjecture the use of others' descriptions to potentially provoke cognitive perturbations was productive in this task due to the communicative nature of the mathematics at hand. We conjecture providing students with repeated opportunities to first generate their own descriptions and then interpret descriptions that communicate more or less effectively and efficiently increases the chances students would experience positive affective experiences and cognitive perturbations that could result in their reorganizing their meanings for organizing space. There are likely other concepts that rely heavily on communicative goals such as conventions, in which students could be supported in learning via the use of others' solutions. We call for additional research exploring this possibility. This and other research could build on prior work showing how examining others' solutions to mathematical problems can support learning (Barbieri et al., 2023; Webb et al., 2014). Further, such research could leverage the constructs outlined here to provide explanations for *how* examining others' solutions can lead to learning.

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Exploring Gender Differences in Tertiary Mathematics-Intensive Fields: A Critical Review of Social Cognitive Career Theory

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Social Cognitive Career Theory (SCCT) has been extensively employed to elucidate the enduring gender differences in mathematics-intensive fields, with a particular emphasis on the complex interplay of motivational factors and extra-personal influences contributing to the underrepresentation of women. Although a plethora of empirical studies corroborate SCCT, three crucial aspects for refinement have come to the fore. First, the theory should place a more substantial emphasis on how cultural and contextual diversity influences academic choices. Second, given the dynamic nature of motivation, which evolves over time, more longitudinal analyses are imperative to capture their temporal trajectory, in contrast to the predominantly cross-sectional empirical studies. Finally, considering the intricate interplay between emotion and motivation, integrating the dimension of emotion into SCCT would significantly augment its explanatory power and provide a more comprehensive understanding of academic selection processes.

Keywords: Gender Differences; Mathematics-intensive Academic Fields; Social Cognitive Career Theory; Women in STEM

Gender disparities in tertiary mathematics-intensive fields and subsequent career trajectories remain a pervasive issue in most Western countries, with women consistently underrepresented. This persistent challenge is particularly evident in tertiary education, where women's participation in disciplines such as geoscience, engineering, economics, mathematics, computer science, and the physical sciences (chemistry and physics) is notably lower compared to their male counterparts (Ceci et al., 2014). Alarmingly, women's representation in computer science and electrical and electronic engineering has not only stagnated but has declined over time in Spain (Olmedo-Torre et al., 2018). Moreover, the gender gap transcends the boundaries of academia and permeates the workforce, with women's participation in math-intensive careers ranging from a mere 14% to 24% across various countries (Piloto, 2023), further underscoring the systemic nature of this disparity.

The persistent gender disparity in mathematics-intensive domains within tertiary education and its far-reaching implications for women's career trajectories necessitate a comprehensive exploration of the underlying factors perpetuating this inequality. Specifically, the underrepresentation of women in math-intensive fields serves as a critical antecedent to their subsequent underrepresentation in related career paths, exerting a profound influence on various aspects of society, including economic growth, wage equality, and the effective utilization of women's potential (Card & Payne, 2021). This underutilization of women's potential is particularly alarming, given that girls often outperform boys at most educational levels (OECD, 2015). To effectively address this multifaceted issue, Social Cognitive Career Theory (SCCT) has emerged as a valuable framework for elucidating the persistent gender gap, providing nuanced insights into the complex interplay of individual, social, and institutional factors that influence academic selection and career trajectories.

SCCT offers a robust framework for understanding academic choices and career development among boys and girls; however, empirical evidence indicates that further

refinement is necessary to enhance the theory's explanatory and predictive capabilities. This paper critically elucidates the limitations of SCCT, particularly its applicability across diverse cultural contexts, its reliance on cross-sectional empirical research despite the theory's dynamic nature, and its underestimation of emotional factors in academic decision-making. By identifying these limitations and proposing potential improvements, this theoretical paper aims to contribute to the ongoing evolution of SCCT, ensuring that it remains a relevant and effective tool for understanding individuals' choices and performance among boys and girls in increasingly complex and diverse environments.

Social Cognitive Career Theory: Key Concepts and Relations

Social Cognitive Career Theory, developed by Lent and Brown (1994), is a comprehensive framework that elucidates the intricate processes involved in academic selection and career development. Rooted in Bandura's (1986) social cognitive theory, SCCT emphasizes the interplay between individual agency and environmental factors in shaping academic choices and career trajectories. The theory posits three interconnected models—interest, choice, and performance, each centered around the core constructs of self-efficacy, outcome expectations, and goal representations. These constructs interact dynamically with personal factors, contextual influences, and learning experiences to shape career-related behaviors and decisions.

The Distinctive Roles of Self-Efficacy and Outcome Expectations in SCCT

Self-efficacy and outcome expectations are two key constructs in Bandura's (1986) social cognitive theory that play crucial roles in shaping individuals' selection processes. Self-efficacy refers to individuals' appraisals of their capability to organize and execute the requisite actions to attain desired performance outcomes. Crucially, self-efficacy is distinct from objectively assessed abilities. Social cognitive theory posits that human capabilities are malleable rather than fixed, asserting that successful performance in complex or demanding tasks typically necessitates not only foundational skills but also a robust sense of efficacy to effectively deploy those skills (Bandura, 1991). While self-efficacy focuses on individuals' confidence in their ability to execute actions, outcome expectations pertain to the anticipated consequences of those actions. Bandura (1986) classified outcome expectations into three categories: physical (e.g., tangible rewards), social (e.g., external validation), and self-evaluative (e.g., self-satisfaction). These expectations can exert a significant influence on career-related decisions and behaviors.

Bandura (1986) acknowledged the dual role of self-efficacy and outcome expectations, but he contended that these two forms of belief often differ in their relative potency, with self-efficacy serving as a more influential determinant of behavior. To illustrate this point, there are numerous instances in which individuals may anticipate valued outcomes resulting from a particular course of action but refrain from pursuing such action if they harbor doubts about their capabilities. However, in situations where outcomes are only tenuously linked to performance quality, outcome expectations may contribute to motivation and behavior independently (Bandura, 1989). This scenario may be especially pertinent to career development, as the intricacies of academic and career environments often yield only imperfect connections between performance quality and outcomes. For instance, an individual with a high ability for mathematics might opt to eschew mathematics-intensive career fields if they lack the support of significant family members; their anticipated outcome is far away from math.

The Structure of SCCT: Interest, Choice, and Performance Model

Interest model. Self-efficacy and outcome expectations, which incorporate values, are crucial in forming interests. Through repeated activity engagement, modeling, and feedback, children and adolescents refine skills, develop performance standards, and establish self-efficacy in specific tasks while acquiring outcome expectations (Lent & Brown, 1994). The development of self-efficacy and outcome expectations contributes to the formation of interests both directly and indirectly. These emerging interests lead to intentions for further activity exposure, increasing the likelihood of task selection and practice. Activity involvement results in performance attainments, such as successes and failures, which prompt the revision of self-efficacy and outcome expectancy estimates (Lent & Brown, 1994).

Choice model. The choice model builds upon the foundation of the basic interest model, extending it to encompass the developmental aspects of academic or career decision-making. This model conceptually divides the choice phase into several interconnected processes (Lent & Brown, 1994). Firstly, individuals express a primary choice goal, selected from their major interests. This choice goal represents the intention to pursue a specific career path or engage in a series of actions, such as aspiring to become an engineer. Secondly, individuals take concrete steps to implement their choice, which may include enrolling in relevant training programs or selecting an appropriate academic major. Finally, the model incorporates a feedback loop, wherein subsequent performance attainments, such as academic successes or failures and admission outcomes, shape future career behavior and choices.

Performance model. The performance model presented here adopts a broad definition of performance, encompassing both the level of achievement (e.g., academic grades) and indices of behavioral persistence (e.g., stability in a chosen major). This model emphasizes the significant roles played by ability, self-efficacy, outcome expectations, and performance goals while omitting interest as an intermediate mechanism. Lent & Brown (1994) posit that interest is more closely linked to selecting career or academic activities rather than setting performance goals. As the choice model suggests, choice goals contribute to determining the performance domains (e.g., work tasks) that an individual will pursue; however, the quality of performance attained may be partially contingent upon the level of one's performance goals. For instance, students who opt for an engineering major (choice goal) also establish goals pertaining to their grade performance in the requisite courses. Aspiring to achieve an A in math (performance goal) serves to regulate one's subsequent course-related behaviors (e.g., study time allocation).

Sociocultural Influences on Gender Disparities in Career Choice and Performance

The SCCT provides a robust framework for understanding how sociocultural factors contribute to gender disparities in career choice and performance from a motivational perspective. Specifically, the theory emphasizes the influence of personal factors, such as ethnicity and gender; contextual factors, including cultural norms, gender role expectations, and emotional support; and experiential factors, like personal achievements, vicarious learning, social persuasion, and physiological states, on career-related outcomes (Lent & Brown, 1994). These factors serve as antecedents of motivational variables, moderators of key relationships, or direct facilitators or barriers to career development. Furthermore, SCCT posits that gender differences in motivation are primarily influenced by social contexts. While acknowledging the role of innate aptitudes and interests, the theory suggests that these predispositions can be molded by environmental factors throughout an individual's life, highlighting the significance of the social contextual factors in shaping career-related outcomes and contributing to gender disparities.

Social Cognitive Career Theory: Empirical Evidence on Gender Perspectives

Gender Differences in Motivation

A growing body of research consistently demonstrates the predictive validity of self-efficacy and outcome expectations in relation to interests, choices, and performance, particularly in mathematics-intensive domains (Lent et al., 2017; Sheu et al., 2010). These findings underscore the central role of motivational variables in shaping individuals' math-related behaviors and outcomes. However, research also indicates that these motivational variables do not operate in a vacuum; they are influenced by broader socio-cultural factors, including gender-role identities. For instance, due to societal expectations and the internalization of gender norms, girls often exhibit lower self-efficacy and outcome expectations, as well as reduced interest in math-intensive fields compared to their male counterparts (Luo et al., 2021; Lv et al., 2022).

The Impact of Family and School on Gender-Based Motivational Differences

As essential carriers of socio-cultural influences, family and school have been widely recognized as critical determinants of choices and performance. The family, as the primary social unit, provides children with their initial exposure to gender-specific expectations and roles. This early socialization within the family context lays the foundation for the development of gender-specific beliefs and behaviors. Schools, on the other hand, provide opportunities to refine these beliefs and behaviors. Therefore, the complementary relation between family and school is crucial in the evolution of children's career aspirations.

Family factors. The family factors, including socioeconomic status (SES), parental beliefs, and parental behaviors, exert a profound influence on individuals' academic achievements and career aspirations. SES, a multidimensional construct encompassing income, education, and occupation, has been consistently linked to children's academic choices. Gao et al. (2024) demonstrated that SES accounted for a substantial 50% of the variance in adolescents' mathematics achievement in New Zealand. Mathematics achievement serves as a critical prerequisite for students' selection of math-related majors (Wang & Degol, 2013). Higher SES families tend to provide greater resources, support, and extracurricular activities that foster the development of mathematical skills (Bandura & Bussey, 2004; Wang & Degol, 2013), thereby fostering the formation of interests and influencing future choices. Conversely, low-SES families often prioritize meeting their children's fundamental daily necessities over fostering the pursuit of mathematics-related competencies, with girls' academic performance and development receiving even less attention and support.

Beyond SES, parental beliefs about gender also exert an important influence on children's confidence and career aspirations. Empirical evidence suggests that parents' perceptions of their children's abilities can have a more significant impact on their offspring's self-beliefs and career goals than actual performance (Fredricks & Eccles, 2002; Parsons et al., 1982). More specifically, parents who endorse gender stereotypes in math tend to underestimate their daughters' abilities while overestimating their sons' abilities, and they may even place less importance on their daughters' participation in tertiary math-intensive majors (Parsons et al., 1982; Nicoletti et al., 2022). Moreover, parental behaviors often vary based on gender, inadvertently reinforcing gender disparities. Research has revealed that parents are more inclined to engage in scientific discussions with their sons and provide them with related resources while unintentionally limiting opportunities for their daughters (Crowley et al., 2001; Meece et al., 2009). This differential treatment can perpetuate gender stereotypes and contribute to the underrepresentation of women in math-intensive areas (Jacobs et al., 2005).

School Factors. Alongside family influences, schools and tertiary institutions also play a crucial role in shaping gender-specific beliefs and influencing students' motivation and career aspirations. Professors, teachers and peers are key agents in this process, with their beliefs, expectations, and interactions contributing to gender disparities in math-related participation. Specifically, teachers' beliefs and expectations about gender-specific stereotypes can significantly impact students' motivation. For example, teachers may emphasize girls' efforts over abilities in mathematics, leading to lower expectations and diminished confidence among girls (Patrick et al., 2007; Meece et al., 2006). However, the effect of teacher support can vary by grade level, with elementary school environments often favoring girls, while secondary schools tend to emphasize performance goals that may disadvantage them (Wang & Degol, 2013).

Peers also play a significant role in influencing math-related achievement and participation. Girls in supportive peer groups tend to excel in mathematics and science; however, the Big-Fish-Little-Pond Effect suggests that while being surrounded by high-achieving peers can provide academic stimulation and motivation, it may also lead to a diminished self-concept. This phenomenon occurs because individuals tend to compare their performance to that of their peers, and in a group of high achievers, even objectively strong performers may perceive themselves as less competent, thus lowering their academic self-concept (Leaper et al., 2012; Marsh, 1984). In addition to the influence of teachers and peers, it is important to consider the potential impact of single-sex schooling on girls. Docherty et al. (2020) discovered a peculiar discrepancy in engineering enrolments at the University of Canterbury (New Zealand) between 2005 and 2017. The study revealed that the proportion of female enrolments in engineering from single-sex high schools was disproportionately high, accounting for 56% of all female students in engineering, despite the fact that only a small proportion of girls (15.73%) attend single-sex schools (MoE, 2023). This finding suggests that single-sex settings may play a role in shaping girls' academic choices. Moreover, Smith and Evans (2024) reported that girls attending single-sex schools perform better in mathematics and science than their co-educational counterparts in New Zealand. They suggested that single-sex environments may reduce stereotype threat for girls by providing them with a more supportive and nurturing learning context, thereby fostering their confidence and interest in traditionally male-dominated fields. However, the research on the effectiveness of single-sex schooling is complicated, with some studies suggesting that single-sex schools do not have any significant advantages for girls in pursuing math-related goals compared to co-educational settings (Gao et al., 2024).

Social Cognitive Career Theory: Limitations and Pathways for Future Development

The aforementioned studies offer considerable empirical evidence that substantiates the SCCT, especially concerning its capacity to elucidate gender disparities in mathematics-intensive domains. Nevertheless, while these studies corroborate the SCCT framework, it is imperative to acknowledge that certain research also underscores the theory's limitations. These constraints encompass the inadequate emphasis on cultural diversity, the predominant reliance on cross-sectional research, and the absence of an emotional dimension. The limitations identified through our summary provide valuable guidance for future scholarly efforts to refine and expand the SCCT framework, thereby strengthening its explanatory capacity and applicability across diverse contexts.

Placing Stronger Emphasis on Cultural Diversity to Strengthen SCCT

The SCCT underscores the crucial role of sociocultural factors in shaping gender differences, but it falls short in providing a comprehensive analysis of the social environment due to its

Western origins, which may limit its universal applicability across diverse global contexts. Despite distinguishing between the social and biological concepts of gender and ethnicity, with a primary emphasis on the former, the SCCT has not adequately addressed the influence of cultural backgrounds in different countries on individuals' motivation and career choices. Recent cross-cultural studies investigating the application of SCCT have yielded inconsistent results, underscoring the complex interplay between cultural factors and academic choices. For example, Wang et al. (2023) identified significant gender disparities in the application of SCCT within the Chinese educational system, with girls exhibiting lower interest and requiring greater confidence in mathematics compared to boys. Conversely, Gao et al. (2024) found no such gender discrepancies in New Zealand. Moreover, Michaelides et al. (2019) observed that in some Eastern European and Islamic countries, girls outnumbered boys in math-intensive fields and demonstrated higher confidence in mathematics. These divergent findings highlight the need for a more nuanced understanding of the cultural factors influencing academic selection processes and suggest that the explanatory power of SCCT may vary considerably across different cultural contexts.

To address the limitations of SCCT in capturing the complexities of cultural influences on academic and career decision-making processes, future research should prioritize the investigation of cultural background factors and their underlying mechanisms in shaping individuals' educational and vocational trajectories. This endeavor necessitates the development of a substantial body of cross-cultural empirical research alongside the creation of culturally specific models that can effectively capture the nuances of diverse contexts. For instance, in Iran, where women's status is relatively low, science, technology, engineering, and mathematics (STEM) fields are often perceived as pathways to higher income and social status, potentially leading women to choose these fields to improve their social standing (Stoet & Geary, 2018; Michaelides et al., 2019). In contrast, in some Western countries with higher levels of gender equality, girls typically excel across various disciplines and face a wider range of career options. This diversity of choices may result in a lower proportion of girls choosing STEM fields, as they are more likely to make decisions based on their absolute abilities, whereas boys are more likely to make choices based on their relative abilities (Wang & Degol, 2013). As evidence gradually accumulates, it will serve as a foundation for refining and expanding the theory, enabling it to more accurately account for the intricacies of academic selection processes across various cultural landscapes.

Employing Longitudinal Studies to Validate SCCT

The cross-sectional research designs employed by the vast majority of empirical investigations examining the tenets of SCCT pose inherent limitations in their ability to fully capture the dynamic feedback mechanisms central to the theory. Specifically, SCCT posits a complex interplay of feedback loops in which an individual's self-efficacy beliefs and outcome expectations exert a profound influence on their choices and performance. In turn, the consequences of these choices and the resulting achievements serve to reshape future self-efficacy beliefs and outcome expectations, creating a continuous cycle of reciprocal determinism (Lent et al., 1994). However, cross-sectional studies are unable to adequately test these intricate relations of reciprocal determinism.

To fully elucidate the intricate dynamics posited by SCCT, a longitudinal approach is indispensable. Such research designs facilitate the exploration of temporal relations and enable the mapping of individual trajectories across extended timeframes (Ployhart & Vandenberg, 2010). By gathering data at multiple junctures, researchers can delve into the nuanced ways in

which self-efficacy and outcome expectations evolve in response to diverse experiences and interventions, thereby offering a more comprehensive and ecologically valid assessment of SCCT (Navarro et al., 2014). Furthermore, longitudinal studies possess the unique capacity to illuminate the directionality of relations between variables, tackling questions of causality that remain elusive in cross-sectional designs (Zapf et al., 1996).

Integrating Emotional Dimensions to Broaden SCCT

The SCCT has a notable limitation in its insufficient emphasis on the role of emotions in shaping individuals' academic choices. While the theory primarily focuses on explaining these choices from a motivational perspective, highlighting the interplay between self-efficacy, outcome expectations, and interest, it does not adequately address the significant influence of emotions on cognitive functions crucial to academic decision-making. If the value of outcome expectation negatively influences individuals' math achievement, it is challenging to explain using SCCT. However, by incorporating emotion into the SCCT framework, the result becomes plausible. It can be argued that a higher value may induce math anxiety, consequently lowering achievement. More specifically, emotions play a critical role in shaping cognitive processes such as attention, memory, and problem-solving (Lerner et al., 2015), which are essential components of academic-related decisions. Positive emotions have been associated with higher levels of engagement, deeper learning, and improved performance, contributing to stronger self-efficacy and more favorable outcome expectations (Reschly et al., 2008). In contrast, negative emotions can disrupt these cognitive processes, leading to decreased motivation and poorer academic outcomes (Pekrun, 2006). Consequently, the lack of sufficient attention to the emotional dimension within the SCCT framework may limit its explanatory power in fully capturing the complex dynamics of the academic choice process.

To address this limitation, future research should focus on integrating emotional dimensions into the SCCT framework. For example, Pekrun's (2006) Control-Value Theory (CVT) combines motivation and emotion to predict individuals' choices and performance in educational contexts. In CVT, control refers to individuals' beliefs about their abilities, i.e., whether they believe they can complete a task, while value includes intrinsic value (such as interest) and extrinsic value (such as the importance of the task). When individuals perceive their abilities as insufficient (low control) but simultaneously consider the task outcomes to be crucial for them (high extrinsic value), they often experience negative emotions such as anxiety or helplessness, which can adversely affect their academic performance. Therefore, by incorporating emotional factors into the SCCT framework, researchers can more comprehensively explain the dynamic interactions between emotions, cognition, and situations, thus establishing a more realistic theoretical model to describe individuals' academic choices and behaviors. This integration not only expands the theoretical perspective of SCCT but also significantly enhances its applicability to diverse populations and contexts, especially those groups and environments where emotional responses vary significantly.

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Integration by Substitution: A Justification Based on Multiplication, Measurement, and Inverse Proportions

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Researchers have argued that interpreting definite integrals and their notation as summations of small amounts of a target quantity (Adding Up Pieces) is especially productive for applied problems. Jones and Fonbuena (2024) extended research on Adding Up Pieces by providing an analysis of integration by substitution grounded in quantitative reasoning. The present theoretical paper develops a complementary justification that makes three contributions. First, whereas past research on integration has rarely foregrounded explicit, quantitative meanings for multiplication, the analysis presented here is based on connections between multiplication and coordinated measurement with two units. Second, the analysis presented here uses coordinated measurement to explain how, across diverse situations, small amounts of a target quantity can be understood as products. This supports reasoning about Riemann sums and areas as approximations for definite integrals. Finally, the analysis provides a new justification for integration by substitution that relies on rectangular areas and inversely proportional relationships.

Keywords: Calculus, Integration, Multiplication, Measurement

Introduction

A growing body of research has reported on students' interpretation and construction of integrals, especially when solving problems in physics and other applications outside of pure mathematics (e.g., Cui et al., 2006; Jones & Ely, 2023; Nguyen & Rebello, 2011; Oehrtman & Simmons, 2023; Thompson, 1994a). Recently, Jones and Fonbuena (2024) pointed out that this literature has largely overlooked the central topic of integration by substitution, or u-substitution. They then developed a conceptual analysis of the topic grounded in Thompson's account of quantitative reasoning (e.g., Smith & Thompson, 2007; Thompson, 1990) but, in that analysis, stopped short of discussing explicit, quantitative meanings for multiplication. The present report develops a complementary analysis based on (a) connections between multiplication and coordinated measurement with two different units and (b) inversely proportional relationships. The results add to a growing body of studies that have examined how quantitative interpretations of multiplication based on coordinated measurement can support calculus learning (e.g., Beckmann & Izsák, 2024; Izsák, 2023)

The paper is organized into four sections. In the first, I give a brief review of research on definite integrals that includes a summary of the results reported by Jones and Fonbuena (2024). A main point will be that past research has often referred to multiplication in connection with the $f(x) \bullet \Delta x$ and $f(x)dx$ notations without articulating an explicit, quantitative meaning for the operation. In the second, I give a brief overview of research on multiplication with quantities and point to complexities highlighted in that research that have yet to be grappled with directly in research on definite integrals. In the third, I review recent work on a coordinated measurement meaning for multiplication that can provide coherence across a wide range of problem situations and topics, including situations that often serve as applications of integration. A main point will be that the coordinated measurement meaning for multiplication can support using Riemann sums and rectangular areas to approximate definite integrals across diverse situations. In the

fourth, I explain how the coordinated measurement meaning for multiplication can support a justification for integration by substitution that is based on Riemann sums and inversely proportional relationships.

Review of Research on Definite Integrals

Applying definite integrals to solve problems about physical situations requires reasoning in terms of quantities, such as lengths, time, work, electric charge, etc. Past research has provided a variety of perspectives on such reasoning, and much of it has been grounded in Thompson's account of quantitative reasoning (e.g., Thompson, 1994b; Thompson, 2011). One line of work has focused on different interpretations of definite integrals and their notation. Thompson and colleagues (e.g., Thompson et al., 2013; Thompson, 1994a; Thompson & Silverman, 2008) have argued that interpretating integrals as accumulations from rates of change can facilitate reasoning about the Fundamental Theorem of Calculus. In other work, Jones (e.g., 2013, 2015) has found that college students can interpret the $\int_a^b f(x)\,dx$ notation as indicating addition of small pieces (Adding Up Pieces), denotating boundaries of a region in the plane, and signifying a function whose derivative is the integrand. At the same time, researchers have debated whether or not, when using Adding Up Pieces, students should understand those pieces as composed through multiplication (e.g., Ely, 2017; Jones et al., 2017; Jones & Ely, 2023; Thompson & Harel, 2021).

A second set of perspectives has decomposed reasoning about definite integrals into layers that include ones focused on local relationships (e.g., Oehrtman, & Chhetri, 2015; Oehrtman, & Simmons, 2023; Seeley, 2014; Stevens & Jones, 2023; Von Korff & Rebello, 2012). As one example, Sealey articulated the *Riemann Integral Framework*, which is a mathematical decomposition of Riemann sums and definite integrals into five layers that she called the Orientation, Product, Sum, Limit, and Function layers. She reported that university calculus students had the most trouble working in the Product Layer, which is where students interpret the $f(x)dx$ notation. As a second example, Oehrtman and Simmons proposed *Emergent Quantitative Models* as a tool for tracing how students' attention shifts across three scales—named the Basic, Local, and Global models—when constructing definite integrals. A Basic model relates quantities with constant values, a Local model restricts a basic model to small regions where varying quantities are approximately constant, and a Global model is a sum based on Local models. Thus, similar to the Product Layer, the Local model is where students would reason about products of the form $f(x) \bullet \Delta x$.

Research on definite integrals summarized above has consistently highlighted the importance of reasoning about small pieces of a target quantity but, to the best of my knowledge, Jones and Fonbuena (2024) were the first to analyze such reasoning in the context of integration by substitution. They began their conceptual analysis by decomposing Adding-up-Pieces reasoning into three aspects—*partitioning* a quantity into small pieces (precursors to differentials), *determining the target quantity* in each of those small pieces (similar to the Product Layer and the Local model mentioned above), and *summing* the small pieces of the target quantity to get the total quantity (similar to the Global model mentioned above). Their analysis traced the consequences for each of these three aspects when one changes from one input variable to another. They described these consequences as "conversions" of the partition, integrand quantity, and interval of summation.

Although past research on definite integrals has frequently attended to small intervals where relationships between quantities are approximately linear, to the best of my knowledge, none of

these studies has addressed explicitly an associated, quantitative meaning for multiplication. The next section explains why this omission is consequential.

Review of Research on Multiplication with Quantities

In comparison with the literature on definite integrals, theoretical and empirical research on multiplication is much older. For the present analysis of integration, I concentrate on theoretical results. Theoretical perspectives on multiplication have included classifications of problem situations in which one typically multiplies or divides (e.g., Greer, 1992; Nunes & Bryant, 1996; Vergnaud, 1983, 1988); interpretations, or meanings, of multiplication as an operation on quantities (e.g., Davydov, 1992; Izsák & Beckmann, 2019; Schwartz, 1988; Thompson & Saldanha, 2003); and psychological primitives that include repeated addition (Fischbein et al., 1985), splitting (e.g., Confrey, 1994), and units coordination (e.g., Hackenberg et al., 2023; Hackenberg & Tillema, 2009; Steffe, 1988; 1994). The analysis of integration by substitution presented below makes the most direct contact with the strands focused on classification and on interpretations of multiplication as an operation on quantities.

For classification research, I highlight Vergnaud's (1983, 1988) seminal contribution. He has analyzed mathematical structures for multiplying with quantities and distinguished two primary subtypes. *Isomorphism-of-measure-spaces* (I-o-M) situations are ones where the multiplicative structure consists of a simple direct proportion between two measure spaces, M_1 and M_2 (1983, p. 129). Here the direct proportion is often viewed as a rate of change. *Product-of-measure-spaces* situations (P-o-M) are the Cartesian composition of two measure spaces, M_1 and M_2, into a third, M_3 (1983, p. 134). In anticipation of approximating integrals, notice that Riemann sums use rectangular areas, associated with P-o-M situations, to model both I-o-M situations (e.g., integrating velocity to get distance) and P-o-M situations (e.g., integrating force to get work). One response would be to abandon using rectangular areas as generic models for products. Indeed, some researchers (e.g., Jones & Ely, 2023; Thompson et al, 2013) have claimed that the area-under-a curve interpretation for definite integrals is insufficient for modeling physical situations. An alternative response, pursued in subsequent sections of this paper, is to seek a meaning for multiplication that spans I-o-M situations and P-o-M situations. Such a meaning could help students understand how rectangular areas and, thus, Riemann sums can be used to express products and approximate definite integrals across both I-o-M and P-o-M situations.

Research on interpretations, or meanings, of multiplication as an operation on quantities has consistently focused on the role of units and, by implication, measurement. Izsák and Beckmann (2019) organized these interpretations into two categories, those that emphasize unit transformations and those that emphasize measurement. In one example of the unit transformation approach, Schwartz (1988) argued that when addition is used to model situations both addends and the sum refer to the same unit (e.g., 2 miles + 3 miles = 5 miles) but, when multiplication is used, the units of the factors are transformed into units of the product (e.g., 2 miles per hour • 3 hours = 6 miles). In a second example of the unit transformation approach, Thompson and Saldanha (2003) stated: "By convention, the standard unit of the product quantity is defined as that amount made by one unit of each constituent quantity" (p. 103).

Measurement might be implied in unit transformation approaches but is addressed more explicitly in further quantitative interpretations of multiplication. Thompson (1994b) articulated the following definition of a quantity when introducing his theory of quantitative reasoning: A quantity consists of "an object, a quality of the object, an appropriate unit or dimension, and a process by which to assign a numerical value to the quality" (p. 184). This definition implies that numbers are outcomes of measurement: They express how many units, or how much of a unit,

exhaust the extent of the "quality" in question. We often use the term "multiplication" to refer to this fusion of number and measurement—for instance, we might say that a taller building is three times the height of a shorter building when what we mean is that three of short height exhaust the tall height. Such multiplicative language, based on measurement with a single unit, is often taken as evidence of "multiplicative thinking." In such cases, measurement, numbers, and multiplication are intertwined.

Multiplication as Coordinated Measurement and Riemann Sums

Further research on quantitative meanings for multiplication has emphasized coordinated measurement with two different units (e.g., Beckmann & Izsák, 2015; Davydov, 1992; Izsák & Beckmann, 2019). Davydov argued that the essential situation for multiplication is one where it is impractical to measure some quantity of interest with a given unit because the unit is much smaller than the quantity one wishes to measure. The solution is to coordinate the smaller unit with an intermediate unit purposefully chosen to measure the quantity of interest more conveniently. As an example, if one wished to determine the number of teaspoons in a volume of milk, one might use cups as an intermediate unit, determine how many teaspoons are in 1 cup, and determine how many cups are in the target volume. Here the two measurement units for the target volume are teaspoons and cups.

Inspired by Davydov (1992), Izsák and Beckmann (e.g., Beckmann & Izsák, 2015; Izsák & Beckmann, 2019) have explicated a coordinated measurement meaning for multiplication. Figure 1 shows an equation where N, M, and P are each outcomes of measurement, as discussed in the previous section. The product amount is measured simultaneously in base units, P base units, and in group units, M group units. N expresses the measure of 1 group unit in terms of base units. Consider again our milk example, where a volume is measured in both teaspoons (base units) and cups (group units). If 48 is the measure of 1 cup in terms of teaspoons, and 7 is the measure of the target volume in terms of cups, then 336 is the measure of the target volume in terms of teaspoons. Following the meaning in Figure 1, our $N \bullet M = P$ equation would be 48 (teaspoons in 1 cup) • 7 (cups in 1 target volume) = 336 (teaspoons in 1 target volume).

N	•	M	=	P
How many base units make one group unit?		How many group units make one product amount?		How many base units make one product amount?

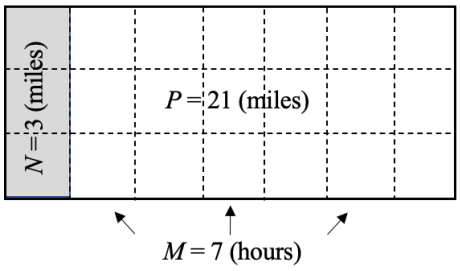
Figure 1. A coordinated measurement meaning of multiplication (Izsák & Beckmann, 2019, p. 91)

Izsák and Beckmann (e.g., Beckmann & Izsák, 2015; Izsák & Beckmann, 2019) have pointed out that the coordinated measurement meaning for multiplication works with positive real numbers, can tie together diverse situations using a single shared quantitative structure, and challenges the field to rethink several established results. One of those results is the partition of multiplication situations into structurally distinct subclasses—for instance I-o-M and P-o-M situations.

Figure 2 shows how the coordinated measurement meaning for multiplication summarized in Figure 1, combined with rectangular area, can fit both I-o-M and P-o-M situations. The same rectangular area represents a product amount measured both in base units (each small square represents 1 base unit) and in group units (each column represents 1 group unit). One can think of these situations as Basic models (Oehrtman & Simmons, 2023) for two common applications of integration. Figure 2a fits a problem about change in location (a walked distance). If each

small square represents 1 mile (1 base unit) and each column corresponds to 1 hour (1 group unit), then the measure of one column is 3, the change measured in hours is 7, and the change measured in miles is 21. The assignment of units to *N*, *M*, and *P* is in contrast to the more usually assignment of units: 3 miles per hour • 7 hours = 21 miles. Figure 2b fits a problem about change in energy. If each small square represents 1 Newton-meter (1 base unit) and each column corresponds to 1 meter (1 group unit), then the measure of one column is 3, the change measured in meters is 7, and the change measured in Newton-meters is 21. Again, this assignment of units to *N*, *M*, and *P* is in contrast to the more usually assignment of units: 3 Newtons • 7 meters = 21 Newton-meters.

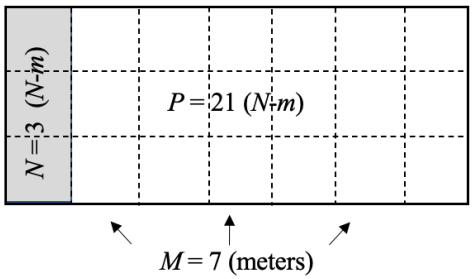
The examples in Figure 2 indicate how attention to two measurement units, made explicit in the $N \bullet M = P$ equation, allows one to see parallels (i.e., there is an isomorphism) between rectangular areas and the two applied situations, one an I-o-M situation and one a P-o-M situation. The same reasoning demonstrated here with a Basic model works equally well with Local models (e.g., Oehrtman & Simmons, 2023). Thus, Riemann sums based on rectangular areas can be used to approximate target quantities in both I-o-M and P-o-M situations. Learning to think about multiplication in terms of coordinated measurement could help students understand how rectangular areas can be used as a generic models for products and why multiplication based on rectangular areas gives the same numerical values as the corresponding multiplication in applied situations. The next section relies on Riemann sums based on rectangular areas to develop a justification for integration by substitution.



Shading corresponds to 1 hour.
Walk is measured in hours (7) and in miles (21).

N	•	M	=	P
3 miles in 1 hour	•	7 hours in 1 walk	=	21 miles in 1 walk

(a)



Shading corresponds 1meter.
Change is measured in meters (7) and in *N-m* (21).

N	•	M	=	P
3 *N-m* in 1 meter	•	7 meters in 1 displacement	=	21 N-m in 1 displacement

(b)

Figure 2. Coordinated measurement and rectangular area. (a) An I-o-M situation. (b) A P-o-M situation.

Coordinated Measurement and Integration by Substitution

The analysis of integration by substitution presented here is based on Riemann sums and, similar to the analysis given by Jones and Fonbuena (2024), is based on a change in the independent variable through function composition. Consider the Moving Truck situation presented below. Speed is taken to be piece-wise constant, and the discussion of Figure 2a above justifies using rectangular areas to represent distances. Figure 3a uses rectangular areas to show distance traveled for each segment of the trip. Think of the graph in Figure 3a as representing a simple Riemann sum.

Three friends are driving a moving truck from Detroit to Boston.

- *Steve takes his turn first and drives 4 hours at a steady speed of 60 miles each hour. At the end of his turn, the friends stop and rest for 1 hour.*
- *Chara takes her turn second and drives 4 hours at a steady speed of 72 miles each hour. At the end of her turn, the friends stop and rest for 1 hour.*
- *Zawadi takes her turn last and drives 4 hours at a steady speed of 54 miles each hour.*

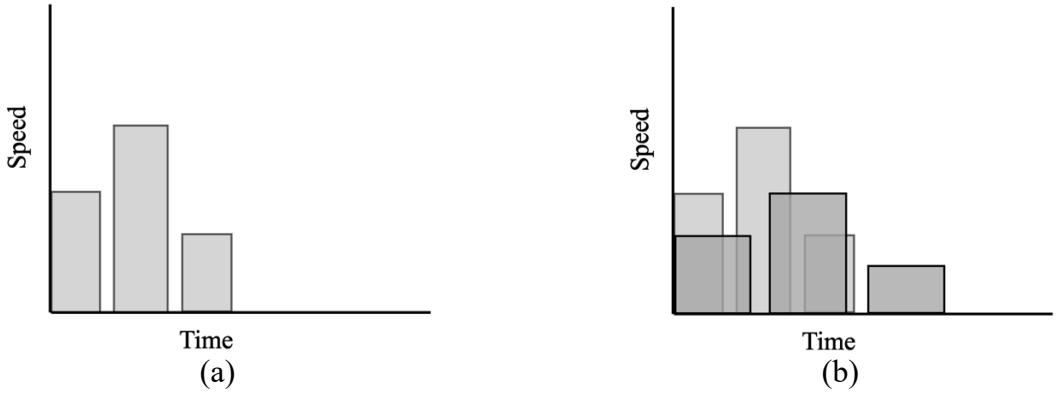


Figure 3.Moving Truck situation. (a) Time measured in hours. (b) Time measured in playlists.

Now suppose that the three friends invent a new unit based on the time it takes to go once through a playlist that they listen to as they drive. Suppose that 1 playlist is 2/3 of an hour. The miles driven in 1 playlist are fewer than the miles driven in 1 hour by a factor of 2/3. Thus, speeds measured in miles per playlist are 2/3 of the corresponding speeds measured in miles per hour. Simultaneously, the number of playlists it takes to measure any given duration of time is 3/2 the number of hours it takes to measure the same duration of time. Figure 3b superimposes onto Figure 3a a second graph of speeds and times for each segment of the trip based on playlists. Think of the graph in Figure 3b as representing a new Riemann sum that expresses distances when time is measured in playlists. Comparing areas when measuring time in hours and when measuring time in playlists, notice that we have an inversely proportional relationship: Shrinking in the vertical direction is offset by stretching in the horizontal direction, the distance for each trip segment measured in miles is the same whether time is measured in hours or in playlists, and the total shaded areas corresponding to the two Riemann sums are the same. Again, reasoning based on a Basic model applies equally well to reasoning based on a Local model.

In the Moving Truck situation, there is a single linear relationship between the original input variable, hours, and the new input variable, playlists. If students used a different unit of time for each segment of their journey, the same argument would work, each rectangle associated with time measured in hours (Figure 3a) would have the same area as the corresponding rectangle with time measured in the new unit. Thus, the Riemann sum depicted in Figure 3a would still have the same area as the transformed Riemann sum depicted in Figure 3b. The statement of integration by substitution is then plausible because the two definite integrals are approximated by the same sequence of values.

Discussion and Conclusion

To the best of my knowledge, the justification for integration by substitution presented here is both original and at odds with more traditional justifications. In traditional calculus courses, integration by substitution is often discussed as the "reversed chain rule," which can be

expressed with the following equation: $\int_a^b f'\big(g(x)\big)\, g'(x)dx \;=\; \int_{g(a)}^{g(b)} f'(u)du$. (A more general statement holds where f is continuous.) The student is tasked with recognizing functions f, g, and g' in the integrand on the left-hand side of equation that afford the transformation to the right-hand side of the equation, an integral one hopefully knows how to compute. One common justification for this transformation replaces $g(x)$ with u and $g'(x)dx$ with du and, thus, associates $g'(x)$ with dx.

In contrast to the traditional justification, which is motivated by how integration by substitution is typically applied, the justification presented here started with a Riemann sum for $\int_{g(a)}^{g(b)} f'(u)du$ and explained why the corresponding Riemann sum for $\int_a^b f'\big(g(x)\big)\, g'(x)dx$ gave the same number. This new justification requires reexamining several common ways of reasoning about integration. I have already discussed how the coordinated measurement meaning of multiplication supports using rectangular areas as generic models for products that span both I-o-M and P-o-M situations. In addition, the new justification consistently interprets integrands as rates. In particular, rather than associating $g'(x)$ with dx, as is common in more traditional explanations, the justification presented here associates $g'(x)$ with $f'(g(x))$ and the entire $f'(g(x))\, g'(x)$ expression is treated as a rate of change. For the Moving Truck example, $g'(x)$ rescaled rates of change represented by the vertical dimension that were caused by a change in the units used for measuring time. This, in turn, supports consistent interpretations of integrands across related topics, including the Fundamental Theorem of Calculus. Finally, although the new analysis is not a proof, it highlights how inverse proportions, a topic from middle grades mathematics, makes the statement of integration by substitution plausible. It is not a rigorous explanation *that* the statement is true but, rather, offers a plausible explanation for *why* it is true.

The analysis of integration by substitution provided here extends recent research on ways that multiplication understood as coordinated measurement with two different units can support coherent pathways from elementary and middle grades topics to calculus topics (e.g., Beckmann & Izsák, 2024; Izsák, 2023). As one example, Beckmann and Izsák argued that the large majority of first-year calculus is the study of situations in which *limiting behavior of co-variation takes place within a structure that coordinates multiplication with two measurement units*. They highlighted two different perspectives on proportional relationships, multiple batches and variable parts, and argued that approaches to rates of change based on multiple batches and unit rates are ill-suited for understanding $\frac{\Delta x}{\Delta y}$ as a single measure that remains (approximately) constant as the differences Δx and Δy tend to zero. Beckmann and Izsák then explained the concomitant affordances of variable parts and measurement rates. Finally, by attending to quantitative meanings for multiplication as a foundation for calculus, the present report resonates with other recent research that has examined sources of students' difficulties with calculus that reach back into middle grades topics (e.g., Boyce et al., 2021; Byerley, 2019; Carlson et al., 2015; Thompson & Harel, 2021). This emerging body of research suggests that a substantial reexamination and revision of how topics related to multiplication are developed in earlier grades could help students be better prepared for studying calculus grounded in quantities.

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The Suggestion of a New Framework of Asian Identities With the Critical Perspective on AsianCrit in Postsecondary Mathematics/STEM Education

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This study proposes the critical perspective on AsianCrit (Asian [American] Critical Race Theory) and suggests a new framework for potential research and practices in mathematics/STEM education regarding Asian identities. This study reviews and critiques the traditional views on AsianCrit for postsecondary mathematics/STEM education. Particularly considering Asians' diverse immigrant and citizenship status in STEM, gender identity, ethnicity, and more social identities, this study addresses the complicated relationships between Asian stereotypes, the Model Minority Myth, the intersectional identities of Asians, and the possible ways of empowerment.

Keywords: Critical Race Theory, AsianCrit, Racism, Equity

Throughout Asian discrimination during the COVID-19 crisis from 2019 to the March 2021 murders of eight Asian women in Atlanta and #StopAsianHate—a slogan from 2021 to alert people of Asian hate crimes, scholars have called the need for research on Asian racism in the context of equity and social justice. Then, did those movements bring anything to mathematics and STEM education? It could or could not.

Generally, people perceive that there are many Asians in STEM. How many Asians are in STEM in the United States (U.S.)? There are 17.3 million Asian Americans who make up approximately 5.6 percent of the U.S. population, and around 10 percent of the undocumented population in the U.S. is Asian American (Curammeng et al., 2017). Compared to this information, it could be true that there are many Asians in STEM. National Center for Science and Engineering Statistics (NCSES, n.d.) states that regardless of fields in STEM, the most significant racial group of color in every field of major is non-Hispanic Asians. There are nearly but less than 40% of non-Hispanic Asian graduates in computer and mathematical sciences and nearly but less than 30% of them in engineering. Overall, the historically minoritized racial/ethnic groups—Hispanic or Latin* ethnicity, non-Hispanic American Indian or Alaska Native, non-Hispanic Black or African American, non-Hispanic Native Hawaiian or Other Pacific Islander—are underrepresented in STEM fields except for social and related sciences, compared to non-Hispanic Asian.

Then, does the Asian representation in STEM mean no racism or discriminatory issues toward Asians? Shortly, this study says "No" to this question. Research in Asian [American] Critical Race Theory (AsianCrit) described discrimination associated with Western perspectives, such as Orientalism and the generalization of exoticism, that Asian populations experience. These experiences include being perpetuated foreigners, treated monolithically, perceived as intelligent but socially excluded, and sometimes threatening American society as "yellow peril." Despite these compelling arguments of AsianCrit, there is still little research exploring or employing AsianCrit that fits racial/ethnic issues in mathematics/STEM education regarding Asian identities. So, this study aims to suggest a new framework for future mathematics/STEM education research and practices geared toward diversity and inclusion, consisting of Asian identities. The research question is "How is AsianCrit dismantled, revised, and incorporated to examine Asian identities in postsecondary mathematics/STEM education?"

What and Why AsianCrit?

AsianCrit is one kind of "spin-off" Critical Race Theory (CRT) that aims to aid in addressing and analyzing Asian racial issues, especially in terms of systemic discrimination and inequities in the U.S. (Curammeng et al., 2017; Iftikar & Museus, 2018). According to the assertion of CRT, race is a social construct and a significant factor in deciding inequity, and that intersects with property rights, giving rise to social, systemic, and educational inequalities among people of color (Ladson-Billing & Tate, 1995). CRT is overviewed by the main tenets (Mensah & Jackson, 2018, See Figure 1), and the various "spin-off" CRTs primarily follow themes of CRT.

Table 1. Overview of Critical Race Theory Themes (Mensah & Jackson, 2018, p. 6)

	Solórzano (1997)	Zamudio, Russell, Rios, and Bridgeman (2010)	Delgado and Stefancic (2012)
Theme 1	The centrality and intersectionality of race and racism	Critique of colorblindness	Colorblindness
Theme 2	The challenge to the dominant ideology	Critique of liberalism	Interest convergence or material determinism
Theme 3	The commitment to social justice	Whiteness as property	The social construction of differential racialization
Theme 4	The centrality of experiential knowledge	Interest convergence	Intersectionality and anti-essentialism
Theme 5	The interdisciplinary perspective	Intersectionality	Unique voice of color or counter-stories

According to CRT principles, "Asian" is a collective identity organized around broad commonalities rooted in various ethnicities traceable to points in Asia (Patton et al., 2016). Throughout the ways Asians are racialized in the U.S., the racial/ethnic identity of Asians has been treated monolithically; nevertheless, the populations as racialized as Asians consist of more than 48 ethnic groups (Curammeng et al., 2017; Iftikar & Museus, 2018). Also, the definitions of Asians, Asian migrants, Asian immigrants, and Asian Americans are complicated and often difficult to differentiate, even though those words are not interchangeable (Chen & Buell, 2018).

Given these complicated identities of Asians and racialized issues treated monolithically, why does this study deal with AsianCrit? First, the focus on AsianCrit addresses the limitations of the general CRT. The most significant limitation of the CRT is the Black and White binary in the early stage of the theories (Alcoff, 2003) and the lack of a nuanced and more profound reality of Asians. Second, the investigation of AsianCrit will allow us to challenge stereotypes and discriminatory perceptions of Asians. The stereotypes toward Asians are unique because they sound like "compliments" for Asians. For instance, one of the most common stereotypes is that "Asians are good at math, and they are smart." This does not seem harmful but a compliment; even Asians believe that. However, these stereotypes are not only inaccurate but also harm Asian identities in various ways. (See "Model Minority Myth" below) Therefore, investigating Asian stereotypes with critical perspectives should be pursued more through AsianCrit. Ultimately, the

advance of AsianCrit will debunk White supremacy and systemic racism toward people of color (e.g., anti-Blackness) and let Asians be included as the constitutes of people of color.

Literature Review on the Traditional View of AsianCrit

Summary of the Traditional Asian [American]Crit

In the context of Asian American jurisprudence, Asian Americans' experiences and perspectives became invisible (Curamment et al., 2017). Also, historically, Western perspectives such as Orientalism and the generalization of exoticism have patronized and dehumanized Asian people in the U.S. (Iftikar & Museus, 2018). This racialization of Asian Americans is little salient in the early development of CRT, but the early stage of the conceptualization of CRT was focused on the Black-White binary (Iftikar & Museus, 2018). Thus, scholars have called more attention to the shift from the Black-White binary to the racial-specific CRT with a more nuanced and deeper inquiry-based reality, and this study focuses on Asians. The significant point is that the argument of the limitation on the Black-White binary does not indicate Anti-Blackness. Instead, the way to challenge the Black-White binary is eventually to debunk White Supremacy because the rationale of the Black-White binary in the early stage of CRT blocks the view on genuine racial issues not just about Asians but also the constitution of people of color.

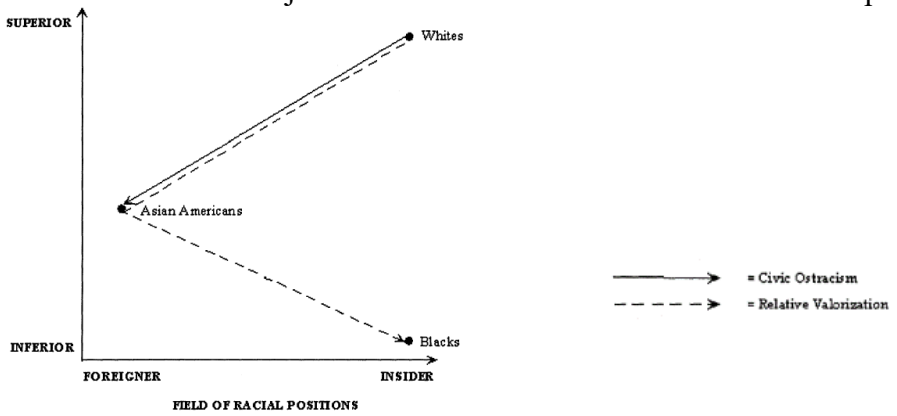


Figure 1. Racial Triangulation of Asian Americans in the United States (Kim, 1999, p. 108)

Grounded on the need for spin-off CRTs for more racial/ethnic specificity, there have been attempts to theorize Asian-specific CRTs. One of the earliest attempts was Chang's (1993) Asian American Legal Scholarship framework. Chang (1993) points out anti-Asian, nativist racist violence and discrimination and the idea of model minorities that exclude Asians from people of color. Also, other attempts to theorize AsianCrit include the law's role in perpetuating forever foreign. Particularly in the literature with educational context, it has been argued that CRT should address the distinguishable racialization between different ethnic Asian Americans and other people of color (e.g., the racial triangulation; See Kim, 1999). Asian Americans are "racially triangulated" by two axes—the superior-inferior axis and the insider-foreigner axis (Kim, 1999, See Figure 4). Kim (1999) argues that this study assumes Asian Americans are racialized in the zone of both civic ostracism—possibly less than Blacks, but still—and relative valorization to make Asian Americans perpetuate foreigners. This racialized position for Asian Americans is distinguishable from the one for Blacks, but the oppressive purpose of this racialization is to reinforce White dominance, privilege, and supremacy.

Model Minority Myth (MMM)

Model Minority Myth (MMM) is a central tenet that AsianCrit criticizes and explains why MMM is problematic. Since the middle of the 20th century, monolithic racial depiction in the U.S. led to MMM (Curammeng et al., 2017). MMM refers to the racial stereotypes usually toward Asian Americans who are hardworking, achieve higher education, and obtain higher socioeconomic status (Poon et al., 2016) than other minority groups such as Blacks or Latin*. Also, MMM characterizes Asian Americans as a polite, law-abiding group who have achieved higher success than the general population through some combination of innate talent and pull-yourself-up-by-your-bootstraps immigrant striving (Blackburn, 2019). So, MMM is strongly interwoven with discriminatory stereotypes and racialized positions from the view of the highest "White" hierarchy in the U.S.

Model Minority Myth in Mathematics/STEM. Despite the harm of MMM, it has rarely been discussed in higher education studies. Poon et al. (2016) found that MMM's original purposes—maintenance of anti-Black racism and White supremacy—are not critically mentioned in research on framing Asian Americans and Pacific Islanders in higher education. Among 112 studies of Asian Americans and Pacific Islanders in higher education in the reviewed literature, only slightly more than 10% of the studies used a critical definition of MMM. Thus, it is crucial to deal with the racialized problems and issues regarding MMM.

One of the most common narratives toward Asians based on MMM is the "Asians are good at math" narrative. Shah (2019) analyzes critically how the "Asians are good at math" narrative imposes the positionality of Asian people in STEM as racialized. The "Asians are good at math" narrative, which is prevalent not just in mathematics/STEM education but also in general society, is little considered as a harmful racial issue but rather a harmless compliment. Nevertheless, the "Asians are good at math" narrative is not a compliment. First, this racialized statement itself is not true—not all Asians are good at math. Second, the "Asians are good at math" narrative does not allow Asians to pursue non-STEM or non-mathematical paths such as arts and sports, which also reinforces another stereotype like "Asians are smart but not creative." Third, the "Asians are good at math" does not normalize Asian students' "failure" experiences in mathematics and STEM. Fourth, the "Asians are good at math" implies non-Asians are supposed to be worse in math than Asians, so it maintains the deficit view on non-Asians of color but also does not threaten Whites' ability and White supremacy.

Grounded on the dismantling of the "Asians are good at math" statement, which is based on MMM above, Shah (2019) states that the type of success of Asian students in STEM indicates a neglected and dehumanized narrative of Asian people and reproduces White supremacy. Instead, it is a compliment; the narrative of "Asians are good at math" presents an ontological cost on Asian people as a threat to personhood (Shah, 2019). Also, the racialization of MMM toward Asians is shifting as "yellow peril," becoming the excuse for a 'new White flight.' Suddenly, Asians, who were once perceived as A+ students and good classmates, are repositioned as "too good at math" and threatening to White students' self-esteem, justifying racial re-segregation. (Chen & Buell, 2018)

Critiquing Traditional View on AsianCrit in Postsecondary Mathematics/STEM Education

This study reviewed the traditional approaches, tenets, and studies on AsianCrit. Still, the general arguments of the traditional views on AsianCrit make sense, but there are limitations of traditional views on AsianCrit. More specifically, the "racially triangulated" experiences of Asian Americans and MMM are intensely relatable to postsecondary mathematics/STEM education. Still, considering the current phenomena and issues in postsecondary

mathematics/STEM education, the traditional views on AsianCrit should be revised toward a "spin-off" version for postsecondary mathematics/STEM education. So, in this part, this study suggests reconsidering AsianCrit for future directions in understanding Asian identities in postsecondary mathematics/STEM education.

First, traditional theories in AsianCrit are limited by the different immigrant statuses of Asian students. Particularly in higher STEM education, involving many Asian international students (e.g., Indian, Chinese, Korean, and students from other Asian countries), how is the AsianCrit applicable without considering international students? Interestingly, Asians involve alternatively larger non-U.S. native-born populations in STEM. Regarding U.S. citizenship status, native-born U.S. citizens are (non-Hispanic) White-dominant; on the other hand, non-U.S. and naturalized U.S. citizens are (non-Hispanic) Asian-dominant. This data result from NCSES (n.d.) indicates that Asians in STEM have a proportionally more "international" student population in terms of U.S. citizenship and immigration compared to other race/ethnicity populations. So, for instance, the view of the "perpetual foreigner" has been criticized with the voice of Asian Americans. Potentially, the "perpetual foreigner" view may create Asian international students' unique experiences to be constantly "others" in their educational experiences, which is not necessarily the same as Asian Americans who were born and raised in the U.S.

The second limitation is that since traditional theories of AsianCrit are more toward pan-ethnic understanding, ethnic diversity within the Asian group is less highlighted. More specifically, understanding postsecondary mathematics/STEM education experiences and dynamics related to ethnic diversity has not yet been deeply dealt with. Also, given the "pan-ethnic" understanding, there are some limitations to traditional investigation or use of the MMM—not just MMM, which is little discussed with the critical perspective in postsecondary STEM education. Still, the pan-ethnic understanding of MMM also has limitations since MMM is applied differently for some Asian ethnic groups. For instance, MMM is applied differently for some certain Asian ethnic groups (e.g., Southeast Asian Americans) as both modeled and deviant (Poon et al., 2016; Iftikar & Museus, 2018).

The third limitation is that the traditional theories of AsianCrit do not necessarily cover issues from multiple social identities of Asians. For instance, AsianCrit may not fully capture how Asian women are modeled and simultaneously racialized and gendered. According to Ong et al. (2011), which is a review of around 40-year scholarships for women of color in STEM areas on postsecondary education by adopting the synthesis method to analyze 116 studies with the intersectionality framework, even though Asian American women look overrepresented in STEM compared to the U.S. population, their representation in the group of academic tenure/leadership positions is the lowest or significantly lower than other demographic groups. There has been little attention that Asian American women are placed in an ironic, paradoxical space in STEM fields, simultaneously overrepresented as Asian Americans and underrepresented as women (Castor & Collins, 2021), so they are "invisible." Additionally, on the gendered issues, AsianCrit was not necessarily used to examine Asian sexual orientations, sexuality, nationality, language, and other social identities.

More specifically, in what ways should Asian identities be investigated? First, this study suggests adopting an understanding of Asian international students and immigrants in AsianCrit. AsianCrit usually refers to Asians as only "Asian Americans." However, due to the increasing number of international students and migrant/immigrant populations in STEM majors in the U.S., the invitation to AsianCrit for Asian international students and migrants/immigrants in higher STEM education is necessary. No consideration of Asians in the context of STEM

postsecondary education is insufficient to address Asian discrimination and racism because "Asian migrants working under the H-1B visa obscure domestic educators' unwillingness to teach students of color, but they also camouflage the failure of domestic educators to provide a rigorous STEM education for students of color" (Chen & Buell, 2018, p. 615). What does Asianness mean in mathematics/STEM in the U.S.? Moreover, what do Asianization and Americanization mean for Asian students in mathematics/STEM in the U.S.? What are the possible tensions within that process? These would be the questions for the future direction of the AsianCrit research mathematics/STEM education.

Second, this study insists on considering more diversity in AsianCrit within Asian groups such as ethnicity. Particularly in mathematics/STEM education, Asian students are usually modeled by the MMM. Still, some partial Asian groups—usually those who have darker skin—are treated as more deficient, such as students who achieve lower than East Asians. This would be coined and explained with "Colorism: [the] practice of discrimination in which those with lighter skin tones are treated better than those with darker skin and [it] occurs not only between groups but within groups as well" (Sue, 2016, p. 184). This means there is some colorism among Asian students. For instance, Asian populations who have darker skin within the Asian group, such as Southeastern Asians and South Asians, are perceived by more deficit views or negative stereotypes. If we disregard the diverse ethnicity of Asian populations, then we cannot genuinely focus on diversity and inclusion in mathematics/STEM education.

Lastly, intersectionality should be adopted actively. Notably, more active use of the intersection between racism and sexism for Asian women in STEM is necessary. The use of intersection is aligned with the argument of examination of Asian diverse ethnicity, languages, and immigrant statuses that I mentioned above. Furthermore, non-heterosexual or non-cisgender, or queer identities should be explored more within the Asian context, as McWilliams and Penuel (2016) assert that queer perspectives invite intersectional perspectives, including race/ethnicity, socioeconomic status, disability status, and age. Based on the suggestions, the following section describes a new suggested framework for future directions for research and practice through the critical perspective on AsianCrit in mathematics/STEM education.

Outline of the Suggested Framework

This study suggests a new framework on Asian identities for postsecondary mathematics/STEM education to address the complicated and interwoven relationships between Asian stereotypes, MMM, systemic racism, intersectional identities of Asians, and the possible ways of empowerment.

Figure 2 illustrates the diagram corresponding to the framework. Asian identities are formed as giant icebergs with their tip and parts underneath the seawater. The tip of ice means the superficial perception of Asians grounded on stereotypes and associated with the monolithic view and MMM. The seawater that hides most of the icebergs illustrates systemic racism and oppression toward Asians, such as nativism, orientalism, exoticism, dehumanization, xenophobia, and White supremacy. The seawater is too cold, deep, dark, and sometimes dangerous, so it does not allow people to explore it deeply. Underneath the water, the hidden but also strongly impacted by racism represent Asian identities, which are diverse and necessary intersectional approaches. The identities consist of the intersections between gender, ethnicity, languages, immigrant/citizenship status, sexual orientation, disabilities, and many more. These hidden Asian identities should rise to the surface to empower Asian identities and debunk the "water."

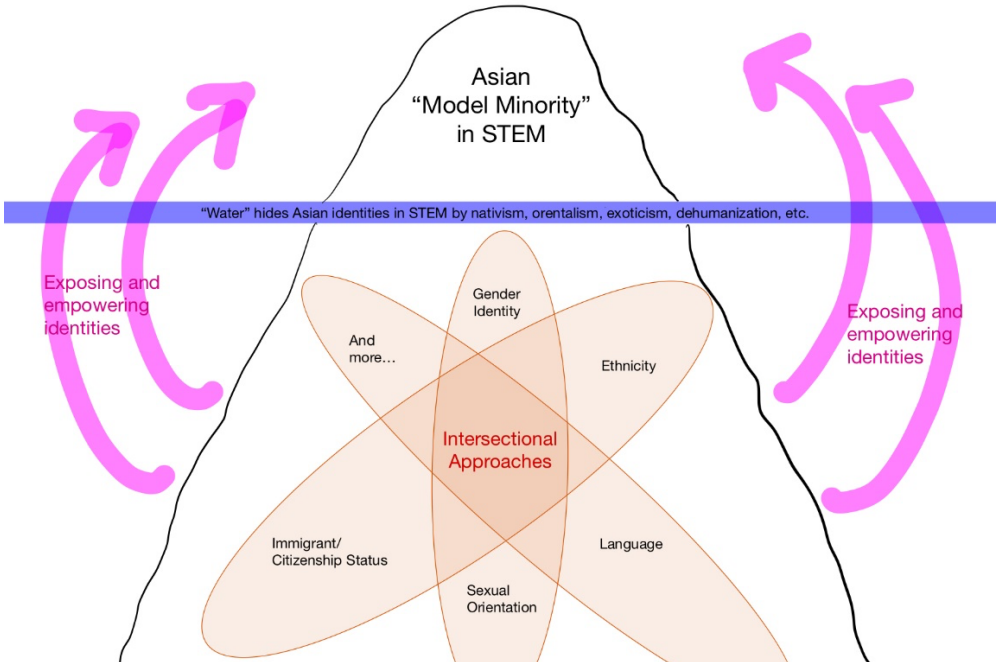


Figure 2. The Suggested New Framework on Asian Identities for Postsecondary Mathematics/STEM Education

Potential Use of the Suggested Framework

Then, what should be the next step in using the suggested framework in postsecondary STEM education? First, the framework can be used to the way to debunk Gap gazing (Gutiérrez & Dixon-Román, 2011), and deficit views frequently happen when Asians are paired with Whites to illustrate the racial achievement gap in STEM. Consequently, Asians are implicated as "honorary Whites," which refers to "signaling that the exits from oppressed minority status and assimilation into whiteness can be incomplete and contingent on fortifying antiblackness" (Martin, 2019, p. 465), and so the racial disparities many experiences are neglected (Poon et al., 2016). Shedding light on multiple interconnections of Asian identities can help dismantle racial disparities and MMM.

Also, not just in mathematics classrooms but also in STEM departmental- and major-wise, student culture should embrace diversity and counter-spaces for minoritized students. To do so, it is suggested to understand intra-racial diversity among Asian students in STEM. An excellent research example is George Mwangi et al. (2016), a study on intra-racial dynamics between native-born and foreign-born black collegians in STEM. According to George Mwangi et al. (2016), native-born and foreign-born Black collegians appreciate peer diversity as a "global" feel provider. Native-born Black students often take a rule to initiate cross-cultural interaction. On the other hand, both native-born and foreign-born Black students reported class dynamics such as native-born Black students' anxiety due to stereotypes regarding Black people in the U.S. and foreign-born Black students' pressure that comes from the academic ability that they should prove or accent or stereotypes of their home country. Based on the findings, George Mwangi et al. (2016) propose that higher education practitioners intentionally provide formalized opportunities for foreign-born and native-born students of color to interact with and learn from each other. Like George Mwangi et al. (2016), there is a potential for understanding the dynamics of Asian interracial students to be necessary for genuine practices for deeper cross-cultural interactions between Asian international students and Asian Americans and other Asian students with diverse histories and backgrounds.

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A Relationship Between Actor-Oriented Transfer and Mathematics Self-Efficacy

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Bandura's self-efficacy theory notes that self-efficacy determines how much time and effort students will put toward their work, meaning higher self-efficacy will lead to students working harder on their coursework. It should thus be in every mathematics instructor's best interest to understand how they can positively impact a student's self-efficacy related to mathematics. This theoretical report examines the intersection between an actor-oriented perspective of transfer of learning and Bandura's self-efficacy theory. I then propose a merging of the two through the concept of actor-oriented self-efficacy, which emphasizes using actor-oriented transfer methods to better understand students' mathematics self-efficacy levels. I conclude with an example of how one can apply an actor-oriented self-efficacy perspective to a study dealing with students' self-efficacy related to proving.

Keywords: actor-oriented transfer; self-efficacy; transfer

Introduction

Ampadu and Anokye-Poku (2022) confirmed that there is a positive relationship between a student's attitude toward mathematics and their achievement in mathematics courses. Since mathematics self-efficacy has been shown to be a reliable mediator between one's mathematics attitude and mathematics achievement (Randhawa, 1993), improving a student's mathematics self-efficacy may be an area of focus for student growth. Note that self-efficacy refers to a person's beliefs regarding how well they can perform a series of actions given a certain situation (Bandura, 1982). Mathematics instructors should then be concerned about how their instructional decisions in the classroom impact their students' mathematics self-efficacy, since this can directly influence how students perform in their courses. As Randhawa (1993) noted, "teachers should do whatever they can to assure all students in mathematics courses that they are capable and should venture forward in mathematics-related areas of study" (p. 47). If instructors can improve students' mathematics self-efficacy, they can improve students' abilities to achieve in their mathematics courses and how encouraged they are to consider pursuing mathematical-related majors and careers.

Mathematics instructors should also be concerned for their students' mathematics self-efficacy as it plays a significant role in students' decisions regarding pursuing STEM-related fields. Hackett (1985) noted mathematics self-efficacy is a reliable predictor of math-related major choice, meaning one's mathematics self-efficacy has a strong impact on if a student chooses a math-related major in college. Additionally, "a lack of confidence in mathematical ability" has been "a major contributing factor for women's decisions to leave calculus but not men's" (Ellis & Cooper, 2017, p. 7). Mathematics self-efficacy may therefore play a key role in women's decisions to stop pursuing mathematics at the calculus level. An instructor's interest in their students' mathematics self-efficacy may go beyond a desire for their students to perform well in their course, as it impacts each student's discernment process when they are deciding if they wish to continue studying mathematics.

While considering student mathematics self-efficacy, note that a personal efficacy belief is the "conviction that one can successfully execute the behavior required to produce the outcomes (Bandura, 1977, p. 193). It thus does not necessarily matter if an observer believes a student's

self-efficacy has changed; rather, it matters if the student believes their own self-efficacy has grown. Therefore, to explore how instructors can positively impact students' mathematical self-efficacy, it is optimal to understand the impacts classroom decisions have on students' mathematical self-efficacy from the perspective of the students themselves.

As I am interested in the classroom decisions and learning situations that can positively impact mathematics self-efficacy, I am also interested in understanding what situations in the classroom raise students' self-efficacy so that students demonstrate this heightened self-efficacy in future situations. In other words, I am interested in how self-efficacy can be transferred. With this in mind, I consider using transfer of learning, which refers to "the ability to apply knowledge learned in one context to a new context" (Mestre 2003, as cited in Karakok, 2021, p. 103). Transfer of learning focuses on identifying how students learn knowledge in one moment, the learning situation, and apply it in another moment, the transfer situation (Lobato, 2012). Thus, there is a similar notion in studying transfer of learning to that of changing mathematics self-efficacy; there are moments where instructors hope to change self-efficacy and there are future moments where instructors hope to see that raised self-efficacy in action. Moreover, as I am interested in students perceived self-efficacy (meaning student perspective), I focus on one of the contemporary approaches to transfer of learning, namely, actor-oriented transfer framework (Lobato, 2012).

I note that there is a precedent for applying transfer of learning theory to other areas, specifically areas that are individualistic in nature, to better understand them, such as Karakok's (2021) discussion of the similarities between transfer of learning and student creativity. This suggests one can better understand student mathematics self-efficacy by first considering the intersection between the theories of self-efficacy and transfer of learning. Furthering this, Karakok (2021) specifically discussed how a researcher may use an actor-oriented transfer (AOT) perspective to study creativity and the transfer of creativity. While I discuss AOT in detail later, I note that it focuses on viewing situations from the actor's perspective where, in this case, actors are students in the classroom. I thus consider how one may use transfer and AOT framework to understand student mathematics self-efficacy and the transfer of mathematics self-efficacy. Moreover, I discuss how an AOT perspective can aid in our pursuit of understanding student mathematics self-efficacy and what pedagogical decisions impact student mathematics self-efficacy. This theoretical report is thus driven by the following research questions:

1. How can an actor-oriented transfer perspective be used to understand student mathematics self-efficacy and the impacts pedagogical decisions can have on student mathematics self-efficacy?
2. What similarities exist between actor-oriented transfer theory and methods used to study student mathematics self-efficacy?

Literature Review

Mathematics Self-Efficacy

I consider mathematics self-efficacy with Bandura's (1982) notion that self-efficacy refers to a person's beliefs regarding how well they can perform a series of actions in a certain situation in mind. As noted, mathematics instructors should be concerned about their students' self-efficacy, as it plays an important role in determining how much students are able to achieve while studying mathematics and if they choose to continue studying mathematics (Ampadu & Anokye-Poku, 2022; Ellis & Cooper, 2017; Hackett, 1985; Randhawa, 1993). Self-efficacy is also a significant piece of each student's mathematical journey as "efficacy expectations are a major

determinant of people's choice of activities, how much effort they will expend, and of how long they will sustain effort in dealing with stressful situations" (Bandura, 1977, p. 194). Each student's efficacy may directly impact the amount of time and effort they dedicate to mastering their coursework. Therefore, mathematics instructors should understand how their pedagogical decisions impact their students' mathematical self-efficacy, as doing so enables instructors to positively influence the amount of time and effort students give toward their studies as well as students' achievements in mathematics.

To understand how to influence students' mathematics self-efficacy, I first discuss potential sources where students can gain self-efficacy. In his self-efficacy theory, Bandura (1977) outlined four principal sources for one's self-efficacy: performance accomplishments, vicarious experience, verbal persuasion, and emotional arousal. Performance accomplishments, noted as the most influential source, rely on personal mastery experiences; self-efficacy rises through one's own continued successes. Bandura also noted that "improvements in behavioral functioning" through performance accomplishments "transfer not only to similar situations but to activities that are substantially different from those on which the treatment was focused" (p. 195). Vicarious experience refers to observing others doing difficult tasks without adverse results, so that an observer more firmly believes that they can successfully navigate similar challenges. Bandura notes that this source is not as strong as performance accomplishments, and that any self-efficacy gained from it has a greater chance of being diminished later when compared to performance accomplishments. Verbal persuasion, involving suggesting to another that "they can cope successfully with what has overwhelmed them in the past", will also be weaker than performance accomplishments; since there is no actual performance of tasks to base these suggestions off, their self-efficacy is less likely to be impacted than from the previous two sources (p. 198). Finally, emotional arousal relates to emotions that come from stressful situations and the effects they have on one's self-efficacy relating to a task.

Note that Bandura's four sources have been used previously in studies on students' mathematics self-efficacy. Regier and Savic (2020) used Bandura's four sources to identify changes to students' mathematics self-efficacy for proving that were caused because instructors implemented teaching actions that aligned with Sriraman's (2005) principles for fostering creativity. Watkins and Middleton (2021) focused on understanding which of the four sources could successfully predict mathematics self-efficacy in Black college first-year students that attended a predominantly Black campus. Usher and Pajares (2009) worked to validate Bandura's theory of self-efficacy's four sources for students studying middle school mathematics. Parsons et al. (2015) focused on which sources of mathematics self-efficacy (Bandura's and others) engineering students identified as having changed their self-efficacy levels, noting that performance accomplishments were found to be the most important source while verbal persuasion was absent. There is therefore a precedent for researching students' mathematics self-efficacy levels that utilizes Bandura's four sources as a theoretical framework.

Instructors thus have the potential to impact students' mathematical self-efficacy by understanding Bandura's four sources; if a pedagogical decision leads to consequences related to these sources, that decision has the potential to change one's self-efficacy levels. This leads to the following question: how can observers successfully identify the impact that choices made in the classroom have on students' mathematics self-efficacy levels with respect to Bandura's sources? To answer, I turn to transfer theory, which focuses on how certain situations in the classroom impact students (with respect to future situations). This draws a similarity to my question, as it is interested in understanding what students take from a situation (with respect to

their self-efficacy) and how their effected self-efficacy impacts their future ability to perform in the classroom.

The Actor-Oriented Transfer Perspective

To understand which classroom decisions impact students' mathematics self-efficacy (and how they do so) with respect to Bandura's model, I consider utilizing the transfer construct, as this relates to how one event impacts a student's knowledge by considering a future event where they apply that knowledge. More specifically, I consider using an actor-oriented transfer (AOT) perspective to identify what impacts, with respect to Bandura's four sources, have occurred.

The actor-oriented transfer (AOT) perspective engages with the actor's (student's) point of view rather than an observer's perspective; instead of focusing on if a student performs a specific action or acts in a certain way (which an observer has pre-determined to be of significance), it examines the student's actions or behaviors related to a task, where knowledge is applied, regardless of their correctness (Lobato, 2012). Moreover, an AOT perspective aims to understand how previous experiences, where knowledge is learned, lead to a student's decisions, rather than focusing on if previous experiences lead to specific, pre-determined choices by the student (Lobato, 2012). While an observer perspective views "successful or improved performance on transfer task" as evidence of transfer, an AOT perspective believes that "regardless of successful or improved performance on transfer tasks, any influence of prior learning experiences is considered as evidence for transfer" (Karakok, 2021, p. 111). Therefore, AOT's framework values understanding students' perspective regardless of normative correctness so that such an approach could provide means to understanding perceived student self-efficacy. An AOT perspective thus enables observers to use each student's viewpoint when considering if certain pedagogical decisions had effects on a student's self-efficacy levels and, if so, which of Bandura's sources these effects may be related to.

The Intersection Between Actor-Oriented Transfer and Self-Efficacy

Note that AOT's view of 'transfer' is concerned with understanding "the processes by which individuals generate their own similarities" between a learning situation, where they are first introduced to a topic, and a transfer situation, where they attempt to apply their knowledge of this topic (Lobato, 2003, p. 18). Bandura's two most impactful sources, performance accomplishments and vicarious experiences, dealt with improving self-efficacy with respect to one situation so that an improved confidence can help an individual perform better in future situations that they perceive as similar to the prior experience (1977). There is thus a notion of transfer present in Bandura's model, meaning AOT framework can potentially be used to identify when there is a 'transfer' of mathematics self-efficacy between situations. Additionally, the AOT perspective "assumes that learners are making connections between situations nearly all the time, guided by aspects of the situation that they find personally salient" (Lobato, 2003, p. 19). An observer thus assumes learning situations *do* have impacts on students that become present in transfer situations. Applying this perspective to students' mathematics self-efficacy with these situations in mind, it is not a question of *if* there is an impact, but rather *what* this impact looks like and *why* it happened. Moreover, the AOT perspective means relying on each student's input to answer the question of what changes to their self-efficacy have occurred between learning and transfer situations. While an observer could believe that there has been a change from observing a student's time in each situation, studying self-efficacy can benefit from focusing on students' perspectives due to self-efficacy encompassing each person's personal beliefs regarding their ability to perform successfully.

The focus on students' beliefs in AOT agrees with other studies that highlight how the most important data comes directly from the students themselves when considering topics that deal with students' personal beliefs, such as self-efficacy and creativity. When gathering data on what teaching actions influence fostering creativity in a Calculus course, Satyam et al. (2022) note that

> these actions are from the students' (not instructor) perspective. We did not corroborate whether instructors in fact did these actions. Given our lens of creativity and that the individual themselves is the ultimate judge of whether they felt creative (independent of their reporting of it to us), students' judgments of what their instructor did that actively contributed to their sense of creativity are most valuable. (p. 542)

This stands as an example of why research topics focusing on personal attributes (as they depend on each student's beliefs), such as creativity, can benefit from gathering data based on students' viewpoints; it does not matter what the instructor intended to do, nor what an observer believes they see, as much as it matters that an actor believes something to be true.

Additionally, when Regier and Savic (2020) discussed the impacts teaching actions related to fostering creativity have on students' self-efficacy for proving, they coded sources of self-efficacy from student interviews "when a potential source (enactive experiences, vicarious role-modeling, verbal persuasion, physiological reaction) was described in relation to a change in student confidence in proving" during interviews with students (p. 5). Thus, there are previous studies that examine students' beliefs by using evidence of changes in self-efficacy or creativity only when students noted that to be the case.

Merging Self-Efficacy Theory with Actor-Oriented Transfer

After discussing the intersection between actor-oriented transfer and creativity, Karakok (2021) outlined her concept of actor-oriented creativity (AOc). AOc blends actor-oriented transfer and creativity research, as it means to "distinguish students' perspectives on creativity and perceptions of their own mathematical creativity from the ones observed or declared by others" (pp. 121 – 122). Actor-oriented transfer thus has the potential to be integrated with another area of research that can possibly benefit from its methods. With this in mind, I take a similar path by constructing actor-oriented mathematics self-efficacy, which means to consider mathematics self-efficacy through an actor-oriented transfer lens.

I consider using an AOT approach to study students' mathematics self-efficacy levels to highlight the student perspective on what pedagogical decisions could positively impact mathematics self-efficacy. When studying mathematics self-efficacy with Bandura's four sources of self-efficacy in mind, researchers may need to identify when a student's self-efficacy has changed and, if this is the case, the catalyst for this change with respect to the four sources. To do so, I propose the concept of actor-oriented self-efficacy (AOSE) which focuses on understanding each student's mathematics self-efficacy from their own perspective, rather than from that of an observer. This means AOSE requires researchers to turn to the student perspective for information on a) when their mathematics self-efficacy has changed and b) what situations, with Bandura's four sources in mind, led to those changes. Karakok (2021) noted a potential outcome of using actor-oriented creativity stems from how "inclusion of students' perspectives on mathematical creativity and their perception of their own mathematical creativity could help us detect nonnormative instances of students' mathematical creativity and provide processes that are not reported in earlier studies" (p. 121). Similarly, AOSE acknowledges that a student's discussion of what has impacted their mathematics self-efficacy can potentially illuminate researchers to instances that they may not have considered.

This AOSE perspective considers the nature of measuring self-efficacy; as self-efficacy is based on one's own beliefs regarding their ability to successfully perform a task, it follows that correctly determining a student's mathematics self-efficacy levels requires introspection on the student's part. This distinction agrees with Lobato's (2003) note on the usefulness of transfer models: "traditional models rely too heavily on the determination of transfer from an expert's point of view and, as a result, can lead to design decisions that are not informed by the specific generalizations that students have formed" (p. 20). An actor-oriented self-efficacy perspective would thus engage with students' specific generalizations regarding their mathematics self-efficacy and put the emphasis on their perspective, rather than that of any observers. While an instructor or an observer can hypothesize about whether a student's self-efficacy has changed and what it was that changed it, a student is the best person to discern what has happened to their own self-efficacy levels. Actor-oriented self-efficacy thus centers on highlighting students' viewpoints on their own mathematics self-efficacy. This can ensure that pedagogical decisions made with the intent to raise mathematics self-efficacy levels are made with students' perspectives in mind.

Applying Actor-Oriented Self-Efficacy to a Case Study

I apply an AOSE perspective to a previous study (Regier & Savic, 2020) conducted on the effects teaching principles of creativity have on self-efficacy levels related to proving. Regier and Savic (2020) created the Self-efficacy for Proving Scale (SEPS) which "consisted of three specific proving statements in which students were asked to rate their confidence of five subtasks related to proving each statement" and had students "rate their confidence in their ability (on a scale of 0% confidence to 100%)" to do five subtasks related to proving (pp. 4-5). SEPS surveys were given at the beginning, middle, and end of the semester so that results could be used to discuss the students' changes in self-efficacy related to each statement by comparing students' SEPS scores from the beginning of the semester to their scores at the end of the semester.

While the SEPS agrees with AOSE when determining if students' self-efficacy levels changed by asking students for their perspectives on their own self-efficacy levels, it does not ask students what events or teaching actions may have led to changes in their self-efficacy. An AOSE approach would try to understand what caused these changes in self-efficacy levels from the students' viewpoints. Note that Regier and Savic (2020) created another survey, the Five Principles Survey (5PS), that students completed at the end of the semester and contained questions regarding students' experiences with respect to each of Sriraman's (2005) five principles of creativity. Therefore, an AOSE perspective's additional goal could be realized by creating additional questions in the SEPS, like those in the 5PS, that ask students if (and if so, how) any of the principles of creativity impacted their self-efficacy levels with respect to each statement they needed to prove. This addition will agree with an AOSE approach by only identifying principles of creativity as having impacted a student's self-efficacy levels if a student declares them to have done so. This would provide data for the second half of AOSE's goals by creating data focused on how students' self-efficacy levels change and based on the students' perspectives.

Regier and Savic (2020) also conducted student interviews at the end of the semester that asked students "questions about their classroom experience, their relative confidence in proving now, and how they gained confidence in their class" (p. 5). Note that these interviews already agree with an AOSE perspective; they gather data on student self-efficacy levels by asking for the student perspective on if and how their self-efficacy changed because of their experiences in

the classroom. I consider expanding this approach by gathering data on how specific situations in the classroom changed student's self-efficacy levels through an AOSE viewpoint.

Since interviews took place at the end of the semester, they focused on students' overall experience in the course and self-efficacy levels for proving. Therefore, students may not have remembered specific instances that helped change their self-efficacy for proving. Drawing inspiration from AOT's theory of learning and transfer situations, I consider doing interviews during the semester periodically and directly after students work on proving statements related to a subcollection of proving tasks that correspond to recent topics discussed in the classroom (for example, self-efficacy related to proving by contradiction). By using the same questions found in end of semester interviews, adjusted for this suggested timeframe, observers may receive data discussing more moments related to principles of creativity that changed students' self-efficacy levels for proving; it would give students a chance to reflect on more recent classroom content that they may have forgotten by the end of the semester. Overall, this approach would exemplify AOSE's viewpoint further, as it would allow students to reflect on situations that may have impacted their self-efficacy with respect to a certain subcategory of proving. This would mean students could potentially identify more explicit moments that changed their self-efficacy levels, thus giving researchers a better understanding (from the students' perspectives) of when and how principles of creativity impacted students' self-efficacy levels.

Future Work

I hope to apply our AOSE model to a study that examines the effects that teaching actions can have on how one's mathematics self-efficacy changes over the course of a semester. Specifically, I look to consider how the teaching actions stated in Satyam et al. (2022) can impact a student's mathematics self-efficacy in a calculus course setting. Calculus has been noted to be a 'gateway' course that impacts students' decisions to study STEM-related majors or not (Moreno & Muller, 1999). Moreover, Ellis and Cooper (2017) found that women were deciding to stop pursuing STEM-related majors due to experiences in their calculus courses. Thus, using an AOSE perspective to investigate teacher actions' impact on students' mathematics self-efficacy in a calculus course could potentially lead to more students deciding to continue studying STEM-related majors by deepening instructors' understanding of what teaching practices can raise students' mathematics self-efficacy levels.

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Does Everything Have to be Gay? Queer Pedagogy in the Undergraduate STEM Classroom

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Queer pedagogy has been underutilized in mathematics learning contexts, particularly at the undergraduate level, despite offering unique perspectives on gender, sexuality, knowledge, and normalcy that are valuable for scientific inquiry and which support queer students in STEM. In this paper I discuss queer pedagogy, present practical examples of its implementation in STEM courses at the undergraduate level and argue for its necessity as a tool for instruction.

Keywords: queer theory, queer pedagogy, mathematical inqueery, critical pedagogy

Shortly after I moved across the country to start working on my PhD, I called home to tell my parents all about my "new life." As we spoke, I described the gay volleyball team I had joined, the local drag show I had attended and emphasized how happy I was to be living in a city with a vibrant queer community. My father, sounding affronted, said, "Not everything you do has to be gay."

Queer pedagogy has been implemented in arts and social science education for decades as a way to offer homonormative representation and to question the normalization of specific types of authorities, dichotomies, and knowledges (Ferguson, 1989; Letts, 2002; Stafford, 1988; Sullivan, 2003); but queer pedagogy appears to be rare in STEM contexts, particularly at the undergraduate level, despite the opportunities it can provide in supporting queer students, highlighting problematic cultural norms, and contributing to scientific inquiry.

In this paper my aim is to provide a concise explanation of queer pedagogy and to coordinate that explanation with examples of queer pedagogy's implementation in undergraduate STEM contexts. It is my hope that by linking the theoretical and practical realities of queer pedagogy, queer pedagogy can become more accessible and coherent as an educational tool, and that more scholars and educators will feel empowered to queer their research and their classrooms.

Queer, Queer Theory, and Queer Pedagogy

In order to discuss queer pedagogy, it is first necessary to discuss what is meant by the word "queer" and to investigate the origination of queer pedagogy as an offshoot of queer theory. However, the complexity and history of these movements renders any brief overview of them incomplete and reductive. For this reason, I want to acknowledge that the definitions, histories, and descriptions provided here represent a limited interpretation of queerness, queer theory, and queer pedagogy as dictated by the focus of this work. Should you find yourself curious about other interpretations and dimensions of these topics, I encourage you to explore the cited authors, particularly Nikki Sullivan's *A Critical Introduction to Queer Theory*, Davey Shlasko's article "Queer (*v.*) Pedagogy", and Deborah Britzman's book chapter "Is there a Queer Pedagogy? Or, Stop Reading Straight."

Queer

While it was originally a descriptive verb meaning "odd", in the early 19th century the word "queer" gained popularity as a pejorative noun for individuals who existed outside of cisgendered and heterosexual expectations of speech, dress, and socialization (Merriam-Webster,

n.d.; Sullivan, 2003). As Western society demonized and pathologized queer behavior in the late 19th and early 20th century, "queer" became synonymous with "deviant" "diseased" or "perverse" and was used primarily as a slur against gender and sexual minorities until its reclamation by queer activists and academics in the mid to late 20th century (Chauncey, 1994; Foucault, 1990; Sullivan, 2003). Now in the early 21st century, "queer" is used colloquially as an umbrella term to refer to anyone within the 2SLGBTQIA+[1] community, and in most instances no longer implies negative connotations. It is now commonplace for researchers to describe sexual and gender minorities within their data as queer (e.g. Cox & Pinheiro, 2024; Leyva et al., 2022; Voigt, 2022), and many 2SLGBTQIA+ individuals, myself included, proudly self-identify as queer.[2]

Within the academic community the word "queer" expands beyond identity. Morris (1998) proposes that the word "queer" as it is used by queer theorists encompasses three primary meanings: queer as a subject position, queer as a politic, and queer as an aesthetic. Queer as a subject position describes anyone who exists outside of society's normative understandings of sex and gender and is inclusive of the colloquial and historical meanings of queer described above. Ironically, the drawback to establishing queer as a subject position is that it often reaffirms "the heterosexual center" by defining queerness in contrast to "normalcy" (Davis & Sumara, 2000, p. 123), creating a new binary of "queer-not queer" (Morris, 1998, p. 276). Thus, queer cannot be restricted to a subject position and must also be defined politically. Queer as a politic questions the existence of normalcy, defies assimilation, and advocates for the deconstruction of the boundaries which police gender and sexuality (Luhmann, 1998; Morris, 1998; Shlasko, 2005; Sullivan, 2003). Often, the political use of the word queer is in the form of a verb, and to "queer" something suggests an act of disturbing, resisting, and reshaping. Lastly, queer as an aesthetic intentionally reinterprets writing, art, music, and other forms of text as "politically radical" (Morris, 1998, p. 276), a practice which "queers both text and reader" (Shlasko, 2005, p. 124).

This paper, like many writings on queerness, queer theory, and queer pedagogy, will use the word "queer" as an umbrella term, a subject position, a politic, and as an aesthetic without always making clear which definition is being referenced. Often, several of these definitions are applicable at once and they cannot be reasonably detangled from each other. Rather than stopping to redefine queer each time it appears in this text, I encourage you to embrace the word's radical and purposeful ambiguity.

Queer Theory

One way to characterize queer theory is as a theory which concerns itself with queer individuals, the queer experience, and queer topics related to gender, sex, and sexuality (Shlasko,

[1]Two-spirit, lesbian, gay, bisexual, transgender, queer/questioning, intersex, asexual/aromantic

[2]It is notable that while queer may describe an individual's gender or sexual orientation, it often erases or neglects other important aspects of an individual's identity, particularly race and class (Barnard, 1999; Sullivan, 2003). This has resulted in a homogenized view of queerness which favors white middle-class portraits of homosexuality (Mercer, 1994) and reproduces racial marginalization (Barnard, 1999; Leyva, McNeill, Balmer, et al., 2022; Sullivan, 2003). While it is not a focus of this paper, I want to acknowledge that queer studies are frequently complicit in ignoring the racialized nature of queerness. For a primer on this topic, see "Queer Race" by Ian Barnard (1999).

2005; Sullivan, 2003; Voigt, 2022). But more broadly, and consistent with Morris' (1998) definition of queer as a politic, Fifield and Letts expressed that:

> Queer theory shares concerns around sociocultural power relations of life in sexed and gendered bodies while troubling attempts to treat sexuality, or anything else, as a stable and knowable marker of identity. Queer theory includes unruly, deconstructivist perspectives that destabilize dominant norms around sexuality, gender, bodies, knowledge, power, and identity by imagining and inhabiting alternatives that resist being fully known or explicable. (2019, p. 68)

Queer theory can focus on queer topics of gender and sexuality, but it also leverages queerness to criticize anything that society has deemed to be normal and to consider how that normalcy can be disturbed. Queer theory also offers opportunities to reflect on the personal and political nature of identity, and, as Voigt (2022, p. 4) described it, "offer[s] alternative ways of thinking and performing in the world".

Queer theory can also be understood as an epistemological stance which questions what is known and what can be known (Dubbs, 2016; Letts, 2002), a stance which resonates with modern explorations of education, learning, and instruction and which I see as foundational to the development of queer pedagogy.

Queer Pedagogy

As with queer theory, queer pedagogy is not solely concerned with queer individuals or queer representation, going beyond "merely adding authentic or likable portrayals of lesbian/gay icons to an otherwise straight—and already overcrowded—curriculum" (Luhmann, 1998, p. 124). Instead, it can be understood as an application of queer theory and queer politic to instruction. Flessas and Zimmerman (2019, p. 96) explained queer pedagogy as "encouraging critical thought and the willingness to delve into complexity; investigating, questioning, and re-evaluating assumptions; and interrogating processes of normalization…" By this definition, I see queer pedagogy as having a home among the critical pedagogies which have gained popularity in the mathematics education community. In fact, Shlasko (2005, p. 125) citing Spurlin (2002) argued that queer pedagogy is at the "intersection of queer theory and progressive pedagogy" and I contend that many of the critical pedagogies that have emerged as part of the sociopolitical turn in mathematics education are implicitly queer.

Rands (2009) described a queer pedagogy of mathematics as mathematical inqueery[3], which "brings queer theory's interrogatory edge into the intersection with pedagogy and mathematics to mathematize the queer and queer mathematics" (Rands, 2019, p. 64). Mathematical inqueery encourages discomfort with abnormality, and its application has been compared to working with alternate number bases or non-Euclidean geometry (Rands, 2019; Sheldon, 2019). Ultimately, queer pedagogy and mathematical inqueery attempt to trouble normative presentations of gender and sexuality while exploring what can lie outside of "binaries such as homo-hetero, ignorance-knowledge, learner-teacher, reader-writer, and so on" (Luhmann, 1998, p. 128).

[3]Mathematical inqueery is sometimes written as mathematical inqu[ee]ry to highlight the intersections of queerness and inquiry and to emphasize a disregard for conventional grammar (Sheldon, 2019). I enjoy this styling of the word but have chosen not to include the brackets as part of an effort to make this paper more accessible to those who use screen readers.

Queer Pedagogy in Undergraduate STEM Classrooms

There are countless ways to apply queer pedagogy to the undergraduate STEM classroom, and the examples provided in this report are neither exhaustive nor irreproachable. In the words of Letts and Fifield (2019, p. 5) in their introduction to *STEM of Desire,* "we suggest you approach particular definitions of queering STEM education as provisional orientations, not as determinative demarcations of proper and improper queering."

My choice to discuss lessons constructed by Helene Götschel and James Sheldon is twofold. First, I feel that their work illustrates the explicit integration of queer pedagogy into a realistic academic context and can serve as inspiration for undergraduate educators interested in queering anything from a single task to an entire course. Second, Götschel and Sheldon share valuable insights on how and why queer pedagogy was an appropriate choice for their classrooms, insights which strengthen my argument for the necessity of queer pedagogy in undergraduate STEM.

Queering Physics with Helene Götschel

In Götschel's (2019) book chapter "Teaching Queering Physics: An Agenda for Research and Practice" she detailed two of her recent attempts to teach physics "in a queering way" by presenting non-normative versions of typical physics learning tasks.

In the first task Götschel (2019) modified a lesson on angular momentum to feature male figure skater Evgeni Plushenko rather than the female figure skater presented by the textbook. To introduce this lesson to her students, Götschel played clips from several of Plushenko's skating routines. The male engineers in her class were "visibly uneasy" while studying Plushenko's skating routines and giggled as they watched the skater perform incredible feats of athleticism while using his body in "feminine" ways (p. 136). Götschel attributed the discomfort displayed be her students as a result of the masculine body that is expected and upheld in physics classrooms.

In the second task, Götschel (2019) had students work through a standard kinematics exercise, calculating whether or not an antelope running from a lion could escape its predator over a set course of time. Once this calculation was finished, Götschel showed her students a short film featuring several lionesses working together to hunt a zebra and used this film to prompt a discussion of variables like acceleration, change of direction, and the use of teamwork, aspects which are typically neglected in rectilinear presentations of the task. She described this film as giving students the opportunity not just to think about additional kinematic variables, but also as a chance to question the notions of active masculinity and passive femininity as they play out in physics.

Both of these tasks queer physics through the use of queer pedagogy because they "make the familiar strange" and disrupt the culture of physics classrooms and curricula without sacrificing the rigor of the learning objective (Götschel, 2019, p. 126). In the first task, Götschel's (2019) choice to highlight the masculine body in an artistic setting forces students to sit with the discomfort of their own gendered assumptions and to question how physicists evaluate movement scientifically, politically, and aesthetically. In the second task, students are given the opportunity to question what is lost when physics tasks focus on unrealistic and rectilinear presentations of speed and distance while deconstructing the dichotomies of male-female and individualism-collectivism. Götschel's instructional choices within these tasks demonstrate how queer pedagogy does not have to overshadow a lesson's primary focus but can offer a layer of radicality with subtle alterations to existing curricular materials.

James Sheldon's Queer Curriculum of Infinity

In Sheldon and Rands (2013), the authors discussed the queerness of many mathematical topics, naming infinity, topology, imaginary numbers, complex analysis, and decomposition as topics ripe for the queer pedagogy of mathematical inqueery. A few years later, Sheldon (2019) chose to focus on infinity as an inherently queer topic and used queer pedagogy and the queer aesthetic to explore infinity in a five-week course on mathematical problem solving. This course included lessons on counting, modular arithmetic, set theory, and other topics which required an understanding of mathematical infinity, but rather than teaching infinity as a singular lesson at the beginning or end of the course, Sheldon embedded discussions of infinity into every lesson. Sheldon described using the *Hotel Infinity* videos from Dr. Tova Brown, and historical accounts taken from *The Mystery of the Aleph* by Amir D. Aczel to show students the complicated and messy human history surrounding infinity rather than presenting it solely as a mathematical object. As Sheldon described it, he "invited students to feel the sense of the mystery, fascination, power, intrigue, drama, and contradictions that infinity entails. … entangling them in infinity's web rather than straightening out their experience of queer mathematics" (Sheldon, 2019, p. 316).

Sheldon's approach to queer pedagogy, and specifically the use of mathematical inqueery, was embedded in the course's design, shifting the student's focus from *definitions* of infinity to the *history* and *story* of infinity, a choice which I consider abnormal in standard mathematical instruction. This instructional choice is queer precisely because it offers an approach to learning mathematics that is aesthetically oriented while still conveying the necessary material. Sheldon's queer curriculum of infinity also gave students an opportunity to question how definitions and theorems are created and accepted by the mathematical community and to understand mathematics as a human process rather than a scientific eventuality.

The queerness of Sheldon's curricula may seem less obvious than Götschel's, having no direct references to gender, sexuality or identity, but the course's design mirrors epistemological stances in queer theory; it asks what students "cannot bear to know" (Britzman, 2014, p. 214) and invites them to sit with the queer tensions between humanity and infinity.

The Necessity of Queer Pedagogy in Undergraduate STEM: "Queer Students Deserve Nothing Less" (Shlasko, 2005, p. 125)

As Dubbs (2016, pp. 1041-1042) stated, "the ability to consider sexuality irrelevant in the mathematics context is a heteronormatively privileged position; for queer students, their queerness, indeed *other*ness, is intersectional across social contexts." The teaching and learning of mathematics hinges on identity (Gutiérrez, 2013) and every classroom contains queer students who deserve to see themselves represented within the curriculum. STEM subjects are socially, culturally, and politically motivated and reproduce the inequities present within society as a whole; as long as queerness remains marginalized within society, queer students and perspectives will remain marginalized within STEM. I assert that queer pedagogy provides a vital opportunity for students of all genders and sexualities to feel more supported in STEM classrooms, to problematize societal constructions of normalcy, and to reacquaint themselves with the joy of scientific inquiry. In the remainder of this section, I explain how I see queer pedagogy accomplishing these things and specify the importance of this undertaking in an undergraduate setting.

Marginalized students deserve additional support in traditional mathematics classrooms, and queer students are no exception. In a recent quantitative study of queer-spectrum students in undergraduate mathematics courses, Voigt (2022) found that sexual identity had a significant

relationship with a student's mathematical learning experience and their ability to access learning environments when compared to non-queer identifying peers. The study also revealed that when using a regression model, the effect sizes of sexual identity were similar to the effect sizes of race, gender, and first-generation college student status (Voigt, 2022). Additionally, sexual minorities have been found to leave STEM degrees at a higher rate than their heterosexual peers (Hughes, 2018) and in some instances view the pursuit of STEM degrees as dually alienating since they face pressure to remain invisible within the STEM classroom while they are simultaneously underrepresented in queer discourses (Toynton, 2007). These findings support an understanding of queer students as marginalized individuals in the academic environment and suggest a need for practices which support queer students, queered STEM curricula, and queer pedagogy as a visible discourse in STEM.

Consider Götschel's (2019) tasks on queering physics. To my knowledge, students in the classroom, queer or otherwise, were not aware of the instructor's intent to use queer pedagogy. However, because Götschel's tasks were designed to encourage students to face implicit biases in their understanding of gender, movement, and society, I see that she created an opportunity for queer students to connect more deeply with the topics being studied and to see their peers interact positively with topics which are often left outside of the classroom (namely gendered presentation and gendered bodies). Societal expectations regarding gender, presentation, desire, and sexuality oppress and affect *all* students (Götschel, 2019; Gunckel, 2019; Rands, 2013) and I posit that by confronting these topics within the physics classroom (Götschel, 2019), students of all identities were given the opportunity to question the normalcy of these social constructs in a safe and healthy environment without deviating from important curricular topics such as speed, acceleration, and angular momentum.

An additional benefit of using queer pedagogy in undergraduate STEM classrooms is that I see queer pedagogy as being compatible with the tenets of scientific inquiry. Queer pedagogy questions knowledge and reality, exploring alternative ways of knowing and understanding the world, even when those understandings differ from the norm. The same can be said of scientists and mathematicians who, for the sake of argument, I will characterize as creators of tools for learning about and understanding reality. When these mathematicians and scientists take part in the creation of math and science, they are queering (disrupting/interrogating) normalcy, knowledge, and constructs as we currently understand them. In discussing queer theory, Britzman (2014) described what one can know and "cannot bear to know" (p. 214), and to me these words feel just as relevant to scientific inquiry as they do to the field of queer studies.

Sheldon's (2019) queer curriculum of infinity is an excellent example of queer pedagogy upholding the principles of scientific inquiry. His implementation of queer pedagogy encouraged students to question mathematical concepts and definitions, a curricular choice which I feel reintroduces the joy of creation and discovery which is foundational to the study of mathematics. By inviting students to linger in the nonbinary space between knowledge and ignorance (Luhmann, 1998) I posit that Sheldon's students were allowed the rare privilege of acting as mathematicians themselves while learning about mathematicians of the past.

Finally, I must address why my argument is constructed specifically as an appeal to undergraduate educators rather than to educators as a whole. While I firmly believe that queer pedagogy is valuable to students of any age (particularly because queer pedagogy often does not concern queer topics), it is only at the undergraduate level that the long held (and fallacious) argument that discussions of gender and sexuality are not "appropriate" for children (Fla. Legis. House Bill 1557: Parental Rights in Education, 2022) is rendered moot. There is also evidence to

suggest that college students who have positive interactions with queer students are more likely to have positive attitudes towards queer relationships (Liang & Alimo, 2005). To me, this implies that undergraduate classrooms are ideally situated to implement queer pedagogy and provide young people the opportunity to learn tolerance, find acceptance, and expand their scientific mind. Yet, when reading relevant literature on the use of queer pedagogy in STEM, K-12 contexts appeared to be more common. Sheldon (2019) and Götschel's (2019) work were two of a limited number of studies which explicitly cited queer pedagogy in an undergraduate STEM context. For this reason, I find it necessary to encourage the conscious expansion of queer pedagogy into undergraduate STEM curricula, as I see a gap in the field of queer pedagogy at this level.

Conclusion

Queer pedagogy encourages "studying the risks of normalization, the limits of its own practices, and the im/possibilities of (subversive) teaching and learning" (Luhmann, 1998, p. 130), and as mathematics education researchers continue to advocate for more equitable teaching and learning practices, queer pedagogy has much to offer. I contend that queer pedagogy is a powerful tool for studying gender, sexuality, identity, knowledge, and authority, topics which are relevant within the undergraduate STEM curriculum as demonstrated by Götschel (2019) and Sheldon (2019). Additionally, queer pedagogy provides the opportunity to uplift queer voices, support critical perspectives, and encourage students to participate in the interrogative cycle of scientific discovery.

In the future, I intend to study the intersection of queer pedagogy and other critical discourses, and to uplift racialized aspects of queerness which are frequently neglected in the queer community (Leyva, McNeill, Balmer, et al., 2022). To that end, I invite researchers to consider how queer pedagogy might be compatible with their work, and to reflect on how,

> disassociating race, gender, and sexuality … can lead to the production of accounts of race that are (at least implicitly) sexist and or/ homophobic, theories of gender that are (at least implicitly) racists and/or homophobic, and analyses of sexuality that are (at least implicitly) racist and/or sexist (Sullivan, 2003, p. 66).

It is impossible to isolate gendered and sexualized experiences from other experiences of marginalization and I hope to contribute to the coordination of these discourses alongside other scholars who have done exemplary work discussing the racialized and gendered realities of undergraduate classrooms (e.g. Brockenbrough, 2016; Leyva et al., 2021; Leyva, McNeill, & Duran, 2022; McNeill et al., 2022)

I also intend to use queer pedagogy in my research as a tool for queer readings (Britzman, 2014) of proof, queer proof validation, and queer guided reinventions through the use of realistic mathematics education (Freudenthal, 1972). Additional topics which may benefit from the use of queer pedagogy are studies which concern student identity, mathematics identities, and studies which question the construction of mathematics as a stable and fixed.

"Not everything you do has to be gay." At the time, my father's words left me chagrinned, and I was quick to reassure him that my queer experiences represented only a fraction of who I was and what I did. Now I have a better answer to his indictment: Does everything "have to be gay?" Maybe not. But I challenge the idea that there is any value in trying to dissect my life into categories like "gay" and "not gay". My experience of mathematics is a queer experience of mathematics, and my life, my teaching, and my research is all the better for it.

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A Framework for Dimensions of Student Mathematical Thinking in Instructional Planning and Teaching

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Effective teaching relies on understanding and responding to students' mathematical thinking, enabling educators to build on student prior knowledge and deepen comprehension. While extensive research has established this link at the K-12 level, the research base for undergraduate mathematics education lacks a cohesive framework that integrates student thinking into instructional practices like planning and teaching. This paper addresses this gap by proposing a comprehensive framework designed to embed student thinking more deeply into undergraduate mathematics instruction, with the goal of improving teaching effectiveness and student learning outcomes. We conclude by discussing the implications for both instructional practice and research, as well as outlining the future directions for empirical validation.

Keywords: Student Thinking, Theoretical Framework, Undergraduate Mathematics Instruction

Effective teaching hinges on the ability to understand and respond to how students conceptualize and think about mathematical ideas. This understanding allows educators to tailor their instruction in ways that build on students' existing knowledge and foster a deeper, more robust comprehension of mathematical concepts and skills. Every teacher however, has a unique subset of beliefs that shape their planning and teaching practices (Stipek et al., 2001). These beliefs are crucial as they influence every aspect of the educational process, especially in undergraduate educational contexts (Vishnubhotla et al., 2024). Further complicating this, undergraduate instructors face constraints such as limited time for planning and developing new lessons and activities, and institutional pressures to adhere to a set syllabus that leaves little room for reform-based approaches (Johnson et al., 2013). This makes it difficult to classify particular types of instruction (e.g., inquiry, lecture) as "effective" or "more effective than the other." Rather, we propose that we can address "effective instruction" by addressing how instructional practices are reflective of and/or consider *student mathematical thinking*.

In the complex landscape of undergraduate mathematics education, both researchers and instructors understand the importance of attending to and using student thinking during instruction (Kuster et al., 2024). Anthony and Walshaw (2009) synthesized research literature to develop a research-based "set of principles that underpin the kinds of pedagogical approaches found to develop mathematical capability and disposition within an effective learning community" (p. 148). Some examples of these principles are facilitation of classroom dialogue focused towards mathematical argumentation, building on student thinking, and selection of tools and representations to provide support for students' thinking. Within these examples (and others provided by Anthony and Walshaw (2009)), student thinking is a major underlying theme.

Despite the critical role student thinking plays in instruction, much of the research connecting student thinking with teaching practices has predominantly focused on K-12 education, leaving a gap in the literature at the undergraduate level. Additionally, while there is a great deal of research on how student thinking can be supported during instruction, currently there is not an understanding of how student thinking unfolds in all stages of instruction nor is there consistency with the ways in which student thinking is utilized for the act of teaching by an instructor. In this

theoretical report, we briefly describe how student thinking is situated in literature, and propose a framework that integrates student thinking into the planning and teaching phases of instruction.

Background Literature and Our Goal

Over the past decade, a significant body of research has emerged emphasizing the critical role of student thinking in mathematics education. For our purposes in this section, we briefly outline research in two categories; planning and in-the-moment instruction. In their discussion of planning activities that support effective instruction, Akyuz and colleagues (2013) highlight the importance of five practices: preparation, reflection, anticipation, assessment, and revision. Supporting this work, in a comprehensive literature review regarding lesson planning, anticipating students' thinking and learning was tied to better learning outcomes and improved lesson quality (Cevikbas et al., 2024). Additionally, Vale and colleagues (2019) discuss how anticipating solutions during planning, allows teachers to better elicit student thinking, and to notice and attend to variation in students' responses during instruction. Datnow and colleagues (2021) discuss how gathering evidence of student thinking continually throughout instruction is ultimately valuable for informing planning. Their work also shows how teachers can use evidence of student thinking as data, and reflect on it in conjunction with formal assessment data to identify ways they can better support student learning. This body of work provides a foundation for describing the role student thinking plays in the planning phase of instruction.

Research discussing the importance of student thinking during in-the-moment instruction has focused on specific aspects such as facilitating discussions (e.g., Leatham et al., 2015; Leatham et al., 2021; Smith & Stein, 2011), recognizing productive student ideas (e.g.,Sherin et al., 2010), engaging students in rich mathematical thinking (e.g, Henningsen & Stein; 1997; Jackson et al., 2013) and providing support to early career teachers (e.g., Amador et al., 2022; Stockero et al., 2015). Student thinking can also be thought of more concretely, as data that can be used to inform instructional decisions (Datnow et al., 2021). In line with this, the notion of *responsive teaching* focuses attention on student thinking as an implicit guide to the practice of teaching (Dyer & Sherin, 2016). Responsive (sometimes called adaptive teaching (Gallagher et al., 2022; Parsons et al., 2018; Vaughn et al., 2021)) is a student centered teaching style that involves teachers observing student thinking and adapting their teaching based on their students' needs and abilities. This premise is similar in nature to the principles of generating student ways of reasoning, building on student contributions, and developing a shared understanding from Inquiry Oriented Instruction (Kuster, et al., 2018, 2019). It is clear student thinking plays a central role in a variety of teaching practices as they relate to in-the-moment instruction.

While the current literature base forms a strong foundation, making large strides toward understanding and implementing instructional methods that focus on student thinking, it also presents a few challenges. For instance, there exists a wide array of terms that describe instructional practices related to student thinking — such as eliciting, monitoring, building on, attending to, using, leveraging, understanding, scaffolding, and interpreting — but how these terms are defined or fit together across the field is often unclear. This variation in terminology reflects the multifaceted and complicated nature of the relationship between student thinking and teaching, while also highlighting a lack of clarity and coherence across the research literature. Within undergraduate education, the application of these research concepts is even less clear. Many instructors acknowledge the importance of student thinking with regard to instruction, but in our experience struggle to systematically incorporate student thinking into their instructional

practices. Moreover, there is often inconsistency in the use of these practices even across instructors that report having similar goals and using similar strategies (Kuster, et al., 2024).

Despite the communal value for using student thinking to inform instructional practices, the mathematics education community has yet to fully articulate a cohesive framework that delineates the role student thinking plays across the different aspects of the instructional process. Such a framework would ideally provide a consistent, research-based approach for integrating student thinking into instructional practices and goals, ensuring more effective teaching and improved learning experiences and outcomes in undergraduate mathematics education.

This paper aims to bridge this gap by exploring the intersection of student thinking and instructional practices in undergraduate mathematics education. We propose a framework that builds on current research to embed student thinking into every stage of the instructional process (i.e., planning to teaching). By offering a comprehensive framework and actionable insights, this paper seeks to not only contribute to the academic discourse but also empower educators to make informed, student-centered instructional decisions. When reflecting on our goals as undergraduate mathematics educators, prioritizing an environment that places student thinking at the forefront can unify the discussion regarding 'best practice' of teaching. Ensuring that our teaching and planning practices consistently aim to support and enhance student thinking is critical to effective teaching. Ultimately, our goal is to foster a learning environment where student thinking drives mathematical understanding and success at all educational levels.

Student Mathematical Thinking Theoretical Framework

Before discussing our framework of student thinking, we first provide our definition of Student Thinking. ***Student Mathematical Thinking*** *is any output (verbal or written) where students provide artifacts from which one can determine or infer how they thought or are thinking mathematically; it is the process of thinking, not the product from thinking.* For example, "my answer is 4" is not an example of student thinking whereas, "it cannot be 4 because that violates the axiom" is an example of student thinking. Student mathematical thinking refers to the ways in which students process, understand, and engage with mathematical concepts, problems, and reasoning. It encompasses students' cognitive processes as they solve problems, make sense of mathematical ideas, and develop strategies for approaching tasks.

Our framework on student mathematical thinking describes the ways in which student thinking can be used by an instructor in lesson planning or teaching. Importantly, we view knowledge and learning from the emergent perspective and interpretive framework (Cobb & Yackel, 1996) where knowledge is both individual and collective. Therefore, our framework considers student mathematical thinking both in terms of the individual and collective, but we do not attempt to have our framework delineate a dimension as individual or collective as the use of student mathematical thinking while teaching and planning is both individual and collective.

The framework emerged from previous work on the lesson planning practices of mathematics faculty (Author, 2024). Our findings revealed that though faculty often aligned with a particular teaching style (e.g., inquiry-based, lecture, or a blend of both), this label did not necessarily align with their actual practices. Vishnubhotla and colleagues (2024) had similar observations. We developed this framework centered on student thinking as a means for addressing this issue by being able to characterize key aspects of instruction separate from teaching style.

In our framework we have different dimensions, each is titled with verbs that modify the phrase 'Student Thinking'. For example, Determining Student Thinking. The framework features three dimensions within Lesson Planning, Designing, and Reflecting (Determining, Modeling,

and Preparing) and five dimensions within Teaching (Eliciting, Interpreting, Scaffolding, Assembling, and Reflecting on). This framework evolved from our combined experiences as teachers and researchers across various educational settings. It reflects a fluid approach to both planning and instruction, with each closely intertwined; many in-the-moment teaching decisions are also impromptu planning actions. This framework does not adhere strictly to a linear or circular sequence, as not all steps are always executed, depending on the instructional context. With this framework it is our goal to identify foundational actions in which teachers engage, from which many other practices are composed.

Our full framework can be found in Table 1. Following Table 1 we expand upon each dimension and support it with appropriate research literature.

Table 1. Student Thinking Framework.

Category	Dimension	Definition
Lesson Planning, Designing, and Reflecting	**Determining**	Deciding on the learning outcomes of the lesson, in terms of the student mathematical thinking in which one wants the students to engage related to the topic. Note: We assume course learning objectives have already been decided and agreed upon by the university.
	Modeling	Developing possible ways in which students might mathematically think about the content at various waypoints throughout the progression of the lesson based on past experience, resources, and/or literature.
	Preparing with	Developing and/or ordering a list of possible questions, tasks, lecture points, etc. that can be utilized to elicit or progress student mathematical thinking along this (modeled) pathway.
Teaching	**Eliciting**	Engaging students in tasks, problems, and/or questions (things they do not have the answer or solution method for) with the intent of bringing their thoughts and ideas to the public space.
	Interpreting	Actively listening to students with the intent to figure out and understand their mathematical thinking and how they are using their understanding about a particular concept.
	Scaffolding	Building scaffolds that match with possible pathways students may take based on interpreted student thinking (e.g., how the students are thinking about the concepts) and how it fits with the instructional goals.
	Assembling	Having scaffolded the students' mathematical thinking might develop and lead to more sophisticated understandings, the instructor attempts to move students along that progression through careful consideration of activities that are intended to evoke current thinking in ways that lead to new and more complex understandings.
	Reflecting on	Considering how students thought about concepts and ideas, how those ideas developed throughout the lesson (for them), and how the instructor could have scaffolded questions or tasks that better aligned with that progression.

Determining Student Thinking is the initial step in using student thinking, and involves identifying a lesson's desired learning outcomes (Stephan et al., 2016; Nabb et al., 2018). This goes beyond determining the concepts to be understood and the skills that need to be mastered, and includes identifying targets related to the specific student thinking and learning one wishes to help their students develop;. To determine these goals, instructors may consult research on the topic, engage in thought experiments, or ask colleagues about their prior experiences.

Modeling Student Thinking involves making the implicit reasoning processes of students explicit by developing a blueprint for the ways students are likely to understand and reason with particular concepts or approach certain tasks. Additionally, teachers consider how students may reason with these ideas to construct understandings of more complex ideas. For instance, while students must understand and develop skills concerning limits, function, and average rate of change before they can robustly understand instantaneous rate of change, they must also think about those foundational concepts in particular ways to develop a rich and robust understanding of instantaneous rate of change, or how this can be represented as a function. In the research literature this is often referred to as anticipating student thinking (e.g., Stein et al., 2008). We have chosen the modifier *modeling,* to highlight the need for active engagement with the student thought process and the role modeling often plays in the construction process. Developing foresight into how students might understand and reason about and with a concept or idea is central to this dimension. In the research literature, modeling student thinking is emphasized as a critical aspect of lesson planning, particularly in how it informs the design of instructional tasks, the sequencing of content, and the explicit illustration of how mathematical thinking develops and can be supported (e.g., Carpenter et al., 1999; Jacobs et al., 2010; Ball et al., 2008; Peterson and Leatham, 2009).

Preparing with Student Thinking involves intentionally designing instructional activities that support students in progressing from their current levels of understanding to more advanced levels. These instructional activities (e.g, tasks, questions, lectures) are designed using the previously developed models and understanding of student thinking and learning. During the preparation phase, teachers are thinking forward and backward; forward in the sense that one must create lesson materials that support the students in engaging in the desired thinking and reasoning, and backward in the sense that one must identify lesson activities that support the thinking and reasoning that serve as waypoints to that destination. With our calculus example, teachers that use student thinking to prepare lessons are considering not only the tasks concerning the desired ways of reasoning about instantaneous rate of change, but also sequences of tasks that would support students in thinking about function, limits, and average rate of change in ways that foster student development of instantaneous rate of change. This preparation of lesson activities is discussed in the literature at both the individual lesson level (Stephan et al., 2016) and at the course level (Rasmussen & Keene, 2019).

Eliciting Student Thinking involves drawing out and making students' thought processes, ideas, and understanding observable. The goal is not to simply evaluate what students know but to engage them in deeper reflection and reasoning, helping them to articulate their thinking. Eliciting student thinking takes many forms, such as using questioning, discussions, and tasks designed to engage students in thinking about concepts (Kuster, et al., 2019). Smith and Stein (2011) discuss the importance of using cognitively demanding tasks for supporting mathematically rich student thinking and reasoning. With regard to in-the-moment decision making, teachers cannot identify productive ways to later scaffold or assemble student knowledge without first having access to their current knowledge (Hufferd-Ackles et al. 2004;

Leatham et al. 2015). Along this same vein, they cannot determine the accuracy of their models, or how to best utilize their prepared activities without having an understanding of their students' current thinking.

Interpreting Student Thinking is a critical aspect of effective instruction, as it allows teachers to understand and subsequently respond appropriately to students' conceptualizations and reasoning processes. Research underscores the importance of interpreting students' responses to adjust teaching strategies and support learning. For example, Jacobs and colleagues (2010) highlight how teachers' professional noticing—observing, interpreting, and responding to students' mathematical thinking—enables them to provide targeted support and modify instruction based on real-time insights into students' understanding. This practice is crucial for formative assessment (Franke et al., 2009), and is a key practice in Inquiry Oriented Instruction (Kuster, et al., 2019). Boaler (2016) also emphasizes that interpreting student thinking can uncover diverse problem-solving approaches and reasoning, allowing teachers to adjust tasks and discussions to better meet students' needs. By accurately interpreting students' mathematical thinking, educators can better identify where students are along the path of achieving the intended thinking goals, and what and how student ideas can be combined to support their progression along that path.

*Scaffolding Student Thinkin*g is an important aspect of in-the-moment teaching and is characterized by making decisions based on information from both the planning phase and teaching phase. That is, having interpreted the students' thinking, instructors then make decisions about how to guide and manage the lesson based on the models they developed and the materials they have prepared during the planning phase. When Scaffolding, teachers consider how students are thinking in-the-moment about a mathematical concept, and make in-the-moment teaching decisions regarding how that thinking relates to the instructional goals and how it may be utilized. They may even revise the models they developed or the materials they created in lieu of the new information they received while in the process of interpreting. Effective scaffolding requires the teacher to deeply understand students' thinking (Interpreting), anticipate how that thinking will be able to develop (Modeling), and then making instructional decisions that support that development (Franke et al., 2007; Jacobs et al., 2010).

Assembling Student Thinking can be roughly thought of as the enacting the scaffold; the actions taken by the instructor to utilize the scaffold and build the desired student thinking. This may be actualized with the utilization of a variety of instructional methods such as whole class discussions, intentionally designed task sequences, teacher-directed techniques, selecting and sequencing student presentations, etc. Effective teaching practices involve recognizing and extending students' ideas, creating productive learning environments, and promoting connections between mathematical concepts (Peterson & Leatham, 2009). The flexibility of the specific instructional methods that foster these connections is intentional, but to be Assembling Student Thinking the approach must not only incorporate students' existing knowledge, but also help them to advance their understanding through thoughtful and strategic instructional design.

Reflecting on Student Thinking is a critical aspect of effective mathematics instruction, as it plays a vital role in enhancing teaching practices and supporting student learning. By carefully analyzing students' responses and thought processes, teachers are able to make informed decisions about instructional strategies and adapt their teaching to better meet students' needs. This reflective practice helps identify where students are struggling and allows for adjustments to be made to address these challenges. Carpenter and colleagues (1996) emphasize that understanding students' cognitive processes enables teachers to align instruction with students'

existing knowledge, thereby improving the effectiveness of teaching. Reflecting on student thinking provides teachers with the necessary insights to offer specific, actionable feedback that helps students build more robust and complex understandings. Additionally, Lampert (2001) suggests that this practice supports teachers' professional growth by offering insights into their own instructional practices and identifying areas for improvement.

Conclusion, Implications, and Next Steps

In this paper, we proposed a framework integrating student mathematical thinking into planning and teaching. Our framework emphasizes the importance of Determining, Modeling, and Preparing with Student Thinking during lesson planning, and Eliciting, Interpreting, Scaffolding, Assembling, and Reflecting on Student Thinking during instruction. We propose these as core building blocks from which one can characterize instructional practices across an array of pedagogical approaches. By placing student thinking at the center of instructional decision-making as opposed to teaching style, we hope to enable educators to more effectively foster deeper understanding and more meaningful engagement with mathematical concepts.

Along these lines the framework is intended to offer flexibility, recognizing that instructional practices vary widely across classrooms and teaching styles, while also providing a coherent structure for incorporating student thinking into any teaching approach. By doing so, it empowers educators and researchers to move beyond a reliance on traditional classifications of teaching methods (e.g., inquiry-based vs. lecture) and instead focus on the central goal of supporting student learning; fostering and advancing student thinking.

Despite its potential, the framework also presents several challenges and opportunities. One of the key limitations is the lack of a formal component dedicated to evaluating learning outcomes, a crucial aspect of instruction. Additionally, research is needed to explore how this framework can be adapted and applied across various instructional contexts, content areas, and student populations. We also acknowledge that our framework does not prescribe specific methods or concrete steps, but rather serves as a guiding principle. Future work should focus on empirically validating the dimensions of this framework, as well as exploring the ways in which they interact and evolve during real-time instruction.

The implications of this work are profound, particularly in its potential to unify the discourse around "best practices" in teaching mathematics. Rather than focusing on the form of instruction, we suggest that the mathematics education community should prioritize instructional practices that are steeped in student thinking. This reorientation allows for a more flexible, adaptable approach to teaching while ensuring that student thinking remains central to the instructional process, something research seems to indicate as crucial to effective teaching. As we continue to refine and test this framework, we hope it will contribute to both the research literature and practical teaching strategies, ultimately supporting educators in creating learning environments that promote deep mathematical understanding and success at all levels of education.

Future research should focus on examining how student thinking is operationalized across different types of mathematical content and instructional settings. Additionally, studies that explore how instructors in higher education can develop the skills and knowledge needed to effectively engage with student thinking will be crucial for further developing this framework. Our aim is to inspire both researchers and practitioners to place greater emphasis on student thinking, not just as a peripheral consideration, but as the driving force behind instructional planning, decision-making, and evaluation. By doing so, we can create more effective and equitable learning experiences for all students.

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Integrated Cognitive-Instrumental Model: A Dual Framework for Analyzing Cognitive Processes

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Influenced by Piagetian constructivism and Vygotsky's sociocultural theory, contemporary research has explored both individual cognitive development and the role of tools to mediate learning. In this theoretical report, we propose the Integrated Cognitive-Instrumental Model (ICIM), which synthesizes two prominent frameworks: APOS Theory (grounded in constructivism) and the Instrumental Approach (rooted in sociocultural theory). ICIM combines the strengths of both frameworks, where APOS Theory explains how learners move through stages in their cognitive development, and the Instrumental Approach shows how tools and artifacts shape and support this development. Together, these frameworks allow for a more comprehensive understanding of students' cognitive development by bridging the gap between Piagetian and Vygotskian principles and highlighting the interplay between individual cognitive processes and the socio-cultural environment. ICIM offers a robust, multi-theoretical approach that provides a new perspective into student thinking with the potential to inform both research and instructional practices.

Keywords: APOS Theory, Instrumental Approach, Constructivism, Sociocultural Theory

Introduction

Understanding student thinking has been the cornerstone of mathematics education research for over a century, dating back to the late 19th and early 20th century with Behaviorist theory focusing on observable behaviors and the external conditions under which learning occurred (Thorndike, 1922; Woodworth & Thorndike, 1901). While these early studies laid the groundwork for systematic educational research, they overlooked the internal cognitive processes that are fundamental to meaningful learning and understanding how to support such learning (Erlwanger, 1973). As the field evolved, the focus gradually shifted toward understanding the cognitive mechanisms that underpin students' learning processes, a shift that was groundbreaking in fostering a deeper comprehension of how students construct and internalize mathematical concepts. This shift in focus was significantly influenced by the work of theorists such as Jean Piaget and Lev Vygotsky. Piaget's writings, along with von Glasersfeld's interpretations (e.g., von Glasersfeld, 1984, 1987, 1995), introduced the constructivist approach, which emphasizes the active role of the learner in independently constructing knowledge/reality through a process of action and reflection. Vygotsky's sociocultural perspective, meanwhile, highlighted the critical role of social and cultural contexts in shaping cognitive development, particularly through language, tools, and social interactions with more knowledgeable others (Confrey, 1995; Pugh, 2017).

Drawing on these two theoretical perspectives, we present a framework with the goal of providing a straightforward mechanism to characterize students' in-the-moment cognitive activity. To achieve this, we draw on two frameworks that build on and connect Piaget's constructivist theory of learning and Vygotsky's sociocultural perspectives. In the following sections, we summarize the frameworks used, describe the merger of these frameworks into one integrated framework, and conclude this report with implications, considerations, and questions for the field. By integrating these perspectives, we hope to offer a more accessible and multi-

theoretical understanding of student cognition in mathematics, with the potential to inform both research and instructional practices.

Frameworks

The first framework we are leveraging with this work is APOS (Action, Process, Object, Schema) Theory, grounded in constructivist principles focused on the mental actions, processes, and general cognitive structures involved in mathematical thinking. The second framework is the Instrumental Approach, informed by the sociocultural perspective, which emphasizes the role of tools and artifacts in shaping cognitive processes. In the following sections we present a review of the various components of APOS Theory and the Instrumental Approach, outlining their constructs so that they can later be synthesized together in an integrated framework that we are tentatively referring to as the *Integrated Cognitive-Instrumental Model* (ICIM).

APOS Theory

Drawing on Piagetian principles to characterize students' cognitive activity, we draw on the well-established cognitive framework known as APOS Theory (Arnon et al., 2014; Asiala et al., 1996; Dubinsky 1991; Dubinsky, & McDonald, 2001). APOS stands for Actions, Processes, Objects, and Schema, and describes a constructivist perspective built upon Piaget's theory of reflective abstraction. Central to the theory is the hypothesis that mathematical knowledge is the result of an individual's construction of four key components: a) mental actions, b) processes, c) objects, and d) schemas to make sense of mathematical problem situations. Each of the four key components of APOS Theory represent a distinct stage in the development of mathematical knowledge (Dubinsky & McDonald, 2001). Actions represent external transformations of mathematical concepts. These actions can be thought of as the application of explicit algorithms or memorized procedures. Importantly, actions lack justifications and are not typically connected to other forms of mathematical knowledge. When an action is carried out multiple times through repeated stimuli, and reflected upon, it becomes a process. Processes are perceived as internal, implying that they have meaningful connections to an individual's existing mathematical knowledge. This enables individuals to imagine a variety of transformations, omit steps in their conceptualization of the actions occurring, anticipate results, and provide a justification for their processes. An object is constructed from a process when individuals become aware of the various processes that describe a central idea. More specifically, the learner realizes that transformations can act on the central mathematical idea itself, therefore treating the idea as an object that can be acted upon. The literature refers to these objects as higher-level mental constructions that extend beyond the context in which they were originally formed – in more theoretical terms, they are extensions of Piaget's notion of reflective abstraction (Arnon et al., 2014; Dubinsky, 1991). Lastly, schemas represent a coherent collection of actions, processes, objects, and other schemas that relate to a specific mathematical idea or central topic. Schemas are characterized by their interconnectedness, providing the learner with an internal framework which helps them determine whether a given mathematical situation falls within their scope (Dubinsky & McDonald, 2001).

For the purposes of using APOS theory to understand student thinking, the application involves a cyclical process consisting of *theoretical conjectures, assessment design, and empirical data collection and analysis*. The purpose of this iterative cycle is to continually refine conjectures about the mental constructions of the learner, and is necessary for understanding how they might reason about various mathematical concepts central to the topic of investigation. The

theoretical conjecture phase begins with the development of a genetic decomposition, which is a set of mental constructions that students might make to understand a mathematical concept. For example, students may first construct an action-based approach of logical statements by typing logical statements such as 'True and False,' 'False and False,' and 'True and True' into a text-based programming software such as Python or R to investigate the outputs. This action stage may then be followed by a process stage in which the learner identifies the central idea that the 'and' proposition requires two true propositions to produce a True output. Once the learner identifies this central idea, they may then utilize this as a piece of a larger mathematical process, therefore treating the 'and' operator as a mathematical object. In a teaching experiment investigating undergraduate students' conceptions of set theory and logic, students exemplified these action-process-object stages using the 'and' operator in Python as an object to generate the intersection of three sets (Martinez, 2022). This decomposition is based on the anticipated mathematical activity of the students, given the researcher's understanding of the concept, and it is refined through the subsequent iterations of interactions with the learner. In studies focused on the teaching and learning of a specific concept, typically the goal is to guide the learner through the genetic decompositions proposed by the researchers (Arnon et al., 2014). It's important to note that the progression through the various stages of APOS Theory is not linear. Some students, as they are engaging in the action stage, move to the process stage, but may move back to the action stage to investigate a particular case of the concept (Dubinsky & McDonald, 2001). Below I highlight another example of genetic decomposition for the concept of sets and subsets which we will return to in the next section.

Consider the following question: How can we determine the total number of subsets of a set, S, and is there a way to calculate this for any set? The genetic decomposition starts with the action stage; a student might approach this problem by explicitly listing all the subsets of some set, say $S = \{1, 2\}$. They might list the subsets as $\{\}$, $\{1\}$, $\{2\}$, and $\{1, 2\}$. At this stage, the student's understanding is limited to the concrete actions of listing subsets, and they may not yet grasp the underlying combinatorial principles. As the student gains experience and reflects on the process, they might realize that there is a systematic way to generate subsets without listing them all individually. Ideally, they will begin to understand the process of binary choices: for each element in set S, they can choose to include it in a subset or exclude it. This stage involves interiorizing the process of subset formation. At the object stage, the student has a more abstract understanding in that they can perceive subsets as objects themselves and recognize that these subsets can be operated upon. They can imagine performing operations on subsets, such as finding the union or intersection of subsets. This stage involves encapsulating the process of generating subsets into a coherent concept of subsets as objects. At the schema stage, the student has a comprehensive framework for dealing with subsets. They have a collection of actions, processes, objects, and other schemas related to subsets that they can then use to determine when a problem situation involves subsets and use their schema to address various subset-related problems. As demonstrated, APOS theory can provide a structured approach to understanding how students develop mathematical understanding. However, APOS theory (and Piaget's constructivist theory of learning in general) fails to address social factors such as students interacting with their environment (in the Vygotskian sense), as an integral aspect of one's learning. More specifically, the use of tools and their influence on cognition is missing from APOS Theory. In the next section I describe the Instrumental Approach, a theoretical framework that artfully bridges the work by Piaget and Vygotsky to account for constructivist principles, and the use of tools to describe cognitive activity.

The Instrumental Approach

Grounded in Vygotsky's (1978) concept of mediated activity, the Instrumental Approach broadly serves as a framework for analyzing how individuals expand their cognitive abilities through the use of tools to manipulate their environment. While APOS Theory largely developed through mathematics education research in the United States, the Instrumental Approach emerged within the French tradition of mathematics education research (Artigue, 2002; Guin & Trouche, 1998; Rabardel, 1993a, 1993b; Vérillon & Rabardel, 1995; Trouche, 2004). Vérillon and Rabardel's (1995) work was particularly influential, as they distinguished between subject-object interactions and laid the foundation for understanding how instruments develop through the use of tools, or artifacts. As described by Roorda et al. (2016), artifacts can be a physical object such as graphing calculators, geometry measuring tools or other manipulatives, but they could also be graphs, computational programming environments, images, proofs, formulas, or any other objects that can be used to reason about a given mathematics task. As the learner incorporates the artifact into their activity to solve a mathematical task, the cognitive process of the learner interacting with the artifact transforms the process merely from a subject-object interaction (i.e. learners solving a mathematical task without the artifact) into what we call an *instrumented activity situation*, or an expanded cognitive function that would not otherwise be possible without the artifact.

As Martinez (2024) described, "the development of the instrument involves the learned ability to manipulate the artifact itself (proximal action), in order produce an extended, and meaningful, manipulation (distal action) of the environment in which these actions are taking place" (p. 5). The relationship between proximal and distal actions are central to the development of the instrument (Abrahamson & Bakker, 2016; Artigue, 2002; Trouche, 2004; Vérillon & Rabardel, 1995). Take the example of a child learning how to use a spoon to eat soup, the child needs to understand how to manipulate the spoon (proximal action), even as they keep an eye on the effect it has on the soup (distal action). On its own, an artifact is not an instrument. It is only through the interaction and use of the artifact to solve a mathematics problem does it become an instrument through the manipulation of the artifact to enact a meaningful change in the learner's environment. Figure 1 is a representation of the interactions that are taking place as a learner (subject) solves a mathematical task (object) using an artifact to mediate the cognitive activity which leads to the development of an instrument.

The process of transforming an artifact into an instrument is known as *instrumental genesis*, which involves a bi-directional interaction between the learner and the artifact (Artigue, 2002). What this means is that the learner may be afforded creative and interesting ways to think about a mathematics problem given the potential use cases of the artifact while at the same time being limited by the constraints of the artifact itself which shapes the ways in which the learner may go about solving the task. This two-way cognitive interaction is known as the *instrumented action scheme* (Trouche, 2004) and is rooted in Piagetian constructivist principles. As described by Vergnaud (2009) one's scheme is the "the invariant organization of activity for a certain class of situations" (p. 88). In other words, one's scheme describes the process by which learners apply their existing knowledge to familiar situations and how they adapt or reorganize their thinking when faced with new problem tasks.

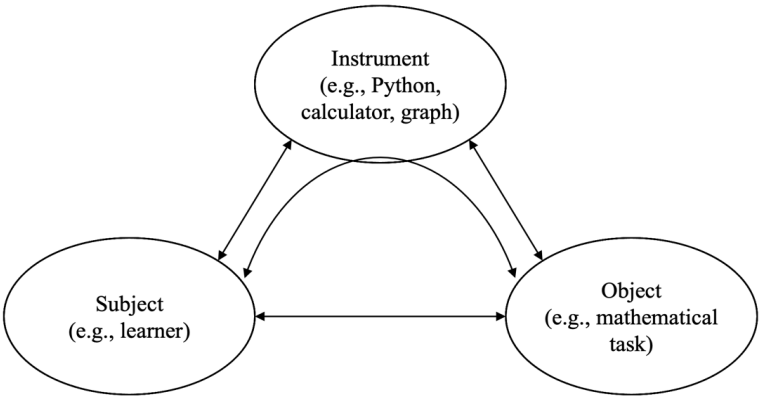


Figure 1. The subject, object, instrument interaction.

In the context of the instrumental approach, schemes consist of four main components (Trouche, 2004; Vergnaud, 2009), paraphrased from Buteau et al. (2020):

1. The goal of the activity: With sub-goals and expectations.
2. Rules of action: Stable behaviors of the learner.
3. Operational invariants: Which can be theorems-in-action (propositions considered as true) or concepts-in-action (mathematical concepts considered as relevant).
4. Possibilities of inferences: These possibilities are essential for the adaptation of the scheme to the specific features of the situation.

As Buteau et al. (2020) explain, a subject's mathematical activity can reveal these elements of their scheme when they state something like, "When I want to do this [goal], I always act like this [rule of action] because I think that [operational invariant]" (p. 1027). While students may not always articulate their reasoning so clearly, we can infer their thought processes through their observable behaviors. For instance, rules of action can be described as approaches the learner comfortably relies on to accomplish a familiar task, which are guided by their beliefs or understanding (operational invariants) of how things work. The operational invariants are divided into theorems-in-action and concepts-in-action. Theorems-in-action are the learner's constructed understandings of how things work, while concepts-in-action relate to the mathematical ideas important for the task. The possibilities of inferences refer to situations where learners modify their schemes by creating new rules of action or operational invariants.

Taken from Martinez (2023), let's consider the same problem scenario from the previous section in which a learner is asked to find all the possible subsets for any given finite set. Let's also imagine that the learner is using a computational programming environment like Python as the artifact which is developed into an instrument. In this case, one potential *rule of action* could be the creation of a loop that creates all the possible subsets. A *concept-in-action* could be the idea that a subset contains only elements from the original set and a *theorem-in-action* could be that sets are unordered, meaning {1, 2, 3} is equivalent to {3, 2, 1}. As for a *possibility of inference*, encountering the empty set (which is not something that a learner using Python may naturally produce) would require the learner to create a new concept-in-action that includes the empty set as a subset.

It is important to note that these conceptual elements, which are inferred by the researcher, must ultimately be supported by the learner's observable actions. Naturally, some elements of a learner's scheme may be overlooked, but the goal is to identify the most relevant concepts tied to each rule of action. Together, these conceptual components form the learner's scheme, but as Roorda et al. (2016) describe, schemes can be observed at two levels of granularity. At the first

level, a scheme may consist of a single rule of action and its related operational invariants, observable through one instance of behavior. At the second level, multiple schemes combine to form an instrumentation scheme related to a specific task (Roorda et al., 2016). In the case of calculating subsets, the learner's overall instrumentation scheme may involve several steps: creating an algorithm, storing subsets in a list, and determining the total number of subsets. Each of these steps can represent individual level-one schemes, while together they form a level-two scheme for solving the subsets problem.

Integrated Cognitive-Instrumental Model

We should note that neither of these frameworks makes claims about the permanence or stability of cognitive behaviors; however, they both provide fine-grained analyses of in-the-moment reasoning at the individual level. That said, from our perspective, each framework has specific weaknesses that could limit its utility in analyzing student thinking. In the case of APOS Theory, the notion of schema – intended to encompass all actions, processes, and objects related to a mathematical topic – can be overly vague. This can make it challenging for researchers, and especially practitioners, to extract practical insights about student thinking. Some researchers have worked around this issue by focusing more directly on epistemological nuances rooted in Piaget's notion of reflective abstraction, such as Sfard's (1991) concept of *reification* and Zandieh's (2000) idea of *process-object pairs*. However, since APOS Theory is grounded in Piagetian principles, we argue that its more pressing limitation is its lack of emphasis on how tools mediate learning, particularly when attempting to provide a holistic view of learners' cognitive processes. In recognizing this gap, we turn to leverage the Instrumental Approach, which accounts for the role that tools and artifacts play in shaping cognitive development. While the Instrumental Approach addresses how artifacts shape cognition, its focus on concepts-in-action and theorems-in-action can sometimes complicate the analysis process. Although valuable in theory, the emphasis on identifying every subtle cognitive detail can make it challenging to construct a coherent account of a learner's thought process as they engage in the instrumental genesis of the artifact.

By integrating both frameworks, we propose a theoretical stance that conceptualizes learning as the cognitive work learners do at the individual level as they move through the Action, Process, and Object stages of APOS theory, in collaboration with the work they do to instrumentalize artifacts to solve mathematical problems. To illustrate how these frameworks can work in tandem, we revisit the subset problem introduced in the previous sections to synthesize the key ideas from both frameworks and demonstrate their natural co-utilization. As a learner engages in the Action stage of APOS theory, they may write out the possible subsets of S – these are rules of action, or stable observable behaviors. The process stage may entail the identification of a combinatorial method of reasoning about the number of subsets a set has – this describes the development of a concept-in-action, for each element in set S, it can either be in a subset or not be in a subset. At this stage in particular, the learner may utilize computational algorithms (their developed instrument) to omit steps or provide additional justification for their ways of reasoning. Finally, as the learner internalizes these actions and processes into their cognitive understanding of subsets, they may encounter a new situation, such as the empty set, where they must determine whether this new situation aligns with their existing understanding, or requires the development of new rules of action and operational invariants. In Table 1 we provide a table that showcases the overlap of the two frameworks, supported by the theoretical perspectives of Piagetian constructivism as well as Vygotskian sociocultural theory.

Table 1. Integrated Cognitive-Instrumental Model.

Component of Learning	APOS Theory (Constructivism)	Instrumental Approach (Sociocultural)	Affordances of ICIM
Cognitive Development	Action-Process-Object stages describe individual cognitive development	Rules of action guide how learners engage with tools, instrumentalizing the artifact to manipulate their environment	ICIM integrates cognitive development with the mediation of tools
Concept Formation and Abstraction	Reflective abstraction leads to the development of schemas for understanding concepts	Operational invariants (concepts-in-action and theorems-in-action) are developed through tool use	ICIM integrates concept formation from both a cognitive and social/mediated perspective
Adapting to New Situations	Accommodation leads to the development of new actions	Possibilities of inferences are influenced by the instrumented action scheme	ICIM highlights how both internal and external factors influence a learner's ability to adapt

As Table 1 illustrates above, the ICIM addresses three main factors of learning: (a) cognitive development, (b) concept formation and abstraction, and (c) adapting to new situations. We argue that integrating the components of APOS Theory and the Instrumental approach across each of these three components of learning provides a more robust characterization of one's cognitive development while remaining flexible enough to account for two theories of learning.

Discussion

We see the ICIM to be particularly useful for researchers wanting to analyze student's cognitive thought processes related to computational thinking. This is particularly relevant given Lockwood and Mørken's (2021) recent call for undergraduate mathematics education research to investigate the role of machine-based computing in the teaching and learning of mathematics. The integration of the two frameworks provides a more holistic view, adding texture to how we can articulate a learner's cognitive thought process, and builds a unique bridge between Piaget's and Vygotsky's theoretical perspectives that do more than merely complement each other. As Simon (2009) described, "The availability of multiple explanatory theories and the use of multiple layers of analysis can, depending on the research, provide a richer set of constructs for accounting for observed phenomena" (p. 484). The ICIM draws on this idea, not only capturing the complexity of learners' individual cognitive thought processes, but also highlighting the crucial role of tools and social context in shaping these processes.

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A Preliminary Design Space for Digital Tools

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The growth of educational technologies outpaces our understanding of how to promote students' mathematical development with such technologies. It has become increasingly important for researchers to problematize digital tool design so that educators seeking to enrich students' mathematical knowledge with the use of digital tools support, and not hinder, student learning. In this theoretical paper, we draw on existing research on digital tools for mathematics learning to propose a framework that situates to-be-designed tools within a "design space". We argue that an initial conceptualization of tools within this design space can help relate and distinguish tools from the literature that would be otherwise difficult to delineate or compare. We describe the spectra that comprise the digital tool design space, give examples of tools from the literature that fit within clusters of design space, and discuss the utility of such a framing.

Keywords: digital tools, digital task design, student thinking and learning

Over the past few decades, new digital technologies—combinations of hardware and software—have been developed to support mathematics teaching and learning. Over time, the nature of these digital technologies has shifted from standalone software packages designed to support students' learning about a particular mathematical topic (e.g., SimCalc) or provide a specific kind of interactive experience (e.g., Logo) toward interconnected webs of digital technologies utilized by teachers to accomplish a variety of functions. Teachers now use digital technologies for organization and implementation of the work of teaching (e.g., assigning work, producing worksheets, displaying student work), to connect with communities of other teachers that share materials and other resources, to individualize assignments and assessments of previously learned mathematics, and to support new ways of doing and representing mathematics (Sinclair & Robutti, 2020; Clark-Wilson et al., 2020).

At the same time, researchers have studied digital technologies in mathematics education from a wide range of theoretical perspectives and educational contexts. Given the fractured nature of the research literature on digital technologies for mathematics learning, it is important to develop theoretical tools that can be used to connect research findings across studies. This paper is part of a larger project broadly focused on the design of digital tools for mathematics learning at the undergraduate level. By *digital tools*, we mean instantiations of digital technology created within a software environment to support a particular learning goal. As part of our work, we recognized a need for a descriptive framework characterizing attributes of digital tools that could be used to identify commonalities between digital tools that were developed in different software contexts, from different theoretical perspectives, and for different pedagogical and mathematical goals. In this paper, we address the research question: *What dimensions of design can be used to characterize digital tools?* We provide a preliminary framework with three dimensions of design.

Research on Digital Tools for Mathematics Learning

Existing taxonomies distinguish between digital technologies for mathematics learning by classifying them according to their potential use in instructional situations (e.g., Dick & Hollebrands, 2011; Drijvers, 2015; Hoyles & Noss, 2009). For example, Dick and Hollebrands (2011) make the distinction between *conveyance technologies*, which allow students and teachers to present, communicate, and collaborate, and *mathematical action technologies* which "can perform mathematical tasks and/or respond to the user's actions in mathematically defined ways" (p. xii). Whereas this distinction concerns what the tools afford students and teachers, Drijvers' (2015) taxonomy classifies tools by the type of mathematical activity they support for students. Drijvers (2015) distinguishes between digital tools for *doing mathematics* or *learning mathematics*. In the category of learning mathematics, Drijvers (2015) further differentiates between tool use for *practicing skills* and for *developing concepts*. Lastly, Hoyles and Noss (2009) present four categories of digital tools that have differing potential to shape students' mathematical cognition: (1) dynamic and graphical tools, (2) tools that outsource processing power, (3) new representational infrastructures, and (4) tools with high-bandwidth connectivity.

These taxonomies can be useful when networking individual research studies focusing on students' learning with digital tools embedded within a specific software environment. For example, Bray and Tangney (2017) used an elaborated version of Hoyles and Noss' (2009) taxonomy to systematically review empirical research on the use of digital technology in mathematics education. "Such taxonomies prompt researchers to define more deeply the technology under scrutiny to support the reliability and generalizability of findings, a factor that is becoming increasingly important due to the growing complexity and interoperability of emerging technologies" (Clark-Wilson et al., 2020, p. 1126).

However, these taxonomies also tend to be used to describe particular software packages (e.g., GeoGebra), types of software environments (e.g., spreadsheets, clicker response systems), or types of hardware (e.g., interactive whiteboards; motion sensors). This makes it challenging to see how or when findings from one kind of digital technology can be usefully applied to the design or analysis of another. For example, Baccaglini-Frank and Mariotti (2010) studied students' conjecturing processes when engaging with open problems in a dynamic geometry environment. They proposed Maintaining Dragging as a mode of interaction with draggable points that has the potential for supporting particular cognitive processes in the context of Euclidean geometry. For researchers studying digital tools that are not embedded in dynamic geometry environments, there is still quite a bit to learn from this study.

To effectively gain such inter-environmental and inter-software insights, we need theoretical tools that can be used to identify the relevant attributes of digital tools in such a way that they are abstracted from the particularities of the software environment. We propose a framing of *design space* to accomplish this goal. We situate our initial conceptualization of design space using examples from the undergraduate mathematics curriculum because of the rich collection of mathematically complex tools constructible from the undergraduate curriculum. We use definite integration as a primary context to delineate regions of design space according to our identified spectra.

Theoretical Framing

Research on digital technology in mathematics education draws on various theoretical frames (Lagrange et al., 2003), with individual studies often poorly connected to prior work outside of the theoretical framework being utilized (Artigue et al., 2009). In an attempt to integrate disparate theoretical approaches to the study of digital technologies for mathematics education,

the TELMA project endeavored to use specific and shareable language that is "not dependent on a particular theoretical frame, allowing us to communicate" (Artigue et al., 2009, p. 220) across studies examining related phenomena from different theoretical perspectives. To accomplish this, the TELMA group introduced the notion of *didactic functionality* (Cerulli et al., 2006) to describe digital tools for educational purposes. The construct of didactic functionality has three components: (1) the set of tool features/characteristics, (2) a specific educational goal, and (3) a set of instructional modalities for employing the tool in an instructional context toward the desired learning goal. The framework we propose in this paper addresses the first component of didactic functionality by providing a descriptive language for the characteristics of digital tools.

In this paper, we utilize Yerushalmy's (2005) notion of *design space* to theorize dimensions of design for digital tools. A design space is a structured way of representing the space of possibilities for the design of a tool. We conceptualize the design space for digital tools as a 3D space in which digital tools can be placed on each of three axes representing the components of our framework.

A Preliminary Framework for Describing Digital Mathematics Tools

In this section, we introduce a preliminary framework for describing digital tools for mathematics learning. The framework has three main components: (1) the Interactive ↔ Inert spectrum, (2) the Static ↔ Dynamic spectrum, and (3) the Specialized ↔ Generalized spectrum. As is implied by the word spectrum, we intend that digital tools can be characterized as somewhere along a continuous scale described by each component. Using the dynamic label simply means that the tool in consideration is closer to the dynamic end of the spectrum than the static end. Some aspects of a so-called dynamic tool will be more dynamic than its other, more static, components. Yet by having at least some dynamic components, the digital tool lies farther along the dynamic end of the static ↔ dynamic spectrum than if it did not have those components. Likewise, having more dynamic components, or having the same components with more emphasis on its dynamic parts, would place the tool even further down the dynamic end of the spectrum. However, while we characterize each of these three components as a one-dimensional scale, the factors that place tools on this continuum are not well-defined in a mathematical sense that would lend these scales to a total ordering. That is to say, we do not intend for this framework, in its current form, to be used for pairwise comparison of all digital tools along each of the components. A fine-grained comparison such as "Tool A, with two sliders and a moving graph, is more dynamic than Tool B, with one slider and two moving graphs" is not the intention of using the term "spectrum."

Finally, we do not propose a hierarchy of digital tool characteristics within this framework. The quality of a digital tool is determined by its coherence with the pedagogical goals and students' current mathematical and technical capabilities. Although digital tools that are dynamic and interactive may afford students novel opportunities to engage with mathematical ideas compared to the paper-and-pencil environment, they are not inherently better or more valuable than static and inert digital tools. This framework is purely descriptive, and for each of the three spectra, neither pole carries a more favorable connotation than the other.

Interactive ↔ Inert

A key affordance of digital technology is interactivity. Kaput (1992) describes the difference between *interactive* and *inert* media:

> We characterize a medium as "inert" if the only state-change resulting from a user's input is the display of that input. Thus the key difference with notations instantiated in

> interactive media is the addition of something new to the result of a user's actions, something that the user must then respond to. In inert media, the user can only respond to what he or she has directly produced. Any external response to the input must be made by a third party, for example, a teacher, tutor, or peer (p. 526)

We extend Kaput's characterization from a binary categorization to a spectrum. Digital tools can be categorized roughly as more interactive or more inert depending on the overall breadth and depth of their reactiveness to a user's input. For example, a video player has the ability to react to user input regarding stopping and starting of playback at various points in the video feed. However, a traditional video player does not have the capability to capture mathematical input and change the contents of the video feed based on that input. Embedding digital polling, or allowing for swapping of video feeds based on user input, would increase the reactiveness of the digital tool and move it farther along the interactive side of the interactive ↔ inert spectrum.

Static ↔ Dynamic

A second affordance of digital technology is the ability to support dynamic media (Kaput, 1992). With dynamic media, objects can change states as a function of time, which means "time can become an information-carrying dimension" (Kaput, 1992, p. 525). However, in static media, the state of objects remains unchanged over time. Kaput (1992) further suggests that a dynamism range exists, where some notions of "dynamic" are more strongly dynamic than others according to "continuous transition." In our characterization of the Static ↔ Dynamic scale, we expand on the scale's dynamic range.

Our Static ↔ Dynamic scale consists of four classes, ordered from most static to most dynamic: Static, Discrete, Weakly Continuous, and Strongly Continuous. The latter three classes refine "dynamic." To illustrate these four characterizations, we use a recurring example of a digital tool whose purpose is to help students understand the relationship between the definite integral and the area under a curve (see Figure 1).

A *Static* digital tool in this motif is a digital image depicting a generic curve with some area shaded under it, paired with the algebraic representation of a definite integral. We note that a static tool can still be interactive, allowing the user to take some action and get a response from the digital tool, but differing states of the mathematical objects on screen do not have temporal variation. The example in Figure 1a shows an interactive static digital tool that can be used to explore the relationship between a definite integral and the area under a curve. Students can click to place values on the upper limit of integration and see both the symbolic and graphical representations immediately update.

To transform this static tool into a *Discretely Dynamic* tool, imagine that instead of the ability to to click to place values on the upper and lower limits of integration students can drag points to adjust the upper and lower limits of integration on the graph (see Figure 1b). However, the motion of the draggable point is constrained to be only integer values. So when a student wants to adjust the upper limit of integration to be a larger value, they drag the point to the right and see the symbolic and graphical representation update through a series of discrete—but temporally organized—states.

To make this tool *Weakly Continuously Dynamic*, we remove the constraints from the draggable points so that students can adjust the limits of integration by dragging the points smoothly along the x-axis.

Finally, to make this tool *Strongly Continuously Dynamic*, it is converted into an autonomous animation rather than allowing user manipulation. In dynamic geometry software such as

Desmos or GeoGebra, this equates to a slider equipped with a play button that will enable variables to run through a prespecified range of values.

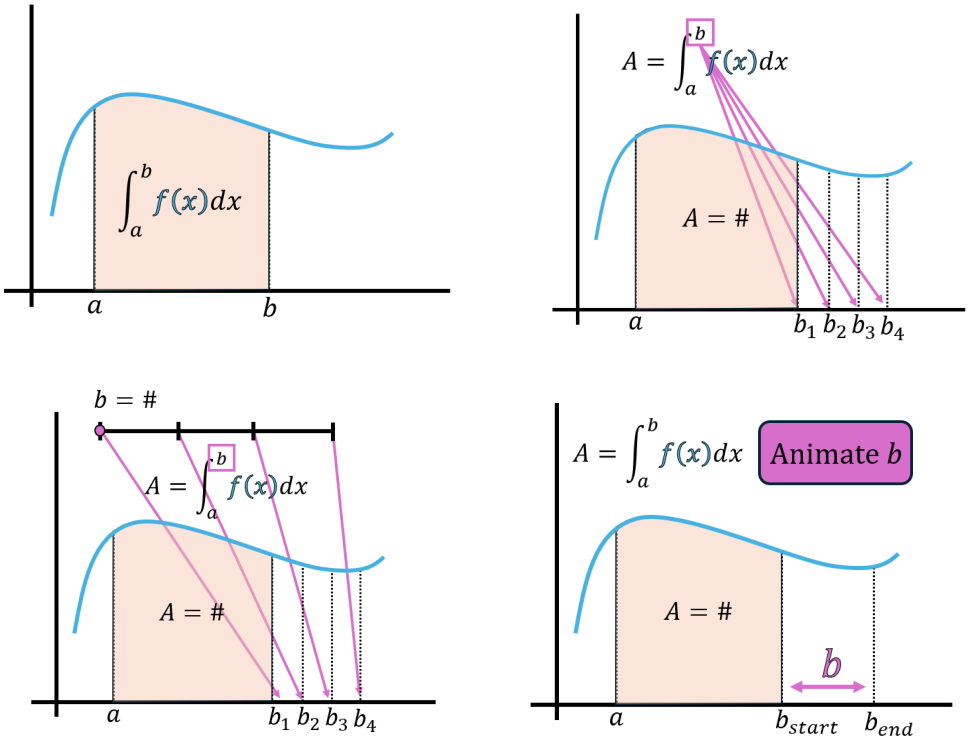


Figure 1: Variations on integration-based digital tools. From left to right: static, static interactive, discretely dynamic, strongly continuously dynamic.

These categories are not mutually exclusive. Each increasingly dynamic category is contained in the preceding static category. For example, a Discretely Dynamic tool has clear Static elements. Each state of the tool is itself a Static digital tool. Thus, when categorizing digital tools along the Static ↔ Dynamic spectrum, we consider the tool's *maximum potential dynamism*—its most dynamic classification within the constraints of its design or software platform.

Specialized ↔ Generalized

Cerulli et al. (2006) argue that digital tools are always developed in relation to some pedagogical goal. We characterize digital tools as *specialized* if they are developed in relation to a specific learning trajectory and *generalized* if they are developed for more general use across a range of pedagogical moments and/or mathematical topics. Examples of highly generalized digital tools are dynamic geometry environments, graphing calculators, and spreadsheet environments. These kinds of digital tools place relatively few constraints on the user's actions and are not necessarily designed to support students in learning a particular mathematical topic. A tool becomes more specialized by placing constraints on students' actions, designing feedback that orients the student toward a particular pedagogical goal, or pairing the tool with a particular mathematical task.

The spectrum of Specialized ↔ Generalized can be elucidated with the example of a definite integral and its connection to the area under the curve. On the specialized end of the spectrum, a digital tool could be hard-coded with a given accumulation function and specific numerical values for the endpoints of the interval over which the area is collected. The graph could further

be linked to a specific context, such as accumulation of distance. Should a designer choose to allow the endpoints to be moved through the frontend of the application, or allow students to choose different accumulation functions, then the digital tool can be used in a broader number of tasks. Yet further down the spectrum, if the accumulation of distance context is dropped, and all possible endpoints and functions are able to be edited by the user, then this can become a highly generalized tool, able to be used in almost any application of definite integrals. However, designing such a tool involves a tradeoff, as it is no longer as directly aligned with particular pedagogical goals for specific tasks.

Discussion and Conclusion

In this theoretical paper, we propose a framework that structures a design space for digital tools to support mathematics learning. Together, all three spectra create a three-dimensional design space in which digital tools can be characterized. To give an idea of how these dimensions coalesce, we give a few prototypical examples of digital tools, and how they would be placed in the framework in a two-dimensional subspace, shown in Figure 2. The Specialized ↔ Generalized axis has been flattened for readability. In the rest of this section, we discuss the implications of the framework and provide an illustrative example of how the framework can support generalization about the design of digital tools across studies from different contexts.

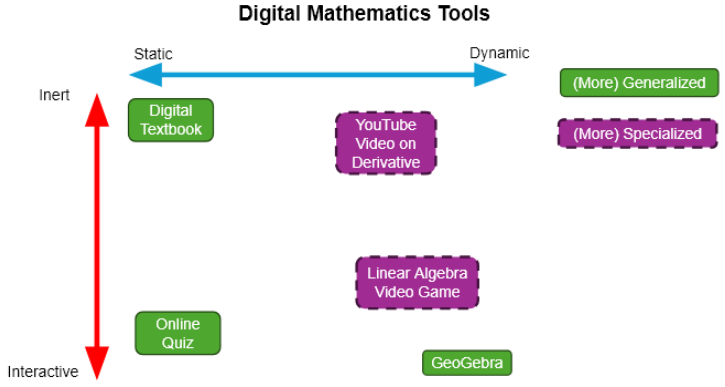


Figure 2. A two-dimensional representation of the design space structured by our proposed framework with examples of digital tool types.

One strength of this framework is its ability to distinguish between digital tools that are quite similar (e.g., built within the same software environment, developed to support the same mathematical goals). The examples shown in Figure 1 are all interactive digital tools that vary in their level of dynamism. According to existing taxonomies of digital tools, these tools would be quite difficult to distinguish. For example, each of the tools could be used to support students in developing concepts (Drijvers, 2015); they are all dynamic and graphical tools (Noss & Hoyles, 2009); and they are all built within a mathematical action software environment (Dick & Hollebrands, 2011).

Despite these similarities, each tool positions students differently with respect to the mathematical ideas at stake. The relationship between a student's perception of time passing and the changing state of mathematical objects embedded within a digital tool is under-theorized but beginning to receive more attention (see Moore, 2024). Early research indicates that the design of digital tools can be linked to students' conception of time as it relates to their conception of a situation involving covarying quantities (Margolis, 2024).

On the other hand, this framework can also be used to draw out similarities between digital tools that are otherwise quite different. Recall that Baccaglini-Frank and Mariotti (2010) wrote about students' interaction with draggable points used to define an interactive geometric diagram embedded within a dynamic geometry environment. Students interacted with the digital tool via a modality called Maintaining Dragging, in which they attended to the trace of the point as they dragged it in such a way that some geometric relationship within the diagram remained invariant.

According to our proposed framework, the digital tools in Baccaglini-Frank and Mariotti's (2010) study would be Specialized, Interactive, and Weakly Continuously Dynamic. In Figure 3, we show a digital tool which was designed to provide students with opportunities to engage in covariational reasoning with graphs and mental constructions of situational quantities (for more details about this task sequence, see Paoletti, 2019). Students are prompted to drag the coordinate point–intended to simultaneously represent the temperature and amount of water flowing out of the faucet–in such a way that only one knob moves at a time. To complete this task, students have to engage in the modality of Maintaining Dragging. And yet, the faucet tool is not built within a dynamic geometry environment and is not designed to support students in making conjectures about Euclidean geometry.

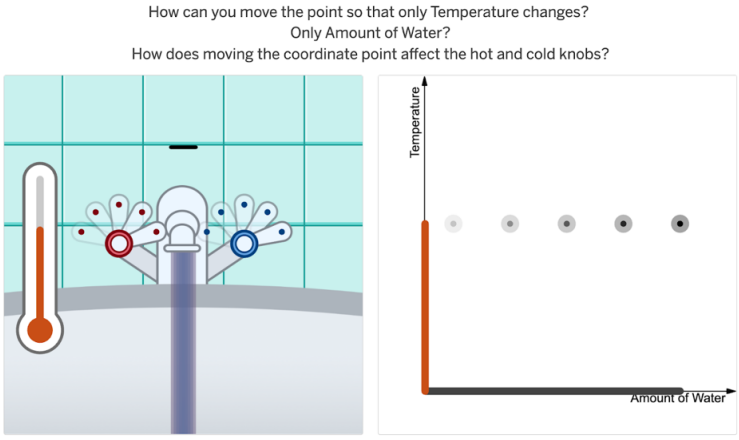


Figure 3. A static image of a student engaging in Maintaining Dragging (i.e., dragging the coordinate point in such a way that the temperature stays constant) with a digital tool.

Although we find this preliminary framework to be promising in our efforts to study the design of digital tools, we do not consider it to be complete. We anticipate that the framework will grow over time as we expand our theorization and analysis of the design of digital tools. For example, we anticipate that a category about the connectivity that digital tools afford communities of learners will be added to the framework soon.

Finally, we consider this framework to be useful for analysis of pre-existing digital tools while also being useful for the process of digital task design. We recommend further research on the affordances of tools across various locations in the design space for students learning.

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Reflections on Creativity and Affect as a Vehicle for Writing in a Mathematics Course

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We explore how one instructor used reflective prompts in a proof-based Elementary Number Theory course to add a writing component to the course. Students consistently reflected on their creative process and affective domain throughout the course. We share the design of the course, samples of the reflective prompts, and our deductive coding of them using two codes: creative process and affect. A preliminary examination of written responses from two students indicates that reflecting on their creative process had led them to be more engaged in the proving process in this course and other courses. They also reported that the readings and reflections helped them recognize and validate their mathematical actions, processes, and affective outcomes.

Keywords: Writing in Mathematics, Mathematical Creativity, Affect, Reflections

Writing has long been considered a unique and valuable mode of learning (Emig, 1977). To promote writing at the tertiary level, institutions have utilized two ways of fostering students' writing skills: writing in the discipline (WID) and writing across the curriculum (WAC). In WID courses, "writing is occurring in some form as assignments in subjects or courses in one or more disciplines" (Thaiss & Porter, 2010, p. 538). WAC typically is meant to "assist teachers across disciplines in using student writing as an instructional tool in their teaching" (Thaiss & Porter, 2010, p. 538). Using both types, instructors can tailor writing assignments to develop the specific content of a discipline as well as promote students' curiosity, creativity, metacognition of processes, affect, and expression of target material through a student's individual framework.

In this study, we focus on writing in a proof-based undergraduate Elementary Number Theory course. We report on how one instructor designed and implemented various forms of writing assignments in this course. Writing is particularly suited to this course because it offers students opportunities to write proofs through generating examples, using mathematical definitions and theorems, utilizing various symbolic representations, connecting assumptions to conclusions, and explaining reasoning. While infusing writing in proof-based courses has been researched and documented, we highlight how a focus on students' creative processes and affect can both increase student engagement with content and enhance their emotional value in process, while serving as a vehicle for transforming this course into a writing course. Specifically, we address the following research questions: *How to infuse written reflective assignments on creativity and affect into a proof-based math course? How do students interact with and perceive these assignments?*

Background Literature and Theoretical Framing

Writing in Mathematics

It has been reported that writing in mathematics courses provides students with a deeper understanding of mathematical concepts (Rose, 1989; Beidelman et al., 1995; Burns, 2004). Burns (2004) stated, "Writing in math class supports learning because it requires students to organize, clarify, and reflect on their ideas" (p. 30). Burns added, "when students write, their papers provide a window into their understandings, their misconceptions, and their feelings about the content they're learning" (p. 30). Existing literature shows that writing in mathematics courses (e.g., Urquhart, 2009; Taylor & McDonald, 2007; Graham et al., 2020) can take various

forms. Writing assignments can range from short reflective questions (e.g., El Turkey et al., 2023) to daily journals in which students keep their reflections and insights on the course topics (Borasi & Rose, 1989). Other forms include summary writing, writing portfolios, and expository essays. For example, Omar et al. (2019) used writing portfolios in a Combinatorics course where students communicated their solution attempts as well as their reflections on their problem-solving process and creativity. Expository essays are usually assigned so students explain mathematical concepts to others, most often to the teacher.

Many studies explored the use of writing in various mathematics courses. Some discussed writing assignments in general education mathematics courses (e.g., Pinter, 2014), History of Mathematics courses (e.g., Loats et al., 2014), Abstract Algebra (Czerwinski, 1994), Modeling (Linhart, 2014), Complex Analysis (Habre, 2019), or Combinatorics (Omar et al., 2019). In Omar et al.'s study (2019), students were prompted to reflect on their creative process, but writing was not the focus of the study. Beyond that study, we did not find literature focused on infusing writing in a mathematics course using reflections on creativity and affect.

We situate our discussions of using reflections and writing as modes of learning and integratory tools within expressionism (Elbow, 1991), as writing provides space between thought and expression to process the emotional value of information (Flower et al., 1992; Overstreet, 2022). This emotional value is essential to both motivating students to examine target material, and in spurring them to creative thought within a discipline (Auger & Woodman, 2016).

Mathematical Creativity

The creative process for problem solving was described by Wallas (1926) through a four-stage process: preparation (understanding the problem and gathering relevant information for a proof), incubation (taking a break and letting the mind work on a proof subconsciously), illumination (AHA moment, insight, or generating an idea), and verification (verifying the idea is a solution or a proof). Later, Guilford (1950) developed three components of creativity: fluency (number of ideas), flexibility (changing ideas or approaches), and novelty (unique or original ideas/approaches). Torrance (1974) added a fourth component of creativity, elaboration (describing or elaborating on those ideas). These earlier studies led to a plethora of definitions and perspectives of mathematical creativity (Mann, 2006). Some focused on the creative process (Kozbelt et al., 2010) while others discussed a product view (Runco & Jaeger, 2012). Additional studies discussed mathematical creativity that is relative to students versus absolute creativity relative to the field of mathematics (Beghetto & Kaufman, 2013; Leikin, 2009).

In this paper, we adopt a process-oriented relativistic perspective that defines mathematical creativity as a process of offering new solutions or insights that are unexpected for the student, with respect to their mathematics background or the problems they have seen before (Liljedahl & Sriraman, 2006; Savic et al., 2017). Thus, we acknowledge that students "have moments of creativity that may, or may not, result in the creation of a product that may, or may not, be either useful or novel" (Liljedahl, 2013, p. 256). This view of mathematical creativity aligns with the developmental theoretical perspective emphasizing that a person's creativity can and does develop over time and a person's process and actions, as opposed to the product created, should be the focus of investigation (Kozbelt et al., 2010).

Affect

We approach affect broadly in this study following recommendations from the literature (e.g., Tang et al., 2022). Grootenboer and Marshman (2016) acknowledged the difficulty in defining the affective domain but conceptualized it largely as beliefs, values, attitudes, and

emotions that overlap. For this study, affect encompasses feelings (McLeod, 1988) and other constructs such as beliefs, attitudes, emotions, motivation, engagement, and so forth (Middleton et al., 2017). We also adopt Tang et al.'s (2022) definition of an affective outcome as a "collective of any emotions, beliefs, attitudes, and values that students reference" (p. 31). Tang et al. additionally highlighted the importance of examining the intersection of students' mathematical creativity and their affect, which we undertake in this study through the design and implementation of reflective prompts attending to both constructs.

Methods

Study Setting

The study took place in an Elementary Number Theory course at a comprehensive northeastern university in the United States. Ten students were enrolled in the course which met three times a week (50 minutes) on Zoom in Fall 2023. The instructor (first author) characterizes the classes as interactive lectures, each of which began with time for student questions, a brief introduction of new content (on OneNote using a stylus), followed by elicitation of target material through student discussion. In this course, the instructor created and assigned three types of written assignments: pre-class assignments (PA) (N=35), homework assignments (HW) (N=10), and a complete proof portfolio for all proofs from class (N=1). In addition, there were two exams and a final exam. Five of the PAs (PA 8, 11, 18, 23, 35) included reflective questions whereas each HW assignment had the last question as a reflection prompt. Reflective questions were graded based on completion, with some comments from the instructor.

Data

The data collected for this qualitative study includes the assignments, students' work on them—including their answers to reflective questions—and end-of-term interview data with two students. We focus in this paper on the assignments themselves to explore the instructor's design intentions as our unit of analysis to document how the instructor promoted writing within this course. Additionally, we share sample written responses from the two interviewed students.

Data Analysis

Reflective assignments. Using deductive coding (Patton, 2002), we coded the reflective questions (N=15) on the pre-class assignments and the HW assignments. The codes used were creative process and affect. The creative process code encompassed any prompts that referenced actions from Creativity-in-Progress Reflection Tool (Savic et al., 2017) (7 prompts), mathematical creativity in general (3 prompts), Wallas's four-stage process (2 prompts), and AHA/memorable moments (2 prompts). The affect code encompassed any prompts about feelings (6 prompts), attitudes (1 prompt), math story/journey (1 prompt), and teaching actions related to affect (2 prompts). It is important to note that some prompts were coded with both codes. Below we provide a sample of reflective questions with their codes in brackets.

Students' written responses. We report on written data from two student participants who also volunteered for the end-of-semester interview: Amity[1] (pronouns she/they; mathematics major, senior, final grade A) and Smiley[2] (pronouns she/her; actuarial science major, senior, final grade B). In a word count of all their responses to the reflective questions, Amity used around

[1] Self-selected pseudonym

[2] Self-selected pseudonym

2700 words while Smily used around 1600 words. These were in addition to the mathematical writing that they submitted on the various assignments. In the following, we share our deductive coding of their written responses to three reflective questions. We underline certain parts of their responses to capture the creative process theme and use *italics* to capture the affect theme.

> *HW 9* [Creative Process & Affect]: Becoming more aware of creativity and how it functions can lead to deeper immersion in problem-solving or proving processes. Has reflecting on your creative process made you feel more engaged in these processes? Explain (Using specific examples is encouraged.)

In response, both students highlighted how reflecting on their creative process has led them to be more engaged in the proving process. Amity wrote, "trying to find different creative ways to do problems definitely makes me *feel more engaged* in trying to figure out the solutions or proofs and helps me understand it better." Smiley explained, "reflecting on my creative process has made me *feel more engaged* in the proving process. Sitting and actually thinking about how I am solving problems makes me *feel like I am actually apart* [sic] *of the proving process* instead of just skating through it." She then added, "Reflecting gives me the chance to edit and revise my proving process… For example, I now sit in the incubation stage more when I am proving, and I have found that it helps out a lot."

> *PA 11 (part 5)* [Affect]: Please discuss an affective outcome you felt while working on a problem or during class.

On this assignment, the students were asked to read parts of the Tang et al. (2022) article about students' affect when doing mathematics. In response to the above prompt, Amity wrote, "I *feel a lot of enjoyment* while doing work for this class. Figuring out how to prove something always feels like a puzzle to me, and when I finally figure it out it's *so satisfying* and *makes me excited* to do more work for the class." Smiley wrote, "I have *felt confidence* in class when you ask me a question that I am actually able to get right. It makes me *feel good* to know that I am understanding the concepts that are being taught."

> *PA 35 (part 5)* [Creative Process & Affect]: Do you think that creativity affected your actions in the course, e.g. what you do during or out of class when working on math, how you problem solve or prove, etc?

Both students indicated how being aware of their creative process helped them in the course as well as other courses. For example, Amity responded, "the creative mindset I've developed from this class has assisted me in my other math classes as well." She elaborated, "In my Complex Variables class, I have found myself becoming much more flexible in my problem solving and finding solutions when I get stuck on something." Smiley wrote, "creativity affected how I problem solve the most. Creativity allowed me to try numerous ways to solve a problem whether it was attempting different proof methods, finding counter examples, or just using trial and error." She added, "As I do problems in the future, I definitely feel like I will be more creative because of this course."

In one of the questions on this last reflective assignment, the instructor asked students about some takeaways they learned from the reflective questions. Amity wrote, "I learned that there are names for the different mental processes I use and how to better make them work together." Smiley stated, "my biggest takeaway was learning about the four stages of solving a problem." She then elaborated on the preparation and incubation stages of Wallas's four-stage creative process. She concluded, "I will definitely use this four-stage process in my other courses."

Preliminary Results

In addition to fostering students' writing in an advanced mathematics course, two main design intentions or instructional goals were captured in the reflective questions: 1) increasing students' awareness of their creative process; and 2) providing students with opportunities to recognize and acknowledge their affect while doing mathematics. The written responses from the two students indicated a recognition of their own creative process. They elaborated on how reflecting on their processes allowed them to better understand the content and be more engaged in the proving process. Their recognition of the creative process through reflections has also promoted flexibility in trying different attempts or proofs, which has assisted them in other math classes. In terms of affective outcomes, both students reported positive feelings such as enjoyment, excitement, feeling good, and confidence. They also linked the creative aspect of writing proofs to their positive feelings which aligns with findings in Tang et al.'s study (2022). We claim that our reflection prompts provided students an opportunity to closely examine their affect while doing mathematics, a practice that is uncommon in mathematics teaching and learning.

Discussion

The instructor designed numerous reflective prompts utilizing the Creativity-in-Progress reflection tool, Wallas's four-stage creative process, and affect. Through the various assignments, the instructor attended to both WID and WAC components of writing courses. While the context of the study was Elementary Number Theory, we hypothesize that the reflective prompts can be adopted and adapted to any upper-level mathematics course that involves problem solving or proving. It is important to note that the consistency in assigning reflective prompts throughout the semester established a norm and an expectation for students to 'sit and actually think' (Smiley) about how they solve problems. In addition, most of prompts were very specific and aimed at focusing students' attention on a specific mathematical action, an affective outcome, or a moment in the course or their mathematical journey.

In terms of limitations, the online nature of the course added a layer of difficulty in having in-class discussions about creativity and affect. While the instructor integrated conversations about creativity and affect during the Zoom class sessions, there would have been more engaging conversations if the class was in person. Additionally, only two students volunteered for interviews, which limited capturing various students' perspectives about the reflective questions.

Conclusion

Writing skills are central to most professional careers, especially in STEM disciplines. According to the CUPM Curriculum Guide, "Clear communication in oral and written forms is essential; students should develop these skills as undergraduates both through writing courses and by preparing written class and project reports" (CUPM, 2015, p. 47). As a mechanism to encourage writing in mathematics, reflections play an important role in mathematics learning as just completing tasks is insufficient (Wheatley, 1992). With these goals in mind, this study explored the design goals of reflective questions that prompted students to read, think, and write about their mathematical creativity, creative process, and affect. In the future, we will examine the students' interviews to learn about how they perceived those reflective questions. We also plan to provide some recommendations for instructors who plan to integrate writing into their mathematics courses.

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How do Mathematicians Explain Conditionals? Diagrams and Explanations for 'if $P(x)$ then $Q(x)$' and '$P(x) \Rightarrow Q(x)$'

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At the introduction to proof, Venn or Euler diagrams are commonly used to illustrate concepts in logic. Standard diagrams show $P(x) \wedge Q(x)$ as the intersection of truth sets for $P(x)$ and $Q(x)$, and $P(x) \vee Q(x)$ as their union. But textbooks rarely include a diagram for the conditional, and those that do use a diagram that differs from the standard form. Does this reflect typical mathematicians' thinking? What do they draw when asked for a diagram to represent the conditional, and does this depend upon sentence formulation? This preliminary report describes a pilot study in which mathematicians and mathematics PhD students were asked to draw a diagram and provide a short accompanying explanation for either 'if $P(x)$ then $Q(x)$' or '$P(x) \Rightarrow Q(x)$'. Results show that they all draw essentially the same diagram, representing a true universal conditional. I illustrate variety in the provided explanations.

Keywords: conditional, diagram, implication, logic

Introduction and Theoretical Background

Venn diagrams are common at the introduction to proof, where they appear in discussions of set theory and logic (Antonides et al., 2024). Diagrams for conjunction and disjunction appear in Figure 1, where the circles represent truth sets for $P(x)$ and $Q(x)$, so the shaded areas contain values of x for which the conjunction $P(x) \wedge Q(x)$ and the disjunction $P(x) \vee Q(x)$ are true.

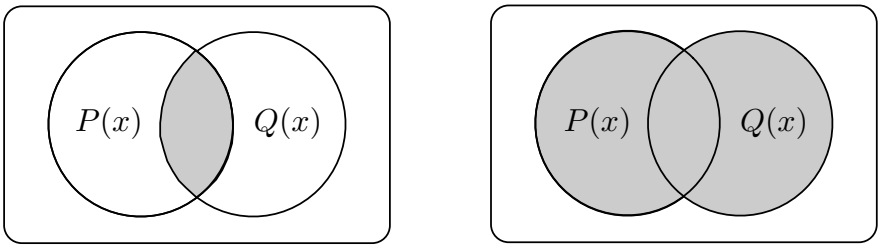


Figure 1. Venn diagrams illustrating conjunction and disjunction.

These diagrams reflect the truth values in Table 1, which additionally shows the truth table of the material conditional 'if $P(x)$ then $Q(x)$', as used in mathematics.

Table 1. Truth tables for conjunction, disjunction, and the material conditional.

$P(x)$	$Q(x)$	$P(x) \wedge Q(x)$	$P(x) \vee Q(x)$	if $P(x)$ then $Q(x)$
T	T	T	T	T
T	F	F	T	F
F	T	F	T	T
F	F	F	F	T

The material conditional is false when $P(x)$ is true and $Q(x)$ is false, and true otherwise, making 'if $P(x)$ then $Q(x)$' equivalent to 'not-$P(x)$ or $Q(x)$'. This leads to a Venn diagram for the conditional as in Figure 2 (Nickerson, 2015) (henceforth, a *truth-value diagram*).

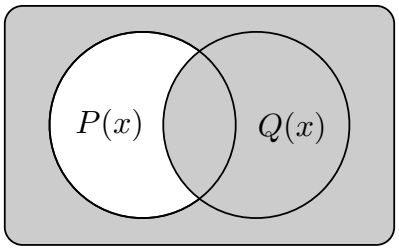


Figure 2. Truth-value Venn diagram for the conditional 'if $P(x)$ then $Q(x)$'.

In everyday language, the material conditional is natural in that we would assert 'if P then Q' when we would assert 'either not P, or P and Q' (Jackson, 1979). But people commonly judge not-P cases as irrelevant to the conditional rather than as cases in which it is true (Newstead et al., 1997). Moreover, they commonly interpret a conditional as asserting an inferential link between its antecedent and consequent, so that 'missing-link' conditionals ('If bats have wings then Paris is in France') sound like nonsense (Bourlier et al., 2023; Douven et al., 2020).

In mathematics, the defined propositional conditional is the material conditional (Durand-Guerrier, 2003). Yet diagrams like that in Figure 2 are rarely seen. Of the 17 introduction-to-proof textbooks most commonly recommended in the UK and the US (Alcock & Sa, in press), only two include a diagram in their explanations of conditionals, and neither use a truth-value diagram. Instead, both give Euler diagrams in which the truth set of $P(x)$ is contained in the truth set of $Q(x)$, as in Figure 3 (henceforth, *inclusion diagrams*).

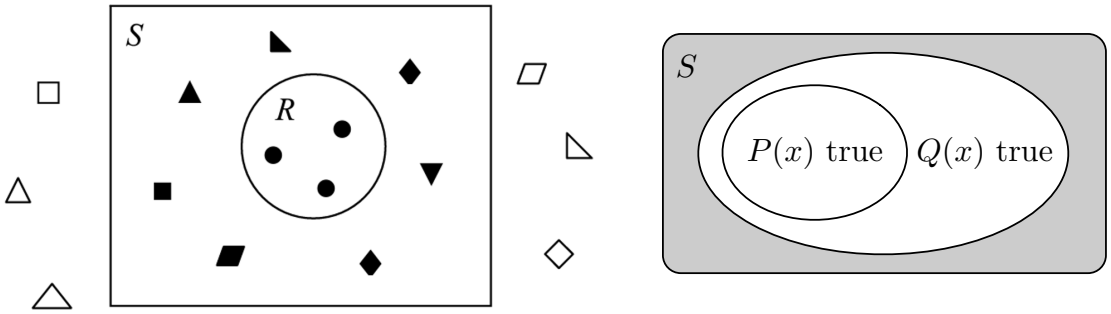


Figure 3. Diagrams for conditionals, formulated as $R \Rightarrow S$ (round ⇒ solid) in Lay (2014, p.5) and as $P(x) \Rightarrow Q(x)$ in Stewart & Tall (2015, p.134).

In inclusion diagrams, the idea of representing all possible combinations of truth values for $P(x)$ and $Q(x)$ is abandoned in favour of a diagram in which the conditional 'if $P(x)$ then $Q(x)$' is universally true (there are no x values for which $P(x)$ is true and $Q(x)$ is false). The reason for this is suggested in the explanation offered by Stewart and Tall:

> "We are really interested only in the case where $P(x) \Rightarrow Q(x)$ is true for all x. In this case, if $P(x)$ is true, so must $Q(x)$ be; that is, if $a \in \{x \in S | P(x)\}$ then $a \in \{x \in S | Q(x)\}$ which means $\{x \in S | P(x)\} \subseteq \{x \in S | Q(x)\}$." (Stewart & Tall, 2015, p.134)

The claim that mathematicians are primarily interested in true universal conditionals is arguably reflected in other textbooks' use of 'P implies Q' or '$P \Rightarrow Q$' as straightforward alternatives to 'if P then Q' (Alcock & Sa, in press). The language of implication puts mathematicians at odds with philosophers, who usually denote the material conditional '$P \supset Q$' ('P hook Q'; Edgington, 2020) and are wary of ambiguity around '⇒' and 'implies', reserving these for argument forms or avoiding them altogether (Smith, 2020). But the mathematical use corresponds comfortably

to natural language for true universal conditionals: where 'for all x, if $P(x)$ then $Q(x)$' is true, $Q(x)$ can be deduced from $P(x)$ (e.g., Epp, 2004).

Of course, it could be that the diagrams in Figure 3 are idiosyncratic: perhaps mathematicians do not typically represent these constructs in this way. Or it could be that formulation matters: perhaps '$P(x) \Rightarrow Q(x)$' invites an inclusion diagram but 'if $P(x)$ then $Q(x)$' would be more likely to elicit a truth-value diagram. I therefore intend to investigate diagrams produced by a wider sample of mathematicians, along with explanations they provide for those diagrams, and features that they and students value in diagrams and explanations produced by others. This preliminary report describes a pilot study addressing three research questions:

1. What Venn/Euler diagrams do mathematicians produce for the conditional?
2. Does this vary between the formulations 'if $P(x)$ then $Q(x)$' and '$P(x) \Rightarrow Q(x)$'?
3. How do mathematicians explain their diagram(s)?

Method

Mathematicians were recruited at the 2024 British Mathematical Colloquium, where attendees are mathematicians and PhD students working largely in pure mathematics. They participated voluntarily in response to invitations in an email and a talk. Each participant read an information sheet and signed a consent form that asked for their main area of mathematical work (pure, applied, statistics/data science, other) and their experience level (PhD student, postdoc, lecturer[1], other). Fourteen people participated, of whom:

- 9 were PhD students in pure mathematics (PhD P in Table 2);
- 1 was a PhD student in applied mathematics (PhD A);
- 1 was a postdoc in pure mathematics (PD P);
- 3 were lecturers in pure mathematics (Lec P).

Participants responded (under no time constraint) to the prompt:

> Please draw a Venn/Euler diagram to represent 'if $P(x)$ then $Q(x)$' ['$P(x) \Rightarrow Q(x)$'].
> Then add a short explanation suitable for undergraduate students.

Half received the 'if $P(x)$ then $Q(x)$' prompt and half the '$P(x) \Rightarrow Q(x)$' prompt; all had most of a page in which to respond. Their responses were analysed qualitatively as below.

Analysis and Results

To address Research Question 1, I first analysed the diagrams. This was a quick process because all participants drew inclusion diagrams. Two then added a truth-value diagram, and one of these offered two further alternative diagrams; these I will present at the conference. The inclusion diagrams varied in three respects:

1. Some labelled the sets $P(x)$ and $Q(x)$ and some just used P and Q;
2. Some framed the sets within a universal set and some did not;
3. Some included extra labelling or shading.

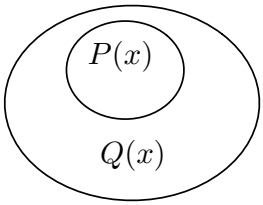
Table 2 summarises the features of each inclusion diagram and shows which participants included a truth-value diagram. It also shows that the answer to Research Question 2 appears to be that formulation of the conditional does not affect the Venn/Euler diagrams that mathematicians draw, at least not in the way anticipated. Participants responded with the same (first) diagram regardless of the prompt, and the two truth-value diagrams appeared in response to the formulation '$P(x) \Rightarrow Q(x)$', not 'if $P(x)$ then $Q(x)$'. Of course, this is a small sample.

[1] In the UK, the term 'lecturer' is equivalent to the North American 'professor', and people specialise early so that a PhD is a research-only degree.

Table 2. Features of inclusion diagrams: TV = provided a second, truth-value diagram.

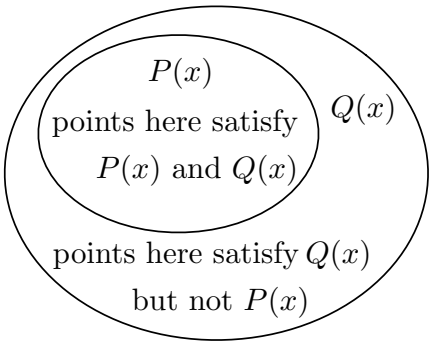
Prompt	Participant	Labels	Univ	Extra labels/shading	TV	Explanation
if-then	PhD P	$P(x)$				Basic
if-then	PhD P	$P(x)$				Converse
if-then	PhD P	$P(x)$		P shaded		Background
if-then	PhD P	P	Yes			Converse
if-then	PhD P	P	Yes	specific points		Converse
if-then	PD P	P				Basic
if-then	Lec P	$P(x)$				Background
⇒	PhD P	$P(x)$				Converse
⇒	PhD P	P				Converse
⇒	PhD P	$P(x)$		'x for which $P(x)$ holds'	Yes	Basic
⇒	PhD P	P	Yes			Basic
⇒	PhD A	$P(x)$		'this area is a subset of this area'		Background
⇒	Lec P	$P(x)$		'points here satisfy $P(x)$ and $Q(x)$'		Converse
⇒	Lec P	$P(x)$		$Q(x)$ shaded, $P(x)$ cross-hatched	Yes	Basic

To address Research Question 3, I analysed the particpants' explanations for the inclusion diagrams (Table 2, last column). These varied primarily in whether and how they went beyond a basic explanation of the diagram's structure. Some gave just that basic explanation, as in Figure 4. Some additionally noted that the converse of the conditional might not be true, as in Figure 5. Some provided comments on the background assumed when mathematicians read and write conditionals (cf. Dawkins & Roh, 2024), as in Figure 6.



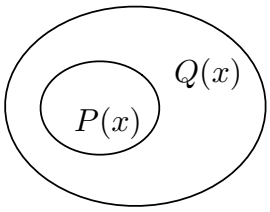
"if $P(x)$ then $Q(x)$" means that whenever P is true, Q is also true. So every x in the "P circle" has to be in the "Q circle".
Therefore, "P circle" should be inside of "Q circle".

Figure 4. Diagram with basic explanation.



A 'bubble' represents the set of points which satisfy a property. In this case, $P(x)$ or $Q(x)$. Since $P(x) \Rightarrow Q(x)$, all points who satisfy $P(x)$ also satisfy $Q(x)$, hence the inclusion, but there might be points satisfying $Q(x)$ which do not satisfy $P(x)$.

Figure 5. Diagram with significance of area outside inner set.



Note that "$P(x) \Rightarrow Q(x)$" is not a statement (or sentence) as it contains a free variable. So, given a suitable structure and a point, say α, in that structure we can test whether the premise $P(\alpha) \Rightarrow Q(\alpha)$ holds in the structure or not. Namely, whether property P holding at α "implies" that Q holds at α.
Also note that by adding a quantifier we can make this into a statement; for example: "$\exists x\ P(x) \Rightarrow Q(x)$" or "$\forall x\ P(x) \Rightarrow Q(x)$". Normally, or most commonly, we look at the latter, that is
"$\forall x\ P(x) \Rightarrow Q(x)$".
This, in a given structure, is to be interpreted as for all elements property P implies property Q. In other words, the elements that satisfy property P form a subset of those elements satisfying Q.

Figure 6. Diagram with explanation including assumed background.

Discussion

This pilot study investigated diagrams mathematicians draw for the conditional, whether these diagram are affected by formulation of the conditional, and what explanations mathematicians offer for their diagrams. Regardless of formulation, participants all drew inclusion diagrams first, with a small minority adding truth-value diagrams. Both diagrams and explanations reflected those offered in Stewart and Tall's (2015) book in that mathematicians appear to think primarily about true universal conditionals; some additionally explained non-equivalence of the converse and some added material on background assumptions about propositions and quantification.

I plan to extend this work in two directions.

First, I will conduct a study in which a larger number of mathematicians respond to the same prompts, except with a note that they can, if they wish, include more than one diagram. I expect this to confirm that inclusion diagrams are favoured. I will then ask both mathematicians and undergraduates to use comparative judgement to rate the quality of (a selection of) the resulting explanations, coupling the results with a content analysis to identify and compare features of diagrams and explanations that are valued by the two groups (cf. Davies, Alcock & Jones, 2021).

Second, I will ask mathematics students in introduction-to-proof courses to select which of a range of diagrams best represent '$P(x) \wedge Q(x)$', '$P(x) \vee Q(x)$' and 'if $P(x)$ then $Q(x)$'. Asking about all three connectives will permit more sensible inclusion of a wider range of diagrams and might reveal non-standard interpretations. For '$P(x) \wedge Q(x)$', one can imagine that students might select the diagram on the right of Figure 1. For 'if $P(x)$ then $Q(x)$', they might select an inclusion diagram with $Q(x) \subseteq P(x)$ – this would be natural if they imagine restricting attention to P cases and asking whether these must also be Q cases, as in the Ramsey test (Ramsey, 1929) (one participant in this pilot study drew such diagram then crossed it out). This work might also reveal consistency or otherwise in how students interpret shading in relation to truth sets.

I will report progress of research in both directions at the conference.

Acknowledgments

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A Systematic Review of the Teaching and Learning of Quantum Mechanics: Implications for Mathematics Instruction and Research on Undergraduate Mathematics Education

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In this work, we review the literature on the teaching and learning of quantum mechanics situated in chemistry courses with the aim to provide a tool for education researchers and practitioners. We found that much of the research on quantum mechanics in the chemistry context involves students at the secondary level of instruction. Along with this, much of the research leans on mathematics and physics education research to support the claim that students have difficulties with mathematics; however, these claims require further investigation within chemistry education research.

Keywords: Quantum Mechanics, Chemistry, Interdisciplinary

Introduction

Physical chemistry is an upper-division course in the undergraduate chemistry curriculum that is notoriously challenging for students as it broadly focuses on three fundamental chemical concepts: kinetics, thermodynamics, and quantum mechanics (Holme, 2023). Briefly, kinetics is the study of rates of chemical reaction, thermodynamics is the study of energy and its movement in chemical systems, and quantum mechanics is the modeling of atomic and molecular structure. Many studies have been conducted investigating how students learn about thermodynamics (Bain et al., 2014) and kinetics (Bain & Towns, 2016) and have been reviewed to provide a tool for instructors and researchers teaching physical chemistry. In the review on thermodynamics, Bain et. al (2014) found that the literature investigating thermodynamics often posed alternative conceptions for the laws of thermodynamics, along with factors related to success in physical chemistry. Another finding in this study was that many papers focused on the mathematics used in physical chemistry, posing the need for interdisciplinary research collaborations. In Bain & Towns (2016), when reviewing chemistry education research on kinetics, they found that kinetics studies largely focus on secondary-level students' reasoning about chemical concepts. Along with this, they once more call for cross-disciplinary research, specifically between chemistry and mathematics, on students' reasoning of related rates and graphical representations (Bain & Towns, 2016).

The remaining topic in physical chemistry, quantum mechanics, has yet to be reviewed with a focus on the teaching and learning in chemistry contexts; however, outside the context of chemistry, quantum mechanics is a content area that has been widely studied in physics education. For example, a review of the physics education research regarding quantum mechanics found that students have difficulties understanding chemical concepts such as wave-particle duality, wave functions, and atomic structure. Along with this, students were found to have conflating ideas about classical and quantum mechanics (Krijtenburg-Lewerissa et al., 2017).

With this in mind, we aim to synthesize the chemistry education literature that investigates the teaching and learning of quantum mechanics in a chemistry context, with the goal of sharing our implications for practitioners and researchers. In this review, we identify the areas for further research and provide a detailed outline of the mathematical concepts and skills that are utilized in physical chemistry to provide insight to undergraduate-level mathematics instructors.

Data Collection and Analysis

Our approach for identifying articles to include in this review followed guidelines for reporting systematic reviews (Page et al., 2021). Articles were required to be peer-reviewed, empirical research studies (i.e., with research questions/aims, methods, and findings) that primarily focused on quantum mechanics topics, including both secondary and tertiary educational contexts. An overview of the data collection is provided in Figure 1. We searched the ERIC database, *Chemistry Education Research and Practice (CERP),* and the *Journal of Chemical Education (JCE)* using the search terms "chemistry" and "quantum mechanics", with our search last updated on January 13, 2025. After removing duplicates and articles that did not meet the inclusion criteria, we iteratively reviewed the citations and included additional articles that were within the scope of the review. In all, N = 50 articles were included in the review.

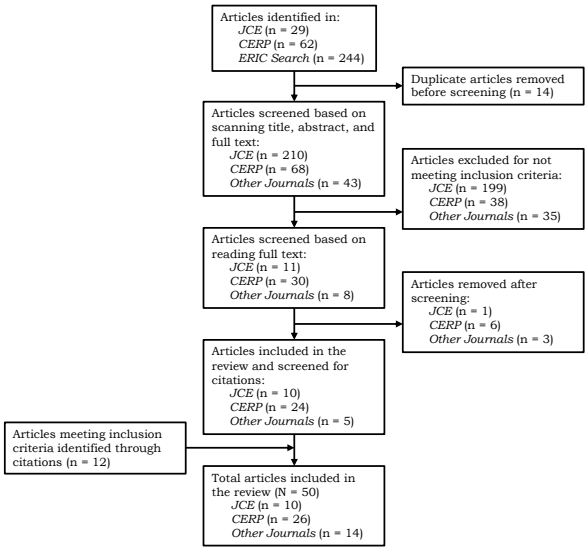


Figure 1. A PRISMA flow diagram detailing the article selection process. JCE is the Journal of Chemical Education, CERP is Chemistry Education Research and Practice, and ERIC is the Education Resources Information Center.

Once the dataset was finalized, we read the articles and grouped them based on the main topics of the articles. Throughout the process of reading the included articles, we took detailed notes and memos on the key findings to identify preliminary categories that arose. The first three categories that emerged were related to the quantum chemical concepts that the articles investigated: (1) Students' reasoning related to atomic and molecular structure; (2) Students' reasoning related to spectroscopy; and (3) Students' reasoning related to commonly discussed quantum mechanical models (e.g., particle-in-a-box, quantum mechanical harmonic oscillator, etc.). Any article that mentioned multiple categories (i.e., students' interpretations of the particle-in-a-box quantum mechanical model and atomic structure) was included in both relevant categories. In consideration of space, we will only discuss findings related to students' interpretations of quantum mechanical models, contextualizing it for the mathematics education research community.

Preliminary Findings

Overview of the Data

The findings presented in this section discuss the preliminary categories stemming from analysis of the data. To begin, we briefly provide an overview of the sample. We found that 48% (n = 24) of the articles used qualitative methods, 6% (n = 3) of the articles used quantitative methods, and 46% (n = 23) were mixed-method. Along with this, we identified the country that where the research was conducted, along with the associated participants' level of instruction. Most of the studies in our review were conducted in the United States (40%, n = 20) with many conducted in Greece (28%, n = 14). As for level of instruction, we found that articles typically involved students in introductory chemistry courses (Figure 2); however, a notable number of the articles investigated students' reasoning at the secondary level.

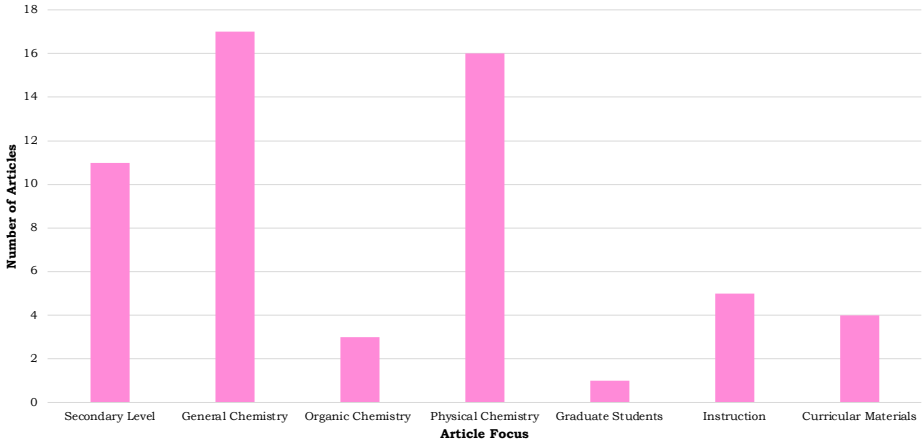


Figure 2. A bar graph showing the focus for each article. The total number of articles in this graph is greater than number of articles in the review, since each article may have multiple datasets.

Students' Reasoning About Commonly Discussed Quantum Mechanical Models

We discuss the key findings related to how students reason about common quantum mechanical models that are presented in a physical chemistry course. Though there are few, for clarity, we structure our discussion around the quantum mechanical model that is under focus.

Particle in a Box. Three articles in the sample focused on students' interpretations of the particle-in-a-box quantum mechanical model (Beck et al., 2020; Muniz et al., 2018; Muniz & Oliver-Hoyo, 2014). This model is often used in physical chemistry courses when introducing quantum theory to students as it describes the translational movement of a particle in a theoretical, one-dimensional box. To describe the particle's movement, students are taught their first application of the Schrödinger Equation, which is an eigenfunction that involves the use of partial derivatives. This eigenfunction takes the sum of the particle's kinetic and potential energy, called the Hamiltonian, and multiplies it by a wavefunction to produce the energy eigenvalue of the model. However, for the particle in a box model, the particle is assumed to have no potential energy, only kinetic energy. During classroom discussions, it is common in physical chemistry courses to discuss the mathematical treatment necessary to normalize and square the wavefunction to find the probability of locating the particle in its one-dimensional box which can be difficult for students. This typically includes walking through partial derivatives and integrations of the wavefunction with students. When students are learning about the particle-in-a-box model, there are many difficulties that have been identified in the literature. The primarily reported challenge is students' conceptualizing that more complex models are more suited, or more accurate, than simpler models (Muniz et al., 2018). Students were also found to struggle with the representations that are used in this model (e.g., eigenfunctions, etc.)

along with conceptualizing the meaning of quantum number *n* (Muniz & Oliver-Hoyo, 2014). It was also observed that students tended to conflate the particle-in-a-box model with other models, which leads to misapplications of the model to real-world phenomena (Beck et al., 2020; Muniz et al., 2018).

Quantum Mechanical Harmonic Oscillator. There were a few studies that investigated students' interpretations of another model, the quantum mechanical harmonic oscillator (Beck et al., 2020; Muniz et al., 2018). This model is another fundamental quantum mechanical model that describes the vibrational movement of particles, which is often taught after students learn about the particle-in-a-box model and requires a similar mathematical treatment. The main difference with this model from the previously described particle-in-a-box model is that vibrational movement requires students to consider how to model kinetic and potential energy with the Schrödinger equation. As discussed previously, students have shown difficulties with model application and model conflation (Beck et al., 2020; Muniz et al., 2018). Along with this, students tended to avoid mathematizing models when possible (Muniz et al., 2018), and when they do use math, they focus primarily on surface features of quantum mechanical models, both lexical (e.g., the language used when describing the model) and complex (e.g., the intensity of the mathematics that is shown in the model) (Muniz et al., 2018). When describing the kinetic and potential energies of the harmonic oscillator, students used classical mechanics to build explanations (Beck et al., 2020).

Two-Dimensional Rigid Rotor. There were a few studies that investigated students' interpretations of the two-dimensional rigid rotor model (Beck et al., 2020; Muniz et al., 2018). The two-dimensional rigid rotor is taught sequentially after the quantum mechanical harmonic oscillator. This model describes the rotational movement that a particle can experience. This model requires a similar mathematical treatment of the Schrödinger equation, including the use of the potential energy component of the Hamiltonian. The main difference in the mathematical treatment of this model is that it includes polar coordinates in the wavefunction. In these studies, students were found to conflate vibrational and rotational motion when applying the model (Beck et al., 2020; Muniz et al., 2018). When using the two-dimensional rigid rotor system to predict and explain molecular rotation, students were found to focus on incorrect aspects of the model causing misapplication of the model (Beck et al., 2020).

Discussion and Next Steps

The three quantum mechanical models described above are not the only models that physical chemistry students learn. After instruction on these introductory models, students move on to find the analytical solution to the hydrogen atom using the Schrödinger equation, discuss the challenges with quantum mechanics that arise with the three-body problem, and learn approximation methods that allow for exact solutions to the Schrödinger equation when modeling multi-electron atoms. As can be seen in our findings, there is a need in the literature for more studies that investigate students' interpretations of the models in physical chemistry.

The models that are taught in physical chemistry require a robust amount of mathematics skills. However, the included studies rarely investigated students' conceptualizations or reasonings regarding the mathematics that they are applying in physical chemistry. When comparing our quantum mechanics review to the quantum mechanics reviews that exist in physics education literature (Krijtenburg-Lewerissa et al., 2017), there is a jarring difference in the way mathematics is discussed. In the physics education reviews on quantum mechanics, findings from investigations conducted on students' use of specific mathematical operations are discussed in detail whereas these findings are largely absent from our review. Instead, the articles

in our sample rely on this prior work as support for understanding and explaining physical chemistry students' difficulties with quantum mechanics.

Although physics and mathematics education research has provided a clear pathway and foundation for this research, there is a need for more research investigating chemistry students' interpretation and use of the mathematics present in physical chemistry courses. Considering the interdisciplinary nature of the topics in physical chemistry, it is necessary for more interdisciplinary education research to be conducted on these topics. With more interdisciplinary research collaboration, a better understanding of students' difficulties with the content in physical chemistry may provide new pathways for instruction. For instructors, we have presented on the mathematics concepts and skills that are relied upon in physical chemistry to clarify how they are used in chemistry courses. The next step for our project includes analyzing the remaining categories that emerged in our study to further review the chemistry education literature related to the teaching and learning of quantum mechanics.

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Can Interleaving Enhance Student Outcomes in Freshman Calculus?

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In contrast to traditional homework sets that typically group problems by topics, an approach known as blocking, interleaving consists in alternating the topics for each problem set. Studies in middle school mathematics and college physics report important benefits of interleaving on students outcomes. The goal of the present study is to investigate whether similar benefits can be observed in the context of college calculus. Students in two sections of a Calculus for Life Sciences course received either blocked or interleaved weekly homework. On two surprise quizzes given in the middle and at the end of the quarter, the interleaved group outperformed the blocked group (d=0.18 and d=0.33 respectively). On a quiz given one month after the end of the course, the interleaved group outperformed the blocked group on the second half of the quarter but not on the first half. Limitations of these results are discussed.

Keywords: student achievement, calculus, interleaving effect.

The United States faces an urgent need to improve mathematics and science education in order to meet the future demand of science, technology, engineering and mathematics (STEM) workforce (President's Council of Advisors on Science and Technology (PCAST), 2012; National Research Council, 2013). Over the years, there has been many calls for the development and adoption of evidence-based instructional methods in STEM courses, especially at the college level (e.g., National Research Council, 2000; American Association for the Advancement of Science, 2009). Poor performance in introductory math courses such as calculus has been linked to students' decisions to leave STEM majors (Chen, 2013; Ehrenberg, 2010; Hunter, 2019). Calculus is often considered a "weed out" course (Weston et al., 2019) and constitutes one of the most challenging hurdles for students pursuing STEM majors. Improving student outcomes in calculus is therefore of great importance.

There is strong evidence that active learning can improve student learning compared to traditional methods (e.g., Freeman et al., 2014; Code et al., 2014; Hake, 1998). However, active learning does not by itself guarantee that all students will automatically perform better or earn top grades. It is therefore necessary to investigate other teaching methods that can be used either by themselves or in combination with other instructional approaches such as active learning.

One component of instruction that is often overlooked by teachers and students is how to organize and sequence learning opportunities. A common way of organizing a course is to divide topics into blocks that are then studied and practiced individually, an approach known as blocking (Rohrer, Dedrick, & Hartwig, 2020). In contrast, interleaving consists in mixing the practice of different skills together. Thus, when completing an interleaved homework set, students practice both the individual skills in the current section and also the ability to identify which method (from across prior units) is appropriate to solve a given problem. This is not trivial because similar looking problems may require different strategies (e.g., $\int x(e+1)^x dx$ requires integration by parts whereas $\int e(x+1)^e dx$ requires a U-substitution) and choosing an appropriate strategy is often challenging for students (Ziegler & Stern, 2014). In addition, exams are typically interleaved as they cover a range of different topics and students are not explicitly told which concept or method they should use to solve a given problem. Students are therefore

expected to exhibit on exams a skill that has hardly (if at all) been practiced. There is a growing body of literature from cognitive science and education psychology showing the potential of interleaving to support student learning and performance. The goal of this study is to investigate how interleaving can be concretely implemented in a college Calculus for Life Sciences course and what its impact on student performance is.

Theoretical Background and Relevant Literature

The first study comparing interleaving and blocking for mathematical tasks was conducted by Rohrer and Taylor (2007). Participants in this study learned to compute the volumes of four little-known solids following either a blocked or interleaved schedule. In the blocked group, the participants read a tutorial on how to compute the volume of a given solid and then completed four practice problems on that solid. They repeated this pattern for the four solids. In contrast, in the interleaved group participants first read the four tutorials and then completed the practice problems that were interleaved, or in other words, mixed. In both groups participants received feedback after each problem in the form of the correct answer. The participants took part in two practice sessions that were one week apart. During these sessions the blocked group showed greater accuracy with an average of 89% versus 60% for the interleaved group. However, on the test that took place one week after the second practice session, the scores dramatically reversed with the blocked group having an average score of 20% and the interleaved group 63%.

Two non-mutually exclusive mechanisms have been proposed to explain the benefits of interleaving: the *spacing effect hypothesis* and the *discriminative contrast hypothesis*. Since interleaving consists in intermixing the practice of different topics, an interleaved schedule automatically creates some temporal spacing between two practice problems or practice sessions on a given concept. Many studies have shown the benefit of spacing repetitions over time rather than massing repetitions (e.g., Cepeda et al., 2006; Donovan & Radosevich, 1999; Bjork, 1994). The *discriminative contrasts hypothesis* proposes that juxtaposing problems on different concepts next to each other promotes the comparison between the concepts and makes their discriminative features more salient. Several studies support this hypothesis (e.g., Wahlheim et al., 2011; Sana et al., 2017) and two in particular (Taylor & Rohrer, 2010; Kang & Pashler, 2012) suggest that the benefits of interleaving do not solely come from the spacing effect.

Large-scale studies conducted in the classroom with authentic classroom materials have reported a large positive effect of interleaving. For example, Rohrer et al. (2020) conducted a study with 54 seventh-grade classes involving 787 students. The students practiced four different types of mathematical problems during 8 practice sessions. Each class either followed a blocked or interleaved practice schedule. Students then completed a test 33 days after the last practice session. On that test, the interleaved group significantly outperformed the blocked group with an average of 61% versus 38% (Cohen's d=0.83). While showing a large effect, this study focused only on four topics and was conducted in middle school. One question is thus whether a similar positive effect of interleaving can be observed in a course where all topics throughout the course are interleaved and where the material is more complex.

Samani and Pan (2021) addressed this question by interleaving or blocking homework sets in an undergraduate physics course. During four weeks, students would either complete blocked or interleaved homework sets. Then for the next four weeks they would switch to other the type of homework. Students received a surprise test two days after the end of each stage. On both tests the interleaved group outperformed the blocked group (Cohen's d=0.40 for the first stage, and d=0.91 for the second stage). One should note all problems were computational in nature with a numerical final answer (with units) and involved recalling the appropriate formula. One should

also note that no statistically significant differences were observed on the course exams. In this study students completed three homework sets per week for a total of 27 problems, which is more than the typical one homework set per week, and over 75% of them were seniors.

The goal of the present exploratory study is to test whether interleaving could bring similar benefits in an introductory Calculus for Life Sciences course in which students are mainly freshmen and sophomores and complete one homework set per week. Moreover, questions should also include non-computational questions such as writing the equations of a mathematical model based on the description of a biological system or interpreting a bifurcation diagram. More specifically, I answer the following two research questions:

- What is the impact of interleaving on student performance in a Calculus for Life Sciences course in which students complete one weekly homework?
- What are the benefits of interleaving homework sets after a one-month delay?

Methods

This study was conducted in two sections of a Calculus for Life Sciences course at a large research-intensive public university in the southwest with a total of 491 students. The vast majority of the students (94%) were either in their freshman or sophomore year and over 90% planned to major in biology or the life sciences. The course focused on modeling and how to use mathematical models to better understand biological systems. The instructor used active learning methods in the form of clicker questions with peer instruction (Crouch & Mazur, 2001) and small-group activities. The students also had weekly coding labs led by teaching assistants. During these labs students solved one question similar to a homework exercise in groups, and then spent rest of the time learning to code in SageMath. The course was identical for both sections (same instructor, same lecture materials, etc.) except for the structure of the homework. The ten-week quarter was divided into two four-week stages (the topics of weeks 5 and 10 were not part of the study). During the first stage, one section received blocked homework sets while the other one received interleaved homework sets. Each section then switched to the other type of homework for the second stage.

Blocked and interleaved homework sets were created in the following way. Each lecture was divided into two topics and two or three homework exercises were created on each topic. In the blocked homework sets the topics appeared in the same order as they had been studied in class and the exercises on a given topic were grouped together. For the interleaved homework sets, once a topic was studied in class, one exercise on this topic would appear on the next three homework sets (except for the last two weeks of each stage). The interleaved homework sets thus contained topics from different weeks and two exercises on a given topic would never appear one after the other. Importantly, in each stage students received the exact same homework exercises, only organized differently. Students submitted their homework weekly and homework exercises were graded on a 3-point scale (very good, sufficient, insufficient).

Students received a surprise in-class quiz on the Thursday of weeks 5 and 10. They also had a midterm on the Monday of week 6 (just before the beginning of the second stage) and a final exam on the Monday following week 10. In order to assess the impact of interleaving after a one-month delay, students took an in-class quiz during the first day of the following quarter. This quiz took place 28 days after the final exam. Students were encouraged to do their best on that quiz by being told that it would help the instructors assess their current level of understanding of the material was. They also received a bonus point for completing the quiz. The quiz contained

two questions on material from the first stage and two questions on material from the second stage. The midterm, final exam and quizzes were graded by the teaching assistants of the course.

Results

Quizzes and Exams During the Quarter

For the first surprise quiz, students in the interleaved group outperformed the students in the blocked group with an average of 66.11% for the interleaved group and 62.14% for the blocked group ($p=0.047$, Cohen's $d=0.18$). A similar outcome was observed on the second surprise quiz that took place on the last day of the course with an average of 57.24% for the interleaved group and 50.54% for the blocked group ($p<0.001$). This time the effect size was larger with Cohen's $d=0.33$. One should note that while the second half builds on the first half, the questions asked in each quiz targeted concepts from the corresponding stage. A student can do well on a stage 2 question (e.g., determining the long-term behavior of a given model using linear stability analysis) without necessarily doing well on a stage 1 question (e.g., writing down the equations of a model based on the description of a biological system or estimate the future value of a variable using Euler's method). It thus seems reasonable to assume that the results of the second quiz are primarily (if not only) impacted by the interleaving or blocked condition in stage 2.

For the midterm and final exam, there was no statistically significant differences between the two groups. Specifically, on the midterm the interleaved group had an average of 79.81% and the blocked group of 79.42% ($p=0.78$). For the final exam, restricting the analysis to the questions covering the second half of the quarter, students in the interleaved group outperformed students in the blocked group with an average of 87.68% for the interleaved group and 85.11% for the blocked group. However, this difference was not significant ($p=0.092$). The difference between the two groups on the questions covering the first part of the quarter was not significant either.

Although the lack of statistically significant differences between the two groups on the midterm and final exam may suggest a limitation of the impact of interleaving, two other elements may explain this result. First, it is possible that the unannounced quizzes helped students identify topics they had yet to master and thus prompted them to thoroughly review these concepts. This hypothesis is consistent with results on topics that appeared both on quizzes and the subsequent exam. For example, for the question on the stability of equilibrium points the average was 43.57% on the end-of-quarter quiz and 89.14% on the final exam. Second, even without the unannounced quizzes, students tend to review very intensely shortly before the exam. On surveys students indicated having spent on average 16.63 and 21.63 hours specifically reviewing for the midterm and the final exam respectively. One should also note that there is no indication that study behavior changed between the two groups. For example, the perceived difficulty of the homework reported by the students was not significantly different between the two groups. This suggests that the differences seen on the quizzes does not come from different study behaviors but from interleaved structure of the homework itself.

Quiz After a One-Month Delay

The quiz that was given at the beginning of the following quarter contained four questions, two on material from the first stage and two on material from the second stage. On the first stage there was no statistically significant difference between the two groups ($p=0.79$).

On the second stage the interleaved group outperformed the blocked group with an average of 56.71% for the interleaved group and 51.75% for the blocked group ($p=0.044$, Cohen's $d=0.20$). There is some similarity with the results of the quizzes given during the quarter with

interleaving having a stronger impact on the second half of the quarter than on the first half. This is also in line with the results reported by Samani and Pan (2021) in college physics.

Discussion and Future Work

The results suggest that interleaving can help student performance in calculus (although perhaps modestly) and are consistent with previous studies in middle-school mathematics (Rohrer, Dedrick, Hartwig, et al., 2020; Rohrer et al., 2015) and college physics (Samani & Pan, 2021). These three studies have reported larger gains and one question is therefore where this difference may come from. The first study by Rohrer et al (2015) focused only on two types of problems, graph and slope problems. One interesting feature is that the wording of each type of problem was always the same ("Graph the equation…" and "Find the slope of the line that passes through the points…"). It is reasonable to assume that the problems given on the test again used the same phrasing. While not explicitly mentioned, it seems probable that this same feature was used by Rohrer and collaborators in their (2020) study. This characteristic leaves open the possibility that interleaving may help students better identify keywords or key features of problems but is not necessarily as powerful when problems on a given topic can take different formats. Based on the examples accompanying their article, Samani and Pan (2021) also seem to have used problems that had the exact same wording except for the numbers and names of the atoms involved in the problems. Since their tests involved some problems that combined two topics, it is hard to determine whether the wording of the test problems was similar to the homework problems. However, since in the present study some problems on a given topic could have different format, this could explain the lower effect sizes observed here.

Another notable feature of Samani and Pan's (2021) study is that all problems involved recalling the correct formula(s) and then using it to get to the correct numerical value and units. When looking at their memory sub-measure (whether students could recall the appropriate formula), they found effect sizes of d=0.41 for the first stage and d=0.96 for the second stage, while when looking at the correctness of the answers sub-measure, the effect sizes were d=0.25 for the first stage and d=0.40 for the second stage. While the authors explain that it is not uncommon in introductory physics courses to have a low level of correct numerical answers due to the conceptual and computational complexity of the questions, the difference in effect sizes may suggest that interleaving really helps students recall the appropriate formulas but is not as potent in supporting their problem-solving skills.

One common feature between the current study and Samani and Pan (2021) is that the interleaving effect is stronger for the second half of the course. There is no obvious explanation for this phenomenon, however, one may tentatively hypothesize that interleaving helps students more when the complexity of the material increases. This is something that needs to be further tested. Another question that needs further investigation where do the differences in effect sizes between the current study and the studies by Rohrer et al. (2020; 2015) and Samani and Pan (2021) come from. Is it due to the differences in the format of the problems used as noted above or do other factors also play a role? Finally, future research should further look into why benefits observed on unannounced quizzes do not carry over to exams and if this is indeed due to students cramming before an exam or due to other characteristics of the exams.

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Mathematicians' Goals for Their Upper-Level, Proof-Based Mathematics Courses

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Despite significant research efforts focused on the development of student-centered instruction, most mathematicians continue to teach through lecture. Consequently, researchers have called for further examination of the reasons why mathematicians appear to believe in their current teaching practices. Our study builds on previous research addressing this call by examining how mathematicians' in- and out-of-class instructional decisions in upper-level, proof-based mathematics courses reflect their pedagogical goals. Four mathematicians participated in a series of three, one-to-one interviews examining their instructional design of a course they had recently taught. Data analysis is ongoing and is being guided by a deductive form of thematic analysis. In this paper, we provide an overview of literature on mathematicians' pedagogical goals, describe the study design and plans for data analysis, and present preliminary findings in relation to one participant mathematician.

Keywords: Upper-Level Courses, Goals, Resources, Orientations, Instructional Design

Introduction and Background Literature

In their recent review, Melhuish et al. (2022) distinguished between two forms of in-class instruction of undergraduate, proof-based, mathematics courses: those based on a lecture-based pedagogy and those based on a student-centered (or inquiry-based) pedagogy. Many mathematics education researchers have expressed dissatisfaction with lecture-based instruction, particularly its operation on a transmission-based model (Melhuish et al., 2022). Student-centered instruction, which emphasizes engaging students in doing mathematics during class time, has been suggested as an alternative (e.g., Laursen & Rasmussen, 2019). To assist instructors in adopting student-centered instruction, there has been significant research on the development of research-informed curricula for various mathematical content areas (e.g., Larsen et al., 2013; Leron & Dubinsky, 1995). However, "the influence of this research so far has been modest" (Melhuish et al., 2022, p. 4). Most mathematicians continue to teach through lecture (Johnson et al., 2018), despite the alternatives on offer. Additionally, the relatively small number of mathematicians who choose to adopt student-centered instruction are more likely to develop their own materials than to adopt research-informed curricula (Fukawa-Connelly et al., 2016).

The lack of uptake of these materials, despite the significant research efforts that focused on their development and dissemination, led to the call for research to "better explore the reasons that mathematicians appear to strongly believe in their current practice" (Fukawa-Connelly et al., 2016, p. 281). Subsequent research addressing this call (Woods & Weber, 2020) concluded that, in relation to lecture-based pedagogy, mathematicians choose to lecture as they believe in its efficacy as an instructional method and feel it aligns well with their pedagogical goals.

Over the past decade, numerous other studies have investigated various aspects of lecture-based instruction. This means we now have a good understanding of what happens in the 'typical' undergraduate mathematics classroom (Melhuish et al., 2022). However, the time instructors spend in class with students accounts for only one component of instruction. Many mathematics educators agree that, regardless of the format of in-class instruction, the majority of students' learning of advanced mathematics will take place outside the classroom (Rupnow et al., 2021). Therefore, an important aspect of teaching undergraduate mathematics is preparing

students to work independently outside of class (Lew et al., 2022; Rupnow et al., 2021; Wu, 1999). This aspect of instruction has received much less attention in the mathematics education research literature. Rupnow et al. (2021) were among the first to examine how mathematicians assign homework problems in upper-level courses. Six mathematicians participated in task-based interviews in which they were asked to select homework problems from an abstract algebra textbook and justify their choices. The mathematicians' pedagogical goals guided their assignment of homework problems, and they perceived students' engagement with these problems as essential to their learning.

The studies of Woods and Weber (2020) and Rupnow et al. (2021) both address Fukawa-Connelly et al.'s (2016) call by providing insight into how mathematicians' decisions in relation to in-class instruction and homework are guided by their pedagogical goals. However, to date the various components of undergraduate mathematics instruction have mostly been examined independently, with a large emphasis on the in-class component. The goal of our study is to build on existing work and examine mathematicians' goals and pedagogical decision-making in relation to *all* instructional components of their upper-level, proof-based mathematics courses, along with the factors that influence these goals and decisions. Prior research that has surveyed mathematicians, has identified potential influences on their teaching practice, such as their experiences as a teacher and student (Johnson et al., 2018). However, research that has employed qualitative methodologies has demonstrated that mathematicians' teaching goals can vary on the level of the individual (e.g., Johnson et al., 2013; Pinto, 2019). Based on this research, we decided to adopt a qualitative study design to examine mathematicians' potentially varying goals and pedagogical decision-making in relation to all instructional components of their courses. In summary, our study addresses the following research questions:

- **RQ1:** What are mathematicians' pedagogical goals for their upper-level, proof-based mathematics courses and how do their orientations and resources inform these goals?
- **RQ2:** How are mathematicians' pedagogical goals reflected in the instructional decisions they make?

Theoretical Perspective

Given that research has indicated that mathematicians' instructional decision-making is primarily guided by their pedagogical goals (Rupnow et al., 2021; Woods & Weber, 2020), we decided to use Schoenfeld's (2010, 2011) theory of decision-making to inform our study design and data analysis. Schoenfeld proposes that the pedagogical decisions that a teacher makes can be understood as a function of their goals, resources, and orientations. Goals are described as "the things that people consciously or unconsciously set out to achieve" (Schoenfeld, 2011, p. 459) and can vary in terms of size. For instance, an instructor will likely have overarching goals for a course they are teaching, in addition to goals for an individual lesson. Schoenfeld suggests that a teacher's decision-making involves the selection of goals that are consistent with their orientations and resources. Orientations is used as an "inclusive term to encompass beliefs, dispositions, values, tastes and preferences" (Schoenfeld, 2010, p. 15). For example, over the course of their teaching career a mathematician may develop understandings or perceptions about the nature of mathematics, teaching, or students. Schoenfeld also proposes that the availability of resources also influences a teacher's decision-making. He elaborates that "what a teacher can or cannot do in the classroom is clearly a function of what he or she knows, what material and other resources are available, and what constraints are in place" (Schoenfeld, 2010,

p. 10). Schoenfeld's theory emphasizes the interaction among an individual's goals, orientations, and resources and notes that, typically, orientations and beliefs are slow to evolve and change.

Methodology

The study adopted a qualitative, phenomenological (Willig, 2021) research design. Four mathematicians participated in a series of three, one-to-one interviews in summer 2023. The interviews were designed to examine the mathematicians' instructional design of an upper-level, proof-based, mathematics course they had recently taught. Data analysis is in the initial stages.

Participants

Four research mathematicians, employed in a large, Irish, PhD-granting university, participated in the study. We employed purposive sampling (Braun & Clarke, 2013), inviting mathematicians with over 20 years' teaching experience who had taught an upper-level, proof-based mathematics course in the last three years to participate. We endeavored to have a balance in the research areas and institutions of undergraduate study of participants, as well as a gender representation that reflects that of the discipline of mathematics in the university. In this report, we present preliminary findings in relation to one participant to whom we assign the gender-neutral pseudonym Doolan.

Course Context

The four participants in the study were each interviewed about an upper-level, proof-based mathematics course they had recently taught. Given this paper's focus on findings related to Doolan, we provide further contextual details about their course. Doolan teaches a proof-based course covering topics from analysis to third- and fourth-year mathematics students. The in-class component of the course consists of 36 hour-long classes. Doolan lectures in roughly 30 classes, the remaining six classes are recitation classes (referred to as tutorials) in which the homework problems are discussed. There are four homework assignments in the course. Students are provided with a problem sheet consisting of around ten problems and are asked to submit solutions to two problems of their choosing. Each homework assignment is worth 5% of a student's overall grade, with a two-hour closed-book examination accounting for the remaining 80%.

Data Collection

All four mathematicians participated in a series of three, one-on-one interviews with the first author in summer 2023. Each interview lasted an average of an hour and was audio-recorded and transcribed. To add specificity to our interview questions, we asked each mathematician to select a section of content from their course consisting of at least one of each of the following: definition, example, theorem, and proof. We asked that the section of content they picked be reflective of the main focus of the course (rather than introductory material), and feature in the homework assignments.

The participants provided the lecture notes, homework assignments, and past examination questions relevant to their chosen section of content. These were used in the design of the interviews, with the first interview focused on the design of the lecture notes and lectures, the second on the homework assignments, and the third on the assessment component of the courses. As the focus of our preliminary findings is on homework, we provide further detail about the second interview. Based on the interview protocol used by Rupnow et al. (2021), for each problem related to the relevant content topic we asked why the mathematician had decided to

include the problem and what they hoped students might learn from engaging with it. We also asked participants more general questions about how they operationalize homework in their courses and how they view the role of homework. Additionally, the mathematicians were asked if they hold recitation classes on homework problems and, if applicable, to describe the role and structure of these classes.

Data Analysis

Data analysis is in the initial stages. We plan to use a deductive form of thematic analysis, utilizing a method called template analysis (Brooks et al., 2015). Schoenfeld's (2010, 2011) theory of decision-making will inform the coding of the data. We also plan to incorporate the findings of Rupnow et al. (2021) and Woods and Weber (2020) in relation to mathematicians' goals as *a priori* codes (Brooks et al., 2015). To date, the authors have familiarized themselves with one participant's (Doolan's) interviews, with notes written on points of potential analytic interest. Analytic notes, and associated quotes, from the second interview were examined for evidence of pedagogical goals, as well as resources and orientations influencing these goals. This process informed the preliminary findings that we present in this paper.

Preliminary Findings

The aim of our study is to examine mathematicians' goals for their upper-level, proof-based mathematics courses, factors influencing these goals, and how their goals are reflected in their pedagogical decision-making in relation to all instructional components of their courses. In this paper, we present preliminary findings in relation to one of Doolan's pedagogical goals: encourage and support students in engaging with homework problems. We emphasize that data analysis is in the initial stages. Our primary purpose in presenting these findings is to illustrate the nature of our data and to present some of our initial analytic thoughts.

Doolan's reasons for assigning students homework problems broadly align with the findings of Rupnow et al. (2021). Like participants in Rupnow et al.'s study, Doolan downplays summative assessment as a purpose of assigning homework and highlights the learning opportunities engaging with problems can provide. Doolan emphasizes problem solving as a skill that should be developed in an undergraduate mathematics degree, stressing the importance of learning to struggle and persevere with non-trivial problems:

> *Doolan:* The whole point of it is that you hit these problems that you haven't met before, and you don't know if you can solve them. And if all you do is spend like a minute thinking about it and saying, you know, 'I can't do it' then, you know, you're not going to be a mathematician.

Given the importance they attribute to students' engagement with problems, it is perhaps unsurprising that one of Doolan's goals as an instructor is to encourage and support students in their work on homework problems. In this paper, we analyze one of Doolan's instructional decisions that we feel is consistent with this goal: the decision to allow students to select two homework problems from each of the four problem sheets in the course to submit for assessment. This decision appears to be primarily motivated by Doolan's pedagogical beliefs. Doolan proposes that by not assigning students particular problems, they will be encouraged to read, and engage with, all the problems to assess their relative difficulty:

> *Doolan:* If I give two particular problems as the assessment on that chapter, the students might just read those two problems and not bother about the rest. If they read the entire

sheet, even if it's with the viewpoint of finding the easy ones, they do have to think well, 'Is this hard or is this easy?' So, there's a little bit of learning going on just from that.

Additionally, by not requiring students to submit all problems for assessment, Doolan is able to include more difficult problems in the homework assignments. Acknowledging that it is likely that not all students will be able to complete these harder problems, Doolan believes there is still value in students attempting and thinking about them: "it would be a problem that I'd encourage students to think about, but I wouldn't expect them to hand it up for their assessment."

Doolan's decision to allow students agency in selecting their own problems also appears to have been influenced by their own experience as a student. Doolan recalled that as a student in a Ring Theory course, they became captivated by one question on a problem sheet: "one kind of intrigued me and I spent weeks like looking at it. I had a whiteboard at home and I was trying every kind of way." This formative experience as a mathematics student has left Doolan with the belief that "just throwing yourself into one problem like that, at the right time, can be really instructive. And, if it works out for you in the end, gives you great confidence." Doolan acknowledges that the decision to not assign students particular problems may be partly about "allowing a little bit of freedom" so that students have the chance to become similarly interested in problems of their own choosing.

We note that Doolan's decision to have students submit just two homework problems reflects the fact that they are constrained in terms of the time they have available to grade students' work: "practical things come into play. It takes quite a lot of time to grade. Yeah, so two is fair." We also highlight that Doolan's practice in relation to allowing students to select homework problems for assessment has evolved over the course of their teaching career. Doolan mentions that in previous iterations of the course they randomly assigned students particular problems to submit for assessment. The students at the time felt "it was a bit unfair that everyone had different problems to hand up." Based on this feedback from students, Doolan's understanding of students' perceptions of instructional practices developed, and their teaching practice in relation to assigning problems was adjusted accordingly.

Discussion and Conclusion

In this paper, we have outlined some of our initial analytic thoughts on one participant's goals and decision-making in relation to homework. While emphasizing the preliminary nature of our results, we note the apparent alignment between Doolan's instructional decisions and their pedagogical beliefs. This is consistent with prior research indicating that mathematicians can generally offer sophisticated rationales for their pedagogical actions (Melhuish et al., 2022). The implication of this finding is that attempts to change mathematicians' practice are likely to fail if the alignment between instructional interventions and mathematicians' goals is not considered (Melhuish et al., 2022; Woods & Weber, 2020). Consequently, developing a robust understanding of mathematicians' pedagogical goals and decision-making in relation to both in- and out-of-class instruction is important in terms of informing the development of instructional interventions. In this regard, our ongoing work on this study builds on previous research by examining mathematicians' goals and pedagogical decision-making in relation to all instructional components of their upper-level, proof-based courses.

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Mathematical Induction: Which Written Proofs or Visual Proofs are More Convincing?

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This study uses a survey to investigate mathematical convictions in tournament-style comparative judgements for written and visual proofs. The survey uses four theorems that can be proven using mathematical induction with each having four proofs: two written and visual. The visual proofs were equally convincing, and the respondents indicated that they believed these visual proofs to be mathematical proofs but not prove their theorems for all values n. The written proofs by mathematical induction were often the most convincing proofs—winning the overall tournament. Familiarity and content seemed to be prevailing factors for judging whether one proof was more convincing than another.

Keywords: Mathematical induction, comparative studies, visual proofs, mathematical conviction

Mathematical conviction (Brown, 2018), the ways in which one is persuaded about the truth of a theorem given a proof, has been deemed a nebulous concept posing challenges to empirical study. This study seeks to gain new insights by exploring how students respond to two types of proofs with specific features: visual proofs and written proofs by induction. Though visual proofs have a contested status as proofs (Brown, 2008), they may offer useful tools for investigating what types of proofs may be convincing (Alcock & Simpson, 2004; Relaford-Doyle & Núñez, 2021). Mathematical induction (MI) has been well-documented as being difficult for students to understand (Ernest, 1984) where issues may arise at any step (Stylianides et al., 2016), but it may also provide interesting insight about when and how students are convinced by it (Brown, 2014; Stylianides, 2007). The purpose of this study is to investigate how these two features influence student proof convictions using comparative judgments (c.f., Brown, 2018).

Background Literature

Most literature focuses on whether students are convinced by proofs or something other than proof (Stylianides et al., 2017). In this study, I am concerned with the features of proof that make them more convincing in comparative judgement tasks.

Components for this Study

There are a variety of studies on mathematical convictions which report a multitude of factors influencing whether and how a mathematical proof is convincing. Common aspects influencing this judgement are familiarity with the content and method (Brown, 2018; Chazan, 1993; Healy & Hoyles, 2000), example-use (Healy & Hoyles, 2000; Weber et al., 2022), and whether one understands that proof (Brown, 2018; Selden & Selden, 2003).

Visual elements may also influence mathematical convictions because those visual elements may convey the same features that written proofs can (Relaford-Doyle & Núñez, 2021; Zhen et al., 2016). This can be done perceptually (Harel & Sowder, 1998; Miller et al., 2022) and might be salient enough for students to gain insight into the argument (Weber, 2010). There is also evidence that visualizations and convincingness interact (Alcock & Simpson, 2004, 2005). Combining recursion and visual elements can also be helpful for becoming convinced by a mathematical argument (Knuth, 2002).

Recursion can provide an entry point into MI (Harel & Sowder, 1998). MI is susceptible to cognitive gaps (Norton et al., 2022) which may be alleviated through example-use (Stylianides et al., 2016). So, combining visual aspects, in the form of proof, and mathematical induction provides the foundation for this study's methods because MI can be formulated to reveal recursive relationships thereby influencing mathematical convictions.

Methods

I sampled from a discrete mathematics 2 course at a large southern university in the United States using a survey which received 15 responses. For four theorems, four proofs were created or adapted from prior literature. Two of these were visual proofs, one that demonstrated how different cases were related (visual induction-promoting proofs, VIPPs), and the other showed a single case to be taken as general, often when $n = 7$ (visual general-promoting proofs, VGPPs). The other two proofs were written proofs: an MI proof (WMIPs) and a non-MI proof (WNMIPs).

Instruments

The survey features a tournament-style comparative judgement design with four theorems with each of these having the previously mentioned types of proofs. Every written proof avoided prompting phrases for proof methods such as "this is a contradiction" or "by the principle of mathematical induction." Figure 1 below shows the general structure for the tournament.

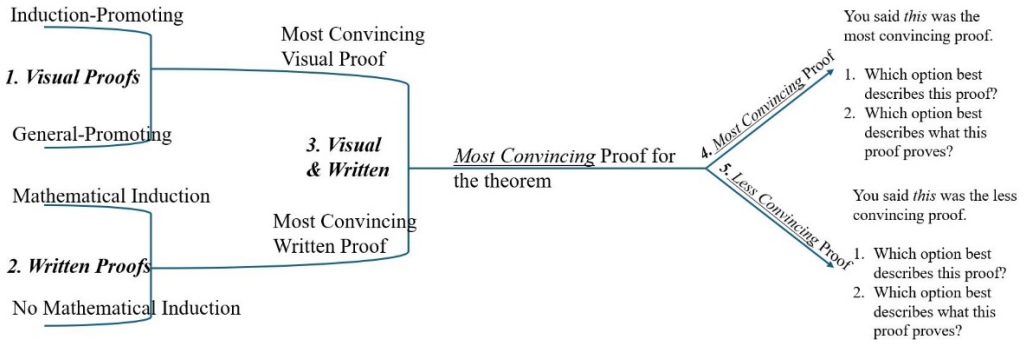


Figure 1. The General Structure of the Survey for the Four Theorems.

In Figure 1, the numbers indicate the order presented within the survey. Visual proofs were always compared first, written proofs second, and then a visual and a written proof third. The order of all answer choices was randomized to prevent a priming effect. The exact phrasing for prompting students to select the more convincing proof was "which proof, in your opinion, is the most convincing? In other words, which proof better persuades you of the truth of the theorem?" (Brown, 2018, p. 8). Students were allowed to select an option that stated both proofs were equally convincing. After the third bracket, the students were shown their most and least convincing proofs and asked what type of proof they were and what they proved. Borrowed from Brown (2018) and Relaford-Doyle and Núñez (2021), these questions confirmed student engagement and how they interpreted the proofs (the type and how general).

The first and second theorem (see Table 1) were intended to be familiar and were often used as either examples or homework problems in ITP courses. The third and fourth theorems were intended to be unfamiliar. That is, these theorems were likely not seen as homework problems or examples in class. All WNMIPs were direct proofs except the one for Theorem 1 (see Brown, 2018, p. 11, Argument 1B). Table 1 presents the four proofs for one of the four theorems.

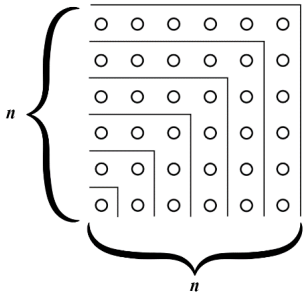
Table 1. Theorem 2 and its Proofs

Theorem 2: Let n be a positive integer, then $1 + 3 + 5 + \ldots + (2n - 1) = n^2$

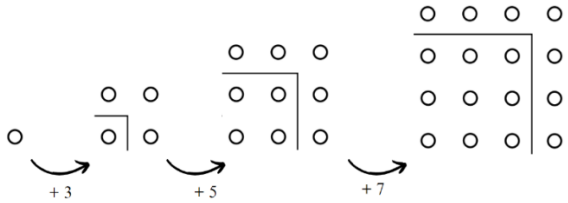
Proof 2A (WNMIP)

$$\begin{aligned} 1+3+5+\cdots+(2n-1) &= \sum_{k=1}^{n}(2k-1) \\ &= 2\sum_{k=1}^{n}(k) - \sum_{k=1}^{n}(1) \\ &= 2\frac{n(n+1)}{2} - n \\ &= n^2 \end{aligned}$$

Proof 2B (VGPP)[a]



Proof 2C (VIPP)[a]



Proof 2D (WMIP)

Let $n = 1$, then $2(1) - 1 = 1^2$.
Suppose that the theorem is true for some positive integer k.
It follows that,
$1 + 3 + 5 + \cdots + (2k - 1) = k^2$.
Thus,
$1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) = k^2 + 2k + 1 = (k + 1)^2$.
It follows that for all natural numbers n,
$1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

[a] Adapted from Brown (2008, p. 35)

Results and Analyses

There were several preliminary analyses performed. Table 2 was created to specifically see how convincing a type of written and visual proof were for theorems that could be proven with MI. Table 3 was created to follow students' decisions and which proof type was the most convincing. Table 4 was created to investigate how students understood the visual proofs. The current sample size cautions against inferring too much from these frequencies, but I illustrate the analysis to be performed on the larger sample set being gathered.

First and Second Bracket Results

The first and second brackets compared two visual proofs and two written proofs, respectively. These results are summarized in Table 2.

Table 2. First and Second Bracket Results Across the Four Theorems

	First Bracket		Second Bracket		
	VIPP (15)	VGPP (15)	WMIP (15)	WNMIP (15)	Total (30)
Theorem 1	7	5	11	4	27
Theorem 2	6	9	8	6	29
Theorem 3	5	9	11	3	28
Theorem 4	12	3	11	2	28
Total	30	26	41	15	112

The numbers appearing next to the headers indicate the maximum number possible for an entry in that column. The entries in the columns do not always add to 15 because some students said both proofs were equally convincing. In the first bracket, the visual proofs were nearly chosen as often as each other. Theorems 1 and 4 had higher instances of the VGPP chosen as more convincing than the VIPPs whereas Theorems 2 and 3 were the other way around. In the third bracket, the WMIPs were chosen nearly three times as often as the WNMIPs with those proofs

being chosen almost as frequently as each other for Theorem 2. These findings suggest that students had preferences regarding which visual proofs were more convincing, but they did not always align to the distinction between VIPP and VGPP across the tasks. By contrast, students found the MI proofs more convincing, at least within this limited sample.

Overall Tournament Results

The students made a total of 180 selections across the four theorems. These results are summarized in Table 3.

Table 3. Overall Tournament Results Across the Four Theorems

		Student															
		A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	Total
Third Bracket Only (4)	VIPP	0	0	3	0	1	2	0	0	0	0	0	1	3	3	2	13
	VGPP	0	1	0	0	0	2	0	2	0	1	0	0	1	1	1	9
	WMIP	3	1	1	3	1	0	2	1	1	3	3	2	0	0	1	23
	WNMIP	1	0	0	1	0	0	2	0	1	0	0	1	0	0	0	6
Totals Across All Brackets (12)	WMIP or VIPP	8	6	10	9	8	7	5	5	5	8	7	7	6	7	9	107
	WNMIP or VGPP	4	4	2	3	0	5	6	6	5	4	3	2	4	5	3	56

The numbers appearing next to headers indicate the maximum number possible for that row. In the third bracket, students more often selected the WMIPs as the winner of the tournament. This is interpreted as WMIP being the most convincing proof for these 15 students across the four theorems regardless of its content. The last two rows were used to help determine whether the WMIPs or VIPPs were in fact more convincing, in general, than their counterparts. The results showed that, when given the chance, the students chose those proofs as the more convincing nearly twice as much. This also shows that 10 of the 15 students chose the WMIPs or the VIPPs more than half the time while only one student chose the WNMIPs or VGPPs more than half the time.

Students' Descriptions of the Proofs

One component of this study was to determine how students viewed the visual proofs. The fourth and fifth components of the survey structure were used for this (see Figure 1).

Table 4. Students' Descriptions of the Proofs

Question Asked	Option on Survey	VIPPs (34)	VGPPs (26)	WMIPs (45)	WNMIPs (15)
Which option best describes this proof?	Direct Proof	14	9	7	5
	Contrapositive Proof	2	1	1	2
	Proof by Contradiction	1	0	1	2
	Proof by MI	8	7	32	6
	This is not a proof.	2	4	1	0
	Something else.	4	4	2	0
	All values of *n*.	19	17	37	12

Which option best describes what this proof proves?	Specific values of n.	10	8	5	3
	Something else.	2	2	2	0

In Table 4, the numbers in the parentheses next to the headers indicate the number of times that type of proof entered the third bracket therefore entering the fourth or fifth component of the survey. The two most common descriptions for the visual proofs were direct and MI proofs. A few students stated that the visual proofs were not proofs or described them as something else. These latter responses consisted of either uncertainty about the status of the proofs and picture or visual proofs. Notably, the number of responses indicating the visual proofs are mathematical proofs and those indicating that the proof proved the theorem for all values of n do not match. This was interpreted as students being convinced by something that they identified as mathematical proof while providing evidence that it did not meet the generality of a proof.

Discussion and Conclusion

In the first bracket, students chose the visual proofs with almost the same frequency. So, it cannot be yet determined whether one type is more convincing than the other. However, Theorems 1 and 4 had higher instances of VIPP while Theorems 2 and 3 had higher instances of VGPP. On one hand, the students may have "seen" induction (Brown, 2008) in the VIPPs for Theorems 1 and 4 but not Theorems 2 and 3. On the other hand, Relaford-Doyle and Núñez (2021) noted students might understand and reason about a diagram purporting to promote mathematical induction in ways that separate the diagram and their knowledge of mathematical induction. So, the result might be that the diagram can be helpful or convincing but for reasons unrelated to mathematical induction or other formalized methods of proof. Nevertheless, initial findings suggest that students have preferences based on some features of these visual proofs.

In the second bracket, the WMIPs were most often chosen as the more convincing proof. This can be interpreted as students' instruction on MI heavily influencing their choices on matters of conviction. That is, it is easily identifiable and familiar (Brown, 2018). The students' propensity for choosing the WMIP as the winner of the tournament can again be interpreted as students' familiarity with this proof method as a deciding factor for the second and third brackets. Notably, the WNMIPs for Theorems 2, 3, and 4 were not so often chosen. These proofs featured almost exclusively summations, manipulations, and few words. This might suggest that the idea that students value familiarity sanctioned in their classrooms despite prior research reporting its technicalities and difficulty (Ernest, 1984; Stylianides, 2007). In addition to this, the WMIPs and VIPPs were also chosen almost twice as often as WNMIPs or VGPPs throughout the survey. However, WMIPs constituted almost 60% of these selections. This again lends to the idea that the familiarity of written proofs by MI enhanced their convincingness.

For how students viewed the visual proofs, most students either viewed them as direct proofs or proofs by MI. However, not all those who selected an explicit proof type (50 instances) also said the proof proved the theorem for all values of n (36 instances). This is interpreted as those students possibly holding varying definitions or conceptions about mathematical proof (Morris, 2002, 2007; Recio & Godino, 2001). In other words, students may be convinced by something other than what many communities call "mathematical proof" even in more advanced mathematics courses such as discrete mathematics 2 Because these results and analyses had a limited sample set, I am gathering a larger data set this Fall.

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Conceptualizing Language for Mathematical Problem Types Across Domains

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In this preliminary theoretical report, we problematize the language used in mathematical problems – in particular, with respect to the objectives and the directives of problems. We use this to conceptualize a set of eight problem types, described in three clusters, that run across mathematical domains. This paper explores the motivation for such work, as well as a rationale for why existing problem classifications in mathematics do not accomplish our intended aims; we identify potential applications of the framework, as well as challenges, for discussion.

Keywords: Problem types, Mathematical domains, Language, Mathematics register

Much of the study of mathematics is reducible to the problems students are asked to solve. While university syllabi identify the content of a course in terms of concepts and topics, the list of tasks assigned operationalize what students must learn and do. Problem solving, and the posing of mathematical tasks, are both central themes of mathematics education (Schoenfeld, 1992); as is the language used within them – which may be referred to as part of the mathematics register (Halliday, 1975; Pimm, 1987) or even its own genre (Bakken & Andersson-Bakken, 2021). However, if we consider the role of language in posing tasks to students, two key issues arise. By the time students take university mathematics they have years of experience with the genre of mathematics problems. Yet, whereas mathematical terms are often both well-defined and consistently used, the language used for posing mathematical problems is neither. For instance, the directive "find" in one problem context might mean perform an operation (e.g., "Find $f(2)$ when $f(x) = x^2$"), in another provide a singular example (e.g., "Find an x such that $f(x) = 4$ if $f(x) = x^2$), and elsewhere solve an equation (e.g., "Find all x such that $x^2 = 4$"). This suggests that decoding the nature and objective of a task could serve as part of a hidden curriculum students must navigate. To lay the groundwork for an investigation of the relationship between the language used in posing mathematics tasks and the actual objective being asked of students, this preliminary theoretical paper considers: *How can we categorize mathematical problems, across a variety of domains, that might help specify the nature of the task assigned?* We wanted categories that applied across domains because students experience the language of mathematics tasks across different courses. Although the current paper – a preliminary theoretical report – does not compare whether the same language is used for different tasks, or whether one task is expressed in various language, it lays the foundation for such investigation.

Motivation and Background

As a somewhat humorous illustration, consider the common meme reproduced in Figure 1 – of a student's solution to a mathematical problem. Here the student interprets the imperative "find" in an everyday sense, rather than according to any mathematical sense(s).

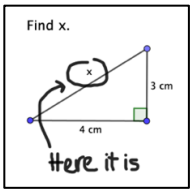


Figure 1. A student solution to the problem's directive to "Find…"

A mathematical problem uses language to give a directive (sometimes in question form) and tends to have a particular kind of mathematical goal associated with it. The problem's directive – e.g., the imperative verb – is supposed to signify this objective to the reader. Yet, we often do not give clear meaning and definition to these problem-posing terms. So, on one side, they are left ambiguous with *no* mathematics-specific meaning, partly because they are part of a meta-language about the doing of mathematics. On the other, we avoid defining them because, as we illustrated above with "find," they are polysemous and may be used to express a variety of goals, depending on the specific task (e.g., Schleppegrell, 2007). That is, an imperative verb can have *many* mathematics-specific meanings. In Figure 1, the task used the imperative "Find…" but the goal is to solve for the real-numbered value(s) of x given the properties of the figure – it is a problem about a solution set.

Rationale, Goals and Assumptions

Why do we need another way to categorize mathematics problems? Naturally, mathematics educators have developed categorizations, which reasonably tend to relate to 1) learning goals, 2) students' prior knowledge, 3) intended solution methods, or 4) expressions of epistemologies or mathematical practices. These features often make the categories specific to a mathematical topic (features 1-3), contextual within an instructional setting (features 1-3), or situated within broader social practice (feature 4). For instance, Schoenfeld's (1992) famous distinction between exercise and problem relates to the solver's prior knowledge. Stein et al.'s (2009) framework for cognitive demand similarly takes student experience into account. "Open" versus "closed" problems (c.f., Bahar & Maker, 2015) tend to account for solution methods, whereas Stanic and Kilpatrick (1989) elaborated five ways that problems can be a means to other ends: problems get used as justification, as motivation, as recreation, as vehicle (for learning), and as practice. This primarily categorizes problems in terms of their learning goals. Of course, one might also specify learning goals in other ways. Many frameworks categorize problems based on content area – whether broad (e.g., algebra, geometry) or more specific (e.g., systems of equations). Others categorize goals in terms of cognitive activities, such as recall or analyze, intended to capture processes. Some do both (e.g., the TEDS-M study, Tatto et al., 2008).

Such frameworks help us understand certain aspects of problems – often informative in textbook analyses (e.g., Gracin, 2018; Herbel-Eisenmann, 2007; Li, 2000). However, although we can imagine similar uses, we instead want categories of what students must produce in response to the problem, regardless of solution method, that can be applied across contexts and without reference to broader practices – which existing frameworks do not accomplish. These choices express our interest in understanding the *nature* of the problem posed in relation to the *language* used in the problem. We do not mean to deny the importance of all of these factors in how students interpret new tasks, as though the language used to express a problem can be isolated from all of these other factors. Rather, we seek to understand what information is (and is not) conveyed by the specific imperative or interrogative language used in a problem. It may be that most of the information about what the problem requires must be inferred from other sources of information, such as prior experience with similar problems. But this would imply that the language used to express a task remains largely inert and could be made more productive.

In setting down this path, we had four underlying assumptions that guided how we developed our framework. First, we wanted a framework that was broadly applicable across domains, which meant we did not want domain-specific problem types (e.g., "Differentiate"). The aim was for problem types to be comprehensible in any area of mathematics. Doing so affords looking at underlying connections across domains, and also allows us to compare across domains.

Second, our goal was to establish a framework that characterizes the core "objective" that students are asked to accomplish in a problem. As such, we were not necessarily concerned with how students might do so – the procedures used to get there. Accordingly, we focused on the general nature of a problem's solution rather than the path to that solution.

Third, we wanted to aim for a coding framework that was less subjective. That is, guessing at the particular ways students might accomplish a problem is challenging, as is guessing at an author's rationale for a problem. We wanted to classify more objective aspects.

Fourth, we sought a framework that would reduce the linguistic complexity in the posing of mathematical problems. We are dubious of being able to exhaust all problem types given the diversity of interesting tasks out there. Since we ultimately cared about the linguistic genre of mathematics problems, we restricted ourselves to common types even if we missed special cases.

Framework Definitions and Descriptions and Examples of Each

Table 1 provides our initial framework of 8 problem types, along with working definitions. We approached this task with a perspective of mathematics as being oriented around sets of objects, and operations on those objects[1]. For collections of objects, we also might talk about descriptions of or relationships between them. This orientation means we need to have clarity about the nature of the elements within those sets to consider problems with respect to our categories. We can understand the 8 types in terms of three clusters depending upon the requested response: something related to a solution set, the outcome of a procedure to be applied, or a statement or justification. We also attempt to convey each category as not being domain specific (i.e., they would cover many problems in most undergraduate mathematics courses).

Table 1. Problem type framework and definitions

Cluster	Problem Type	Definition
Solution set	SOLVE	Respond with *the complete solution set of mathematical objects* that satisfies a given set of properties; the complete solution set may come in a particular representation, such as a variable expression or a graph
	EXEMPLIFY	Respond with *an example solution of a mathematical object* – or demonstrate the existence of a solution – that satisfies a given set of properties; or show no such solution exists
	COUNT	Respond with how many; this may be the *cardinality* of the solution set
	APPROXIMATE	Respond with *an approximate solution* that satisfies a given set of properties; an estimate, sketch, or hypothesis of the solution
Operation output	EVALUATE	Respond with the outcome of a procedure or operation; simplify, compute, or determine the outcome of a specific process
Statement or argument	CONJECTURE	Respond with a mathematical statement about – possibly the answer to a question about – given objects, properties, or relationships between them
	JUSTIFY	Respond with a justification or explanation for a mathematical statement about given objects, properties, or relationships between them
	DERIVE	Respond with a mathematical statement about given objects, properties, or relationships between them [CONJECTURE], and provide a justification or explanation for the truth of that statement [JUSTIFY]

SOLVE, EXEMPLIFY, COUNT, APPROXIMATE

With these four, problem responses can be framed as being chosen from a larger set of objects, with the objective being to identify what objects satisfy a given set of properties. These four problem types specify four different objectives with respect to a solution set. SOLVE means

[1] Granted, we can talk about an operation as also being reducible to a set, but we kept both ideas here.

to specify the entire solution set; EXEMPLIFY to specify one element of the solution set; COUNT to specify how many are in a set; and APPROXIMATE to approximate a solution.

In mathematics, one might think of numbers as the default set of objects under consideration. But in combinatorics it is a set of outcomes; in graph theory it is a graph (a collection of vertices and edges); in geometry it is collections of points, lines, and planes on a particular kind of surface or space; and so on. Solutions to problems, then, might consist of very different types of objects. We classify a problem as a SOLVE problem if it requests all solutions be found. Equations and inequalities common to the domain of algebra (e.g., $x + 2 = 2x - 7$), which specify a relation between two expressions, are common ways we find problems about solution sets, but they are not the only way. In graph theory, for instance, we might see, "Find all graphs (up to isomorphism) with vertex-degree sequence $\{2,2,2,2,2,2\}$." The critical point is that, amongst a larger collection of objects, a given set of properties is being specified and the objective of the problem is related to identifying objects that maintain those properties.

Notably, when we talk about a set of objects, or a solution set, the way in which those objects are represented is not especially important. The problem would still be about a solution set regardless of whether we chose to represent a solution as a decimal or fraction; or an expression (possibly with variables) or a visual representation of that solution as a point on a number line or a Cartesian graph. The representational form of the solution is not a key differentiator.

EVALUATE

In contrast to the first four, this problem type is centrally about an operation. Here, the task instructs students to apply some procedure rather than specifying the properties the final response must satisfy (as do problems of the first four types).

Operations is a broad category indeed. Symbolic expressions in algebra (rather than equations and inequalities), followed by an equals sign, are a common way EVALUATE problems are posed (e.g., $5 + 7 =$, or $\int_0^4 x^2 =$), though certainly not the only way. Here, the "=" is indicative of the directive to perform the operation and record its result. Nonetheless, it puts basic arithmetic in the same category as some calculus problems. There are also many other types of operations in mathematics for which we tend to use words (not symbolic expressions). For example, problems that ask to "Simplify" an algebraic expression or "Transform" a geometric figure similarly indicate an operation for the solver to perform on an object. Frequently, there are reasonably clear procedures to accomplish these operations, but the critical point is that the objective of the problem is to produce the intended output of that operation.

CONJECTURE, JUSTIFY, DERIVE

Especially in collegiate mathematics, students are asked to produce mathematical statements or justifications (in this context, this includes explanations, arguments, and proofs). In most any mathematics course, students can be asked to explain or conjecture – though the proportion of such tasks is often low. The last three categories of problems are distinct because the response should be a statement or justification. Responses to CONJECTURE tasks should be statements specifying properties or relationships, to JUSTIFY tasks should be justifications, and to DERIVE tasks should be a combination of both. Opposite of the first four, in these tasks a (smaller) collection of objects is often given, and the request is to identify distinguishing properties.

Justifications are prominent in proof-based courses such as analysis or algebra, but problems inviting statements and justification can abound in every mathematical domain. In graph theory,

for example, one might be asked to "Justify that a tree with n edges must have $n + 1$ vertices." Providing explicit rationale for any claim is essential across all of mathematics.

Discussions, Opportunities, and Questions

As we hope was evident from the examples, these 8 problem types provide distinguishable objectives that a mathematical problem might set for a student, while remaining general enough to be applicable across the various mathematical domains. In what follows, we comment on the opportunities this theoretical perspective might offer the field, as well as raise some preliminary tensions and questions.

This framework helps (objectively) describe and classify the objective of different types of mathematical problems. It gives a defined set of language that we can compare to the very messy collection of imperative verbs used in posing problems. Such theory would help us inquire into questions about the intersection of language and mathematical problems, such as: *What everyday language is used to convey these particular problem objectives? Is the variety and variation of problem posing language a source of confusion for students in general, or on particular problem objectives?* (In an initial use of the framework coding textbook problems, there were dozens of verbs used for EVALUATE problems, and the imperative verb "find" could have signified *any* of the problem types!) Since the framework is not domain-specific, another opportunity provided by the theory is to inquire into the types of problems commonly asked in various domains. That is, *are there differences between mathematical domains in terms of the types of objectives posed in problems?* (e.g., is it true that algebra is really "solving"-heavy, or geometry more skewed toward "justification"?) The critical point is that we find potential value in pursuing this theoretical development because it affords opportunities to answer questions around the nature of problem prompts and language that do not exist quite yet.

Beyond broad questions for the audience in this preliminary theoretical report (e.g., *What challenges or opportunities do you see with the framework?*), here we highlight two tensions, with the purpose of leading into some more specific audience questions. First, consider a simple equation such as $2x + 5 = 8x - 1$. Very apparently, this is a search for a solution set – a SOLVE problem; which $x \in \mathbb{R}$ have this property? However, one might also conceptualize this as a search for (x, y) *pairs* (in $\mathbb{R}^2$) that are shared amongst the linear solutions sets of $y = 2x + 5$ and $y = 8x - 1$, the x-values of these pairs being solutions. Furthermore, there are also standard algebraic procedures to solve – an EVALUATE. If we consider an equation as our "base" object (and not a real number), then we might simply be performing an operation(s) on that object – and the outcome of performing that operation(s) is indeed the solution set. This points to the need to clarify "base object" in problems with this framework; or, alternatively, to preference one coding over another in specific situations. *Does having to specify base object go against the aim of less subjectivity? Or, alternately, is having to specify those objects in talking about the goals of problems a further benefit to communication?* Second, the inclusion of COUNT seems reasonable, given that there are many interesting questions about "how many" across mathematical domains – e.g., how many solutions to a polynomial of degree n? how many distinct triangles with lengths 4, 5, and 6? etc. However, enumeration would also be very closely aligned with the field of combinatorics. *Does this make the category type too domain-specific? Are all counting problems COUNT, or are some EVALUATE – and, if so, what is the distinguishing line? Moreover, if COUNT is a category, should there be a category that captures, for instance, the essence of the field of probability, which is about assigning likelihood to an event set (e.g., solution set) within a broader landscape of outcomes?*

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Exploring Students' Reasoning With Function Composition Using Graphical Representations

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In this study, we explore how students reasoned on function composition tasks with and without a dynamic graphical representation. Fifteen graduate students individually participated in the task-based and semi-structured interviews to sketch composite functions given their graphs, equations and dynamic graphical representations. One student's responses were analyzed in this paper. Preliminary results suggest that: (1) incorporating multiple reasoning strategies, especially graphical reasoning, and (2) utilizing the dynamic graphical representation, helped build the student's conceptual understanding of function composition.

Keywords: function composition; graphical reasoning; dynamic graphical representations

Starting from secondary school (cf., CCSSM, 2010), students are exposed to, and taught about, function composition via algebraic substitution. Yet, such procedural approaches are limited in their ability to develop sufficiently deep mathematical understandings of function composition, and sometimes even have detrimental effects (e.g., Ayers et al., 1988; Moore & Bowling, 2008; Chen et al., 2023; LaPlace et al., 2024). In this study, we try to explore the ways students reason about the composition of two functions through graphical and other lenses.

Research Question, Background, and Framework

Specifically, we seek to answer the following research question: *How might students reason about function composition with and without the dynamic graphical representation?*

Function Composition in Extant Literature

As the primary operation in function arithmetic, *function composition* $[g \circ f = g(f(x))]$ is a key content topic in algebra through calculus that helps undergraduate students learn about functions effectively (Ratliff & Garofalo, 2006; LaRue & Infante, 2015). Previous research emphasized the importance of functions and function composition in advanced mathematical topics, such as the chain rule which was described by Webster (1978) as "crucial" and "fundamental" in calculus and its applications (e.g., Abramson, 2021; Clark et al., 1997; Gordon, 2005). However, limited studies have focused on function composition. Studies that have explored students' meaning for function composition across algebraic rules, tables, and graphs largely observe an overreliance on algebraic procedures when engaging with function composition (Bowling, 2009; Engelke et al., 2005; Miller et al. 2015), while also recording the limitations of algebraic approaches (Chen et al., 2023; Meel, 2003; LaPlace et al., 2024).

Dreyfus and Eisenberg (1983) pointed out that without a formula, the graphical representation of function has very little meaning for most students entering calculus. In recent years, increasing numbers of studies have argued that multiple representations might contribute to better conceptualization of function and related topics (i.e. composition), and suggested more attention to graphical perspectives including animated graphs (e.g., Ainsworth, 1999; Artigue, 1992; Chen et al., 2023; LaPlace et al., 2024; Parr, 2023; Van & White, 2004). Furthermore, researchers have claimed that the graphical interpretation would be beneficial for students' understanding of function composition (in chain rule and other calculus topics) (Gordon, 2005; LaRue & Infante, 2015), yet studies have not explored in what ways students might reason about

function composition other than algebraic substitution. In conclusion, there is a dearth of literature on function composition, especially through a graphical lens.

Graphical and Algebraic reasoning

Educators in secondary and college mathematics courses typically approach function composition through algebraic equations $(g \circ f)(x) = g\big(f(x)\big)$. However, traditional instruction through the algebraic substitution of equations does not provide much insight about composing functions when students are provided with graphical representations. Graphs tend to make more visible some properties of functions, such as domain, range, symmetry and continuity. For this study, we differentiate reasoning on equation-based characteristics of functions, which is related to algebraic reasoning (Cuoco et al., 1996; Kriegler, 2008), from reasoning based on graphical characteristics of functions, which is likely similar to geometric reasoning (Crowley, 1987; Cuoco et al., 1996). To help exemplify this difference, a function, algebraically, is often characterized by its equation, e.g., $f(x) = 2x + 1$ (i.e., the structure of the set of points included in the relation); whereas, graphically, we depict a function with a graph (typically a geometric curve in the Euclidean plane), e.g., $f(x)$ is a line.

Methods

To explore how students reason about function composition through graphical and other lenses, with and without dynamic graphical representation, this study employed a qualitative multiple case study through face-to-face and one-on-one task-based and semi-structured interviews between the researcher and participants. As a desirable sample of this study to collect rich and informative data, it would be preferable for the participants to have basic understanding of function composition to be able to reason during the interview. Therefore, university graduate students (n=15) chosen from purposive sampling in a mathematics education program were interviewed.

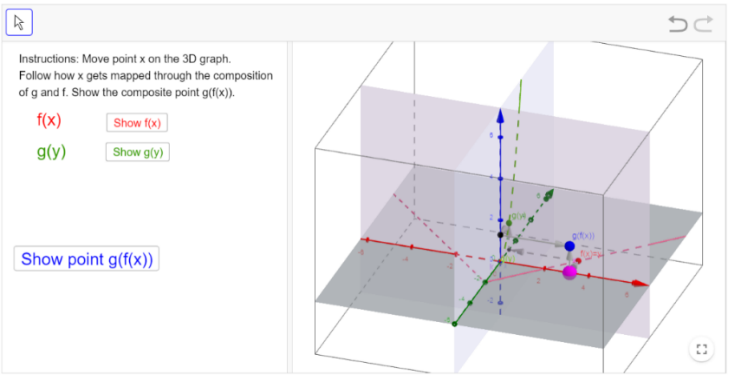


Figure 1. Dynamic Graphical Representation in GeoGebra

The dynamic graphical representation in this study (displayed in Figure 1) is a three-dimensional simulation applet of function composition in GeoGebra. The set up demonstrates a point-by-point analysis of function composition: the user can drag the pink point along the $x-axis$, while the corresponding value of $f(x)$ (absolute value function in red in Figure 1 as an example) will be plotted on the $y-axis$ by an arrow simultaneously; this point is also the input of the function $g(y)$ (where $y = f(x)$). Meanwhile, the value of $g(f(x))$ will be shown on the $z-axis$. Coordinating this point's movement along with the movement of the point on the $x-axis$ (the blue dot in Figure 1) produces the composite function. I aim to explore the affordance of this three-dimensional dynamic graphic representation on students' reasoning, and to study the

possible learning opportunities as well as obstacles when using this representation on solving function composition problems.

The task-based interview contained three rounds. The applet was introduced to participants after the first two rounds, and before the third round where students are asked to use the dynamic representations. Each round of the interview consisted of four function composition problems (due to space constraints, one problem is displayed as an example in Figure 2), and participants were asked to sketch $g(f(x))$ functions given static graphs (Round 1), equations (Round 2) and dynamic graphical representations as well as static graphs (Round 3). There were some shared problems across the rounds in order to make students' responses comparable across the three rounds of interviews. In each round, the researcher asked participants to think aloud while sketching, and students were interviewed at the end of each round with follow-up questions. Interviews were recorded, and video as well as audio data were transcribed. The qualitative data sources included participants' written responses, students' actions when solving the problems, and the transcription of their oral responses.

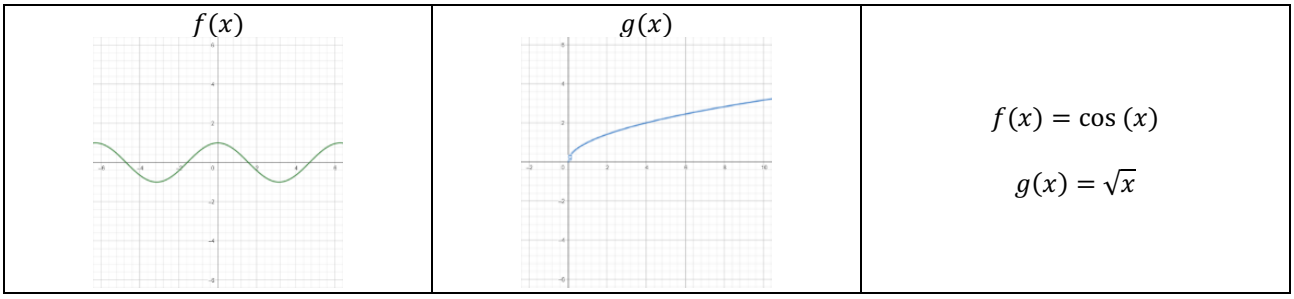


Figure 2. Example: Round 2 Problem 3 (equations only given) and Round 3 Problem 4 (graphs only given)

After initially reading through the interview data, I focused on one participant, Jade (pseudonym), who exemplified rich and interesting strategies of graphical reasoning and other reasoning patterns. The analysis concentrated on identifying the approaches and patterns of reasoning across the problems as well as describing any distinct ways of reasoning graphically in each problem. For the preliminary stage, this study adapted the themes presented in Chen et al., (2023) for the ways students reason function composition. Participant's written and oral responses were coded into categories of: (1) equation related *algebraic reasoning*; (2) graph related *graphical reasoning*; and (3) reasoning through *discrete points*. These were then listed as reasoning steps in chronological order. For example, the participant responded: "I plot some points, 2, 4, 8…and I find it (the graph of $g(f(x))$) grows slower than this one (pointed at the graph $g(x)$), so the graph (pointed at the $g(f(x))$ graph) is like this." This statement above was coded to a step of *discrete points* followed by a step of *graphical reasoning*. In addition, the detailed properties of graphs in the participant's graphical reasoning steps were also identified.

Preliminary Findings

In general, Jade successfully sketched 10 out of 12 composite function graphs. This was an extremely high success rate, especially given results from prior students (e.g., LaPlace et al., 2024). Figure 3 provides an overarching picture of the chronological solving process Jade went through for each problem in each of the three rounds.



Figure 3. Jade's solving process with reasoning steps (algebraic reasoning in yellow, graphical reasoning in blue and discrete points in green)

In most problems (11 out of 12), Jade applied more than one way of reasoning, and half of the problems (6 out of 12) were solved utilizing all three types of reasoning. Figure 3 reveals that, for Jade, different reasoning approaches in one problem were not isolated from each other, instead, the interplay among ways of reasoning drove the solving process forward. In particular, the pattern of alternately using graphical approaches and discrete points is typical (e.g., Round 1 Problem 3). In this pattern, Jade usually repeatedly applied the strategy of: (1) observe certain properties of the general function shape, and then (2) examine her estimation or specify the precise location of the shape by checking specific points on the function graphs.

Jade was an interesting case because graphical reasoning was a leading approach across three rounds (63%, 68 out of in total 108 steps), especially in round 1 and 3 when f and g were given initially as graphs (not equations). It is not surprising that Jade approached problems through algebraic thinking more frequently in round 2 when given equations. Even so, graphical reasoning still appeared in parts of those problems. Jade applied graphical reasoning through the general shape, domain and range, continuity, symmetry, concavity, and periodicity of function graphs. For instance, for domain and range in one problem Jade reasoned: "The first thing I know is because $x \geq 0$ (pointed at g(x)), so we only pick the point above x-axis (pointed at and meant the output of $f(x)$)…" This graphical observation critical for informing for which x the composite function is defined.

Unlike existing literature (e.g., Miller et al., 2015), Jade did not show an overreliance on algebraic procedures. From Figure 3 it can be concluded that her attempts at reasoning algebraically mostly happened at the beginning of the solving process, and then Jade paid more attention to the other two reasoning strategies. This is probably because – particularly with the intentionally unusual and unfamiliar combinations of parent functions in many of the problems – the equation of the composite function is still difficult to use to quickly generate its graph. We do note, however, that Jade incorrectly sketched the composite function to be the line $y = x$ without considering the domain issue, when she algebraically figured $f(x) = \ln(x)$ and $g(x) = e^x$ (Round 2 Problem 1) simply "cancelled out with each other". This mistake coincides with what

the literature raised about the negative impact of relying solely on an algebraic approach (e.g., Chen et al., 2023).

Compared with the rounds without dynamic graphical representations, Jade solved the tasks with fewer reasoning steps in round 3 when she was supported with the dynamic graphical representation (see Figure 3). Moreover, the fewest number of algebraic steps appeared in round 3 as Jade only tried to reason from equations once. The dynamic representation supported her reasoning about function composition without the aid of equations. Jade also developed more sophisticated conceptual understanding of function composition while interacting with the dynamic graphical representations. In Round 2 Problem 1, Jade could not graphically explain why two curves ($f(x) = \ln(x)$ and $g(x) = e^x$) combined to a straight line ($y = x$ on the positive domain). By comparison, with the same problem for Round 3 Problem 1, using the dynamic graphic representation Jade concluded and explained the composition of inverse function will make a line of $y = x$ because: "…when the blue ball moves (point of $g(f(x))$), the pink (point of initial input x) and green ball (point of $g(y)$) moves with the same pace". At the end of Round 3, Jade shared: "…It's like (the dynamic representation is) more direct for me to find (how) the x of $f(x)$ and the y for $g(y)$ make the final composition of the function." From the experience with the dynamic graphical representation, she changed her opinion from previous rounds and concluded that "…for the shape of the final graph, I think we need to consider both two (parent function) graphs."

Discussion and Future Research

This novel study investigated patterns in the way a student reasoned about function composition when given static and dynamic graphical representations. It reported, for the first time, a student's detailed graphical reasoning approaches applied on composing two functions.

The results of Jade's case suggest the benefits of incorporating diverse reasoning approaches. Particularly, one prominent and effective strategy Jade implemented was approaching properties of function graphically, and alternately confirming or refining the sketch of a composite function's graph with discrete points. This finding demonstrates the feasibility and the advantages of understanding function composition beyond algebraic substitution.

Jade's responses also revealed the promising potential of graphical reasoning, as it is fully possible to sketch the composite function graph by reasoning exclusively through graphical lenses without knowing or using the function equations. Jade's reasoning path in Round 3 Problem 4 is one example, where even with her previous experiences with the equations in Round 2 Problem 3, Jade still addressed this problem solely through graphical approaches.

Lastly, yet importantly, the comparison between Jade's reasoning with and without the dynamic graphical representation highlighted the possibility of using this representational form as a viable tool. Almost no algebraic reasoning approach appeared in round 3, which indicates the potential of the dynamic graphical representation to foster graphical reasoning, and to support students in overcoming an overreliance on algebraic equations. More significantly, with the dynamic graphical representation Jade showed a higher-level of understanding on the behavior of function graphs and how the graphs interact with each other when being composed.

I am interested to learn from audience: (1) how can college instructors introduce function composition through a graphical lens? (2) how do you think this approach can benefit or hinder students' learning about advanced calculus related topics, such as the chain rule for derivatives?

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Female Minoritized Students' Sense of Belonging in College Algebra and Precalculus Classrooms at a Diverse Minority Serving Institution

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Sense of belonging is an essential factor in supporting student success in undergraduate STEM courses. Female minoritized students, in particular, are less likely to report feeling a sense of belonging due to experiences of isolation, bias, racial and gender microaggressions, and stereotype threat. Although researchers have investigated sense of belonging at the campus or department level at PWIs, HBCUs, and research institutions, less is known about students' mathematics classroom belonging at open-access, racially diverse institutions. In this report, I present the preliminary qualitative results from a mixed methods study regarding the aspects of classroom experiences that foster sense of belonging for Black and Latina female students in college algebra and precalculus classrooms at a Minority Serving Institution. My qualitative findings indicate that sense of belonging is influenced by mathematical microaffirmations, mathematical microaggressions and mathematics self-efficacy.

Keywords: Belonging, Intersectionality, Gender, Race, Minority-Serving Institution

Sense of belonging at the classroom level has been identified as a key factor in supporting student's engagement, persistence, academic confidence, and achievement in college (Wilson et al., 2015). However, minoritized female STEM students are more likely than any other demographic group to feel a lack of sense of belonging to their campus (Johnson, 2012) and to their STEM major (Rainey et al., 2018). Researchers argue that minoritized female students face an additional burden in STEM introductory courses, as they navigate both gendered and racialized experiences due to the "double bind," i.e., their intersecting racial and gender identities (Leyva et al., 2020; Museus et al., 2011). Intersectional work that investigates minoritized women's experiences in STEM exist (Borum & Walker, 2012; Leyva, 2016; McGee & Bentley, 2017), but literature addressing minoritized women's sense of belonging remains underdeveloped especially in diverse introductory mathematics classrooms.

Introductory mathematics courses such as college algebra and precalculus can serve a gatekeeping function, as it is one of the places where students decide whether to stay or leave STEM, especially in open-access institutions (Seymour & Hunter, 2019). There are limited studies on mathematics classroom-belonging that consider race or gender, and even fewer that address the intersection of these dimensions in diverse mathematics classrooms, particularly for female minoritized students. Understanding female minoritized students' sense of belonging, therefore, is a key aspect in better understanding their academic experiences in gateway mathematics classrooms, and by extension, understanding how to better support their persistence and achievement through classroom structures and pedagogies that foster rather than hinder students' sense of belonging (Hurtado & Alvarado, 2015; Museus et al., 2017; Strayhorn, 2018).

Consequently, I examined how minoritized female students with intersecting racial and gender identities experience belonging, in gateway mathematics classrooms in a minority-serving, open-access institution with a racially and socioeconomically diverse student population, a context seldom considered in belonging studies. The following research question guided this study: What aspects and experiences in college algebra and precalculus classrooms

support or hinder Black and Latina female students' sense of belonging at a racially and ethnically diverse open access institution?

Theoretical Perspectives

To explore female minoritized students' sense of belonging and experiences in gateway mathematics classrooms, I framed my study using the two conceptual frameworks: (a) sense of belonging and (b) intersectionality. The importance of *sense of belonging* as a conceptual framework has been well established in the literature (Ostrove & Long, 2007; Strayhorn, 2018). Researchers have connected sense of belonging to student's identity, types of institution, interaction with peers, relationship with faculty, campus climate, engagement, and positive student outcomes (Bollen & Hoyle, 1990; Hurtado et al., 2015; Strayhorn, 2013; Wilson et al., 2015; Zumbrunn et al., 2014). For this study, I used Strayhorn's (2018) definition of sense of belonging: "Students' perceived social support on campus, a feeling or sensation of connectedness, and the experience of mattering or feeling cared about, accepted, respected, valued by, and important to the campus community or others on campus" (p.4).

My second framework, *intersectionality*, is related to one of the core elements of Strayhorn's (2018) theoretical model of belonging: social identities intersect and affect students' sense of belonging, and students experience belonging in different ways. Crenshaw (1989, 1991) is credited with introducing the term "intersectionality," in order to explore how Black women experience double-discrimination on the basis of race and sex. For my study, I used Collins and Bilge's (2020) working definition of intersectionality: "Intersectionality investigates how intersecting power relations influence social relations across diverse societies as well as individual experiences in everyday life. As an analytic tool, intersectionality views categories of race, class, gender – among others – as interrelated and mutually shaping one another" (p. 2).

Background Literature

Research has indicated that sense of belonging is correlated with positive outcomes such as academic achievement, retention, persistence and mental health (Strayhorn, 2018; Gopalan & Brady, 2020). Conversely, a lack of sense of belonging is the primary cause of student opting out from their STEM major, particularly among minoritized and female students, even when achievement is high (Strayhorn, 2018; Good et al., 2012). Frequent validating interactions with supportive faculty, engaging pedagogies, positive interactions with diverse peers, a welcoming campus culture/climate, racial/gender diversity, living on campus, and positive perception of one's cultural identity positively contribute to minoritized students' sense of belonging (Strayhorn, 2008; Hurtado et al., 2015; Gasman & Nguyen, 2016; Hurtado & Carter, 1997; Johnson, 2007; Lee & Davis, 2000; Locks et al., 2008; Museus et al., 2017; Nguyen et al., 2021; Velasquez, 1999). Moreover, when students feel a sense of belonging in their educational environment, they are willing to take risks, challenge themselves, and commit to their major (Booker, 2016).

Although most belonging studies are situated at the campus level, because the classroom is the primary context where student and faculty interactions take place, it serves a critical role in promoting sense of belonging and may even determine which students decide to remain in STEM majors (Riegle-Crumb et al., 2019; Seymour & Hunter, 2019). Studies have linked classroom belonging to self-efficacy, academic motivation, engagement, confidence, achievement, and persistence (Freeman et al., 2007; Kirby & Thomas, 2022; Wilson et al., 2015; Zumbrunn et al., 2014). Moreover, students feel more connected in classes with instructors who are perceived as caring, open, supportive, competent and incorporate active learning pedagogies

that encourage peer collaboration (Wilson et al., 2015; Kirby & Thomas, 2022; Booker, 2007). Despite the importance of sense of belonging, the research remains underexplored for female minoritized students in diverse settings (Booker, 2016; Johnson, 2012; Rainey et al., 2018; Strayhorn, 2018), particularly in the mathematics classroom context.

Recent research point to faculty microaffirmations as a positive factor in students' sense of belonging in STEM (Demetriou et al., 2023; Estrada et al., 2019). Cawley and Wilson (2023) defined *mathematical microaffirmations* as subtle actions or statements that can support students' positive mathematics identity by affirming that they belong in mathematics. Mathematical microaffirmations are more significant when they come from faculty or others who have influence in a learning context and are most powerful when they are specific to the individual student. In contrast, mathematical microaggressions are subtle ways in which instructors convey the message that students don't belong in mathematics which can have profound negative effects (Su, 2015).

Methods

This study was conducted at an open-access, four-year, public college in the Southeast region with an ethnically diverse population of about 12,000 students (12% Asian/American, 32% Black, 27% Latinx, 24% White/ 59% Female and 41% Male). Approximately 40% of the entering freshman class are first-generation students and over 50% are eligible for a Pell-grant. The college has also earned two specific designations under the Minority Serving Institutions Program: Asian American and Pacific Islander Serving Institution and Hispanic Serving Institution. College algebra or precalculus is usually the first math course which STEM first-year students take at the institution.

This report is part of a larger mixed-methods study which integrated both quantitative (pre- and post- sense of belonging surveys) and qualitative methods (mathematics autobiography and semi-structured individual interviews). In the first quantitative phase, 1,135 students completed the presurvey at the beginning, and 639 students completed the post-survey at the end of the Fall 2023 semester in 44 sections of college algebra and 14 sections of precalculus. This paper reports on the second qualitative phase, in which I interviewed 13 students (two Black-Latina students, five Black students, and six Latina students) at the end of the Fall 2023 semester. Of the 13 interview participants, 11 also completed mathematics autobiographies. Each hour long semi-structured interview was audio recorded and transcribed; all students names are pseudonyms.

I analyzed the transcripts and mathematics biographies using Braun and Clarke's (2006, 2021) six-phases of thematic analysis using MAXQDA. I began the coding process using a deductive approach using my two theoretical perspectives, sense of belonging and intersectionality. I then shifted to an inductive state of coding (Esposito & Evans-Winters, 2022) in which I: (a) read through the transcripts, (b) engaged in descriptive coding by assigning one word or short phrase to key moments, and (c) engaged in in vivo coding with the participants' words/phrases. Through this process I constructed themes from combining or rearranging codes. I also kept a research journal to write down reflections, emergent ideas, questions and research progress.

Findings

In examining how Black and Latina female students described their experiences, two major themes emerged as pivotal in affecting their sense of belonging: mathematical microaffirmations/ mathematical microaggressions, and mathematics self-efficacy. Quantitative results indicated

that five students' sense of belonging to their mathematics class increased, four students' belonging stayed relatively the same, and four students belonging' decreased.

Mathematical Microaffirmations and Microaggressions

Three of the four students whose sense of belonging increased, discussed feeling uncertain of their sense of belonging at the beginning of the semester, but felt encouraged by their mathematics professor which helped them believe that they could do the mathematics. Alexa said, "I was just like, I don't know what I'm doing but, she came over and she was like, you're on the right track…you're doing this perfectly…I know what I'm doing and I'm doing this correctly. So, all of this is doable." Marlina also credited her professor for making her believe that she was capable of being successful in her precalculus class:

> He would ask a question and he would look straight at me and be like, Marlina, you're on the right track. And I'd be like, I don't know. And he'd be like, "Okay, well, what are you thinking? And then I'd tell him what I'm thinking. He'd be like, "Yes, that's exactly you are-- you're on the right track. And I always felt encouraged. And then it also made me feel good to go up on the board and do it. I didn't feel like I was bad at math anymore. I felt like if I applied myself, then I could accomplish things… I felt I didn't want to hide and disappear into the back of that wall.

Mathematical microaffirmations also involves listening to acknowledging and validating students' mathematical questions and concerns (Cawley & Wilson, 2024). Tyanna shared how she struggled with mathematics test anxiety and her professor pulled her aside after class to encourage her and give her strategies to deal with it. Tyanna stated, "I felt seen because I didn't know it was a thing. I thought it was just based, like, anxiety about anything because I always have a little bit, but it just felt like, oh, other people experience it, and it's not like I'm being dramatic." Moreover, Tyanna appreciated how her professor was patient and understanding when someone asked a question on a problem even it was "easy."

On the other hand, examples of discouraging mathematical microaggression phrases that students shared which made them feel that they didn't belong in the class were, "Everyone should know this," "This is high school," "This is easy," "You should already know this, so we're just going to skip this step". Participants shared that they felt "stupid," or not as smart as other people in the class, and not wanting to ask any more questions. Anela described how she felt discouraged after experiencing a mathematical microaggression.

> I mean, like, if somebody is telling you over and over in an aggressive tone, this is review, I feel let down because maybe I should know this, but even though everybody in this class is struggling with review topics, it's not something that you should keep calling out we should know this. How about, let's practice this or let's go over this real quick since I see that majority of you are having a problem with it… I felt like maybe I wasn't smart enough because she kept over saying like, oh, you don't get this. Well, maybe you should drop out of the class right now. I take it seriously when it's coming from a professor because I'd rather do the smart thing, you know? So, I did feel a little academically insecure.

Mathematics Self-Efficacy

During the interviews, seven students (three students whose belonging decreased, all four students whose belonging was constant, and one student whose belonging increased) shared that they felt behind, did not feel prepared for their mathematics course, or felt helpless even though they tried their best. For instance, Jess explained that she did not learn key mathematical

concepts in elementary school, which made her feel as if she was getting judged in her college algebra class for struggling. Therefore, Jess did not feel belonging in the class explaining, "everybody would get a question down. I'm still trying to figure out how do you even go about it. It made me feel left behind. I would just slowly move on from the question and going on to the next, so it won't seem like I'm stuck on this question forever during groupwork. If I don't understand it, I'll let someone else take over." Similarly, Faith also shared that everyone else seemed to understand, which made her refrain from engaging in class activities with her classmates. Likewise, Britteny also felt discouraged when she just could not understand a problem: "It wasn't more like anyone made me discouraged. It was like an internal thing. Like wow, how come I don't understand this? You just feel a little bit down, the way he explained it is so easy, but I'm just not getting it, you know?" By the end of the semester, Ana had given up on her college algebra course: "the more that I knew that I couldn't get it, I couldn't do it, it's kind of just discouraging, to where one gives up."

In contrast, all four students whose belonging increased, viewed themselves as capable of doing well in their mathematics class. When asked what made her feel belonging in her precalculus class, Alexa, pointed to "the validation that I can do the work and understand it as much anyone else." Another student, Leslie, described herself as "really good at math," found her professor easy to understand and shared, "I felt like I belonged." For Marlina, at the beginning of the semester it was dreadful going to class, but that slowly changed for her. "I think my sense of belonging increasing definitely made me do better, because if I had continued to feel like, I hate this, I don't want to go to class, and if I continue to have that shadow on myself, I wouldn't have done well in the class. But because I felt like I can do this, like other people are encouraging me, and then I was able to do a lot better in the class, and I enjoyed it, ultimately. I was proud of myself in the end."

Discussion

My study differs from past studies, as Black and Latina female students did not discuss experiencing racial or gender microaggressions explicitly; rather, the participants expressed appreciation of seeing diversity in their classrooms. They did however raise mathematical microaffirmations/microaggressions and mathematics self-efficacy as key to their sense of belonging. Therefore, to increase students' belonging in the classroom, mathematics faculty should consciously include mathematical microaffirmations in their interactions with their students, by noticing and encouraging their efforts, and creating space for students to make mistakes, present their work in small groups, and discuss their thinking out loud with peers during class. They should also avoid unintentionally discouraging students with mathematical microaggressions as it decreases students' sense of belonging.

Although mathematical microaffirmations is critical in increasing students' sense of belonging, it is not sufficient. Students' mathematics self-efficacy is also related to students' sense of belonging, although the directionality between self-efficacy and belonging is less clear. However, preliminary analysis suggests that faculty can influence students' self-efficacy which can support or hinder belonging and vice versa. Participants discussed the following ways in which their professors can help students develop mathematics self-efficacy: (a) frequent informal feedback aside from assessment, (b) approachability and availability of professors outside of class, (c) engaging activities that include peer-collaboration, (d) a classroom environment where questions are encouraged, and (e) opportunities for students to experience small successes in the classroom. These findings provide evidence of the significance of faculty's role in fostering students' belonging through mathematical microaffirmations and mathematics self-efficacy.

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A Design Principles Framework for DEI Professional Development for Mathematics Graduate Teaching Assistants

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Diversity, Equity, and Inclusion (DEI) has become an increasingly important topic in mathematics education and in the professional development (PD) of Mathematics Graduate Teaching Assistants (MGTAs). Currently, little is known about the types of initiatives that are effective in supporting MGTAs' use of equitable and inclusive teaching practices. Based on data from 45 MGTAs enrolled in our DEI-focused PD program, we propose a preliminary framework to elucidate the characteristics of activities that support MGTAs' engagement in DEI issues. We illustrate the utility of the framework by highlighting examples of activities in alignment with the framework, as well as the memorable impact of these activities on the MGTAs who completed coursework in our PD program.

Keywords: DEI, Professional Development, Graduate Teaching Assistants

In recent years, scholars have made numerous calls for improving Diversity, Equity, and Inclusion (DEI) in mathematics education (AMS, 2019; MAA; 2019; Voigt et al., 2023). This involves addressing issues on all levels: societal, structural, departmental, and within the classroom. At the classroom level, enacting teaching practices that advance DEI efforts typically require appropriate training within a supportive system. With respect to Mathematics Graduate Teaching Assistants (MGTAs), the literature is clear that most MGTAs would benefit from more professional development (PD) that enables them to test and adopt new teaching practices (Deshler et al., 2015; Beisiegel, 2019; Speer et al., 2010). The professional development of MGTAs is one of many important areas to focus on in undergraduate mathematics education since MGTAs often lead multiple recitations for large lecture courses or teach lower level mathematical courses as the lead instructor (Selinski & Melbourne, 2015).

Although many institutions publicly state their support of DEI efforts, Cumming et al. (2023) discovered that only 21% of 83 universities that they polled met their DEI goals. This indicates that despite being actively engaged in DEI efforts, there is much to learn about what types of initiatives are effective in supporting DEI efforts, including the adoption of equitable and inclusive teaching practices. While studies on DEI training have increased over the years, the field is still investigating the efficacy of such training (Wang et al., 2024). We investigate one facet of DEI PD, namely design principles of effective DEI PD activities. The goal of this paper is to explicate a preliminary Design Principles Framework for individual activities that promote MGTAs' abilities to enact equitable and inclusive teaching practices, and to argue for the utility of the framework through a) examples of activities that employ the Design Principles Framework, and b) excerpts of MGTAs' experiences with and takeaways from those activities.

Background

At our respective universities, we employ the ELITE PD program which consists of a pre-fall term orientation, a sequence of three courses (each lasting one term, semester or quarter), and

a peer mentoring experience. This professional development program involves developing and advancing MGTAs' teaching practices that foster student engagement and active learning, equity, and inclusivity in the classroom. The authors are the instructors of these PD courses and have observed and reflected on how our MGTAs interacted with the curriculum.

The Design Principles Framework

Based on the literature on PD for teachers, our reflections on teaching, and our analysis of MGTA feedback from surveys and exit tickets, we provide four *Design Principles* when considering whether an activity will be productive in advancing MGTA's abilities to engage in DEI issues. It should be noted that not every activity will 100% adhere to these principles. Instead, we claim that these are useful guidelines that one should keep in mind when preparing an activity for MGTAs. Figure 1 summarizes the Design Principles Framework.

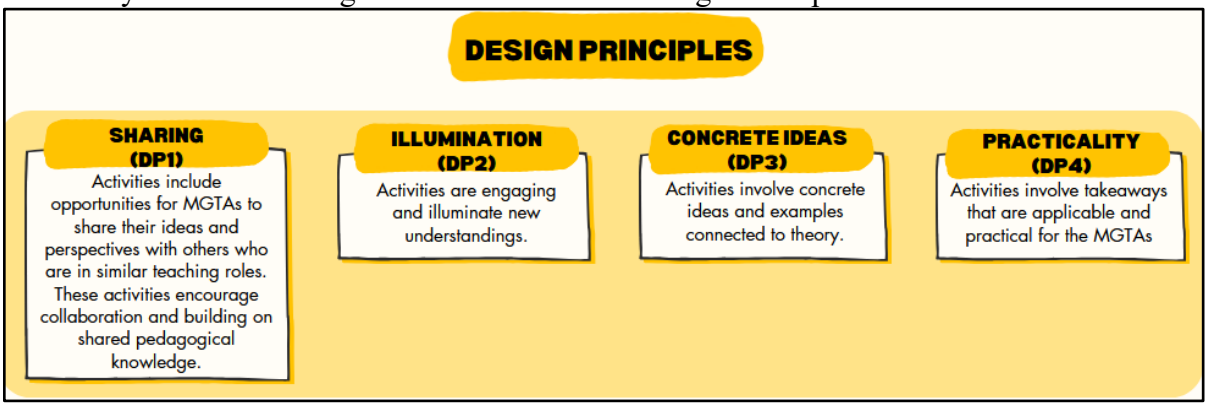


Figure 1. The Design Principles Framework

DP1 Community and Collaboration: Activities should include opportunities for MGTAs to share their ideas and perspectives with others who are in (or have been in) similar teaching roles. These activities should facilitate collaboration and building on shared pedagogical knowledge.

DP1 highlights the importance of the MGTAs as a cohort of educators and learners. Since the MGTAs have a unique composition of backgrounds and experiences, professional development activities should leverage those by providing spaces for MGTAs to share and discuss ideas as fellow educators (Quaisley, 2023; Oehrtman et al., 2009). We have consistently found that the MGTAs appreciate the opportunities to learn not only from us as the instructors, but also from their fellow MGTAs who are in similar positions. Practically speaking, this means that the kinds of discussions questions should be open-ended and in an environment that is safe for MGTAs to share. Additionally, since MGTAs usually have similar teaching assignments, providing the space to discuss their classrooms furthers their sense of community as a cohort of educators.

DP2 Awareness Leading to Enactment and Engagement: Activities should provoke engagement and illuminate new understandings. (How we intend MGTAs will experience and interact with these activities).

DP2 describes how activities should be oriented to productively engage MGTAs in reflecting on issues in DEI. We have observed that MGTAs have stronger recollections of DEI issues when the activities involve self-discovery, illumination of different perspectives, interaction, and/or a form of novelty. In essence, productive PD should not have MGTAs be passive learners, instead activities should have them take an active role (Soto & Marzocchi, 2021; Van Veen et al., 2012). For example, telling MGTAs to "do group work in their classes" and giving them a long list of methods has not been effective in shifting their pedagogy. Instead, we have MGTAs engage with

a few active learning techniques, thoroughly discuss how they can be employed, and create examples of implementation from the courses they teach. Additionally, we also provide hypothetical situations focused on specific equity issues, such as how to support women who are consistently underestimated by their male peers (Grunspan et al., 2016), when engaging in group work activities. These types of activities should support MGTAs in understanding the perspective and experiences of marginalized populations in mathematics.

DP3 Concrete Ideas and Examples Connected to Theory: Activities should not just be about theory, they should involve concrete ideas and examples connected to theory.

DP2 and DP3 are interconnected principles, where DP2 refers to how MGTAs should interact with the activity and DP3 refers to the substance that composes those activities. Specific examples help MGTAs process complex ideas from theory by providing a mental anchor they can associate with these theoretical topics (Darling-Hammond et al., 2017). For example, one activity we use involves MGTAs reading quotations from disabled students about their university experiences. These narratives allow students to learn from actual students about access and design issues that might have been previously imperceptible to them, which allows them to become more aware about the access needs of all students. In essence, effective activities in DEI PD should not be purely lecture and abstract. Instead, providing examples and sample narratives aids MGTAs in understanding the complexities of DEI issues.

DP4 Practicality and Actionable Takeaways: Activities should involve takeaways that are directly applicable and practical for the MGTAs.

While activities should be engaging and involve specific examples, there also needs to be tangible ways to shift their pedagogy. In other words, the activities should meet MGTAs where they are and provide something they can try out or consider in their classrooms (Sims & Fletcher-Wood, 2021). MGTAs need such opportunities to practice new techniques and get appropriate feedback. For example, in one of our classes, we assign MGTAs one active learning technique to learn about (they are given the resources to do so) and then report back on what the activity entails, what situations it is useful for, and a short sample of how they might implement it in their own class. Our MGTAs often reported back to us that they tried the techniques they learned about and mentioned that they were successful in making their classroom more interactive and engaging for the students.

Examples of Implementation

In this section, we provide samples of activities from the PD program that we have observed the MGTAs as finding memorable and impactful. We bolster our claims from data we collected from our 45 MGTAs by means of Exit Tickets from individual classes, End-of-Course Surveys, and Clinical Interviews with the MGTAs throughout the academic school year. To demonstrate the Design Principles in action, we provide two examples of activities that we have employed at our three universities. We provide a summary of the activity, sample discussion questions, and highlight how the Design Principles are employed.

Vignettes

The usage of vignettes, short descriptions of scenarios or teaching clips, has long been utilized in advancing teacher education (Angelides & Gibbs, 2006; Herman, 1998). Throughout the entire ELITE PD program, we employ vignettes to discuss specific topics in equity and issues pertinent to our local institutions. For example, we utilize Yong's (2018) vignette that focuses on an issue prevalent in STEM where women deal with the "chilly" climate by being undervalued and spoken over (Simon et al., 2017, p. 300) (Figure 2).

Rhea said that she found herself the only woman in a group with three men. The students were asked to answer questions on the back of a worksheet, which involved parsing some definitions on the front of the worksheet.

What do you think happened?

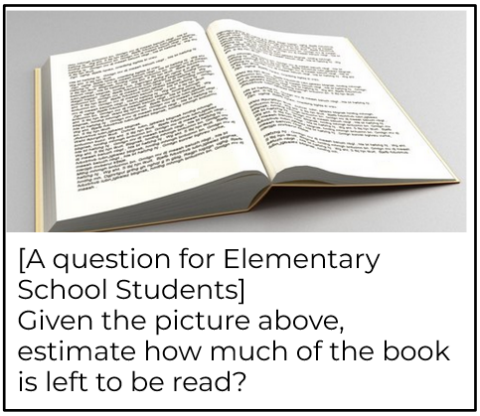
The three men chose to forge ahead on the questions without reading the definitions first, referring to them when they needed them. She wanted to read the definitions first and understand them before moving on. They left her behind and she basically worked on the assignment by herself.

How do you think student Rhea felt? How could the TA handle the situation?

Figure 2. Sample Vignette Targeting Equity Issues in STEM

With all the vignettes, our intent is to provide a realistic situation [DP3] that provides insight into a particular area or specific DEI issue [DP2]. Usually we leave open-ended questions such as "what do you think happened?", "how could the TA handle the situation?", and "what could the TA do next time to prevent this issue?" which allows for the MGTAs to share their unique perspective with their peers [DP1] as well as engage in thinking practically about what they would do as the instructor [DP4]. There is a wealth of sample teaching vignettes available online such as the works of Saul (2019) and Yong (2018), but we also advocate for drawing on the MGTAs' experiences in creating scenarios of issues pertinent to the local context.

Problematic Assessments



[A question for Elementary School Students]
Given the picture above, estimate how much of the book is left to be read?

Figure 3. The Book Approximation Problem

In this activity, we have our MGTAs discuss racially biased questions and reflect on what it is trying to assess and who it actually benefits. We start with a problem that we believe our MGTAs would find innocuous (Figure 3). From our experience, the MGTAs would answer in a normative way because of their accustomization in reading left to right, however some elementary school students with international backgrounds answered it incorrectly due to their cultural influence of reading right to left. When told this information, MGTAs start to realize their assumptions (one being the myth that mathematics is purely "objective"). We then move onto other racially biased questions, where we discuss what the question is trying to assess, what it actually assesses, who benefits, and what might an alternative assessment be.

In this activity, the MGTAs go in not knowing the conclusion, which enables them to be surprised by how the book question could be biased and provides an opportunity for them to reflect on their assumptions [DP2]. We have found that the book example as an introduction to problematic assessments has been memorable to our MGTAs in helping them realize different people's backgrounds [DP3]. Additionally, we guide the discussion to give them practical tools

to help them reflect on their usage of assessments. The questions we provide, "What does this question try to assess?" and "Who does this favor?" are easy questions for the MGTAs to ask themselves before deciding on a form of assessment [DP4].

MGTA Reception and Discussion

After implementing the ELITE PD program across several years with 45 MGTAs, we gathered feedback on the program and specific activities through surveys and interviews. Our preliminary data analysis indicates how the Design Principles had positive impacts on the MGTAs in relation to their pedagogy. These excerpts come from the MGTAs responses to questions such as "How has the ELITE PD program helped you thus far?" and "Are there things that really stand out as having a big impact?".

> *The things that had a big impact was I am able to kind of implement within the classroom. For instance, one group talked about series and how they could make that more interactive. So I found that something I could implement within my classroom. In another example, our group talked about the first day of class and how to make an inclusive environment like by asking open-ended questions and validating everyone's opinions that kind of break that math anxiety*. [DP1 & DP4]

> *One thing that really stood out to me was when something was that I hadn't thought about and I'm kind of aware of it. Things like that is like the way the ideas were presented was in a way that halfway through the activity I go "oh, this is important, I should think about that more".* [DP2]

> *We had so many kinds of good examples given, so with each of the lessons, I found that I would find a new aspect of it that I hadn't thought about before. And I think that really is the key of a program like this, it's not about checking things off. But the entire point is to make you think more introspectively, that we learned how to do these things and how to approach them and kind of opening your mind to new forms of pedagogy and examining the ways that we've approached it.* [DP2 & DP3]

Concluding Remarks

Overall, our preliminary data indicates the activities adhering to the Design Principles Framework was effective in making lasting impressions on MGTAs. The majority of the MGTAs stated that the specific examples [DP3] and opportunities to learn from each other were impactful in aiding them to reflect on DEI issues [DP1]. Additionally, the MGTAs valued the ways in which the activities demonstrated how issues of identity and power play out in the classroom [DP2] and that there were clear takeaways they could employ in their classrooms [DP4].

While the feedback from the MGTAs evidence the effectiveness of the Design Principles Framework in a DEI-focused PD, we are cognizant of the need of better assessments to track how MGTAs' teaching shifts in the long-term. However, we see our preliminary Design Principles Framework as contributing to the literature, by taking an initial step of illustrating the potential impact the framework could have on future MGTA PD development.

Questions for the Audience

1) What metrics might capture impactful changes in MGTAs' equitable teaching?
2) What seems important for productively engaging MGTAs in conversations about DEI?

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Eliciting Students' Informal Ideas About Series: The Partial Sum Sequence Game

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Series play key roles in calculus and other disciplines, yet research highlights several challenges with the topic, underscoring the need for curricular innovations to support conceptual understanding. In this paper, we introduce the Partial Sum Sequence Game and argue that it, with the guidance of a teacher-researcher, supported a pair of undergraduate students to make important connections between sequences, sequences of partial sums, and series convergence.

Keywords: Calculus, series, sequence of partial sums, guided reinvention

Series play essential roles in calculus, computer science, physics, and other disciplines. However, several reasons are cited in the research literature for why this topic can be challenging for learners. González-Martín (2010) found that their participating instructors (and the textbooks they used) focused on decontextualized, algorithmic procedures rather than promoting conceptual understanding of series. Additionally, learners may need support in distinguishing between sequences and series (Akgun & Duru, 2007; Kung & Speer, 2013) or viewing them both as particular functions with natural numbers as their domains (Martinez-Planell et al., 2012). Some learners describe series as running totals without coordinating them with index values (natural numbers) (Eckman & Roh, 2022; Martinez-Planell et al., 2012). Such a view explains students' difficulties with series convergence (Eckman & Roh, 2022) and solving problems that require viewing series as a sequence of partial sums (Martinez-Planell et al., 2012).

Because series is a challenging topic, there is a need to improve instruction to promote conceptual understanding of series. Toward this goal, scholars have recommended supporting students in obtaining the first few terms of the sequence of partial sums with different representations (Martínez-Planell et al., 2012) and using concrete examples (Morrel, 1992). This paper discusses a task (game) that aligns with these recommendations. We argue that the game, with the guidance of a teacher-researcher, supported a pair of undergraduate students to make important connections between sequences, sequences of partial sums, and series convergence.

Guided Reinvention and the Partial Sum Sequence Game

We drew on Realistic Mathematics Education's heuristic of guided reinvention (Freudenthal, 2005; Gravemeijer, 1999; Gravemeijer & Doorman, 1999) to frame the instruction during our experiment. In particular, we aimed to support "a process by which students formalize their informal understandings and intuitions" (Gravemeijer et al., 2000, p. 237). The "guided" in "guided reinvention" emphasizes that students are not expected to formalize their informal understandings on their own; the students are guided by strategically designed tasks that feature a well-chosen context problem as well as the instructor's moves. Gravemeijer and Doorman (1999) explain that this context problem should "offer opportunities for students to develop informal, highly context-specific solution strategies" that can "function as [...]catalyst for curtailment, formalization or generalization" (p. 117). The context problem does not necessarily have to draw on "everyday-life contexts" but should be meaningful or "experientially real" for students (Gravemeijer, 1999, p. 158). The Partial Sum Sequence Game, is the context problem that we designed to elicit students' informal ideas about series and series convergence (see Figure 1). We intended to design the game to promote informal thinking of series as sequences

of partial sums and series convergence/divergence. In particular, we engineered the game so that students could be supported to determine that a sequence $\{x_n\}$ won if and only if the series $\sum x_n$ converged.

Game:
We are conducting an experiment in which we move an object a certain number of feet north per day. A sequence $\{x_n\}$ will tell us how many feet x_n to move the object north on any given day n. To win the game, accurately predict the object's location if the experiment continues indefinitely!

Playing the game with $\{n^2\}$:

n	1	2	3	4	5	…
n^2	1	4	9	16	25	…
Total distance from starting point	1	5	14	30	55	…

$\{n^2\}$ loses the game since we cannot accurately predict the object's location

Figure 1. Partial Sum Sequence Game.

As mentioned, instructors' task facilitation also provides essential guidance (Kuster et al., 2017; Rasmussen & Marrongelle, 2006; Stephan et al., 2014). A crucial part of the instructor's role is eliciting and building on students' informal ideas. Kuster et al. (2017) explain that teachers "determine which [elicited] ideas (correct or incorrect) are important and relevant to the development of the mathematics, and which ideas can be leveraged to build student thinking that is commensurate with the mathematical goals of the lesson," and importantly, "use these contributions to inform the lesson" (p. 6). Hence, when considering the implementation of the task, we consider not just features of the task that support student thinking but also the instructor's guidance.

Methods

The data for our study comes from a teaching experiment (Steffe & Thompson, 2000) with two first-year undergraduate students - Lara and Stella (pseudonyms), a teacher-researcher - the first author, and an observer/camera operator. Lara was enrolled in Calculus 2 and Stella in Statistics at the time of the study. Both students had passed Calculus 1 the previous semester. There were 11 teaching experiment sessions that were each 1.5 hours. The sessions were recorded in such a way that synched the students' collaborative digital work with a video recording of the students' discussion and gestures.

We focus this report on Sessions 8 and 9 when the students played the Partial Sum Sequence Game with various sequences that they had previously created and explored. In particular, before these sessions, the students generated different sequences and organized them in an Euler diagram with different sequence properties such as increasing, bounded, and convergent (Vroom et al., 2024). As they used this diagram and examples of sequences to make general conjectures about sequences, they also defined sequence and the various sequence properties. Stella and Lara considered a sequence a function with the natural numbers as the domain, defining it as "a list of only natural numbers as inputs, where there is always an output, and the relationship can be defined as some mathematical function." After Sessions 8 and 9, the students wrote a "cheat sheet" for playing the game, which included several conjectures related to different tests for divergence, such as the comparison test and the divergence test (Davis & Vroom, 2024; Vroom & Davis, 2024).

We began our data analysis for this paper by re-watching the video and re-reading the transcripts. As we did so, we noted key moments, which we took to be instances when students articulated an idea that seemed connected to ideas cited in the research literature as important or challenging. For instance, instances when students' discourse suggested seeing series as sequences of partial sums (Martinez-Planell et al., 2012). We then identified how (if at all) we

thought the task and/or the teacher-researcher's moves guided the students to develop these ideas. We present some of these key moments in the following section.

Results

The game with the teacher-researcher's guidance elicited students' informal ideas about series, and in particular, the students' informal ideas about (1) sequences of partial sums, and (2) the convergence or divergence of sequences of partial sums. We discuss these next.

Eliciting Informal Ideas about Sequences of Partial Sums

Playing the game involved Lara and Stella creating various sequences of partial sums, which the students termed total distance sequences. The students (with the teacher-researcher's guidance) (a) identified these as particular types of sequences and (b) represented the mathematical relationship between the daily distance sequence and the total distance sequence in a couple of different ways. A key moment that led the students to identify the total distance sequences as particular types of sequences occurred when the teacher-researcher introduced the students to the game and suggested playing it with $\{n^2\}$. As she asked the students how far the object was from the starting point on various days, she recorded their responses in a table tracking days with the corresponding total distance. After recording a few days of total distances, the teacher-researcher asked if this table could represent a sequence. The students responded yes, and referenced their definition for sequence that they constructed (see Methods for their definition), checking that it could be "modeled by a mathematical function" and the inputs were natural numbers. The table, along with the teacher-researcher's question, supported students in making explicit connections between their work in the context of the game and the concept of sequences. As they continued to play the game, they further emphasized this connection by using various function representations (e.g., tables, graphs) to represent total distance sequences.

The game context also seemed to support the students in representing the mathematical relationship between the daily distance sequence and the total distance sequence in a couple of different ways. A key moment occurred immediately after Lara and Stella decided that the table of total distances for $\{n^2\}$ could represent a sequence. Lara indicated that the mathematical function representing the total distance sequence was $\{n^2\}$. After the teacher-researcher pointed out that on the fifth day, they were 55 feet from the starting point rather than 25 like n^2 would suggest, the pair continued to brainstorm how to represent the relationship. They focused on how each term was the sum of all the previous terms of $\{n^2\}$, as Stella noted: "So we need to add together all the previous days, squared. I don't know how to write that." To further support the students in articulating their idea, the teacher-researcher asked them to write out how they found different total distances on given days. The students then indicated that they added previous terms, such as $1^2 + 2^2 + 3^2 + 4^2$ for the total distance on the fourth day. To further promote connections between the daily distance sequence and the total distance sequence, the teacher-researcher continued the conversation:

> Teacher Researcher: I thought you were doing something a little bit different whenever you were actually calculating. If we were like looking at the eleventh day [it] didn't seem like you were doing $1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 \ldots$ all the way up to, you know, [11] days. What were you doing?
>
> Lara: Adding the previous total distance?
>
> Stella: Because the previous total distances are all the previous days added together. So, we technically still like are adding all the numbers, it's just shorter.

This led the students to write this strategy as $t_n = t_{n-1} + n^2$. Here, the teacher-researcher gave the students important guidance by (a) supporting them to see that the daily distance sequence representation (i.e., $\{n^2\}$) did not match the values of their total distance sequence and (b) pointing out how their representation did not reflect the strategy they used to find each value of the total distance sequence. After several rounds of the game, the students generalized their idea and noted that the total distance sequence could be represented by $t_n = t_{n-1} + x_n$ where x_n represents terms of the daily distance sequence.

Eliciting Informal Ideas about Sequences of Partial Sums Divergence or Convergence

For Lara and Stella, determining if a sequence won or lost the game was equivalent to determining whether the sequence of partial sums converged. With guidance, the students explained that (a) if a total distance sequence diverged, then the daily distance sequence lost the game, and (b) if a total distance sequence converged, then the daily distance sequence won the game. One key moment that led the students to develop the idea that a sequence loses the game if its total distance sequence diverged occurred as they played the game with $\{n^2\}$. After the students discussed how the total distance sequence was a sequence (see previous section), the teacher-researcher asked if $\{n^2\}$ won the game. Initially, Stella responded, "Yes," but Lara pushed back, saying, "So yeah, wait, no? Well, it's gonna be infinity," and the object was going to be "beyond space." Stella clarified, "So like given a day, yes, we can. I feel like that's what I meant. So, like if they were on day 300 we could figure it out." The students then decided that if the experiment continued indefinitely, then $\{n^2\}$ would lose. Here, we see the students made a key connection: winning or losing the game required them to predict the object's end behavior, not just the location of a given day. The experiment continuing *indefinitely* was an important game feature that supported this idea.

As the students continued to play the game, they articulated that, in general, a sequence loses the game if its corresponding total distance sequence tends to infinity. A key moment that led to this generalization occurred when the students, in response to the teacher-researcher asking the students which sequence they wanted to play the game with next, indicated that they did not want to play with $\{1, 2, 3, 4, 5, 15, 15, 15, 15, \ldots\}$. The students explained:

Stella: Yeah, I don't think we can win this one because you're adding 15 on every time, because it would be going… it would be infinity.

Lara: Yeah, that would be divergent.

Stella: It would be infinity again.

Here, we see the students suggesting that they shouldn't play the game with a sequence that has a total distance sequence that diverges to infinity. The teacher-researcher then continued to support the students in generalizing this idea further, specifically, that a sequence would lose the game if its corresponding total distance sequence diverged in general, not just by tending to (positive or negative) infinity. To do so, she suggested that they play the game with the sequence defined by n^2 if n is odd and $-n^2$ if even. Before calculating the first few terms of the total distance sequence, Stella conjectured that it would lose the game because she expected the total distance sequence to "decrease infinitely." After the teacher-researcher suggested that they calculate the first few terms, Stella noticed that the terms were alternating between positive and negative values and stated, "I take back what I said." She indicated that she still thought the sequence would lose the game because they "would not be able to predict where it's gonna end up" however, it was not because it tended to negative infinity, as Lara elaborated, it was "because it's oscillating, and it's not oscillating by [a smaller amount] … we can't predict it". Here, we see

that by supporting the students to play the game with sequences that would have various types of divergent total distance sequences (e.g., tending to infinity, oscillating), the teacher-researcher reinforced the notion that a sequence loses the game when its total distance sequence diverges.

The students were also guided to see that if a total distance sequence converged, then the daily distance sequence won the game. A key moment that led to this occurred after determining the sequence $\{n^2\}$ lost the game. The teacher-researcher requested that they play the game with $\{½^n\}$. After calculating the first ten terms of the total distance sequence, Stella said, "Dang it, we lost again," presumably noticing that the total distance sequence would not obtain its limiting value of one. The teacher-researcher pushed back, saying "You think so?" and Lara responded, "I don't think so because we're still like converging to one, I guess. So… eventually…" The teacher-researcher connected convergence back to the game, saying, "So if the experiment continues indefinitely, would you be pretty confident to say…" and Stella finished her sentence by saying, "It's just like one." Here, we see that by pushing back, the teacher-researcher supported the students in realizing that a total distance sequence did not need to obtain its limiting value to win the game; it only needed to converge. The part of the game that indicated that the experiment ran *indefinitely* also seemed to support this idea.

The students continued to attend to whether the total distance sequence converged to determine if the daily distance sequence won. For instance, as the students played the game with $\{1/n\}$, the corresponding total distance sequence caused them to pause. The teacher-researcher asked, "What are you trying to notice?" and Stella responded, "If it's going to a certain number," as she tried to determine if it "flattened out." Additionally, when they played the game with the sequence defined by n^2 if n is odd and $-n^2$ if even, the students noticed that they could have a winning sequence that had an oscillating total distance sequence, as Lara explained, "so as long as it narrows into a number." In other words, the oscillating sequence converged. Similar to how playing the game with various losing sequences supported the students in seeing that they generated divergent total distance sequences, playing the game with various winning sequences supported them in seeing that they generated convergent total distance sequences.

Discussion

In this paper, we presented a task that we argue supported, with the teacher-researcher's guidance, a pair of students to make important connections between sequences, sequences of partial sums, and series convergence. We shared how, in the context of the game, the students developed informal ideas related to concepts that have been cited as particularly challenging for students in the research literature. In particular, the students developed informal ideas related to sequences of partial sums and their convergence/divergence.

Further, we highlight how the features of the task and/or moves by the teacher-researcher supported the students in developing these ideas. For instance, to support students in seeing sequences of partial sums as a sequence (a function with the natural numbers as its domain), the teacher-researcher strategically recorded the students' ideas in a function table; Rasmussen and Marrongelle (2006) have conceptualized moves like these as transformational records. We also found that the game experiment ran indefinitely as an important feature to elicit informal ideas about series convergence and divergence. The wording supported the students to connect winning (losing) the game as equivalent to the sequence of partial sum converging (diverging).

The data we presented here are promising first steps towards promoting conceptual understanding of series. In the future, we aim to implement the task in a whole-class setting to continue revising it and suggesting implementation.

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Characterization of Student Learning in a Transition-to-Proof Course

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This research examines the ways students in a transition-to-proof class learn what a mathematical proof is and how to read, construct, and write proofs. We interviewed four university students who had successfully completed a transition-to-proof course about what experiences they thought most influenced their learning about proofs. We then interviewed two instructors of the course about their goals for student learning about the nature of proof and how they devised their instruction to advance those goals. We first describe the activities students engaged in and that the instructors provided for students that supported their apprenticeship into proof-based mathematics. We further analyze how students' characterization of these activities influenced their views of proof.

Keywords: transition-to-proof, cognitive apprenticeship, abstract algebra

The National Council for Teachers of Mathematics includes reasoning and proof as one of five process standards that should be present in curriculum for K-12 students, suggesting that students learn to select and execute a variety of proof structures (NCTM, 2000). Undergraduate students majoring in mathematics or mathematics education must become proficient in both reading and writing proofs in preparation for careers in research and/or education. This proficiency is accompanied by an understanding of what constitutes mathematical proof. This study seeks to add to the growing body of literature on proof by examining the ways students learn what a mathematical proof is and how to read and write proofs. Four university students who had successfully completed a transition-to-proof course and two instructors of the course were interviewed in an effort to answer the following research questions:

1. How do students characterize their learning about what constitutes a mathematical proof in a transition-to-proofs course on abstract algebra?
2. How do instructors characterize student learning about what constitutes a mathematical proof in such a course and adjust instruction to facilitate that learning?

Literature Review

Stylianides proposed a three-part definition of proof that relies on the mathematical community within which a proof is considered (2007). A proof is an argument with the following characteristics: (a) the argument uses statements that are accepted as true by the community; (b) the argument relies on modes of argumentation that are accepted by the community; (c) the argument is represented in a way that is understood and appropriate for the community. We adopt this definition of proof because it attends to varying expectations for the presentation and acceptance of proofs in different communities. It is critical to understand the norms established within the community from which a proof originates, especially for students transitioning from courses that previously focused primarily on computation, modeling, or other forms of mathematical activity. Instructors of the same course at the same university may even have differing goals and beliefs about norms that should be adopted by students, but as suggested by Dawkins & Weber (2017) we anticipate the values that underly those norms are similar.

Universities with mathematics programs often designate a course supporting students' transition into proof-centered mathematics courses. These transition-to-proof (TTP) courses

teach logic rules, proof structures, and set theory (David & Zazkis, 2018). Some TTP courses study specific topics in mathematics (e.g., algebra or analysis) after completing several units about standard proof techniques. A smaller percentage of universities choose to embed all proof instruction within a content area, most frequently analysis.

Theoretical Framework

We consider proof to be a social language, which Bakhtin characterized as "a way of speaking characteristic of a particular group in a particular sociocultural setting (Wertsch, 1991, pp. 95)." One method through which individuals learn a social language is through participation in a *community of practice,* a group of individuals dedicated to refining their practice and training new members (Wenger, 1998). Through interactions with more experienced members, the student engages in a *cognitive apprenticeship of* social expectations, heuristic strategies, and efficiency techniques (Collins & Kapur, 2022).

We adopt Vygotsky's *zone of proximal development* (ZPD) as a theoretical framework for data analysis of sociomathematical development in this study. The standard interpretation of the ZPD is the interplay between what an individual can do on their own and what they can do with the help of a more knowledgeable other (Vygotsky, 1987, pp. 208-209). Vygotsky stresses that all cognitive functions first occur between individuals in a social context (inter-psychological). As an individual becomes more comfortable and knowledgeable about the function, it moves to a psychological context (intra-psychological) (Wertsch, 1991, pp. 88-89).

A complementary framing of the ZPD is based on the interplay of development of spontaneous and scientific concepts (Vygotsky, 1987, pp. 214-222). *Spontaneous concepts* develop through everyday experience, usually without the aid of formal instruction. Ideas within a spontaneous concept are specific and lack conscious awareness, volitional control, and generalization. *Scientific concepts* are those learned through mediated interaction with their meanings, such as instruction. They are more general and systematized but lack the accessibility and applicability of spontaneous concepts. As an individual develops both spontaneous and scientific concepts, the complementary strengths of each support the development of the other.

Methods

Undergraduate and graduate research subjects were recruited for interviews after their successful completion of an Introduction to Abstract Algebra course. This course serves as the transition-to-proof course for the large southwestern university where the study took place. We recruited students who had successfully completed the course within the previous 2-3 years. Four students participated in the research study: Alex, Carly, Dana, and Bennett (pseudonyms). Alex (undergraduate) completed the course 7 months before the interview. She had not taken any other mathematical proof courses. Carly (undergraduate) also completed the course 7 months before the interview. Carly had been enrolled in the same course during a prior semester but withdrew after 10 weeks due to personal reasons. Since her successful completion of the course, Carly had taken introduction to analysis. Dana (graduate) completed the course 7 months before the interview. She took the courses concurrently with introduction to analysis and had successfully completed intermediate analysis at the time of the interview. Bennett (undergraduate) completed the course 19 months before the interview. He had since taken introduction to analysis, two advanced calculus courses, and cryptography.

Each research subject completed one interview lasting 45-60 minutes. Interviews were conducted and recorded via Zoom. Each student interview was divided into 4 parts.

Part 1: Students were asked questions to determine their general understanding of the TTP course's goals as well as specific moments in the course that provoked learning.

Part 2 : Students discussed noteworthy features of two proofs, the first provided by an instructor of the course and the second by the student as an exemplar of their work in the class.

Part 3: Students answered questions about the general nature and purpose of proof, including the context of a proof and the potential evolution of their understanding of proofs over time.

Part 4: The researcher explained the research questions to the student and asked them to describe the process of learning what a proof is and how to read and write proofs.

Instructor interviews contained parallel versions of parts 1, 2, and 4 of the student interviews. Dr. King & Dr. Williams had both recently taught Introduction to Abstract Algebra. Dr. King is an experienced professor who finished teaching the TTP for the first time two months before the interview. Dr. Williams is an assistant professor at the beginning of her career.

All interviews were transcribed for analysis. We reviewed each transcript multiple times, searching for commonalities and differences between student learning experiences. We conducted open and axial coding to identify common themes in the data (Corbin & Strauss, 1990).

Results

Throughout the interviews, students described activities that positively impacted their ability to learn what proof is and how to read and write proofs. The instructors described ways they believed students learn proof and how they facilitated such activities. Each type of activity is described below with examples of how they specifically impacted students' learning of proof.

Conversing with a More Knowledgeable Other.

All four students reported talking to another individual about specific proofs while they were taking the course. In most cases, the more knowledgeable other was the instructor or a student who had been successful in the class before. Alex and Bennett frequently visited their professor's office hours where they had meaningful conversations about homework problems. Bennett claimed that office hours were extremely helpful if an individual prepared for them beforehand. He reported attempting homework problems and going to office hours to discuss specific lines in the proof where he was unable to continue. Bennett's professor tailored his conversations to Bennett's current conception of proof during office hours. The ensuing conversations helped Bennett complete missing pieces of his proof. Carly formed a small study group with another student enrolled in the course. She claimed that simply verbalizing her proof ideas and drafts to her friends often helped push her to more proficient proof production. Carly would put an outline of her proof on a whiteboard and then verbally explain the connections she was going to make throughout the proof. The act of sharing her proof draft with a peer highlighted the importance of making valid mathematical connections in a proof. Carly's social interaction with her peer supported her psychological development of proof writing techniques.

Attending to Written Feedback from the Professor

The students, particularly the three undergraduates, described the utility of feedback from their professor. Most referred to feedback on their homework, which included both a score and comments about the student's work. The inter-psychological experience of obtaining feedback helped support the intra-psychological development of proof concepts. Bennet contrasted the

feedback he received in his first proofs course to feedback in other computation-heavy mathematics courses; he interpreted feedback from non-proof classes as an indication that he was doing something incorrectly. In contrast, he described feedback in his transition-to-proofs class as "the best thing you can be getting." Bennett described the struggle he had adopting the language norms of proof-writing, such as specific sentence structure or vocabulary. He felt that reading instructor feedback taught him many of the linguistic expectations of proof writing.

Carly reported that the instructor of her first transition-to-proofs course provided very little feedback. The second time she took the course, the feedback included edits to her proofs, such as replacing one word with another, which helped Carly understand the language norms of proof writing. If the feedback was unclear, she would speak with the professor to make sure she understood the instructor's intent. Dana claimed that both written and verbal feedback from her professor helped change her mental image of proof.

Dr. King provided extremely detailed feedback to students via Canvas, personalized to each student. He contrasted this with his Calculus courses, where he finds students typically making the same small set of mistakes. The idiosyncratic proof-writing of students in Dr. King's proof course required him to provide feedback that was more detailed repeatedly throughout the semester. His feedback often focused on components missing from student work, such as the presence of a predicate or a variable, and he noted that his students incorporated the feedback to write better proofs by the end of the course.

Consistent Class Attendance

Each of the four students commented about the importance of attending class regularly. Carly took rigorous notes that included the professor's board work and sidenotes referring to the context of classroom discussions. She also reported asking clarifying questions regularly throughout class and claimed that clear instruction and added logical commentary significantly improved her learning experience. Dana learned the value of scratch work alongside a written proof and found it particularly useful when her professor would model such aspects of proof development in class. This instruction helped her shift from memorizing and re-producing a select number of proofs to more engaged problem-solving.

Dr. King claimed that his general approach to instruction in a TTP course was the same as it was in a Calculus course. He discussed familiar concepts, taught them a new concept, and facilitated opportunities for students to experience the new concept productively. In other words, he reminded students of spontaneous concepts, taught a new scientific concept, and then supported the unification of the two. Before engaging in logical mathematical arguments, Dr. King engaged students in considering the logical nature of everyday statements, including using logic to determine the truth value of a statement. He hoped that analyzing non-mathematical statements in a formal, logical way would prepare students to analyze mathematical statements logically.

Exposure to Many Proofs

Each student talked about interacting with many different proofs. This interaction took on a variety of forms including example proofs from instructional materials, proofs found online, and proofs being studied in another course. Alex predominantly worked on her homework independently and in office hours. While working independently she found that reading proofs on the same topic improved her ability to recognize the utility of specific theorems or algorithms. She would search for proofs online that used the theorems she was studying in class. This helped her see how the theorem could be applied in the context of a proof. The perusal of proofs from

online sources also provided her with the opportunity to repeatedly see what a proof looks like and how a proof is structured. She often tried to mimic the structure of proofs found online in her own homework assignments. In preparation for exams, Alex would attempt to prove a statement she had previously proved successfully. She would then compare her proof to one she had previously written, searching for specific missing components in her newly constructed proof. Dana found it challenging when the concepts in her analysis course assumed understanding from the transition-to-proof course she was concurrently enrolled in, but also found it helpful to practice her logic and proof writing skills consistently.

Dr. King felt students needed exposure to many proofs to refine their proof skills. In preparation for exams, he provided extensive study guides with 40-50 problems. His problems were designed to expose students to many types of proof within a wide range of categories, such as contrapositive proofs and proofs by cases. Dr. King said that successful students often worked through these problems and discussed them with him.

Discussion

The activities described above represent ways students are apprenticed into proof-based mathematics. The personalized nature of instructor feedback to students helped students refine their conceptions of proof to more closely align with the norms upheld by the instructor. As instructors made small corrections to student language and mathematical claims, students came to understand the mathematics of proof and the appropriate ways to communicate. Instructors' identification of specific proof components missing in a student proof reinforced the need to attend to a new set of norms of criteria for a valid solution.

Modelling and mentoring played a pivotal role in these students learning of proof. They benefited from instructors modeling not only the structure of a final product but also the problem-solving that accompanies such a product. As students adopted these problem-solving techniques, they grew in their ability to recognize a proof and produce one of their own. Proofs students encountered in other settings, such as proofs found online, also acted as models for what constitutes a proof.

Both students and instructors were aware of the differing needs of students in TTP courses. The needs of students in the TTP course were greater and different in nature than those of students in a computation-heavy courses. Instructors facilitated opportunities for students to converse with more knowledgeable others through office hours, study groups, and written feedback. Instructors were also mindful of the quality and the quantity of proofs students experienced as part of the coursework.

As we continue this vein of research, we will consider data collection methods that investigate social interactions between the student and others, including the instructor and peers. It is perhaps easy to anticipate the positive effects of frequently conversing with the professor of the course. However, it seems that for some students a conversation with a novice may be equally beneficial as a conversation with an expert. When conversing with another novice, the student is perhaps more likely to articulate basic reasoning, which can lead to more concrete formulations of mathematical concepts. Analysis of these critical social interactions will be included in future studies.

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Student Validations of Author-free AI-Generated Proofs

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We report on a small-scale clinical interview study of students' validation of AI-generated proofs. During these one-on-one interviews, students were tasked with using ChatGPT to generate proofs and validate those purported proofs. Consequently, neither the interviewer nor the participant knew if the justifications being validated constituted a proof before evaluating it. As such, no two participants validated the same proof. This required the development of novel post-interview analysis methods, which led to some valuable insights into students' proof validation processes. We view the primary contributions of this work to be both the methodology we developed, and our insights gained into how AI-generated proofs provide a different lens for students' validation and error detection.

Keywords: ChatGPT, proof validation, introduction to proof, logic

Proof validation has been a common line of inquiry in proof education research (e.g., Selden & Selden, 2003; Alcock & Weber, 2005; Panse et al., 2018). Typically, participants are presented with some purported proof and asked if it is valid. The justifications utilized within these studies usually fall into one of three categories: 1) researcher-generated justifications that are produced with validation in mind when being constructed (e.g., Selden & Selden, 2003), 2) mathematician-generated justifications that are created for non-validation purposes and then re-purposed by researchers (e.g., Czocher & Weber, 2019), and 3) student-generated proofs that are re-purposed for proof validation work (Moore, 2016). Notably, in some of these studies, the purported authorship of justification misaligns with the actual authorship. For instance, a researcher might generate a justification for participant consideration, but indicate that the justification under consideration has a student author (e.g., Davies et al., 2023).

One notable difference between on-the-spot ChatGPT-generated justifications and the human-generated justification typically found in proof validation studies is the error rate. In many of these studies, the justifications presented by researchers include only relatively few errors. For instance, a twelve-line purported proof presented by a researcher may contain one error, while the number of errors generated by ChatGPT are not dependent on the length of the purported proof and the nature of the errors is not predetermined. It is not uncommon for a short ChatGPT-generated proof to have a dozen errors and for the subsequent proof generated to be error-free. Consequently, when the justifications generated by ChatGPT do not constitute what we would consider a valid proof, there tends to be significantly more opportunities for students to notice a subset of those errors and deem the proof invalid.

In contrast, fewer errors inserted in human-generated justifications create a limited opportunity for students to notice those errors. With these proofs, student validation might hinge on whether a student notices a single error, fabricated by the researcher, that links two lines in a twelve-line proof. In studies that use low error-rate proofs, some researchers have reported that although students do not initially notice an error, when directed to examine a line where an error occurs, students tend to find it without further guidance (e.g., Alcock & Weber, 2005; Selden & Selden, 2003). We argue that expecting students to find a single intentional error in a dozen line purported proof is an unnecessarily high standard. Additionally, noticing such an error is a poor indicator of a student's skill in validating proofs. By using AI-generated proofs, which may

contain significantly more errors, we can better articulate what students view as errors and how adept they are at finding them.

Methods

Six Anonymous State University (ASU) participants participated in one-on-one clinical interviews. Each participant was either currently enrolled in or had completed a transition-to-proof course at the time of the interview. Each interview lasted approximately 60 minutes and was video and audio recorded. Participants were given access to a computer with ChatGPT open to engage with the interview tasks. Students received a printout of the ChatGPT prompt associated with each interview task, but otherwise blank. The handwritten annotations of all participant annotations were scanned to create digital artifacts.

Interview Tasks and Data Collection

Each interview involved the participant using ChatGPT to generate proofs of pre-prepared prompts and subsequently validate them. Each participant received two statements (shown below), one related to number theory and the other to set theory.

1) Show that $7^n - 2^n$ is divisible by 5 for all n.
2) Show that if A and B are subsets of X, then $(A \cap B)^C = A^c \cup B^C$.

Participants were instructed to prompt ChatGPT to generate three proofs to validate—two of statement one and one of statement two. Additionally, we stipulated that the two proofs of statement one use different proof methods. The participants were asked to use their own phrasing for each statement when prompting ChatGPT. Once ChatGPT provided an output, the participant was instructed to study the proof and to think aloud wherever possible. Finally, the participant was invited to speak about their thoughts on the proof's correctness and completeness. They were asked to explicitly cite the features of the proof that led to their decisions. At the end of each interview, the interviewer saved and downloaded the interaction. This downloaded interaction included video recordings of the interview, user inputs entered by the interviewee, and the subsequent outputs generated by ChatGPT. Afterward, these interactions were analyzed.

Data Analysis

ChatGPT generated the purported proofs in this study in direct response to the interviewees' actions. As such, each of the purported proofs generated was unique and dependent on the students' phrasing and its own programming. So, before studying the participants' validation skills, it was necessary to examine each purported proof irrespective of the participant's validation. This timing minimized the influence of the participants' perceptions on those of the expert. This process began with the second author studying each purported proof generated by ChatGPT to glean whether it was valid (from the research team's perspective) and document each error the proof might contain. We documented errors in calculation, grammar, and syntax. Finally, each proof was judged valid (from the research team's perspective) if the logical structure was sound and if it was free of errors that detracted from the strength of the argument. For example, a proof which contained algebraic errors upon which a proof by induction was based was deemed invalid. By contrast, a proof that misattributed a set theory principle to De Morgan's Laws was considered valid because the principle was correct and well-substantiated.

After the research team assessed each purported proof, we examined each participant's engagement with the proofs via the interview recordings. As the participant thought aloud during their assessment of each proof, we specifically noted any instance in which the participant noted a fault in the proof as well as the nature of the error. We also noted the participants' final

decision and justification about each purported proof's validity, correctness, and completeness. After complete annotations of the expert and participant perception of each proof were available, we compared and contrasted them with a particular focus on the errors identified by each and the final decisions on the proof's overall validity. While the expert's search for errors in each proof was meant to be exhaustive, we did not make a similar request to the participants.

Results

In general, students read each ChatGPT-generated proof and noted any errors they found. They commonly terminated their validation process if they found a single significant error or an accumulation of multiple errors that they deemed sufficient to deem the ChatGPT proof invalid. As such, we comment on the proportion of errors found by participants in relation to those found by the research team up until the point the participants stopped reading. The participants identified 25 of the 55 errors – 45% of the errors identified by the research team. This rate is similar to rates reported in validation studies with human-generated proofs (e.g., Alcock & Weber, 2005; Selden & Selden, 2003). As discussed earlier, those human-generated purported proofs typically contain one or zero errors. So, we view that statistic as gesturing toward a parallel between our results and the results of those studies.

Table 1 below compares the research team's validation to the student participants' validation. The instances where the participant and expert disagreed are signified with boxes. In all participant interviews, the expert deemed the first two proofs (of Statement #1) invalid and the third proof (of Statement #2) valid. As can be seen from the table, the participants' final validity assessments were consistent with the research team on 16 of the 18 proofs (89% agreement). Note that on one such disagreement, the participant made their decision based on the weak link between inferences in the proof. The researcher initially noted this lack of detail yet deemed the proof valid. The above agreement rate is notably higher than similar statistics noted in validity studies with human-generated proof tasks (Alcock & Weber, 2005; Selden & Selden, 2003, Panse et al., 2018). Given our small sample size, we cannot make broad statements regarding the statistical significance of this difference. However, we believe this is a result of the higher error rate in the AI-generated proofs. As such, we argue that AI-generated proofs may be more appropriate for studying students' error identification process, whereas human-generated proofs may be more appropriate for studying overall validation.

Table 1. Validation decisions of participants regarding ChatGPT-generated proofs

✓ = Valid Proof	✘ = Invalid Proof	☐ = Disagreement with Expert

	Student #1	Student #2	Student #3	Student #4	Student #5	Student #6
Pf #1a	✘	✘	✘	✘	✘	☑
Pf #1b	✘	✘	✘	✘	✘	✘
Pf #2	✓	✓	☒	✓	✓	✓

In general, participants read proofs looking for any perceived errors and did not appear to be focusing on a particular error type. Their validation processes included highlighting stylistic, logical, and calculation errors. Below is an excerpt from Student #2 investigating an argument based on induction in the first proof of Statement #1. The proof's printout can be found in Figure

1 (OpenAI, 2023). The numbers in square brackets correspond to portions of the proof indicated by Student #2 as he was speaking.

Student #2: It's doing the right process. This [1] is right… It's doing the $m + 1$ case, and then it's going to show that the m^{th} case implies that the $(m + 1)^{\text{th}}$ case is true…
Interviewer: So, where are you?
Student #2: Okay, so I think it lost a term here. I think this completely fell apart. Because we have the 5 + 2 here [2], but then the two term is missing [3]…No, that doesn't make sense because this [4] is a multiple of 7, 7 times 7 times 7 times 2. Those are prime. This [5] is 5 times 2 times 2 times 2 and 2, which are prime, so there's no way that you can make these [5] out of these [2] multiplications. So, I'm going to say that this is an invalid proof.

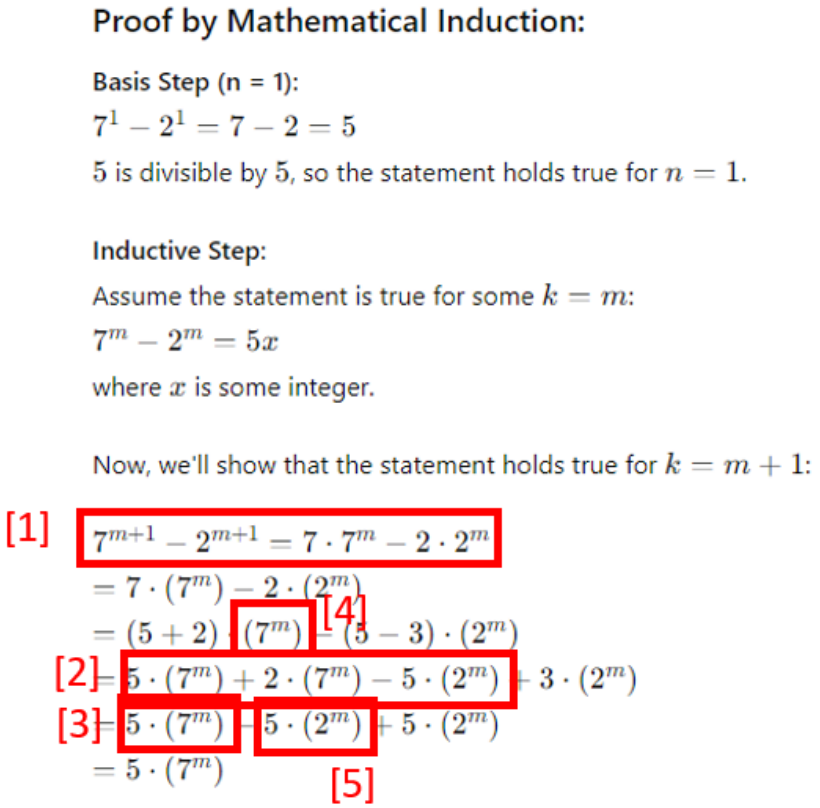


Figure 1. ChatGPT-generated proof #1 from interview with student #2

At the beginning of the transcript, Student #2 attends to the (overall) logical structure of the proof (Selden & Selden, 1995). He affirms that induction is an appropriate proof framework to prove the statement and then proceeds to examine the details of the proof. When he attends to the calculations in the sequence of inferences in the inductive step of the proof, he finds algebraic errors. Since the argument relies on those calculations, Student #2 deems the purported proof invalid. More specifically, since the calculations are what the proof uses to show the "the m^{th} case implies that the $(m + 1)^{\text{th}}$ case," a flaw in that part of the argument causes the purported proof to, "completely fall apart". Notably, the purported proof validly asserts that $7^m - 2^m = 5x$ for some integer x. However, this x, and by extension, the overall induction hypothesis, is not

utilized within the purported proof. This error was noted by the research team but not attended to by the student. Since he stopped reading the purported proof at the end of the transcript, we are unable to comment on whether he would have attended to it if he had been directed to that specific line of the proof (e.g., Selden & Selden, 2003, Alcock & Weber, 2005).

Discussion

We presented students with purported proofs generated on-the-spot by Chat-GPT. This has several consequences: 1) The participants knew that each purported proof was, in a sense, author-free. 2) Neither the participant nor the interviewer knew whether any given purported proof was valid at the time it was generated, and 3) Given the current nature of AI, the purported proofs under consideration tended to have more errors than human-generated proofs used in proof validation studies. The confluence of these consequences creates advantages and disadvantages in studying proof validation.

The method we reported has some notable disadvantages. First, researchers need additional work to assess whether each justification students see is valid before examining students' processes. This contrasts with work that shows the same purported proof to multiple participants across multiple interviews. Additionally, the uniqueness of each AI-generated proof makes comparing work across students more complicated, which is why much of what we discussed here focused on summary statistics rather than cross-interview comparisons.

Despite the potential drawbacks, AI-generated proofs present significant advantages by providing a lens into proof validation that is arguably not colored by authorship, or at least colored in a very different way. It is known that purported authorship of a justification influences validation processes (e.g., Davies et al., 2023; Harel & Sowder, 1998; Weber & Mejía-Ramos, 2011). For instance, a justification purported to be generated by a mathematician is likely to be viewed by participants with less suspicion than a comparable proof purported to be generated by a student. This authorship influences the field's understanding of participants' proof validation processes. For instance, consider what might happen if Student #2 had been given the same proof discussed in the results section but was instead told a mathematician generated it. In that case, it would be reasonable for them not to attend to the proof framework and check for its validity since a mathematician is unlikely to (unintentionally) use an inappropriate proof framework. Instead, we see student #2 determine whether the overall logical structure is appropriate before checking the details. That is, Student #2 examined the proof for various errors that could have occurred, which may be different than the subset of errors they might have looked at with different information about the author of the proof.

Notably, our overall error identification rate is similar to the correct validation rates reported in studies with human-generated purported proofs (Selden & Selden, 2003; Alcock & Weber, 2005; Panse et al., 2018). However, our reported correct validation rate was noticeably higher (ibid). These rates point to the intuitive finding that increasing the number of errors contained within a purported proof increases the likelihood that a student identifies enough of those errors to deem the proof invalid. Additionally, we argue that AI-generated purported proofs have significant potential for studying students' error identification processes, as there are more identifiable errors per line. Such exploration can increase the field's understanding of the relationship between students' overall error identification process and final validity assessments.

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Transformational Reasoning in an Introduction to Proofs Course

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The goal of this preliminary report is to demonstrate students engaging in transformational reasoning in an introduction to proof class. Utilizing transcript data and operationalizing Simon's (1996) definition of transformational reasoning, these preliminary results include descriptions of transformational reasoning in an advanced math context, as well as how students used transformational reasoning in order to accomplish their mathematical goals.

Keywords: transformational reasoning, introduction to proof

Inquiry-oriented instruction (IOI) provides a classroom environment in which students can share and continually refine their thinking. In IOI the classroom mathematical agenda is advanced by building on students' informal ways of reasoning to develop more formal mathematical concepts. One such type of informal way of reasoning that provides a powerful foundation for more advanced reasoning is transformational reasoning (Simon, 1996). Transformational reasoning involves (1) mentally (or physically) enacting operations on objects which result in the objects being transformed and (2) observing the result of the operation, which results in one coming away with a sense of how something works (Simon, 1996).

Scholars have studied transformational reasoning in a variety of math contexts. These include pre-calculus and calculus (Carlson et al., 2002; Johnson, 2012), algebra and geometry (Ellis, 2007; Simon, 1996; Zandieh et al., 2008), and proof and justification (Harel & Sowder, 1998).

Theoretical Perspective

Simon (1996) coined the term transformational reasoning to describe a type of mathematical reasoning that he noticed students were spontaneously engaging in, that resulted in the students coming away with a sense of how something worked. Simon defines transformational reasoning in the following way:

> Transformational reasoning is the mental or physical enactment of an operation or set of operations on an object or set of objects that allows one to envision the transformations that these objects undergo and the set of results of these operations. Central to transformational reasoning is the ability to consider not a static state, but a dynamic process by which a new state or a continuum of states are generated. (1996, p. 201)

This definition includes (a) an object (or objects) that is (are) being transformed, (b) the operation(s) that affect the object (or objects), and (c) results of the operation(s) on the object(s). The process of transformational reasoning involves identifying (a) and (b), which allows one to envision the transformation taking place, and once (c) is attained one can examine them together to draw conclusions about the entire process to understand why it works.

To get a better feel for transformational reasoning, consider the *bisection method*. The idea of the bisection method is to generate (via bisection) a sequence of intervals of decreasing length. One application of the bisection method is to approximate roots of continuous functions with a sign change. In this application, the intervals all share the property that the function outputs have opposite signs at the endpoints, i.e., positive output on the left and negative output on the right or vice versa. The sequences of the right and left endpoints converge to a root of the function.

One might engage in transformational reasoning in this context by focusing one's attention on the intervals as defined above, i.e., the object. The operation is that the intervals decrease in length at each step. The result is increasingly small intervals, the lengths of which converge to zero. The resulting sense for how the bisection method works comes from realizing that the interval length shrinking to zero means that the endpoints will match, and since the function is continuous these endpoints must be a root.

As noted above, it is critical that students consider a dynamic process in order to engage in transformational reasoning. Simon goes on to state that this dynamic process could be imagined or concrete (Simon, 1996, p. 202). Since transformational reasoning results in figuring out how something works, Simon argued that this was different from following a chain of reasoning (i.e., deductive reasoning) or noticing and describing a pattern (i.e., inductive reasoning) (p. 197) but that it may overlap with either deductive or inductive reasoning depending on the circumstances (p. 204).

Transformational reasoning supports the ascertainment of evidence when one is proving. Harel and Sowder (1998) report on a series of proof schemes that explain the ways in which students might engage in proving, one of which is the transformational proof scheme. For the authors a proof scheme "consists of what constitutes ascertaining and persuading for that person" (Harel & Sowder, 1998, p. 244). Thus, a proof scheme contains two parts: (1) how an individual convinces themselves (ascertaining), and (2) how an individual convinces someone else (persuading). Convincing oneself necessitates understanding how something works, and transformational reasoning can support this. Therefore, transformational reasoning supports ascertainment of evidence regardless of the math activity in which one is engaging.

In the present study, students in an introduction to proof class were engaged in the guided reinvention of fundamental real analysis concepts and I explored the students' activity by investigating two research questions: (1) what does transformational reasoning look like in an inquiry-oriented introduction to proof class? and (2) how do the students use transformational reasoning to accomplish their math goals?

Methods

Data for this study come from a large NSF-funded design project ASPIRE in Math (DUE# 1916490). Briefly, the ASPIRE in Math project sought to support students in transitioning to advanced math while also supporting instructors to shift to inquiry-based mathematics teaching. A whole class teaching experiment, in the form of an introduction to proof class, was conducted in Spring 2020 at a public university in the Pacific Northwest of the United States during the beginning of the COVID-19 pandemic. The goal of the teaching experiment was to finalize a curriculum and to begin articulating instructor support materials. The goals of the curriculum included engaging students in the guided reinvention of fundamental concepts in real analysis and developing advanced math practices including defining, conjecturing, and proving. The task sequence for the curriculum involved students proving the Intermediate Value Theorem by repurposing the bisection method for approximating a root of a continuous function (see Larsen et al., 2022).

Each of the 20 class sessions lasted 110 minutes. The research team included a professor of mathematics education, and two doctoral students (one of which is the author of this report). Each class was captured using Zoom recording and screen recording technology. Other data collected included student generated artifacts from various tasks that were recorded on Google Docs, slideshows, and Jamboards. The students (all names are pseudonyms) would record

conjectures, informal definitions, informal proofs, and images created or screenshots in these documents.

Data analyzed for this preliminary report came from transcripts of two different class days, and associated Google Docs. These two days were selected because of the content of the mathematical discussions. The episodes analyzed include students identifying sequences generated by the bisection method, students conjecturing about properties of the sequences, and students justifying that the left endpoint sequence converges.

I conducted a two-phase analysis. In the first phase I sought to ascertain and describe instances of transformational reasoning in terms of the components (i.e., objects, operations, results) and the resulting "sense of knowing how things work". Within student utterances I coded for (1) objects that make up a model (Lesh et al., 2000), (2) operations that transform the objects in the model, (3) observations about the outcomes of the operations, and (4) mentions of structural features (e.g., referring to a sign change, knowing that the midpoint becomes an endpoint of a subsequent interval, etc.) of the model. Structural features were used to identify when students were primarily explaining the process for implementing the bisection method. The second phase involved analyzing how students used this reasoning to address their mathematical goals.

Initial Results

Students used transformational reasoning to identify sequences, conjecture about sequences, and justify that the left endpoint sequence converges.

Identifying Sequences Generated by the Bisection Method

The class session began with a demonstration of how the bisection method can be seen to generate sequences, by showing how the left endpoint sequence is generated, followed by formally defining *sequence*. Then the students were invited to come up with as many sequences as they could (see figure 1).

1. Sequence of left-endpoints, a_n.
2. Sequence of right-endpoints, b_n.
3. Sequence of function values at c_n.
4. Sequence of midpoints, c_n.
5. Sequence of $f(a_n)$ values.
6. Sequence of $f(b_n)$ values.
7. Sequence of step numbers.
8. Sequence of the number of digits at each step in the midpoint.
9. Sequence of $b_n - a_n$
10. Sequence of the distance c_n is from the root (sequence of errors).
11. Sequence of number of matching digits after each iteration.

Figure 1. List of sequences identified by the students in class

Leo offered sequence 8 (see figure 1). Leo's explanation showed signs of transformational reasoning. He stated that "when you do the bisection method, you add, decimals are added at each step, and if you have a calculator that could keep up, then you could count those." Here Leo's object of focus was the midpoint sequence (i.e., the root approximation at each step in the method). I conjecture that the transformation he was envisioning is the computation of the midpoint at each step, and the outcome of this transformation is the addition of new digits in the midpoint at each step. Leo's description implies that he had a feel for how the midpoint sequence

works and that allowed him to identify a sequence that counts the number of digits in the midpoint at each step.

Conjecturing About Properties of the Sequences

Later, students were asked to conjecture any properties about the sequences that they noticed or limits the sequences might have. Maya and her groupmate Kevin worked together to produce a lengthy document with several observations about each sequence. A sample is provided in figure 2.

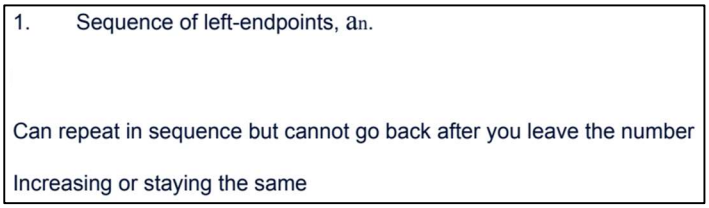


Figure 2. Sample conjectures from Maya and Kevin

Maya and Kevin's description of the sequence behavior indicated that they were engaging in transformational reasoning. Maya and Kevin identified the object they were focusing on in the bisection method as the left endpoint sequence. In the remark "can repeat in sequence but cannot go back after you leave the number", the pair identified an operation that takes place as the left endpoint sequence is generated in one step. I conjecture that the transformation they envisioned is one step in the process, and they observed that the left endpoint sequence either stands still ("repeat") or moves forward, but "cannot go back". Then, as a consequence of this one step, I further conjecture they envisioned generating the entire left endpoint sequence which resulted in an understanding of how the left endpoint sequence works which they encode as either "increasing or staying the same".

Justifying that the left endpoint sequence converges

A few class-days later, the students come to the conclusion that the left endpoint sequence converges, and then they are asked to articulate why this is the case. Maya's explanation builds on her previous transformational reasoning discussed above.

> *Maya*: So, when you're doing this method, each next term of the left endpoint sequence that you're making is either going to be the same as the term you were on because the other one changed, or it's going to go halfway to the other term, right? So um, you're either adding zero to it if it stayed the same, or you're adding the difference between it and b sub the same n.
>
> *Teacher*: And what is the remaining difference?
>
> *Maya*: So, it's $\frac{a_n + b_n}{2}$, or if you're thinking about it in terms of what you're adding to a_n, it's $\frac{b_n - a_n}{2}$, and the limit of that as n goes to infinity becomes zero, so you end up adding smaller and smaller things. So, you know that long run a_n has to stop moving or have a limit.

Maya's use of transformational reasoning results in a justification for why the left endpoint sequence converges. She begins by utilizing the transformational reasoning she arrived at in the

previous section. However, Maya was not imagining the formation of the left endpoint sequence in the same way. Her description was more specific. It went from "can repeat in sequence but cannot go back" (see figure 2) to "it's either going to be the same as the term you were on… or it's going to go halfway to the other term". Then, since she had a feel for how the left endpoint works, she transitioned to focusing on a new object that is within the left endpoint sequence, namely what gets added onto the previous term: 0 or $\frac{b_n - a_n}{2}$. I conjecture that the transformation she envisions is that the terms in the left endpoint sequence are growing because they are accumulating small amounts. The result of which is that the sequence converges because the amounts that are accumulated become smaller and smaller. By understanding how the left endpoint sequence works, she was able to home in on the feature of the sequence that explained her intuition that the sequence converges.

After this activity, students were asked to generalize their justifications into conjectures about general sequences. This was done by identifying the properties that were used to justify their thinking and written in the form "If [property or properties] then x_n converges." Then these conjectures were evaluated for plausibility with examples and counterexamples after which they would work to prove some of the conjectures. For example, the class established that Maya's argument could work but only if it is made more specific by comparing the amount "added on" to the terms of a geometric series.

Concluding Remarks

My results demonstrate two main things: (1) what transformational reasoning looks like in an intro-to-proof context, and (2) how students use transformational reasoning to engage in advanced math activity. In terms of the first research question by identifying objects, operations, and results in student utterances, I was able to describe how students came to know how things worked. This method provides a means for other researchers to unpack transformational reasoning in other math contexts.

Engaging in transformational reasoning supported students to accomplish their math goals. Students identified sequences that are generated by the bisection method and conjectured and justified properties that the sequences have. My results indicate that students might use a two-step approach to engaging in transformational reasoning that allows them to advance their ideas further, as in the case of Maya. Maya and Kevin's two-step process involved first imagining how to go from one term to the next, and then using the results of that reasoning to further reason about the generation of the entire left endpoint sequence. Maya's justification that the left endpoint sequence converged has a similar structure.

Maya's activity demonstrated that transformational reasoning could support the identification of key ideas (Raman, 2003, 2004). Maya began by articulating how the left endpoint sequence worked. Once she knew it had to converge, she used her sense of understanding to ascertain *why*. The key idea for Maya what gets added on to the previous term converges to 0.

The instructional materials used in the teaching experiment here also engage students in algorithmatizing, defining, and proving general conjectures. The next phase of my analysis will be examining these other mathematical activities for instances of students engaging in transformational reasoning. The presentation will include an expanded data set.

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The Effect of Mathematics Anxiety Interventions on Mathematics Achievement: A Meta-Analysis

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Mathematics anxiety is detrimental to student achievement in math. This study uses a meta-analysis to investigate the effectiveness of various mathematics anxiety interventions on improving mathematics achievement of college students. We include intervention studies on college math classrooms with quantitative measures of both mathematics anxiety and achievement. Only RCTs and QEDs written in English are included. We use ERIC as a database. 8 studies were included in the extraction involving 800 participants. Studies generally indicated a moderately positive effect of math anxiety interventions on math achievement (p = 0.00124).

Keywords: mathematics anxiety, classroom interventions, meta-analysis

Mathematics anxiety is common, normalized, and has the potential to sharply change the career trajectory of those who have it. Mathematics anxiety is the feeling of tension or fear provoked by math, mathematical problems, or mathematical situations (Ashcraft & Moore, 2009; Jiang et al, 2021). Individuals with math anxiety suffer from physiological tension and discomfort under mathematical situations (Ashcraft & Krause, 2007; Ramirez et al, 2013). As a result, they display avoidant behavior both in the short term and long term, sacrificing accuracy to rush through math tests, and avoiding math-heavy courses, college majors, and even career paths (Alkhateeb, 2002; Ramirez et al, 2013; Ashcraft & Krause, 2007). Neurological research shows that even brief exposure to mathematical symbols triggers a neural response associated with threat avoidance (Weed et al, 2021).

Students often make pivotal career decisions during college, taking different classes or trying different majors before deciding where to take their career. This gives a unique significance to math anxiety for college students. On the one hand, a negative experience in an introductory math course may turn students away from STEM careers or careers involving math. On the other hand, a positive experience with a math anxiety intervention may convince a student to stick with a STEM career choice. The purpose of this meta-analysis is to investigate the degree to which mathematics anxiety interventions improve the mathematics achievement of college students.

Literature Review

Theoretical Perspective: Mathematics Anxiety and Mathematics Achievement

Literature repeatedly demonstrates a negative correlation between math anxiety and math achievement (Ashcraft & Krause, 2007; Ashcraft & Moore, 2009; Samuel & Warner, 2021). There are three main theories on the cause of this relationship (Codding, 2023). *Cognitive interference theory* suggests that high math anxiety induces excessive cognitive load on the student, taking up working memory and therefore lowering their capacity for reasoning through mathematical tasks (Ashcraft & Krause, 2007; Ashcraft & Moore 2009; Ramirez et al, 2013). *Deficit theory* suggests that the relationship is instead caused by a perceived achievement deficit (Carey et al., 2016; Gunderson et al., 2018; Wang et al., 2015). Under this model, the student performs below her own expectations on some mathematical task, and begins perceiving herself as mathematically incompetent. The student then develops negative feelings around

mathematical activities. The final theory suggests that the relationship is *bidirectional*, with math anxiety detracting from a student's math performance, and subpar achievement causing further anxiety for the student.

Present Study

Interventions on Mathematics Anxiety

Given the three theories above on the relationship between math anxiety and math achievement, interventions aiming to mitigate the effect of math anxiety on student achievement in math can either target math anxiety or math achievement (Codding, 2023). Interventions that target anxiety (behavioral interventions) include affirmations, breathing exercises, stretching, music, and any non-content related activity that can help relax the student's nerves. Interventions that target math achievement (knowledge interventions) include instructional strategies, study strategies, and other interventions that help students break down and understand the content more effectively. By observing how these two types of interventions affect math anxiety, we can contribute more evidence to discern the relationship between math anxiety and achievement.

Research Questions

Q1: How do mathematics anxiety classroom interventions affect mathematics achievement in college students?

Q2: Do combined type interventions affect mathematics achievement differently from behavioral interventions?

Methods

Eligibility Criteria

We use the PICOS framework to define eligibility criteria for this study. The *population* for included studies is college or university students. We choose this population because of their closeness to decision-making moments in their careers. For *interventions*, we choose studies with classroom math anxiety interventions. "Classroom interventions" refers to interventions associated with a course, like beginning lectures with affirmations, incorporating writing exercises in homework, or using group work as part of the class structure. These were chosen because they can be flexibly and cleanly applied by instructors. To accept an intervention as a "math anxiety" intervention, we just check that the authors refer to it as such. For *comparison*, we require a control group that did not receive the intervention by the time the post-intervention measure is taken. For *outcomes*, we check if the study includes quantitative measures for both math achievement and math anxiety. For *study design*, we check that studies are intervention studies with Randomized Control Trials or Quasi-Experimental Design to maintain a baseline for quality. We did not limit the publication date or location of the study, but we did require that studies be written in English.

Database search

We searched ERIC for studies of interventions on math anxiety in college or university math classrooms. We limited the number of screened studies to only the first 150 studies (out of a total 283 found) in the ERIC search results. These included published studies as well as dissertations. With additional time, we would have searched ProQuest for more gray literature to further minimize publication bias. The following search string yielded the results:

(university OR college) AND (mathematics OR math) AND anxiety AND (intervention OR program OR strategy$ OR treatment$) AND ((classroom$ OR school$ OR education$) OR (pedagogy$ OR teaching$)) AND (achievement$ OR performance$ OR test score$ OR outcome$)

The studies were moved to Zotero, where an initial PDF search was conducted. Next, they were moved to Rayyan (https://www.rayyan.ai/) to begin screening.

Screening

The screening was done in two stages. First, 150 studies were taken through title and abstract screening. At this stage, the first author checked the language, population, intervention, and outcomes. 103 studies were excluded, and 47 studies passed to full text screening. For this stage, the first author checked the study design and inclusion of a control group in addition to the rest of the criteria. Only 8 studies passed this screening into extraction (denoted with a * in the citations).

Extraction

We extracted descriptive information on studies (title, authors, and publication year), effect size information (means, group size, and standard deviations on the quantitative achievement measure for each treatment and control group). We also coded the type of treatment (behavioral, math knowledge, or combined), and who delivered the intervention (researcher, teacher, computer, or other) for use in the moderator analysis. A nested structure was used for effect sizes. For example, suppose a study included two treatment groups and a control group, and then measured the effect with three exams. Then the two treatment groups would be coded as "group 1" and "group 2," and each would have an effect size for "outcome 1," "outcome 2," and "outcome 3," and all of these would fall under a single "study ID".

Analysis and Results

Effect Size Analysis

The interventions in these studies vary in type, delivery, date, and location, so there are factors other than sampling error affecting extracted effect sizes. We use a random effects model to account for that in the meta-analysis. We use Cohen's d analysis with the Hedges' g correction to analyze pooled effect sizes. A multivariate model is used for the entire analysis since some studies include multiple treatment groups or multiple achievement measures, causing dependencies between the effect sizes.

We use the metafor package in R with the robust standard error correction from clubSandwich to compute the result. This produced a pooled effect size of 0.373 with a standard error of 0.076 ($p = 0.00124$, $df = 12$). The extracted studies therefore show a significant, moderate effect of the interventions on student math achievement. The heterogeneity test shows significant heterogeneity in the sample, affirming the choice of the random effects model.

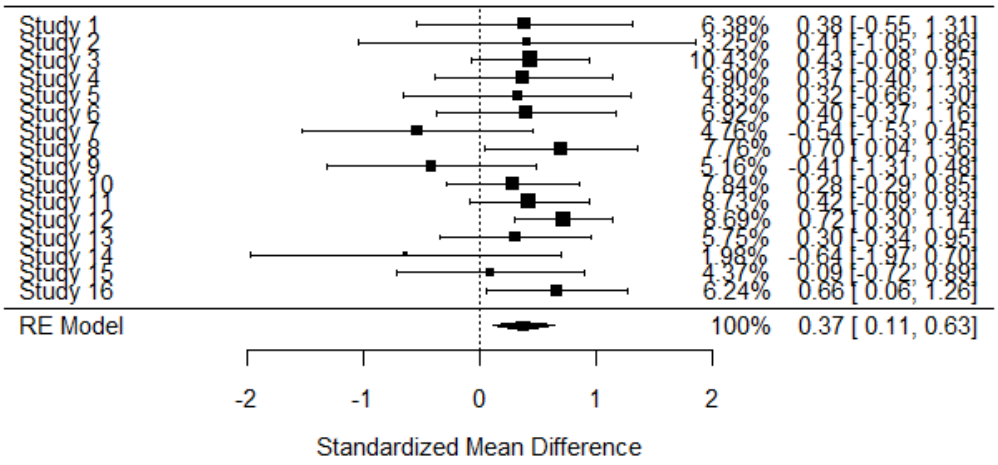


Figure 1. Forest plot of the means in the meta-analysis.

Moderator Analysis

The moderator analysis is performed through a meta-regression. There was not enough data to analyze effectiveness on mathematics anxiety or who delivers the intervention as moderators, but treatment type was analyzed. The test of moderators showed that treatment type was indeed a moderator in this study ($p < 0.05$). In particular, while "behavioral only" and "combined type" interventions both had moderate positive results on student achievement, the effect was only found to be statistically significant for the combined type. There were not enough "math knowledge only" interventions for that effect to be analyzed. This is likely due to the eligibility criterion requiring the interventions to be math anxiety interventions.

Table 2. Descriptive results of meta-regression.

Treatment Type	Effect Size	SE	Z-Val	P-Val
Behavioral only	0.3483	0.2083	1.6718	0.0946
Combined	0.3814	0.1765	2.1603	0.0307
Regression coefficients	$\boldsymbol{\sigma}_1$	$\boldsymbol{\sigma}_2$		
	< 0.0001	0.0863		

Descriptive Results

The studies analyzed were published between 1982 and 2020, with a total of 800 participants. These studies were generally set in a university course, with an intervention applied throughout a semester. Post-intervention results are taken after the end of the course. All studies found a significant effect of the intervention on mathematics anxiety. Of the 16 effect sizes, 10 were based on a combined treatment type and 6 were based on a behavioral-only intervention. Studies generally found math anxiety interventions to have a positive effect on math achievement.

Answers to Research Questions

Q1:How do mathematics anxiety classroom interventions affect mathematics achievement in college students?

A: Mathematics anxiety classroom interventions have a moderately positive effect on mathematics achievement for college math students.

Q2: Do combined type interventions affect mathematics achievement differently from behavioral interventions?

A: Treatment type does moderate the effect of math anxiety interventions on math achievement. In particular, combined math-knowledge + behavioral interventions have a more significant positive effect on mathematics achievement than behavioral-only interventions.

Discussion and Implications

The results of the meta-analysis demonstrate that classroom math anxiety interventions have a moderately positive effect on math achievement, with the result for combined behavioral + math knowledge interventions being more significant than behavioral-only interventions. Therefore, we can recommend that college math instructors implement combined-type interventions to break the hold math anxiety has over their students. This paper suggests that even students taking their one required math course in college can improve their math performance and be relieved from their math anxiety through a classroom intervention. Students impacted by math anxiety can be better prepared to choose and succeed in STEM majors and math-heavy career paths (Ashcraft & Krause, 2007; Ramirez et al, 2013).

There are some important limitations for this study. First, the screening and extraction was done by one person only, so mistakes are inevitable. The three stages of screening help mitigate this problem, and the first author reviewed decisions for each stage of screening and the extraction. However, two more papers had to be excluded during extraction due to full-text screening errors. It is possible there are more mistakes that were not caught. Second, with so few studies, most of which come from published journals, there is a plausible potential for publication bias. Publication bias is the phenomenon where studies are more likely to get published if they find statistically significant results, especially small studies or studies with lower power (Palmer et al, 2008). This can bias the results of a meta-analysis to produce a higher effect size.

The final major limitation is the number of studies included. The goal of the moderator analysis was to provide additional data to inform the theories on the relationship between math anxiety and math achievement. While a more statistically significant effect was observed for combined treatment than behavioral-only treatments, the effect sizes themselves were close (0.34 vs 0.38). It is possible that behavioral-only interventions simply have a wider confidence interval (larger standard deviation) and would show significance with a larger sample. A significant effect of behavioral-only interventions on math achievement would corroborate interference theory, as targeting math anxiety without math knowledge improves expressed knowledge (Ashcraft & Krause, 2007; Ashcraft & Moore 2009; Ramirez et al, 2013). Moreover, no math knowledge-only studies were included, and this was because of the eligibility criterion requiring math anxiety interventions. Expanding the pool of studies may resolve this, but it may be more effective to instead search interventions more generally, and code the treatment type on all of them (with the same behavioral, math knowledge, and combined types as codes).

Overall, while the results of this study are promising, a study amending the above limitations would be well-positioned to give better recommendations on types of interventions a college instructor might try with her students as well as a deeper understanding of the relationship between math anxiety and math achievement.

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Is This a Proof? Students' Initial Criteria

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Criteria for proof are important because future teachers will need to evaluate students' arguments as proofs or nonproofs. This preliminary report examines how a group of prospective and practicing teachers in a reasoning and proof course began to collectively construct criteria for proof. Given a set of hypothetical student arguments, their initial criteria differed, despite common mathematics coursework, including an introduction to proof course. Ongoing analysis uses documenting collective activity to understand how collective criteria and definition were constructed.

Keywords: proof definition, prospective teachers, practicing teachers, proof criteria, argumentation

Proofs are an essential component in doing, communicating, and recording mathematics (Schoenfeld, 1994). Hersh (1993) contrasted proofs from a mathematical point of view and in a classroom setting. Within the creation and development of mathematics, proofs are social constructs for convincing others such that "without complete, correct proof, there is no mathematics" (p. 396). From a humanistic view, a proof is an explanation. Considering the role of proof in a classroom setting, Stylianides and Stylianides (2009) described proof as "general, valid, and accessible to the members of the community" (p. 239). Notably, Stylianides (2007) viewed proofs as mathematical arguments with certain properties: 1) proofs contain statements which are familiar to all students in a classroom, 2) the reasoning behind the steps of the proof should be valid for and known to all students in the class; and 3) the representation of the proof should be accepted by the classroom community.

The National Council of Teachers of Mathematics (NCTM, 2000) asserted that reasoning and proof "should be a consistent part of students' mathematical experience in prekindergarten through grade 12" (p. 56). Along with the consideration of proof within a classroom setting, the Common Core State Standards for Mathematics (CCSSM, 2010) further emphasized that students need to build a logical progression of statements to explore the truth of their conjectures by justifying their conclusions, communicating and responding to the arguments of others, and recognizing and using counter examples. Notably, students are expected to develop a proof conception that involves "a person's ability to prove and analyze proofs and their beliefs about proof and justification" (Conner, 2007, p.11). Thus, teachers are asked to embed proof as a core practice in their daily mathematical instruction (NCTM, 2000).

However, previous research has shown that secondary school students, collegiate students, and pre-service teachers mostly focus on empirical arguments while constructing proofs (e.g., Knuth, 2002a, 2002b; Reyhani et al., 2012; Sears, 2019). Despite their beliefs about the limitations of empirical examples (e.g., Stylianou et al., 2015), students have difficulties in distinguishing proofs from non-proofs. They have a tendency towards considering generic arguments and pictorial representations as non-proofs (e.g., Lesseig et al., 2019), and they consider the role of proof as mostly a verification of the truth of a mathematical statement (e.g.,

Lesseig et al., 2019; Likando & Ngoepe, 2014; Varghese, 2009). Though research has shown that some practicing teachers considered proof as socially constructed (Knuth, 2002a), some prospective teachers did not consider proof as a product of collective engagement (e.g., Sears, 2019; Varghese, 2009). On the other hand, research has shown that given education, students' and prospective teachers' conception and comprehension of proofs can be developed (e.g., Belin & Karagöz Akar, 2020; Miller, 2020). These results align with Stylianides et al. (2017):

> "There is a great need for interventions to introduce students of different ages to an appropriate set of criteria for judging whether an argument meets the standard of proof…Unless the field of mathematics education designs effective ways to instill in students expandable as well as developmentally and mathematically appropriate views of what proof means, it is unrealistic to expect that students will make sustained progress in their learning of proof during their mathematical education" (p. 257-258).

Taking this as a challenge and building on the previous studies, in this study, we further investigated how prospective and practicing mathematics teachers come to collectively develop criteria for proof, by which we mean criteria to determine whether an argument can be considered a proof. Acknowledging that defining proof and identifying proofs and non-proofs are important components of knowledge for teaching proof (Steele & Rogers; 2012) and considering that teachers are key figures in the education of future generations (NCTM, 2000), we specifically investigated the following research question: *How do practicing and prospective teachers come to collectively construct and use a community definition of proof?*

In this paper, we show how teachers and prospective teachers who had a significant amount of mathematics coursework prior to the course, often disagreed as to whether an argument was a proof or not. We include an excerpt of our data that depict how at times their written and verbal work seemed to disagree. These data exemplify why it is important for a class to have explicit collective criteria for proof as they continue to evaluate arguments. We are interested in how different individual warrants and criteria for proof evolved into a collective understanding or definition. At the beginning of the course, we observed that when presented with the same student work, teachers can have very different understandings of what constitutes proof; ultimately in the course, they constructed a collective definition. We are in the process of analyzing how the teachers moved from the initial activity shared here to the collective criteria.

Methods

We collected data from a reasoning and proof class that the third author taught in the fall semester of 2024. The class was comprised of 12 students who were prospective or practicing secondary mathematics teachers. The participants had previously completed an introduction-to-proofs course as well as several other mathematics courses. The class met weekly alternating between 165 minute in-person classes and asynchronous classes. The asynchronous classes were weeks 1, 3, 5 and 6, and the in-person classes were weeks 2, 4, and 7. We only collected data from the three in-person classes. The data consisted of videos of three groups of four participants each (except for week 2, in which one group had three), videos of whole class discussions, pictures of participants' written work, and pictures of whole class board work. Because we are focused in this paper on the first part of our research question regarding the construction of a collective definition of proof, we only describe the activities and tasks that preceded the agreed upon class criteria for proof.

In the first class, the participants completed three tasks. For the first task, we gave them 16 solutions to the question "How many 3 x 3 squares are there in an n x n square?" and asked them to discuss in their groups whether they were convinced by each solution and what modifications

they would make to the solution. For the second task, we gave the students 23 arguments for "Prove that when you add any 2 odd numbers, your answer is always even" and asked them to sort the arguments according to their underlying ideas. For each of their sorted groups, we asked them to determine if the underlying idea had merit for proving the statement and to select the best argument in the group. The participants had written their own solutions to these two problems in preparation for class, so they were familiar with the problems. The third task we gave included 10 arguments for the statement "the sum of two odd numbers is even" and asked students to decide if the argument is a proof and write a rationale for their decision. An example of the conversation around one of these arguments is presented below.

At the end of the first class, the participants were asked to write down, as a group, criteria for determining if a given argument is a proof. These criteria were shared with the whole class and compiled by the instructor into a list of 14 criteria. The participants were then asked to refine the class list to include fewer criteria, which they were unable to finish due to time constraints. We began the second in-person class (week 4) by asking them to revisit the class list and to consolidate the list into a set of three criteria. As a result of small group and whole class discussion, the participants agreed on the following criteria for proof: 1) makes logical connections using mathematical and community-appropriate evidence, 2) accounts for all cases, and 3) comes to the conclusion of the initial conjecture.

Our analysis examined the participants' arguments as they discussed each presented argument. We used Rasmussen and Stephen's (2008) Documenting Collective Activity (see also Cole et al., 2012; Rasmussen et al., 2015) to examine how the class collectively constructed their understanding of criteria for proof. In this method, the data, claims, and warrants are identified and recorded; three criteria are then used to examine when ideas begin to function as if shared in the classroom community. To ascertain when an idea or way of reasoning is functioning as if shared in a community, we used three criteria: 1) when the warrant was initially present for a claim but is no longer necessary; 2) when a part of an argument serves a different function in a later argument (i.e., a claim originally needed justification but is later used as data or warrant for another claim); 3) when an idea is used as data or warrant for different claims in multiple arguments (Rasmussen & Stephen, 2008; Cole et al., 2012). Our research team members individually examined participant small group discourse, making tentative codes for data, claim, and warrant. We then came together and constructed argumentation logs of each small group and whole class discussion, discussing and reconciling differences. Note that our analysis was conducted from the perspective of a participant, so we assigned ideas the function that we believed most closely aligned with the participant's intent. Thus, warrants sometimes preceded claims, and statements were not assigned a function based on how we would construct an argument but by what the participants seemed to intend with their utterances. At times, the participants included qualifiers and rebuttals in their arguments, which we captured in our argumentation log. We also included participants' written arguments in our argumentation log. These were usually easier to identify because we specifically asked for claims and reasoning.

Results

Participants were given 10 arguments purporting to prove that the sum of two odd numbers is even, each from a hypothetical student. The following excerpt is from one small group's discussion of Student E's argument (see Figure 1), which was the fifth argument in the list.

Student E

Any odd number can be written as 2x + 1. So let's add two odd numbers.

2x + 1 + 2x + 1 = 4x + 2
4x + 2 is even since 4 and 2 are both even.
Or 2(2x + 1) shows that 4x + 2 is even.

Figure 1. Student E's work shared with participants for their evaluation of whether the argument is a proof or not.

1 *Lindsey:* Nooo! (shaking her head and mouthing in response to reading Student E's response)
2 *Chloe:* I feel like it is almost there.
3 *Debbie:* It is almost there.
4 *Lindsey:* Yeah, but it is adding two of the same number.
5 *Chloe:* Mm-hmm (affirmative).
6 *Lindsey:* They also don't tell you what *x* is.
7 *Chloe:* I feel like if they use *x* and *y* (inaudible).
8 *Lindsey:* Yeah, yeah.
9 *Chloe:* If you used two different variables. And they don't really say why 4*x* + 2 is even.
10 Well, I guess they say because four and two are even. Having that four times *x* is
11 like they don't really say what that represents.

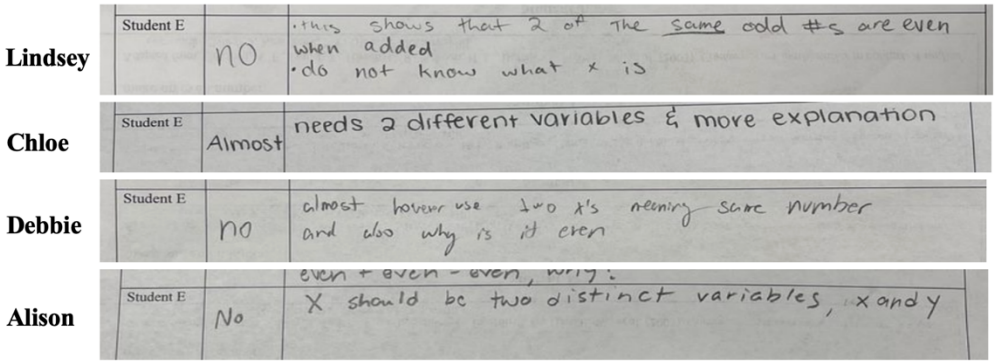


Figure 2. Participants' written responses to Student E's work.

We inferred that Chloe and Lindsey had two different claims. Notably, Chloe claimed that Student E's work was "almost there" (line 2), and Debbie supported her claim (line 3); Lindsey was very assertive, saying "noooo" (line 1), to claim that Student E's work was not a proof. In lines 4 and 6, Lindsey provides warrants for her claim that this was not a proof; the other members of the group seem to agree and accept this as a warrant for their claim that it is almost a proof. That is, each one of them seemed to be interpreting the same warrant as a warrant for their personal claim (potentially without realizing that the other person was using the warrant in their own way). Analysis of the participants' written work (Figure 2) further suggested that they did not come to consensus on whether Student E's work was a proof or not. Interestingly, although Debbie agreed with Chloe during their discussion, in her written response, she made the claim that Student E's work was not a proof. She made a definite claim, but she also explained in her warrant "almost however use two x's meaning same number" (see Figure 2).

Discussion and Conclusion

Our preliminary conclusions about the participants' initial criteria include that participants came into the course with their own somewhat idiosyncratic criteria for whether a student argument was a proof. At times, it seemed that participants translated their own criteria for proof into an assessment of a student argument by suggesting that they could tell what the student "meant" by something that was not necessarily correct. They also wondered if the grade level of a student impacted whether an argument could be considered a proof, suggesting that even if the argument would not be a proof for themselves, it could be for a student. This is consistent with the perspectives of other prospective teachers who have examined student arguments (Suominen et al., 2018).

In our analysis, we have begun to see groups of participants use previous claims as they assessed additional students' arguments. Although we have not completed our analysis of how ideas began to function as if shared, we have preliminary evidence of such ideas even within small group discussions. We believe it is important to document that the participants began their discussions with differing conceptions and criteria for proof. Otherwise, coming to consensus and shared criteria would not be necessary. In our presentation, we will include additional evidence of initial difference as well as ideas that began to function as if shared, culminating in our whole class discussion of criteria for proof. We will engage attendees in examining initial criteria for proof from our data and ask attendees to discuss the following questions:

- What criteria for proof would you expect undergraduate students to use at the beginning of a course?
- How are criteria from advanced mathematics classes translated to expectations for students' proofs?
- Have you attended to students' initial and developing criteria for proof?
- On the surface, students can seem to have a consensus, but in our case, students had different claims (even with a similar warrant). How do you engage with these kinds of situations in your own practice?

In this study, we are investigating how prospective and practicing teachers come to collectively construct and use a community definition of and criteria for proof to evaluate different student arguments. We shared some data from the beginning of the study to show that the participants did not necessarily begin with a shared set of criteria for proof. In fact, our analysis of the data showed that using the same explanations (warrants), the participants made different claims about the same argument. We suggest that this lack of consensus provides a rationale for investigating how a collective definition and set of criteria can be constructed. The Committee on the Undergraduate Program in Mathematics (CUPM, 2015), appointed by the Mathematical Association of America, emphasized that mathematics undergraduates, including prospective teachers, "should learn to read, understand, analyze, and produce proofs at increasing depth as they progress through a major" (p. 11). This further provides motivation for mathematics teacher educators to consider how a learning environment may influence practicing teachers' and prospective teachers' conceptualizations of proof for different audiences. This is especially important because teachers have difficulties in distinguishing proofs from non-proofs (e.g., Lesseig et al., 2019), and some do not consider proofs as a product of collective engagement (e.g., Sears, 2019; Varghese, 2009).

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Where is the Calculus in Calculus-Based Introductory Mechanics: A Textbook Analysis

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This paper describes a study of where and when calculus concepts and skills appear in the first semester of introductory physics. Following similar work in differential equations, engineering, and chemistry, we have adopted the Calculus Concept Framework and used the framework to identify which concepts and skills appear in each section of the first twelve chapters of a standard introductory physics textbook. Each section of each chapter was coded independently by three researchers and results were collated and discussed.

Keywords: physics, calculus in disciplines, textbooks

Calculus is a prerequisite for many courses in introductory physics. The sequence of courses taken by physical science and engineering students is frequently referred to as 'calculus-based' and has two or three semesters of calculus as pre- and co-requisites. (There is a parallel sequence of courses that do not require or use calculus, sometimes described as 'algebra-based' or 'trig-based.' These courses are generally taken by life science majors and pre-health professional students.) However, the nature of the relationship between calculus and introductory physics is undergoing renewed examination by scholars in both fields.

Introduction and Previous Research

There has been a significant body of work examining the use of mathematics in physics courses, for both introductory and more advanced levels. Redish and Kuo note that mathematicians and physicists use mathematics differently: while physicists load physical meaning onto symbols and equations, mathematicians tend not to (Redish and Kuo 2015). Thompson's quantitative reasoning framework describes the representation of measurable quantities with units (Thompson 2011). However, Biza and colleagues suggested that calculus in the disciplines could be a *filter* or *scaffold*: in the former case, calculus is used to determine who is qualified for later coursework; in the latter, it is useful as supporting knowledge to enhance success in later coursework (Biza et al., 2022).

Several prior studies have investigated student understanding of concepts and skills from calculus in introductory mechanics. (Here we focus on work in first-semester introductory mechanics, noting that there is a substantial body of relevant work in later courses.) For example, researchers have described a relationship between a limited understanding of definite integrals and difficulties in physics, in contexts including kinematics graphs (Beichner 1994, McDermott et al., 1987, Bajracharya et al., 2023). RUME scholars have reported that the conceptual understanding of integration that students need in physics is often not an outcome of standard calculus instruction (Orton 1983, Ferrini-Mundy & Graham 1994). Jones and colleagues have shown that most students leave calculus with a conceptualization of integrals as area under the curve (Jones 2015). However, more than half of the integrals in US physics textbooks used an adding-up-pieces conceptualization, and under 10% used area (Piña and Loverude 2019).

Several authors have reported that students perceive differences between practices in physics and calculus. Hitier and Gonzalez-Martin (2023) reported that "in mechanics, ready-to-use formulas are provided, so students can solve these problems without using knowledge from derivatives nor thinking in terms of covariation." Noah-Sella, et al. (2023) described a number of

differences in the ways that students perceive related problems in physics and calculus, including the perception that physics problems require 'more thought' than calculus.

In the related field of engineering, faculty assessed the value of calculus "for mathematical maturity more than just the actual calculus" content, because "the way [the engineering course] is taught, you can do it without calculus." (Ferguson 2012, Faulkner, et al. 2019). Tague et al. (2013) collected data from engineering faculty on mathematical expectations and reported areas of priority and perceptions of students' competence. Areas of priority included evaluating solutions, using units and dimensions, knowing how to create and interpret graphs, performing algebraic manipulations, conveying engineering relationships mathematically, and using symbols instead of numbers. Faulkner et al. (2019) noted that faculty teaching engineering courses having mathematics courses as prerequisite or corequisite agree that their students have difficulties with: using and interpreting mathematical models; choosing and manipulating symbolic and graphical expressions; and, using computational tools. Participants identified students' struggles to "set up integrals and derivatives to describe physical situations" and described students' perception of calculus as just equations, preventing them from "capturing the underlying physical principles".

Scholars in several disciplines have posed the question of where exactly calculus is used and what calculus concepts and skills are used in courses for which calculus is pre-requisite. Many prior studies have adopted the Calculus Concept Framework [CCF] (Sofronas 2011), which was developed as the result of several interviews with experts. The CCF identifies a list of concepts and skills that are essential to calculus understanding and that are generally taught in a calculus course sequence (the concepts and skills are the first two sections shown in Figure 1, respectively). Czocher used the CCF to investigate what concepts and skills from calculus were used in a course in differential equations (Czocher et al., 2013). The work was replicated in additional contexts, including introductory engineering (Faulkner et al., 2020), and general chemistry (McAfee and Rodriguez, 2024). No similar analysis exists in introductory physics, which this study seeks to remedy.

Research question and methodology

For the sake of this study, we sought to investigate the research question: What concepts and skills from calculus are encountered by students in introductory calculus-based physics, and when are they encountered? For this initial effort, we made two choices to limit the scope of the project, allow for preliminary discussion of results, and establish a baseline for future research.

First, we chose to limit ourselves to the topics covered in the first semester of introductory physics, which is generally limited to classical mechanics. Most students take a sequence of introductory physics courses that are two- or three-semesters or a three- to five-quarters. Mechanics is generally the first course in the sequence, with electricity and magnetism, optics, and modern physics coming in subsequent courses. Some of these topics (notably electricity and magnetism) have different mathematical demands than the first-semester course and results for those will vary. In subsequent work we will extend this analysis to later topics.

Secondly, we chose to use the presence or absence of concepts and skills in the textbook assigned for the course as a measure. We recognize that focusing on the textbook is only a proxy for what students might realistically encounter in the course, as instructors might increase or decrease the emphasis on mathematical formalism in lecture or homework exercises. Future work will seek to characterize instructor practice as well and document what calculus concepts and skills students actually encounter in class and in homework.

We chose a standard physics textbook (Resnick et al., 2014). We chose to code chapters 2 through 12, which are the chapters generally covered in the first semester course. We omitted chapter 1, as it is purely a discussion of units and includes no calculus. We stopped after rotational motion and equilibrium. We excluded fluids or thermal physics, which are not covered in the first-semester course at our university but are sometimes included at others. The edition chosen was a matter of convenience; a later edition of the text is currently in use, but the older edition was more readily available. A quick comparison of selected chapters suggest that the newer edition would lead to nearly identical results.

We created a spreadsheet with the concepts and skills from the CCF and cells for each section of the chosen chapters for the first semester course. For early meetings, we chose a single chapter to code and the following week to compare results. When any of the three coders disagreed, we discussed the coding and how each coder was interpreting the prompt. While many codes were uncontroversial, some needed further discussion. Based on this discussion, we negotiated and clarified coding practices. For example, we had to decide whether the codes should reflect the use of concepts and skills in homework exercises at the end of a section, or just the narrative and sidebars. We chose to code based on both homework exercises and section text, noting that this might lead to over-estimating the experience of students. While a limitation, this also suggested avenues for further research that we are beginning to pursue.

We also observed that aspects of mathematical practice that were adjacent to calculus were present. As a result we added several extra categories that were not present in the CCF. While these are not calculus concepts and skills, they were clearly elements of the practice in using mathematics in the context of the physics textbook. These included the presence of graphs or figures, the use of function notation, the use of the vector concept and unit vector notation, and the explicit attention to units or dimensions. These should be considered supplemental to the main focus of calculus topics, but also fruitful topics of further study.

After a few weekly meetings focused on a single chapter, the researchers moved on to multiple chapters within a single week, and reconvening to assess how in agreement we were. As we came into alignment of definitions there was an increase in agreement with the topics covered in each chapter. By the end of the first-semester analysis, the agreement between coders very high. For example, of 243 cells in the Chapter 11 spreadsheet, all three coders independently coded the same result on 236 (97% agreement). Remaining disagreements generally included the presence of a concept or skill ‘hiding’ in a homework exercise.

Results

We describe the results in tabular form below. In the table, the numbers in each cell refer to the number of sections in which the concept or skill appears. Since chapters have different numbers of sections, the shading indicates frequency relative to the number of sections in a chapter, with solid green indicating a high fraction of incidence and solid red indicating none.

Below the table we make some observations and claims that summarize the results of our coding. Broadly speaking, in the calculus concepts and skills there are more red entries than green. The three most-frequently coded items from the CCF are arguably not even calculus: algebraic manipulations; facility with definitions and notation; and trigonometric manipulations.

Concepts	**Total Count (%)**	Ch2	Ch3	Ch4	Ch5	Ch6	Ch7	Ch8	Ch9	Ch10	Ch11	Ch12
Derivative	30.65	4	0	6	0	0	1	2	1	3	2	0
Integral	19.35	3	0	0	0	0	2	1	3	3	0	0
Limit	14.52	3	0	2	0	0	2	0	0	2	0	0
Approximation	8.06	1	0	0	0	0	0	1	2	0	0	1
Sequences/Series	3.23	0	0	0	0	0	2	0	0	0	0	0
Riemann Sums	6.45	0	0	0	0	0	0	0	1	2	0	1
Parametric and Polar Equations	0.00	0	0	0	0	0	0	0	0	0	0	0
Continuity	0.00	0	0	0	0	0	0	0	0	0	0	0
Optimization	0.00	0	0	0	0	0	0	0	0	0	0	0
Transformation	1.61	0	1	0	0	0	0	0	0	0	0	0
Skills												
Derivative Computations	6.45	4	0	0	0	0	0	0	0	0	0	0
Techniques of Integration	4.84	3	0	0	0	0	0	0	0	0	0	0
Limit Calculations	3.23	0	0	2	0	0	0	0	0	0	0	0
Sequences/Series	4.84	1	2	0	0	0	0	0	0	0	0	0
Area / Volume	14.52	1	3	0	0	1	1	0	2	0	0	1
Parametric Equations	0.00	0	0	0	0	0	0	0	0	0	0	0
Polar Coordinates	0.00	0	0	0	0	0	0	0	0	0	0	0
Trigonometric Manipulations	46.77	2	3	3	2	2	4	2	4	1	5	1
Epsilon Delta Proofs	0.00	0	0	0	0	0	0	0	0	0	0	0
Listening/Reading Comprehension	0.00	0	0	0	0	0	0	0	0	0	0	0
Manipulating Algebraic Expressions	98.39	5	3	7	3	3	6	5	9	8	9	3
Facility with Logarithms and Exponentials	0.00	0	0	0	0	0	0	0	0	0	0	0
Facility w/ Definitions & Notation	93.55	2	3	7	3	3	6	5	9	8	9	3
Additional Concepts & Skills												
Function notation	14.52	4	0	1	0	0	2	2	0	0	0	0
Something is a function of something	100.00	6	3	7	3	3	6	5	9	8	9	3
Units / dimensions	98.39	5	3	7	3	3	6	5	9	8	9	3
Vector concept	64.52	4	3	5	3	2	3	0	6	3	8	3
Unit vector notation	17.74	0	3	1	1	2	2	0	0	0	2	0
Graphs	33.87	5	3	7	2	0	1	1	1	1	0	0
Diagrams	82.26	3	3	5	3	3	4	4	8	6	9	3

Figure 1. The prevalence of calculus concepts and skills in the introductory physics textbook. The color indicates fractional prevalence, ranging from solid red (no occurrence) to solid green (present in all sections).

There is considerable variation in the frequency of calculus skills and concepts. The most prevalent calculus concept in our sample was the derivative. Derivatives appeared in 8 of the 11 chapters that we coded, and in 20 of the 62 sections (just over 30%). Integrals appeared in 5 of the 11 chapters, and 11 of 62 sections (just under 20%). In contrast, there was not a single epsilon-delta proof. Limits appeared in only four chapters (and 9 sections) and sequences and series in two sections of a single chapter. (The latter does become important in later courses, see Loverude 2024.) While the concepts of derivative and integral were somewhat prevalent, the skills of computing derivatives and techniques of integration were less frequently encountered. Even in calculus-based physics, derivatives and integrals are less-frequently encountered than algebra and trigonometry. Algebraic manipulations was coded for in every chapter, and 61 of 62 sections. Trigonometry was coded for in every chapter but one (chapter 2, one-dimensional kinematics) and appeared in 27 different sections.

The sequence of topics in introductory mechanics is mismatched to the sequence in calculus prerequisites, in terms of the use of calculus skills and concepts. Depending on prerequisite structures, students may be enrolled in first-semester calculus at the same time they are taking introductory mechanics. However, the most calculus-intensive sections of the text come very

early. In the text, students encounter integrals in Chapter 2, which would come in the first or second week of the course. Integrals are covered later in the calculus sequence. Dot and cross products appear in first-semester physics not until Calculus III.

The calculus concepts and skills in use vary considerably across the semester. Not all chapters present the same level of calculus usage. Chapter two (one-dimensional kinematics) was one of the highest in terms of CCF concepts and skills coded, and chapter 4 had the highest prevalence of derivatives, though fewer limits and no integrals. The chapters on Newton's laws (5 and 6), in contrast, did not include a single limit, derivative, or integral. After chapter 2, students do not see another integral until chapter 7.

There are mathematical skills and concepts in physics that are not part of the CCF. While some of the skills and concepts identified in the CCF appeared rarely (or never appeared) in our analysis, we found that the mathematical practices were not fully captured by the CCF. As a result, we expanded our analysis to include additional categories that were relevant to the course but not included in the CCF: function notation, reasoning involving units, and unit vector notation, as well as the presence of graphs and diagrams. Notably, units and vectors were considerably more prevalent than core CCF concepts of limits, derivatives, or integrals. Units appear in nearly every section. Vectors are common as well, including dot and cross products.

Function notation is used early in the text, and then largely disappears. As the table shows, function notation is used in beginning sections, but the use of this notation diminishes, even though functions are present throughout the course. In later sections, function notation disappears entirely, even though expressions are complicated and identifying independent and dependent variables is not trivial. There is some reason to believe that different usage of function notation is a tension point for students in physics (Taylor and Loverude 2022).

Discussion

The results show that many sections of the introductory physics textbook do not use calculus skills or concepts, and the presence of calculus is not uniform. This raises some questions about the role of calculus prerequisites for this course, and whether those serve as scaffold or as filter. If calculus is to serve as scaffold, rather than simply a filter, instructors and textbook authors might wish to consider the mismatch between calculus and physics practices and the sequences of topics. We intend to compare our results with an analysis of texts that do not explicitly require calculus to see the extent to which these differ.

We note that the frequency of these concepts and skills in the textbooks might not reflect the experiences of students. Calculus concepts and skills appeared in derivations of equations or in certain examples and homework problems. Depending on instructor preferences, student experiences might vary greatly. Some students might see calculus in derivations but never need to evaluate a derivative or integral if relevant problems are not assigned. We have begun to probe the extent to which these data match the experiences of instructors and students at our university.

By focusing on the first semester, we have not included the significant presence of calculus (including multivariable and vector-valued functions) in the electricity and magnetism portion of the course. This will be another focus of future analysis.

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Metacognitive Processes of Undergraduate Students During Mathematical Proof Construction

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This phenomenological case study explores the metacognitive knowledge of undergraduate students that they accessed while constructing mathematical proofs, and the related actions they take in their proof process with that knowledge. Two pairs of undergraduate mathematics majors completed two proof-construction task-based video-recorded interviews and engaged in follow-up qualitative interviews asking them to watch clips of their work and reflect on their thinking. This preliminary report discusses the qualitative data analysis process that has explored the control actions taken related to metacognitive knowledge of one pair of students during the first proof task.

Keywords: Metacognition, Problem Solving, Proof

In mathematics, conjecturing and proving new theorems is how humanity's mathematical knowledge is expanded and communicated (Ernest et al., 2016; Hamami, 2022). Undergraduate mathematics proof-based courses aim to imbue the necessary mathematical knowledge, writing and reasoning skills for students to read, construct, and analyze proofs. Development of these skills is not confined to individual courses but is an overarching goal of undergraduate mathematics programs (Committee on the Undergraduate Programs in Mathematics [CUPM], 2015). Students' proof skills impact and are impacted by both communication with others and their own reasoning about mathematics (e.g., Hanna & Barbeau, 2008). Previous studies note a relationship between one's metacognitive skills and success in problem-solving tasks (e.g., Mamona-Downs, 2002; Schoenfeld, 1994; Zhao et al., 2019). This study seeks to expand the body of mathematics education research by considering metacognitive processes of undergraduate mathematics students during the proof construction process.

I consider *metacognition* as "any knowledge or cognitive activity that takes as its object, or monitors, or regulates any aspect of cognitive activity; that is, knowledge about, and thinking about, one's own thinking" (Lerman, 2020, p. 608). The meta-level (where metacognitive knowledge resides) and object-level (anything external to the meta-level) interact through monitoring and control. *Monitoring* occurs when the information taken in influences the meta-level whereas *control* occurs when metacognitive knowledge impacts external cognitions and actions (Nelson & Narens, 1994).

I define *proof construction* to be the process of using a series of logical assertions deduced from a set of premises and previous assertions, and/or a set of instructions that when followed are representative of these deductions, where this collection of assertions and instructions results in an argument that "justifies a claim to oneself and others" (Blanton et al., 2003, p. 113). *Proof* is then defined to be this collection of assertions. Research has explored cognition related to the proof construction process (e.g., Hanna, 2020; Inglis & Alcock, 2012; Moore, 1994; Weber, 2001). What remains is how individuals examine their own thoughts and skills related to proof, and how their judgements impact their decisions during the proof construction.

Literature on metacognitive processes in mathematics education has predominantly focused on problem solving. It is established that metacognitive skills play an important role in students' development as problem solvers (e.g., Downs, 2002; Schoenfeld, 1992, Zhao et al., 2019).

Students employ metacognitive skills during all stages of the problem-solving process including the orienting, planning, executing, and checking phases (Carlson & Bloom, 2005).

Views differ among researchers on how problem solving and proving are related (Mamona-Downs & Downs, 2011). Some mathematics education researchers consider proving to be a subset of problem solving and use problem-solving frameworks to study proof construction (e.g., Savic, 2015; Selden & Selden, 2013; Wulan et al., 2021). However, research also suggests that some components of the proof construction process are not inherently addressed in problem solving. It remains unclear if and in what ways metacognition in problem solving compares to that in proof construction. Specifically, mathematical proofs constitute their own genre (Selden & Selden, 2013) and the language, deductive reasoning, and formatting can differ from other problem-solving tasks. These set proving apart as a specific case of problem solving, suggesting that proving requires a separate line of metacognitive research (Raman, 2003; Savic, 2015).

To gain insight into students' metacognition, I seek to identify students' metacognitive knowledge and processes during proof construction. More precisely, with this phenomenological case study, I aim to address the following research questions: What declarative, procedural, and conditional metacognitive knowledge do students demonstrate or claim to have and use during construction of a mathematical proof? And what monitoring and control processes do undergraduate mathematics students engage in during construction of a mathematical proof?

Theoretical Perspective and Methodology

I approach this research with a social constructivism theoretical perspective, concerned with individuals' constructed conceptions while also viewing their experiences and dialogue with self and others as integral to their learning process. This concern for social and cultural contexts distinguishes social constructivism from other forms of constructivism (Patton, 2002). I specifically considered intramental and intermental thinking, (i.e., the internal and external respectively) (Ernest, 2010). I claim that metacognition is a key component of intramental thinking as it centers on an individual's interactions with their own cognition.

Vygotskian social constructivism asserts the appropriation and use of signs (language) and tools (strategies) (e.g., Ernest, 2010; Vygotsky, 1987). The process of appropriating sign and tool use allows restructuring of conceptions and impacts others' knowledge through the use of tools and signs (Grossman, 1999). I claim that the process of restructuring one's own conceptions requires metacognitive monitoring, and access to metacognitive knowledge regarding what one already knows, so that alterations can be made. Additionally, the metacognitive process of control informing the individual's use of signs (language) and tools (strategies) in practice, mediate the collective appropriation of such knowledge towards agreed upon social practices.

A phenomenological study examines the "meaning, structure, and essence of the lived experience" (Patton, 2002, p. 104), in this case the phenomenon being studied is individuals' cognitive and metacognitive processes during the construction of mathematical proofs. I conducted a qualitative phenomenological case study to examine two pairs of undergraduate students' metacognitive processes as they construct proofs, through in the moment verbalization of their thinking and later reflections on their proof construction experience. The four undergraduate student participants were from proof-based undergraduate courses at a mid-size university in the United States. Prior to interviews, students were paired based on similar mathematical backgrounds and experience working on mathematics collaboratively together.

Data Collection and Analysis

As individuals' conceptions "are formed through their interactions with each other as well as by their individual processes" (Ernest, 2010), this study had pairs of participants engage in collaborative proof-construction tasks using think-aloud protocols. Each pair was given version A or version B of a task and asked to prove or disprove the claim. Version A provided a true claim, and version B provided a falsified version of the true claim. Students were audio and video recorded and remotely monitored via Zoom©, and Livescribe© pens were used to record written data. Once participants completed the task, I returned to the participants to ask follow-up questions based on my observations during the task. I then determined portions of video where metacognitive processes may have occurred by utilizing the Model of Sign Appropriation and Use (Ernest, 2010, p. 44) for possible social location and ownership of utterances. Participants were invited to an individual follow-up interview a couple days after the paired interview to watch these selected video clips, reflect on their thinking during that part of the task and answer reflection questions about their experiences.

The preliminary results presented in this paper are the analysis of Lola and Harper's experience with version B of the first task. Figure 1 is an abridged version of the task, adapted from Gilbert and Larson (2012). Note that due to a typo, I had to step in to make a minor correction to the claim that only one of k or l had to be greater than 1.

> **Task 1 version B prompt.** A gamekeeper gives player A a positive integer. The players then take turns changing the integer they are given to a smaller integer, and then pass that smaller integer back to their opponent, player B. If the integer you received and are changing is even, the two legal moves are:
>
> ○ Subtracting 1 from the integer, *or* Dividing the integer by 2
>
> If the integer you received and are changing is odd, the two legal moves are:
>
> ○ Subtracting 1 from the integer *or* Subtracting 1 from the integer, then dividing this result by 2
>
> The game ends when the integer reaches 0, and the player who makes the last move (changing an integer to zero) wins the game.
>
> *Prove or Disprove the Claim:* Assume the starting integer is given to player A. Any starting integer that when factored is written as $2^k \cdot l$, where k and l are both odd integers greater than 1, will guarantee that player A wins the game.

Figure 1. Task 1 version B

Both the task and follow-up interviews were transcribed. Data analysis and use of each code is shown in Figure 2. I first used deductive methods, keeping in mind the Model of Sign Appropriation and Use (Ernest, 2010, p. 44), to determine whether each transcript excerpt described internal thought processes (coded as Intramental), or other external happenings (coded as External). The intramental data was analyzed to determine if the subject of each excerpt was their own thinking (coded as metacognitive) or was about the object-level (coded as cognitive).

From the metacognitive data, I then sought to determine actions taken due to metacognitive reasoning, as this would be indicative of a control process. To do this, I separated the metacognitive excerpts into two categories, one for excerpts that were followed by an action (coded as having a relevant action), and one for any metacognitive knowledge stated apart from any action taken (coded as stand-alone).

From the relevant action data, inductive coding was used to categorize each indicated action. Each piece of data was individually coded by a description of the action taken, and then codes that were similar were combined. Because these actions were taken following the indication of metacognitive knowledge, they were coded as control actions.

Apart from relevant actions, I also used deductive coding on all data coded as metacognitive to determine if the metacognitive knowledge indicated was declarative, procedural, or conditional. At this point all data was placed into case files for each participant based on whose metacognitive knowledge was indicated in the transcript, whoever initiated the discussion of their reasoning.

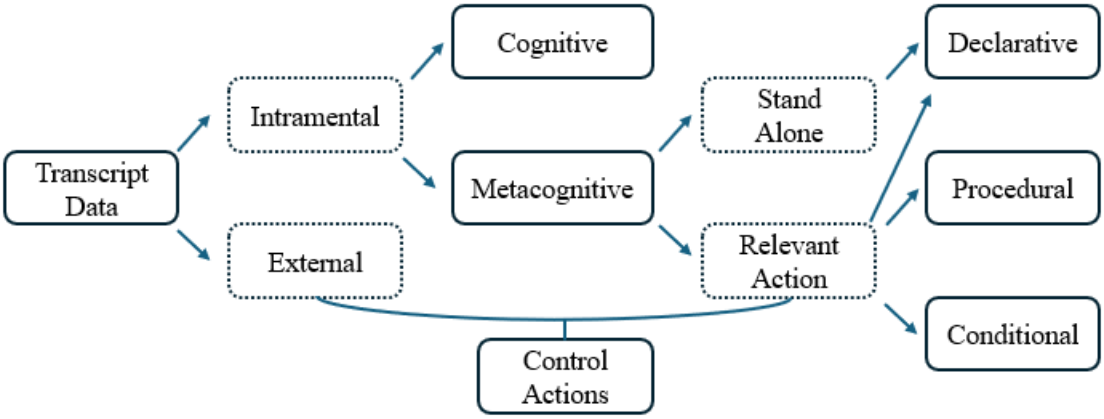


Figure 2. A Flow Chart of the Data Analysis Process

As an example of the data analysis, consider this quote from Lola, "I don't know, the most efficient way to write this? Let's - let's write out the tree and then. Yeah, so B can give two." Lola and Harper then had a lengthy discussion of how to order their disproof from the stages of their tree diagram. Since this excerpt describes Lola's internal thinking about writing, this was coded as Intramental. Because the subject of her statement was her own knowledge, uncertainty that she knew the most efficient method, this was coded as metacognitive. There was a relevant action in this statement the decision of her proof organization.

Preliminary Results

As Lola and Harper began, Harper read the task aloud. Lola then began writing generalized versions of odd numbers for k and l. Harper suggested trying some examples, with each of them as the two players in the game. Lola, in her follow-up interview, noted that she often likes to write down the given information and relevant definitions when beginning a proof. In her follow-up interview, Harper stated she often begins by trying concrete examples. In the paired interview, Lola commented trying examples might be a way to determine if a counterexample exists. They decided on the starting value six, and played the game for a couple of rounds, to work out possible outcomes from allowable moves. They determined six was a counterexample to the claim. Working through other examples, they found a starting number of eight is also a counterexample. Harper suggested drawing tree diagrams of each possible decision in the game. Lola added arrows to these diagrams to indicate which player was "giving" a number that round. Lola was the first to indicate a desire to write a proof that the claim is false using their examples. When unsure how to begin writing, Lola suggested they follow the order of how each turn played out in their tree diagram. Harper ended up being the one to write the proof on paper, but both Harper and Lola went back and forth stating what should come next in their written proof.

Throughout the task and follow-up interviews there were instances of metacognitive knowledge and related actions that were inductively coded as types of control actions. The

control actions from Lola and Harper included Reading Aloud, Re-reading the Prompt, Checking Work, Re-working, Evaluating Strategy, Generalizing, Using Math Drawings, Organizing Proof, Writing Proof. A few of these codes are in Table 1 with descriptions and corresponding example quotes. Both participants engaged in most of these actions with a few exceptions. Only Harper demonstrated Reading Aloud, and only Lola demonstrated Generalizing and Writing Proof.

Table 1. A table of a few demonstrated control actions participants took.

Metacognitive Control Process Actions During Proof Construction		
Control Actions	Description	Example Transcript Data
Re-Working	Reviewing why their work was useful	"wanting to go back and run through the logic again, remembering what I was trying to prove"
Evaluate Strategy	Actively considering the value or effectiveness of their current approach	"I don't know if this general tree thing is ever going to end…Any pattern…has probably already showed up and we just didn't notice it. Let's go back to actual numbers."
Organizing Proof	Determining what will be included in the proof or the order	"I don't know, the most efficient way to write this? Let's hit let's write out the tree and then. Yeah, so B can give two…"
Writing Proof	Deciding to move from "scratch work" to writing down a proof or part of a proof	"…I write chaos all over the place on a page just to give myself a sense of what's going on…then I…write it out using more formal language and a series of steps that you can follow…"

There were notable patterns in the metacognitive knowledge utilized in each task that will be more thoroughly verified in continuing data analysis. Most often the metacognitive knowledge related to Harpers actions was procedural, related to possible strategies usefulness in this context. While metacognitive knowledge was not always clearly demonstrated, Lola frequently initiated a pause for various forms of reflection during the task. These intentional pauses may be indicative of further metacognitive monitoring and control processes that are not explicit in the data.

Discussion and Questions for the Audience

The preliminary results indicate that undergraduate mathematics students are accessing metacognitive knowledge during proof construction, and their metacognitive knowledge impacts their choices during the proof construction process. The two proof specific control actions observed indicate that there are control processes unique to proof construction. This preliminary result solidifies the need for considering metacognition in proof tasks specifically. The exploration of the declarative, procedural, and conditional metacognitive knowledge is ongoing with the second pair's data. I will complete the analysis with constant comparison of each case. My questions for the audience are: were there any instances of data in the presentation you felt should have been coded as metacognitive that were not? In your teaching and research experience, have you observed metacognitive knowledge or actions specific to proving?

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Exploring Discrepancies in Mathematicians' Use of Programming in Teaching and Research

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We present preliminary findings from an ongoing investigation into the discrepancies between how university-level mathematicians incorporate programming into their research versus their teaching. Motivated by earlier studies that highlight a gap in the use of programming between research and teaching, we explore this tension through interviews with mathematicians at a university recognized for supporting programming in science and mathematics teaching. Specifically, we draw on Engeström's Cultural Historical Activity Theory to analyze the interview with one mathematician who illustrates this discrepancy. Our preliminary findings suggest connections to more deeply held epistemic beliefs about which tools and purposes are considered legitimate in mathematical practice.

Keywords: Mathematicians, Programming, Mathematical Practices, Activity Theory

Background

In this paper, we present preliminary findings from an investigation into the underlying causes of the discrepancies in how university-level mathematicians use programming (writing and executing computer code) in their research versus their teaching. A motivation for this project is a survey of 302 Canadian mathematicians by Buteau et al. (2014), where 43% reported using programming in their research compared to 18% in their teaching. In a follow-up interview study, Broley et al. (2018) found institutional factors commonly cited as barriers to using programming in teaching (e.g., students do not know how to program). However, other research (e.g., Iannone and Simpson, 2022) indicates that mathematics students generally have access to courses in programming, suggesting the significance of factors beyond institutional issues.

In an interview study with six mathematicians, Lockwood et al. (2019) found that the mathematicians mainly used programming in their research to gain intuition into problems or to provide numerical results where analytic methods fell short. This suggests that the discrepancy between the use of programming in teaching and research could be linked to a previously established tension between the high-value mathematicians place on intuition in research and the comparatively lower emphasis on intuition in teaching (e.g., Burton, 2004; Hersh, 1991).

To explore this question, we conducted interviews with 15 mathematicians at a site with long-standing and recognized institutional support for the use of programming in teaching. While it was immediately clear that the interviews confirmed the existence of a discrepancy between the use of programming in teaching and research, we aimed to probe deeper for underlying causes by drawing on Engeström's Cultural Historical Activity Theory (CHAT). Here, we present an analysis conducted on one of the interviews that exemplified this discrepancy.

Specifically, we explore the following questions:

- What features do we see in the data when using CHAT as an analytic lens to compare how a mathematician describes their teaching and research activity systems as they relate to the use of programming?
- Is additional data needed to further support these findings?
- Should we modify or further refine our operationalization of CHAT?

Theoretical perspective

Engeström's Cultural Historical Activity Theory (CHAT) builds on the activity theory of Vygotsky and Leon'tev (Engeström, 2015). Given the scope of this paper, we focus on operationalizing Engeström's extended activity triangle (Figure 1).

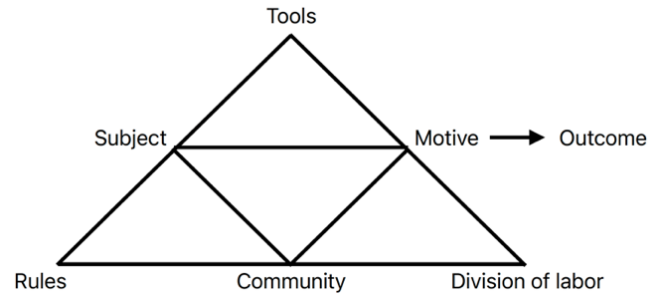


Figure 1. The extended activity triangle (Engeström, 2015).

Engeström's activity triangle expands Vygotsky's mediating triangle by emphasizing the collective aspects of activity (rules, community, division of labor). Activities are defined as enduring, motive-driven, situated, and collective human endeavors, such as mathematical teaching and research. To provide an analytic lens, we follow Mwanza and Engeström (2005) in operationalizing Engeström's triangle as a checklist of questions to ask the data (see Table 1).

Table 1. Operationalization of the nodes of the activity triangle.

Node	Example of questions
Subject	What role does the subject identify with as part of the activity?
Motive	Why is the subject pursuing the activity?
Tools	With what means does the subject pursue the activity?
Rules	What rules and norms regulate the subject's pursuit of the activity?
Community	Who does the subject see as supporting their pursuit of the activity?
Division of Labor	What power structures affects who does what in the activity?
Outcome	What is the 'output' of the activity as perceived by the subject?

Following Leon'tev, Engeström views the motive as the activity's reason for existence and a force that drives all other components. Our analysis, therefore, begins by focusing on the motive, using the other nodes to add layers of detail. The aim is to identify certain cultural tensions, called dialectic contradictions, that serve as historical driving forces, such as the tension between applied and pure mathematics. While we see the nodes as acting like the diagnostic tools of a physician, dialectic contradictions are not issues to be 'cured' but mechanisms to be understood and leveraged for change and innovation within the activity.

Methodology

We conducted semi-structured interviews with 15 mathematicians at the department of mathematics of a major northern European research university, who were sampled purposefully to represent a wide variety of mathematical research orientations (pure, applied, and computational). Each interview lasted 1.5 – 2 hours and addressed programming in their research and teaching, as well as institutional issues. All interviews were audiotaped and transcribed. To ensure a high level of communication, the interviews were conducted by the first author, who has

a background as a research mathematician. Additional data, such as CVs, examples of publications, and webpages of taught courses, were consulted to support the analysis.

In this paper, we analyze the interview of an applied computational mathematician, given the pseudonym Emil. He was selected because he uses programming extensively in research but much less in teaching. Using our operationalization of CHAT over repeated readings of the full interview transcript, we generated descriptions of his teaching and research practices. We then compared these descriptions, looking for tensions indicating dialectic contradictions that relate to the discrepancy in the use of programming exemplified by Emil. These findings were discussed among the authors and compared with prior analyses of the complete data set.

To protect Emil's anonymity, we have altered quotations to obscure specific details about his research focus and the courses he has taught.

Findings

To save space, we combine our interview analysis with a description of Emil's teaching and research activity systems. The main points are summarized in Figure 2.

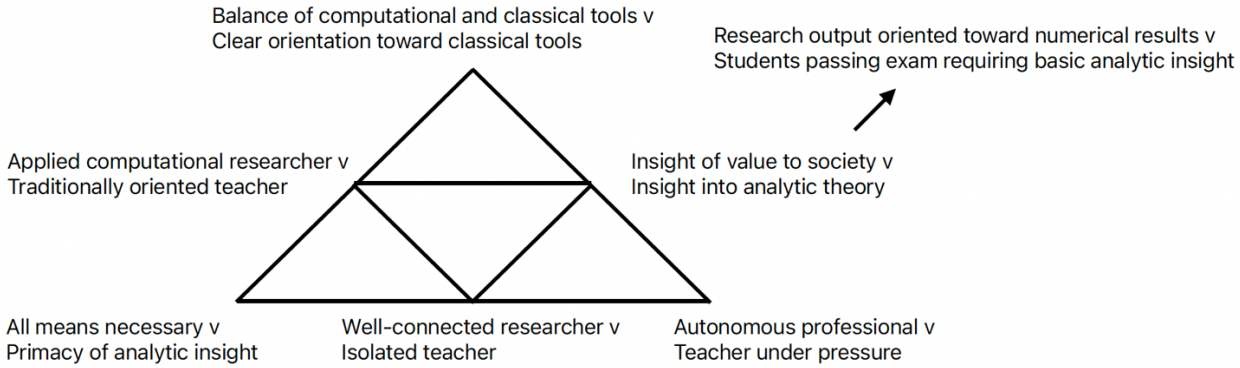


Figure 2. The main features of Emil's research (top) and teaching (bottom) activity systems.

Description and comparison of Emil's two activity systems

Subject. As a researcher, Emil explicitly identified as an applied computational mathematician. As a teacher, his preference to avoid programming in introductory courses suggests a more classical focus where students are to work analytically with pen and paper.

Motive. Regarding research, Emil stated that "the end product is that I want a deep understanding of how a physical phenomenon behaves, based on an approach that integrates an analytic and numerical approach with the use of real-world data." In discussing his teaching, Emil gave additional details, saying that, ideally, he would want students to have the same combination of abilities so that they can "piece them together into an end product that is useful for the industry and society." This motivation was tempered, however, by his low expectations of the capabilities of first-year students: "I have to say that the challenge for students to get through the first-year course that I teach, it is not their ability to use a computer, on the contrary, it comes down to their ability to understand how basic mathematical objects behave."

Tools. As a researcher, Emil said, "If I did not have computers and programming as tools, then I would not be able to do anything." He explained that analytic approaches to applied mathematics ran out of steam 50 years ago, and since then, progress has been entirely reliant on computers and programming. As we noted in the motive node, Emil was skeptical of whether it was possible to leverage programming in the same way in teaching first-year students. The following quote indicates that not being able to tap into this synergy bothers him:

> I think that being able to use a computer to represent the mathematics you do is very advantageous. If you can do it in concert with your ability to represent the same things using your head, paper, and pencil, I think you will get further. […] I wish our students would reach such an advanced level that we could begin to talk about synergies. However, [that is] aiming far higher than what we are able to get in [my course].

Rules. In his research, Emil views computers as indispensable for doing research in applied mathematics. However, when talking about first-year students, he stresses the primacy of analytic mathematics over the use of programming. That is, he seems to insist that any use of programming in mathematics must be guided by analytic intuition or insight - it does not work the other way around. He explains this as follows in his own words:

> … programs are, to a larger degree, acting like black boxes. And there is no reason to believe that these researchers have written their own programs either. They just sit and use it as a black box that someone else has programmed. Then, I get worried because I feel that they miss some of the deeper analytic insight that I am used to having. So, personally, I prefer that you not just show the result of the numerical simulation but also have some deeper analytic understanding. And maybe, I think it is, there is something to implementing a simple model rather than just implementing the full equation governing the physical system. Because then you can focus on what a simple model... how it behaves, as I think, deep down, can give you more insight.

Community. In his research, Emil collaborated with fellow researchers, contacts in the industry, and MSc and PhD level students. This is corroborated by a look at his publication record and personal webpage. As a teacher, the level of collaboration with colleagues seems less pronounced (which is also a recurring theme in the other interviews).

Division of labor. Emil expressed significant autonomy concerning his teaching and research. However, there are some discrepancies. Although he interacts with a network of collaborators in his research, he adopts a more independent stance in his teaching. In his research, Emil expressed the absolute necessity of using computers. In his teaching, he expressed the difficulty of using programming based on his impression of his students' analytic proficiency level. He also pointed to continuous reorganizations, both within the university and externally, as taking so much energy that he did not feel the strength to look at the 'big picture.'

Outcome. In his research, the outcome is research papers of interest to the industry, with numerical simulations at the forefront. In his teaching of first-year students, the outcome is that mathematics students pass exams that test what Emil considers basic analytic skills.

Discrepancy between activity systems

In the above analysis, we see that Emil, as a researcher, desires insight into physical phenomena based on a combined analytic and numerical approach that allows him to work with real-world data. He does mathematics to contribute to our understanding of the real world and he does it using programming, as it would be impossible otherwise.

While Emil wants students to have 'the whole package,' he is concerned that their lack of analytical mathematical ability makes it difficult for them to pass courses in their first year. Therefore, he teaches classical analytical mathematical content using pen and paper, without involving computers, except for using ready-made tools for visualizations. He believes this approach gives students the necessary insight required to work numerically.

We interpret this as a discrepancy related to tensions in the epistemic dimensions of the motive of the activity and by what means the activity is pursued. We represent this in the 'epistemic matrix' shown in Figure 3. We note that while Emil would like students to use 'the whole package,' he seems to feel pressured by circumstances outside his control (such as his perception of students' low analytic proficiency) to adopt another, more classical, stance concerning his teaching.

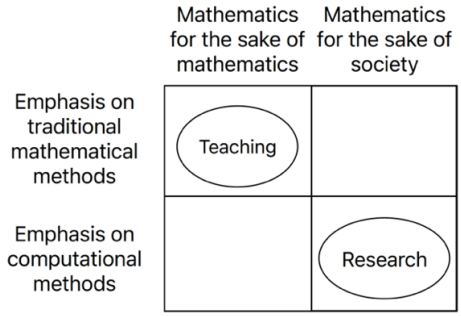


Figure 3. An 'epistemic matrix' representing the discrepancy between Emil's research and teaching activities.

Discussion

Using CHAT as an analytic lens, we are naturally led to focus on Emil's motives for research and teaching, highlighting discrepancies between the 'reasons why' of the nodes. The most conspicuous discrepancy concerns the use of programming. While Emil views his mathematics as something that should serve society, with programming as a necessary tool to handle real-world data, he believes that any use of numerical methods must be guided by analytical ability. For Emil, students must first learn basic analytic skills before being tasked to program anything. Thus, he emphasizes traditional methods in teaching, not because he devalues programming, but because he sees analytic ability as a prerequisite for insightful use of programming. This observation resonates with and connects the findings of Buteau et al. (2014), Broley et al. (2018), and Lockwood et al. (2019), which we discussed in the background section. We propose the epistemic matrix as a tool to explore the gap between the use of programming in mathematics research and teaching. Our findings suggest that this gap stems from a tension between Emil's belief in the primacy of analytic ability and his perception of first-year students' skills.

A first natural question is whether the dimensions of the epistemic matrix are strongly correlated. This does not seem to be the case, as preliminary analyses of other interviews suggest that there are mathematicians corresponding to other areas of the matrix. A second question is whether the dimensions of the epistemic matrix represent dialectic contradictions (i.e., historical driving forces of the activities). To understand this, analyses of additional interviews, a closer look at institutional factors, and a review of related results in the philosophy and history of mathematics are necessary.

While one might argue that using CHAT to analyze a single individual is a contradiction in terms, as it is a socio-cultural theory meant to analyze *collective* activities, we maintain that if the goal of the analysis is to uncover epistemic dimensions, then the relevant collective component is chiefly how the subject *perceives* the collective dimension, which may be different to how peers report the collective dimension.

We conclude with potential questions for the audience:

1) Does our use of CHAT seem useful and appropriate as we have operationalized it?
2) Do you see the epistemic matrix as having a potential to inform mathematics education?
3) Are there other frameworks than CHAT that are better suited for our analysis?

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The Guided Reinvention of Equivalence Classes and Equivalence Relations

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We present a preliminary local instructional theory of students' guided reinvention of equivalence classes and equivalence relations that has resulted from engaging in one iteration of the design research cycle. We leverage the instructional design theory of Realistic Mathematics Education (Freudenthal, 1991) to design a task sequence in an experientially real setting that can guide students to reinvent equivalence classes, equivalence relations, and their properties. In this paper, we present our revised task design describing the rationale, task implementation in a classroom setting, and analyses of student reasoning. Further, we conclude by discussing our plan to reimplement the tasks in another iteration of the design research cycle.

Keywords: Realistic Mathematics Education, Equivalence, Guided Reinvention

Equivalence is ubiquitous in mathematics, particularly in abstract algebra. Students are introduced to notions of equivalence, like equality, as early as elementary school (e.g., Alibali et al., 2007; Kieran, 1981; Knuth et al., 2006). As they progress through their mathematical careers, students encounter more complex notions of equivalence like congruence and isomorphisms in abstract algebra. Researchers have explored undergraduate students' interpretations of notions of equivalence (e.g., Cook et al., 2021; Mirin, 2019) and reasoning about equivalence across domains (Cook et al., 2022). More domain-specific research has investigated productive ways of thinking about equivalence in combinatorics (Lockwood & Reed, 2020) and the notion of sameness in the context of isomorphism and homomorphism in abstract algebra (e.g., Rupnow, 2019). Despite this recent work, little is known about how students reason about equivalence relations and equivalence classes in abstract algebra. We aim to contribute to this literature through our design research-informed (Gravemeijer et al., 2003) local instructional theory of students' guided reinvention of equivalence relations and equivalence classes. Guided reinvention (Freudenthal, 1991) is an instructional design principle from Realistic Mathematics Education involving "an instance of psychological explication whenever students already possess some intuitive or less formal understanding from which they construct formalized meanings" (Dawkins, 2015, p. 67). Scholars have investigated students' reinvention of various abstract algebraic concepts, like groups (Larsen, 2009, 2013), quotient groups (Larsen & Lockwood, 2013), isomorphisms (Larsen, 2009, 2013), rings (Cook, 2012), integral domains (Cook, 2012), fields (Cook, 2012), (ir)reducible elements (Serbin et al., 2024), and unique factorization domains (Bae et al., 2024). However, scholars have not yet explored students' guided reinvention of equivalence relations and equivalence classes, which we address. In this preliminary report, we present our revised task sequence and analysis of student reasoning after one iteration of the design cycle. We address these research questions: *Which ways of reasoning do students engage in as they reinvent equivalence relations and equivalence classes? How does the task sequence in this local instructional theory evoke these ways of reasoning?*

Theoretical Perspective

Design research (Gravemeijer et al., 2003) involves a cycle by which researchers design an instructional task sequence, implement it in an instructional setting with a teacher-researcher

guiding students, analyze the student reasoning that results from engagement with the task sequence, and revise the sequence to improve its subsequent use. Design research can be used to create a *local instructional theory* (LIT), "the description of, and rationale for, the envisioned learning route as it relates to a set of instructional activities for a specific topic" (Gravemeijer, 2012, p. 107). Realistic Mathematics Education (RME) is an instructional design theory that can inform such design research (Freudenthal, 1991; Gravemeijer, 1999). RME has three design heuristics that researchers consider as they design task sequences: experientially real context problems, guided reinvention, and emergent modeling. RME-informed task sequences have problems set in experientially real contexts that have accessible entry points for students to engage in intuitive mathematical activity. These tasks should also provide the opportunity for students to create a model (e.g., a physical object, manipulative, or symbol) of their mathematical activity that they can use as a model for further abstract mathematical reasoning. This shift in the emergent model, accompanied by progressive mathematization, supports the student in reinventing a mathematical concept. Students "reinvent" a mathematical concept with the guidance of a teacher by progressively formalizing their informal, intuitive understandings. Overall, this design theory of RME informed the design of our LIT.

Methods

Our design research (Gravemeijer et al., 2003) began with designing the RME-informed task sequence to engage students in the guided reinvention of equivalence relations and equivalence classes. We aimed for them to develop intuitive, informal understandings of those concepts and transition to formally define those concepts. We conducted a teaching experiment (Cobb, 2012), where we implemented the task sequence in a class setting, analyzed students' reasoning about the tasks that were evident in recordings of class discussions, and revised the task sequence and its rationale to potentially better support students in reaching the intended learning outcome.

We collected data in a Mathematics for Secondary Teachers course at a large public university in the southern US during the Spring 2024 semester. This course contained 23 students who were junior and senior mathematics majors. The task sequence was administered in the beginning of the hybrid course over four classes; two in-person and two on Zoom. These classes were recorded and transcribed. The recordings could not capture group work discussions, so we only analyzed the student reasoning shared in whole-class discussions. The projected/ shared screen was recorded. Pictures of student work on whiteboards were also collected. We performed a qualitative video analysis (Lesh & Lehrer, 2012), in which we wrote content logs of the class discussion videos and memos of students' ways of reasoning evident in the videos. These memos also included reflections of how the tasks seemed to evoke these ways of reasoning. Each author individually wrote these content logs for the four classes, and we met to compare our interpretations of the videos and our analyses of student reasoning. We discussed how effectively the tasks guided the students' reinvention and decided how to revise them.

Results

We present our revised task sequence based on this analysis. We describe each task, its rationale, and examples of student reasoning exhibited on the task. We then present additional tasks 6-11, which were added to the task sequence during the revision, along with their rationale.

Task 1: Coloring task

Our LIT begins with a number line coloring task, which leads students to create a set of equivalence classes under congruence modulo 3 on the set of integers. The task states that each

number on the number line is colored either red, blue, or green and outlines two rules: 1) The negative (opposite) of a red number must be colored blue and 2) the sum of two blue numbers (not necessarily distinct) must be colored red. Students are tasked to find a coloring of the number line that meets these conditions. A coloring using all three colors and meets these conditions will successfully divide the number line into a repeating green, red, blue (or green, blue, red) pattern (see Figure 1). This task engages students in an experientially real task setting of coloring an integer number line. We aimed for students to use the colored number line as an emergent model of their activity that they could mathematize to reinvent equivalence classes.

In line with the exploratory nature of the task, students used a trial-and-error method, leading to correct and incorrect strategies. A common incorrect strategy was to start by making all negative numbers blue and all positive numbers red (or vice versa). When using this strategy, students noticed that it led to problems and contradictions like: "We chose red to be the positive numbers and then the blue ones to be negatives, and when we add two negative numbers, we can't end up getting positive." The group noticed that since they colored the negative numbers blue, their sum should be colored red (condition #2) which means the sum should be a positive number. However, given that the sum of two negative numbers will always be negative, this leads to a contradiction. A successful strategy that was used by a couple of groups was to build up the coloring based on the two conditions, as one student explained it: "We started with 1 and from there we went and said 1 plus 1, since the sum of it is 2, then it's red [condition 1]. So since 2 is red, then the opposite has to be blue [condition 2], so -2 would be blue." This system resulted in some numbers not being assigned blue or red, so the group decided to color these numbers green. Once students shared this strategy, the instructor had the rest of the groups use it to color their number lines to develop the class's shared understanding of the strategy.

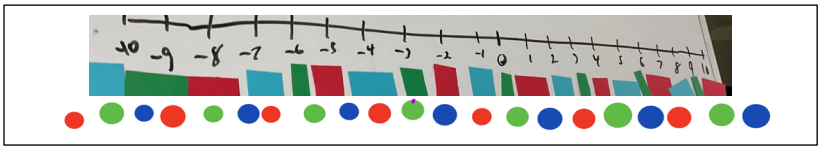


Figure 1. Students' patterned colorings (green-blue-red or green-red-blue) of the integers number line in task 1.

Task 2: Make a rule for each color

Task 2 guides students to develop rules to determine what color any integer will be according to the pattern. This task anticipates the formalization of the equivalence classes since the sets of integers with green dots, blue dots, and red dots are each equivalence classes on $\mathbb{Z}$. We saw two strategies for articulating a rule. First was a student's "jumping the number line" strategy:

> Whenever it starts at 0, you can just add 3, so it would be 0, 3, and then plus 3, 6, plus 3, 9, plus 3, and so on and so forth. However, if you start at 1, then 1 plus 3 would be 4, which is red. 4 plus 3 is 7, which is red. And then 7 plus 3 is 10. And the same thing works for 2. So, 2 plus 3 is 5, 5 plus 3 is 8, and so on.

Using the number line model, the student considered the starting point for a color as 0, 1, or 2 and made jumps of size three to find the next number with the same color. Others mathematized this idea with set notation, writing $\{3k|k\in\mathbb{Z}\}$ to represent green, $\{1+3k|k\in\mathbb{Z}\}$ for red, and $\{2+3k|k\in\mathbb{Z}\}$ for blue. The second student strategy involved modular arithmetic. Students noticed the numbers could be distinguished by their remainder when divided by 3. As their rule, one group wrote "*Let $n \in \mathbb{Z}$, for this to find the nth color we use, green: (n mod 3) = 0, red: (n mod 3) = 1, blue: (n mod 3) =2*". To develop a shared understanding of these rules, the task next had students use these rules to determine the color 534, 2156, and -74 (green, blue, red, respectively).

Task 3: Develop an informal definition of equivalence and equivalence class

For the next task students develop a personal definition of equivalence with the goal of 1) connecting the dot activity to the notion of equivalence and 2) begin transitioning to more formal mathematics. Most students' personal definitions of equivalence involved the idea of sameness; for example, "Classified as the same under some condition" or "They produce the same results with certain conditions." The instructor prompted students to describe what their definitions had in a common and a student suggested "sameness." The instructor then asked students to elaborate on whether elements in the same class had to be exactly the same. A student said, "It doesn't necessarily have to be the same exact thing…we have multiple different objects that you can have that will satisfy these conditions. As long as those conditions are met, you have an equivalence class." The instructor connected students' informal notions of equivalence to the number line coloring activity, noting that dots of the same color are equivalent, so integers of the same color are in the same equivalence class. The instructor then wrote an informal definition of equivalence class as *the set of elements that are equivalent to each other under some rule*.

Task 4: Establish a rule for two integers being equivalent modulo 3 and use it

After the instructor claimed students' green, blue, and red-dot equivalence classes $\{3k|k\in\mathbb{Z}\}$, $\{1+3k|k\in\mathbb{Z}\}$, $\{2+3k|k\in\mathbb{Z}\}$ were sets of elements that are equivalent modulo 3, task 4 prompts students to complete the claim, "For any $a, b \in \mathbb{Z}$, a is equivalent to b modulo 3 if and only if:". The goal of the task is for students to create the equivalence relation for equivalence modulo 3. Students contributed a variety of answers like "a and b have equal remainders [when divided by 3]" and "3 divides $(a - b)$." After students shared their ideas, they chose a rule and used it to determine whether 216 and 487 were equivalent modulo 3. To ensure students could use these rules, the task prompted students to find two integers that are not equivalent modulo 3 using their rule. Students successfully found a non-example and a correct justification. One group used 11 and 13 and explained that the remainder is not the same when dividing by 3. This task helped students recognize that equivalence classes of integers with the same color contained integers that were equivalent according to some rule, which will be formalized as an equivalence relation.

Task 5: Relating Equivalence Classes and Partitions of $\mathbb{Z}$ and Defining Partition

For Task 5, students discuss what they notice about red, green, and blue dots. The goal of this task is for students to use their number line model to develop an intuition for the properties of equivalence classes on $\mathbb{Z}$, namely that they form a partition of $\mathbb{Z}$. This activity anticipates the formal definition of partition: A partition of a set X is a collection of nonempty subsets of X such that (i) the union of those subsets is X and (ii) the subsets in the collection are pairwise disjoint. Students first noticed the pattern of the colored dots on the number line repeating green, red, blue. They noticed the cardinality of each of the sets of {red dots}, {green dots}, and {blue dots} was infinite. The instructor elaborated: "We noticed that this pattern goes on for all of the integers, and so that means if I put together all those sets, of the greens, the reds, and the blues, I get all of the integers." She prompted students to restate that idea using more precise language or notation. A student claimed, "the union of equivalence classes makes up the set of integers." Another student said: "each set is distinct from each other since one is just red, one is just blue, and one is just green." The instructor revoiced that indeed the sets are distinct and disjoint, and she pointed out that this is a special property she wanted them to notice. She mentioned that this has to do with the intersection and asked what the intersection of these sets is. Three students replied that it is the empty set, so the sets have no common elements. The instructor claimed that these two criteria were axioms in the definition of a partition and guided students to define

partition. She asked why the color dot sets form a partition of $\mathbb{Z}$. Students replied it was an assumption of task 1 that a number cannot be more than one color, which implies the sets are disjoint. Also, since all integers could be assigned a color, the union of those sets is $\mathbb{Z}$.

Newly Designed Tasks

We designed new tasks to conclude the sequence: *Task 6)* Find examples and non-examples of partitions of $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{N}$. *Task 7)* Use the following rules on $\mathbb{Z}$ to create subsets of elements that are equivalent under the rule. Do these subsets form a partition of $\mathbb{Z}$? If yes, verify using the definition of partition. If not, state what part of the definition is not satisfied. $x \sim y$ iff a) $x^2 = y^2$, b) $x = ky, k \in \mathbb{Z}$, c) $x \neq y$, d) $xy \geq 0$. *Task 8)* Rules with corresponding equivalence classes that form a partition of $\mathbb{Z}$ are called equivalence relations. Part 1: What properties do these rules need to have to be equivalence relations? Part 2: Create a definition for an equivalence relation. *Task 9)* Now that we have defined equivalence relation, we can refine our definition of equivalence class to make it more formal. *Task 10*) Consider this collection of subsets for the given set. Does this collection form a partition? Could these subsets form equivalence classes? Use the definitions of equivalence relation, equivalence class, and partition to justify your claim. Ex: $\{1,2\},\{2,3\} \subset \{1,2,3\}$. *Task 11)* a) What do you notice about the relationship between equivalence relations and partitions? b) Write a conjecture that formalizes what you noticed.

These new tasks guide students to 1) reinvent and formalize the definition of equivalence relation and 2) articulate the connection between equivalence classes and partitions of a set. Task 6 gives students an opportunity to become familiar with the definition of partition that is established in Task 5. Task 7 begins students' reinvention of equivalence relation by exploring different rules that do and do not satisfy the three properties of equivalence relations (reflexive, symmetric, and transitive). In Task 8, students reflect on this work and articulate these three properties, which they then use to write a definition of equivalence relation. In Task 9 the instructor uses this definition to redefine equivalence class formally (*Let ~ be an equivalence relation on A. Let $x \in A$ The equivalence class of x is $[x] = \{y \in A | \, y \sim x\}$).* In Task 10, students explore subsets of a set and are guided to notice how subsets that fail to be partitions also fail to be equivalence classes since their corresponding relation fails properties of equivalence relations. Task 11 guides students to articulate this and establish the Fundamental Theorem of Equivalence Relations: Let A be a non-empty set. 1) *if ~ is an equivalence relation on A then the set of equivalence classes of ~ forms a partition of A.2) if $\{A_i \, | i \in I\}$ forms a partition of A then there is an equivalence relation on A whose equivalence classes are the sets $A_i, \; i \in I$.*

Discussion

This preliminary report shares our findings from only one iteration of the design cycle in our attempt to create a local instructional theory on the guided reinvention of equivalence relations, equivalence classes, and partitions. Our analysis of student reasoning on the tasks informed our revisions and additions to the task sequence. The added tasks serve to further engage students in defining (Zandieh & Rasmussen, 2010), which was absent from the first implementation of the tasks. To further refine the tasks, we will implement the tasks in a classroom setting again as part of a new iteration of the design cycle. We will provide a theoretical contribution to the field regarding characterizations of key ways of reasoning that are milestones in reinventing equivalence classes, equivalence relations, and partitions. We hope for instructors to be able to use our task sequence to guide students to reinvent those concepts. We pose this discussion question to the audience: What feedback do you have for our design of the task sequence?

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Reasoning Beyond Content-Specific Objects: A Case Study of One Student's Structural Thinking

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In all levels of mathematics, especially beginning at the undergraduate level, structural thinking is pivotal for deeper understanding or more complex reasoning about the relevant mathematical concepts. In this preliminary research study, we investigated the structural thinking of one advanced undergraduate mathematics student as he conjectured the definitions of various concepts in graph theory. We observed various affordances and obstacles to his structural thinking. Affordances included his attention to a general structural property and his use of analogical reasoning with corresponding structures in group theory and point-set topology. Obstacles included utilizing a constrained example space and relying on his familiarity with previously encountered mathematical concepts.

Keywords: Abstraction, Analogy, Graph Theory, Structural Thinking

The ability to think *structurally* in undergraduate mathematics is an invaluable skill. Spurred by David Hilbert's insistence on conforming to a structural approach to studying mathematics in the early 1900s, the evolution of mathematical thinking to conform to the flexible recognition of underlying structure of objects has been a purposeful endeavor in modern research (see Hausberger, 2018). The shift toward structural thinking also played a significant role in the development of the mathematical subjects we offer to undergraduate students today. Take, for example, the subjects of abstract algebra and point-set topology. Before the expansion of structural thinking in the early 1900s, such subjects were difficult to access for both students and professional mathematicians alike. Despite the modern focus on structure in mathematics, educational research has revealed that students struggle to articulate certain concepts often associated with structural thinking, such as quotient group (Dubinsky et al., 1994). However, we wondered about the development of students' *structural understanding* as a separate phenomenon as opposed to their content-specific understanding. In this paper, we present a preliminary investigation of what one student, Isaiah (a pseudonym), understood about structure after having taken courses in introductory abstract algebra and point-set topology.

Literature Review and Conceptual Framework

At the center of this research study are the notions of structural thinking, analogy, abstraction, and the deep connections between them. Mac Lane (1996) stated, "Structure is essentially a list of mathematical operations and relations and their required properties…, and often so formulated to be properties *shared by a number of possibly quite different specific mathematical objects*" (p. 174, emphasis added). In this rough definition, there is already implicit reference to a type of relational structure between mathematical objects that share certain properties.

To investigate structure more directly, Linchevski and Livneh (1999) introduced the notion of structure sense, which aimed to capture one's ability to recognize the same structure flexibly and creatively across various contexts, specifically in the context of high school algebra. Novotná and Hoch (2008) extended this concept to structure sense in abstract algebra, further elaborating on the structure inherent to the set and binary operation of algebraic structures such as groups. Recently, Hicks and Flanagan (2024a) further extended structure sense to more

targeted structures, including sub-structure sense, sense of structure preservation, and quotient structure sense (see also, Hicks & Flanagan, 2024b). Additionally, to provide a way to analyze students' reasoning across related structures (e.g., subgroup and subring), we introduced analogical structure sense, which aimed to "parse students' attention to structure during analogical reasoning" (Hicks & Flanagan, 2024a, p. 4).

Hicks, Flanagan, and Park (2023) investigated the relationship between analogy and abstraction, which culminated in two ways that students may abstract structure during the process of analogical reasoning (see also, English & Sharry, 1996). This suggested that a certain level of abstraction may be required when engaging with analogical structure in widely different domains, such as between groups and topological spaces. This preliminary study aims to further contribute to this growing body of literature on the connections between structural thinking, analogy, and abstraction, by investigating how one advanced undergraduate student reasoned about developing structures in graph theory. To do this, we proposed the following two research questions:

1. What were the affordances and obstacles that an undergraduate student elicited while conjecturing the definitions of structures (sub-structure, structure preservation, quotient structure) in a new context of graph theory?
2. In what ways did the student utilize his prior experience with analogous structures in group theory and topology when conjecturing the definitions of structures in graph theory?

Methods

Because our goal was to investigate students' structural thinking after having taken courses in higher mathematics, we solicited participation from all students in an introductory point-set topology course in which abstract algebra was a prerequisite. Of the eight students that were contacted, only Isaiah responded. The data for this study was collected from a one hour, task-based interview with Isaiah, conducted by both authors digitally over Zoom. The interview was intended to serve as a pilot for future data collection and comprised of four distinct parts: a brief introduction to the concept of a graph and a determination of what Isaiah already knew about graphs from his previous coursework, followed by requesting Isaiah to create each of the concepts of subgraph, graph homomorphism, and quotient graph. During the interview, we asked clarifying and probing questions when needed.

The interview was subsequently transcribed and analyzed by both authors. While Isaiah was creating his graph-theoretic structures, we found that he presented several affordances and obstacles to his structural thinking. Affordances for Isaiah's structural thinking included features of his thinking that helped drive him forward in creating structure or reasoning about underlying structure. By contrast, obstacles included features of Isaiah's thinking that led to impeded creation of structure or resulted in incomplete or inconclusive reasoning about underlying structure. We note here that we did not directly attend to Isaiah's graph-theoretic thinking, meaning that we were not looking for affordances and obstacles to his understanding of concepts in graph theory. Rather, we wished to capture the affordances and obstacles to his more general thinking about the underlying structure of the graph-theoretic concepts. In this way, it was possible to identify affordances for Isaiah's structural thinking that would perhaps be considered obstacles to his graph-theoretic thinking. Likewise, certain aspects that might be affordances to

his graph-theoretic thinking (i.e., recalling what he knew about graphs from his previous coursework), might have also resulted in obstacles to his structural thinking.

Results

The analysis revealed two primary affordances and two primary obstacles to his structural thinking about the three graph theoretic concepts. We provide individual evidence for each of these affordances and obstacles. As stated above, our analysis attended to structural thinking rather than his content-specific thinking about each of the relevant graph theoretic concepts.

Attention to a General Structural Property as an Affordance

Throughout the interview, Isaiah attended to a general, overarching property that described the structure of graphs. For example, when developing the concept of a subgraph, Isaiah stated, "So, I think there would have to be... vertices still being adjacent in the subgraph." This was the first indication that Isaiah identified adjacency as the overarching structural property for graphs. The act of establishing this property proved to be an affordance to Isaiah's structural thinking throughout the interview, specifically by helping him to make headway with establishing access for creating the subgraph, graph homomorphism, and quotient graph structures. For example, when working on the graph-theoretic homomorphism concept, Isaiah stated:

> So, when I hear of homomorphism or homeomorphism, it's the idea of structure preserving, or you know, isomorphism. So, in a graph, I guess this structure is what stuff is adjacent to, if that makes sense, I think. So, I'd have to think about how to define it so it actually makes sense.

Furthermore, present in Isaiah's structural thinking was also a fair amount of flexibility (see Lee & Heid, 2018) for what constituted the general structural property for graphs. Specifically, he was open to revising his understanding of the general structural property as he continued his constructions of graph-theoretic concepts. While he initially focused on adjacency as the primary property, he slightly modified this property as he explored the graph homomorphism concept:

> I'm trying to think, because I probably want to just look at stuff, what it's adjacent to, and then have to work through the graph in order to define the homomorphism. So, I would have to send one vertex to a vertex in another one [graph] such that it has the same number of neighbors... Yeah, I guess that's, that makes sense. I think, if I say, a graph is homomorphic if there a function, say f , going from the vertices of one to the vertices of another one such that, and then I'll say, if, I've seen this notation before, so if the [number of] neighbors of V is equal to the [number of] neighborhoods of f of V , so $N\,V$ [$N(V)$].

Thus, Isaiah's structural thinking was afforded by first establishing a central structural property for defining graphs which subsequently undergirded his graph-theoretic constructions. We note here that although Isaiah was technically shifting away from the normative graph-theoretic property of graph homomorphism itself, his structural thinking was afforded by his attention to the general structural property, as well as his openness to revising the criteria as he progressed.

Analogical Reasoning as an Affordance

During his reasoning about graph-theoretic concepts, Isaiah's structural thinking was afforded through analogical reasoning with group theory and topology. While constructing the subgraph concept, Isaiah noted the following:

> But I guess because I mean in both of them, it's all the elements are coming from whatever, a group, a subgroup, all the elements are also elements of the group. In the

> topological space, at least the set, all the elements are coming from the set... I think this [subgraph] is more similar to a subgroup type of definition where that makes sense.

Here, Isaiah is recognizing that in the same way that the subgroup and subspace concepts are defined by a subset relation, the subgraph concept also possesses the subset relation. In this way, Isaiah had constructed the subgraph concept by first identifying that there should be some notion of a subset involved. However, Isaiah also attended to specific differences, further emphasizing the role of analogical reasoning in his structural thinking. In particular, he made an observation regarding how the subgraph was related to the parent graph structure. He stated the following:

> So, in a subgroup, the binary relation, you're still going to have that I guess. And then a subgraph, you don't need to include all the edges, I think, so it's a little bit different because you can just remove one edge from the graph and that still should be a subgraph. But, in a group, you can't remove a binary [relation], you can't remove one of the relations, I guess, and still have it be; it wouldn't be a group then. So, I guess there's a little bit more freedom in a subgraph than in a subgroup or a subspace.

Isaiah appeared to make an analogical connection between the edges between vertices in a graph and the binary operations between elements in a group. By attending to this, he observed that there was "more freedom" when determining subgraphs as opposed to determining subgroups. The fact that Isaiah not only leveraged similarities between contexts, but also made fruitful observations about the differences between contexts, meant that Isaiah's analogical reasoning served as an affordance to his structural thinking.

Constrained Example Space as an Obstacle

Throughout the interview, Isaiah appeared to rely on a constrained collection of examples for graphs. In particular, whenever there was more than one graph present in his reasoning, he almost always chose the second graph to be a subgraph of the first graph. This proved to be an obstacle for his reasoning about the structure of graphs, and specifically the meaning of structure preservation. While testing his definition of graph homomorphism, which he conjectured to require preservation of the number of neighbors, he stated:

> Let me try and see if this definition even holds up with what I'm thinking exactly...Maybe I'll try and do a subgraph of this. I think that may be a better approach...Actually, I think I should change this definition a little bit to be bijective too.

Since he only looked at examples with subgraphs, his current working definition of graph homomorphism would always be bijective. In particular, by attending to only the specific examples including subgraphs, Isaiah was no longer thinking about the underlying structure of graphs, but rather only considered a special case where bijectivity was always present. Even though this was correct reasoning from Isaiah in the context of this example, he did not consider how this addition of structure impacted his definition beyond those cases where subgraphs were present. Instead, he appeared to forcibly add an extra condition due to the result of a single constrained example. Thus, this presented an obstacle to Isaiah's structural reasoning because rather than flexibly considering the validity of his original definition, he chose to add conditions to his definition to satisfy his limited set of examples.

Prior Mathematical Familiarity as an Obstacle

Multiple times during the interview, Isaiah would reference mathematical concepts he had encountered before. Although prior familiarity with concepts can act as an affordance for structural thinking, such as was the case with Isaiah's analogical reasoning about previously known concepts in the domain of group theory and topology, we found that an obstacle also

arose from his desire to inject familiar concepts into unfamiliar situations. A primary example of this included when Isaiah was working through his definition of graph homomorphism. He recognized the importance of preserving the structure of adjacency but kept returning to the number of neighbors because he had "seen this notation before." For instance, he stated:

> Say you have two edges that are both adjacent in the one graph and you apply the function, I think they should end up being adjacent in the other graph as well. I think I could probably prove that by. Yeah, no. So, I don't know. I think the structure and graph is probably pretty, very related to I guess the number of neighbors that stuff has.

Similarly, when working through the concept of quotient graphs, Isaiah expressed how he had seen the "power set of graphs" before, which primarily guided his construction. Unlike Isaiah's reasoning by analogy, which involved calling forth a more tightly connected collection of structural knowledge about group theoretic and topological structures, we found that Isaiah's reasoning in the above examples relied on singular concepts disconnected from any structural foundation. Thus, Isaiah's structural thinking was limited by the concepts he used to conjecture the definitions that were called from memory with no appeal to their structural purpose, causing his prior mathematical familiarity to act as an obstacle to this kind of thinking.

Discussion

The results and analysis from this preliminary study shed further light on the importance and value of structural thinking, especially at the undergraduate level. Beyond this, the results provide evidence that content-specific reasoning and structural reasoning do not always coincide. Particularly, an affordance to one's structural thinking may be an obstacle to their content-specific thinking, and conversely. Isaiah used meaningful, constructive structural reasoning without the correct graph-theoretic definitions. We conjecture that an affordance to both one's structural and content-specific thinking would help generate effective, meaningful learning. However, a more detailed study would be required to address this. Moreover, this study further supported the value of students leveraging analogical reasoning when engaging with new-to-them mathematical concepts (Hicks & Flanagan, 2024a; Lee & Sriraman, 2011). Also, if students can meaningfully construct an abstracted structure for certain concepts, such as sub-structure, they can use that structure to reason structurally in a new context (Hicks et al., 2023).

The results of this study also provided support that learning and understanding concepts in abstract algebra and point-set topology are difficult for undergraduate students (e.g., Larsen, 2009; Melhuish et al., 2020; Serbin, 2023), especially quotient concepts, like quotient groups (e.g., Hicks & Flanagan, 2024b). However, in contrast, these results also revealed that students can generate productive structural reasoning, despite having challenges with the content-specific definitions. In addition, the results highlighted the importance of a flexible example space for structural reasoning, especially in the context of conjecturing new definitions (e.g., Lynch & Lockwood, 2019). When reasoning through concepts with example-related activity, it is imperative that there exists coherence between an individual's structural reasoning within the examples and the general structural property for the concept itself. Finally, Hazzan (1999) claimed that undergraduate students tend to reason with "familiar objects," which can sometimes be an unproductive strategy for them (pp. 76-77). The finding from this study that prior mathematical familiarity can be an obstacle to structural thinking supports this claim.

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Using Math Literacy Strategies in Calculus: Amplifying Undergraduates' Understanding of Partial Derivatives

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Mathematical literacy is a critical component of teaching mathematics as its own language. In this paper, we share the use of a writing to learn mathematics (WTLM) activity to help college students communicate their mathematical thinking and understanding. We conducted an embedded single-case study with two sections of undergraduate students enrolled in a Calculus II course focused on integral and multivariable calculus topics. Three WTLM activities were integrated into the course to provide opportunities for students to explain their symbolic, procedural, and conceptual understandings of partial derivatives and related topics. In this paper we share the framework and tools developed for this study and the data analysis plan and preliminary analysis steps.

Keywords: partial derivatives, concept definition, concept image, math literacy

Traditional teaching practices often limit how students are asked to communicate their mathematical understanding. When considering mathematics as a language, and teaching it as such, we can use practices like those used in language learning to develop mathematical communication skills by paying attention to notations as our alphabet, concepts as our ideas, and mathematical statements as sentences (Alfaro Viquez and Joutsenlahti, 2020; Bruun et al., 2015; Roepke and Gallagher, 2015). Considering mathematics as a language allows instructors to focus on developing students' ability to communicate mathematically by "unpacking" and "repacking" mathematical concepts, procedures, and representations (Viquez & Joutsenlahti, 2020). This ability is one of the components of mathematical literacy, which is "an individual's capacity to formulate, employ, and interpret mathematics in various contexts[including] reasoning mathematically and using mathematical concepts, procedures, facts, and tools to describe, explain, and predict phenomena" (OECD, 2018, p. 67).

In this study, we examine the use of writing to learn mathematics (WLTM) tools to provide opportunities for students to share their concepts beyond traditional representations and procedures and build their mathematical literacy. We focus specifically on the concept of partial derivatives as introduced and used in a combination Calculus II and III course. This topic is important to address in the mathematics classroom because of how applicable partial derivatives are in both math and other STEM disciplines (Becker and Towns, 2012; Bucy et al., 2007; Emigh and Manogue, 2017). We addressed the research questions: (1) What are students' concept definitions and concept images when initially learning about partial derivatives in calculus? and (2) How do topics that use partial derivatives add to students' conceptual understandings? This work is part of Cuadra's larger study using three distinct WTLM activities as data collection sources. In this paper, we share the framework and the data collection tools that we have designed for this study as well as the analysis plan.

Framework

One way of strengthening students' understanding of mathematical phenomena is by associating a concept definition or properties with a strong concept image. A *concept definition* is "a verbal definition that accurately explains the concept in a non-circular way" (Vinner, 1983,

p. 293). A concept definition is similar to the usual notion of a definition in mathematics, except the concept definition includes attributes and properties that the writer of that definition deems important. A *concept image* "a set of properties together with a mental picture" and generally refers to a visual representation or depiction of a particular concept (Vinner, 1983, p. 293). This image may be an illustration, diagram, graphic, or any visual medium that conveys and communicates the essence of a specific idea or abstract notion. Focusing on concept definitions and concept images, we can restructure the teaching of mathematics to become conceptual as well as procedural. As Tall (1992) states ``the individual's method of thinking about mathematical concepts depends on more than just the form of words used in a definition" (p. 496). Hence, students' experiences factor in when they are developing ideas about a mathematical concept. By attending to concept definitions and concept images, students can develop a deeper and more meaningful connection with the material.

Zandieh (2000) created a framework to study students' concept definitions and concept images of derivatives in a first-year calculus course. Her framework combined different representations of derivatives (e.g., graphical, verbal, symbolic, and physical) and process-object layers of the derivative. The latter come from Sfard's (1992) idea of processes as operations on previously established objects. Zandieh included process-object layers of the concept of derivatives (e.g., ratio, limit, and function) that can be viewed both as dynamic operational processes and as static structures objects.

Emigh and Manogue (2017) used Zandieh's framework to analyze students' interpretations of partial derivatives in undergraduate math and physics courses through interviews. Two common interpretations from the students were that partial derivatives relate to slope and change. Roundy et al. (2015) also modified both the content and the structure of Zandieh's framework to study students' concept images of partial derivatives and included content-specific items like experimentally measured quantities. Their framework was developed using what they learned about experts' concept images of partial derivatives.

Following the work done by Roundy et al. (2015), we adapted Zandieh's original framework by using the representations and process-object layers related to partial derivatives as students first learn about them in calculus. To determine the different process-object layers, it is helpful to think of the underlying mathematical concepts that support an understanding of partial derivatives. As partial derivatives are an extension of single-variable derivatives, the original process-object layers for single-variable derivatives found in Zandieh's (2000) framework are still applicable for partial derivatives, namely (a) ratio, (b) limit, and (c) function. In addition, we consider (d) single-variable derivatives as an important added process-object layers of partial derivatives. One key modification is the incorporation of covariation reasoning into the existing framework. as this is a critical underlying concept that supports students' understanding of partial derivatives. Mkhatshwa (2021) used a covariational reasoning framework when studying students' procedural understanding of quantities represented by partial derivatives and found that students tended to shift from interpreting them as rates to interpreting them as amounts. The framework in Figure was used to guide the research design and analysis for this study.

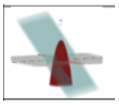
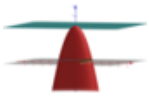
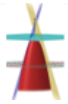
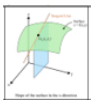
Process – object layer	Graphical	Verbal	Symbolic / Numerical	Physical
	Slope	***Rate of Change***	***Independence of Variables / Ratios of Change***	***Optimization***
Ratio		**Average rate of change**	**$\Delta f/\Delta x$ numerically**	
Limit		**Instantaneous rate of change**	**$\Delta f/\Delta x$ with Δx small**	
Function		**Rate of change at any point/time**	**$\partial f/\partial x$ depending on x**	**Extrema analytically**
Derivative		**Rate of change of f as x changes**	**Treat y as a constant when applying the differentiation rules**	**Extrema graphically**
Covariation	**Direction**	**Coordination**	**Quantitative Coordination**	
Non-layer	**"slope"**	**"rate of change"**	**"independence of variables"**	**"optimization"**

Figure 1. Framework guiding the research design for this study

Methods

Participants for this study included 43 undergraduate students enrolled in an applied calculus course that combined topics from both Calculus II and Calculus III curricula. Cuadra served as the teacher-researcher for two sections of this course.

This study is designed as an embedded single-case study to examine students' understanding of partial derivatives (Yin, 2003). The single case of focus is calculus students' understanding of the concept of partial derivatives. This study is bounded by the topics covered within the Applied Calculus II course and the students participating in the course. Since mathematics is a cumulative subject, it is important to look at the building blocks that encompass partial derivatives, such as single-variable derivatives and functions of several variables, as embedded cases that provide richer information on students' understanding of partial derivatives. In addition, topics that are dependent on deep understanding of partial derivatives including higher-order partial derivatives, extrema(s) of two variable functions, Lagrange multipliers, and double integrals also serve as avenues of inquiry as these applications of the concept can influence understanding. Students used the data collection tools below for each of these building blocks and application topics as well as for the specific concept of partial derivatives. We have a complete set of data from 43 students collected over the course of a semester.

Data Collection

Data was collected using three WTLM math literacy activities intended to help students communicate their mathematical thinking in math courses (Porter & Masingila, 2000). WTLM gives students an opportunity to reflect on their own thinking about the mathematical ideas covered in the class and to communicate their understandings. For this study, Cuadra developed three tools: (a) "The Important" Prompt, (b) an Aspects of Mathematical Phenomenon chart, and (c) learning logs. Roepke and Gallagher (2015) adopted the "The Important" prompt from

Margaret Wise Brown's *The Important Book* (1949) for usage in mathematics classes. In Brown's book, she takes everyday objects and describes what makes each object special. Roepke and Gallagher (2015) adopted this idea in the following activity. First, the students complete the sentence "The important thing about (insert topic) is..." by writing at least five things they found important about that mathematical topic and then write what they found "most important" about the topic. This activity allows students to think beyond the typical "compute" exercises and analyze an idea beyond computation.

The second WTLM tool is one we are calling the *Aspects of Mathematical Phenomenon (AMP)* Chart. It is based on the vocabulary development tool known as the Frayer model (Frayer et al., 1969) which had four quadrants surrounding a featured vocabulary word written in the middle. In mathematics classrooms, scholars have adapted the Frayer model quadrant labels to include things like definitions (in your own words), facts, pictures, examples, and nonexamples (Kenney et al., 2005; Roepke & Gallagher, 2015). Since students are not accustomed to being asked to share a concept image, we have extended this kind of model by creating the AMP Chart to allow students to unpack the aspects of a mathematics concept across six cells and captures a snapshot of their concept definition and concept image. Figure 2 shows an example of a completed AMP chart.

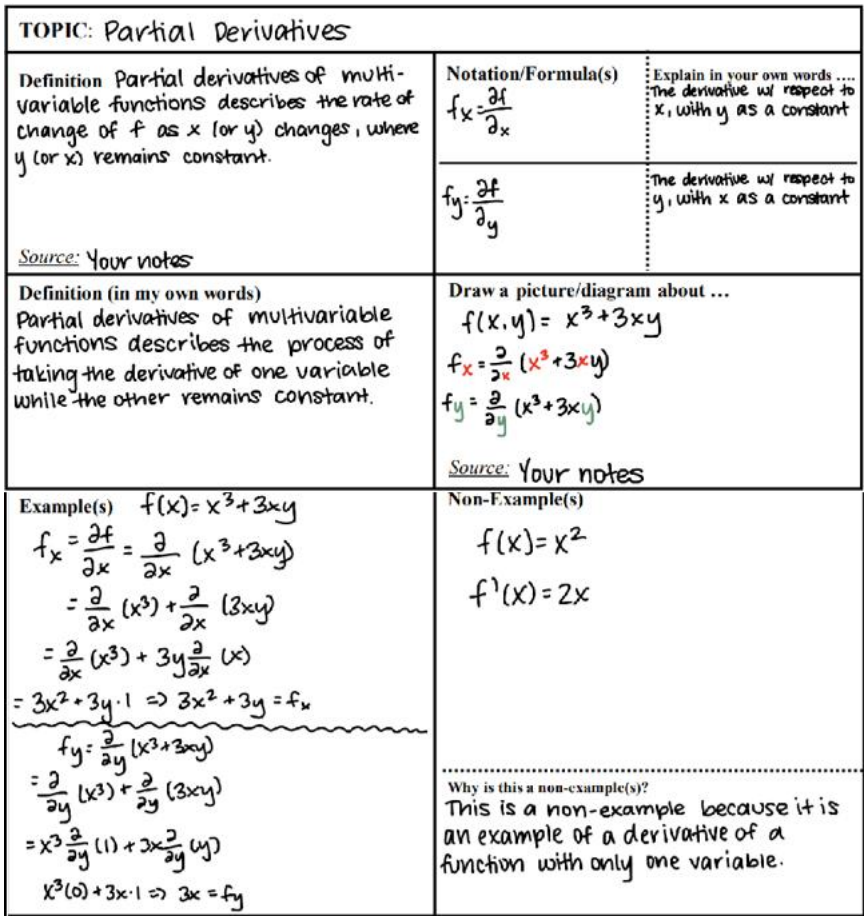


Figure 2. Example of a completed AMP chart by a Calculus II student.

The final WLTM tool used were learning logs, or journals that serve as a collection of student-generated ideas that examine the thinking processes and conceptual understanding of the

students (Carter et al., 1993, p. 87). A key reason to use learning logs that they create connections between language and mathematical literacy (Carter et al., 1993). Learning logs here are reflection prompts that students write to share how their understanding of partial derivative changes after they engage with different representations and applications of partial derivatives.

Data Analysis Plan

As this is a preliminary report, data analysis of the significant amount of data collected is ongoing. We intend to analyze the WTLM activities that students did for partial derivatives and then for topics dependent on partial derivatives (i.e., Lagrange multipliers, higher-order partials, extrema, and double integrals).

We are using a *concept mapping* approach for analysis (Butler-Kisber & Poldma, 2010; Daley, 2004; Moreno, Guthrie, & Strickland, 2023). For the AMP charts, for example, we cut up the chart into six individual cells and are making a concept map for each cell, looking for connections and distinctions across the students' responses. This approach to analysis allows us to consolidate the data into manageable pieces (Daley, 2004) and map our interpretations of the data as these emerge (Butler-Kisber & Poldma, 2010). We deconstruct the students' responses to see what connections the whole group (as our single case) is making to the mathematics topics. Looking across the resulting concept maps for each of the cells of the AMP charts then helps us build a picture of the concept images developing surrounding partial derivatives. A similar process will use the Important Prompt to build a representation of the concept definition of partial derivatives that emerges when students first learn about it. Finally, the learning logs allow us to look beyond a single snapshot of this concept definition and image and see how they are evolving as new topics related to partial derivatives are presented in the course. Figure 3 shows an example of one initial concept maps that we have generated after examining the "Definition" cell of the AMP Chart. Using the Framework described above in Figure 1 allows us to see which process-object layers the students are attending to in these cells.

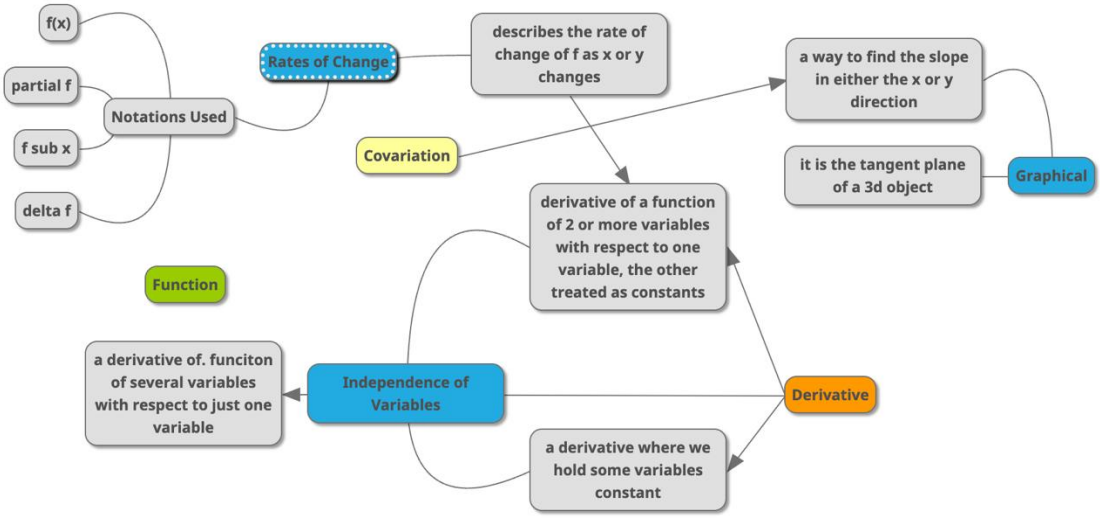


Figure 3. Concept map of students' definitions

Though data has been collected, we are currently just beginning analysis of the data and are eager to discuss ideas about this interesting work with the RUME community.

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Here's What "I" Do: Investigating the Actions Students Take in Proof Comprehension Tasks

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Literature around the reading of proof has focused on experts and the strategies they use to understand the texts they are reading (Fang & Chapman, 2020; Paul, 2018) or in comparing students to experts in what the reader focuses on as they read using eye-tracking technology (e.g., Inglis & Alcock, 2012). Few have investigated the students and their strategies (Weber, 2015). In this preliminary report, we describe the most frequently seen actions taken by participants during various proof comprehension tasks. Our findings indicate that students use a variety of possible strategies to make sense of proofs they are reading. Some of these possible strategies are found to be used by mathematicians and secondary mathematics teachers, while other possible strategies are found to be used in general reading practices.

Keywords: Proof Comprehension, Reading Proof, Student Perspective

Investigations into what students do as they attempt to understand proof can provide insight into how students think about proofs, such as what it means to understand a proof, as well as identifying what strategies students commonly rely on when they arrive at a confusing part of a proof. Both math education and literacy scholars have investigated various ways in which mathematicians read proof (e.g., Fang & Chapman, 2020; Inglis & Alcock, 2012; Paul, 2018; Weber & Mejia-Ramos, 2011), though few have investigated how undergraduate students read proof (Inglis & Alcock, 2012). In proof comprehension, scholars focus on assessing or improving students' comprehension of proof (e.g., Hodds et al., 2014; Mejia-Ramos et al., 2017; Roy et al., 2017), with less attention on what the students do as they read proof for understanding (Dawkins & Zazkis, 2021; Weber, 2015). Considering the student and the actions they take can provide important information in the development of interventions. These interventions can be aimed at further developing these skills by building on student starting points. In this preliminary report, we set out to answer the following research question: What actions do students take to help with their understanding of a given proof during proof comprehension tasks?

Literature Review

Scholars have investigated various ways in which mathematicians and students read mathematical text. These could be through comparing what experts and novices attend to while they read proof using eye tracking for proof validation tasks (Inglis & Alcock, 2012), how mathematicians determine the validity of a proof (Weber, 2008), why mathematicians read proofs (Mejia-Ramos & Weber, 2014; Weber & Mejia-Ramos, 2011), or in what mathematicians do as they read a proof about unfamiliar material (Wilkerson-Jerde & Wilensky, 2011).

Literacy experts have also investigated the reading habits of mathematicians and secondary education teachers. Fang and Chapman (2020) discuss a case study of the reading strategies employed by one mathematician as they read a journal article. Paul (2018) describes the different reading strategies used by various secondary education content teachers, one of which is a mathematics teacher. They both report various strategies such as annotating, evaluating and verifying, corroboration (such as using prior knowledge or other texts), and strategic reading (such as rereading, sectioned reading, or skimming).

The studies described above focus on comparing mathematicians to students or focus purely on experts in how they read and the strategies they may employ as reading. Little has been conducted to see what students do as they read mathematical text. One exception is Weber (2015) who investigated what strategies students use as they read proofs. Weber (2015) and Samkoff and Weber (2015) report seven strategies used by the students or recommended to be used by mathematicians. Mathematicians recommended readers should make sure they understand the definitions of all terms (Samkoff & Weber, 2015), while students were reported to see if the proof can be broken into independent parts, identify what proof methods are being used, and see why confusing statements are true with a particular example (Weber, 2015).

Theoretical Framing

Afflerbach et al. (2008) defines reading strategies as "deliberate, goal-directed attempts to control and modify the reader's efforts to decode text, understand words, and construct meanings of text." (pg. 368) Within the realm of reading proofs, we define a strategy as deliberate acts from the reader with the goal of understanding some subset of a given proof. Thus, we define a proof comprehension strategy as any action of the reader that is done outside of the proof's text with the goal to understand some portion of a given proof (inclusive of the whole proof). By outside of the proof's text, we mean the student is engaging in furthering their understanding through actions that are not done or recommended within the proof. In this preliminary report, we focus on the actions the students take that may make up a proof comprehension strategy.

Methods

The study described here is part of a larger dissertation study investigating the strategies students use to understand proof. In this section, we describe the methods pertinent to the results. Eight participants were recruited for this study, each of which had completed an introduction to proof (ITP) course the semester prior to data collection. Table 1 shows participant information using participant chosen pseudonyms.

Table 1: Participant Information

Participant	Race/Ethnicity	Gender	Grade Received in ITP
Adrian	White or Caucasian	Non-binary	B
Claire	White or Caucasian	Female	B
Erik	White or Caucasian	Transman	A
KTRM	Latin/Hispanic	Female	C
Marie	Latin/Hispanic	Female	A
Olivia	Mixed Race	Female	B
Tifanni	Latin/Hispanic	Female	A
Zeus	Latin/Hispanic	Male	B

Each participant was met with for seven semi-structured interviews. Four of those interviews involved a proof-comprehension task where participants were asked to read a provided complete and correct proof and to think-aloud as they tried to understand the proof to the best of their ability. Table 2 shows the four theorems for context of the study (proofs have been excluded due to space). After finishing the task, participants were asked various questions on their beliefs of their understanding of the provided proof, the difficult parts of the proof, and about the actions taken as seen by the interviewer.

Interviews were video and audio recorded. Video recordings were open coded (Strauss & Corbin, 1998). Instances were first coded by the first author based on ostensive cues (verbal,

written, or gestural indicators). Through the initial open coding, code names and descriptions were given to each instance. Once all instances were open coded, a second round of coding was done by the first author to refine code names (actions), and subcodes were also given during this time to further differentiate the actions taking place.

Table 2: Theorem Statements for Proof Comprehension Tasks

Theorem	Theorem Statement
1	If $a, b \in \mathbb{N}$, then there exists integers k and l for which $gcd(a, b) = ak + bl$.
2	Suppose R is an equivalence relation on the set A. Then the set $\{[a] \mid a \in A\}$ of equivalence classes of R forms a partition of A.
3	Let $f: A \rightarrow B$ be a function. Then the inverse relation f^{-1} is a function from B to A if and only if f is bijective. Furthermore, if f is bijective, then f^{-1} is also bijective.
4	The sets $\mathcal{P}(\mathbb{N})$ and $\mathbb{R}$ have the same cardinality.

Results

Through open coding of the instances, 34 different actions were found amongst the 8 participants across all four proofs. Due to limited space, we focus our results to the most frequently used actions. To determine this, we set a lower bound of 5% of the total final actions coded (1096) and required the actions to be done by at least five out of the eight participants. Seven actions were found to meet these requirements: Asks for Meaning, Creating Warrant, Identifying Proof Structure, Making Connection, Marking Proof, Rereading, and Writing Information. Table 3 shows the frequency of each action per participant, with each student's highest frequency bolded.

Table 3: Action Frequency per Participant

Student	Coded Action						
	Asks for Meaning	Creating Warrant	Identifying Proof Structure	Making Connection	Marking Proof	Rereading	Writing Information
Adrian	11	**39**	9	13	0	14	9
Claire	6	**36**	5	13	1	11	25
Erik	5	**20**	18	6	4	5	19
KTRM	4	**15**	0	4	2	6	7
Marie	5	**22**	14	15	9	14	17
Olivia	11	15	3	6	11	16	**25**
Tifanni	7	11	8	8	10	**32**	8
Zeus	10	12	16	4	**41**	9	7
Total	59	170	73	69	78	107	116

The creating warrant action was the most frequent action taken by all but three participants (Olivia, Tifanni, and Zeus). In these instances, participants focused on justifying an explicit or implicit claim made in the proof. The justifications presented by the participants were coded even if the justification was incorrect or incomplete (such as deciding to move on from the claim without finishing their justification). The data used to warrant a claim varied amongst instances, such as filling in skipped algebraic steps, using a previously self-made conjecture, using the definition of a concept, using a self-made diagram, using a self-made example, using their own reasoning, using a previous part of the proof, or using their own previously made written work.

The most frequent action for Olivia was writing information, which was among the top seven actions for all remaining participants (except Adrian). This action consisted of a participant

writing information either on a separate sheet of paper or on the sheet of paper containing the proof. This action was coded specifically when a participant wrote information that was already presented to them (such as a definition that was given to them by the interviewer on a separate slip of paper, the defining of a set or function in the proof, a claim made within the proof, or the rewriting of an example they created) or when the participant wrote a conjecture of their own. Further variety was seen through the reasoning behind the writing of the information. Participants commonly stated they wrote information for two reasons, either to keep track of important information for ease of referencing or to help them understand information presented to them. Figure 1 shows the notes Olivia took during their time reading Theorem 4's proof. In these notes, Olivia wrote the function f that is used, initially doing this to keep track of the function and have an easier time referencing it. She later used this written work to make sense of the function in what values n and a_n could be.

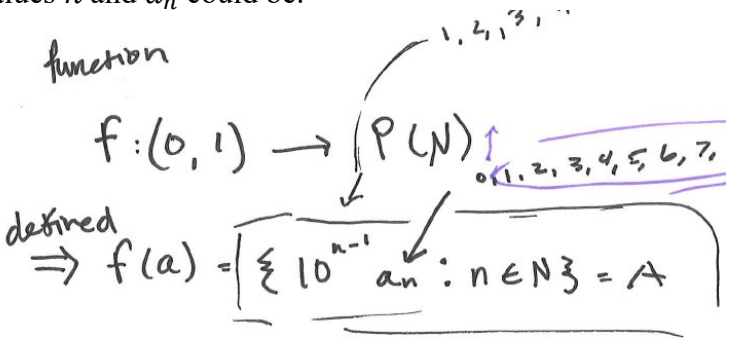


Figure 1: Some of Olivia's Notes for the Proof of Theorem 4

The most frequent action for Tifanni was rereading, which was among the top seven actions for all participants. These instances varied in what was being reread, such as rereading some part of the proof or a provided definition. If rereading some part of the proof, participants could have reread the whole proof (including or excluding the theorem statement) or a line, paragraph, or theorem of a proof. Participants would at times also have specific styles of rereading. Two such styles were rereading but stating the highlights, as if creating a summary of the proof (categorized as a subcode), or purposefully rephrasing (subcode) a line of the proof for ease of reading. For example, as Tifanni's read the proof to Theorem 3, she reads the line "Let $b \in B$" as follows: "Let bee exist in bee, oy [sic], let little b exist in big b." Here Tifanni is making a distinction in the letters noting "Which b am I talking about? Like in my head. Because of course I'm seeing them, but since I'm reading it, I'm like, so I'm talking about *this b*, not *this other b*" (emphasis added). Participants frequently reread to better understand what they just read, either immediately or after having established more information from later text or a definition.

Marking the proof was the most frequent action for Zeus and within the top seven for Marie, Tifanni, and Olivia. This action commonly consisted of participants underlining, dotting, or boxing parts of the proof. This was typically done to draw attention to key claims made in the proof – such as in Theorem 3's proof with the final claim of an argument being that f is onto. Participants would also use these markings to highlight parts of the proof they were still confused about. Erik, Marie, Tifanni, and Zeus had identifying proof structure in their top seven most used actions. Identifying proof structure actions varied between what the participants were identifying. These ranged from identifying independent parts of the proof, what the proof needed to show to prove the theorem, what proof method was being used, or what purpose a part of the proof was playing towards proving the theorem.

Making connections was in the top seven most frequently taken actions by all participants except for Olivia and Zeus. When making connections, participants would identify a relationship (typically through similarities and differences) – or lack thereof – between various texts or concepts. Participants commonly discuss a noticing between the proof and some other text – such as an example the participant created, another part of the proof, the theorem statement, or a definition of a concept. For example, Theorem 2's proof uses a contrapositive argument to show that equivalence classes are disjoint. Participants frequently discussed the differences between part of the definition of partition (that if two sets are not equal they are disjoint) and the start of one of the paragraphs of the proof (assuming two equivalence classes are not disjoint).

Lastly, asking for meanings was in the seven most frequently taken actions by all participants except Claire, Erik, and Marie. Participants would ask the interviewer for the meaning of a definition, notation, or theorem referenced in the proofs. Participants commonly discussed needing the definition to remind themselves of a concept or to help make connections between the proof and the definition. For example, participants frequently stated they needed to use the definition of partition to help see the structure of Theorem 2's proof, as they commonly could not recall (or were unfamiliar with) the definition of partition.

Discussion

Participants took a variety of actions as they attempted to understand the four proofs they were shown. Similar to Weber (2015), participants created warrants for claims, identified different proof structures, or attended to the meaning of terms used within the proof. Though, in our study we see more distinctions in the warranting of claims and expand on these distinctions. First, our participants warranted claims in a variety of ways. Either through examples, diagrams, definitions, their own reasoning, or using prior parts of the proof. This shows students are using more than just examples to make sense of claims they do not see immediately. Additionally, participants not only attended to the meaning of terms within the theorem statement (as described by Samkoff & Weber, 2015), but this extended into the proof itself. Participants frequently ask for definitions they were able to (and did) recall, as a way to ensure their thinking was correct. We also identified actions novel to Weber (2015), such as rereading, marking proof, writing information, and making connections.

Though novel to Weber's study on undergraduates, these actions can be seen being used by mathematicians and secondary mathematics teachers in literacy studies. Fang and Chapman (2020) describe their mathematician participant corroborating (making connections), rereading, as well as evaluating (creating warrants). Paul (2018) describes secondary mathematics teachers rereading, annotating (writing information and marking proof), and corroborating (making connections). Our study shows that students are doing actions similar to those found in the literature by taking general reading strategies and using them while reading proof. That is, novice provers bring a rich array of strategies for comprehending proofs. By identifying the strategies students bring into the proof setting, we are taking the first step in developing a theory of what strategies students use, what contexts lead to the use of certain strategies, and when and how particular strategies serve to be productive for students to generate insights. This can lead to targeted instruction and support for students to develop their emerging proof comprehension strategies. Future research from the larger dissertation study will be to describe these strategies in what kinds of strategies students use, how they are used, and how they may differ between uses. Limitations to this study are that participants saw a finite set of proofs and the actions they took may have been due to the types of proofs or in their familiarity with the content as they had just completed an introduction to proof course.

Acknowledgements

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Instructors' Vulnerability, Curiosity, and (Dis)trust of Data Informing Critical Transformations in Mathematics

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This preliminary report examines the reactions of mathematics instructors in Networked Improvement Communities as they engaged with student experience data to inform critical transformations within mathematics. Data-driven decision-making is promoted for improving equity; however, without a specific critical focus such efforts will likely fail. Examining data critically requires instructors to grapple with, at times, painful experiences of students, and their role in shaping these experiences. Using critical ethnography, we analyzed field notes to identify ways that instructors responded to data in meetings held from Spring 2023 to Summer 2024. Findings show that instructors' reactions to data often fall into two categories. Ego threat, characterized by reactions such as discomfort and defensiveness, effectively delegitimized student data. In contrast, some instructors exhibited curiosity and vulnerability, viewing students as experts of their experiences. Ongoing research will explore how ego threat shapes critical efforts and how fostering openness can enhance critical data-driven decision making.

Keywords: data, decision making, equity, critical transformations, mathematics instructors

Although the term "data-driven" is prevalent in research, data do not drive themselves (Bensimon et al., 2007); data are collected and analyzed by people with their own experiences, perspectives, agendas, and biases. In the context of education, instructors and institutions hoping to engage in data-driven decision making (DDDM) may inadvertently reinforce ideologies that perpetuate existing inequities in education (Datnow, 2017; Dodman et al., 2021; Taylor Jr., 2020). As such, Dodman et al. (2021) argue that a critical approach to DDDM is necessary in equity-focused transformation efforts. However, because a critical approach requires challenging dominant practices and interpretations, educators may struggle to enact a critical approach to DDDM. The purpose of this study is to examine the data use practices of mathematics instructors as they worked within Networked Improvement Communities (NICs) that specifically encouraged DDDM to identify critical transformations needed within their mathematics programs (Hagman et al., 2024). NICs (Martin et al., 2020) are networked communities of people using data to work toward a common improvement goal. For this preliminary report, we asked: How did mathematics instructors respond to data about student experiences in the context of making critical transformations in mathematics?

[1] Authorship is based on four levels: Bennett and Funk contributed equally as first authors (Level 1); Fantin-Hardesty and Tremaine contributed equally as Level 2 authors; Callahan, Quaisley, and Smith contributed equally as Level 3 authors; and Courtney, Hill, and Hagman contributed equally as Level 4 authors.

Theoretical Perspectives

Dodman et al. (2021) argue that a critical approach to data-driven decision making practices (CDDDM) is necessary to help achieve social justice and equity aims. Within their framework, a teacher's data use practices are viewed to be a function of their identities and beliefs about data, students, and their own agency or efficacy to address inequity in and outside the classroom. As such, CDDDM demands that educators reflect upon and challenge their identities and beliefs; this can be a vulnerable and uncomfortable process, similar to the process that educators must go through when learning to engage in equitable teaching practices (e.g., Tuitt et al., 2018). Indeed, engaging in critical transformation work demands vulnerability (Gilson, 2011). Drawing from work on whiteness and emotional politics, we view vulnerability as the capacity for "hearing, seeing and feeling (being affected by and affecting) experiences of oppression" (Wale, 2019, p. 1194). In the context of this study, we specify that vulnerability may come into two forms: (a) pedagogical vulnerability is the capacity to be open with a group of peers collectively engaged in critical transformation efforts informed by data and (b) emotional vulnerability is the capacity for being affected by and affecting the experiences told by students through data. Both forms of vulnerability necessarily require understanding one's own role (and at times complicity) in these experiences: people have to be in a "brave space" (brown, 2017) of admitting the status quo is not good enough to themselves and to each other--their teaching is not good enough, their department is not good enough--to engender more equitable mathematics teaching and learning.

Pedagogical vulnerability resides in the interactions among instructors (Mangione & Norton, 2023). Due to the vulnerable nature of critical work, effective NICs must foster a culture that supports vulnerability and encourages critical action. NICs do not automatically result in pedagogical vulnerability; such a state is achieved only when the members intentionally build a space of trust where they have a willingness and openness to explore their vulnerability with others (Kelly, 2013). Within the NIC literature, relational trust is a strong indicator of health of a NIC, and crucial to supporting exploration, experimentation, and change (Henrick et al., 2017).

Because engaging in CDDDM may bring discomfort, some people may experience "ego threat", a threat to one's self-image or self-esteem, (Gregory & Mebane, 2020) when they encounter data or situations that challenge their professional self-efficacy or teaching identities as "good" teachers. Common responses to ego threat are denial (devaluing the meaning of the data), avoidance (ignoring or dismissing data), defensiveness (making excuses for the discrepancy between role expectations and the data outcome), or attack (challenging the validity of the data or confronting the person sharing the conflicting opinion). Within a community of fellow mathematics instructors, discomfort may also bring about efforts to comfort one another or absolve each other's role in the negative experiences of students - we view this as similar to what Applebaum (2017) describes as the practice some teachers engage in of comforting discomfort students may face in a social justice classroom.

Methods

This study uses data from the ACT UP Math project (NSF DUE-2201486) involving university mathematics departments with NICs engaged in critical transformation efforts. Kappa University, a private doctoral-granting institution in the Southeastern United States, and Tau University, a public doctoral-granting institution in the Southern United States, are the two institutions of focus for this analysis. Our research utilizes critical ethnography (Palmer & Caldas, 2015) as its methodological approach. As part of a research-practice partnership (Coburn et al., 2021), we are deeply immersed in the data, acting both as ethnographic researchers and participants. Since Fall 2022, we have collaborated closely with the NICs to gain insights into

their institutional environments and objectives concerning critical transformations toward diversity, equity, and inclusion, while systematically documenting all interactions. While NIC membership evolved over two years, the Kappa and Tau NICs maintained 8-11 members each composed of mathematics instructors, administrators, and students.

To determine the critical transformations needed to make their introductory mathematics programs more equitable, each NIC explored institutional data and engaged in a critical, data-driven decision-making process. The NICs began collecting new institutional and classroom-level data in Spring 2023, and continued to discuss data collection and analysis as they met in subsequent semesters. For this analysis, we initially reviewed the structured field notes from all monthly NIC meetings between Spring 2023 and Summer 2024 to identify instances in which data about student experiences were explicitly discussed. We engaged in multiple rounds of open and theory-driven coding (Maxwell, 2005). Each of the field notes were initially open coded by two researchers; then, we independently aligned our open coding with the theory referenced above and met to discuss salient themes. The themes of instructor responses included: curious, vulnerable, reflective, questioning, devalue/dismiss, defensive, doubt/skepticism, attack, disappointment, (un)surprise/shock, and planning/action. We sorted the themes into larger categories and discussed the wide range of responses and the multiplicity of particular codes. We present these themes in our findings and later discuss multiple avenues for continued analysis. All names used in the following sections are pseudonyms.

Findings

We sorted our themes of NIC members' responses to data into two categories. Several of the reactions of NIC members to data could be classified as responses typical of ego threat (Gregory & Mebane, 2020) and other reactions of NIC members conveyed open dispositions (curiosity and vulnerability) toward the data. We elaborate on these two overarching themes below.

Related to Ego Threat and Comforting Discomfort

We first present two conversations among members of the Tau NIC and the Kappa NIC that illustrated doubt, dismissal, and devaluing of the data. The first exchange occurred during the Tau NIC's first summer meeting in 2024, as members viewed the survey data collected from their students in the spring semester. The survey included items about engagement during and outside of class, sense of belonging, expected final course grade, and demographic information.

> *Jordan (Instructor)*: If we look at the average [of student engagement data…], it seems to contradict what we see from final exam grades. Students say, "We are engaged. We are doing the work." I put some questions in the [review] that are the same as the final exam, and I was surprised to see that the majority of students failed those questions on the final. So this contradicts what I see. This [data] is aspirational, not real.
> *Olivia (Admin/Instructor)*: It's our data!
> *Jordan*: Students are not giving their true data.

In this conversation, Tau NIC members expressed distrust, either of the students providing the data or of the representation of the data. Viewing the quantitative data as "not…true" or "aspirational" justified their responses to dismiss the student feedback presented.

The second conversation occurred among Kappa NIC members in Fall 2023 as they examined institutional data about student experiences in mathematics courses gathered through a large-scale survey. Senna questioned how useful or legitimate the data were:

> I'm not saying it's invalid data, but I think we've tried– made a lot of efforts since then to try to fix things? Obviously, they're still not where we want them to be, but I wonder if […] student morale was even worse then.

The discussion continued as other instructors asked when the data were collected and suggested that they administer the original survey again, since "that was all pre-COVID" and "a long time ago." When presented with data that indicated low student morale, the Kappa NIC dismissed this finding by acknowledging the data was older and possibly not reflective of more recent reform efforts. Senna, an instructor, implied that student morale has improved in light of their efforts, while also admitting that their efforts are "still not where we want them to be."

In both the Kappa and Tau NICs, there were instances where data from student focus groups were dismissed due to assumptions about students. During one Kappa NIC meeting in Spring 2024, one member described students' perspectives from a student focus group as "ridiculous" and not valuable when those suggestions advocated for dispelling certain normative practices in mathematics education, such as administering exams or collecting homework. Some of the Kappa NIC members continually asserted that "students love to complain," and in turn, they dismissed some of the focus group students' experiences and ideas. Similarly, during a Tau NIC meeting, one NIC member (an instructor) shared that "students said that other students should take easy classes along with mine," other NIC members consoled him that "students always complain" that their class is too hard. We found these instances noteworthy as both NICs pushed back on the suggestions from students and sought to comfort each other when facing critiques.

Related to Curiosity and Vulnerability

In some instances, data exploration fostered curiosity about student experiences and department practices. For example, Tau members recorded their own teaching and shared their observations with the NIC. Olivia reflected on their receptivity to make data-informed changes: "I think it may be impacting the rest of the NIC to be able to see someone else [another instructor] being vulnerable and being curious and willing to try something different in practice." Olivia noted the influence of NIC members' attitudes about the data on one another. At Kappa, student focus group data revealed that some students struggled to take calculus with peers who had already taken the course before. This prompted curiosity from instructors about department practices (e.g. "How does the department decide what to recommend students take?").

At times, NIC members displayed vulnerability in responding to data, particularly as it connected to their own instruction. For example, during a Tau meeting, Garrett presented a qualitative analysis of his students' responses to a survey administered in Fall 2023. In response to some of the data, he shared "I got my feelings hurt." But Garrett also shared that his approach to data had shifted from the prior semester, in which he reacted defensively to focus group data: "the purpose of this [data analysis] is not to confirm I'm doing a great job. [I] wish I had that mindset with the focus group [data collected in Spring 2023]." These quotations from Garrett represent a shift in perspective about data - from expecting that data would "confirm" his beliefs, to using data as a tool for questioning teaching practices. At Kappa, an undergraduate student NIC member shared an anecdote about the way in which students bond over "hating on math." Skylar said that "I'm into bonding. I am happy to be the brunt of jokes for bonding." Her sincerity in welcoming such bonding demonstrates her willingness to validate students' perspectives and experiences, even when it leaves her as "the brunt of [their] jokes."

The following conversation further highlights how NIC members remained open to the data through receptiveness and a willingness to (re)consider their roles in conversation with the data:

Connor (Instructor): I don't see anything revelatory in this data. Waiting for you all to disagree with me.

Natalie (Instructor): I would love to. Partially using this as confirming what I expected. Not sure how this informs what we should do. Maybe we're seeing it wrong - maybe it's about us not seeing what we want in the data.

Connor: What did we do?

Natalie: This is like a control, baseline. What can we change to change these data?

Jordan (Instructor): I would expect the STEM students to outshine the non-STEM majors, but it's not that way.

Natalie: I'm going to go back to the questions - the data is not revelatory. I'm wondering about - maybe we should stop looking at the confirmation of what's working. I'm wondering if there was one thing I did, verbally or in class work, or in feedback on exams, how could I, if I intentionally went in to increase students' sense of belonging.

In this exchange, Tau NIC members exhibit curiosity about what the data are saying and even question their own assumptions. They think about how the data from students can inform their practice and seem receptive to making changes.

Questions for Discussion and Directions for Future Work

Our preliminary findings illustrate how mathematics instructors reacted to data with responses characteristic of ego threat (Gregory & Mebane, 2020) and curiosity and vulnerability. One future direction of this work is to use an ego threat framework as a way to address how threats to professional self-efficacy, self-esteem, and the identity of being an mathematics instructor manifest in the delegitimization of data within equity-oriented reform efforts. There are legitimate reasons for distrusting or being skeptical of both qualitative and quantitative data; however, we wonder to what extent responses that align with ego threat hinder the progress of the NICs to make data-informed decisions. In our previous work which focused on students as experts of their own experiences and as partners in institutional change efforts (Hagman et al., 2024), we noted how some instructors easily delegitimized data about students' experiences when students were viewed as dishonest, "complainers," naive, or, in other words, not as experts. We plan to continue this work by examining how anti-deficit perspectives influence the positioning of students as partners and how that impacts the progress of the NICs. Other directions for future exploration related to CDDDM include:

1. How can instructors be supported in being vulnerable to what students share about their experiences during data collection?
2. What data do instructors find compelling or convincing and for what purposes (e.g., to inform implementation)?
3. What are the implicit arguments, beliefs, etc. shared by instructors as they engage with and make use of the data?

Dodman et al. (2021) suggested that CDDDM is influenced by an instructor's identity and beliefs. We plan to analyze other sources of data (e.g., interviews, journal entries) to study the connection of beliefs and the arguments made by instructors as they work toward CDDDM. In this proposed preliminary session, we would like to engage in dialogue with fellow scholars to brainstorm ideas for further exploring these questions.

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An Analysis of the Conceptual and Authentic Opportunities in Calculus I Textbook Tasks

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Gatekeeper mathematics courses, including calculus, historically prevent marginalized students from pursing STEM majors by often focusing on procedural fluency over conceptual depth. Efforts to better prepare students for future career pathways include using tasks that emphasize authentic application contexts. In this study, we analyzed application tasks within the first half of the derivatives unit for two commonly used undergraduate Calculus I textbooks. Our analysis identified both the authenticity of the application context, and the conceptual understanding required to solve each task. By juxtaposing these task features, we found notable discrepancies between the degree to which tasks presented conceptual opportunities within authentic, real-world applications. These preliminary findings inform the need for further research investigating the relationship between conceptual understanding and authenticity in application tasks. This work can inform the design of novel tasks which appeal to both features, thus better preparing students to succeed in both calculus and future degree pathways.

Keywords: calculus, textbooks, authentic applications, conceptual understanding

Introduction

Entry-level collegiate mathematics courses, such as Calculus I, often function as gatekeeper courses because they are historically documented to prevent students marginalized by their racial and/or gender identities from pursuing STEM majors (Battey et al., 2022; McNeill et al., 2022; 2023; Leyva et al., 2021). Considering the importance of such courses for students' success in a variety of career pathways, there is a need to investigate approaches to make such courses mathematically rich, contextually relevant, and accessible. One approach to address this problem is to understand if tasks within these courses present opportunities for conceptual mathematical engagement within authentic applications (i.e., tasks with a real-life context where a concept from a mathematical course is being used in a way that is relevant to that context). While there have been studies about the opportunities for conceptual learning available to students in gatekeeper courses such as Calculus I via textbooks (Palm et al., 2006; Tallman et al., 2021), there has not been an analysis that identifies those opportunities as presented within authentic applications. Identifying if those opportunities exist, even just within textbooks, can impact whether undergraduate students engage with mathematics concepts as relevant within contexts outside the mathematics classroom. Furthermore, such an analysis can shape curriculum writing and assessment development to address the problem of gatekeeper courses. Therefore, this paper attends to this problem by asking the following research question: *In problems available within the derivative lessons of Calculus I textbooks, how do the tasks in textbooks compare in terms of conceptual understanding within tasks with real-life applications?*

Literature Review

Much of the existing research on gatekeeper courses focuses specifically on calculus, which is often considered to represent the thinking necessary to thrive in ambitious, STEM-forward

careers. The body of literature suggests different interpretations of what is considered a relevant and meaningful task for students within the calculus context. First, calculus courses vary in terms of which standards are covered and how those standards are sequenced, which has led to high attrition rates (Barr et al., 2022; Schmidt et al., 2022). Second, among those standards covered, calculus courses often focus on learning outcomes that exceedingly prioritize procedural accuracy (NCES, 1995). Both structural elements of calculus courses contribute to *weeding out* students who are assumed unable to thrive, demonstrating the current role of calculus as a gatekeeper course (Ellis et al., 2015; Larsen et al., 2016; Leyva, 2021). Finally, although calculus courses are deemed critical steppingstones for students to access STEM careers, there is evidence that minoritized groups are underrepresented in the enrollment and completion of such courses (Adelman, 1999), and that the coursework does not include clear connections to students' interests or lived experiences (Adiredja, 2021; Alzaga Elizondo & Larsen, 2024; McNeill et al., 2022). Together, the combination of inconsistent content standards and mathematical expectations leads to course-wide structural decisions that, intentionally or unintentionally, push students out. This, alongside evidence of the under-enrollment of minoritized students, suggests a need to intervene in how calculus concepts are shared.

One place to intervene is to investigate whether the mathematical tasks available in gatekeeper courses can challenge students' prior experiences with conceptual and application-based tasks. In elementary and secondary mathematics classes, students develop habits to find and complete computational steps on tasks that include applications without considering or acknowledging the context provided (Schoenfeld, 1989). After secondary mathematics courses, students further develop habits to ignore how the numbers, units, and operations within real-world problems conceptually relate to the challenge presented within context (Reinke et al., 2024; Schoenfeld, 1989; Verschaffel et al., 2000). Even the intention to interweave conceptual ideas to real life applications can be irrelevant if the tasks can be solved by only attending to the associated procedural computations. Although not a complete solution, understanding what tasks are commonly available for professors to assign students provides an initial step toward mitigating courses like Calculus I from gatekeeping students.

Methods

To identify opportunities for students to engage in conceptual understanding within authentic applications, we focused on the sections introducing one core calculus concept, derivatives, across two textbooks. We selected the textbooks by surveying publicly available syllabi to create a sample which reflects different approaches used across calculus courses, being a traditional textbook (Rogawski, 2011) and a problem-based textbook (Boelkins, 2013). For the analysis, we examined problems specifically within sections that (1) introduced to the derivative and (2) introduced to the concept of the derivative as a function. To select problems to answer our research question, we examined whether problems included text outside of the expression, equation, or directions that was situated in some real-life context. We adapted two frameworks (Greene & Shorter, 2017; Palm, 2008) into deductive coding schemes (Fereday & Muir-Cochrane, 2006) to measure conceptual richness and authenticity, which were used to assign each for conceptual rigor, based Greene & Shorter's (2017) conceptual understanding weighting system (CUWS), and another for the authenticity of the application, based on Palm's (2008) research on authentic applications (AA).

To rate a task's conceptual understanding (i.e., CUWS score), a task could earn a point for the *skills* required (i.e., if the task did not involve only computational or algebraic skills or memorized facts, and required understanding of the concept), the *method* required (i.e., if the

task could not be answered completely using a method that the student might have developed in prior coursework), and the *conceptual rigor* required (i.e., if the task involved interpreting the meaning of mathematical characteristics, making connections between different representations of mathematical characteristics, and/or proving or justifying mathematical statements), with a maximum of 3 points available. To rate the authentic application (i.e., AA score), a task could earn a point for each of the following: the *event* (i.e., if the real-life scenario described in the task was an actual event, and not a statement about a relationship between two variables in a context), the *question* (i.e., if the question inherent to the task was one that might be reasonably asked outside of the classroom), the *purpose of the task content* (i.e., if answering the question about the event would provide clarity regarding the event), the *language use* (i.e., if the language specific to the event did not hinder answering the question), and *the existence, realism, and specificity of data* (i.e., if the information provided contained realistic values that could be considered close to corresponding values in real life), with a maximum of 5 points available.

Each task was coded by the first and second author in three stages: first, using the original adaptations of the two frameworks for each task; second, adapting the framework based on any discrepancies identified by the coders (Campbell et al., 2013); and third, to complete consensus coding (Watts & Finkenstaedt-Quinn, 2021).

Preliminary Findings

Between the two analyzed textbooks, there was a difference in the prevalence of tasks with some application context. For the traditional textbook, 15 out of 198 tasks examined (7.5%) included an application context; for the problem-based textbook, 29 out of 66 tasks examined (44%) included an application context. After analyzing the conceptual richness and the authentic application of each task including some application context, we additionally found differences in the average CUWS and AA scores between both textbooks (Table 1). First, the central tendency descriptions of the CUWS scores for tasks within each textbook varied, with the traditional textbook having a mean and median CUWS score in the lower third range and the problem-based textbook in the upper third range. However, there was not a noticeable difference in the mean and median AA scores. This suggests that both textbooks presented similar opportunities for authentic applications, but with varying conceptual richness between textbooks.

Table 1. Analysis of Derivative Tasks in the Traditional and Problem-Based Calculus I Textbooks

		CUWS Score		AA Score		Absolute Difference	
Textbook	Number of Tasks	Mean	Median	Mean	Median	Mean	Median
Traditional	15	1.27	0	3.33	3	2.27	3
Problem-Based	29	2.52	3	3.58	4	1.31	1
Difference		1.25	3	0.25	1	0.96	2

Contrasting Characteristics: Absolute Difference between CUWS and AA Score

The differences most pronounced in the average absolute difference between a task's CUWS and AA score (Table 1). This value emerged as we collected the combinations of CUWS and AA scores, noticing that the range between one score and another indicated an important variation between problems that happened to have the same CUWS score (for instance) and a different AA score (and vice versa). In other words, the absolute difference denotes if there exists a difference in the intensity of a task's CUWS and AA score, where larger values indicate that a task is lacking in either its conceptual complexity or its authentic application, despite having a stronger score in the other category. In what follows, we illustrate the differences in the contextual richness of two tasks that cover similar Calculus I content with a similar degree of authenticity.

The two tasks described represent the variation possible in CUWS score for tasks with the same AA score. One of the tasks is drawn from each textbook; the tasks were in similar sections in which the derivative had been defined and introduced but not yet calculated. The task in the traditional textbook, Task A, had a 0-3 score (CUWS-AA) and asked students to use the symmetric difference quotient (SDQ) to approximate the derivative at a certain point within the context of vapor pressure and water temperature. The task in the problem-based textbook, Task B, had a 3-3 score (CUWS-AA) and focused on the US census in Grand Rapids, MI, asking students to compute the average rate of change over an interval that gets smaller. Task A was given a CUWS score of 0 because the task only involved computational and algebraic skills; the task provided the formula to compute the SDQ as a more accurate way to estimate the derivative than the difference quotient, and the context of vapor pressure was irrelevant as long as the student used the given temperature values. Task B, on the other hand, was given a CUWS score of 3 because the task could not be completed using only computational skills or a method from a prior class; instead, the task required students to interpret the meaning of the derivative within the context of population growth. Like Task A, Task B provided formulas to compute the average rate of change and model the population as a function over time. However, the task required students to not only identify the average rate of change but to situate that change across multiple intervals and describe how their calculations contextualize what might have been happening in Grand Rapids within the different time periods. Despite the differences in CUWS scores, the tasks received similar AA scores because they were situated in authentic contexts in which estimating the derivative would provide useful information about the given application.

Contrasting Characteristics: Likelihood of Particular Score Juxtapositions

Across all tasks identified in both textbooks, we also identified the juxtaposed pairs of CUWS and AA scores (Table 2). In the traditional textbook, the most common juxtaposition found was 0-3 (CUWS-AA). Note that this score was given to 8 (53%) of the tasks analyzed in the traditional textbook; in contrast, for the problem-based textbook, this score was given to only one task analyzed (3%). For the problem-based textbook, there were nearly an equal percentage of tasks that were scored 3-5 and 3-4, for a total of 17 tasks (59%). These scores were less common in the traditional textbook, given to three problems analyzed (20%).

Table 2. Analysis of the Most Common Score Juxtapositions of Tasks in the Traditional and Problem-Based Calculus I Textbooks

CUWS Score	AA Score	Tasks in Traditional (N = 15)	Tasks in Problem-Based (N = 29)	Total
0	3	8	1	9
2	3	2	2	4
3	3	2	4	6
3	4	1	9	10
3	5	2	8	10

Discussion

The preliminary findings from our analysis indicate potentially impactful differences between traditional and problem-based calculus textbooks. In the textbook sections introducing the derivative concept, we observed differences in the overall prevalence of tasks including an application context and the degree to which such tasks (1) required conceptual understanding and (2) lent authenticity to the application context. Our findings additionally suggest that discrepancies often exist, for individual tasks, between the degree of authenticity and the degree of conceptual understanding required. While providing an authentic context can enhance students' perceived relevance of calculus, such tasks must *also* engage students' conceptual understanding to improve student learning. Together, these findings suggest the potential impact of the textbooks that calculus instructors select for teaching their courses. Importantly, including more authentic, conceptually rich tasks can create learning environments where students can both recognize the relevance of calculus for their future careers and engage with tasks that support their conceptual understanding.

In the future, we intend to expand our preliminary analysis to include more textbooks and to encompass additional calculus concepts for a more thorough exploration of calculus tasks. This will include investigating the relationship across tasks between the CUWS score and different components of the AA score (i.e., event, purpose, etc.). We will use the combined framework focusing on both conceptual understanding and authenticity of application to provide a structure for developing new tasks across mathematical domains that are both conceptually rich and authentic. Careful attention to the nature of assessment tasks along theses dimensions will provide one pathway to help transform gatekeeper courses, such as calculus, into *gateway* courses that can support students with achieving success in their chosen career pathways.

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Student Challenges in Proof by Mathematical Induction: A Preliminary Analysis of a Generalizing Study

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Many studies have investigated the challenges students face during proof by mathematical induction (PMI). However, the majority of these studies are drawn from a small sample size. In order to generalize the findings from previous literature, we draw from a study of over 1000 proofs to develop an analytical framework for identifying common student challenges in PMI. In this preliminary analysis, we report our finding from analyzing 180 written responses for three types of PMI tasks.

Keywords: Proof by Mathematical Induction, Student Challenges

It is well documented in the literature that students often find proof by mathematical induction (PMI) particularly difficult (Styliandines et al, 2007; Norton and Arnold 2019; Baker 1996). The majority of these studies have used qualitative techniques to analyze students' written proofs, student interviews, and/or classroom recordings and observations from a small sample of data. However, to advance the field forward in terms of the generalizability of previous findings (Melhuish and Thanheiser, 2018), a study on student challenges in PMI using a larger data sample is yet to be conducted. In light of this, in this preliminary study, we document our efforts in analyzing student challenges of PMI by analyzing a subset of data from a larger sample. We first discuss existing findings on student challenges in PMI, report our study where we analyzed 180 out of 1019 proofs, and finally discuss implications of our findings and our future steps.

Empirical Background

Many scholars have investigated students' cognitive gaps surrounding proof by mathematical induction using a variety of methods (e.g., Dubinsky, 1986; Styliandines et al., 2007). For example, in a study with 40 high school and 13 college students, Baker (1996) reported that students displayed a variety of challenges including in mathematical content and deductive knowledge. For instance, Baker (1996) reported that students had difficulty in connecting the significance of proving the base case with the formulation of the inductive hypothesis, students applied the inductive step to all cases of the inductive statement, and students relied mostly on procedural techniques over the conceptual understanding of the proof strategy. Through a classroom-based study with pre-service teachers, Davis et al (2009) found 4 main challenges students experienced during PMI: problem-solving, connecting information from multiple areas, validating others' proofs, and infinite iteration. In a different study, through analyzing 67 student responses and interviewing 8 of those students, Taitra (2023) found that while most students in the study were able to set up the inductive step, the students found proving the inductive step to be challenging. Reid's (1992) study on students' understanding of proof by induction revealed a key finding: students struggled to connect induction with recursive thinking. Studies have also shown that students often take a results-oriented approach to writing induction proof (Harel, 2001). That is, students may use examples to verify the truth of statements (Baker, 1996; Harel,

2001), use a backward strategy while writing proofs (Baker, 1996), or treat proof by induction as a list of steps that need to be checked (Harel, 2001; Styliandes et al, 2007; Knuth, 2002). Building on our prior work (Authors, Year), we intend to contribute to and extend the research on student challenges in PMI by analyzing 1019 student responses to proof by inductions surrounding divisibility, summation, and recursive relations. In this preliminary study, we address the following research question: *What challenges did students face in PMI as identified from our analysis of 180 written responses across three types of PMI tasks?*

Methods

Data for this study comes from a larger project, where the project goal was to measure the effectiveness of completing Proof Blocks, a software that enables students to construct proof by arranging pre-written statements, over reading a book chapter (see Poulsen et al., 2024). As part of the larger project, in total we collected 1019 written responses for the proofs shown in Figure 1 from students enrolled in a discrete mathematics course (The full data set can be found in Poulsen, 2024).

Divisibility: For any natural number n, $2n^3 + 3n^2 + n$ is divisible by 6.

Recurrence: Suppose that $g : \mathbb{N} \to \mathbb{R}$ is defined by
- $g(0) = 0$
- $g(1) = 4/3$
- $g(n) = 4/3\, g(n-1) - 1/3\, g(n-2)$, for $n \geq 2$.

Use induction to prove that $g(n) = 2 - 2/3^n$ for any natural number n.

Sum: Claim: $\sum_{p=0}^{n}(p \cdot p!) = (n+1)! - 1$, for all natural numbers n.

Figure 1. PMI tasks

In this preliminary analysis, we analyzed 180 proofs (60 divisibility, 60 summation and 60 recursive) with the aim to develop an analytical framework that can be applied to the entire data set. To identify student challenges in proof by mathematical induction, we initially coded our data using an *a priori* framework derived from the literature. Four research team members independently applied these codes to sections of the data and convened weekly to address any discrepancies. Resolving these disagreements often resulted in adjustments to the existing codes or the development of new ones. Adjustment to existing codes took the form of expanding the definition of each code, condensing multiple codes into one. When aspects of student work could not be coded by existing code, a new code was created and defined. The new codes and modification of existing codes were checked against the already coded data set to ensure consistency in our use of the codes. In the following section, we present the codes as findings of student challenges in PMI that we have developed up to this point of the study.

Findings

We present the overview of the themes, the corresponding sub codes, and their definitions that emerged during our analysis in Table 1. Due to space constraints, we illustrate four of the sub-codes.

Table 1. Label

Major Themes	Sub-codes	Definition
Quantification Related challenges	The induction variable, k, is used without quantification	Proof uses a variable that represents the specific value the inductive hypothesis (IH) to be true up to, without defining that variable beforehand

		(e.g. indicators include the proof not having explicit language such as "for some k..." or "there exists k...")
	Conflating between the roles of n and k.	Uses the same variable in the inductive hypothesis to represent an arbitrary value and a specific value.
IH Related Errors	Assuming the IH true always (instead of for a particular value)	Assumes the IH without quantifying the variable or allows the variable to be any natural number. The proof must have some IH in order for this code to apply.
	Mismatch between values assumed in IH and value proved in inductive step (IS).	Value proved in IS is not the successor of the value assumed in the IH.
	IH is to be proven, not assumed	The proof shows no indication that the IH is something that is assumed, and instead claims that either have proved it or will prove it.
	Using IH inductive hypothesis without assuming it	No assumption is made for the statement being true. Proof jumps right into proving k+1, with no basis for k being true.
	Confuse recursive definition with IH	Proof assumes the definition of the recursive function rather than assuming that the recursive function is equal to the closed form equation for some k.
Base Case Related	More base cases than necessary	Proved for 2 or more base cases when only needed 1, or 3 or more when only needed 2
	Not enough base cases	Proved for 1 base case when they needed 2.
	Not proving the property is true for base cases of the recurrence relation	The proof does now show that the recurrence relation is equal to the closed form for base cases. In particular, there is no evidence shown in the proof that the closed form expression for 0 and 1 are equal to $g(0)$ and $g(1)$ respectively as defined for the recurrence relation.
Domain-specific knowledge	Showed knowledge of PMI outline, but not filled in	Proof outlines the complete steps of PMI, but didn't provide details.
	Developing an understanding of the summation sign	Incorrect use of the summation operator in the proof.
General Logic Errors	Treating an expression or function as a proposition	The proof treats an expression which does not have a truth value in place of a proposition which does have a truth value.
	Backwards proof	Starting with the equality to be proved and manipulating both sides (rather than starting

	from one expression, and manipulating it to show it equal to the other)
Incorrect lower bound for induction variable	The lower bound for the inductive variable doesn't allow the IH to be soundly applied to complete the proof.
Proof by example	No rigorous proof is given. Proof is instead given by a few examples, implying that proving few examples proves the entire statement.
Assumed only $g(k)$ when they also needed to assume $g(k-1)$	The proof only prioritizes one recursive relationship in the IH and does not assume the IH to be true for the previous two recursive relationships.

Implicit Quantification of the Induction Variable

In this vignette, we provide an example proof (Figure 2) where the induction variable k is used without quantifying it. Here, the student seems to understand that, in the IH, the equation holds true for all natural numbers up until an arbitrary value, (represented by the variable k), but the proof does not define this variable. While the student's intentions of using k in this context may not be difficult to determine, the proof shows no evidence that the student may have an understanding of the significance of defining k in PMI.

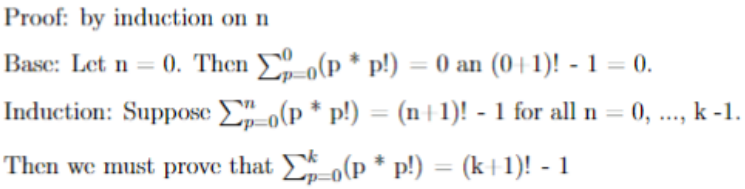


Figure 2. Student proof for summation task where k is used without quantification

Incorrect Lower Bound for Induction variable

In this vignette, we provide an example proof (Figure 3) where an incorrect lower bound for the induction variable is used. In the proof in Figure 3, the base case is proven for when $n = 1$. Therefore, the induction variable k, should iterate between the 2nd case to the nth case. However, in the proof, the student sets the lower bound for k as n, in their IH statement.

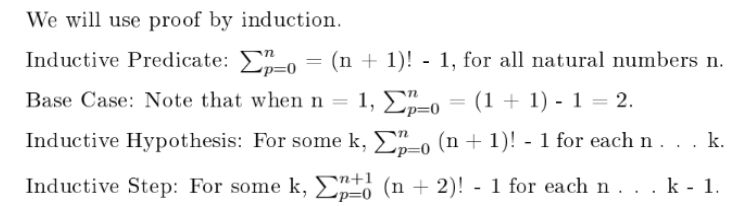


Figure 3.Student proof for summation task with incorrect lower bound for induction variable.

Mismatch Between Values Assumed in the Inductive Hypothesis and Values Provided in the Inductive Step

In this proof (Figure 4), the value shown in the inductive step is not the successor of the value assumed in the inductive hypothesis. The proof assumes that the inductive hypothesis holds true for values from 0 to k. However, while the goal is to prove the statement for $k + 1$, the proof only proceeds to prove it for k, as indicated by the "need to prove" statement.

next, suppose the hypothesis that for some k, $\sum_0^n$ (n * n!) = (n + 1) ! - 1
in k = 0, 1.....k

we need to prove that $\sum_0^k$ (k * k!) = (k + 1) ! - 1

Figure 4. Student proof for summation task where there is a misalignment in values

Assuming the Inductive Hypothesis is True Always (Instead of for a Particular Value)

In this proof (as shown in Figure 5), the inductive hypothesis is assumed to work for all natural numbers. This assumption makes the proof concerning $n + 1$ obsolete, as $n + 1$ is an element of all natural numbers, which was already assumed.

Base case: n = 0. $\sum_{p=0}^{n}(p \times p!) = 0 = (n+1)! - 1$.

Inductive hypothesis: Assume that the claim works for all natural numbers n, and prove that it works for natural number $n + 1$.

Inductive steps: $\sum_{p=0}^{n+1}(p \times p!) = (n+1)(n+1)! \times \sum_{p=0}^{n}(p \times p!) = (n+1)(n+1)!((n+1)! - 1)$. Then, we can rewrite the function, so it will be $(n+2)! - 1$.

Figure 5. Student proof for summation task where inductive hypothesis is true always

Discussion and Conclusions

In this report, we presented our preliminary findings from analyzing 180 written proofs that were drawn from a larger data set comprising 1019 proofs. Our objective is to utilize the analytical framework presented here for the remaining proofs, while continuously refining and enhancing it to ensure the development of a comprehensive and rigorous analytical framework. Some of the findings reported here comport with previous findings from the literature. For example, Baker (1996) also found that his students used specific examples to complete PMI. Similarly, Stylianides et al. (2007) found that the pre-service teachers in their study relied on a list of steps to prove PMI. This, we believe is similar to what we identified as "showed knowledge of PMI outline, but not filled in," re-affirming students' procedural over conceptual understanding of PMI. Additionally, we also found some student challenges in PMI that were not reported elsewhere. For example, developing and understanding of the summation sum, mismatch between values assumed in IH and value IS, using IH inductive hypothesis without assuming it, and treating an expression or function as a proposition were unique to our data set. These newer themes may be attributed to our sample size, type of proofs (i.e., proofs that require summation operation and recurrence relations), and students' mathematical background. However, we acknowledge certain limitations to our findings. First, the data sources were only written proofs. These data sources are limited for uncovering the thinking behind students' mathematical decisions. Therefore, our claims may not accurately reflect students' understanding of PMI. Second, to determine whether or not a code/theme truly represents a common challenge faced by students during PMI, we need information about the prevalence of that code/theme. Our future study includes examining the frequency of all codes. Adopting an asset-based perspective (Adirejda, 2019), we echo other scholars (e.g., Norton et al., 2023) in emphasizing the need to explore the underlying reasons for students' difficulties and further advocate for research that examines the effective strategies instructors of PMI can employ to address these challenges.

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"Why Would a Woman Put Themselves in This Situation:" A Discourse Analysis of Gender and Pronouns in Solving a Hat Problem

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The language we use to talk about, do, and learn mathematics informs how we socially construct our understandings thereof. This preliminary report discusses an instance from a different study where the use of feminine pronouns to describe smart logicians in a particular mathematical problem resulted in various reactions from participants. I analyzed participants' interactions with the given task using an approach to discourse analysis that examines how the grammatical structures of language are used to construct particular meanings. Preliminary findings showcase how participants' responses to the task reflect broader cultural understandings of who can be a smart logician. These findings are linked to the need for representation in curricular materials and the use of hat problems in the learning of logic and problem solving.

Keywords: word problems, logic, pronouns, gender, discourse analysis

Language is our primary medium for talking about, doing, and learning mathematics; as such, the mathematical language we use informs how we socially construct our understandings of mathematics (Planas et al., 2021). Knowing this, we may strive to use language to forward particular goals in mathematics. When teaching, we may use established formal mathematical language to enculturate students into a community of professional mathematical practice, or we may create new mathematical language to be culturally responsive to the backgrounds of our current students. When constructing mathematical word problems, we may choose to vary the demographic characteristics of characters to subvert the white cisheteropatriarchal and masculinized nature of mathematics (Leyva, 2017, 2021). One such variation through the subtle deployment of the feminine pronoun *her* is the catalytic element for this manuscript.

While conducting another study (see Bernier, 2024), I asked participating undergraduate students to engage with mathematical tasks from a collection by Roeder (2018). The focal task discussed in this manuscript is excerpted here (emphasis added):

> Three *smart logicians* are standing in a line, so that they can only see the *logicians* in front of them. A hat salesman comes along and shows the three *logicians* that he has three white hats and two black hats. He places one hat on each *logician*'s head and hides the remaining hats.
>
> He then says to the logicians, "Can anyone tell me what color hat is on *her* own head?" No one responds.
>
> He repeats, "Can anyone tell me what color hat is on *her* own head?" Still no answer.
>
> A third time: "Can anyone tell me what color hat is on *her* own head?" One of the logicians speaks up and gives the correct answer.
>
> Who spoke, and what color hat is on *her* head? (Roeder, 2018, p. 21)

When I designed the study, the fact that the problem used feminine pronouns to refer to the smart logicians was not noted as significant or unusual. Yet, as students arrived at this task in their

respective interviews, they reacted to it in ways that seemed notable and reflective of broader mathematical and societal discourses.

After completing the earlier study, I decided to examine this phenomenon further. In this manuscript, I report preliminary findings from a discourse analysis examining how participants used gendered language in discussing and solving this problem. Before I discuss my analysis procedures and preliminary findings, I briefly discuss hat problems and the matter of representation in curricular materials.

What Color Hat is on Her Head? – Hat Problems

Hat problems, like the one included in Roeder (2018), are a common type of logic puzzle used across various works and contexts (Brown & Tanton, 2009; Howse et al., 2018). These problems involve the use of logical deduction and mathematical induction to determine the color of a hat on a particular character's head through the progressive dissemination of information and elimination of impossible scenarios. They can be made increasingly complex by altering the quantities of participants, colors, and hats, and the means of dissemination of information (Brown & Tanton, 2009). The iteration of the problem used in Roeder (2018) is isomorphic to one that has existed at least since 1995, with a previous iteration using male prisoners as the characters in place of smart logicians (Gerald & Mario, 2000).

Representation in Curricular Materials: It Matters

All learners deserve to see themselves and others reflected in the materials they use to learn (Gutiérrez, 2017). Indeed, many scholars have discussed at length the need for curricular materials to carefully and appropriately reflect a diversity of identities in order to promote equitable learning (Hickman & Porfilio, 2012; Macintyre & Hamilton, 2010; Mills & Mustapha, 2015). Given that cultural images of mathematical professionals are overwhelmingly masculine (Mendick et al., 2008), mathematical tasks which depict women as mathematically capable could be useful in encouraging women to participate in mathematics. The task as published in Roeder (2018) may (or may not) have been written with that intent, given its logical equivalence to but narrative distinction from earlier-stated hat problems. The problem was originally a reader submission (Roeder, 2017), but the provenance of the reader's version is unknown.

Discourse Analysis

For this manuscript, discourse is taken to be the use of language, spoken or written, to communicate ideas and actions (Johnstone, 2018). Analyzing discourse is a worthy endeavor because discourse is both shaped by and shapes people and social structures (Johnstone, 2018; Planas et al., 2021). Learning in undergraduate mathematics classrooms involves discourse (Hetrick et al., 2023; Rios, 2023) – both the discourse occurring in the room and the Discourses (Gee, 2014a) which comprise what it means to be a student, a learner, or a mathematician.

Of course, given the broad definition of discourse above, most modes of qualitative analysis constitute an "analysis of discourse." Indeed, many different forms and approaches to discourse analysis have been developed and pursued across the social sciences and within educational research (Johnstone, 2018; Warriner & Anderson, 2017). I largely adopt Gee's (2014a, 2014b) approach, where the structure of the discourse is critical to its analysis and where "meaning [is] an integration of ways of saying (informing), doing (action), and being (identity), and grammar [is] a set of tools to bring about this integration" (2014a, p. 8). In particular, I draw on the "tools" which Gee (2014b) characterizes as "specific questions to ask of data" (p. 2).

Methods

This manuscript examines a phenomenon that occurred while I was researching another phenomenon (see Bernier, 2024). As such, the data for this manuscript was not collected with the analysis presented here in mind. The relevant aspects of that study are described here, as well as the analytic approach I used for this manuscript.

Study Context and Data Extraction

The larger study from which the relevant data is extracted sought to deepen our understanding of how undergraduate students can engage in play when solving puzzles and mathematical problems (Bernier, 2024). To do so, I conducted a series of four clinical interviews (Hunting, 1997) one-on-one with six undergraduate students (Table 1). In these interviews, participants either played digital puzzle games or solved mathematical tasks taken from collections of "mathematical puzzles." The second interview included three problems from Roeder (2018), including the task discussed in this manuscript.

Table 1. Summary of participants

Pseudonym	Gender Identity	Pronouns
Batman	Woman	she
Daniela*	Woman	she
David Wallace	Man	he
Pattakoby	Non-binary	he/they**
Shane	Man	he
Timothy Burns	Woman	she

*All participants chose their own pseudonym, except Daniela who declined to do so; the name Daniela was chosen arbitrarily.
**For readability, I use only masculine pronouns for Pattakoby.

For this manuscript, I pulled data from the transcripts already used for prior analyses. I located within each participant's second interview transcript any discussion of the relevant task and extracted all such lines in chronological order. I then placed the extracted segments into a single Microsoft Word document for analysis.

Analysis Procedures, or, What's in my Toolkit?

My approach to data analysis here was primarily guided by four tools from Gee's (2014b) toolkit: the Deixis Tool (p. 16), the Fill In Tool (p. 18), the Why This Way and Not That Way Tool (p. 53), and the Figured Worlds Tool (p. 177). The Deixis Tool involves identifying and examining the shifting references (deictics) such as pronouns used by speakers. Since the gender of the logicians in the task was communicated via deictic, these seemed especially salient. I identified each use of a gendered pronoun and changed the font to an assigned color – orange for he/him/his, turquoise for she/her/hers, and purple for they/them/theirs. I also underlined other gendered words which took on "deictic like properties" (Gee, 2014b, p. 16), like guy and person. This process identified many (though not all) utterances worthy of further analysis.

While the Deixis Tool allowed me to identify critical moments in the data, the Fill In Tool and Why This Way and Not That Way Tool allowed me to get at the deeper meaning. According to Gee (2014b), these two tools examine the unstated implications of speakers' grammatical choices. With the Fill In Tool, I examined different segments for "[w]hat is not being said overtly, but is still assumed to be known or inferable" (p. 18). The Why This Way and Not That Way Tool led me to consider how else a participant might have said or phrased the same sentiment, and what the implications are of their chosen phrasing. To apply these tools, I read through the extracted segments, highlighting and annotating moments where the implications of their speech and choices were apparent. For example, this included highlighting each instance of an incongruous pronoun being used (e.g., he or they being used to describe a singular logicians). Comments then discussed the apparent implications of these grammatical choices.

Finally, I linked the local discourses in my data to larger discourses by applying the Figured Worlds Tool. To use the Figured Worlds Tool, one must "ask what typical stories or figured worlds the words and phrases of the communication are assuming and inviting listeners to assume" (Gee, 2014b, p. 177) In this case, the relevant Figured World seemed to be that of doing logic/mathematics and/or being a logician/mathematician

Preliminary Findings

Apparent throughout all interviews was a tendency of all participants – and myself as an interviewer – to at least sometimes refer to the logicians using incongruous pronouns. In some cases, these are fairly well explained by considering the aesthetics of the utterance. For example, Shane made the utterance: "there's three people, five different color hats, and one of them knows what color hat they have. In this utterance, Shane's use of them is congruous with the logicians being women, because it is being used as the third-person plural. The use of they, on the other hand, is incongruous, because its referent is "one of them," a singular smart logician. But, saying "one of them knows what color hat she has" could feel awkward or unintuitive.

On the other hand, some of the instances where participants used incongruous pronouns seem less innocuous. Pattakoby, for example, read the task aloud – but when he did, he verbally replaced "her" with "their" initially, and later used masculine deictics:

> He places one hat on each logician's head and hides the remaining hats. He then says to the logicians, "Can anyone tell me what color hat is on their own head?" No one responds…
>
> …and because the one in the middle knows that the guy standing in front of him has a white, possibly the guy in the middle can guess…

Pattakoby and Shane did not ever verbally acknowledge the use of feminine pronouns in the task description. Applying the Why This Way and Not That Way Tool and the Figured Worlds Tool leads me to conjecture that they did so because in their figured worlds a "smart logician" would be a man – perhaps to the point of not even accurately parsing the words in front of them, or implicitly assuming that the use of feminine pronouns was a mistake.

Several of the other participants did make note of the use of feminine pronouns, with many of them questioning the meaning of said use. Batman and David Wallace both phrased their questions similarly, with Batman asking, "Who is her?" and David Wallace asking "Why does he say her own head?" My responses to Batman and David Wallace did not acknowledge the gender of the characters and were intended to clarify the problem:

> Yes, they were supposed to see if they can figure out what color hat is on their own head. So each person in that line is trying to, from whatever they can see, figure out if they know what color hat is on their head, which they can't see what color hat is on their head.

Using the Fill In Tool and Why This Way and Not That Way tool, it seems I misinterpreted these questions during the interview as being requests for clarification of the task goals. But, Batman and David Wallace both made the use of the word her central to their question, which implies that they were particularly confused about the use of a feminine pronoun to refer to the smart logicians. My own use of gender-neutral pronouns in my response is likely reflective of not having carefully considered the task's use of pronouns prior to the interviews and internalized subconscious biases I hold within my own Figured Worlds.

Finally, Timothy Burns and Daniela both noted the gender of the smart logicians explicitly. Timothy Burns asked the question in the title of this manuscript: "Which type of scenario is this and why would a woman put themselves in this situation?", while Daniela initially thought the use of her was meant to supply some sort of clue regarding the solution to the problem:

> It does say that the person who spoke up and gave the correct answer was a girl because it says, who spoke and what color is on her head? I don't know if that's important though because it doesn't mention anything about gender besides that. Or, wait, actually they might all be girls because he says, the salesman asks all of the logicians. Can you tell me what color is on her own head?

Applying the Fill In Tool, both Timothy Burns and Daniela observe that the logicians are meant to be women. However, neither seems to be evaluating this problem as a plausible scenario in the first place, with Daniela seeming to interpret the task like a language riddle while Timothy Burns seemed to reject the whole problem as absurd. In other words, their reaction to the task did not relate as much to their understanding of *who* can do mathematics, but instead, of *what* features of a mathematics task are and are not important.

Limitations and Implications

Given preliminary findings based on a secondary and unplanned-for analysis to observe a phenomenon, I must be extremely cautious in suggesting implications of this work. Especially important is my lack of use of the Frame Tool (Gee, 2014b, p. 44), which notes the importance of considering ever-increasing layers of context when conducting discourse analysis. I have not fully considered, for example, how what I know about participants' backgrounds and prior experiences may have informed their reactions to the task, nor how the interview context might be different than the classroom context in informing these reactions.

Still, I believe this analysis illustrates a broader notion that work to subvert inequities in mathematics is never easy or free. I cannot speak to the intent of whoever decided that the logicians in the problem should be women, but I know that I myself have made similar changes in producing curricular materials. The intent of this, on my part, has been that small changes to make materials more representative might help to undermine established cultural images bit-by-bit. What we see in this analysis, though, is that changes which are too small may be overlooked or ignored. While it is compelling to not want to make a "big deal" about representation – and a desire to avoid tokenism or otherwise "singling out" representation of women and other marginalized groups – it seems there is a danger too in being too subtle. At a minimum, instructors and researchers should be careful – more careful than I was – in examining the potential implications and anticipating the reactions of learners when interacting with tasks that include characters from marginalized groups and subvert cultural expectations or norms.

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All My Work is Right, Why Didn't I Get the Right Answer? When Logarithmic Identities Fail

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When learning about logarithms and solving logarithmic equations, students often misapply logarithmic identities. In some cases, misapplying an identity leads to the deletion of solutions or the introduction of extraneous solutions. This results from applying an identity without first checking that all logarithms in that identity are defined on the same domain. In this preliminary work, we examined 14 websites that College Algebra students might use when learning how to solve equations using logarithms, looking for mentions of the domains of definition whenever identities were stated or applied. Of the 11 websites that stated a logarithmic identity, only 4 mentioned domains of definition. Of the 11 websites that showed an example logarithmic identity application, zero mentioned domains of definition. We hypothesize that this lack of emphasis on domains of definition in teaching and learning logarithmic identities may lead to confusion when working with logarithms and in mathematics as a whole.

Keywords: Logarithms, Identities, Domain, College Algebra, Solving Equations

Understanding logarithms and using logarithms appropriately are common stumbling blocks for students, starting in introductory algebra and often continuing into calculus and beyond. Although some research has investigated undergraduate students' understandings of the concept of logarithm (Berezovski & Zazkis, 2006; Leopold & Edgar, 2008; Liang & Wood, 2005; Park & Choi, 2013; Rafi & Retnawati, 2018), little research has focused on undergraduates' knowledge related to logarithmic identities (sometimes also called *logarithm properties* or *logarithm rules*).

Consider the equation $\log(x^2) = 2$. Simply applying the definition of the base-ten logarithm, we obtain that $x^2 = 100$, and thus find that $x = \pm 10$ gives all solutions to the original equation. Checking both solutions in the original equation confirms that both values are indeed solutions to this equation.

However, a common solution path for students taking a course in College Algebra might be to apply the logarithmic identity $\log(m^n) = n\log(m)$ to the original equation to obtain $2\log(x) = 2$. From this point, dividing both sides of the equation by 2 and applying the definition of the base-ten logarithm suggests that $x = 10$ is the only solution to this equation, though we know from the previous example that this equation has at least (indeed, exactly) two solutions.

A student might wonder: "If everything I did was correct, then why didn't I get the right answer?" But the expert observer will note that there is a caveat to the logarithmic identity applied by our hypothetical student: it is true that $\log(m^n) = n\log(m)$, but only when $m > 0$. In particular, it is true that $\log(10^2) = 2\log(10)$, but it is not true that $\log[(-10)^2] = 2\log(-10)$, as the function $f(x) = \log(x^2)$ is defined for all $x \neq 0$, but the function $g(x) = 2\log(x)$ is defined only for $x > 0$. While experts are likely to notice this error, students who are still learning about logarithms and their properties are likely to overlook the solution $x = -10$, and they may even reject it outright due to the fact that one "can't take the log of a negative number."

Our research begins to answer the following research question: How do resources targeted toward College Algebra (or high school algebra) students attend to issues related to function domains when teaching about logarithmic identities? In this study, we explored websites College Algebra students might use as reference materials when learning how to solve problems using

logarithmic identities. We focused our attention on the extent to which these websites acknowledge the importance of the domains of the logarithmic functions involved in making a logarithmic substitution during problem solving as in the example presented above. We present some preliminary findings and discuss implications for the teaching and learning of logarithms as well as mathematics more broadly.

Students' Understanding of Logarithms

Logarithms are notoriously challenging for students, both in concept and in application. Park and Choi (2013) argued that learner's difficulties with logarithms are often overlooked due to prevalent calculator usage. Indeed, Leopold and Edgar (2008) found that only 21% of university students enrolled in a second-semester introductory chemistry course responded correctly to mathematics questions about logarithms on a standard, calculator-free, university assessment.

Berezovski and Zazkis (2006) proposed interpretive frameworks to explore high school students' understandings of logarithms, based on Confrey's (1991) interpretive frameworks that were designed to explore learners' understandings of exponents and exponential functions. These frameworks were designed to investigate the different conceptualizations and approaches students may use when simplifying expressions or solving problems related to or requiring logarithms. In a sample of 17 students' responses to the task of simplifying $\log_3 54 - \log_3 8 + \log_3 4$, 12 students reached the correct answer, but 9 of those unnecessarily applied the change-of-base formula, suggesting that a calculator was used to reach their final answer. It is therefore suggested that only 3 of 17 students in this sample understood the meaning of logarithms well enough to solve this problem without the aid of a calculator.

There is some evidence to suggest that the correct application of logarithmic identities is challenging for students. Liang and Wood (2005) found that students commonly assumed that $\frac{\log(x)}{\log(y)} = \log\left(\frac{x}{y}\right)$ and that $\log(a+b) = \log(a) + \log(b)$, a finding echoed in the work of Rafi and Retnawati (2018). Liang and Wood (2005) also observed students treating the "log" notation as a sort of "common factor" that could be canceled when it appeared in the numerator and denominator of a fraction, as in the example $\frac{\log(100)}{\log(10)} = \frac{100}{10} = 10$. Indeed, this may suggest that students sometimes do not view logarithms as functions at all or understand the significance of logarithms as functions.

Little other research has investigated students' understandings of logarithmic identities and applications of logarithmic identities in solving equations. Furthermore, in light of the misconceptions outlined above, the accuracy and clarity of the resources students use to support their learning is paramount. Our work begins the process of examining those resources to gain a more complete picture of the information students may consume.

Methods

We performed a preliminary content analysis (Cohen, Manion, & Mason, 2013) on 14 websites. For this introductory study, we chose to focus our attention on websites (as opposed to textbooks or classroom curricula) as these resources are available to all learners regardless of their instructor or the physical resources made available to students. Future studies will explore other sources of content.

To select these websites, we created a Python web scraper that would fetch the top 8 Google results for any given search term(s), as we assumed that most students would select websites from among the top search results for a given search query. We then determined 5 phrases that a

student new to the topic of logarithms and logarithmic identities might search to gain a deeper understanding, and we used these as search terms. These search terms were as follows: (a) "Properties of logarithms", (b) "How to use a logarithm", (c) "Logarithms in college algebra", (d) "Logarithmic identities", (e) "Logarithm rules". For this preliminary investigation, we excluded any apps, comment-based websites such as Reddit and StackExchange that lack a systematic content review process, and video resources such as YouTube and KhanAcademy, as we felt that these kinds of resources deserved a more thorough treatment than the preliminary exploration provided in this report. These resources will be explored in more comprehensive future research on this topic. Of the websites left, we chose the top 3 for each search result, yielding 15 websites that we further examined for their treatment of domains of logarithms.

Next, each of the 15 websites was analyzed for the following events: (a) logarithm defined, (b) logarithmic identity stated/defined, and (c) logarithmic identity invoked (e.g., when working through an example). Whenever one of these codes was applied, we then applied one of the following subcodes: (a) domain(s) explicitly defined/stated, (b) domain(s) not mentioned, or (c) only 0 addressed as an exclusion from the domain. To validate the analysis technique of a single author, each author individually coded the same 3 websites, and codes were compared and discussed until 100% agreement was reached on all codes. Note that we only analyzed the specific web pages that were identified by our web scraper. We did not include in our analysis any web pages that were linked to within those web pages.

Table 1. Codes used in our analysis, with possible subcodes corresponding to domain mention.

Codes Examined for Domain			
Codes	Logarithm Defined	Log. Identity Stated	Log. Identity Invoked
Subcodes	Domain explicitly defined	Domain not mentioned	Only 0 explicitly mentioned

Originally, we also looked at occasions where graphs were shown, or extraneous solutions were acknowledged, as these could be considered as implicit mentions of domains of definition for the logarithmic functions involved in a given problem. However, both events occurred very few times (2 each) and these websites did not explicitly discuss these aspects of the problem in relation to function domains. Furthermore, neither of the websites that acknowledged extraneous solutions attributed their emergence to the expansion of a function's domain due to the misapplication of a logarithmic identity; they simply verified that these solutions were extraneous with no discussion of their origins.

After each of the 15 websites was analyzed, we excluded one that defined logarithms once, did not explain the domain on which they are defined, and did not define or use any logarithmic identities. This left us with a total of 14 websites included in our analysis; we present the results of this analysis below.

Results

Once we analyzed the 14 websites in our sample for the above events, we calculated relevant statistics.

Table 2. Coding scheme and instances of occurrence within the 14 websites analyzed

Results			
	Logarithm Defined	Log. Identity Stated	Log. Identity Invoked
Domain mentioned	5	4	0
Domain not mentioned	3	7	11
Only 0 explicitly mentioned	1	0	0
Totals	9	11	11

Most websites from this sample defined a logarithm, and even more stated or invoked logarithmic identities. A small majority (55.55%) of websites that defined a logarithm also stated the entire domain on which logarithms are defined ($x > 0$). One website (11.11%) that defined a logarithm only stated that logarithms are not defined at $x = 0$, but did not indicate that logarithmic functions are undefined for negative x-values. The remaining websites that defined logarithms (33.33%) did not mention any domain.

Although a website in the sample was equally likely to state and invoke a logarithmic identity (78.57%), some websites only had one event or the other. Of all 14 websites, 8 (57.14%) of them both stated and invoked an identity. Given a website that stated a logarithmic identity, it was 36.36% likely the website also stated the domain on which that identity holds. Similarly, given a website that stated a logarithmic identity, it was a 63.64% chance no domain of definition was mentioned.

Finally, given a website invoking a logarithmic identity, there was a 100% chance that the website did not acknowledge any possible changes to function domains that might have been introduced through the application of that identity. This means that, in a website containing an example application of an identity, there was never any discussion about why the identity was (or was not) valid to invoke or how a change in the domain might introduce extraneous solutions or delete solutions. Two websites that invoke a logarithmic identity (18.18%) applied an identity that altered the domains of the functions in the equation. This produced extraneous solutions, which were then discarded. For instance, one website solved the equation $\log(2x + 3) + \log(x) - \log(12) = 0$ using the product and quotient identities for logarithms. This yields one solution, $x = 2$, and one extraneous solution, $x = -3$. The extraneous solution was generated through the application of the product rule to combine $\log(2x + 3) + \log(x)$ into $\log(2x^2 + 3x)$, which changed the domain of the left-hand side of the equation from $(-\frac{3}{2}, \infty)$ to $\left(-\infty, -\frac{3}{2}\right) \cup (0, \infty)$. Websites that did not include examples of extraneous solutions did not mention that they were a possibility. Furthermore, no websites gave examples in which the application of a logarithmic identity caused the deletion of a solution, as in the example we gave in our introduction.

Overall, our data suggest that when a website's goal is simply the definition or introduction of the concept of a logarithm, the domain of the logarithmic function is likely to be included. However, when the focus of the conversation is on logarithmic identities, the domains of the functions involved are unlikely to be mentioned.

Discussion

Our work suggests that websites College Algebra students might use as reference materials when learning about logarithmic identities do not place much importance on the domains of definition of the logarithmic functions used in those identities. It is important to note that logarithmic identities are theorems, and that the satisfaction of the hypothesis of a logarithmic identity requires checking the domains of the functions to be used in that identity. In our earlier example, we can (and perhaps should) restate the power rule for logarithms as follows: "If m $>$ 0, then $\log(m^n) = n\log(m)$". Although we cannot confirm or refute that $x > 0$ in the equation $\log(x^2) = 2$ before we know the solutions, the formal statement of the theorem serves as a warning that, if this equation has negative solutions, applying this identity will cause us to delete them.

As suggested by Berezovski and Zazkis (2006), learners' grasp of logarithms and logarithmic identities is often tenuous. For the hypothetical student who asks, "Why did I get the wrong answer if all my steps were correct?", it may be frustrating to learn that their first step of applying a logarithmic identity led to an error caused by inattention to the domains of the functions involved in the substitution.

In upper-division proof-based courses, a strong emphasis is placed on checking that the hypothesis of a theorem is met before applying the theorem. However, in College Algebra courses, theorems like these are often not even stated as theorems, and students are often instructed to apply them with no regard to their hypotheses. We argue that this sort of "fast and loose" way of doing mathematics at lower levels can lead to confusion at higher levels and reinforces the belief held by some students that mathematics is simply a set of arbitrary rules for moving symbols around on paper.

Future Work

This work is the first step in a program of research that will probe College Algebra students' understandings of logarithms. A key consideration of this research is what information is presented to students in the resources they use to support their learning. Future work will explore the contents of algebra textbooks, comment-based websites, and video resources such as YouTube and KhanAcademy. We will investigate College Algebra students' understandings and perceptions of the applications of logarithmic identities when completing various types of routine exercises given in College Algebra courses. We will also explore instructors' ability to detect these sorts of errors in student work and explore the feedback they provide to their students.

We will also expand this work to attend to other situations in which equality is sometimes claimed by experts or students, but domains are often left implicit, including trigonometric identities and applications of the "factor-cancel" method for simplifying rational functions. We will explore the extent to which domains are acknowledged when using these algebraic and trigonometric tools and how experts and learners react when they encounter unexpected results that arise from their applications.

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(Re)Humanizing Mathematics for Historically Minoritized Learners: Narratives of Community College Students in Noncredit Bearing Developmental Math

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While many community college (CC) students come to higher education with specific career or life goals, these aspirations may be forced to change if students are not able to get past the gatekeeper of developmental mathematics. Racially minoritized students are disproportionately tracked into developmental mathematics classes upon entering CC, and often become trapped in a financial and emotional cycle of take-fail-repeat, with student debt linked back to developmental coursework totaling over $1.3 billion per year across the U.S. (Jimenez, Sargrad, Morales, & Thompson, 2016). This report presents a subset of data from a larger narrative inquiry project that focuses on humanizing stories from an intimate group of CC students, uncovering how they came to know themselves as learners and doers of mathematics. Findings provide insight into how mathematics learning experiences can function to dehumanize, while also paving the way to creating more humanizing undergraduate mathematics learning spaces.

Keywords: mathematics identity, narrative inquiry, developmental / remedial mathematics, community college

Supporting mathematics educators to engage with and act on issues of social and racial justice requires an understanding of the lived experiences of their students, particularly as these experiences relate to mathematics learning and doing. One key aspect we can turn to as a first step to developing these understandings are the storied and lived experiences of historically minoritized mathematics learners as they choose to tell them. This report features data from a larger project that explores the mathematics narratives of an intimate group of community college students who have taken non-credit bearing developmental mathematics coursework, as they describe their experiences learning mathematics, their ideas about what mathematics is, and their self-perceptions as influenced by these experiences and ideas. By centering the self-narrative, we can focus our attention on the following research question: How do current practices in undergraduate mathematics learning spaces function to humanize or dehumanize our students as learners and doers of math?

The larger project from which this study comes focuses on community college (CC) students in noncredit bearing developmental mathematics (NCBD) coursework for a few key reasons. Firstly, these students' specific career or life goals may be forced to change if students are not able to get past the gatekeeper of developmental mathematics coursework. Secondly, racially minoritized students are disproportionately tracked into NCBD mathematics classes upon entering college, and often become trapped in a financial and emotional cycle of take-fail-repeat, with student debt linked back to developmental coursework totaling over $1.3 billion per year across the 50 states and D.C. (Jimenez, Sargrad, Morales, & Thompson, 2016). And thirdly, despite this, mathematics as a field of study has been positioned as more important than ever before for a diverse pool of learners, all while inequitable access to quality mathematics-learning experiences prevails.

This positioning of mathematics as a high stakes subject within schooling systems begins in elementary school and continues throughout grades K-12 and into undergraduate education. School mathematics requirements have evolved historically out of a movement to privilege

mathematics as a field of study because of its ties to national, capitalistic goals. As such, mathematics has gained a reputation as being necessary for the formation of citizens who are able to contribute to the economic success of our country in innovations requiring knowledge of science, technology, engineering, and mathematics (STEM) (Gutstein, 2003; Schoenfeld, 2004; Secada, 1989; Vashchyshyn & Chernoff, 2018; Washington, et al., 2012).

Theoretical Framework and Methodology

Because students' descriptions of their mathematical learning experiences are central to understanding how they understand their mathematical selves, I call upon the theoretical frame of narrative mathematics identity (Langer-Osuna & Esmonde, 2017; Larnell, 2016), taking a humanizing approach to the exploration of CC students who are currently enrolled in NCBD mathematics coursework. Using narrative inquiry as a tool to learn about students' identities as mathematics learners and doers, I explore narratives of an intimate group of five community college students (including four BIPOC students and one white, disabled student), uncovering how they came to know themselves as learners and doers of mathematics. While the broader study contains stories from these five diverse individuals, this report focuses on excerpts from a single participant, Marty, in order to provide a deeper view of one participant's story within the scope of this paper, as opposed to just grazing the surface of all five participants' experiences.

The self-narrative provides insight into how experiences and actions related to school mathematics starting in elementary school and evolving over the life course through the undergraduate years may shape what students see as possible for themselves. This humanizes the stories about how not just undergraduate developmental mathematics, but school mathematics more broadly as it exists today, can serve as a filter, removing certain students from educational opportunities.

With the idea of humanizing research at the heart of this project, participants were recruited through voluntary response sampling via in-class announcement, as well as a YouTube video posted to the course websites in NCBD mathematics courses within a large community college system in a small Midwestern City. Recruitment took place in 2021-22, at a time when most mathematics courses at this particular community college were being offered virtually, but with live instruction, due to the COVID-19 pandemic. I also had IRB permission to recruit via social media, so I posted recruitment flyers on Facebook and sent emails to my personal networks with permission to reshare. Through this, I recruited six participants, but five followed through and completed the full study. Participants took part in both a survey and two interviews, which were semi-structured in nature, leaving space for various lines of questioning and a more natural conversational flow, rather than just sticking to a script.

Interviews were offered either in person or virtually, not only in light of the COVID-19 pandemic, but also for participant accessibility and preference. Students were given a small token of appreciation for their participation.

After interviews were sound recorded, transcribed, and de-identified, member checks were performed to support participants in telling their stories, and to be sure the story that is re-told reflected what they would like to put forth.

To analyze participants' stories, I call upon Danny Martin's Central Themes of Mathematics Identity (2000), as well as honoring the themes that arise from within the narratives of my participants. Martin's themes include:

> (a) references to the instrumental value of mathematics, (b) references to motivation to attain mathematical knowledge, (c) references to opportunities to learn mathematics, (d) references to strategies to learn or participate in formal and informal mathematics

contexts, (e) references to constraints or barriers on participation within mathematics-learning contexts, and (f) references to one's own capacity to perform in mathematics-learning contexts. (2000)

Transcriptions were coded according to these themes, as well as a few themes that arose from the ground up and appear in data from the larger study (not within the scope of this paper). Participants were sent copies of the analyses of their stories and were given opportunities to provide any feedback they'd like to offer on thematic analysis of their narratives. Pseudonyms have been used to de-identify and protect participant identity, with participants being given the option to choose their own pseudonyms.

Select Interview Data and Analysis: Marty's Story

Apparent throughout my conversations with participants were examples of how students have been trained to identify mathematics quite narrowly, limited only to practice involving numbers, formulas, equations, and the like, whereas creative problem solving, guessing and checking, or other less "formal" methods would go unrecognized as mathematics. Take for instance the following quote from my interview with "Marty" (pseudonym), my focal participant within this report.

Marty: I told a professor years ago that I'm afraid, or I want to learn how to build properly, but I want to learn how to build in a way that I don't have to use mathematics regularly, because I'm not good at it, and he straight up told me that he never does math, and he totally understands, because he hates it. (Interview 2)

Marty identifies as a Latina furniture maker and woodworker who, at the time of the interview, was in her early 30s. At the time of our interview, she had been accepted into art school and was in her first year after transferring. Her narrative focuses on her mathematics identity through activities and feelings she has around mathematics learning and doing over the life course. Marty knew from a young age that she wanted to be an artist and identified furniture making, woodworking, and teaching these trades as her desired goals, all which she believed were possible for herself through art and in spite of mathematics.

Marty was born to Cuban immigrants and grew up in South Florida with one brother. As she shared, she always envied him for being "stupidly good" (Interview 1) at math, as she describes him; a reflection of her own perceived capacity of her ability to perform in mathematics-learning contexts (Theme F). Marty says she struggled with school mathematics growing up, but she was tracked into the "normal classes" (Interview 1). This reflects Marty's own perception about her capacity in mathematics (Theme F). It also reflects how she understands the opportunities she had to attain mathematical knowledge through coursework (Theme C). Marty's self-perceptions of her lower ability in mathematics reflect her mathematics identity. As she worked her way through grades K-12, she started approaching high school graduation knowing that she wanted to go into the arts, and knowing that she wanted to teach one day, which meant college, and specifically art school, was the route she would need to take.

Marty: I decided that I wanted to go to community college because I still didn't know where I wanted to go to art college, and so while making that decision um, it was also like, let me do this. Get all of the credits that are necessary, but unnecessary really, out of the way, and um, and I'll just be able to focus on making my work later. (Interview 1)

I asked her what she meant by "necessary, but unnecessary," and Marty describes how required general education coursework, such as mathematics, science, or language arts, are often centered around art when you take them in art school. Rather than an arts-based interdisciplinary education being an incentive for Marty to attend art school straight out of high school, she says

this was instead a deterrent, along with the high financial burden of paying for general education coursework at an art school.

> *Marty:* …they're [general education courses] really expensive to take at the university, so I decided to do it at the community college. And when you take those classes at an art school, they shape the class around art. Like, the science class would be, um, like, on color, you know. And I remember seeing some of the homework, and I was like, that's kind of dumb. Like, I enjoyed my, like, my science classes where I took, and it was an unbiased like, art science. It's like science science. And mathematics is math. And English is English, and it's not based around art. (Interview 1)

Marty's differentiation of science, mathematics, or English classes that are contextualized around the arts as being "kind of dumb" got me thinking: does she prefer to learn science, mathematics, or English as decontextualized? And what would that look like or mean for her, and why does she see "purer" versions of these content areas as more valuable? This line of thinking gave me some insight into how Marty has been trained to perceive mathematics and science as higher level when it is context-free or abstract.

Marty demonstrates in the above quote that she was motivated to obtain not only mathematical knowledge (Theme B), but also knowledge in a variety of general academic content areas. Marty also wanted the opportunity to learn mathematics (Theme C) at the community college, rather than doing a more content-specific mathematics course centered around the arts within a 4-year art university.

Marty goes on to describe her avoidance of math while problem solving in her work.

> *Marty:* I don't really use it [mathematics] anymore. As a woodworker, that's kind of a funny thing to say. Um, I just...when I do mathematics now, on the very rare occasion, it's really hard. (laughs) Um, yeah, cuz it's art math, which is a funny thing, cuz I was like, I don't wanna do art math.
>
> *Ranza:* What's art math?
>
> *Marty:* It's just complica...like, when you're trying to come up with something that...like, I was working on these weavings, and it's like, there was art mathematics that was involved in figuring out the scale, and I was like, I just can't figure this out! It's not working. (laughs) And it took me so much longer. Um, but, yeah. (Interview 2)

We see here a common occurrence demonstrated by Marty, as she categorizes her rich mathematical practices as "figuring it out without the math." This reflects not only Marty's narrow perceptions of what counts as legitimate mathematics, but also the limited ways that students are often trained with a classroom setting to identify what types of problem solving are mathematical and which are free of mathematics.

Implications of this Study

Findings of this study provide invaluable information for mathematics teachers, not only about how we can support the development of positive mathematics identities within school settings, but also how certain mathematics experiences may serve to de/humanize students by limiting their perceptions of what "counts" as legitimate mathematics, in turn creating situations that lead them to believe their experiences, interests, and problem solving do not count as mathematics. This leads students to believe that they are not learners and doers of mathematics, when in fact they very well are. It is our responsibility as mathematics educators to support an expansion of what we center as legitimate mathematics within our classrooms and schools and support students in seeing themselves and their interests as mathematical. For example, in Marty's situation, we see how engrained her invalidation is as a user of mathematics—her

experiences and messaging over a lifetime have led her to believe that the mathematics activities in which she engages are not legitimate mathematical practices. This tells us one story of how important it is to value students' mathematical knowing and doing, even when their mathematics may look different than that within the narrow gaze of the U.S. mathematics textbook.

Outcomes of the larger study help us to gain an awareness of both the patterns and unique qualities that are present within students' lived experiences around mathematics; to use this awareness to continue exploring and centering student voice as a means to creating knowledge regarding how mathematics teachers can champion their students in accomplishing the goals they set out for themselves; and to use this awareness to continue experimenting with and developing appropriate mathematics pathways and support structures so that mathematics coursework acts as a humanizing mechanism in students' lives.

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The Use of Proof Comprehension as a Self-Assessment Tool in Undergraduate Proof Writing

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Research has shown that self-assessment techniques can improve understanding and academic performance for mathematics students. As proof construction, validation, and comprehension have proven to be facets of mathematics with which students can use more learning opportunities, Mejía-Ramos et al. (2012) present an assessment model for proof comprehension. For our project, this framework is utilized in understanding proof self-assessment, with the goal of aiding students in the process of proof construction and validation. In this preliminary report, we provide results from the first phase of our project, in which we surveyed seven university mathematics faculty about their approaches to proof construction and self-assessment. Results from this study are being used to generate a proof self-assessment tool through the lens of proof comprehension that undergraduate students may use to improve their proof writing strategies. Pedagogical suggestions are also provided based on these results.

Keywords: Self-Assessment, Proof Writing, Proof Comprehension, Evaluation, Criteria

Introduction

Recently, there have been many research studies conducted on undergraduate students' understanding of proofs, including the construction and writing of proofs (e.g., Azrou, 2015; Chamberlain & Vidakovic, 2021; Lew & Mejía-Ramos, 2019; Mejía-Ramos & Weber, 2019), validating proofs (e.g., Ko & Knuth, 2013; Powers et al., 2010), proof comprehension (e.g., Chamberlain, 2017; Dawkins & Zazkis, 2021; Selden & Selden, 2015; Weber & Tanswell, 2022; Yang & Lin, 2008) and assessment of proof comprehension (e.g., Mejía-Ramos et al., 2012; Cooley et al., 2024). Melhuish et al. (2022) and Stylianides et al. (2017) provide comprehensive reviews of research in many areas of proof in undergraduate mathematics education. Yang and Lin (2008) express it is important while constructing a proof to "have diverse ideas come to mind at the 'right time,' reorganize these ideas, and formally write the proof," (p. 73). The reorganization of these ideas is a cycle of writing, validating, reading and comprehending, and evaluating one's own proof. Selden and Selden (2015, 2017) discuss the interaction and comparisons of these various aspects of proof writing in undergraduate mathematics and point out there has been little research on how proof comprehension, construction, validation, and evaluation are related. As such, in this study, we seek to investigate how proof comprehension strategies may be used as a self-assessment tool in proof construction and evaluation.

Research shows students' academic performance can benefit from self-assessment and self-monitoring techniques (Andrade & Valtcheva, 2009; Beumann & Wegner, 2018; Kemp et al., 2020; Poehner, 2012; Stallings & Tascione, 1996) and students' proof writing can benefit from proof validation tasks (Powers et al., 2010) and self-explanation practice (Hodds et al., 2014). Further, students who use self-monitoring techniques have higher self-efficacy and self-assessment accuracy (Andrade & Valtcheva, 2009; Poehner, 2012; Stallings & Tascione, 1996). However, in some cases where students feel as though a grade is emphasized in the self-assessment, students may evaluate their work based on effort or focus on what may give them partial credit on an assignment (Kemp et al., 2020; Stalling & Tascione, 1996). Notably, in Kemp et al., (2020), the researchers describe a shift in how the participant uses her own criteria as she is evaluating her responses versus a peer's responses to proof comprehension questions.

These authors suggest future research can investigate how proof comprehension assessments for students influence their criteria in self-evaluation and understanding of proof writing. Beumann and Wegner (2018) investigates self-assessment in functional and real analysis on homework related to proofs. The authors present evidence that students can benefit from self-assessment tasks depending on several factors, and they note positive outcomes for self-assessment in small group tasks where the group can achieve full credit by hard work. They call for more research on and sophisticated implementations of self-assessment in higher mathematics education.

Beumann and Wegner (2018) advocate for self-assessment in that it should allow students to complete meta-tasks that are not marked by an instructor and help generate appropriate beliefs about mathematics, as self-assessment reinforces that checking the validity of a proof requires careful work and may depend on personal taste (i.e., the strategy used or amount of details given). Research shows students are better able to self-assess when given criteria on which to base their performance (Andrade & Valtcheva, 2009; Poehner, 2012), but Weber (2012) finds that mathematicians can greatly disagree on the validity of even simple proofs. When comparing mathematicians' and undergraduate students' beliefs about mathematical proofs, Lew and Mejía-Ramos (2019) report the following trends: (a) mathematicians believe proof language should obey conventions of academic language while students often either do not know these conventions apply to proof writing or are unaware they exist at all; (b) students face obstacles in understanding how mathematicians introduce objects in proofs; (c) students may not be aware of how important it is to focus on the context of a proof to decide how formal it should be. Since the criteria being used to evaluate a student proof can vary depending on who is evaluating the proof (e.g., an instructor, a peer) and what the purpose of the evaluation is (e.g., for effort, an assessment, peer-evaluation), it makes sense that students may not be aware of or understand the criteria under which they are being evaluated or may face obstacles in self-assessing and monitoring their progress (Stallings & Tascione, 1996; Poehner, 2012).

Based on the researchers' experiences and results from relevant research, our overall project seeks to investigate in what ways self-assessment through the lens of proof comprehension can improve proof construction and evaluation skills of undergraduate students. Specifically, we proceed with the assumption that self-assessment in proof writing necessarily requires proof comprehension skills (as one is reading their own proof), proof evaluation skills (as they are deciding if what they have written is logical and "complete"), and an awareness of what criteria, if any, is being used to evaluate their proof. This report provides preliminary results on the following research questions after a first phase of data collection investigating the perspectives of mathematics faculty on proof self-assessment: (a) What strategies and/or criteria do mathematicians use to self-assess proofs as they write them? (b) In what ways can proof comprehension assessment strategies be used in undergraduate self-assessment of proof writing?

Theoretical Framework

We focus in this preliminary report on the proof comprehension and assessment models presented in Yang and Lin (2008) and Mejía-Ramos et al. (2012). Yang and Lin (2008) provide a model of proof comprehension for proofs in high school geometry. The researchers use mathematicians' and mathematics teachers' perspectives to develop facets of reading comprehension in geometry proof. The authors emphasize that when students learn mathematical proof, there is often a gap between conjecturing and formally verifying, and their model aims to overcome this gap or eliminate limitations in proof writing. Mejía-Ramos et al. (2012) presents an assessment model for proof comprehension in undergraduate mathematics, which builds off of Yang and Lin's (2008) and Conradie and Frith's (2000) models, as well as trends found from

interviews with mathematicians and other mathematics education literature. They provide a multidimensional model with seven types of proof comprehension assessment, divided by *local* and *holistic* types of assessment. The local types are *meaning of terms and statements, logical status of statements and proof framework,* and *justification of claims.* The holistic types are *summarizing via high-level ideas, identifying the modular structure, transferring the general ideas or methods to another context,* and *illustrating with examples*. The authors emphasize there may be interesting relationships between these and call for further investigation into these relationships. For this report, we present results from adapting the Mejía-Ramos et al. (2012) model to analyze mathematics faculty responses to a survey in which they described their own approaches to self-assessment in proof writing. As this is a preliminary report, the ultimate goal of the results from this survey is to design a tool faculty can provide to students to help prompt the use of proof comprehension in self-assessment and to improve overall proof writing skills. We present and describe this adapted model in the Results section and provide examples of prompts meant to provoke these types of self-assessment via proof comprehension strategies.

Methodology

Several studies about proof comprehension and self-assessment emphasize the importance of the collaboration of mathematicians and mathematics educators, as mathematicians contribute an expertise in proof reading, an understanding of what is reasonable for students to understand, and they are more likely to embrace research results when they are actively involved (Cooley et al., 2024; Mejía-Ramos et al., 2012; Yang & Lin, 2008). Our current project takes this perspective, as we surveyed tenure-track and tenured mathematics faculty at a Hispanic Serving Institution (HSI) who regularly teach or have recently taught a mathematics course involving proof writing. We note the researchers on this project include a Hispanic male undergraduate student and a white female faculty member. The Institutional Review Board (IRB)-approved online survey was conducted anonymously, and out of the 14 faculty members recruited via email in the department at the institution, seven volunteered to participate. The 22 item survey (20 of which were open-ended) included two mathematical statements for faculty to consider: (a) For $a, b \in \mathbb{R}$, prove that $|a + b| \leq |a| + |b|$, and (b) For all integers a, b, c, if $a, b,$ and c are odd, then the equation $ax^2 + bx + c = 0$ has no solution that is a rational number. Faculty were asked questions about their own proof writing approaches for these statements, proof self-assessment strategies, other reflection questions, and then were prompted to assess a sample student proof for each of the statements. The data collected from the survey was analyzed first using open-coding (Richards & Hemphill, 2018) techniques in several passes by the researchers, and these codes were negotiated throughout. Trends emerged regarding ways that mathematicians approach self-assessment in their proof writing and the criteria faculty use for themselves and their students to evaluate the success of a proof. Many of these codes and subcodes resembled the model presented in Mejía-Ramos et al. (2012), and so we re-analyzed our data through this lens. For this report, we focus on Faculty A's responses to two open-ended questions from the survey that prompted participants to describe their self-assessment strategies and evaluation criteria in proof writing. We are using the results from this data as guidance to design a tool students may use to help with self-assessment of their proof writing. We plan to eventually interview students while using this tool to assess its effectiveness.

Results

Below, we provide the types of assessment presented in Mejía-Ramos et al. (2012), adapted for self-assessment as it was used in our analysis. We note that 1-3 below relate to *local* and 4-6

relate to *holistic* types of self-assessment (we omit the adaptation of the seventh type of assessment from Mejía-Ramos et al., (2012), *illustrating with examples*, as we found this code came up primarily in relation to constructing a proof rather than self-assessment).

1. *Use of terms and statements* - To assess clarity and use of key terms and statements, including explanations of these
2. *Logical status of statements and proof framework* - To assess validity of statements in the proof and the logical relationship between them and the given proposition
3. *Justification of claims* - To assess comprehensibility of the chaining logic in the proof and use of other assertions and assumptions
4. *Presentation via high-level ideas* - To assess presentation and communication of the main ideas of the proof and the overarching approach
5. *Logical flow with the modular structure* - To assess the logical flow and proof structure in terms of its main components/modules and the logical relationship between them
6. *Other Contexts and Criteria* - To assess "completeness" by comparing and adapting ideas from other proving tasks to evaluate the methods or procedures of current proof

In Table 1 below, we provide Faculty A's responses to two open-ended questions about both statements on the survey (listed in Methodology). Note as the survey was conducted anonymously, we do not have demographic information of Faculty A. We select this participant's responses for this preliminary report because they are thoughtful, representative of the trends found among the seven surveyed faculty, and they illustrate each of the self-assessment types in our adapted model. In this table, we utilize the numbering system above to connect each statement made by the faculty member to these self-assessment types. We describe how these results can be leveraged pedagogically in the Discussion and Conclusion section.

Table 1. Faculty A's responses to selected open-ended questions from survey.

Questions (open ended)	Responses from Faculty A with superscripts connected to adapted model
How did you assess your own proof as you were writing/sketching it? (What things do you look for in your own proof to decide if it is "going in the right direction"?)	Statement (a): [Can I justify the next line given the previous lines?][3] [Am I making unwarranted assumptions (i.e., if pressed could I explain my reasoning to a skeptic?)?][1,2] [Will I understand and agree with my own proof later?][4] Statement (b): [Are there specific details I can leverage or latch on to or are my givens still too vague.][1,3] [What can I do to get a handle on them?][3] [Can I simplify this to look like a previous or easier problem?][6]
Once you feel you are done writing a proof of this statement, how do you assess it for correctness? What things do you look for in your own proof to decide if it is valid?	Statement (a): [For a proof by cases like this one: Are my cases exhaustive?][5] [Am I making too many assumptions?][2] [Can I follow it from one line to the next using sound and valid logic?][3] [Would I be able to explain myself rigorously to another (vs. just making a persuasive argument)?][4] Statement (b): [Logical flow][3,5], [validity][1,2,4], [proof structure (in this case, exhaustive cases and good proof by contradiction structure).][5,6]

When specifically asked about the criteria that Faculty A would use to assess their own proof work, they said, "Validity, soundness, clarity, comprehensibility. Depending on use case, I might add explanatory power if it's meant to demonstrate reasons behind a mathematical principle, or

brevity if searching for 'elegant' proofs." This is consistent with Beumann and Wegner (2018) in that self-assessment should include a consideration for "personal taste," depending on the writer and the intended reader. For example of analysis, when Faculty A said, "Are there specific details I can leverage or latch on to or are my givens still too vague[?]," we interpret this as the faculty member evaluating if they have leveraged all of the given conditions appropriately in their proof, specifically to improve clarity. Thus, this question is classified as self-assessment types 1 and 3 and is focusing on the faculty's criterion of "clarity".

Discussion and Conclusion

Recall our research questions regarding mathematicians' proof self-assessment strategies and ways to incorporate proof comprehension in self-assessment. Using the Mejía-Ramos et al. (2012) model adapted for proof self-assessment, we analyzed seven faculty responses to open-ended survey questions about their self-assessment process and criteria. We used these responses and the model to guide the development of a draft of a self-assessment tool using proof comprehension strategies. Below, we provide examples of prompts that target these self-assessment types that students may use as a tool in proof validation and self-evaluation.

1. *Use of terms and statements* - Did I consider/address all conditions in the proposition? Did I use each concept from the proposition with the appropriate assumptions?
2. *Logical status of statements and proof framework* - What is the purpose of each statement? What justifies making each assumption? How does each statement contribute to the proof framework I am using?
3. *Justification of claims* - Can I justify this line given the previous lines in my proof? Am I leaving out any information that is necessary for a later implication?
4. *Presentation via high-level ideas* - Conceptually, what is the "big idea" of my proof and is it communicated clearly? Is the overarching approach of my proof sound?
5. *Logical flow with the modular structure* - What are the main modules or components of my proof? What is the purpose of each of these components? Do the components work together logically as intended for the type of proof being used?
6. *Other Contexts and Criteria* - Does my proof resemble a familiar proof of a similar statement (in structure)? Does this proof fit the relevant criteria for its intended use or evaluation? Could the intended reader understand and follow my proof?

We suggest the use of these prompts in undergraduate proof writing courses as an activity to encourage self-assessment and opportunities for self-reflection on proof writing. We believe these may improve student self-efficacy in proof writing and communication about evaluation criteria. Future research could investigate the use of this tool in classrooms, specifically in small group proving tasks, as Beumann and Wegner (2018) suggest. We are still refining this tool through deeper analysis of our faculty data, including the consideration of faculty criteria when evaluating student proofs. Our main goal is to assist students with the proof self-assessment process through the lens of proof comprehension. As part of our larger project, we plan to collect data from student interviews using the self-assessment tool to investigate its possible influence on student self-efficacy and proof writing skills. Obtaining other perspectives from additional undergraduate mathematics faculty would also help in improving this adapted model and tool.

Questions for the Audience

1. What self-assessment strategies or activities have you used in proof writing and what obstacles do you feel your students face (or would face) in this self-assessment?
2. What suggestions do you have to improve our adapted model and prompts?

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Inquiring About What? Student Questions in Inquiry-Oriented Abstract Algebra

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While extensive research has been conducted on curriculum development, teaching strategies, and student thinking in inquiry-oriented instruction, there is little focus on the nature of students' inquiry as it relates to question-posing; we know almost nothing about how and why students ask questions in these courses. Since inquiry-oriented mathematics teaching aims to shift mathematical authority to be shared between teacher and students, we wonder about the role of student inquiry in this shift. We present preliminary findings from an inductive analysis of students' questions and the mathematical authority relations that arise during classroom interactions where students ask questions in an inquiry-oriented undergraduate abstract algebra class.

Keywords: Questions, abstract algebra, authority, assessment

The "inquiry" in inquiry-oriented instruction (IOI) has a two-fold conceptualization in research: teachers inquire into students thinking while students inquire into advanced mathematics (Kuster et al., 2017). While extensive research has been conducted on curriculum development, teaching strategies, and student thinking in IOI, there is little focus on the nature of students' inquiry itself; we know almost nothing about how and why students ask questions in these courses. If one of the goals of inquiry-oriented mathematics teaching is to shift mathematical authority to be shared between teacher and students, what is the role of student inquiry in this shift? Further, if asking questions is a predominant mode of engagement for students across undergraduate mathematics courses, what is the mathematical nature of this activity? We present preliminary findings about students' questions and the mathematical authority relations that arise during classroom interactions involving student questions in undergraduate abstract algebra.

Questioning in the mathematics classroom has primarily been explored at the K-12 level, with just one notable study concerning undergraduate mathematics. Paoletti et al. (2018) explored the nature of 11 lecturers' questioning in advanced mathematics courses, finding that students were most often asked to recall facts or produce the next step in a proof, but questions requiring other kinds of mathematical activity were infrequently asked. These results were consistent with the findings of K-12 literature on teacher questions (Kawanaka & Stigler, 1999). K-12 research has also explored how teacher questions can be posed to impact student thinking (Martino & Maher, 1999; Van Zee & Minstrell, 1997), which questions promote students' mathematical development (Kawanaka & Stigler, 1999; Melhuish et al., 2024), and how pre-service teachers develop questioning practices (Gaspard & Gainsburg, 2020; Moyer & Milewicz, 2002; Nicol, 1998). The how and why of student questioning in math class is scant in both K-12 and undergraduate mathematics classrooms. Outside of the mathematics education literature, Van Zee et al. (2001) found that students posed questions in science class when they were invited to do so, during conversations in familiar contexts, or in discourse environments that prioritized understanding each other's thinking (p. 182-183). While Van Zee et al. (2001)

focused on the conditions that may lead to students' questions, our analysis explores the role of students' questions in class discourse.

Theoretical Framing

On the surface, the authority relation when students ask questions in math class seems obvious: students may simply be deferring to the mathematical expertise of the teacher to provide an answer. However, the framework described in this section proposes a finer grain for parsing this relation to describe the modes of authority in which students are engaged during questioning activities. Authority is often conceptualized in terms of perceived status or positionality of actors within a situation, both colloquially and in the literature (e.g., Gerson & Bateman, 2010; Otten et al., 2017). In contrast, we emphasize the actions of the actors in the situation, thus deemphasizing traditional authority roles. Like Hicks et al., (2023) we see mathematical authority as "rooted in the activities within a classroom that contribute directly to the mathematics, as well as who gets to partake in those activities" (p. 826). Bishop et al. (2022) and Hicks et al. (2024) identified the following modes through which teachers or students can take up mathematical authority. The primary contributors of the main mathematical idea under consideration are in authority for authorship. Those who are publicly communicating or extending ideas through writing or speaking are in authority for animation. Finally, participants are in authority for assessment when they are evaluating contributions made in service of an authored idea.

In this report, we frame the activity of posing questions as an authoritative action in order to investigate the authority relations that arise during such interactions. Further, we explore the conditions under which the activity of questioning transitions from being an indicator of student authority for animation, to being an indicator of student authority for authorship, or even assessment. In viewing questioning as a mathematical activity through which members of a classroom community negotiate authority, we expand both the notion of mathematical authority-in-activity (see Bishop et al., 2022) and our scientific community's understanding of questioning in the advanced mathematics classroom.

Methods

This study arose as part of an ongoing project exploring the activity of authoritative assessment in a semester-long inquiry-oriented abstract algebra class, using video data collected by the fourth author. While developing this theoretical construct, we identified several instances of student questioning which caused debate as to whether the utterance made by the student was authoring or assessing an idea.

In line with the AAA framework, we analyzed segments, stretches of classroom activity possessing consistent mathematical focus and participatory structure. We focused specifically on segments in which students' activity included posing questions to develop inductively a theory of questioning as an authoritative activity. We identified a question as any utterance that requested information of some kind. To determine the influence of students' questioning on the overall authority relations within a segment, we attended to not only the mathematical content of their questioning but also the source of that content. We asked, "Whose idea is the subject of the question? What mathematical contribution does it make? Does the contribution become the focus of classroom discourse?" Using the method of constant comparison (Corbin & Strauss, 2015) to develop and refine our definitions, we first made the distinction between questioning activity that contributes new mathematics to the discourse and that which does not. For instance, we differentiated questions regarding a previously spoken idea, uttered by the teacher or another

classmate from those about a student's own ideas. This distinction between questions that clarify and those that contribute does not value one type over the other, but rather foregrounds the public mathematical activity involved so that the influence of these interactions on the classroom authority relation can begin to be understood. Within these two types, distinctions arose in the data that warranted further development into two subcategories for each type, dependent on the scope of the mathematical activity involved. We develop our classification below.

Findings

In this section, we present preliminary findings in the form of four types of questioning that arose throughout the semester, describing how students' questioning garnered them authority for authorship, animation, and assessment. Figure 1 summarizes our classification.

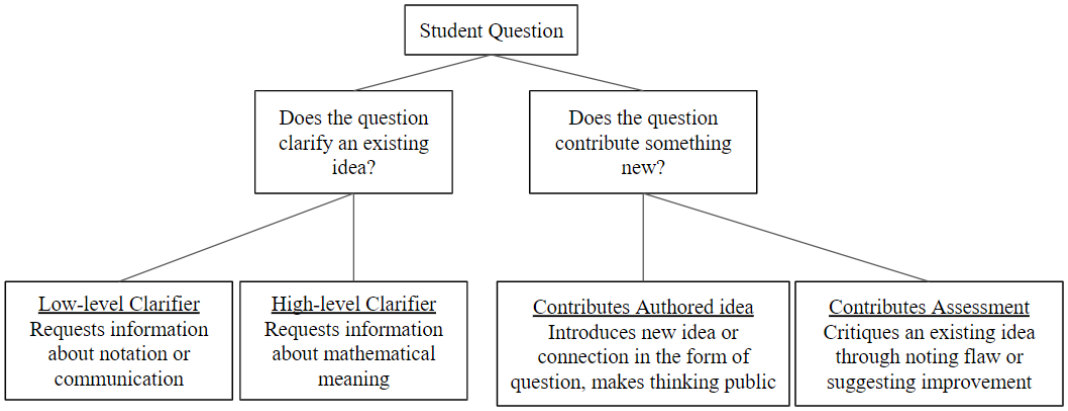


Figure 1. Classification of Student Questions as authoritative activities.

Clarifying Questions

Students regularly asked questions which aimed to clarify an element of the classroom discourse. We saw a distinction between questions that clarified notation and questions that clarified mathematical content. In the first type, which we designated low-level clarifiers, the question asked serves to clarify the visual and audible communication of mathematics, rather than the creation of it. These questions attend to notation or vocabulary within the class discourse and generally are not mathematically substantial. For example, when the teacher had finished presenting a proof, a student asked, "Is that the letter U or V used to represent that set?" Questions like this are focused on an aspect of communication rather than the mathematical content itself. While we see it as likely that this student was engaging in meaningful sense-making of the teacher's proof during this interaction, such activity was not publicly available and so did not contribute to the authority relation of the classroom beyond simple animation, i.e., the student's utterance garnered authority to animate an existing idea through speaking.

On the other hand, we also saw students ask clarifying questions that attended to the meaning of the mathematical content at hand. These questions, which we called high-level clarifiers, made students' reasoning about the mathematics public, usually by alluding to a new idea or connection. For example, while the class was investigating why a non-trivial homomorphism from the group of rotations of the square into the integers (Z) could not be constructed, the following interaction occurred:

Ella: Could you do equivalency classes instead?

Teacher: The problem is that I'm mapping to Z.

Ella: Not Z mod 4?

Teacher: Yeah, right? So I can't map to equivalence classes because the elements here aren't equivalence classes.

Ella's question suggests attention to the operation-preserving property of a group homomorphism and an attempt to rectify the issue. While the utterance was posed as a clarifying question about whether the codomain can be changed to preserve the operation between groups, the question also served as a vehicle for an idea. We saw activities like this as being more mathematically substantial than a low-level clarifier because they involve public student reasoning in ways that questions aimed to clarify simple communication often do not. However, while high-level clarifiers contained new ideas or perspectives, these did not become the focal idea of the conversation in a way consistent with authorship authority. Therefore, for both types of clarifying questions, students had authority for animating the mathematics through speaking.

Contributing Questions

In some cases, students posed questions that indicated an existing idea or belief and requested the teacher's judgement to verify. When this happened, we saw the student's question as contributing an authored idea. To exemplify this, we present the following exchange where Levi and Cammie ask about another student's solution on the board.

Levi: Does it hold that like, we don't have to check any elements in H? So if you checked the coset {a, b, d} to make sure that the left and right are equal, we don't have to do that because H is a group so any left- or right-product of d and H is gonna be in there?

Teacher: Yeah, if H is a subgroup, then I know it's closed, so I don't have to check any of the things in there when I'm checking my left- and right-cosets. Yeah?

Cammie: So for cH you could also use eH or fH?

Teacher: Yes, I could use any name I wanted for this.

In the course of asking his question, Levi introduces the idea that the closure property of the subgroup H simplifies the process of constructing cosets and explains his reasoning. The novelty of Levi's idea together with the public justification for the thinking within the question earned credit for authorship because the question was substantive enough to become the focal mathematical idea of the segment.

Although Cammie's question displays some reasoning about the choice of coset representatives, we see this question as a high-level clarifier rather than a contributing question because she appeals to an idea authored in a previous discussion by the teacher. Observe that Levi's question authored mathematics while Cammie's question animated existing mathematics through speaking. We see similarity between Cammie's question in this excerpt and Ella's high-level clarifier in the last section because neither student elaborates on nor justifies their thinking, though both clearly demonstrate sense-making of the mathematics at hand.

Another kind of contribution that arose through students' questioning activity came in the form of authoritative assessment. Questions that *contributed an assessment* typically contained a critique of an authored idea through noting a flaw or suggesting an improvement. In the following example, a student's question about a proof the teacher has presented does both.

Justin: Why do we need to do eH? Why can't we just do H times gH?

Teacher: I could've done H●gH.

Justin: Right.

Teacher: Yeah. Because H is the identity coset, it's the same name as eH. And I didn't use that e for anything. I was trying to highlight for you guys that this was the identity.

Consistent with our framing of assessment, we noted that the teacher's response to Justin's question defended her choice of notation when she said, "I was trying to highlight for you guys that this was the identity," indicating that the question was playing the role of a public critique within the discussion.

Implicit within Justin's question is an attention to the technique of proof-writing, with some implicit regard to whether it is more appropriate to express the idea of multiplying by the identity coset using fewer symbols. Viewing proof-writing as a cultural adaptation to the discourse of mathematicians provides a potential reason for why Justin would phrase this critique as a question; perhaps he experiences some uncertainty about whether mathematicians would find his expression more appropriate. As the teacher points out, his technically-more-concise statement has the potential to obscure for someone reading the proof that the coset used is in fact the identity. This discussion, though subtle, arises from Justin's critique and contributes something of mathematical importance to the discourse, so we see his question as a way of taking up the authority to assess the teacher's mathematics.

Discussion

In our preliminary work, we have seen how student questioning activity may move beyond authority for animation to become authority for authoring or assessing mathematics. Rather than treating all questions as students appealing to the authority of the teacher for authorship or assessment, we attend to the public nature of students' questions and the mathematical activity involved in formulating and communicating them to parse ways that students' questioning creates an authority relation. In viewing questioning as an authoritative mathematical activity, we expand the scope of possible outcomes in interactions when students question and explore the boundaries between animation and authorship.

We distinguished Ella's high-level clarifying question from Levi's contributing question in our analysis because Levi's question became the mathematical focus of the segment while Ella's did not, although both involved the students bringing their own mathematical ideas into the class discourse. In this example, the difference between a high-level clarifier and a contributing question lies in two places: the mathematical scope of the segment and the teacher's response to the question, how it was taken up or developed. Levi's question occurred in a segment where the classroom activity was narrowly focused on students asking questions after the presentation of the teacher's proof, so the introduction and justification of a new idea garnered Levi authority for authorship. In contrast, Ella's question occurred during a rich discussion of a particular example, so her idea had less influence in the context of the broader conversation. Similarly, the teacher's respective responses to Ella and Levi influenced the way the authority relation in each segment was created. The teacher validated Levi's thinking and formalized the language used, indicating that Levi was leveraging the properties of a subgroup rather than a group. In response to Ella, the teacher pointed out a flaw in Ella's suggestion. In both cases, the teacher took up authority to assess the students' ideas. We wonder whether this is consistently the case: does the teacher's (or other students') response determine the authoritative nature of students' questions?

Questioning is not the only activity during which the authority relation can shift from animating to authoring or assessing mathematics. While we looked at questioning in this preliminary analysis, we wonder about the circumstances required for such a shift when students are engaged in other classroom mathematical activities like validating proofs, writing mathematics publicly on the board, or working together in small groups. Knowing where this boundary lies in these varying contexts may further illuminate how authority relations are formed across activities.

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Converging Paths: A Survey of Calculus Students' Course Experiences

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This preliminary study reports on a brief survey of students' course experiences in introductory university calculus. In particular, we compared the responses of students who took precalculus and calculus at the same institution with those who took precalculus in high school or other settings. Findings highlight significant differences in students' reported calculus experiences between the two groups of interest, which indicates areas for improved alignment of course structures in undergraduate precalculus and calculus, especially around academic support. Building on this initial study, the data are being used to inform efforts to improve instruction and curriculum of both precalculus and calculus courses at the research site, starting with a new pilot calculus course in Spring 2025.

Keywords: calculus, precalculus, affective experiences

Calculus is widely recognized as a challenging course in the undergraduate curriculum that often serves as a barrier for students pursuing STEM majors (e.g., Steen, 1988; Chen & Soldner, 2013). The combinations of barriers to STEM achievement are formidable, with only 35% of high school seniors who intended to pursue a STEM major earning an undergraduate STEM degree (Seymore & Hewitt, 1997). Yet, despite longtime efforts to improve retainment of students in the STEM pipeline, undergraduate programs continue to struggle with attrition. For example, Chen and Soldner (2013) found that 48% of students pursuing bachelor's degrees and 69% of those pursuing associate degrees in STEM switched to a non-STEM field or left college without a degree.

STEM attrition is particularly pronounced among students from historically marginalized and underrepresented groups, including students identifying as ethnic minorities and women. Treisman (1992) noted that ethnic minority students often face higher barriers in STEM courses, while Ellis and colleagues (2016) reported that women are one and a half times more likely to leave the STEM pipeline after experiencing Calculus I than their male peers. Additionally, Chen and Soldner (2013) found that students attending less selective colleges were more likely to leave STEM fields compared to their peers at highly selective institutions. Increasingly, undergraduate educators have focused on the role of early coursework and institutional support for high quality resources to support student success in calculus and precalculus courses.

Calls for improved student outcomes in calculus have led to many small and large reforms, including curriculum and assessment overhauls to focus more on student engagement and essential understandings, and the integration of innovative teaching methods, often including technology-enhancements, inquiry-based activities, and adaptive learning systems (Bressoud et al. 2015; Rassmussen et. al. 2019). The good news is that research indicates that the instructional methods used in calculus courses can have a significant impact on student outcomes. For example, a recent landmark randomized trial by Kramer and colleagues (2023) demonstrated that active learning techniques that involve students more directly in the learning process results in improved grades and a deeper student understanding of calculus concepts. Similarly, Freeman

and colleagues (2014) observed positive outcomes of active learning approaches in courses in other STEM disciplines. Yet, despite these advancements, traditional lecture-based instruction remains the primary mode of instruction in many undergraduate calculus classes (Rasmussen et al., 2019).

Sonnert and Sadler (2015) observed significant declines in students' attitudes and desire to continue in mathematics after completing an introductory college calculus course. While taking and excelling in advanced math courses in high school had a strong positive effect on students, completing a precalculus or calculus course in college did not show the same impact, highlighting a disconnect in the transition from high school to college mathematics. Students' experiences in college calculus courses seem to play a crucial role in declining affect (Sonnet & Sandler, 2015) and persistence in the subject (Rasmussen & Ellis, 2013). Notably, Rasmussen and Ellis (2013) reported that 31.4% of students who initially intended to progress from Calculus I to Calculus II, but chose not to do so, cited their experiences in Calculus I as a major factor in their decision. Similarly, Ellis and colleagues (2014) found that students' perceptions of their calculus courses, including the pedagogies they experienced, played a critical role in their decision to (dis)continue studying mathematics. Unfortunately, the characteristics of calculus courses that have been identified as most strongly correlated with maintaining confidence, enjoyment of mathematics, and desire to continue in STEM fields–such as slower pacing, smaller class sizes, more qualified instructors, and more student-centered pedagogies–are more prevalent in high school classrooms than in college settings (Bressoud, 2015).

Though calculus is a central junction in the STEM pipeline, it is necessary to investigate the (dis)continuous experience that students face while both preparing for, and enrolling in, calculus courses. Existing literature that has examined curricular and pedagogical aspects of precalculus to calculus transitions (e.g., Apkarian et al. 2018; Voigt et al. 2019; Wade et al. 2023) highlights a lack of consistency. However, there is a gap in the literature emphasizing the affective experiences of students as they navigate the precalculus-calculus corridor of the STEM pipeline.

In this preliminary report, we investigated student-reported survey data on their experiences in a Calculus I class and how it relates to their other class experiences. We pay special attention to the group of students who completed precalculus courses prior to enrolling at the university research site compared to those students who completed precalculus at the same university in which they took calculus. We hope results of the study can spark conversation about the importance of alignment between course structures in college-level precalculus and calculus courses with those of high school precalculus settings. The guiding question is, "**How can students' reported experiences in university calculus classes inform better alignment of curriculum and instruction of precalculus and calculus courses?**"

Methods

The institutional setting is important to this study. Mountain State University (pseudonym) is a large public university in the western United States. Students typically enroll shortly after graduating from high school in the local or nearby state. Admission is minimally selective (~85% acceptance rate). The undergraduate student body is typical of many mid-to-large U.S. public research universities, including many first generation students (~35%) and students eligible for federal financial aid (~50%). Calculus classes are typically taught by full-time lecturers and adjunct faculty who specialize in lower-level undergraduate mathematics teaching. The calculus curriculum has many of the features of reform calculus programs, including technology-assisted assessment, frequent low stakes individual and small-group assessments, and a focus on applications. Most applied tasks center around physical contexts typical of

introductory engineering and physics courses. Course coordination is moderate, with a schedule of topics, homework, and exams provided on an opt-in basis among instructors, but daily instructional methods vary. Success rates, as measured by the percentage of registered students who earn a letter grade of A, B, or C, have improved in recent years, averaging about 70%, with considerable variability across instructors.

A recent institutional emphasis on first year student experience and retention highlighted calculus as one of a small number of STEM courses targeted for improved equity in student outcomes, and a task force within the department was established to address potential improvements in the course. This study is part of those efforts.

Precalculus courses at MSU are designed for students who are admitted to the university without sufficient preparation for calculus, with very robust systems for supporting student success. The courses are taught by different faculty in a separate unit from the mathematics department using a hybrid of flipped-classroom and emporium formats centered around building students' mathematics self-efficacy (Bandura, 1997). Class sizes are similar to calculus (about 40 students each), but classes are much more coordinated. In all sections, students spend several hours per week in an online program that helps reinforce procedural knowledge, while class time is a mix of small group activities and individualized work time for students to reinforce conceptual knowledge with their instructor available to answer questions. The model has resulted in high success rates (consistently around 80%), and MSU has received two recent national awards highlighting the positive impacts of precalculus systems on student success and retention outcomes (APLU, 2023; NCCU, 2023).

The primary data source for this observational study was an abridged version of the end-of-course student survey from the large MAA National Study of College Calculus (Bressoud et al., 2015). We distributed the survey by email to all 905 students who registered for calculus at MSU during fall or spring of the 2023-2024 academic year, yielding a sample of N = 190 student responses (response rate = 21%). As a preliminary study focused on informing alignment of precalculus and calculus courses, the primary groups of interest were students who reported completing their precalculus courses at MSU (43%) versus those who reported completing precalculus in all "Other" (57%) settings (high school = 49%, two-year or community college = 6%, others = 2%). We computed comparative descriptive statistics of students' reported calculus course experiences in these two groups, emphasizing indications of similarities and differences.

Results

Overall, students who took precalculus at MSU tended to report lower letter grade attainment in their first calculus course (A = 37%, B = 37%, 19%, D/F/W = 13%) compared to those who took precalculus in Other settings (A = 53%, B = 23%, C = 17%, D/F/W = 7%), χ^2 (3) = 14.2, $p < .01$. Similarly, students who took precalculus at MSU tended to report lower levels of agreement on survey items about math enjoyment (MSU = 71%, Other = 86%), confidence (MSU = 61%, Other = 78%), and preparation for calculus (MSU = 53%, Other = 65%). Details are shown in Figure 1.

In questions about their experience of calculus course quality (see Figure 2), students who took precalculus at MSU tended to report lower agreement with survey items about the course increasing their interest in mathematics (MSU = 43%, Other = 55%), getting the support they needed to succeed (MSU = 54%, Other = 80%), and wishing more courses were taught like their calculus course (MSU = 51%, Other = 64%) In addition, students who took precalculus at MSU reported feeling more frequently "lost and unable to keep up" in calculus (MSU = 42%, Other = 26%), although they reported asking questions (MSU = 47%, Other = 35%) and learning through

peer interaction (MSU = 67%, Other = 65%) with similar distributions in their calculus courses (see Figure 3).

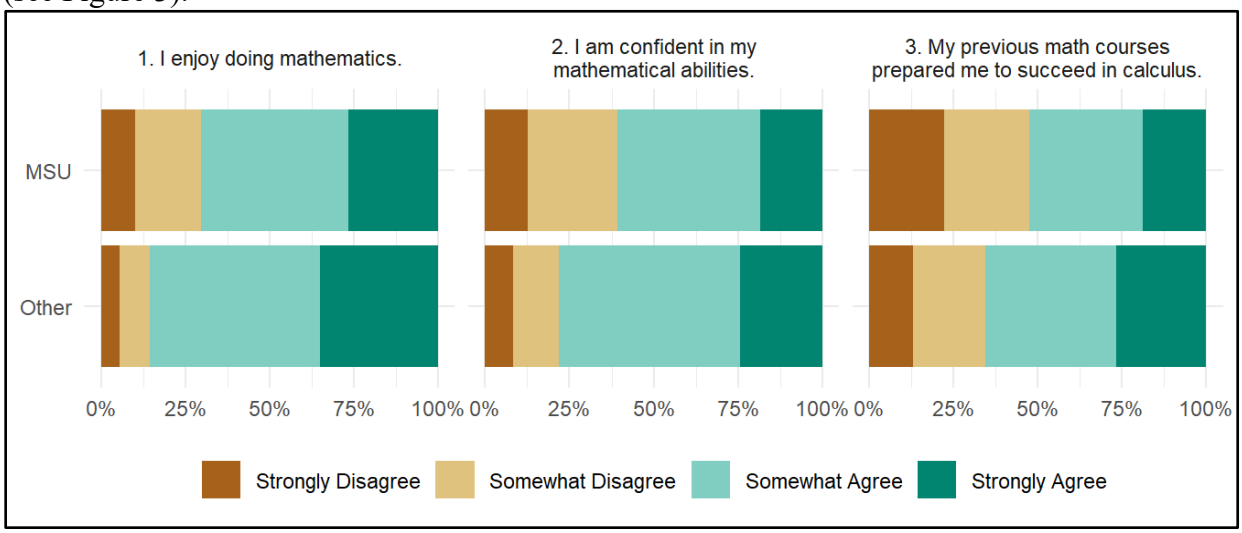


Figure 1. Indicators of calculus students' reported mathematics identity.

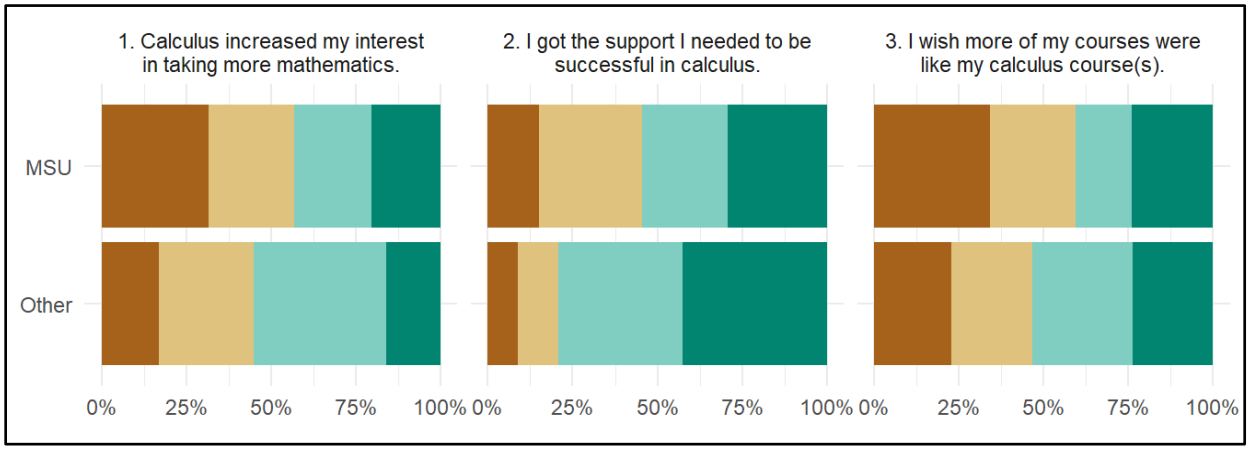


Figure 2. Indicators of calculus students' perceptions of calculus course quality.

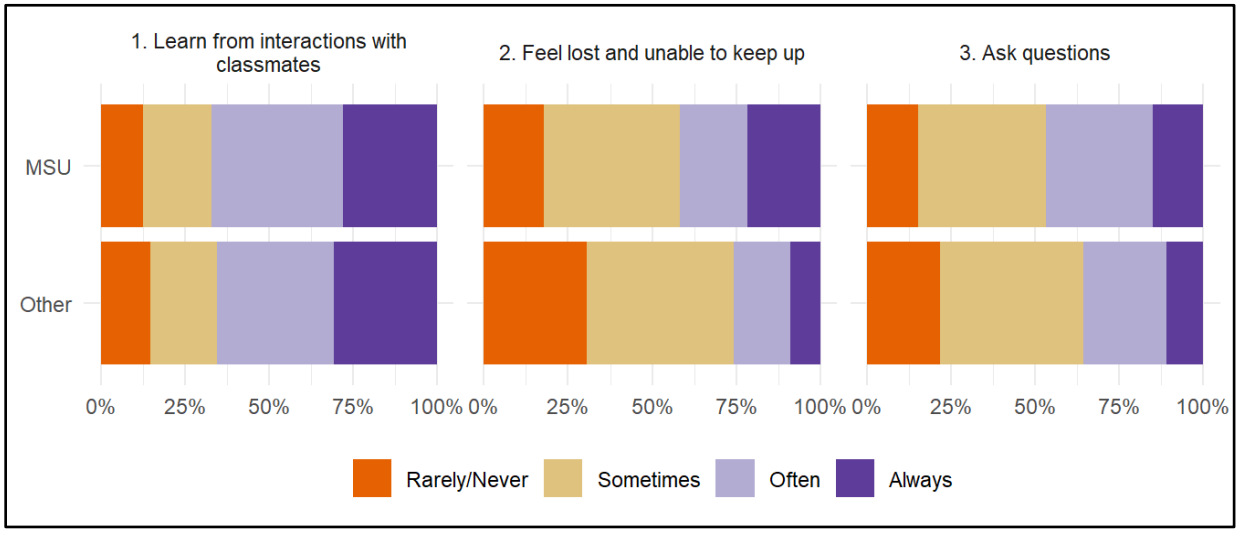


Figure 3. Indicators of calculus students' experiences of calculus instruction.

Discussion

The distributions observed in our survey of students' course experiences suggest substantial differences in the calculus experiences of students who completed precalculus at MSU compared to students who completed precalculus in another setting (typically high school). There is a consistent pattern of students who took precalculus at MSU reporting lower perceptions of instructional quality and support for their learning. This is particularly surprising given that the precalculus course at MSU has been designed to foster student self-efficacy and has garnered national attention for its impacts in supporting student success and retention efforts across the institution.

The results from this study are only observational, thus it is difficult to discern the cause(s) of the observed differences. Nonetheless, the disparities suggest room for improvement in better aligning precalculus and calculus transitions at the university level. More broadly, the results highlight a need for more research into localized, contextual factors which may impact students' affective experiences in calculus and other undergraduate mathematics courses which are taken in sequence at MSU, and all institutions of higher education.

We anticipate that attending to vertical alignment of sequenced courses, especially in terms of instructional methods, assessment strategies, and other course design features could positively affect students' perceptions of course quality. To test this idea, we are currently building a pilot calculus class that aligns with the structure of precalculus classes at MSU to see if there are any differences in student's reported experiences in such a course. This course will launch in the Spring 2025 semester. In addition to survey data, we hope to follow up with focus group interviews with students to better understand their experiences in transitioning from precalculus to calculus. This data will help inform cycles of design research to help improve the pilot calculus class, and other changes to create more consistency in this portion of the STEM pipeline.

Questions for Audience

As a preliminary study, we are particularly interested in the perspectives of other undergraduate mathematics education researchers and university teaching faculty. We want students to experience cohesive, well-designed undergraduate courses, and recognize that the challenges of offering high quality calculus course experiences are widespread. Thus, we welcome input from colleagues about interpreting the survey data, in the context of the scholarship of teaching and learning at the undergraduate level. In particular, we are interested in conversing about the following questions:

1. What changes to calculus might lead to better course experiences for both students who took precalculus at the same university and those who took precalculus in other settings?
2. Which aspects of course design are most important to align across university precalculus and calculus courses?
3. What other aspects of students' calculus course experiences are most important to consider in the context of precalculus?
4. How might we effectively facilitate cross-course collaboration about course design between undergraduate calculus and precalculus instructors? What are the barriers and drivers?

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Use of WeBWorK as an Online Homework System as it Relates to Student Mathematical Identity

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This study explores how WeBWorK, an online homework system, influences student mathematical identity, with a special focus on students at a Native American-serving institution. Mathematical identity refers to beliefs, attitudes, and emotions toward mathematics, impacting motivation and learning. Through an online survey we gathered insight on student perceptions of WeBWorK's role in shaping four identity dimensions: self-efficacy, understanding, confidence, and persistence. Results reveal significant differences in student perceptions compared to prior studies, with survey participants showing lower engagement across these dimensions. Preliminary analysis suggests that WeBWorK may not equally support the mathematical identities of all students, particularly those from minority or first-generation backgrounds. This research contributes to understanding how web-based homework systems affect diverse student populations, providing insights for improving such systems to foster mathematical engagement and identity development.

Keywords: WeBWorK, online homework, mathematical identity

Introduction

In an increasingly technology-heavy world, many institutions of higher education are turning to the use of web-based homework (WBH) to allow for more time to be spent on material outside of class, which may be correlated to increases in student achievement (Cooper et al., 2006**;** Kehoe, 2010). Several types of WBH platforms exist and are used nationwide to supplement or replace traditional homework systems (Kehoe, 2010), including the open-source platform WeBWorK. From the perspective of the instructor, WBH has clear educational benefits, ranging from instantaneous feedback on assignments, an increase in understanding of the material, more consistent time spent studying, and leveling access across general and diverse populations of students (Boudreaux et al., 2022), as well as the economic benefit and objective nature of the system for large class sizes (Wood & Bhute, 2019). Furthermore, use of WBH allows additional time for students to develop a sense of their mathematical identity (Fernandez et al., 2024).

Mathematical identity here refers to a person's beliefs, attitudes, emotions, and dispositions about mathematics and their resulting motivation and approach to learning and using mathematics knowledge (Fernandez et al., 2024; Martin, 2000). There are a myriad of recent studies focused on how WBH affects mathematical identity in STEM minority populations (e.g., girls and women [Fernandez et al., 2024] and race [Nguyen et al., 2006]). Our study contributes to this important effort by looking at how the use of a particular WBH, WeBWorK, affects aspects of students' mathematical identity sorted by demographics including race (with a particular focus on Native American/Alaska Native students) and first-generation status. Our research question is: *How do student perceptions of their engagement with WeBWorK relate to different attributes of their mathematical identity? Moreover, how do these perceptions change with differing student demographics?*

Background

Student interaction with WBH systems has increased and has been particularly well-documented in the past 10 years (Dorko, 2020; Dorko et al., 2023; Hauk & Segalla, 2019; Kehoe, 2010; Serhan, 2019; Valdez & Maderal, 2021). Hauk & Segalla (2005) acknowledge WBH is "only a support tool…and [replaces] the unevenly implemented pedagogical interaction of homework grading with a uniform method of feedback" (p. 233). However, WBH systems are suggested to be at least as effective, if not more effective, than traditional written homework (Boudreaux et al., 2022; Hauk & Segalla, 2005; Wood & Bhute, 2019). Student perceptions of the value of WBH relate to a more positive learning experience in the classroom (Serhan, 2019). Furthermore, perception of the use of WBH is significantly correlated to motivation toward mathematical learning (Valdez & Maderal, 2021). Here, mathematical learning can be placed within the larger context of mathematical identity.

Considering attitudes toward aspects of WBH under the lens of mathematical identity can help us understand how a student may develop individual and group membership within a mathematical community (Cribbs et al., 2015). Mathematical identity is often used as a means for understanding student engagement with the subject (Black et al., 2010). This engagement often translates to growth of a student's conception of self, which has been found to be connected to their choice-making and performance at advanced levels (Mendick, 2006). Some researchers even claim that identity is crucial to the belief that one can be a creative participant in mathematics as a social practice (Boaler & Greeno, 2000).

For our study we will consider four facets of mathematical identity: self-efficacy, learning/understanding, confidence, and persistence. In particular, self-efficacy through a mathematical identity lens refers to a person's internal beliefs about their capabilities to produce levels of performance that allow them to exercise influence over events that affect their lives (Brenner et al., 2017) and their success in a mathematics classroom. Here, we ask how student engagement with WeBWorK relates to different aspects of a student's mathematical identity. As such we are interested in how these aspects relate across demographic axes such as first-generation and minority (with a specific focus on Native American/Alaska Native students) statuses.

Methods

For this study we administered an online survey to students at a public bachelors granting Native American-serving nontribal institution which we've given the pseudonym, Mountain College. The survey focused on student engagement with the web-based homework platform WeBWorK. WeBWorK is an open-source homework system supported by the Mathematical Association of America and the National Science Foundation. Created in 1996, WeBWorK describes itself as an open-source online homework system for STEM courses with an Open Problem Library of over 35,000 homework problems (WeBWorK, n.d.). At the time of the survey, WeBWorK was being used in all math classes at Mountain College from College Algebra, through the calculus sequence, and beyond in courses such as linear algebra and differential equations. WeBWorK had also been used for a number of previous terms.

Mountain College is located in the Southwestern United States serving a diverse student population. The majority of the student body identify in a minority race category (54%), with 27% of the total population as Native American or Alaska Native students. There is also a high percentage of first-generation college students (44%).

Overall the survey consisted of twenty five 5-scale Likert questions and five open responses covering four domains: accessibility, attitudes and engagement, mathematical identity, and

community. For this preliminary report, we will be focusing on the results pertaining to mathematical identity. There were six survey questions aimed at student engagement with WeBWorK and mathematical identity which can be seen in Table 1 below. All questions were asked on a 5-point Likert scale from 1-Strongly Agree to 5-Strongly Disagree. Whenever possible the wording of the questions matched questions used in previous investigations reported in the literature. These questions are identified in Table 1.

Table 1. Survey questions relating WeBWorK to students' mathematical identity

	Survey Question	Dimension of Identity
Q1	Given any WeBWorK problem I could start to answer it, or at least part of it, right away.	Self-efficacy
Q2[a]	Doing my homework using WeBWorK enhanced my learning of mathematics.	Learning/Understanding
Q3	Using WeBWorK has helped me understand material in other courses.	Learning/Understanding
Q4[a]	The use of WeBWorK has helped me develop confidence in doing mathematics.	Confidence
Q5[a]	Solving questions using WeBWorK prepared me for the paper-based class exams in the class.	Confidence
Q6	Doing homework using WeBWorK discouraged me to take more math classes.	Persistence

[a] (Serhan, 2019)

The survey was administered in Spring of 2024 inviting all Mountain College students with experience with WeBWorK at Mountain College to participate. In total we received 109 responses, 97 of which responded to all six of the identity questions. These 97 responses will be analyzed using primarily quantitative methods, specifically through a series of t-tests. Quantitative analysis will include descriptive statistics, comparing our means to results in existing literature, comparing means of individual questions sorted by various demographics, and exploring the correlation within and between the identity-focused questions and other dimensions captured by the survey.

Results

Of the 97 complete responses, 55% of students identified as a minority with 32% identifying as Native American/Alaska Native. Furthermore, nearly 31% (30.9) of the respondents identified as first-generation college students. Therefore, we can say the demographics of our survey participants are representative of the demographics of the student body of Mountain College.

Mountain College Compared to Previous Findings

Comparing the survey results from Mountain College to preexisting results found in literature yielded significant findings. Half of the questions focused on identity were taken from previous studies and their results are listed in Table 2 below.

Table 2. Means for survey questions relating WeBWorK to student mathematical identity

	Survey Question	Previous Findings	Survey Results
Q1	Given any WeBWorK problem I could start to answer it, or at least part of it, right away.	*N/A*	2.3
Q2	Doing my homework using WeBWorK enhanced my learning of mathematics.	*2.4*[a]	2.8**
Q3	Using WeBWorK has helped me understand material in other courses.	*N/A*	3.6
Q4	The use of WeBWorK has helped me develop confidence in doing mathematics.	*2.4*[a]	3.1**
Q5	Solving questions using WeBWorK prepared me for the paper-based class exams in the class.	*2.3*[a]	3.2**
Q6	Doing homework using WeBWorK discouraged me to take more math classes.	*N/A*	2.8

[a](Sehran, 2019) Note: Literature results were recalibrated to accommodate for a reverse-scoring Likert scale
** indicates a p-value less than 0.001

Every question that had a previously reported mean was found to be statistically significant compared to previous findings. For each of the questions a one-sample t-test was run using the mean from the literature. Our analysis found statistical evidence that Mountain College students' average scores not only differed from the previously reported average scores but were lower than all the previous findings. Note that since the Likert scales were presented with 1-Strongly Agree and 5-Strongly Disagree, a higher mean value indicates stronger disagreement with the question statement. A series of one sample t-tests with alpha = 0.05, two-tailed, were used and each difference of means was found to be statistically significant yielding $p<0.001$. These results suggest that at Mountain College WeBWorK may be supporting students' mathematical identity less than at other institutions.

Mountain College Findings by Demographic Group

Table 3 contains results from our survey data. Participant self-reported race is separated into three categories: white, non-minority identifying (White), n = 45; Native American/Alaska Native (NA/AN), n = 31; and non-NA/AN minority (Other), n = 16. Non-responses to the race demographic question (n = 5) were not included in the table or analysis. First-generation status is separated into two categories: (Yes), n = 30; and (No), n = 65. Non-responses to the first-generation demographic question (n = 2) are not included in the table or analysis.

Table 3. Results for survey questions broken out by student demographics

Survey Means by Demographic Groups					
Survey Question	Overall Mean	Race		First-Generation	
Question 1	2.2	White	2.2	Yes	2.2
		NA/AN	2.2	No	2.4
		Other	2.5		
Question 2	2.8	White	2.8	Yes	2.9
		NA/AN	2.7	No	2.8
		Other	2.6		
Question 3	3.6	White	3.6	Yes	3.5
		NA/AN	3.5	No	3.6
		Other	3.4		
Question 4	3.1	White	3.1	Yes	3.1
		NA/AN	3.1	No	3.2
		Other	3.0		
Question 5	3.2	White	3.1	Yes	3.3
		NA/AN	3.2	No	3.2
		Other	3.1		
Question 6	2.8	White	2.9	Yes	2.7
		NA/AN	2.8	No	2.9
		Other	3.1		

A preliminary series of one-way ANOVA tests have been run to explore differences within question responses between demographic groups within questions and thus far no statistical differences have been found. Furthermore, the p-values related to race ranged from 0.323 to 0.785 whereas the p-values related to first generation status ranged from 0.259 to 0.936.

In the presentation we will continue reporting on our analysis including a conversation focused on the correlation between the identity questions and also how the responses to the identity questions correlated with the other dimensions captured in the survey.

Discussion and Implications

In our preliminary analysis, we are noticing that while student responses at Mountain College are statistically different from previous studies, there is very little difference in survey responses based on categories within and across race and first-generation status. This invites questions about considering the data through a different lens. As such, we are considering analyzing the survey results through breaking down responses into cohorts based on both the types of classes taken and initial mathematics course placement. Furthermore, we ask how this free resource might work for development of student mathematical identity.

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Student Conceptualizations of Confidence Interval Attributes

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Interval estimation has become one of the preferred means for communicating the statistical findings of a study. Students, and even experienced professionals, often misinterpret frequentist confidence intervals. As part of the broader study in which this work is situated, we developed an instrument to assess undergraduate students' statistical literacy surrounding confidence intervals. We conducted think-aloud interviews with six students who had recently completed an introductory statistics course. In this work, we qualitatively capture the meanings that students construct about attributes of confidence intervals to inform the results from the instrument. We use grounded theory to model how students conceptualize attributes of confidence intervals and propose evidence of interactions between interrelated ways of thinking. For this paper, we focus on one student's conceptualizations about the confidence level and error attributes of confidence intervals to illustrate our modeling process in detail.

Keywords: confidence intervals, grounded theory, interaction effects, statistics education

Introduction

Hypothesis testing is often presented as a rote multi-step procedure (Horton, 2015; Lock et al., 2017). This training has led many research communities to falsely believe that only statistically significant results are meaningful. The standards adopted by entire disciplines that require statistically significant results for publication perpetuate this belief. As a result, researchers strive to achieve statistical significance, which can lead, and has led, to research malpractice (Collins, 1984; Wasserstein et al., 2019).

Moreover, the outcomes from hypothesis testing are frequently misinterpreted by students and professionals (Hoekstra et al., 2014). Interval estimation has been proposed as a more acceptable alternative since confidence intervals are accessible, readily display the precision of the interval construction method, and are not restricted to a specific null hypothesis (Hoekstra et al., 2014, 2018; Kaplan & Roland, 2022a). However, individuals treat confidence intervals like significance testing when they focus on whether the null value falls in the interval (Wasserstein et al., 2019). As such, it is insufficient for just the *presentation* of the findings to shift from a p-value to a confidence interval; the *interpretation* of the findings must shift as well from that of desiring statistical significance to that of desiring transparency of results. More broadly, there must be a shift from procedural treatment of statistical tools towards building and working with deep, productive meanings and exhibiting quantitative reasoning (Hatfield et al., 2022). Thus, studying individuals' understanding of confidence intervals is imperative.

Literature Review

Individuals, even experienced researchers, grossly misinterpret confidence intervals (Hatfield, 2024; Hoekstra et al., 2014, 2018; Roland & Kaplan, 2022; Saldanha et al., 2024). In fact, even statisticians and authors of introductory statistics textbooks disagree regarding how to

interpret a frequentist confidence interval (Hoekstra et al., 2018; Kaplan & Roland, 2022a). These misinterpretations are worsened by the ambiguity surrounding the word 'confidence' (Kaplan & Roland, 2022a, 2022b). Morey et al. expand upon these misinterpretations of confidence intervals by identifying the most common confidence fallacies: fundamental confidence fallacy, precision fallacy, likelihood fallacy, and fallacy of acceptance. They describe these misinterpretations as fallacies since they are in opposition to Neyman's theory of confidence intervals (Morey et al., 2016). The fundamental confidence fallacy deals with making probability claims about the resulting confidence interval. These claims reference the probability or confidence that any realized interval contains the parameter (Hoekstra et al., 2018; Saldanha et al., 2024). These claims place the confidence in one particular interval, rather than the method where it rightfully belongs (Hoekstra et al., 2014, 2018; Saldanha et al., 2024). The precision fallacy deals with belief about precision based on the width of the confidence interval (Hoekstra et al., 2018; Morey et al., 2016). The precision fallacy manifests itself in the belief that a narrow interval gives more knowledge or suggests that the statistic is a better estimate of the parameter.

The likelihood fallacy is thinking that the confidence interval presents a set of reasonable or plausible values for the parameter. If a value falls inside the interval, it is reasonable, otherwise it is unreasonable (Hoekstra et al., 2018; Morey et al., 2016). Thoughts that align with the likelihood fallacy include assigning probability to individual values or thinking that values in the interval are more likely values for the parameter (Saldanha et al., 2024). When interpreting a confidence interval, individuals naturally want to accept a set of values, which is known as the fallacy of acceptance (Hoekstra et al., 2018; Mayo & Spanos, 2006; Morey et al., 2016). Accepting a value as the true parameter because it falls in the interval is an indicator of the fallacy of acceptance. Other problematic meanings of confidence intervals include believing the interval is a descriptive statistic (Hoekstra et al., 2014), is a set of reasonable values for individual cases (Hatfield, 2024) or for the sample statistic (Kaplan & Roland, 2022b).

After discussing some problematic meanings, we now present a productive meaning of confidence intervals. A person with such meaning would view the process of constructing a confidence interval as entailing randomness, leading to variation in the resulting intervals when the process is repeated. They would also have the ability to conceptualize and articulate the distinction between the procedures for, and results from, obtaining a sample and constructing a confidence interval (Hatfield, 2024). Therefore, in a productive interpretation, the confidence is in the method, not in a particular interval, and there is recognition of uncertainty involved in the process (Saldanha et al., 2024).

There are a few assessments in the statistics education literature to measure understanding of confidence intervals (delMas et al., 2006, 2007). For example, the Comprehensive Assessment of Outcomes in Statistics (CAOS) measures statistical literacy, reasoning and thinking on major topics covered in the introductory statistics curriculum, including confidence intervals (delMas et al., 2007). Conventional practices in assessment development resemble a one-factor-at-a-time design in which each item assesses one idea at a time, independent of the other ideas (see delMas et al., 2007; Sabbag et al., 2015; Ziegler, 2014). From a cognitive perspective, ideas do not necessarily develop in isolation of each other. Rather individuals often coordinate multiple ideas; thus, evaluating the interactions between multiple ideas is crucial (Thompson et al., 2014).

Methods

The work we present here is part of a larger project which investigates students' statistical literacy surrounding confidence intervals after an introductory course. In an on-going feasibility

study, we are developing, validating, and analyzing an instrument to evaluate interactions among attributes of confidence intervals. This paper intends to provide qualitative insight into the instrument by addressing the following research question in support of these broader aims: How do undergraduate students conceptualize, and coordinate their meanings for, confidence level and error in the context of confidence intervals?

Drawing upon the literature (e.g., Hatfield, 2024; Hoekstra et al., 2018; Lock et al., 2017) and our own expertise, we identified several attributes of confidence intervals that are often taught in an introductory course: confidence level, error, sample size, p-value, and method. For this paper, we focus on the first two attributes. We operationalize confidence level as the proportion of times the process of constructing a confidence interval results in an interval that captures the parameter of interest (i.e., success rate) and error as a measure of variability associated with the data and with the sample statistics.

Theoretical Perspective

In this study, we adopt an interpretivist epistemology, in particular, radical constructivism (Stinson, 2020), which we use to guide the formation of models of student conceptualizations about attributes of confidence intervals. We consider knowledge to be constructed by students based on their interactions and experiences, both in life and in the classroom (Stinson, 2020; Thompson, 2008; von Glasersfeld, 1995). A radical constructivist approach acknowledges that students construct different meanings or ways of thinking (Goldin, 2000), reflecting student autonomy.

Interview Protocol

Cognitive interviews are used for a multitude of purposes in mathematics and statistics education (Clement, 2000; Goldin, 2000; Theobold et al., 2023). One benefit is that they are useful for observing students' mathematical/statistical thinking processes and approaches (Case & Jacobbe, 2018; Clement, 2000; Goldin, 2000). In this study, we conducted think-aloud interviews with six students who recently completed an introductory statistics course. We interviewed these students separately, on Zoom, for one hour. We asked students to read and discuss their thinking aloud as they interacted with our items. As needed, we asked follow-up questions. We audio recorded the interviews to obtain detailed transcripts.

Data Analysis

To model how students thought about confidence intervals, we used grounded theory. Grounded theory is a qualitative method for deriving a theory to describe a phenomenon of interest in a manner that reasonably stems from the observed data and has a clearly outlined scope of generalizability (Strauss & Corbin, 1990). We used a generative approach in that we developed our codes/theories based on students' demonstrated thinking about confidence level and error rather than having a priori hypotheses about the meanings students have for these attributes (Clement, 2000). To start, we read the transcripts and flagged sections (sentences, paragraphs) corresponding to each of the five attributes. Starting with the confidence level attribute, we identified codes (concepts) for each section to represent the thinking that students conveyed. We drew upon existing literature and our understanding of students' statements as a basis for our theoretical sensitivity and to inform the open coding process (Strauss & Corbin, 1990). Examples of our codes include: confidence level refers to confidence in the method, fundamental confidence fallacy, confidence level is a success/capture rate, articulates the

normatively accepted relationship between confidence level and interval width. We grouped (categorized) the codes for each student to create higher order descriptions, encompassing multiple concepts (Strauss & Corbin, 1990). We developed a list of defining properties for each category by iteratively revisiting the original sections and making note of instances which supported (or not) the proposed relationship. Elaborating on the nature of this variation enabled us to construct a more nuanced theory describing the phenomenon of interest (Strauss & Corbin, 1990). For this paper, we focus on just one student (whom we call S3) to illustrate in depth how we model students' conceptualizations about the confidence level and error attributes.

Results

Consider the following excerpt from S3 which is part of an exchange on why they chose a provided option that indicated the width of a 90% confidence interval is narrower than that of a 95% confidence interval (for the same sample) because it has a smaller multiplier (critical z-score), leading to a smaller margin of error. This particular excerpt demonstrates S3's meaning for confidence level: "I'm pretty sure we always just kind of grab multipliers off a table so…I don't really remember…I don't know a whole lot about actually calculating it…based on like confidence." From this excerpt, we observe that S3 has a meaning consistent with a procedural acquisition of the multiplier. This excerpt, along with others coded similarly, lead us to believe that S3 has a meaning consistent with a formulaic treatment of confidence intervals. By this we mean that an individual has an understanding of confidence intervals as a formulaic calculation on individual components. Although this meaning allows S3 to articulate the normatively accepted relationship between confidence level and multiplier (as confidence level increases, the multiplier increases in magnitude), this meaning makes it difficult for S3 to determine how intervals with the *same* confidence level, but constructed using different methods, would differ. Moreover, the following excerpt is in response to an item that asks students to explain, if 100 students replicate a given study, how many of the 100 resulting 90% confidence intervals are expected to capture the parameter of interest. This excerpt reveals that S3 has a meaning for confidence level that is consistent with how we operationalized it:

> Alright so the method in this question method in this case is one we're 90% confident in so…therefore, we would *expect*…for every 100 intervals we would. Well, we would simply multiply the number of intervals by the...rate of correct of a…rate of expected correct, of a[n] expected capture…which would be 90.

Here S3 appears to imagine carrying out a process to produce multiple intervals. Further, S3 connects the confidence level to the notion of a success rate (how many intervals capture the true value). Taking the two instances together, S3 appears to coordinate a repeatable calculational process and anticipated success of the results. This thinking supports S3 in articulating the normatively accepted relationship between confidence level and interval width (as confidence level increases, interval width increases) and meanings for confidence.

Consider the following exchange between S3 and the interviewer on a multiple-choice item that demonstrates S3's meaning for error:

> *S3:* Using the empirical rule, the 95% confidence interval for the mean difference in plant heights would take the following form [statistic $\pm$ measure of error expression]. Okay, the empirical rule...in other words 68, 95, 99 point…something.
>
> *Interviewer:* Mhm 7.

S3: Yeah, yeah one standard error...captures about 68%, 2 standard errors captures 95%. So multiplier is 2 should be 2 in this case. So that…(5 second pause)…okay, margin of error. Right. You said margin of error is the…the radius right? That's what I said. So.

Interviewer: Everything to the right of the plus or minus sign.

S3: Yeah, yeah, exactly. So that's the 2…so…so in other words, that would be…that's the result of multiplying the multiplier and the standard error.

From this exchange, we observe that S3 conveys the margin of error as being a distance for the interval, specifically, a radius, and getting the error via a product of components. There is still a formulaic aspect to S3's meaning for error. S3 routinely comes back to the multiplication aspect as well as the Empirical Rule. While S3 can articulate error as a distance quantity, they tend to focus more on the calculations of and with error than articulating any underlying quantitative relationships involving error. Even so, S3 is able to articulate the normatively accepted relationship between standard error and interval width (as standard error decreases, interval width decreases) but lacks sureness in their meaning of error terms apart from the formulas. We note that S3 conveys that the standard error is a property of the data instead of the statistic, and they imagine deviation as referring to differences between the intervals.

When discussing both confidence level *and* error, S3 appears to coordinate their meanings. Consider the following excerpt from S3 which details their reasoning on a multiple-choice item assessing how the standard error and margin of error for a 90% and 95% bootstrap confidence interval (constructed using the same bootstrapped sampling distribution) would compare:

> So standard error is that a property of the? That's I kinda wanna say standard error is a property of the data itself? So I don't think that would differ between the two methods [method of constructing a 90%, 95% confidence interval] but the margin of error *should* differ because that's a matter of how much leeway we give ourselves, how much confidence we're willing to sacrifice.

S3 appears to use their meaning of confidence level as the method's success rate, and their meaning of standard error as a property of the data, to construct a meaning for error that no longer has the distance aspect previously mentioned. Rather, this meaning for error is in-line with the colloquial use of the word error, as in the interval has the property of making mistakes.

Discussion

A consistent meaning that students appear to have is a formulaic treatment of confidence intervals. It would be of interest to study whether a formulaic approach is an efficient way to solve the respective items and/or whether this meaning is influenced by the presentation of the general formula for a confidence interval throughout the instrument. Moreover, S3's conceptualizations of confidence level and error display the interactions between their ways of thinking about these attributes of confidence intervals. This preliminary evidence of interactions among attributes of confidence intervals justifies the need to efficiently estimate these interaction effects and indicates instances of quantitative reasoning (Hatfield et al., 2022) and potentially covariational reasoning (Carlson et al., 2002; Thompson & Carlson, 2017; Zieffler & Garfield, 2009). There is further insight to be gleaned on this front as we have five other students for whom we can model their conceptualizations of the confidence level and error attributes. We also have three additional attributes of confidence intervals (sample size, p-value, method) to analyze. By understanding how students coordinate meanings for attributes related to confidence intervals, instructors can achieve greater clarity on what changes are needed in curriculum or assessment to foster statistical literacy on the topic of confidence intervals.

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When Different Conceptions Hold Together: A Case of a Student's Coordinate Systems and Graph Reasoning

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We report on a qualitative case study of a college algebra student, Jamie, who exhibited multiple conceptions of coordinate systems (Lee et al., 2020) and forms of graph reasoning (Johnson et al., 2020) during a single task-based interview. Our analysis of Jamie's graph sketches and interactions with digital tools during the interview highlighted connections between her quantitative reasoning and conceptions of graphs. Specifically, Jamie's conception of a quantitative coordinate system was closely linked to her reasoning about changing attributes and their relationships, whereas her conception of a spatial coordinate system was closely linked to her reasoning about the physical motion within the dynamic situation. Jamie's case speaks to the complexity of students' quantitative reasoning and illustrates the interconnectedness of their conceptions of coordinate systems and graph reasoning.

Keywords: undergraduate education, technology, quantitative reasoning, graphing conceptions

Sketching graphs is a productive way for students to represent and reason about relationships between attributes in dynamic situations (Carlson et al., 2002; Thompson & Carlson, 2017). However, the effectiveness of this activity is closely tied to students' underlying conceptions of coordinate systems (Lee et al., 2020) and their graph reasoning (Johnson et al., 2020). Emerging research suggests a potential interconnection between these constructs (Johnson et al., 2024; Paoletti et al., 2022). Our goal for this qualitative case study (Stake, 2000) was to analyze a college algebra student's conceptions of coordinate systems alongside their graph reasoning. This study seeks to delve deeper into the relationship between these constructs by exploring the question: What is the nature of students' conceptions of coordinate systems and graph reasoning within digital graphing tasks involving dynamic situations?

Background

Quantitative and Covariational Reasoning

To frame our study, we draw on Thompson's (1994, 2011) theory of quantitative reasoning. According to Thompson, students conceive of attributes of objects as quantities when they consider them to be measurable. When students conceive of relationships between attributes that are capable of varying simultaneously with one another, they are engaging in covariational reasoning (Thompson & Carlson, 2017). For example, consider a situation in which a toy car travels at a constant speed around a traffic cone (Figure 1). Given this dynamic situation, a student may conceive of the toy car's total distance traveled and distance from the cone as attributes that are possible to measure, thereby conceiving of them as quantities. If the student were to conceive of a relationship between these attributes in which they vary simultaneously with one another, then they would be engaging in covariational reasoning.

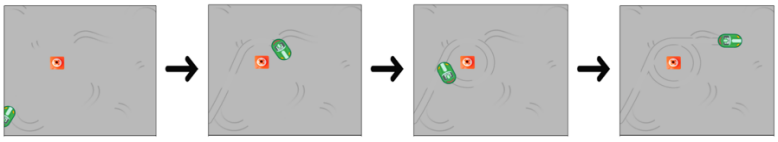


Figure 1. Toy Car Dynamic Situation.

Spatial and Quantitative Coordinate Systems

Lee et al. (2020) distinguished between two kinds of coordinate systems that students may conceive of when representing a situation: spatial and quantitative. A student conceives of a *spatial coordinate system* when they mentally overlay a coordinate system onto a physical or imagined space to determine relative locations within that space. For example, given the dynamic situation in Figure 1, a student may conceive of a spatial coordinate system by imagining a collection of points representing the toy car's position at different moments in time and sketch a graph resembling the toy car's path through the physical space.

A student conceives of a *quantitative coordinate system* when they extract and coordinate quantities from a situation to represent the quantities and the relationship between them in a new space, such as a Cartesian plane. For example, given the same situation, another student may disembed the toy car's distance traveled and distance from the cone, then plot these quantities on a graph with total distance traveled represented on the x-axis and distance from the cone represented on the y-axis. In doing so, the resulting graph would be representing the relationship between the two quantities. Lee et al.'s (2020) distinction offers a framework for analyzing students' conceptions of graphical representations in various contexts.

Graph Reasoning

Johnson et al. (2020) developed a graph reasoning framework to characterize students' conceptions of graphs. The framework includes four broad types of reasoning that encompass students' reasoning about quantities, relationships, and observable aspects of dynamic situations. The first two types of reasoning, *Covariation* and *Variation*, refer to evidence of students' gross coordination of values and gross variation (Thompson & Carlson, 2017). For example, if a student is tasked with sketching a graph of the toy car dynamic situation (Figure 1), they might say, "the distance from the cone decreases at first as the total distance traveled continues to increase, so the graph should start by going down". This response would serve as evidence of Covariation per Johnson et al.'s (2020) framework, as they are attending to the gross change in both attributes. If they only mention the change in one attribute (i.e., "the total distance traveled continues to increase"), their response would serve as evidence of Variation. The third type of reasoning, *Motion*, refers to evidence of students' reasoning about physical motion embedded in dynamic situations (Kerslake, 1977). For example, a student reasoning at a level of Motion might say, "the toy car moves at a constant speed, so the graph should be linear." The fourth type of reasoning, *Iconic*, refers to evidence of students' reasoning about physical features that comprise a dynamic situation (Leinhardt et al., 1990). For example, a student reasoning at a level of Iconic might sketch a graph resembling the path of the toy car, saying, "the toy car's path forms a circle around the cone, so the graph should have a circle too." Johnson et al.'s (2020) framework distinguishes between these four unique ways in which students may conceive of graphs.

Methods

We report on a qualitative case study (Stake, 2005) of a college algebra student, Jamie (pseudonym), who participated in a series of task-based interviews (Goldin, 2000). This study is part of the first author's dissertation, a collective case study conducted as part of a larger National Science Foundation funded project aimed at promoting students' mathematical reasoning by engaging with tasks called "techtivities." Housed in the Desmos Classroom (www.desmos.com), techtivities are interactive tasks based on video animations designed to engage students in quantitative reasoning (Johnson et al., 2020).

The Toy Car Techtivity

In the Toy Car techtivity, students watch an animation of a toy car traveling at a constant speed around a cone (Figure 1). They are asked to consider two attributes: the toy car's total distance traveled and distance from the cone. Next, students are to represent changes in these attributes by dragging "Axis Sliders" along the axes of a Cartesian plane as the animation plays. Students are to sketch a graph relating the two attributes before watching an animation of a computer-generated graph that shows the accurate graphical representation of the relationship.

Data Collection and Analysis

We used purposeful sampling (Gliner et al., 2017) to select Jamie (pseudonym), a college algebra student interested in participating in our study after seeing an announcement posted by her instructor on our behalf. Drawing on teaching experiment methodology (Steffe & Thompson, 2000), the teacher-researcher (TR) conducted exploratory teaching in a series of online task-based interviews (Goldin, 2000), in which Jamie completed four different techtivities. The TR used semi-structured protocols during the interviews.

We analyzed each interview using Wolcott's (1994) description, analysis, and interpretation framework. In the description phase, we documented everything observed during the interviews. In the analysis phase, we inferred Jamie's reasoning based on her actions and utterances. Finally, in the interpretation phase, we connected our findings to related literature, including frameworks from Lee et al. (2020) and Johnson et al. (2020).

Results

Our analysis of Jamie's third interview, during which she completed the Toy Car techtivity, revealed that she demonstrated evidence of multiple conceptions of coordinate systems (Lee et al., 2020) and forms of graph reasoning (Johnson et al., 2020). Through sketches and slider movements, Jamie displayed these constructs in a way that was agentic and meaningful to her.

Conceiving of Attributes as Quantities

Jamie's responses and drawings at the beginning of the techtivity serve as evidence of her conceiving of the toy car's total distance traveled and distance from the cone as quantities (Thompson, 1994, 2011). When the TR asked Jamie to describe the toy car's distance traveled, Jamie traced the entire path of the car with a sketch tool and said, "As the car's moving, he's getting further in distance." When the TR asked Jamie to describe the toy car's distance from the cone, Jamie drew a straight line between the toy car and the cone and said, "The distance is that, but if he's like in this area [puts cursor near the cone], the distance from the cone is way shorter."

Representing Different Aspects of the Situation with Sliders

Jamie dragged the horizontal Axis Slider (intended to represent Distance Traveled) from left to right along the x-axis as the toy car traveled. She followed this up by saying, "I just kept moving my slider until it basically stopped moving, and I tried to go at the same speed." This response, along with her previous description of distance traveled ("As the car's moving, he's getting further in distance"), demonstrate evidence of Variation and suggest that Jamie used the slider to represent the change in the toy car's distance traveled using the horizontal slider.

Next, Jamie dragged the vertical Axis Slider (intended to represent Distance from Cone) upwards along the y-axis as the car traveled towards the cone, downwards along the y-axis as the car traveled around the right side of the cone, and then upwards again as the car traveled around

the left side of the cone and continued away from the cone (Figure 2). She went on to describe her slider movements:

> So I assumed since [the slider] was in the middle of the graph, that it would want me to go back down when it did that little loop. So that's what made me move it. Like that's what made me move [the slider] down and then go back up 'cause it does do a little circle.

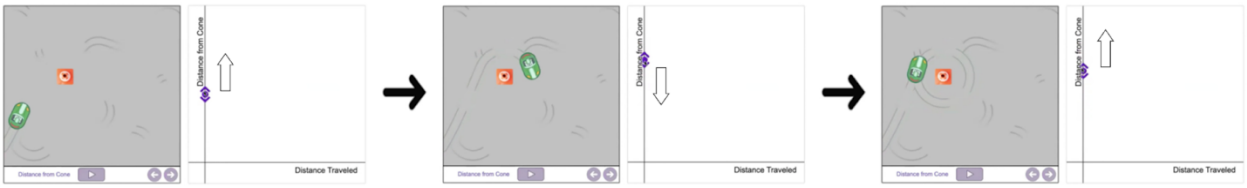


Figure 2. Jamie's Movements with the Vertical Axis Slider (Distance from Cone).

Jamie's movements with the vertical Axis Slider suggest that she was representing spatial aspects of the dynamic situation. She appeared to use the slider to match the toy car's vertical displacement from the bottom of the screen. Jamie's response shows evidence of Motion, as she is describing how her movements with the slider were meant to match the position of the toy car.

Demonstrating Different Conceptions

Jamie proceeded to sketch a graph on the Cartesian plane (Figure 3). The TR asked Jamie to describe her sketch, and she proceeded to talk about different sections of her graph.

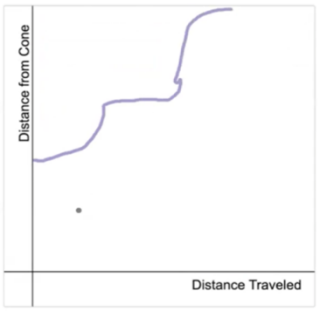


Figure 3. Jamie's Graph Sketch.

> *Jamie*: Yeah I feel like it might be going up, and then possibly, like I don't know if a graph could do this, but plateau out and then like all the way up.
>
> *TR:* And can you talk more about that plateau?
>
> *Jamie:* The plateau is when it's in the circle when it's staying at that one point, but the distance is still moving. So that's why I drew that there. And then after once it gets out of that circle, the distance from the cone goes back up along with the distance traveled.

Jamie's responses highlight multiple forms of graph reasoning. Her first comment, "it might be going up," demonstrates evidence of Motion, as she is attending to the movement of the car in space. Her second comment, "the distance from the cone goes back up along with the distance traveled," demonstrates evidence of Covariation, as she is describing the relationship between attributes.

Jamie proceeded to the last part of the techtivity and watched an animation of the computer-generated graph. When the TR asked Jamie for her initial reaction to the graph, she talked about the difference in how the computer treated the attribute on the y-axis.

> With their graph, they moved that dot down and then they go back up. I didn't think to move it down because when I did my graph, when I did the little move thingies, I more followed the car. Like the car's going up, so I'm going up. And then when the car came back around, I stayed and then I went back up.

Jamie's responses indicate that she situated her original graph (Figure 3) in a coordinate system containing both quantitative and spatial components. The left-most section of Jamie's graph represents the path of the car (i.e., "it might be going up"), indicating a spatial coordinate system. However, the middle ("plateau") and right-most sections represent the gross change in attributes (i.e., "the distance from the cone goes back up along with the distance traveled"), indicating a quantitative coordinate system.

Discussion/Conclusion

We explored the nature of a student's conceptions of coordinate systems (Lee et al., 2020) and graph reasoning (Johnson et al., 2020) within a digital graphing task involving a dynamic situation. Throughout the course of one interview, Jamie demonstrated evidence of both kinds of conceptions of coordinate systems (spatial and quantitative) as well as different forms of graph reasoning (Motion, Variation, and Covariation). Notably, she drew on these different ways of reasoning in ways that supported her goal of graphically representing the dynamic situation. Jamie's case speaks to the complexity of students' conceptions of coordinate systems and graph reasoning.

Our study supports the notion that students' conceptions of coordinate systems (Lee et al., 2020) and graph reasoning (Johnson et al., 2020) are interconnected (Johnson et al., 2024; Paoletti et al., 2022). On the screen with the Axis Sliders, Jamie demonstrated Variation when representing the change in attribute, and Motion when representing spatial aspects of the dynamic situation. While sketching a graph, Jamie demonstrated Covariation in a quantitative coordinate system and Motion in a spatial coordinate system. In other words, Jamie's conception of a quantitative coordinate system related to her reasoning about changing attributes and their relationship, while her conception of a spatial coordinate system related to her reasoning about the physical motion embedded in the dynamic situation.

Our analysis is ongoing. Jamie is one of three students who the TR has conducted multiple interviews with using multiple different techtivities. We are continuing to employ exploratory teaching (Steffe & Thompson, 2000) and use Wolcott's (1994) framework to learn about how students' conceptions of coordinate systems (Lee et al., 2020) and their graph reasoning (Johnson et al., 2020) are related.

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Equity-Based Mathematics Teaching Practices in Pre-Service Teachers' Lesson Plans

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As K-12 classrooms have become more culturally and linguistically diverse, it is important to prepare pre-service teachers (PSTs) to develop their knowledge of designing and implementing instruction based on equity-based mathematics teaching practices (EMTP). We analyzed 41 PSTs' lesson plans designed in a middle school mathematics methods course from five semesters. A theoretical thematic analysis revealed which EMTP categories appeared in the lesson plans. Findings indicate that PSTs seem comfortable and familiar with some categories, but they need support with other categories. In addition, methods courses should focus on EMTP so that PSTs strike a balance between EMTP categories by engaging students in non-routine, high cognitive demand tasks and by supporting diverse learners in mathematics classrooms.

Keywords: Pre-service teachers, Equity-based mathematics teaching practices, Culturally responsive mathematics teaching, Teaching mathematics for social justice

Objectives

In the U.S., the K-12 classrooms have become increasingly diverse and there is a need to prepare mathematics teachers who can cater to the learning needs of these students while maintaining high standards of teaching and learning (Aguirre & del Rosario Zavala, 2013; Association of Mathematics Teacher Educators [AMTE], 2017; National Council of Teachers of Mathematics [NCTM], 2014). In this regard it is important to prepare pre-service teachers (PSTs) to design and implement instruction based on equity-based mathematics teaching practices (McGraw et al., 2023; Moldavan & Gonzalez, 2023).

Equity-based mathematics teaching practices (EMTP) are pedagogical actions that seek to challenge existing educational norms and allow students to actively participate in learning mathematics (Chao et al., 2014). Such teaching builds on students' strengths, connects mathematics learning to students' knowledge and experience, and engages students in solving problems that are relevant to them (Aguirre, 2009; Aguirre et al., 2012; Civil, 2007; Gutiérrez, 2002; Gutstein, 2006; Turner et al., 2012). EMTP can aid in the development of students' positive mathematics identities so they see themselves as doers of mathematics (Berry, 2008; Boaler, 2002; Martin, 2000, 2009; Stinson, 2008). There is a need for teachers to develop classrooms where students can see themselves as mathematicians (Aguirre et al., 2024).

Our study focuses on instances of EMTP included in the lesson plans. The specific research question is: *What EMTP can be gleaned from PSTs' lesson plans designed to connect mathematics and social justice learning goals?* For our study, we analyzed 41 PSTs' lesson plans designed in a middle school mathematics methods course. We engaged the PSTs in developing mathematics lesson plans that included both mathematics and social justice goals.

Theoretical Framework

Mathematical identity can be defined as beliefs that students develop about their own ability to use mathematics effectively and in various contexts, versus being unable to do mathematics (Aguirre et al., 2024). Identity itself can be defined as the stories people tell about themselves or that others tell about them (De Fina, 2015). Students' identities outside of school can play a

powerful role in how they perceive themselves as students, both in general and specifically as mathematics students (Martin, 2000).

Aguirre and colleagues (2013) defined five EMTP that can help develop students' mathematics identities. These practices are: Going deep with mathematics; Leveraging multiple mathematical competencies; Affirming mathematics learners' identities; Drawing on multiple resources of knowledge; and Challenging spaces of marginality. These practices can overlap and work together to support students in feeling a sense of belonging inside mathematics classrooms (Allexsaht-Snider & Hart, 2001; Espinosa, 2023).

Here we provide a brief description of each of the five EMTP. Going deep with mathematics suggests that mathematics lessons include high cognitive demand tasks that allow students to become mathematically proficient. Leveraging multiple mathematical competencies means valuing students' areas of strength in terms of mathematics, and providing learning opportunities where students with various strengths can serve as resources. Affirming mathematics learners' identities refers to instruction that can support the development of a student's mathematical learning identity. For example, the teacher encourages perseverance in problem solving, removes a focus on speed in computation, allows students to fix their mistakes and frames mistakes as opportunities for learning. Drawing on multiple resources of knowledge encourages a teacher to perceive the student as a source of knowledge and experiences and use the student as a resource for teaching and learning mathematics. Students might have mathematical, cultural, linguistic, interest-based, family or community knowledge. The teacher can use this knowledge to connect mathematics to real life experiences and encourage students to solve problems connected to their lives. Challenging spaces of marginality stresses that teachers let go of the traditional mathematics teaching practices with a focus on lecture and also to question biases about who should be in a mathematics classroom. According to this practice teachers should acknowledge the various ways in which students can be mathematicians and expand their own vision of what it means to do mathematics.

Methods or Modes of Inquiry

We employ Phenomenology to identify instances of equity-based practices in the PSTs' lesson plans. Phenomenology allows us to analyze the lesson plans to unpack PSTs' perceptions about EMTP based on the activities included in their lesson plans (Moran, 2000). In other words, we analyzed our data while PSTs' were not explicitly asked to design their lesson plans based on the EMTP. By doing so, we were able to obtain spontaneous, not intentional, patterns of PSTs' use of EMTP in their lesson plans.

The study took place at a mid-sized university in the Mid-Atlantic region of the United States inside a middle school mathematics methods course. The course was geared towards pedagogy of mathematics and specially focused on culturally responsive mathematics teaching and teaching mathematics for social justice. PSTs were asked to develop lessons that included both mathematics and social justice goals. They were asked to include at least one social justice standard (Learning for Justice, 2022) as a social justice goal. PSTs were also required to include supports for emergent multilingual learners (EMLs) in their lesson plans.

Participants were 41 PSTs enrolled in the methods course during Spring 2021, Fall 2021, Spring 2022, Fall 2022, and Spring 2023 semesters. PSTs' lesson plans from these semesters were analyzed. Since our research question is explicitly related to specific teaching practices, theoretical thematic analysis was employed to report patterns within our data (Braun & Clarke, 2006). Due to our focus on equity-based practices, the lesson plans were analyzed based on

categories of Aguirre et al.'s (2013) five equity-based mathematics practices. We perceived each of the practices as follows:

Table 1. Description of five equity-based mathematics teaching practices.

Category	Description
Going Deep with Mathematics	Student engagement in high cognitive demand tasks, generating questions, using tasks connected to student interests, engaging students in problem solving and applying mathematical concepts.
Leveraging Multiple Mathematics Competencies	Tasks with multiple entry points, engaging students in problem solving, using various mathematical representations, discussions, collaboration and learning from strengths of other students.
Affirming Mathematics Leaners' Identities	Activities that allow student's confidence in their math ability to increase, developing persistence in problem solving, providing various ways for students to demonstrate their math prowess.
Drawing on Multiple Resources of Knowledge	Using students as a resource for learning mathematics, e.g., using data, using students' own knowledge to come up with ideas, themes, conjectures, working in pairs or groups, using peers as a resource.
Challenging Spaces of Marginality	Encouraging marginalized students to become confident math learners, positioning students as math experts, making students feel welcomed in math class, incorporating student interests.

Results

In this section we provide an account of the EMTP that we gleaned from the body of the lesson plans, where we did not specifically ask the PSTs to use these practices. We share what each practice looked like, and provide examples of what was included in the lesson plan. We noticed that Drawing on multiple resources of knowledge was evident in a majority of the lesson plans (29), followed by Leveraging multiple mathematical competences (21), and Affirming mathematics learners' identities (12). We only noticed a few instances of Going deep with mathematics (5) and Challenging spaces of marginality (4).

Drawing on Multiple Resources of Knowledge (29)

PSTs' lessons that included this practice, positioned peers or students themselves as learning resources, followed by an academic resource like a book or handout. Some examples of activities where students employed their peers as a resource included: Think-Pair-Share; students will listen to other students' letters and engage in conversation about free and reduced lunch; teacher will ask if any other group solved the problem differently, if so have them share. These examples take the focus away from the teacher as the only resource of knowledge, guidance or support, and students can learn to depend on their peers to solve problems. Likewise, examples of students seeing themselves as decision makers were: students will be tasked with creating their own mini poster based on at least one of their ideas; students will participate in mathematical modeling and use their own ideas and views to create problems about the picture on the Jamboard; students will create a blueprint of an outdoor space of their choosing … they will use

transformations to make images of the shape around the blueprint; students will research local organizations and choose one that the money in the girl scouts activity will go to; students will get their data for this activity from the United States Interagency Council on Homelessness website, students will decide what data they want to use for their argument. These examples demonstrate the ways in which PSTs perceived classroom activities that allow students to see themselves as experts who are able to take charge when solving a problem.

Leveraging Multiple Mathematical Competencies (21)

PSTs included various forms of activities in their lessons to draw on students' mathematical competencies. For example, allowing students to work independently on activities such as the number line activity, writing ideas on an index card, collecting their own data and then analyzing it. PSTs also planned for students to engage in group work to discuss ideas, represent algebraic expressions, do practice problems, come up with multiple representations or compare their own work with a partner. Planning for both group work and individual work may allow students who prefer a specific working style to have a chance to participate in class. Other activities that could allow the teacher to leverage multiple mathematical competencies were: making and presenting posters, students coming up with various mathematical representations, direct instruction, making comparison between graphs to name a few.

Affirming Mathematics Learners' Identities (12)

PSTs' lessons paid attention to students' interests and we coded instances of student interest as Affirming mathematics learners identities. An example of such an activity is the Minecraft activity where students research and brainstorm if squares and cubes are the best shape for Minecraft, create 3-5 argumentative points that defend their side, then draft a letter of persuasion to Minecraft creators to defend their ideas. Another example includes students choosing a resource from a town map that they wish to move to benefit more people in their area, then students recreate the town map with the resource in its new location (with correctly identified and plotted ordered pair) on a poster or Google Slide, and defend why that resource needs to be placed in that position. These examples point towards PSTs employing students' interests in real life scenarios to help them engage in doing mathematics. In addition, PSTs also supported student participation in the classroom by encouraging students to ask questions and have discussions, assigning roles during group work so that all students get a chance to participate.

Going Deep with Mathematics (5)

While we assumed this would be the most commonly observed equity-based practice, we only coded five instances of Going deep with mathematics. The lesson plans included high cognitive demand tasks, real-world applications, and productive struggle to be in this category. While all the lesson plans included real-world applications we only coded instances where the tasks were designed to maintain a high level of cognitive demand for the students.

Challenging Spaces of Marginality (4)

Although we were expecting to notice little to no instances of this EMTP in the PSTs' lesson plans, we coded four instances of Challenging spaces of marginality. Examples for this practice include: teacher will give some students a differentiated version of the worksheet; teacher will ask guiding questions if students are struggling; students will individually work on their project, however, they will engage in mathematical conversations with their peers and encourage one another through the process; students should be working to get through the math and reflection

with one another as mathematicians do when they are attempting to solve a problem. We noticed that all instances in this category were ways to differentiate instruction.

Discussion and Conclusions

We had sought to learn what PSTs' lesson plans reveal about their use of EMTP. The lessons included mathematics and social justice goals. In addition, PSTs were asked to include supports for EMLs. The EMTP that PSTs included in the body of the lesson plans reveal teaching practices that they were comfortable and/or familiar with. The highest instance of Drawing on multiple resources of knowledge points towards PSTs' knowledge of this practice. Likewise, including activities that would leverage multiple mathematical competencies was another practice with which PSTs seemed to be familiar.

Affirming mathematics learners' identities was an area that PSTs needed support with. This means PSTs need to learn about supporting the development of students' mathematical identities. PSTs' coursework should highlight that developing students' mathematical identity requires teachers to pay attention to students' lived experiences, their feelings towards mathematics and their sense of belonging in mathematics classroom (Darragh, 2013; Kaplan & Garner, 2017). Teachers need to maintain a balance between teaching the content and supporting the development of students' mathematical identities by catering to the learning needs of students and creating inclusive classroom environments (Miller-Cotto & Lewis, 2020).

Our PSTs also needed support with both Going deep with mathematics and Challenging spaces of marginality. According to the Culturally Responsive Mathematic Teaching framework developed by Zavala and Aguirre (2023), teachers need to provide opportunities for students to engage in non-routine, high cognitive demand tasks that foster the development of their analytical and critical thinking skills (Smith & Stein, 1998). Developing deep understanding of mathematics is a major goal of equity-based mathematics teaching practices (Aguirre 2009; Gutstein 2006). Students, in particular EMLs, might need various supports and scaffolded instruction for successful engagement with such tasks (Anhalt, 2014; Turner et al., 2024).

With regards to Challenging spaces of marginality we found that our students' lessons included supports that could best be seen as differentiation. Differentiation can be understood as ways in which teachers make instructional decisions based on students' readiness, their interests and learning needs (Tomlinson, 2017). This includes developing a classroom community where teacher pays attention to excellence in instruction, uses assessment to make instructional decisions regarding students' learning needs and pays attention to tasks that are meaningful to the students, connecting the content to the real world (Hocket & Doubet, 2021). While there is overlap between differentiation and Challenging spaces of marginality, there needs to be a special focus on making mathematics classrooms inclusive spaces for marginalized students (Aguirre et al. 2024). This includes teachers becoming aware of their own biases and engaging in activities that challenge "deficit thinking, segregation and harm" (p. 65).

Our findings indicate that a greater focus on EMTP in PSTs' methods courses is needed to help them intentionally include these practices in their lesson plans. Devoting time in the methods courses to unpack these pedagogical ideas could allow the PSTs to ask questions and learn how to identify these practices during their field experience. We noticed that PSTs included aspects of EMTP that were highlighted in their courses. Guided by the findings, our team will focus on developing method course resources especially focusing on equity-based teaching practices that were missing, or not well attended to in these lessons. We will also focus on supporting field experience personnel to learn about these pedagogical ideas so they can highlight these practices in their classrooms for PSTs during field experience.

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Building Hallways to Collaborative Reasoning in Inquiry-Oriented Linear Algebra Activity

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This study examines how students engage in collaborative reasoning while working through an inquiry-oriented task on null space in linear algebra. Three students participated in a series of five online group interviews, followed by an individual interview. The task sequence involved exploring subspaces and null spaces using pathways in a school map. We analyzed interactions and identified moments where students built on or connected to one another's ideas, influenced each other's reasoning, or left space for others to contribute. Our findings suggest that students' collaborative reasoning evolved over time, with instances of adopting and leveraging peers' methods. We highlight the role of positioning in group dynamics and how students' contributions to collaborative reasoning influence their understanding of mathematical concepts. The conclusions emphasize the importance of collaborative learning environments in facilitating deeper mathematical engagement and reasoning.

Keywords: collaborative reasoning, positioning, null space, inquiry-oriented instruction

Active and collaborative mathematics classrooms have been found to benefit students' overall mathematical experiences (Freeman et al., 2014; Laursen et al., 2014; Rasmussen et al., 2020), to improve student attitudes toward math (Laursen et al., 2014; Zakaria et al., 2010), and to decrease math anxiety as compared to that related to lecture-based courses (Gholamali Lavassani et al., 2011). Active learning often involves group and whole class discussions and thus comprises students sharing their ideas and engaging with others. Indeed, collaborative math classrooms using group work have been seen as promoting equity in learning (Esmonde, 2009). However, group work can position particular students with authority in their own learning (Langer-Osuna, 2011). Within these positionings, some students can be seen as holders of mathematical knowledge and thus have their ideas explored more often and thoroughly. In contrast, others might not have their ideas explored as often. In this study, we examine the ways three students reason collaboratively in an inquiry-oriented group interview. We investigate the ways students interact with each other's ideas and the inherent positioning in those interactions. We are interested in the ways students position themselves, each other, and each other's ideas while working toward a common goal: understanding null space in the context of an experientially real task.

Conceptual Framework

Inquiry-oriented instruction is one subset of collaborative learning centering students' ways of reasoning and discourse (Kuster et al., 2017). Making students' ideas central to math learning involves navigating differences in reasoning, making these ways of reasoning public, and students interacting with and engaging in each other's ideas in small-group and whole-class discussions. Additionally, it is intended that students work collaboratively to explore, reason about, and connect these different ideas. Students work together to eventually generalize and formalize the math concept of interest.

However, by selecting which ideas to bring forward and how to sequence them, students and the instructor are implicitly assigning value to particular ways of reasoning (Langer-Osuna, 2017). In small groups, the ideas students explore and choose to bring to the class are often

considered the most mathematically correct, which positions students in the group as either having mathematical knowledge or not. The ways that people are positioned can affect which ideas are elevated during problem solving (Langer-Osuna, 2016, 2018). In addition, how a teacher evaluates the merit of a student's work can affect how the group positions and assigns authority to that student. Students' positionings, whether positioned by themselves or others, are informed by factors such as social standing, academic standing, race, gender, SES, and more. For instance, a student who is seen as holding more mathematical knowledge might have their ideas privileged or leveraged more often. However, if the latter student is considered socially popular, then there is potential for their ideas to be taken up more frequently by their peers.

Collaborative learning has been defined as "the form of classroom organization in which students work together, in small groups, on a shared activity and with a common goal" (p. 57, Sam & Tan, 2011). For this study, we focus our attention on collaborative activities such as students' engagement with others' ideas, the connections students make between their ideas, and whether certain ideas were leveraged. This is because we are interested in the ways that students privilege certain ideas over others when engaging in collaborative activity. This inherently involves positioning because if there is collaboration, then students will always be saying something about their own or others' ways of reasoning, whether implicitly or explicitly. These activities have not been explicitly explored as part of collaborative reasoning. Thus, we examine students' positioning within the context of an IOLA task sequence. In our analysis, we seek to answer one research question: *How do students reason collaboratively about null space within an inquiry-oriented task sequence?*

Methods

Three students from across the Eastern United States were interviewed over the course of three weeks by the second author during five collaborative online Zoom interviews. All the students were white males who had recently completed a course in differential equations. Two of the students, Henry and Tyler, had taken linear algebra previously, and Spade had not. Each of the interviews was recorded and transcribed. PDF scans of students' written work were collected via email after each interview.

During the first four interviews, the students participated in a task sequence on subspaces designed for the IOLA curriculum (Andrews-Larson et al., in prep; Plaxco et al., 2021) that supported a conception of vectors as pathways in a school map. The end goal of the sequence was to have students investigate the null space of a linear transformation. In the task sequence, the linear transformation is conceptually represented as a matrix that transforms camera data vectors into room population change vectors. The null space of this linear transformation can be conceived as all the camera data vectors that result in no change in the population of all four rooms, which can be thought of as loops in the school map. The interview on the fifth day was designed to be a playful task where the students were asked to take the original map, rearrange the rooms and hallways, and determine the effect on the null space.

The interviews were transcribed using Zoom. For our preliminary analysis, each author reviewed each video and identified episodes of collaborative activity. These episodes were then reviewed by the group, and collaborative reasoning actions within each episode were identified.

Findings

In this section, we illustrate instances of the three ways that students collaboratively reasoned (and bold those instances) about null space while interacting with each other throughout the interviews. We begin by providing examples from the first day because all three students were

still developing a working relationship with each other and the interviewer. In later interviews, all three students became more familiar with each other and their mathematical ideas. However, we still share findings from the first teaching interview specifically because our students were exhibiting collaborative activities despite the fact they were still getting comfortable with the interview format and each other. Then we provide additional examples from the later days choosing examples that best illustrate the ways our three students reasoned collaboratively.

Day One: Representing All Pathways Between Rooms

In this section, we will discuss the three main collaborative activities in which we saw students engage: one student's reasoning influencing another's, one student connecting their way of reasoning to another's, and students leaving space for another to talk. The students were asked to find a way to represent all possible solutions to movement around the West Wing. The interviewer allowed students time to think on their own and then to talk to each other. For context, Tyler was not present for about 30 minutes of the interview due to a bad connection.

Henry began by saying it reminded him of "a solution set to a matrix or something, so I'm thinking we need to have two linearly dependent vectors and then add two weights to them." Beyond connecting his reasoning to more formal linear algebra concepts, Henry was also able to situate these ideas in the context of the task. He explained that to represent movement from room C to room C, they could do the square loop and the triangular loop and that some combination of these loops would always get them from C to C. However, for A to C, they needed to add on a vector showing movement in cameras 1 and 2 in addition to any amount of loops.

Spade starts his explanation by saying he was "a little lost" but still explained how he was reasoning about the problem. He mentioned doing some sort of n-1 approach but voiced not knowing how to explain it. Henry asked if he meant representing "it as a summation or something." Spade explained that the ways you go from C to C are all the way around, so using the vector <1 1 1 0 1> as many times as you go around that path. Henry started, "Yeah, I think that's actually, I don't want to interrupt you but–actually, just keep going," **making space** for Spade to continue his explanation and contribute to the pair's collaborative reasoning. Spade explained that you just keep adding the vector as you do the path more times, again mentioning n-1 without elaboration. Henry then **made a connection between their ways of reasoning**, explaining that they both used vectors to represent the loop paths and that you can do those as many times as you want. Doing these loops continuously led to needing scalars. This interaction ends with Spade agreeing with Henry. Spade ultimately adopted Henry's way of reasoning about the hallway movement, using vector expressions to represent loops. When Tyler joined the interview again, Spade explained that each vector in the expression representing solutions to movement from C to C represents a loop one could take around the hallways. Spade added that you can do those loops multiple times, introducing the need for scalars. His explanation showed that **Henry's reasoning influenced his own** regarding pathways between rooms.

Day Two: Connecting Camera Data Vectors to Population Change Vectors

During the Day 2 interview, the students began associating specific camera data vectors (e.g., <5 3 5 4 7>) with the corresponding population change vector. For reference, the population change vector associated with the <5 3 5 4 7> camera data vector should be <6 2 -6 -2>, which means 6 students enter classroom A, 2 students enter B, etc. Spade offered his population change vector first, <6 2 -2 -2>, but qualified his response with, "I think I'm a little lost in my work." Tyler said he "almost got that but had different ones for C and D" Tyler then explained his approach to determine why his vector was different from Spade's. Tyler's approach involved a

system of equations that emerged from thinking about the "inputs and outputs" for each classroom, and he explained that this also helped in determining the population change of each classroom, shown in Figure 1. Henry commented that the sum of all the entries of any population change vector should be zero, even though Tyler initially mentioned (before explaining his approach) that he knew his solution could not work because he was "adding students to the system." When moving to the next problem, Henry wanted to represent Tyler's system of equations as a matrix **demonstrating how Tyler's reasoning influenced Henry's**. Spade said he was "trying to just wrap my head around it…I understand what Tyler did, that's what I was trying to do pretty much," demonstrating that **Spade was also influenced** by Tyler's input/output approach. It is worth noting that even though Spade said he was still trying to make sense of the connection between the camera data vectors and population change vectors, he was the first to volunteer a solution to both problems.

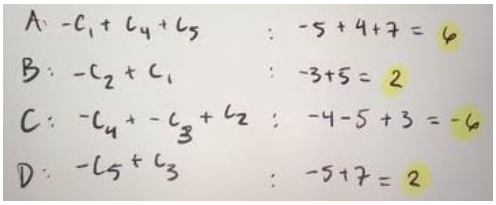


Figure 1: Tyler's system of equations approach to determine the population change vector corresponding to the given camera data vector.

Before moving on to the second camera data vector, **Henry explained a connection between Tyler's system of equations** and the corresponding matrix representation, with which Tyler agreed. It appeared that both Henry and Tyler were interested in using matrices to determine the population change vector for the second camera data vector. For reference, the population change vector associated with the camera data vector <5 5 3 2 3> is <0 0 0 0>. At the end of their working time, Tyler was the first to respond saying that he "did it Henry's way," which implied that **Tyler was influenced by Henry's matrix reasoning**. Again, Tyler did not initially use matrices but used matrix multiplication for the second camera vector (Figure 2a). Spade then explained "I did it your way Tyler…Yup, I got zero for all of them," demonstrating again that Spade was **influenced by Tyler's reasoning** (Figure 2c). When asked who wanted to volunteer to share after Spade finished his explanation, Tyler responded, "Either way, we'll probably say the same thing," **leaving space for Henry** to explain his idea. Henry proceeded to explain his matrix multiplication approach, **making connections** to Tyler's use of systems of equations (see Figure 2(b)). It should be noted that Tyler recognized that he should have placed the coefficient matrix to the left of the camera data vector (Figure 2a) and proceeded to give a thorough explanation of the relationship between the matrix equation and his initial systems of equations approach, demonstrating **a connection to Henry's approach** and his initial approach.

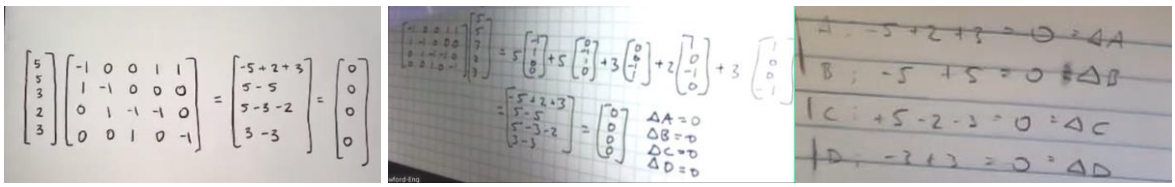


Figure 2: a. Tyler's approach using matrices based on Henry's explanation, b. Henry's matrix multiplication approach that corresponds to Tyler's system, c. Spade's system of equations approach based on Tyler's explanation.

Day Four: Developing Maps from a Matrix

The students were asked to construct maps corresponding to the matrix (Figure 3).

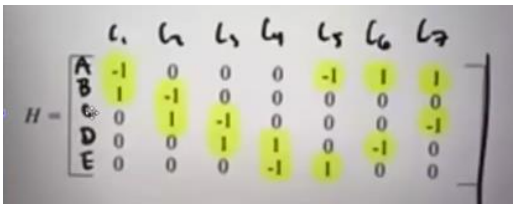
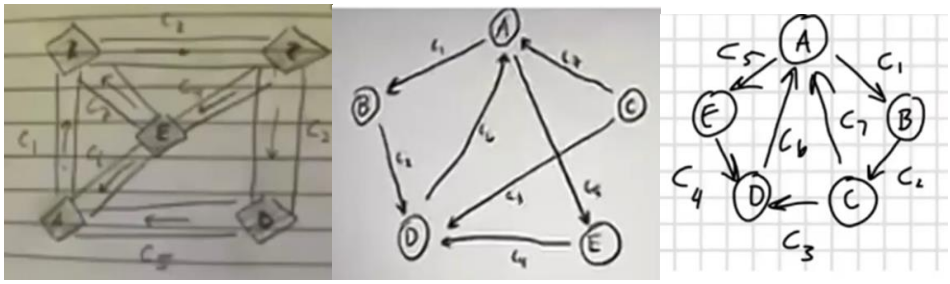


Figure 3: Matrix with Tyler's highlighting.

Each of the students worked independently and then were asked to share their work. Spade began by sharing that there was an additional room and hallway when compared to the previous map. Thus, he took the previous map, added a room in the center, and connected the hallways (Figure 4a), saying that he just "kinda did it." Tyler created a map (Figure 4b) by considering the entries in each column, that "the negative one" was people "leaving somewhere" and "the positive one" was people "arriving somewhere." This idea gave him a way to connect rooms in his map. Henry's map (Figure 4c) was similar, and he **explicitly connected his reasoning** to that of Tyler except Tyler's "went counterclockwise." After listening to Tyler and Henry explain their map, Spade redid his map, **building his own reasoning off of Tyler's reasoning**. When the students were sharing out, Henry went last, making space for the other students to share first.



4a. Spade's original map *4b. Tyler's original map* *4c. Henry's original map*

Figure 4: Student-generated hallway maps for the given matrix.

Conclusion and Discussion

In the first day of the interviews, students began to reason collaboratively by engaging in and connecting each other's ideas, even as far as to adopt another's way of reasoning. Henry was eager to make connections between his and Spade's ways of reasoning but instead made space for Spade to continue his explanation. These key ways of collaboratively reasoning continued throughout the interviews. For instance, Tyler tried Henry's method of developing a matrix equation, while Spade adopted Tyler's method of setting up a system of equations. We see from these examples that all three students reason interdependently, where students were connecting, building off, making space for, and sometimes adopting each other's ways of reasoning. These activities played a pivotal role in how students reasoned collaboratively and developed their understanding of null space. In the next stage of our analysis, we plan to examine our data more thoroughly to find additional episodes of collaborative reasoning. We are interested in further exploring the implications of students making space, or not, for each other, in addition to our other key ways of collaborative reasoning.

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Derivative Process-Object Preferences of Post-Secondary Teachers

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Derivatives are often considered the central part of introductory calculus (Kidron, 2019). While there has been consideration of how non-mathematicians use the derivative differently than their mathematical colleagues (Dray et al., 2019), there has been no survey data to aggregate these differences over many respondents. This study begins to explore how such data can be gathered and analyzed using a survey generated by the researcher to determine the process-object conception of the derivative (Dray et al., 2019; Zandieh, 2000) favored by various post-secondary teachers. In its current form, the survey distributed showed few significant differences between favored process-object levels in the framework of Dray and colleagues (2019). Survey improvements are discussed.

Keywords: calculus, derivatives, STEM education

Calculus is often used as a "gatekeeper" or "filter" for STEM majors (Bressoud et al. 2015). In other words, students must first pass calculus before entering upper-level non-mathematics classes. Ostensibly, this is to ensure that students are prepared to use calculus concepts in non-mathematics classes. However, researchers have found that calculus may not be fulfilling this preparatory purpose, with many students seeing their mathematics classes as disconnected from their future careers. (Black & Hernandez-Martinez, 2016; Brown et al., 2008; Di Martino, 2019; Hernandez-Martinez & Vos, 2018; Rellensmann & Schukajlow, 2017; van der Wal et al., 2017; Vos, 2018). Graduates of various engineering programs have "perceived their mathematics courses as islands, with no relation to the rest of their education" (van der Wal et al., 2017, p. S98). Similarly, others have noted that "For too many students, [calculus] is either an insurmountable obstacle or—more subtly—a great discourager from the pursuit of fields that build upon the insights of mathematics" (Bressoud et al., 2015, p. v).

Many studies have explored how to improve calculus instruction, most notably the 2015 Mathematical Association of America (MAA) national study of college calculus (Bressoud et al., 2015). These studies have often focused on instructional methods, with less emphasis on aligning content with the needs of non-mathematics classes. Some researchers have sought to understand how different professionals understand and use derivatives (e.g. Dray et al., 2019), but there have been few large-scale examinations of the needs of students entering non-mathematics classes. A notable exception to this was a MAA workshop in 1986 to discuss how calculus could better serve students pursuing non-mathematics majors (Douglas, 1986). However, with more recent research showing students struggling with mathematics concepts in their major coursework, more work is needed to refine calculus curriculum to better meet students' mathematical needs.

The goal of this study was to pilot a survey to determine the aspects of the derivative most valued by teachers of non-mathematics courses and to compare those data with those of the mathematicians teaching calculus courses. The survey focused on the derivative since the derivative is the central concept covered in introductory calculus classes (Kidron, 2019). If this survey can be refined, validated, and deployed to a national or international audience, the results could be used to offer recommendations for how calculus curriculum could be improved to better serve all students.

Theoretical Framework

This survey was designed to align with the framework for representations of the derivative developed by Dray et al. (2019). These researchers sought to design a framework to describe the differences that had been observed in how mathematicians and non-mathematicians use the notion of a derivative outside of an introductory classroom context. This framework was itself an expansion of a framework developed by Zandieh (2000) to organize the facets of student understanding of the derivative.

The redevelopment of Zandieh's (2000) framework by Dray and colleagues (2019) stemmed from observation made by the researchers while mathematicians and non-mathematicians were asked to find a derivative on a "partial derivative machine." Using this machine, participants could adjust strings on one part of the machine to change the position of weights on another part of the machine. Participants were asked to find a derivative on the machine. The non-mathematicians suffered no difficulties in doing so and were able to quite easily fulfill the prompt. Mathematicians, however, struggled to develop a limiting process that could be used to replicate the limit definition of the derivative.

This difference led Dray and colleagues (2019) to define the notion of a "thick derivative," where non-mathematicians use "good enough" approximations as stand-ins for the theoretical limit process. This notion of a thick derivative emphasized how scientists use an average rate of change over a small change in an input value to approximate an instantaneous rate of change, something antithetical to the mathematicians who worked on the partial derivative machine in Dray and colleagues' study.

This notion of a thick derivative (Dray et al., 2019) influenced their modification of Zandieh's (2000) framework for understanding of the derivative. In both frameworks, the concept image (Vinner, 1983) of the derivative held by a person is divided into multiple process-object layers as well as different representations. The process-object layers represent how, as initially noted by Sfard (1991), a mathematical process is reified into an object that is later used by a person in other mathematical processes. In the case of the derivative, in the various representations of the derivative, a rate of change is first found using a ratio of changes in an output variable to those in an input variable. This process is then reified into an object of which a limit is taken to find an instantaneous rate of change. The result of this limit process is what mathematicians often term a derivative (Dray et al., 2019). These limits are reified into an object that is then used to generate derivative functions. Thus, this derivative understanding framework breaks the derivative down into ratio, limit, and function layers. These layers and then used within different representations of the derivative to understand and solve various problems. Dray and colleagues included five representations of the derivative in their framework, namely graphical, verbal, symbolic, and numerical, and experimental representations. Zandieh (2000) had only four representations, namely the graphical, verbal, symbolic, and physical representations.

Dray and colleagues argue that scientists use an approximation of the limiting process to generate a thick derivative that can be used as part of their work (2019). Thus, instead of a derivative only being defined at the "limit" layer of their framework, they argued that the *ratio* layer, over a small enough change in the input variable, is sufficient to produce a thick derivative that can be used in applications. As such, they argued for an increased focus on reasoning with differentials in calculus classes. This survey aimed to validate this call as well as use their framework to better understand what aspects of the derivative are most valuable for non-mathematics majors.

Research Question

This study sought to begin answering the question of whether there is a difference between how mathematicians and non-mathematicians value different process-object layers (Dray et al., 2019; Zandieh, 2000) of the derivative.

Significance of this Study

As noted by Dray and colleagues (2019), scientists may place their concept image of a derivative within the ratio layer of their framework, rather than in the limit layer as is done more formally within mathematics textbooks. If respondents in this survey produced placed more emphasis on the ratio and function layers, this would lend credence to the arguments of Dray and colleagues that a greater emphasis should be placed reasoning with differentials rather than with limits when discussing derivatives. Additionally, resulting differences between subject areas could inform institutional decisions on developing siloed or major-specific calculus classes, an initiative that has seen some success in increasing engagement and persistence in calculus classes (Ellis et al., 2021; Klingbeil & Bourne, 2013; Luque et al., 2022; Voigt et al., 2020).

Methods

This study was carried out using a survey designed by the researcher. On the survey, respondents were asked to name their primary subject area (i.e. mathematics, physics, chemistry, etc.) and then to respond to several Likert-scale items rating the importance of various items an instructor may find important for a student to understand at the end of an introductory calculus course. On each item, respondents could rank the item as, "Not at all important," "Slightly important," "Moderately important," "Very important," or "Extremely important." The items were broken up into sections according to the representations of the derivative discussed in the framework of Dray et al. (2019). Within each of these sections, there was one item corresponding to each process-object layer of the derivative. This resulted in five questions corresponding to each of the process-object layers within the framework. For each respondent, the five questions for each subject layer were then averaged to create a score on their value for each of the ratio, limit, and function process-object layers. These scores were considered scale variables, as they came from combined Likert-style questions (Field, 2018).

Data Analysis

I examined the relationship between subject area and the value respondents ascribed to the different process-object layers. I treated subject area as the independent variable, with the three different process-object layer scores as dependent scale variables. These variable types led to data being analyzed using multivariate analysis of variance (MANOVA) (Field, 2018; Huck, 2012). To find significant differences between responses of individual subject areas, Tukey's post-hoc test was used. Additionally, a one-way analysis of variance (ANOVA) was conducted to determine if there were significant differences between the values respondents in general ascribed to the three different process-object layers, as well as within specific subject area groups.

Participants

Wyoming post-secondary faculty were surveyed as part of this study. Specifically, faculty in majors requiring calculus were invited to participate. Although this does not represent a national random sample, Wyoming colleges all conduct nationwide hiring searches that require faculty to have similar educational credentials as those for other colleges throughout the United States

(Higher Learning Commission, 2019). Additionally, Wyoming state law requires that colleges throughout the state have identical course names and course outcomes (Common College Transcripts, 2018), ensuring that instructors from similar disciplines but at different institutions would have similar prerequisite calculus knowledge required for their courses.

Results

After the survey responses were cleaned to remove incomplete responses, there were 75 complete responses that were analyzed as part of this study. Since there were only one or two respondents from some of the original subject areas included on the survey, respondents were grouped as is seen in table 1.

Table 1. Respondents Grouped by Subject Area

Subject Area	Number of Respondents
Mathematics	23
Physics & Chemistry	8
Biology	6
Engineering & Computer Science	12
Other	26
Total	75

Reliability Results

Table 2 shows the reliability scores on each of the sets of items used to assess the ratio, limit, and function layers. As is seen in the table, the values for Cronbach's alpha were relatively high for each layer, suggesting high internal reliability in the results.

Table 2. Reliability Results

Layer	Cronbach's Alpha
Ratio	0.769
Limit	0.848
Function	0.876

Test Results

Based on Roy's Largest Root, the results of the MANOVA were statistically significant, showing that there is a difference in the value of the three different process-object layers by teachers of various subject areas, $V = 0.294, F(4, 70) = 5.146, p = 0.001$. However, when results were broken down using Tukey's post-hoc test, the only significant difference found between subject are groups were between the "other" group and the mathematics group on each layer. When one-way ANOVA tests were run on the overall ratings for each process-object layer, the results were not statistically significant. This lack of significance suggests that overall, teachers value each of these three layers equally.

Discussion

Although the MANOVA yielded a statistically significant result, the small number of significant differences between individual subject groups is disheartening. However, as this was a pilot study for the survey used, the results suggest several ways the survey can be improved. Additionally, the results that were significant offer clues to how this research could be applied.

Survey Improvements

As noted previously, this study piloted the survey used. Since the intergroup significant values were found only between the mathematics group and the "other" group, the options for subject area on the survey need to be expanded. Some respondents to the survey noted in a comment section that subjects such as economics, geology, and chemical engineering were not included in the suggested subject areas. The intent of the survey was to have respondents who marked "other" on the survey state their subject area in a text box, but as no respondents filled in that text box, this feature was not employed properly.

There is also no validity data for this survey at this point. Interviews should be conducted with respondents to the survey to make sure respondents understand the intent of the questions. These cognitive interviews would help improve questions to better assess which of the layers the respondents value more. Additionally, in discussions with some other researchers, there are concerns that the survey questions may not have aligned as closely with the framework of Dray et al. (2019) as was originally intended. Refinements to questions will help better delineate differences in values in these layers.

Most importantly, a larger sample size needs to be collected to better validate these results. While the current sample size of 75 is enough to run some analyses, a larger sample size would allow the use of tools such as factor analysis to determine validity (Field, 2018). With a larger sample size, significant differences between the values ascribed to the three process-object layers may emerge, allowing for more specific feedback on the suggestion of Dray and colleagues (2019) to focus more on differentials in calculus classes.

Possible Applications of Current Results

With the understanding that these results come from a pilot study, some suggestions can be made from these data. Firstly, since significant differences were observed between the mathematics and "other" group, the nature of the known members of the "other" group can inform some instructional choices. Some members of the "other" group, as seen in some comments in the survey, came from subject areas such as accounting and business majors. These students often take some form of "business calculus" where certain aspects of calculus, such as trigonometric derivatives, are removed from the mainstream calculus curriculum. These results, where the mathematicians valued all the process-object layers more than those in the "other" group, seem to confirm that such a reduced calculus curriculum may be desired by teachers in accounting and business majors.

However, as the sample size for this survey was so low, and the known number of accounting and business majors in the survey was so low, such claims are preliminary at best.

Future Applications of this Survey

Once this survey is validated and refined, this survey could be deployed to a larger audience of post-secondary teachers to better make recommendations for how calculus curriculum can be modified to better suit the needs of the non-mathematics students in calculus classes. This larger audience could include college faculty, research scientists, and professional engineers just to name a few. Such results could help calculus better serve students entering STEM fields, rather than serving as a barrier to success.

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A Case Study on Undergraduate Students' Interpretation of Misleading Graphs in Public Media

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Contemporary news media frequently utilizes graphs to simplify the presentation of complex information. However, prevalence of misleading graphs on public platforms often makes readers susceptible to false information. This raises pertinent questions regarding the efficacy of our education system in adequately preparing students to interpret graphs. In this paper we present a case study that aims to explore undergraduate students' interaction with misleading graphs and unpack their mathematical reasoning as they do so to identify patterns that can explain their graph interpretation abilities. The study's findings suggest that students who focused on the data in a graph, rather than its shape, demonstrated better proficiency in graph comprehension and in identifying misleading elements.

Keywords: Graphs, Media, Misinformation, Critical mathematics

Today, most of the news media uses different forms of visualizations to simplify the presentation of complex information. Our brains better process data when they are presented in visual forms, such as graphs and tables, enabling us to discern significant patterns that might otherwise be overlooked (Tufte, 1985). While graphs can significantly improve information processing, the prevalence of misleading graphs in news and public media exposes readers to false information and poses a significant threat to democracy (Cairo, 2014). The risk is heightened during critical events such as pandemics and elections, as misleading graphs can directly influence public perceptions and decision-making regarding health and voting behavior. Under such circumstances, the question remains to what extent are students prepared to interpret graphs and identify misleading graphs published on public platforms?

While prior research has reported students' struggle with identification of misleading visualizations, particularly graphs (Ramly et. al., 2021), there exists a lack of understanding of the reasons behind such struggle. With the aim to fill the gap, in this paper we present a case study that investigates the question, how do undergraduate students interpret misleading graphs in public media, and what patterns in their mathematical reasoning can explain their ability or inability to identify inaccuracies in these visualizations? We hypothesize, once we identify these patterns, we can introduce them in our teaching and learning of graph comprehension and foster within students a habit of critically assessing the accuracies of graphs.

Theoretical Framework

This study is developed on the theories of critical mathematics literacy (Frankenstein, 1994) and graphicacy (Wainer, 1992). While critical mathematics literacy views mathematics as a socio-political tool for identifying and questioning societal injustices (Frankenstein, 2001), graphicacy focuses on understanding and interpreting graphical data (Wainer, 1992). By combining these frameworks, we argue that during this age of prominence in visual data, students' graph comprehension skills would enable them to uncover hidden patterns or biases embedded in visualizations and prompt them to ask relevant questions about data that would develop their fundamental critical analytical skills and enables them to comprehend the complexities of society.

Researchers have proposed various frameworks for evaluating an individual's graphical comprehension; this study utilizes Curcio's (1987) framework, which assesses graph comprehension through three aspects: reading the data, reading between the data, and reading beyond the data. While extracting data from graphs is emphasized in reading the data, identifying relationships between variables presented in graphs is the focus of reading between the data. The advanced level of graph comprehension, reading beyond data, involves making future predictions based on given trends in data. In this study, students interpreted graphs extracted from news media and we used the framework of reading the data, reading between the data, and reading beyond the data to measure their graph comprehension skills.

Method

Worksheet

To investigate students' interaction with misleading graphs, we asked them to complete a worksheet containing graphs with some mathematical inconsistencies. In the first bar graph on federal welfare (Figure 1- Left) presented by the Republican side of the Senate Budget Committee, the y-axis was truncated, and in the second "job loss by quarter" graph (Figure 1- Right) extracted from Fox News, the scales along the x-axis and the y-axis were inconsistent.



Figure 1. Left - Graph on US Federal Welfare; Right- Graph on Job Loss by Quarter

To measure students' ability to comprehend graphs, each graph was accompanied by three questions assessing students' capacity to read the data, read between the data, and read beyond the data (Curcio, 1987). For instance, with the "job loss by quarter" graph, to investigate students' ability to read data we asked them to identify the number of people who lost jobs in September 2008. To measure students' capacity to read between the data, they were provided with a hypothetical situation where a local news agency claimed that joblessness increased at a constant rate between December 2007 and June 2010 and asked them if they agree with the claim. Finally, students were prompted to predict future job losses based on current trends to evaluate their ability to read beyond the data. Students were asked to write justifications for their answers to help the research team gain insights into their mathematical reasoning and identify patterns that can possibly explain why some students are better able to identify misleading graphs, while others are not.

Participants

This research was conducted in an urban private research university in the North-Eastern Region of the United States. Nineteen undergraduate Business Administration students enrolled in a Quantitative Reasoning Course participated in the study. The first author administered an anonymous graph worksheet to the students, which was independently answered by the students.

The research team performed open coding (Strauss & Corbin, 2004) to analyze the students' responses and the written justifications. In this paper, we will present our findings on students' interaction with the misleading graphs and will uncover patterns in their mathematical reasoning that could explain their comprehension of graphs published on public platforms.

Results

Students' responses to the three questions accompanying the two misleading graphs revealed that for both the graphs, students could correctly read the data. However, they experienced struggles while reading between and beyond the data from the graphs, especially for the graph on job loss numbers (Table 1).

Table 1. Table showing students' responses to the three levels of reading data for the two misleading graphs on Federal Welfare and Job loss numbers

Graphs		Graph 1 (Federal welfare)		Graph 2 (Job loss)	
Type of responses →		Right	Wrong	Right	Wrong
Type of Questions	Read the data	19	0	18	1
	Read between the data	17	1	9	10
	Read beyond the data	19	0	2	17

To unpack the reasons behind this challenge with reading between and beyond the data, we analyzed their written justifications. To measure their ability to read between the data for the "job loss by quarter" graph (Figure 1- Right), we asked them, "A local news agency claimed that joblessness increased at a constant rate between December 2007 and June 2010. Do you agree with the claim? Justify." Ten out of 19 students answered incorrectly, asserting that job loss increased at a constant rate in the given time period. Upon examining their rationales, we found that all the students attributed their conclusions to the shape of the graph. They made statements such as "The line is pretty straight which means it is steady and constant" and "I do agree with the claim due to the linear slope of the graph indicating the constant rate", indicating that students relied on the linearity of the graph to infer the constant increase in job loss numbers.

On the contrary, eight students correctly discerned that job loss numbers did not increase at a constant rate. On analyzing their written justifications, we found that the students consulted the numbers presented in the graph and performed mathematical computations. For instance, one student calculated "the slope between Sept 2008 and Mar 2009 is 0.75 (13.5 - 9 mil / 6 months) and the slope between Mar 2009 and Jun 2010 is about 0.1 (15 - 13.5 mil / 15 months)." They said, since "slopes do not equal, then the growth is not constant." Hence, for the "job loss by quarter" graph, students who concentrated on the shape of the graph, missed the discrepancy between the actual data and the graph's shape. In contrast, students who focused on the numerical data recognized that the job loss numbers given on the graph do not support the linear shape of the graph and thus identified the graph to be misleading.

A similar pattern emerged from the "US Federal Welfare" graph (Figure 1- Left). To assess students' ability to read between the data, we asked them, "A local news agency claimed that the number of Americans who received federal welfare doubled from the first Quarter of 2009 to the fourth Quarter of 2009. Do you agree with the claim? Justify." Only two students failed to identify the misleading nature of the graph and asserted that the number of Americans who received federal welfare during the fourth Quarter of 2009 doubled from the first Quarter. Upon analyzing their written justifications, we found that both students focused on the relative height

of the bars to reach the conclusion. One student said, "The size of the bar for 2009 in the fourth quarter could be considered double what it looked like in the first quarter." Conversely, students who answered correctly mainly focused on the numbers given on the graph and engaged in some form of mathematical calculation. For instance, one student who answered correctly to the question, wrote, "Find the bar that says 2009 Q1 and the bar that says 2009 Q4 on the x-axis and identify the number of Americans by reading the y-axis. Find the difference between the two data. In 2009 Q1, the Number of Americans was about 97,000,000 and in 2009 Q4, it was about 101,000,000. There was only a 4 million difference."

To assess students' ability to read beyond the data presented in the "job loss by quarter" graph we asked them "With the current trend following, what would be the total number of job losses in the next quarter?" Given that the graph was misleading, we hypothesized that students' responses to the previous question would influence their approach to this one. If students acknowledged in the prior question that job loss did not increase at a constant rate, it might lead them to infer that it is not feasible to predict the future job loss numbers. To streamline the coding process, we decided, students who would not make any predictions would be considered as having answered the question correctly. All other responses were marked as incorrect.

In the previous question, ten students who failed to recognize the discrepancy between the job loss numbers and the shape of the graph, made some predictions about future job loss numbers. Upon examining students' prediction strategies, we found that most students calculated the job loss numbers in the three given quarters, found the average job loss numbers and used the value to predict the job loss number in the next quarter. During the process although they found that in each quarter job loss numbers did not increase constantly, they did not make any explicit comment on the nature of the graph. Among the nine students who correctly discerned that job loss figures did not exhibit a constant increase, two expressed their inability to forecast future job loss numbers. One of them said, "I can't say because the progression of the numbers is not going at a constant rate." The seven other students who recognized that the graph was misleading, projected some future job loss numbers. They made assumptions about the rate of job loss from the previous three quarters and conducted mathematical computations to reach their conclusion. For instance, one student predicted that the cumulative job loss number by the end of the next quarter would be 16.66 million because, between March 2009 and June 2010, job loss numbers increased by 11.11%, and "considering 11.11% increase rate, the number of job losses in the next quarter (Q3'10) would be approximately 16,666,500".

Upon reviewing students' responses to the "reading beyond the data" question, we realized that, although our intention was for students to recognize that the misleading graph was not a reliable tool for making predictions, the way we phrased the question might have unintentionally limited students' thinking about the inconsistencies present in the graph. Most students focused on a hypothetical situation and made assumptions about the current rate of job loss to make future predictions about the same. Further, upon analyzing each student's predicted job loss numbers, we found that many of them made errors in their mathematical calculations. However, the goal of the study was not to measure the mathematical correctness of the students' calculations; rather focus on their strategies that could explain their graph comprehension.

Discussions

Based on students' responses to the graph worksheet, this study draws the following conclusions. Students showed greater proficiency in reading data from the graphs compared to reading between and beyond the data. Students who failed to identify misleading elements of the graphs relied more on the shape of the graphs rather than the actual data. Conversely, students

who closely examined the data provided on the graphs and conducted mathematical calculations, used their mathematical outcomes as evidence in support their answers. The findings of the study, mainly the gap between students' potential to reading the data and reading between and beyond the data is particularly worrisome. It suggests that as news consumers, while we may excel at extracting data from graphs, we often fail to thoroughly analyze the relationships between variables depicted in visualizations and think beyond what is presented to us. As a result, if confronted with a misleading graph, such as the one on job loss numbers, we would be prone to accepting the information at face value rather than questioning the accuracy of the data. Since the study had no scope to get deeper into students' responses through interviews, we are uncertain of the reasons behind the gap between students' ability to read the data and critically analyze the data.

Consequently, this study proposes the following recommendations to enhance students' graph comprehension. First, educators should encourage students to focus on the data presented in graphs rather than being distracted by their visual shapes or design elements. Second, graph literacy should be introduced through authentic, real-world graphs rather than hypothetical ones, as understanding graphs requires more than just knowing mathematical rules-they must be interpreted within relevant contexts. Lastly, fostering graph comprehension in real-life scenarios involves cultivating students' habit of critical questioning prompting them to ask who the graph represents and excludes, and what biases might be embedded in the graphs.

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A Preliminary Scoping Literature Review on Mixed Methods in RUME

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Through a scoping literature review of 40 RUME studies from 2019 to 2024, we examined how researchers use and describe mixed methods. We categorized these studies by country, subject area, and self-reported implementation process (e.g., sequential, parallel), methodological priority (qualitative or quantitative),and study design (exploratory, explanatory, triangulation). While mixed methods are used in RUME, we found that only half of the studies explicitly define their methodological framework, often missing opportunities to maximize the potential of integrating multiple data sources. Our findings highlight the importance of adopting more nuanced and deliberate mixed methods approaches to fully exploit the strengths of both qualitative and quantitative research in advancing the field.

Keywords: mixed methods, RUME

Much of the research in RUME uses qualitative methods. Artique (2021) critiqued the field's "excessive predominance of small-scale qualitative studies, involving a very limited number of students or teacher" (p. 14). Our recent survey of the field's creation and use of quantitative measures suggests that RUME scholars do a substantial amount of quantitatively oriented work as well (Melhuish et al., 2024). What may be the case is that qualitative and quantitative researchers are operating in separate spheres. This is sensible when thinking about what kinds of research questions are answerable (broadly) by these paradigms. Questions of comparison or prediction cannot be paired with cognitive constructivist teaching experiments, for example. However, sometimes the weaknesses and biases in one approach (e.g., surveying college instructors about their practices) can be mitigated by the strengths of another approach (e.g., observing classrooms). So, the separation does lead to calls for increasing complementary approaches to investigating phenomena of interest to RUME (e.g., Weber, 2013).

Artigue's critique zeroes in on the *size* of the research sample, which is only one aspect of generating trustworthy information about a research topic – one that matters less for qualitative work than for quantitative. What does matter, quite a bit, is using the sample to get the right kinds of information to address a research question or to obtain convergent information about an issue. Quantitative methods are useful for measuring and aggregating patterns across individuals; qualitative methods aid in interpreting patterns of meaning to individuals. Mixed-methods approaches can draw on both paradigms to "provide a more complete understanding of a research problem than either approach alone" (Creswell, 2014, p. 4). They can answer questions like, "Which group performed better, and why?" or "To what extent does the frequency of negative teacher feedback reflect students' perceptions of their own competency?"

Yet, we do not frequently see these complex, multi-perspective questions posed in the RUME research literature. We became interested in detecting the extent to which RUME researchers had responded to calls to integrate quantitative methods in their work. In this preliminary report, we share a scoping literature review aimed at identifying and classifying the types of mixed methods approaches RUME researchers report they use. We share both how researchers describe their own work and the degree to which they are explicit and specific about the methods they use.

Background

An historical definition of mixed-methods research is an inquiry that includes "at least one quantitative method (designed to collect numbers) and one qualitative method (designed to collect words) (Greene et al., 1989, p. 256). Contemporary descriptions recognize that qualitative methods are not merely about words-as-data and that there can be many configurations for how the qualitative and quantitative information are treated, relative to their level of integration and relative to their centrality to the research question. For example, qualitative inquiries in RUME include questions about the meanings teachers hold for equitable participation in a mathematics classroom, gestures that calculus students use to express changing rates, or the mental operations that produce novel quantities. Some methodologists have attempted to classify mixed-methods research approaches based on whether qualitative or quantitative data sources are given more prominence, whether data are sourced concurrently or not, and whether the analysis treats data from different sources in parallel, in sequence (and if so, in what order), in an integrated manner, an embedded manner, or seeks to convert one form to another (Buchholtz, 2019; Creswell, 2014). Each of these descriptive criteria for the methods used in a study is important because they correspond to analytic decisions which govern which research questions can be answered.

Types of Mixed Methods Approaches

Here, we focus on major descriptive features (Creswell et al., 2011); for the implementation process (concurrent, parallel, conversion), purpose of the study (exploratory, explanatory, development, triangulation, conversion, initiation, etc…), and priority of the methods (qualitative or quantitative). Priority is often indicated by how the researchers frame the study. A study meant to test relationships among variables that were created from written responses likely gives priority to quantitative methods. A study that uses a survey to identify individuals who hold certain beliefs about the nature of mathematics, and then conducts oral histories to understand how those beliefs may have formed gives priority to qualitative methods. In each case, one data source is subservient to the other. A related concept in mixed methods is the purpose of the study and how it uses the different data sources. When data sources are used in sequence, the study is usually described as exploratory sequential or explanatory sequential. Exploratory sequential may use qualitative findings to develop an intervention, and then test the intervention's efficacy. This can be a useful approach when variables to include in the intervention are not known a priori. Explanatory sequential first collects and analyzes quantitative data, then digs into particular cases qualitatively to explain why quantitative patterns might exist. One can easily imagine that an ANCOVA comparing an IBL approach to teaching proof to a "business as usual" classroom would seem rather dry without examining paradigmatic cases of student reasoning that lead to one classroom outperforming the other. Data sources might also be analyzed in parallel, where the purpose is to establish multiple threads of evidence in favor of a claim. For example, measurement validity studies often seek multiple sources of information like open and closed form responses and clinical interviews about the items to ensure that items are behaving well. In other writings, this is *convergence* or *triangulation* (Flick, 2014). Evaluation studies might choose to use qualitative methods to analyze processes in parallel with quantitative methods to measure outcomes. In such a case both pieces may be given equal priority and address separate but related research questions about how an intervention is working in a setting. A particular approach to mixed methods research that is common in education and the social sciences is *conversion* of one data type to another data type. When the conversion is from qualitative to quantitative, the process is referred to as *quantizing*, where nominal or ordinal values are assigned to data that was initially conceived as non-numerical (Sandelowski et al., 2009). For

example, a classroom lesson could be analyzed using content analysis or constant comparison according to a conceptual framework, then assigned a count-based or time-based frequency and analyzed quantitatively.

Uses of Mixed-Methods in Mathematics Education Research

Mixed methods are not uncommon in mathematics education research or education research more broadly. Several researchers have examined the prevalence in particular journals including JRME (36% of articles from 1995-2005, Hart et al., 2009; 34% from 2006-2010, Ross & Onquegbuzie, 2012), ESM (17% from 1995-2005, Hart et al., 2009) and JMTE (22% 1995-2005, Hart et al., 2009). These researchers also argued, "Mathematics education researchers appear to utilize lower levels of quantitative analysis complexity, when used alongside qualitative procedures" (Ross & Onquegbuzie, 2014, p. 71) with majority of the mixed methods articles using only descriptive statistics. A cursory look at RUME proceedings from 2023 reflects a smaller number of mixed methods studies. Of the 151 papers, 9 stated they used mixed methods of some sort. However, we caution this small number but may reflect that proceedings may contain portions of studies.

Methods

Although typologies of mixed methods are useful for organizing and contrasting various approaches, especially for researchers new to thinking about mixing methods, they have limited utility for guiding mixed methods research due to the emergent needs for adjusting the procedures as the data sources are collected and analyzed (Poth, 2023). Nonetheless, we use a few dimensions of common typologies to help organize the RUME literature that purports to use mixed methods. For our scoping study, we limited our literature search to a five-year span (2019-2024) and to the twenty mathematics education research journals identified by (Williams & Leatham, 2017). To focus more on RUME, we also included the *International Journal of Research in Undergraduate Mathematics Education*. Our main search term was "mixed methods" anywhere in the text. We used Publish or Perish citation software to pull all 459 matching articles. We then preserved any article that explicitly purported to use mixed methods, included quantitative *and* qualitative data sources, and used a RUME sample. By RUME sample we mean undergraduate mathematics students or instructors. A total of 40 articles met these criteria. For each article, we then used a simplified blend of typologies from Teddlie and Tashakkori (2006) and Creswell & Clark (2011) to classify articles according to the following simplified criteria: (1) whether they cited a mixed methods article or methodological text to justify their approach, (2) which implementation process (sequential, parallel, conversion), (3) gave primacy to qualitative or quantitative methods (4) the research design (explanatory, exploratory, triangulation, complementary, experimental design). We note, we did not infer these designs nor evaluate if their goals were met, but rather report what was shared by the research team.

Preliminary Results

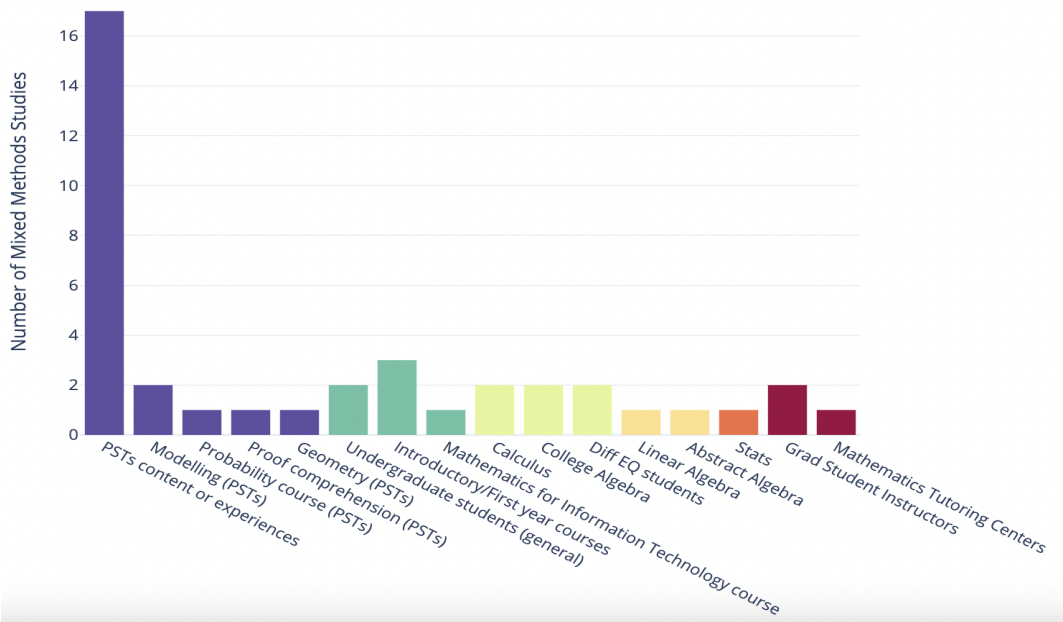


Figure 1. Distribution of $N = 40$ mixed-methods studies in RUME, across content areas and populations

We identified forty articles that met the criteria of being self-described as mixed methods and studying a population of undergraduate mathematics students or instructors. Slightly over half of the studies were conducted in the United States (21) with the remaining 19 spanning 12 countries (Iran, Finland, Australia, New Zealand, Vietnam, Germany, Turkey, Spain, Canada, Israel, South Korea, and Norway). The studies' focal content (which often entails a population/sample) can be found in Figure 1. Note that pre-service teacher content and experiences dominate the mixed methods studies. We conjecture that the mixed methods may be more reflective of teacher education researcher methods than the RUME community. If we removed studies with this focus, we would eliminate over half of the manuscripts.

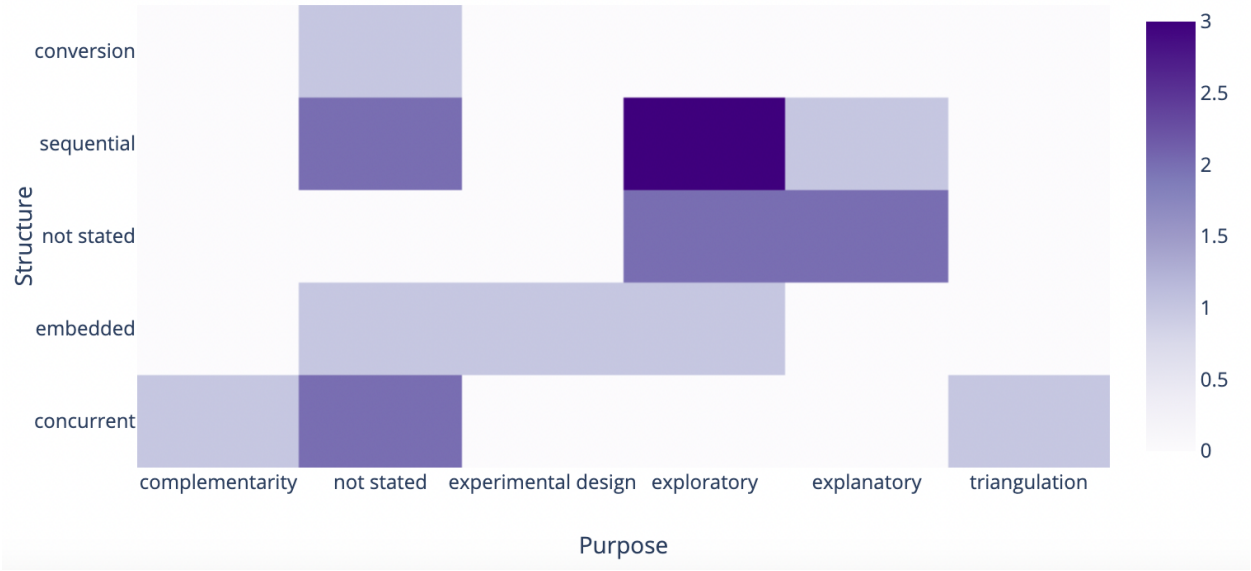


Figure 2. Heatmap of Purpose and Structure for the 20 Studies Explicating That Information

The most cited author was Creswell with 14 citations, followed by Yin with three, and Teddlie and Tashakkori with two. Of the forty articles, only half (20) explicated their mixed methods typology in any way beyond stating it was a mixed methods study. Several studies pointed to the mixed methods being embedded into other methodologies including one instance of instrument validation (Donovan et al., 2024), one "conceptual replication of a quasi-experimental study" (Yudt et al., 2024), and three case studies (Ellis et al., 2021; Galligan et al., 2019; Johns et al., 2023). Two studies indicated their larger approach was "longitudinal" (Woolcott et al., 2021) or "pseudo-longitudinal" (Safrudiannur, et al., 2022).

An overview of the 17 mixed methods studies that addressed their typology can be found in the Figure 2. Not that the most frequent type was exploratory sequential and only occurred three times. Note, an additional three studies mentioned the dominance of qualitative or quantitative data: two equal status and one qualitative dominated. Finally, we note one study (Yee et al., 2024) described their methods as an "integrated generalization design" (Srnka & Koeszegi, 2007), labeled as conversion in the above figure.. This design is two stages and involves qualitatively coding dating, then quantifying these results in order to conduct statistical data analysis. Sometimes this structure is referred to as "quantizing" (Morgan & Hoffman, 2021) where qualitative data is converted to numbers, typically through counting. While it is a common approach in our field, it requires careful attention to sampling and generalization to make inferences from such an approach (Czocher & Melhuish, 2024).

Discussion

This preliminary analysis illustrated that RUME researchers are using mixed-methods, but often with little attention to unpacking key elements of the methodologies. Mixed-methods can be positioned as a way to avoid the limitations of using qualitative or quantitative approaches alone; however, they also introduce tensions and require careful attention to methodological details and theoretical assumptions. Quantitative approaches grew out of positivist and post-positivist paradigms, whereas qualitative approaches typically (though not exclusively) arise from constructionist or interpretivist paradigms. One key distinction is that qualitative approaches, especially naturalistic or interpretive ones, do not share the expectation of objectivity or the supremacy of the researcher's perspective over the research subject's perspective. As Denzin and Lincoln (2005) sharply observed: "The mixed-methods movement takes qualitative methods out of their natural home, which is within the critical, interpretive framework" (p. 9).

Guetterman et al. (2017) looked for Mixed Methods-Grounded Theory studies ("across disciplines") and found that (1) lacking methods detail (2) lack of actual theoretical development (3) most were convergent mixed methods design, as opposed to sequential/exploratory designs. This finding comports with our own perception, as readers of mixed methods studies in mathematics education, that often "mixed methods" implies a coding procedure followed by reporting descriptive statistics for the frequencies of coded segments in the categories. This does not "reach the full potential" (Guetterman et al., 2017, p. 180) of mixing grounded theory, to say the least, let alone reaching the full potential of mixed methods more broadly.

Given the prevalence of grounded theory-based approaches to data analysis in RUME, it seems that exploratory sequential studies (creating a theory and testing it empirically) would be a a natural fit (Smith et al., 2023). Future work can examine the extent to which these kinds of studies are following best practices (Guetterman et al., 2017; Smith et al., 2023) or just following their gut instincts towards the least publishable unit (as discussed in Stewart, 1998).

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Explicit Group Norm Framing by MGTAs

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Considerable attention has been given to the role of Mathematics Graduate Teaching Assistants (MGTAs) in fostering classroom norms. This is especially pertinent in active learning settings where students extensively engage in group work with minimal direct instruction. Social stereotypes can be pervasive within small group dynamics, often hindering participation of students in mathematical discourse. Implementing explicit classroom norms is an effective strategy to promote a more inclusive distribution of learning opportunities. However, more research is needed on how these norms are enacted and supported. Our findings reveal significant differences between MGTAs who consistently frame group norms and those who do not, highlighting the benefits of explicit norm framing for promoting meaningful discussions, structuring group work, and ensuring inclusivity. These results underscore the importance of professional development focused on norm setting to enhance inclusive student engagement in mathematics classrooms.

Keywords: Classroom norms, Inclusion, Teaching Assistants

In recent years, considerable attention has been given to instructors' roles in establishing and fostering classroom norms that promote equitable and inclusive learning environments. The importance of these norms becomes even more pronounced in active learning settings, where traditional lectures give way to collaborative, student-centered activities. In these environments, students engage in problem-solving, discussion, and peer teaching, often working in small groups with minimal direct instruction from the teacher. The shift from passive listening to active participation is meant to enhance student learning processes, but it also brings significant challenges. Without appropriate support, group dynamics can easily mirror broader societal inequities, leading to uneven participation and marginalization of certain students based on gender, race, or other characteristics. Previous studies have demonstrated the pervasive nature of gender and racial stereotypes within small group dynamics, negatively impacting the equitable participation of minoritized groups in mathematical discourse. For instance, studies have shown that women and students from racial or ethnic minoritized groups often have their contributions overlooked, dismissed, or undervalued in group settings, which can lead to decreased confidence, lower participation rates, and reduced learning outcomes (Boaler, 2008; Esmonde, 2009). These patterns of exclusion not only limit the learning opportunities for those students but also reinforce harmful stereotypes about who is "good" at mathematics, reinforcing a cycle of inequity.

To address these issues, educators and researchers have increasingly advocated for the implementation of explicit classroom norms that could promote a more inclusive distribution of learning opportunities (Horn, 2012). Classroom norms can be understood as taken-as-shared beliefs and expectations guiding interactions within the classroom. These norms play a critical role in shaping how students engage with one another and with the content, influencing both the cognitive and social dimensions of learning (Yackel & Cobb, 1996). Rather than being explicitly stated, these norms often emerge through repeated interactions and are implicitly agreed upon by instructors and students. However, when classroom norms are explicitly articulated and consistently reinforced, students are more likely to engage actively and constructively with their

peers, leading to a richer and more inclusive learning experience for everyone involved (Esmonde, 2009).

Despite the growing recognition of the importance of explicit classroom norms, there is still a need for more research on how these norms are enacted by instructors. This is especially true for MGTAs who are often in the early stages of their teaching career. While some studies have explored the use of norms in K-12 settings (Boaler, 2008), less is known about how MGTAs—who may have limited teaching experience and training—understand, implement, and sustain norms in the college-level mathematics classroom. This study seeks to fill these gaps by investigating the use of explicit classroom group norms by 19 MGTAs in a university setting. MGTAs were observed in their classrooms on three separate occasions. Following these observations, three significant interactions between MGTAs and groups of students were analyzed in-depth to characterize the impact of these norms on student engagement. Two main research questions guided our study: 1) To what extent do MGTAs make use of explicit group norms in active learning classrooms? 2) How do MGTAs make use of explicit group norms in active learning classrooms? By examining how MGTAs articulate and reinforce classroom norms, and how these norms shape the dynamics of group work, this research aims to provide insights into effective teaching practices that foster inclusive and equitable learning environments in undergraduate mathematics education. Ultimately, the findings of this study could inform professional development programs for MGTAs and other instructors, contributing to the broader goal of making mathematics more accessible and welcoming to all students.

Methodology

The initial dataset included classroom videos from 19 MGTAs from three different institutions: Beta (5 TAs), Epsilon (5 TAs), and Gamma (9 TAs). During a year of professional development, MGTAs were asked to reflect on the impact of classroom norms on equity and inclusion but were not necessarily asked to give students detailed and explicit instructions on how they should work with peers in groups. Videos from all the MGTAs from a first PD cohort (2022-2023) were included. Due to time constraints, only two MGTAs from a second cohort (2023-2024) were included. TAs' classrooms were video recorded three times over an academic year. The first observation did not necessarily correspond to the initial class meeting. Although this limitation reduced our ability to capture a probable instance when instructors might establish classroom norms, it enabled us to identify moments where MGTAs felt the need to reinforce these initial norms throughout the semester. We hypothesized that MGTAs who explicitly reiterated group norms after the first class would place greater importance on being explicit about these norms than those who only addressed them on the first day.

To analyze such a large amount of data, we focused on three instances in each classroom observation: classroom setup, first interaction, and second interaction. Classroom setup occurred within the first 5-10 minutes of the lesson, during which the instructor set the tone for the class. Instructors outlined general expectations, goals for the session, and any specific guidelines for participation. We identified the end of this setup when MGTAs stopped addressing the whole class and most students were visible working in groups. The two subsequent interactions sampled after the classroom setup were often brief but were considered valuable as we expected that MGTAs might see them as an opportunity to reinforce the expectations given during classroom setup. These interactions were the first two instances 10-20 minutes after the classroom setup where MGTAs approached students to offer assistance. At that point we expected students would have had an opportunity to ponder the concepts or problems of the day.

We conducted three rounds of coding. In the first round, we established our data sample, identifying classroom setup, first and second interaction for each classroom video of each TA. In our second round of coding, we examined the extent to which MGTAs explicitly framed group norms during classroom setup, first or second interactions. Group norm framing was defined here for analysis purposes as instances when TAs not only communicated expectations about how students should work in groups, but also supported and nurtured group work in the classroom by facilitating the formation of groups, interacting with students via their groups, and attempting to answer questions by generating discussion within the group rather than prescriptively between themselves as a single student. In cases where MGTAs' answers were focused on answering individual content-related or logistic questions not intended to promote group work, we did not code the interaction as norm framing. A third round of thematic coding was conducted to characterize instructors' norm framing or instructors' alternative to norm framing when interacting with student groups. First, transcripts labeled as classroom setup or student interactions were coded and grouped into broader categories aligned with the study's research questions. Through iterative review, these categories were refined and collapsed into overarching themes, such as "clear expectations" and "individualized assistance."

Results

The total number of codes associated with explicit group norm framing by MGTAs who had at least four total codes is shown in Table 1.

Table 1. Number of codes by MGTA after second round of coding.

Id	Inst	Cohort	Setup	Interactions	T	Id	Institution	Cohort	Setup	Interactions	T
G1	Gamma	1	16	8	24	G3	Gamma	1	2	4	6
E1	Epsilon	1	11	3	14	G4	Gamma	1	2	2	4
E3	Epsilon	2	9	3	12	E1	Beta	1	0	4	4
G2	Gamma	1	9	1	10	B2	Beta	2	3	1	4
E2	Epsilon	1	6	3	9						

TAs with fewer than four codes total were excluded from the analysis as we considered their norm framing negligible for this study. The distribution of codes per participant in this table shows a low occurrence of explicit framing group norms in our sample with about 20% of MGTAs with higher than ten instances. The main themes identified in the third round of coding of the two MGTAs (G1, E1) with **high explicit framing group norms** were: clear expectations, structured groups, meaningful discussions, and inclusivity. First, at the beginning of the class, MGTAs set **clear expectations** of what group work was going to look like that day. These expectations were focused on communication within groups. For instance, G1 starts their class indicating that group discussions, sharing ideas, and helping each other are all valued during class. This is illustrated in the following quote of the first two minutes of class:

> *G1.* We'll get into groups. We'll take a little private think time and then we will discuss it in groups. So, remember that this is just… time to develop your mathematical thinking, and really share some rough draft thinking [...] It's totally fine if you have questions, if you

don't have an idea, that's OK, your group can help you out, you guys can help each other […] let's take just like, 2 minutes and then… we'll turn our desks and get into groups.

The **structured groups** theme characterizes the MGTAs' efforts to make sure all students were part of a group. MGTAs in this code impose this as a condition to work. The **meaningful discussions** theme is best illustrated in the following quote from G1 who addresses a group of three students (S1, S2, and S3).

S1: We're just not sure we're right.

G1: OK, what was your idea? [while the crouching down to students' eye level]

S1: So our idea was, we have 10 people… [hard to hear].

G1: OK, I like part of that. Let's talk about counting how many handshakes. Now, you're right that everyone [inaudible]. but consider this, put yourself in a room [...] with 20 other people, how many people's hands are you going to shake?

S1 and S2: 19 [while S3 remained silent]

G1: 19, right, 'cause you're not going to shake your own hand, so that's awesome, and you [pointing at S2] are going to share 19 hands, and [says S3's name] is going to shake 19 hands, so what would that lead us to? If every single person is going to shake 19 other people's hands... Discuss for a second, I'll come right back. I think you can get it.

G1 promotes meaningful discussions by building up on students' ideas ("What was your idea?"), validating their thinking ("I like part of that"), and scaffolding their thought with a simplified version of the problem. G1 also gives students time for non-supervised discussion, offering to come back later. The **inclusivity** code characterized instances when MGTAs address students' lack of engagement in a group work. For instance, in the previous interaction, G1 noticed that S3 remained silent and attempts to include them in the discussion while making them part of the example presented to the group. Furthermore, when G1 comes back 3 minutes after the previous interaction, notices that S3 is still not fully included so decides to purposefully stand next addressing them as follows.

G1: [G1 says student's name], how's it going?

S3: I'm good.

G1: Yep, you're good. [The TA stays with S3 and silently reviews the students' work].

G1: So, this is a great idea, I like this. [G1 then validates the S3's ideas and after noticing a mistake asks to explain their thinking, and attempts to guide them to notice it.]

The main themes identified in the third round of coding of two MGTAs (G4, B1) with **low explicit framing group norms** were: brief preamble, unstructured groups, prescriptive feedback, and individuality. The **brief preamble** theme characterized a class start with brief announcements and very little or no time information about the expectations for the day. MGTAs assume everyone will work on the worksheet. Sometimes MGTAs will ask students if they have questions, but in those cases waiting time was brief and students did not volunteer to ask any as shown in the following quote. The **unstructured groups** code represented a freer stance on group work. For instance, G4 in the previous episode made no mention of how groups were to be distributed and later did not force students to be part of a group. Furthermore, individual work is sometimes implicitly or explicitly accepted as shown in the following episode when B1 omit to ask SB1 to join a group.

B1: [Instructor approaches a lone student, asks where their partner is (?)]

SB1: "I was struggling with the integration by parts thing. I was trying to research how to properly do it again. I don't have any questions right now because I'm trying to just wrap my head around the concept."

B1: "Okay. If you've got questions, let me know. [Instructor walks off]

The prescriptive feedback code reflected instances where MGTAs gave detailed instructions or steps to follow, rather than allowing the students to independently explore solutions or discuss different strategies. An example of this can be found in the following episode where B1 approaches a group of four students seated at a table as SC2 asks a question.

SC2: Is that right? {SC2 asks B1 while seated with a group of three students in the same table]

B1: [B1 takes student's paper and quietly reads through problem to himself, and hands it back] Seems fine. I would cancel [something] though.

SC2: [Student clarifies] "I wasn't done with the problem"

B1: [B1 continues monitoring around the room]

B1 directly answered the question of whether SB1's answer was correct, missing an opportunity for instance for the student to explain their thinking. This episode is also an example of the **individuality code.** Although SC2 was in a group of four students, B1 only addressed one of them missing an opportunity to see if peers had a similar concern or make them part of the discussion.

Discussion

This study examined how Mathematics Graduate Teaching Assistants (MGTAs) use explicit group norm framing in small-group settings. Our findings highlight significant differences in the use of this framing. MGTAs who actively engaged in norm framing were able to create more structured and inclusive group dynamics, where students had clearer expectations about participation and communication. For instance, higher norm-framing instructors explicitly outlined the importance of group collaboration and fostered discussions by validating student contributions and addressing misconceptions through scaffolding.

The impact of these practices on student engagement was particularly evident in the inclusivity of students who might otherwise remain on the periphery. One key example from the study was how one MGTA (G1) actively included a previously silent student by personalizing feedback and engaging them in problem-solving, promoting a sense of belonging within the group. This aligns with prior research suggesting that explicit framing of sociomathematical norms can significantly improve student interaction and promote more equitable participation in group work (Esmonde, 2009; Yackel & Cobb, 1996).

Conversely, MGTAs who used lower explicit group norm framing provided minimal direction during group work. These instructors tended to allow students to work individually or without a structured approach to group dynamics. Their feedback often took the form of prescriptive answers rather than fostering collaborative discussions. Such practices, as observed in the study, missed opportunities to promote group cohesion and collective problem-solving. This finding supports previous literature that indicates the absence of structured group norms can perpetuate inequities in participation, particularly for minoritized students (Boaler, 2008; Horn, 2012).

Overall, the study underscores the importance of explicit group norm framing in fostering inclusive learning environments. By providing clear expectations and actively nurturing group work, MGTAs can significantly enhance student engagement and promote equity in classroom interactions. Future research could explore how professional development programs can further support MGTAs in implementing effective group norm-framing practices.

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"Everyone wants everyone to do as well as they can": Norwegian First-Generation College Student Experiences in First-Year Mathematics Courses

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First-generation college students' (FGCS) success in higher education is a growing area of research in the United States (US), but is infrequently recognized in the Norwegian education system; there is a need to investigate how FGCS are successfully navigating first-year mathematics courses to best support FGCS' pursuit of a Science, Technology, Engineering, or Mathematics (STEM) degrees in the US and Norway. This study analyzes six interviews from Norwegian FGCS enrolled in first-year mathematics courses for navigational strategies employed to support their success in mathematics. The preliminary results highlight that the structure of the mathematics courses and the accessibility of resources at the university supported a community of collaboration between peers and instructors. The culture of teamwork across students positively influenced FGCS' success in first-year mathematics courses. We highlight the perspectives of each participant and provide directions for future research to gain a further holistic understanding of supporting FGCS thriving in STEM fields.

Keywords: First-generation college students, first-year mathematics courses, navigational capital

Science, Technology, Engineering, and Mathematics (STEM) fields are pillars of the United States (US) economy and key factors in the US' reputation as a world leader (Langdon et al., 2011). However, there is a "demand rapidly rising for skilled STEM professionals" (Science & Technology Action Committee, 2024) and a perpetual shortage of STEM degree holders (Xue & Larson, 2015). The US is not the only country grappling with the disparity between the number of students graduating with STEM degrees, the increased technical skills required for the workforce, and an excess of STEM jobs (OECD, 2024); Norway is currently experiencing a technical and health sector labor shortage with "occupations that require vocational or higher education are among the most difficult to fill" (OECD, 2024). To meet the continuously growing demand for STEM workers, it is imperative that groups of students who frequently attrite STEM majors (e.g., women, students of color) are retained. One group that could fill this growing demand is the vastly underrepresented group known as first-generation college students (FGCS), or students whose parents did not obtain a bachelor's or higher-level degree.

In Norway, there is minimal research conducted on FGCS, there are no FGCS organizations at institutions, and there is a lack of resources catering to the needs of FGCS. Yet, the demographics of Norwegian higher education are changing with an increase in FGCS enrollment, increasing the need for research centering FGCS. In 2015, Norway reported that 18–24-year-olds without tertiary-educated parents comprise 53% of the population (OECD, 2018, p. 243). The population of 25–34-year-olds in Norway who have attained a tertiary degree has increased by 20% over the last 20 years, meaning that as more students attend university, there will be an increase in FGCS enrollment (OECD, 2022). The Organisation of Economic Co-Operation and Development (OECD) addresses this increase in a report calling for governments to fund FGCS-specific programs (OECD, 2019; OECD, 2020). This study is part of a larger dissertation study drawing on a comparative analysis between FGCS in Norway and the US. This preliminary study aims to investigate Norwegian FGCS' experiences navigating first-year

mathematics courses to better understand how systems and structures in place in first-year mathematics courses support FGCS.

Conceptual Framing

We utilized Community Cultural Wealth (CCW), a student-centered, asset-based theory to capture the complexity and unique qualities of FGCS enrolled in first-year mathematics courses (Yosso, 2005). Community Cultural Wealth was originally developed to highlight capital unique to racially marginalized individuals and breaks down forms of non-dominant capital into the following categories: aspirational, linguistic, familial, social, navigational, and resistant (Yosso, 2005). Due to space limitations, this preliminary analysis focuses on navigational capital, which includes students' knowledge of moving through institutions and gaining social capital (Yosso, 2005). Social capital captures students' "networks of people" (Yosso, 2005, p. 79) and resources students can access, including community, academic, financial, or institutional resources. Within the realm of FGCS majoring in STEM fields, this capital can include students successfully maneuvering through mathematics courses and how students find academic, institutional, and financial resources. In this study, we draw on navigational capital to answer the following question: ***How do first-generation college, STEM-intending Norwegian students discuss navigating first-year tertiary mathematics courses?***

Context of Norwegian University

In Norway, first-year calculus courses (much like in the US) are viewed as gatekeepers for STEM disciplines with reports of first-year calculus failure rates ranging from 21.5% to 43% (Gynnild et al., 2005; Zakariya, 2021). Students pursuing STEM degrees commonly take first-year calculus through their specialized department (e.g. engineering department, mathematics department); yet, the structure of mathematics courses is the same across disciplines, requiring students to complete 70-80% of the course assignments to be eligible for the final exam worth the entirety of their grade (Zakariya et al., 2022). However, the Norwegian university in this study is shifting from high-stakes assessment and has developed multiple attempts, multi-exam first-year mathematics courses to reduce stress and improve student outcomes. Additionally, this Norwegian university has a Center for Excellence in Mathematics focused on improving student outcomes through peer-to-peer tutoring.

Participants

This preliminary study features six participants who were recruited from the same Norwegian institution in Spring 2024. Each participant self-identified as a FGCS (neither parent obtained a bachelor's degree), was enrolled in a first-year mathematics course, and anticipated majoring in a STEM field. Additionally, some participants held other identities or backgrounds that impacted their experiences in transitioning to higher education and first-year mathematics courses. However, some participants did not identify salient identities in addition to their major; as a way to

Table 1. Participant pseudonyms and major.

Participants	
Pseudonym	Major
Ellinor	Mathematics Education
Issa	Software Engineering
Jonas	Mechatronics
Maimona	Artificial Intelligence
Minyguy	Mechatronics
Tito	Mechatronics

ensure anonymity of participants in this preliminary study, we are only reporting the majors provided by the participants (see Table 1).

Methods

Author 1 recruited participants through in-person, in-class presentations, Canvas posts, and emails from second-semester STEM courses, including Calculus 2, Core Elements in Mathematics 2 (modeling with calculus, statistics, and probability), Mathematics 2 (linear algebra, differential equations, and multivariable calculus), and Discrete Mathematics 1 (logic theory, graph theory, combinatorics, and formal models). Six participant interviews were collected and analyzed for this qualitative study. Due to the linguistic diversity of the participants, we provided the option to answer questions in the language of their choice with the option of the interviewer later translating their responses. Each participant chose to complete the interview in English and interjected Norwegian, Danish, Arabic, or Kurdish words when applicable. Each interview lasted for about an hour and participants were compensated with a nominal monetary incentive card.

The in-person semi-structured interviews were recorded via Zoom, transcribed using Microsoft Word, and analyzed to determine the clarity, effectiveness, and overall function of the drafted interview protocol (Clark et al., 2021). We reviewed the transcripts to ensure the transcript accurately reflected the audio as well as to start the data familiarization process. Using qualitative analysis software (MAXQDA), we completed two abductive coding passes. The first coding pass captured instances of participants describing their mathematical background and mathematics course structure. The second coding pass focused on predefined capital (Yosso, 2005) and additional forms of capital that emerged from the data. Through thematic analysis (Braun & Clarke, 2012), codes were grouped and condensed into similarly themed clusters. The quotes within each cluster were summarized to develop emerging themes. An iterative comparison between the codes and summaries occurred which produced the preliminary results found in this paper.

Preliminary Results

Participants discussed how the structure of first-year mathematics courses supported their learning and the collaborative community of peers and instructors they built to successfully navigate first-year mathematics courses.

The Structure of First-Year Mathematics Courses

Mathematics courses at this Norwegian university met twice a week for "an hour and 30 minutes usually [with a] 15-minute break in the middle" (Ellinor). The courses were taught in a lecture-based format where students mainly listened to the instructor and followed along with examples. Tito described his mathematics course in the following way: "the teacher explain[s] basics of what we're going to do and then shows [the] theory behind it, and then maybe some practical examples." Maimona recounted her instructor's didactic practices saying "[the instructor] does go through some tasks. And he asks us, what do you think about [the task]? How do you think you should solve [the task]?" Minyguy similarly described the instructional practices featured in class, "[the instructor] go[es] through different topics on the stand and they explain stuff and you can ask questions, but mostly it's the teacher explaining stuff." Overall, the participants' mathematics courses included the potential for student practice but mainly relied on instructor lectures to convey the material.

Some mathematics courses were paired with a group seminar where students engaged in "practice tasks and the teachers [attended] to answer questions" (Minyguy). The practice seminar was used as additional time for students to engage with the material before assessments which regularly took place throughout the semester. Minyguy elaborated on the assessment structure in the following:

> There are five tests throughout the semester and you have four attempts on each one… I like that over having one major exam because it means that there is less of a last ditch effort right before the exam…You have multiple attempts, so there's not really that much pressure.

However, even with the opportunity to practice outside of class and a more forgiving assessment system, some students found the transition to lecture-based classes to be difficult. For example, Maimona shared "I didn't know how the lecture worked because in high school [I] had [a] different class system." Jonas also expressed how the transition to exam-only assessment proved difficult when learning the material, "I spent the first one or two months pretty much just becoming good at the tests instead of actually getting a grasp on what we were working with." Ellinor confirmed that although the transition to the Norwegian university was "as smooth as possible" until she started second-semester calculus where she "came to the first lecture, and everyone's already a week behind." The twice-a-week frequency combined with seminar space for students to practice and the multiple-attempt, multi-exam assessment structure provided FGCS the opportunity to form a mathematics community.

Creating a Collaborative Community with Peers and Instructors

Two students emphasized their relationships with their instructors in addition to their peers as contributing factors to their success in navigating first-year mathematics courses. Jonas realized building relationships with instructors was key to his success, "if I show engagement, [the instructors] show engagement with me. If I show interest, [the instructors] show interest in me. So that's really good." Ellinor and Jonas noted that instructors were keen on having students ask questions in or outside of class.

> It's just like our professors ask us. Say it. If there's something you don't understand, just say it which makes it a little easier to do. Say, I don't understand that, and then they'll just show it again. Explain it in a different way which helps a lot. (Ellinor)

Instructors encouraged students to speak up during class yet made themselves available after class for any lingering questions from students. Jonas recognized that if he just waited, "most of [his] questions [would] be asked and answered throughout the course, and if they [weren't], [he'd] get a one-on-one with the teacher afterward. [The instructors] put in time for it." The instructors of first-year mathematics courses were seen as approachable and supported student growth through encouraging questions.

In addition to collaborating with instructors, students believed that their friends were "quite gifted" (Issa). Multiple students attributed their success in first-year mathematics courses to working with their peers: "those moments when we collectively understand things is… it's probably one of the most fun things I've ever had in math cause it's just so satisfying" (Tito), "everyone wants everyone to do as well as they can, and that is absolutely a support" (Ellinor), and "I learned the topic better. They learn the topic better. I'm having fun. They're hopefully having fun" (Minyguy). Students mainly collaborated on solving problems from lectures and studying for their mathematics assessments. Maimona noted that she specifically utilized her peers for exam review and corrections because every computer-generated exam was different and the multiple attempts provided more opportunities to practice on a mastery scale. "I feel like

that's actually the reason I'm passing that because I do have multiple tries and other people send me their exams that I get to practice" (Maimona). Tito similarly believed that multiple exams influenced how he studied material. "I sit down with my friends and actually solve the problems and learn a lot from having just that type of exam… that's made me learn more stuff which I wouldn't have done if I'd had an exam at the end of the semester."

The culture of collaboration is normalized by the university's Center for Excellence in Mathematics where students have the opportunity to work together in a dedicated mathematics space and seek help from trained peer tutors. Ellinor described the Center for Excellence in Mathematics as "one of the greatest resources [the university has] that is used a lot." Issa also called the Center for Excellence in Mathematics "a system [at the university] that's really helpful" and the tutors as "really great." Minyguy confirmed that "[the Center for Excellence in Mathematics] is very accessible, it's very low bar of entry." The Center for Excellence in Mathematics is a popular option for students to utilize; when "everyone has finals here and there, it's like you have to fight for the spot at [the Center for Excellence in Mathematics], because so many people sit there to work. And it isn't really like that big of a spot. It's just like a corner of the library" (Ellinor). The accessibility of and normalized nature of seeking assistance through tutoring resources influenced how students collectively worked together.

The collaborative essence of first-year mathematics courses at the Norwegian university encapsulated more than sharing old exams or cooperatively solving mathematics problems, students in these courses were genuinely supportive of each other's growth. For example, Ellinor described sharing notes with peers: "you don't have to give up your notes. It's your notes, but it's like, of course because they would do the same. We always exchange everything. And if someone can't be in class, it's okay, then we take extra good notes." Peer-to-peer support extends to meeting the needs of peers and ensuring everyone in the class has the resources to be successful. "Everyone wants everyone to do as well as they can, and that is absolutely a support. I think this helps everyone in our class. I think it's really important" (Ellinor).

Discussion

We found that FGCS at this Norwegian university primarily navigated first-year mathematics courses by adopting a mentality of partnership or interdependent learning (Ives & Castillo-Montoya, 2020) instead of competition, which is noteworthy given the detrimental impacts of competitive cultures in STEM on FGCS (Canning et al., 2020). The structure of mastery-level assessment and practice seminars provided students with the impetus to collaborate and support each other's academic success in mathematics, which is one of the biggest predictors of retention in STEM disciplines for FGCS (Dika & D'Amico, 2016). Building in-class peer relationships and peer tutor relationships proved to be the most successful navigational strategy for FGCS. However, this preliminary study is just one example of ways FGCS can be supported during their first-year mathematics courses. Additional research needs to be done to address other forms of capital FGCS in Norway are leveraging to persist in first-year mathematics courses. In future work, we anticipate expanding to include all forms of capital captured in CCW (Yosso, 2005) and hope to discover additional forms of capital that are unique to FGCS in Norway. We hope that our findings positively impact structures inhibiting FGCS success in Norway and beyond.

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Measuring the Impact of the Flipped Model in a Pre-Calculus Course

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The flipped model of instruction has been championed as a method for increasing student engagement and enabling instructors to facilitate more active learning within the classroom. Research on the flipped model also suggests that it may also lead to better course outcomes than the traditional lecture approach. Given this potential for student success and engagement, faculty at California State University, Fullerton have adopted the model in many of their courses. In this preliminary report, we measure the impact of the flipped model on course outcomes of a Pre-Calculus course during the Spring 2024 semester. In addition, we compare student perceptions of the course elements that supported their learning between students enrolled in the flipped sections of the course and those in the non-flipped sections of the course.

Keywords: flipped model, active learning, student experiences

Introduction

The flipped model of classroom instruction is a mode of active learning rooted in the constructivist theory of learning (Bishop & Verleger, 2013). In this model, students consume course content outside of the classroom through video lectures and/or course readings and spend in-person classroom time engaging in collaborative learning and problem solving (Lage et al., 2000; Salifu, 2017). Since gaining popularity during the early 2000's, various scholars have championed the model for its effect on student achievement (e.g., Caviglia-Harris, 2016; Connell et al., 2016; Day, 2018; Le & Ichinose, 2022) as well as attitudes towards learning and classroom engagement (Alebrahim & Ku, 2020; Stone, 2012).

Within the past decade, there has been a great shift towards active learning pedagogies (Stone-Johnstone et al., 2024) within the mathematics department at California State University, Fullerton (CSUF). One of the most popular active learning approaches used at CSUF is the flipped model, with many introductory mathematics courses (e.g., College Algebra, and the Calculus series) being flipped by certain professors. In this preliminary report, we evaluate the effect of the flipped model within the Pre-Calculus course at CSUF during the Spring 2024 semester by answering the following research questions:

1. Is there a statistically significant difference in course outcomes between the students in a flipped vs. non-flipped Pre-Calculus course at CSUF?
2. Are there differences in student perceptions of the course elements that supported their learning between the two groups?

Methods

Context

The flipped model, as employed at CSUF, is a three-part model that consists of online interactive content videos, pre-assessments (Ticket in the Door, TITD), and mathematics labs. Traditional lecture time is reconceptualized within this model as a mathematics lab, where

students work collaboratively on activities in their course workbook during in-person class time. Data for this study was collected during the Spring 2024 semester, where there were eight coordinated sections of Pre-Calculus with an average of 35 students in each section. Out of the eight sections, four were flipped and the other four were non-flipped. In the non-flipped sections, students primarily learned course content within the classroom, with some instructors choosing to post optional resources for students to engage within the learning management system (Canvas). All students were assessed using common midterms and final exams.

Theoretical Framework

Given the emphasis on student classroom engagement within the flipped model at CSUF, we view the data through the lens of the Instructional Triangle (Cohen et al., 2003; Lampert, 2001). The instructional triangle is a framework for articulating the process of teaching and learning. This triangle consists of three primary nodes: student, teacher, and content (see Figure 1). The arrows represent the interactions between the nodes. As Cohen et al. (2003) explained, "instruction consists of interactions among teachers and students around content, in environments" (p. 122).

Student-student interactions may occur through in and out-of-class collaboration. Student-teacher interactions include teacher moves for engaging students and supporting their learning (Ball & Forzani, 2009). Student-content represents how students engage in course content, and this may be through course resources (e.g., homework, videos, textbook). Teacher-content interactions represent the instructional choices for content delivery. The concentric circles within Figure 1 represent the varying levels of the environment in which this learning transpires (in and outside the classroom). Given the differing pedagogical approaches within the flipped vs. non-flipped courses, the instructional triangle enables us to explore the student experience by examining the types of interactions that students identified as contributing to their learning.

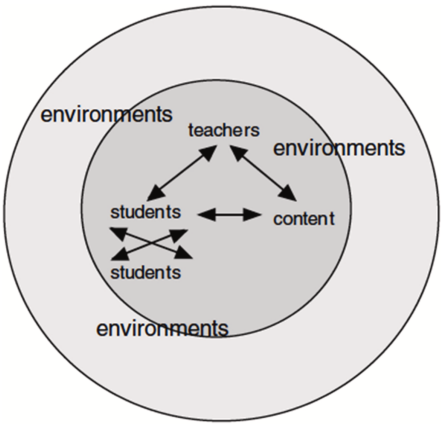


Figure 1. The Instructional Triangle (Cohen et al., 2003).

Data Collection

This study was conducted with approval by CSUF's Institutional Review Board. There were two primary data sources employed to address the research objectives: student course outcome data for all Pre-Calculus students during the Spring 2024 semester and survey data. The authors constructed a Qualtrics survey, consisting of three parts. The first part consisted of Likert-type questions, gauging student perceptions about course structures (e.g., course workbooks, activities, and assessments). The second part consisted of three open-ended questions regarding their perceptions of support and barriers to their learning of course content. The final section

consisted of demographic questions including information about previously attempted mathematics courses, placement scores (for entrance at CSUF), and their projected course grade.

Data Analysis

Employing a mixed-methods approach to data analysis (Creswell, 2012), we answer the first research question by conducting a quantitative analysis. Specifically, we performed a two-sample *t*-test to determine whether there was a statistically significant difference in the course outcomes between flipped and non-flipped courses. Students who withdrew from the course (n = 25) were removed from the analysis since no point value was assigned to a grade of W.

We answer the second research question regarding student perception of the support for their learning of Pre-Calculus, by conducting a qualitative analysis. All Pre-Calculus students were asked to complete the Qualtrics survey during the ninth and tenth week of the semester. Of the 276 enrolled students, 173 completed the survey (approximately 63% response rate): 88 students in the flipped course and 85 in the non-flipped course. For the purpose of this investigation, we focused on students' responses to the open-ended question, "What are two or three components of the course that have supported your learning?"

All student responses were coded using a coding scheme adopted from the Instructional Triangle framework (Cohen et al., 2003). Since students discussed aspects of the course that they found beneficial to their learning, we limited our coding to focus on the Student-Student, Student-Teacher, and Student-Content interactions within the Instructional Triangle. The dataset did not allow for us to explore the Teacher-Content interactions, nor teacher motivations for their instructional choices (i.e., Herbst & Chazan, 2012). Throughout the coding process, a subcode of the Student-Content interaction was created, "External Resources," to describe instances where students relied on learning resources separate from the ones provided by the instructor. A second subcode, "Office Hours," was generated to differentiate between iniclass teacher support and outside-class teacher support, given the additional effort required for the latter.

There were 132 complete student responses to this open-ended question. The first author coded the entire dataset, and 30% of the dataset was double-coded by two other members of the research team. Instances of disagreement were discussed and negotiated, with only two instances of disagreement (95% agreement).

Preliminary Findings

Course Outcomes

The result of the two-tailed independent samples t-test was not significant based on an alpha value of .05, $t(248) = 1.53$, $p = .126$, indicating there was no significant difference in the course performance (course grade point value) between the flipped and non-flipped classrooms. The results are presented in Table 1. While the difference in performance was not statistically significant, the overall mean score in the flipped classrooms was slightly higher. In particular, the mean value of course grades is 0.27 points higher for the flipped modality compared to the non-flipped modality (~ 13.6% difference).

Table 1: Two-Tailed Independent Samples t-Test for Final Grade Point Value by Modality

	Flipped			Non-Flipped				
Variable	M	SD	n	M	SD	n	t	p

Point Value	2.25	1.31	114	1.98	1.36	136	1.53	.126

Perceived Supports for Learning

When comparing the students in the flipped course to those in the non-flipped course, we did not observe meaningful differences in the course components that students identified as supporting their learning. Table 2 illustrates the frequency of the different codes across the two groups of students.

Table 2. Distribution of Code Frequencies between Students in Flipped and Non-Flipped Courses

	Flipped	Non-Flipped
Student-Student	11	5
Student-Teacher	27	22
Office Hours	1	8
Student-Content	57	50
External Resources	2	5

More of the flipped students mentioned Student-Student coded elements as supporting their learning, with one student commenting: "We help out what questions we have to answer as a group while working on the workbook and worksheet." Some students in the non-flipped course also expressed valuing group collaboration, explicitly naming "in class group activities" as supporting their learning.

Students in both groups credited their instructor's in-class explanations and lectures for supporting their learning. However, it is worth noting that eight students from the non-flipped group (compared to just one from the flipped group) specifically mentioned attending their instructor's office hours as a course element that supported their learning.

The majority of the dataset was coded as "Student-Content," with both groups of students identifying the workbook, the online homework, and, for the flipped classroom students, the Ticket in the Doors as valuable to their learning. The primary differences that arose between the two groups were with respect to the types of external resources students credited. Two flipped students mentioned workshops (Supplemental Instruction) as a support, while the five non-flipped students mentioned resources such as YouTube and Khan Academy.

Discussion

The preliminary findings revealed no statistically significant difference in course outcomes between the students in a flipped vs. non-flipped course; however, the data does suggest the flipped model may still have potential benefits for course outcomes. The observed 13.6% higher mean point value of letter grades between the flipped and non-flipped students is promising. Similarly, we did not observe notable differences when comparing the student's perceptions of course supports between the flipped and non-flipped modalities. Students from both modalities primarily credited their learning to course elements such as homework assignments and in-class workbook activities.

Considering the active teaching culture within the mathematics department at CSUF, one could speculate that the non-flipped model might have incorporated more interactive elements compared to a traditional lecture-based course. If so, this could account for the similarities in

student perceptions regarding the course elements that supported their learning between both modalities. The only notable difference in students' perceptions was that non-flipped students reported using Khan Academy and YouTube for video lecture support whereas the flipped students already had instructor-provided videos embedded within the course.

As we continue to explore this data, and the flipped classroom model in general, we invite the audience to consider the following questions:

1. Though we found no significant or noticeable differences between flipped and non-flipped modalities of instruction, how does this finding contend with the current teaching and learning climate?
2. What specific aspects of the flipped and non-flipped course formats might have led to the observed lack of statistically significant difference in student outcomes?

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Is it a Simple Mistake?: Tutors' Perceptions of Student Errors

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Undergraduate math tutors frequently interact with students who are currently at a point of impasse or who have experienced failure and are seeking help. This affords tutors the opportunity to impact students' understanding of mathematics as well as redirect their study efforts. Helping students accurately classify their mistakes is a worthwhile effort, but is challenging and requires training and support. However, to provide this training for tutors, we need to know how tutors view student errors. This report details preliminary results from a survey given to current tutors, asking them to classify authentic student mistakes on a test as "simple" or "not simple." We discuss multiple reasons tutors give for their classifications, including some which stand in contrast to those we might use as researchers.

Keywords: error analysis, tutoring, college algebra, simple mistakes

Experiences in first year mathematics courses influence persistence in Science, Engineering, Technology, or Mathematics (STEM) majors (Seymour et al., 2019). The presence of a math tutoring center, outside the classroom, has been shown to be significantly related to a positive student attitude toward mathematics (Bressoud et al., 2019). A key element of tutoring is identifying student mistakes and addressing students' misconceptions. Consider the following student work from a College Algebra exam.

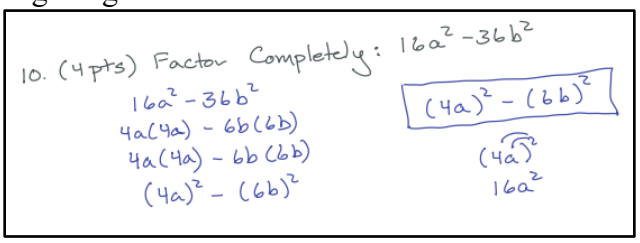


Figure 1. A student's work on a College Algebra exam problem asking students to complete factor $16a^2$-$36b^2$.

When presented with the student work (Figure 1), some undergraduate math tutors classified the mistake as being simple and others did not. A tutor who felt that the mistake was simple stated, "The student recognized that both 16a^2 and 36b^2 are both perfect squares…I think this mistake is just a lack of pattern recognition and can be solved with just more reps and familiarity." In contrast, another tutor felt that the mistake was not simple, stating "I don't think they understood what the question was asking and knowing what to do when asked to factor is important." When people in support roles, such as tutors, provide assistance to students, their perceptions of students' errors can impact the type of assistance they decide to give. However, tutors are often expected to provide guidance and instruction without training in how to analyze student work or respond to it. This preliminary report discusses how tutors classify College Algebra students' errors on an exam, specifically addressing the question, *What justifications do undergraduate math tutors provide when classifying a student's error as simple or not simple?*

Literature and Theoretical Perspective

Errors are typically described as a deviation between "the current state and the desired state" or a lack of adherence to convention (Tulis et al., 2015). Mathematical mistakes may be a result

of misconceptions or "inadequate cognitive schemes" or due to a mental "slip" (Molina et al., 2017). We distinguish between a "simple mistake," which is "one that could be made accidentally, would likely not be repeated, or violates a mathematical convention rather than a rule" and a "not simple mistake" which provides evidence of "lack of conceptual understanding" (Ryals et al., 2020). Prior work has shown that students and teachers view mistakes differently (Lannin et al., 2007), and students are more likely than researchers to categorize their own errors as simple mistakes (Ryals et al., 2020).

Our study is motivated and guided by our valuing the "constructive error climate," which Benecke and Kaiser (2023) describe as "one where both teachers and students acknowledge errors as potentially helpful" and "errors are analyzed and used as learning opportunities" (p. 164). The traditionally negative view of mistakes as something to avoid, is shifting toward a more positive view of mistakes as contributing to learning (Borasi,1987; Dweck, 2006). Reflecting on mistakes helps learners establish correct mental models (Kapur 2010; Kapur 2014; VanLehn et al. 2003). Still, students do not often use errors to learn, as learning from errors is challenging and more demanding than simply looking at a correct solution (Kyaruzi et al, 2020; Heinze et al., 2012). Students need guidance in learning from mistakes. While researchers have shown that beliefs and self-regulation strategies are important, teachers still concentrate primarily on content rather than on problem solving, metacognition, and beliefs (Schoenfeld, 2022). They typically do not direct students' attention to ways to avoid similar errors in the future (Rach et al., 2013). We therefore recognize an important opportunity for tutors to support individual students to analyze, understand, and learn from their errors.

While new concepts are typically introduced during class time, it is during office hours and tutoring sessions that students can discuss their mistakes. Undergraduate peer tutors view mistakes from two perspectives; they recognize their own mistakes as students, and they recognize common mistakes they see in their students' work (Ryals et al., 2022). Peer tutors also work in the environment of a tutoring center where the teacher is not present and the student may be more comfortable discussing their mistakes. While tutors have a unique opportunity to help students understand and learn from their errors, and do so in a positive manner, they would likely need training to do so (Johns, 2022). To provide effective training, we first must determine how tutors view students' mistakes and how those views differ from our own.

Methods

Data Collection

We designed a survey with six student errors from a College Algebra exam, including the one presented above. We selected these examples strategically, wanting to provide a mixture of examples that we, as researchers, considered simple and not simple errors. We also wanted to choose student work that contained one specific error, rather than several, so that attention could be drawn to that one error for analysis. Each problem itself along with the student work was displayed in the survey, and we provided a short description of the error that was made. We asked the tutor to classify the error as Simple or Not Simple and to explain their choice. We did not give the tutors a definition of a simple mistake and did not provide examples of mistakes we would consider simple or not simple. At the end of the survey, tutors were asked to summarize how they distinguish between simple and not simple mistakes and how the type of mistakes students had made on a recent test would influence how they help a student prepare for future tests.

The survey was distributed to all current tutors employed in the campus Math Learning Centers at two institutions in the Spring 2023 term. Eleven tutors at Institution A and 9 at Institution B submitted the survey. Of these 20 submissions, 11 submissions were complete. We omitted responses to the sixth error because we had provided an incorrect description of that error in the survey. Therefore, our analysis focuses on the categorizations of 5 errors by 11 tutors. We have not yet analyzed responses to the final two questions of the survey, where tutors were summarizing their definitions and discussing how they would assist students who had made particular types of mistakes.

Analysis

We began by reading through each tutor's response to each student error. We individually used open coding to begin developing a codebook. We then discussed each response and the codes we had individually developed and came to consensus on which codes to apply for each tutor response. We decided that multiple codes could be applied to a single tutor response. We added these to the first version of our Codebook.

We then individually used this codebook to do a second round of coding and used the constant comparative method to refine and develop new codes (Glaser, 1965). We noted difficulties in applying these codes and met a second time to resolve the codes. In the third round of code development and modification, we began condensing codes and creating subcodes. For example, we had applied the code "Lacking Knowledge" to many tutor responses. We grouped the responses based on similarities and replaced this code with four more specific codes that identified what type of knowledge the student was lacking (e.g., conceptual, specific formulae).

Results

We begin with a discussion of responses tutors gave to support their classification of a student error being simple. Later we will discuss responses given to support not simple classifications, and finally those that could be used for either simple or not simple. The diagram below (Figure 2) shows each code and whether it was applied more often to responses that classified the error as simple or not simple.

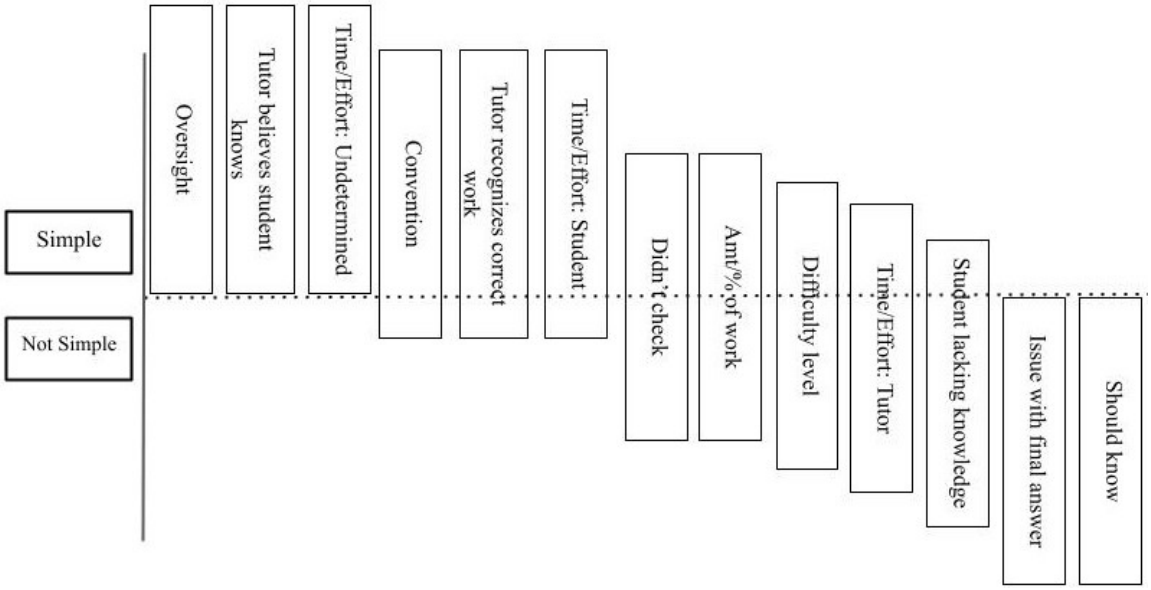


Figure 2. Codes Describing Reasons Tutors Provide for Error Classification

Codes for Simple Mistakes

There were three codes we developed that were applied only to responses where tutors had classified an error as simple. The first was *oversight.* We applied this code when the tutor claimed the mistake was simple because the student had overlooked something or worked too quickly. The second was *tutor believes student knows*. This code was used when tutors stated or implied that despite the error, the student understands the concept or knows the procedure. We also created a code, *time/effort: undetermined* that we used when tutors indicated the mistake was simple because not much time or effort would be required to prevent the error. When this code was applied, it was unclear whether the tutor was referencing time and effort on the part of the tutor, student, or both.

Other codes were used almost exclusively for mistakes tutors classified as simple. Tutors sometimes claimed that the student had not made an error, but rather had failed to adhere to classroom expectations or convention, such as simplifying an expression to a certain point. To these we applied the code *convention*. There were also tutor explanations that focused on the work completed in the problem. We applied the code *tutor recognizes correct work* when tutors explained the student had successfully completed the difficult part of the problem or noted students had demonstrated the knowledge needed to avoid that error somewhere else in the problem. Finally, sometimes tutors focused not on the work done in the problem but what could have been done prior to the exam to prevent the error. Multiple responses indicated an error was simple because the *time/effort* required by the *student* to avoid the error was relatively low.

Codes for Not Simple Mistakes

Several reasons were used by tutors exclusively to justify classifying errors as not simple. The first of these was that the student *should know* how to solve this problem correctly. In cases where we applied this code, tutors discussed their expectation that students in the course should not have made this error or this was not an understandable mistake. A second code, *issue with final answer*, was applied when a tutor claimed an error was not simple because the error caused the student's answer to be significantly different from the correct response.

Many tutor responses attributed errors to the *student lacking knowledge*. We identified four types of knowledge they referenced; two of these were applied only in situations where the tutors were claiming an error was not simple: *lacking conceptual knowledge* and *lacking knowledge of specific content or rules*. When the code lacking conceptual knowledge was applied, tutors described a lack of understanding, reasoning, or fundamental knowledge. The code, lacking knowledge of specific content or rules, was used when tutors had indicated the mistake was not simple because the student was unfamiliar with or had forgotten a rule or formula.

Codes for Both Simple and Not Simple Mistakes

We developed two other codes to use when tutors referenced a lack of knowledge. In the first, *structure and strategy*, tutors attributed errors to an inability to recognize the problem structure or identify an appropriate strategy for solving the problem. Closely related was the code *goal state issues*. This was applied when tutors indicated the student could not identify what the problem was asking or what type of answer they were expected to provide. Both these codes were applied at times to justify classifying a mistake as simple, and others to justify the choice of not simple. Other codes related not to the knowledge tutors believed students had but to student behavior while solving the problem. These include *didn't check* and *amount/percent of work*. Tutor responses with the code didn't check indicated the error would have been avoided if the

student had checked their work. Responses coded as amount/percent of work focused on the amount of work shown or the percent of the problem that was successfully completed. Again, these codes were sometimes used to support classifying a mistake as simple and at other times to support classifying a mistake as not simple.

Two additional types of responses were used to justify both simple and not simple classifications. In the first, tutors would describe the amount or type of assistance they would need to provide the student for them to avoid the error. This was coded *time/effort: tutor*. Not simple errors were ones that would require significant intervention by the tutor. In the second type of response, the tutor focused not on the student's answer but on the *difficulty level* of the problem itself. It was common for tutors to associate problems they perceived as easy with simple mistakes, though this wasn't always the case.

Codes About the Tutors' Reflective Process

The codes discussed above were reasons tutors gave to explain their classification of an error as simple or not simple. However, three additional codes were created that describe the tutors' reflective process as they determined whether students' mistakes were simple or not. While these codes do not directly address the research question, we wanted to capture recurring themes in the tutor approach to analyzing student work that may be useful in future analysis. The first of these codes was *more information*. This was applied to responses where tutors expressed uncertainty about their decision to classify the error as simple or not simple. The second was *tutor notes multiple reasons* and was applied if the tutor gave multiple possible explanations for the student's mistake. Finally, the code *tutor references past experience* was applied if tutors made mention of a mistake being common or seeing this issue with students in the past.

Discussion

In this study we asked undergraduate math tutors to classify student errors on a College Algebra test as simple or not simple and provide justifications for their decisions. We saw that at times tutors provided similar distinctions between simple and not simple errors that we had previously made ourselves (Ryals et al., 2020). For example, tutors typically said that mistakes made in-the-moment due to *oversight* or errors that could be attributed to not conforming to *convention*, such as simplification, were simple. Tutors also tended to classify a mistake as simple if the student had demonstrated the necessary knowledge or skills elsewhere in the problem. In contrast, the tutors were much more likely to consider a mistake not simple if they attributed the mistake to a *lack of knowledge*.

Some of the tutors' distinctions between simple and not simple stood in contrast to our own. For example, tutors sometimes attended to features of the problem itself, rather than the student work. In other cases, they focused on what would need to be done, either by the student or tutor, to prevent the error. If the time and/or effort required was small enough, tutors considered the mistake simple. We also saw cases of the tutor deciding whether a mistake was simple based on how much the error impacted the student's final answer.

We have more data to analyze from the completed survey and would like to distribute the survey more widely. We pose the following questions for discussion:

1. How might we modify the format or administration of the survey in future iterations to collect data we can use to further refine our current codebook?
2. How does the experience level of the tutor impact/influence their categorizations about the simple/not simple distinction? Can we answer this using our current data?
3. How can we use these results to provide training for new and experienced tutors?

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Identifying Pedagogical Proof Moves

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Drawing on the concept of "moves" from English for Specific Purposes (ESP) genre theory, we identified pedagogical proof moves—moves proof authors make for pedagogical purposes—in a corpus of 40 abstract algebra proofs. We share 12 such pedagogical proof moves, presenting three in greater detail.

Keywords: move analysis, pedagogy, proof

There exists an image of mathematics texts as consisting of "pages of information with virtually no repetition, no varying of pace, few topic and concluding sentences, as few words as possible [… where] clarification is achieved by constructing very precise sentences without any extra words" (Tobias, 1989, p. 49). This observation about the terseness of mathematics texts could be made about proofs in particular—and *is* perhaps the image of prototypical proofs (see Dawkins and Weber, 2017, documenting the norm that "irrelevant statements are not presented in a proof," p. 131). Yet, in our experience, proofs, particularly those written for undergraduate students, are *not* solely presented as a chain of deductions without repetition or clarification. Consider the following example from Abstract Algebra lecture notes:

> **Theorem VI.2.** *Let (G,○) be a group and let a be an element of G. Then a has a unique inverse.*
>
> PROOF. Our proof follows the same pattern as the proof of Theorem VI.1, and you'll see this pattern again and again during your undergraduate career. Almost all uniqueness proofs follow the same pattern: suppose that there are two of the thing that we want to prove unique; show that these two must be equal; therefore it is unique. ((This paragraph has the phrase "*key trick!*" written in the margins next to it.))
>
> For our proof we suppose that *b* and *c* are both inverses of *a*. […] ((proof continues))

As we hope the reader will agree, the entire first paragraph is logically unnecessary. But the author's choice to include such "superfluous" elements indicates that they are not superfluous and do serve a purpose. Specifically, the paragraph's purpose appears to be to facilitate learning by distilling and pointing out the key idea behind most uniqueness proofs so that the learner may recognize the given proof as being an instance of this particular proof type.

To describe this first paragraph (and similar pedagogical elements of proof), we draw on a concept from English for Specific Purposes (ESP) genre theory: moves. Moves are functional units of a genre, that is, units which are focused on what a text is doing, not on what a text is saying or its form (Hyon, 2018). Moves have been studied since Swales's (1981) identification of a four-move structure in journal article introductions, later amended to a three-move structure (Swales, 1990) (e.g., establishing a niche). Relevant to our study is that moves: (a) may vary in size and do not need to align with grammatical units, (b) can be optional (i.e., not all moves need to appear in every text), and (c) can be flexibly used (i.e., moves need not occur in a specific sequence) (Bhatia, 1993; Hyon, 2018). With this in mind, we see the proof's first paragraph as realizing a move which one might call *commenting on proving*. Given our focus on the genre of proof and our interest in those moves that appear to serve distinctly pedagogical purposes (i.e., facilitate learning), we seek to address: *What pedagogical proof moves do authors of undergraduate texts make in their proofs?* To better describe these moves in our Results section,

we will also draw on concepts from close cousins of ESP genre theory: (a) intertextuality from Bakhtinian genre theory, and (b) metafunctions from systemic functional linguistics (SFL) genre theory.

Relevant Literature

Mathematics education researchers have termed mathematical proof a genre (e.g., Lew & Mejía Ramos, 2020; Selden & Selden, 2014), a claim that we recently demonstrated aligns with modern genre theory traditions (Küchle & Dawkins, 2024). Since "[a] genre comprises a class of communicative events, the members of which share some set of communicative purposes" (Swales, 1990, p. 58), (slight) differences in communicative purpose can lead to (slightly) different genres. Therefore, we concur with Lai et al. (2012) and Lai and Weber (2014) that it is appropriate to view "proof[s] […] presented to students for pedagogical purposes" (Lai et al., 2012, p. 147) as a sub-genre of proofs.

This sub-genre itself contains some variation. Both Lai and Weber's (2014) and Pinto and Karsenty's (2020) mathematician participants expressed that features of their proofs for pedagogical purposes would depend on whether the proof was presented in lecture or written for a textbook. Both studies showed that mathematicians believed that textbook proofs should be rigorous and complete, whereas lecture proofs—due to lectures' affordances (e.g., reading the audience, gesturing) and constraints (e.g., time)—are allowed to omit details but should incorporate pictures, highlight main ideas, and communicate intuitions. Thus, pedagogical additions to proof (e.g., incorporating pictures, highlighting main ideas) appeared to be more associated with lecture proofs than textbook proofs for the mathematician participants.

Yet, Lai et al. (2012) demonstrated that in textbook proofs written for pedagogical purposes, mathematicians valued more than just rigor and completeness. By asking participants to improve the clarity of two proofs intended for a sophomore- or junior-level mathematics major, Lai et al. (2012) identified three types of revisions the mathematicians made: additions (e.g., adding a picture), alterations (e.g., altering phrasing), and deletions (e.g., deleting irrelevant statements). Further, Lai et al. (2012) confirmed mathematicians' beliefs that the quality of a proof presented for pedagogical purposes could be improved by: (a) "adding introductory and concluding sentences that highlight the proof's framework" (p. 158), (b) "highlighting the main ideas of a proof with the judicious use of formatting" (p. 158), and (c) avoiding the addition of "unnecessary or irrelevant assumptions or computations" (p. 158).

In sum, there is evidence that proofs for pedagogical purposes is a sub-genre of proof and that mathematicians are aware of it. Moreover, mathematicians appear to distinguish between proofs for pedagogical purposes in lectures and those in textbooks. In particular, mathematicians appear to find that several pedagogical ideas (e.g., incorporating pictures, highlighting main ideas) seem to have less of a place in textbooks than in lectures. At the same time, Lai et al. (2012) identified some additions, alterations, and deletions that mathematicians make to textbook proofs to increase their pedagogical merit.

Methods

Data Collection

To build a corpus of proofs, we selected ten random proofs from four Abstract Algebra texts for a total of 40 proofs. The four Abstract Algebra texts were selected to be similar in content but different in style so that pedagogical moves might stand out more. They were: (a) Abstract Algebra lecture notes that the first author remembered for their pedagogical merit (e.g., see the

example at the start of the paper); (b) Gallian's (2017) *Contemporary abstract algebra*, which "[m]ay be the most widely used undergraduate algebra text" (Isaacs et al., 2015); (c) Dummit and Foote's (2004) *Abstract algebra*, chosen for its terseness; and (d) Carter's (2009) *Visual group theory*, chosen for its inclusion of visuals—common in mathematics, broadly, but, in our experience, less common in abstract algebra.

The 40 proofs were randomly selected as follows. For each text, a random number generator was used to select random pages. If a page contained no proof, we moved to the next randomly generated page number. If a page contained one proof, we selected this proof. If a page contained multiple proofs, a random number generator was used to select a proof from this page.

Data Analysis

As Lai et al. (2012) found, mathematicians add, alter, and delete texts (and pictures) from proofs to make them more pedagogical. Given our data (i.e., fixed proof texts), we decided to focus entirely on additively-realized pedagogical proof moves, that is, pedagogical proof moves that we were able to discern because they are realized through the addition of text. (Thus, the moves we eventually identified are a strict subset of all possible pedagogical proof moves.)

To identify these (additively-realized) pedagogical proof moves, we began by using a heuristic, by which we mean an "economical shortcut procedure[] that may not lead to optimal or correct results, but will generally produce outcomes that are in some sense satisfactory or 'good enough'" (Chow, 2015, pp. 977–978). In particular, we began by reading through the 40 proofs and crossing out as much text as possible to still leave a proof we judged to be logically valid. There were several instances in which we were unsure whether text could be crossed out, in which case we erred on the side of crossing it out. (Since we would later study this crossed out text more carefully, this means we erred on the side of overcoding.)

Studying the crossed-out text and asking questions around which text could be omitted to still leave a valid proof brought to our attention two types of moves, which we had felt unsure about crossing out from proofs:

- delimiting (Konior, 1993) moves (i.e., moves which indicate the beginning or ending of [a part of] the proof) (e.g., "Conversely, […]"); and
- justifying-supporting moves (i.e., moves which do not justify, but which provide justifying support by acknowledging that a warrant is needed) (e.g., "As in the proof of Cayley's Theorem, it is easy to verify that […] ((author omits argument))").

With awareness for these moves gained, we next coded each proof from beginning to end for:

- proving moves (i.e., moves which support the logical structure of the argument) (e.g., deducing, instantiating, defining);
- delimiting moves;
- justifying-supporting moves; and
- other (moves).

Given that moves can occur at various grain sizes (Hyon, 2018), our moves range in size from a word (e.g., "Conversely" realizing a delimiting move) to multiple sentences (e.g., the first paragraph of the Abstract Algebra proof at the start of this paper). Further, since text segments can perform multiple functions simultaneously, we allowed for the possibility of text segments realizing multiple moves at once.

We then focused our attention on the "other (moves)" code (to discern pedagogical proof moves) and coded its instances for their function(s) via process coding (Saldaña, 2009), whereby we described their function(s) using a verb or verb phrase (e.g., *simplifying notation*). We are

currently engaging in pattern coding (Saldaña, 2009), that is, we are in the process of combining and consolidating our process codes into a smaller number of pedagogical proof move codes.

Results

Below, we present the three most common pedagogical proof moves in our corpus. We then briefly list the other identified pedagogical proof moves.

Reminding the Reader

The pedagogical proof move that occurred most often (30 times) and in all sources was *reminding the reader*. To further describe this move, we found the concept of intertextuality useful, which refers to "how texts draw upon, incorporate, recontextualize and dialogue with other texts" (Fairclough, 2003, p. 17). We observed authors reminding the reader by: (a) expanding on intertextual references, and (b) making eliminable intertextual references.

In the case of expanding on intertextual references, authors reminded readers of the details of the reference they were making (e.g., reminding the reader of a definition, process, notation). For example: After "Let's recall what the quotient process does to arrows between cosets," the author proceeded to describe the quotient process. This type of reminder saves the reader the potential work of looking up the details of the reference being made.

In the case of making eliminable intertextual references, authors drew a connection for readers to past work (e.g., "This is similar to the proof of Proposition 25"). Thus far, we have discerned two distinct pedagogical uses. First, authors pointed to past work to explicitly or implicitly leave part of a proof to the reader. For example: "This has almost the same proof as Fermat's Little Theorem. I'll leave the necessary modifications as an easy exercise." This way, the author provides the reader with an idea of how the statement could be proved. Second, authors appeared to make eliminable intertextual references to get readers to see similarities between proofs, possibly so that readers might draw on previous experience and/or objectify a proof approach (e.g., the start of the proof presented at the beginning of this paper).

Indicating the Difficulty of an Argument

In all four sources, authors used language that indicated the difficulty of an argument they were making. Yet, there appeared to be different purposes. First, we observed authors using words such as "clearly," "obviously," and "it is immediate" to serve as placeholder warrants (or warrants of self-evidence) (12 times). Given that they were still part of the warrant structure, we coded these as justifying-supporting moves, not necessarily as pedagogical proof moves.

Given our focus on pedagogical proof moves, it was more interesting to us that we also observed words describing effort (e.g., easy, not very hard, only basic algebra, simple) or apparency (e.g., of course, just, straightforward) *not* being used as placeholder warrants (17 times). Instead, they were used to describe the effort a problem would take (e.g., "I'll leave the necessary modifications as an easy exercise"), sometimes alongside an exercise for the reader, sometimes as a description of the upcoming steps. These examples appeared to serve at least one of the following two functions: (a) reassuring the reader, and (b) putting a (sub)proof into perspective. In particular, drawing on the SFL idea of a text's three metafunctions (i.e., ideational, interpersonal, and textual), we see descriptions of effort and/or apparency as possibly intending to reassure the reader (interpersonal function) as well as putting into perspective a (sub)proof so the reader knows where (not) to focus their attention (ideational function). As Halmos (1970) noted, "the reason for saying that [statements] are obvious is to put them in proper perspective for the uninitiate" (p. 137).

Last, motivated by our data, we share an observation—probably familiar to mathematics educators but novel to (undergraduate) newcomers—about the use of "trivial," a seemingly close cousin to the above words of effort and apparency. Although "trivial" can be used to indicate the difficulty of an argument, it also has its very own mathematical meaning as "the most simple example of something" (cringo, 2012) or "the canonical obvious thing" (Brett Frankel, 2012) (e.g., trivial solutions, trivial representation, trivial subgroup). By differing (slightly) from the colloquial meaning, we posit that "trivial" could be received differently by newcomers than intended (see Pimm's, 1987, writing on semantic contamination).

Adding Clarification

The third move we saw all sources engage in was *adding clarification* (20 times). We documented authors adding clarification for two main purposes:

- to add precision (e.g., "We really mean that ($\mathbf{Z}/m\mathbf{Z},+,\cdot$) is a commutative ring."); and
- to "unpack" (or verbalize) symbols and diagrams (e.g., "(In words, T_g is just multiplication by g on the left.)").

Other Moves

There were three other moves we documented in multiple sources: (a) emphasizing (e.g., "it is the **least positive** integer") (12 times), (b) simplifying notation (e.g., "Write $g = \gcd(a,m)$") (3 times), and (c) providing extra information about mathematical objects, processes, or claims (e.g., "Define […] (This mapping is called the *natural homomorphism* from G to G/N.)") (3 times). Furthermore, we identified six moves that each appeared in only one of the sources: (a) visualizing a mathematical idea (10 times), (b) giving an example (6 times), (c) labeling an (set of) equation(s) for possible later reference (3 times), (d) amusing the reader (2 times), (e) pointing out a common student mistake (1 time), and (f) commenting on proving (1 time).

Discussion

As our current findings suggest, there is more to writing textbook proofs for pedagogical purposes than writing rigorous and complete proofs. In particular, the authors of the proofs we studied added a variety of logically unnecessary (but pedagogical) moves to their proofs—in contrast to the image of prototypical proofs. Although our focus was on pedagogical proof moves, this work also raises awareness of delimiting and justifying-supporting moves, which often intersected with pedagogical proof moves and merit further study.

As touched upon, a limitation of our work is its exclusive focus on pedagogical proof moves realized through the addition of text (as opposed to the alteration or deletion of text). Accordingly, we wonder: How could we empirically identify pedagogical proof moves in non-eliminable text? Further, we acknowledge that there are eight "other (moves)" segments that we have been unable to further describe with our current move codes. It is unclear to us if this is an issue with our identified moves or rather a natural finding about the prevalence of text segments without easily discernible purpose. Third, we recognize that although our corpus is appropriate in size for a move analysis, it has been limited to one domain of mathematics. Heeding the warning of Dawkins and Karunakaran (2016), we are in the process of expanding our corpus to Real Analysis and Linear Algebra texts, giving us an opportunity to evaluate our codes in other domains. Finally, although we have presented our work in the typical fashion of highlighting the most common codes (i.e., popular pedagogical proof moves), we wonder: Should this line of work instead highlight less common (and perhaps idiosyncratic) pedagogical proof moves to provide new ideas and possibly disrupt the status quo of (pedagogical) proof-writing?

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Developing Understandings of Accumulation: A Preliminary Research Report

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This preliminary research report examines how undergraduates develop quantitative meanings of accumulation from rate, a foundation for understanding the definite integral and the Fundamental Theorem of Calculus. This study is a pilot study for a larger project. It consists of a pre-/post-assessment and a teaching experiment. The task sequence provides opportunities for students to develop productive meanings for speed, distance, and their relationship as rate of change/accumulation. Preliminary analysis of the pre-assessment indicates that students struggle to conceptualize average and instantaneous speed, which suggests that the teaching experiment can be productive. In this report, I outline the theoretical framework, task sequence, preliminary findings, and ongoing work. I conclude with areas for audience discussion: Seeking feedback on analysis and ideas for expanding upon this pilot study.

Keywords: calculus, definite integral, accumulation, quantitative reasoning, student learning

The definite integral is a concept used for mathematical modeling in many disciplines including physics (e.g., Griffiths, 2013), economics and business (e.g., May & Bart, 2024), and biology (e.g., Garfinkel et al., 2017). Approaches to teaching the definite integral in high school and college calculus courses often promote a meaning for the integral as the area of a region bounded by a function's graph and the x-axis (Huffman Hayes, 2024). However, this meaning of the definite integral is idiosyncratic to the Cartesian coordinate system (e.g., Moore et al., 2014), and it is not productive for modeling all physical contexts (e.g., Sealey, 2006) or for making sense of all mathematical situations, such as backwards integration (Bajacharya et al., 2023).

Over the past decade, a small but growing body of literature has advanced the idea that students should be supported in developing quantitative meanings for the definite integral, i.e., focusing on the meanings of each component of the definite integral in a situation (Jones & Ely, 2023). In their review of literature, Jones and Ely (2023) outline two distinct approaches that researchers have used to develop students' quantitative meanings for the definite integral: Adding up the pieces (AUP) and accumulation from rate (AR). In brief, the difference between the AUP and AR approaches lies mainly in the relationships and quantities that compose the integrand. With an AUP approach, one conceives of the integrand as a multiplicative relationship that forms the target quantity when these "tiny bits" of the target quantity are summed (e.g., Oehrtman & Simmons, 2023; Stevens & Jones, 2023). In an AR approach, the relationship between the integrand and target quantity is explicitly covariational in that it considers how the changing rate of the target quantity affects the total accumulation of the target quantity (e.g., coordinating how total distance driven varies with instantaneous speed; Thompson et al., 2013).

Most of the recent research on students' meanings for the definite integral advances approaches that are compatible with AUP (Jones & Ely, 2023). This emphasis in the literature base may be because the AUP approach is seen as more intuitive and accessible for many calculus students. Intuitive because an AUP perspective can be more easily visualized for many of the bounded, physical contexts that students encounter in their college coursework. Accessible because students often enter calculus with limited understandings of rate (Frank & Thompson, 2021; Thompson 1994), which may make using the AR approach more challenging.

I argue, however, that the AR approach has benefits that outweigh its challenges. The AR approach is derived directly from the Fundamental Theorem of Calculus (FTC), which is the backbone of the discipline and is a major idea in many calculus curricula (Huffman Hayes, 2024; Lanius, 2024). Thompson (1994) emphasized the FTC as "the realization that the accumulation of a quantity and the rate of change of its accumulation are tightly related—is one of the intellectual hallmarks in the development of the calculus." (p. 8). Despite its importance, students have difficulty conceptualizing and operationalizing the FTC, and proponents of AR believe that this approach can coherently guide students to understanding it (Thompson et al., 2013). Moreover, I posit that the AR approach also prepares students for authentic intellectual work in many STEM fields, like engineering, as accumulation from rate is a crucial idea in approximation and predictive modeling (A. M. Clark, personal communication, April 10, 2024).

Although studies have attempted to use an AR approach with students (e.g., Thompson 1994), more research has focused on the way that students develop understandings of the definite integral across a unit via an AUP framework (e.g., Lehmann, 2024; Stevens & Jones, 2023). Thus, there is a need for further examining the viability of using an AR perspective to develop ideas about the definite integral and FTC. This study, which is a pilot study for my dissertation, focuses on investigating how students' thinking about accumulation of rate (as a precursor to the definite integral and FTC) develops over a series of tasks that advances the AR approach. In this preliminary research report I describe the task sequence and theoretical framework underlying it. I then report the results from the pre-assessment and discuss the ongoing teaching experiment. Finally, I conclude with planned next steps for this study and pose areas for audience discussion I anticipate engaging in during the session.

Methods

This pilot study is comprised of two parts: A whole-class pre-/post-assessment and a teaching experiment with a pair of students (Steffe & Thompson, 2000). At the time of submitting this report, the pre-assessment data has been collected and the teaching experiment is ongoing. The teaching experiment will continue across the Fall 2024 semester, and the post-assessment will be conducted in early December 2024. Preliminary analysis of all data will be completed and addressed in my presentation at the 2025 RUME conference. In this section, I describe the setting and participants involved in this pilot study. I also outline the task sequence and the theoretical framework that underlies them.

Theory and Tasks

Carlson et al. (2003) detailed a framework of the requisite mental actions needed to engage with the FTC (Table 1). The mental actions trace the development of an understanding of accumulation from rate that progresses from discrete to infinitesimal intervals. Guided by this framework, I compiled a task sequence that I hypothesize will provide students structured opportunities to develop each of these mental actions.

Table 1. Carlson et al.'s (2003) mental actions of the FTC

MA1	Coordinating the accumulation of discrete changes in a function's input variable with the accumulation of the average rate-of-change of the function on fixed intervals of the function's domain.
MA2	Coordinating the accumulation of smaller and smaller intervals of a function's input variable with the accumulation of the average rate-of-change on each interval.

MA3	Coordinating the accumulation of a function's input variable with the accumulation of instantaneous rate-of-change of the function from some fixed starting value to some specified value.

The pre- and post-assessment items are the *Round-Trip* task and the *Where's the Car?* task (Figure 1), which have the potential to elicit students' thinking about average rate of change and its accumulation in ways compatible with Carlson et al.'s (2003) framework. The purpose of the Round-Trip task (a version of San Diego to El Centro using relevant cities, which are masked for confidentiality; Hackworth, 1994) is to elicit teachers' meanings for average rate of change. It asks:

> A car went from College Town to The Beach, a distance of 90 miles, at 40 miles per hour. At what speed would it need to return to College Town if it were to have an average speed of 60 miles per hour over the whole trip?

The productive meaning for average speed is "the constant speed that the car would need to travel to go the same distance in the same amount of time as the actual trip," though teachers (and likely their students) often use the arithmetic mean to calculate the average speed (Yoon et al., 2015, p. 341).

The *Where's the Car?* task builds on this idea and is intended to provide students an opportunity to engage in MA1 (Table 1). In this task, students must use information about a car's instantaneous speed, provided in a table, to approximate an average rate of change for each discrete interval and determine the accumulated distance. The prompt and corresponding table for this task is shown in Figure 1 below.

The table below shows information about a car's speed at various time after the engine started. Assume that the car travels in a straight line after the engine starts. Approximately how far does the car travel in these 30 seconds?

Time since the engine started (seconds)	**Speed** (meters/second)
0	0
5	11
14	20
30	25

Figure 1. The Where's the Car? task

The teaching experiment consists of a series of four additional tasks. The first task, *Over and Back* (Thompson & Thompson, 1994), is used to further determine students' understanding of speed and develop productive meanings for average and instantaneous speed. Students are presented with an interactive applet of a 200ft race between a rabbit and a turtle. Students can change the animals' speeds to answer a variety of questions designed to promote an understanding of speed as the rate of change of distance (e.g., "Give Rabbit a speed that will make him go over and back in 7 minutes"). Once students have developed meanings for speed conducive to engaging with its accumulation, students will then have an opportunity to develop MA1 by engaging with the *Which Racetrack?* task. This is an extended version of *Where's the Car*? in which students use given speed data to determine which racetrack a car was driving based on its total distance. In *Designing a GPS,* students build to MA2 by gathering instantaneous speed information from a video of a racecar's dashboard to track the ongoing

distance accumulated. By having access to a relatively continuous stream of instantaneous speed data, I will challenge students to "coordinate the accumulation of smaller and smaller intervals" (Carlson et al., 2003, p. 167) in order to calculate better estimates of the distance and begin to explore the idea of instantaneous speed. The final task, *Drag Race,* builds to MA3. This task asks students to use a model of a racecar's speed in a drag race expressed in function notation, i.e., $v(t) = 71 \times (1 - (1/3)^t)$, to determine the car's distance from the starting line (i.e., distance accumulated) at specific times and eventually at any time. By using an explicit function, students can progress from MA2 to MA3 by considering the "instantaneous rate-of-change of the function from some fixed starting value to some specified value" (Carlson et al., 2003, p. 167).

Participants and Analysis

The participants in this study are a group of 12 undergraduate students enrolled in an undergraduate mathematics course focused on covariational and quantitative reasoning at a large, public university in the Mid-Atlantic United States. All students have previously taken Calculus I and thus have been introduced to the integral and FTC, though it is unknown to what extent they have had opportunities to think about these topics quantitatively. The whole group will take the pre- and post-assessments, and a pair of these students are the focal students of a teaching experiment. As a teacher-researcher, I can engage deeply with the thinking of the focal pair of students. In my instructional role for this course, I will also be aware of the progress of the non-focal students over the semester. This dual position will allow me additional insight as I analyze the changes between the pre- and post-assessments and conduct analysis.

During the first course session, the instructor administered the pre-assessment, which was composed of two items designed to probe students' incoming meanings for average rate of change and its accumulation. Students completed the pre-assessment independently. I coded student responses based on my interpretation of the type of quantitative meaning(s) for speed that the student used to solve the problem. The coding scheme for the *Round Trip* task was influenced by the rubric developed for the *San Diego to El Centro* task (Yoon et al., 2015) but was adapted to more closely characterize the meanings exhibited by this group of students. The coding scheme for *Where's the Car?* was developed inductively (Saldaña, 2021) and is presented in the following section.

Preliminary Results and Data in Progress

In this section, I present preliminary results based on students' responses to the two pre-assessment tasks. I also briefly discuss anticipated results from the ongoing teaching experiment.

Pre-Assessment Results

Most students did not exhibit evidence of quantitative meanings for speed that allowed them to productively calculate and/or accumulate average speed in either pre-assessment item.

In the *Round Trip* task, only four (33%) of the responses indicated a final answer that was a value for speed. Of these, only one response correctly identified the necessary return speed as 120 mph. The remaining three responses used the concept of arithmetic mean to calculate the needed return speed as 80 mph, echoing Yoon et al.'s (2015) findings. The final eight (67%) of the responses did not address the written prompt. These responses either did not clearly indicate a final answer (n = 3) or provided a final answer that was not a speed (n = 5). For instance, three students indicated their final answer was "1.5 hours." Notably, this is an intermediate measurement towards calculating the correct answer: 1.5 hours is the time it takes to travel 90 miles at 60 miles per hour.

Responses to the *Where's the Car?* task followed a similar breakdown. Three (25%) of the responses are potential evidence of MA1. These responses used the instantaneous speeds indicated to calculate a distance the car traveled over intervals based on the information in the table and then summed these distances to find the total distance over 30 seconds. A further four responses (33%) presented intermediate calculations that were consistent with MA1 activity within a single time interval, but did not calculate or accumulate the subsequent distances. For instance, some calculated that in the first five seconds the car traveled 55 meters, but they did not calculate the distance traveled in any other interval. Four students (33%) did not exhibit productive quantitative meanings for speed. The most common responses in this category added up the four provided instantaneous speeds and indicated the resulting sum as the total distance traveled. A final student simply responded, "I'm not sure."

Data in Progress/Anticipated Results

The results from the pre-assessment suggest that the students will benefit from further opportunities to develop quantitative meanings for speed (average and instantaneous) and distance as well as the relationships between them. I intend the ongoing teaching experiment to provide these opportunities, and I anticipate that engagement with the task sequence described above will help students develop productive meanings for these quantities in relation to the definite integral.

I have selected two students who did not get either pre-assessment task correct to serve as my focal pair for the teaching experiment. Through working with these students on the task sequence over two course sessions (about six total hours) I anticipate that I will be able to model how their thinking about accumulation from rate of change develops. In particular, I predict that by the end of the teaching experiment I will have clear evidence of the students operating with each of Carlson et al.'s (2003) mental actions.

Potential Areas of Audience Discussion

By the time of the conference in February 2025, I will have collected, compiled, and begun to analyze all the data from this study holistically. This includes the pre-assessment, exploratory teaching, and post-assessment data. Because this is a pilot study for a larger project, getting feedback from an audience at this stage of the research process will be incredibly helpful in refining the findings for presentation to my committee and in brainstorming how this work can be expanded upon for a larger study. I also hope that other researchers who investigate students' thinking about the definite integral will share their reactions to my findings, especially those who focus on quantitative approaches. Additionally, I am interested in further examining the benefits and drawbacks of the AUP and AR frameworks in different situations. Specific questions I intend to ask the audience include:

1. How does the analysis I present compellingly show students' developing reasoning within the AR conception of the definite integral? In what areas could I provide more nuance or further evidence?
2. In what ways do you think this study could be expanded upon? Are there any lines of thinking that would be particularly interesting to follow? Any new areas that would be useful to explore?
3. For calculus instructors: In what ways do my findings resonate with or challenge your experiences teaching the definite integral to undergraduates? What implications might these findings have for your own practice?

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Empowering STEM Pathways: Social and Institutional Support for Hispanic/Latine Students in Higher Education

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While Latine/Hispanic higher education enrollment rates have increased over time, enrollment in STEM, and mathematics in particular, has been slower to change. As part of a larger study, our team interrogates the experiences of those who have successfully navigated the mathematics pathway. Specifically, we look to understand how our research participants leveraged the support of both social (familial, peer, etc.) and institutional (mentors, faculty, administrators) along their academic pathways. We investigate primarily using in-depth qualitative interviews of five individuals who hold bachelor's degrees in mathematics and STEM PhDs, to ascertain both positive and negative influences on their education, utilizing the community cultural wealth framework. While the Latine PhD population is, of course, not monolithic, our findings suggest that social, familial, and navigational capital were instrumental in their ultimately successful navigation of both graduate and undergraduate education.

Keywords: community cultural wealth, mentorship, social networks, student experiences,

The increasing number of Latine students enrolling in higher education institutions has led to an increase in Hispanic Serving Institutions (HSIs) in the United States (Lane et al., 2024; Pawley, 2017). While certainly an increase in Latine/Hispanic student enrollment is an achievement to be celebrated, research on HSIs (Garcia, 2020, 2023) indicates that these institutions are still grappling with what it means to hold the HSI label and often struggle to serve their Latine student populations adequately. It is no secret that US higher education institutions have significant issues supporting, retaining, and graduating students from marginalized backgrounds in STEM and, specifically, in mathematics. According to the National Center for Education Statistics (2023), even with an increase in Latine/Hispanic student enrollment in colleges and universities, only 14.6% graduate with a STEM bachelor's degree, 11.4% obtain a STEM master's degree, and 9.2% are awarded a doctoral degree in STEM.

All stakeholders in the higher education space must understand the current educational landscape better in the context of Latine education if their goal is to truly serve and support Latine/Hispanic students. As part of this imperative, our research team investigates the experiences of those who have successfully navigated the math PhD pathway, focusing specifically on how they leveraged the support of both social (e.g., familial, peer, etc.) and institutional (e.g., mentors, advisors) actors in their academic endeavors. We aim to address the following research question: *What types of resources did Latine mathematics teaching-focused faculty draw from that helped them navigate and persist in the higher education space?*

Theoretical Background

Drawing from Critical Race Theory (CRT), Yosso (2005) developed the *Community Cultural Wealth* (CCW) framework, which takes an asset-based approach by recognizing the cultural wealth held by students who identify as underrepresented minorities (URMs) as valuable forms of knowledge in the education space. The CCW framework is composed of six forms of non-

dominant capital, which include (1) aspirational, (2) familial, (3) linguistic, (4) navigational, (5) resistant, and (6) social. Aspirational capital refers to maintaining hopes and dreams despite educational barriers. Familial capital relies on kinship and family ties that individuals can draw from as they navigate higher education. Linguistic capital refers to the multiple languages and communication styles that URMs can utilize from context to context. Resistant capital can be characterized by the knowledge, skills, and resources individuals can utilize to challenge systems built with dominant cultures in mind. Social capital encompasses the resources that individuals gain as they develop relationships and social networks. The CCW framework reorients cultural capital to a more dynamic model (Sablan & Tierney, 2014), providing an anti-deficit approach to the resources that URMs possess and leverage in their pursuit of a successful education.

In mathematics education, previous research studies have used the CCW framework as a guide for analysis to explore student and educator experiences at varying stages. This framework has been applied to investigate teaching practices (Holbrook, 2021) as well as the experiences of multilingual learners (Pelaez et al., 2023), teaching assistants (Poulsen Stringer, 2023), and others. For this paper, we center our analysis around what forms of capital were attained and when they were leveraged at the undergraduate and graduate levels by current Latine mathematics faculty who have successfully navigated a pathway through STEM. Herzig's (2004) exploration of the experiences of women of color in doctoral mathematics programs and Ong's (2011) study on the challenges and mitigations involved in the transition from undergraduate to graduate education highlighted the pivotal role that supportive environments play in helping these students achieve success in graduate mathematics education. Furthermore, research emphasizes the detriment of chilly and exclusionary climates in STEM departments (Camacho & Lord, 2013), particularly in mathematics departments (Herzig, 2002).

Methods

Emerging from a more extensive study focused on generating equitable hiring procedures for the hiring of Latine teaching-focused faculty (TFF) in STEM departments (Funded by NSF Award #2113355), our project centers on a subset of five TFFs across multiple institutions who completed at least a bachelor's degree in mathematics. We categorize each faculty member as "teaching-focused" if they devote at least 50% of their time to teaching. Four of our participants attained PhDs in mathematics and/or statistics. Of these, three hold faculty roles in mathematics departments, while one participant is a TFF in a digital and computational studies department. The final participant received a PhD in biology and is in a TFF position in a biology department.

We conducted five semi-structured interviews (Bernard, 1988; Zazkis & Hazzan, 1999) that were approximately one to two hours in length. These interviews were held via Zoom, audio recorded and transcribed using Otter.ai, and analyzed with the ATLAS.ti platform. All of our participants were compensated for their time. The interviews hinged on enabling our participants to reflect on their academic journeys and were divided into two parts: (1) experiences leading up to the faculty role and (2) TFF experiences in their current position. For this paper, we focus on the first section of the interviews.

We analyzed the data using a deductive and inductive approach (Corbin & Strauss, 1990; Miles & Huberman, 1994). Our analysis implemented a deductive approach to test the CCW framework using the six constructs of non-dominant capital as a priori codes. Next, we used an inductive approach to generate overarching themes from the data. Researchers collectively coded multiple transcripts until they reached a suitable level of inter-rater reliability (*Krippendorff's alpha = .68*), then coded the remaining transcripts separately.

Results

The nuances in participant experiences allowed us to have a rich dataset that we use to tell their stories and give them a voice. We found that our participants drew from multiple forms of support to navigate their higher education journeys, including peer support, mentorship, and familial support, comprising all the CCW constructs, particularly aspirational, familial, navigational, and social capital. In the subsequent sections, we will explore participants' motivations to pursue mathematics careers and their various forms of support along the pathway.

Familial Support

Familial capital has the potential to serve as a cornerstone for motivating students to seek a STEM career and supporting them along the mathematics trajectory. Our research participants often described family education and expectations as influencing factors in their decisions to pursue higher education. Consider Maria, an Argentine American TFF at an R1 institution whose guardians had varying levels of education. Maria's mother had an accounting degree, her father did not finish high school, and her grandmother did not experience any formal schooling:

> *I have an older brother [attending college] that was, you know, four years ahead of me or five years. So, it [was] clear [that] I was going to go to college…Also, [my family and friends] value [a] faster way to change [social] classes…So [this was] kind of a way of changing and getting a good life so that's really important… – Maria*

For Maria, her family values were an essential factor in her decision to pursue higher education, demonstrating evidence of familial capital. Specifically, Maria expressed that her love for mathematics at a young age encouraged her to pursue the mathematics field.

During our interview with Brian, a TFF at an R1 institution who was a first-generation (FG) college student, he expressed that his parents provided immense support for him to value education as they had high aspirations for their children. Brian reported that his parents made multiple investments to ensure their children could develop a solid educational foundation through private education. Brian and his siblings were expected to perform well in school and to pursue a higher education degree as their "...inheritance was essentially [a] strong education at the get go." The cultivation of substantial familial capital allowed Brian to strengthen his aspirational capital, aiding his navigation of higher education as an FG college student.

Peer and Institutional Support

Without recognizing the influence of peers and institutional actors, no discussion of support systems for those in higher education would be adequate. Fellow students, faculty, advisors, and mentors (both official and unofficial) were instrumental in providing motivation, guidance, and overall support across all our research participants' educational trajectories.

Sergio is a TFF in Mathematics at a doctoral-granting institution in the Midwest; he describes how, as a FG student entering an Ivy League college for his undergraduate he felt a lack of support from faculty in his department but found support outside of his department based on shared cultural backgrounds (ethnicity, FG student status, and both being community college graduates in this case):

> *One thing that I've benefited from was having mentors, but specifically, mentors that have similar backgrounds to me, because you can see yourself in them and think, oh, they succeeded. And I could do it, too. – Sergio*

The efficacy of mentoring based on shared cultural experiences and identity characteristics was a theme throughout our project, and all our mathematics participants described some form of this

in their educational journey. Our women research participants described connecting and feeling supported by other women in faculty positions, particularly those who shared some other cultural touchpoint with them (e.g., international student status, ethnicity, and FG student status). All five of our participants described how, in their current faculty positions, they seek out and participate in mentoring relationships based on shared identity and cultural experiences, often describing students reaching out to them for these reasons.

All our participants highlighted the vital role that peer support networks played in helping them persist in mathematics and STEM. In addition to seeking mentorship from students who were more advanced in their programs and leveraging identity networks (e.g., Latine/Hispanic organizations), emotional and academic support from peers were reported to serve as tools in navigating the process of achieving essential milestones on the pathway, such as high-stakes situations:

> *So, we had our group, and it was a very supportive group. I [joined] with my best friend…without her, I wouldn't have passed my prelims because she was always pushing me to go to study every day, or, you know, asking me if I was ready for X or Y. – Gina*

Social capital in the form of accountability is evident here. Still, aspirational and resistant capital also play a role. Gina, a Costa Rican American TFF at an R1 institution, describes how her best friend encouraged her to remain hopeful about passing her preliminary exams and helped her persevere despite the challenges and pressures. It is clear that social, aspirational, and resistant capital facilitated overcoming a hurdle that graduate students often face.

Tanya, an associate professor of digital and computational studies at a private liberal arts college, reported that her peer networks helped her find institutional support:

> *One of my fellow graduate students was like, well, my advisor is really great. And she asked about you the other day, and I think she would be willing to help…So I reached out to her like, because I was like, oh, she cares about me… So, I reached out and she really got the ball moving on, on that and made sure that that the defense happened. – Tanya*

This use of social capital to navigate difficult situations in their PhD journey exemplifies the critical nature of peer networks and how those can bridge to positive faculty relations. Without this peer network, Tanya would have been left in a difficult situation, without a faculty member to advocate for her during a critical moment in the dissertation phase of her graduate studies.

It is important to note that peer relations are not unilaterally positive, nor are faculty interactions. Our women participants described having to navigate microaggressions from peers and advisors based on various factors. Gina describes being assumed as less competent than her domestic peers based on her country of origin:

> *One of [the professors] told me that he was not expecting [me to do well on my preliminary exams] because of where I was coming from…I hated that he said it, but at least it was in my face, because I know a lot of them were thinking about that…the stereotypes of growing up in a country that allegedly does not have any academia…or thinking that my curriculum was not strong enough because I come from Latin America… that was something that always affected my self-confidence. – Gina*

While peers and institutional actors certainly can provide negative experiences in academia, overall, our participants described the benefits of positive experiences with both institutional and peer actors as a critical form of support along their educational journeys. However, we want to highlight that these individuals' stories only capture one piece of a larger puzzle: those who successfully navigated the 'leaky pipeline' of STEM higher education.

Discussion and Recommendations

In answer to our central research question, we found that social and familial capital support were central to the success of our participants in their academic journeys. Notably, forms of capital were leveraged simultaneously to overcome significant barriers along the pathway, suggesting the dynamic nature of CCW. For example, social and aspirational capital often served as a conduit for access to navigational capital, indicating that these forms of wealth are transmutable. We agree with previous scholars in the STEM education literature (Camacho & Lord, 2013; Herzig, 2004) who have discussed how the chilly climate of STEM departments is a significant barrier to Latine student success within higher education. Our contribution centers on the role of CCW and non-dominant forms of cultural capital that allow students to survive and thrive in these institutions and hostile cultural climates.

In terms of our recommendations for how institutions can effectively leverage and bolster these forms of support, we have two recommendations. The first is cultural, in that universities at a fundamental level need to recognize and acknowledge the funds of knowledge (Moll et al., 2005) that minoritized students bring with them into both the classroom and campus community. Instead of trying to force students to assimilate into the form of a model student by adopting and imitating hegemonic forms of cultural capital, we can encourage and appreciate their unique perspectives and forms of cultural wealth.

Second, our research has detailed the importance of culturally sustaining mentoring through the experiences of our research participants' journeys. This aligns with research on retention and student success in higher education in STEM (Johnson & Elliot, 2020). It cannot be understated that our participants described feeling isolated and marginalized, which was partly remedied by peer and faculty interactions with some form of cultural touchpoint. Our participants may have lacked the support necessary to succeed on the STEM pathway without these relationships. To enhance culturally responsive peer and institutional actor support, we suggest that colleges and universities (1) hire more Latine/Hispanic TFFs to lessen the already heavy load these faculty have on their shoulders, (2) recognize these faculty in tenure and promotion materials for their efforts to support students as agents of change by providing funding for programs they already run and (3) provide support for culturally oriented mentorship programs at the campus level, to move from only admitting Hispanic students to genuinely serving them (Garcia, 2020).

While the Latine/Hispanic experience in higher education is not monolithic, our research highlights the experiences of those who successfully navigated the STEM PhD pathway. However, this ultimately only tells us part of the larger story of Latine/Hispanic graduate education. Some are, of course, pushed out of higher education by institutional or sociological factors. Whether this disparity comes from a lack of mentorship, social/institutional support, discrimination/racism, or other structural factors is not covered by our study. Our suggestion for future researchers is to home in on the population victimized by the ‘leaky STEM’ pipeline to understand the pitfalls better.

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Empowering Course Coordinators: Examining Foundational Math Coordination Orientations

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This study explores the evolving approaches of eight foundational math course coordinators, uncovering key insights into their coordination strategies and mechanisms to enhance their efforts. These coordinators oversee critical courses, including College Algebra, Quantitative Reasoning, Introductory Statistics, Math for Architecture and Construction Management, Precalculus, Calculus, and mathematics courses for prospective elementary teachers. Through a dataset derived from surveys, self-reflections, and professional development workshops, we investigated their perspectives and experiences as coordinators. We analyzed data from both the coordinators and the graduate student instructors they oversee. Specifically, we highlight the integration of instructional routines that promote mathematical reasoning and the development of course-specific dynamic calendar systems, both of which have the potential to improve the instructional effectiveness and coordination of foundational math courses. Our findings offer fresh perspectives on how to better support course coordinators in their crucial role, ultimately benefiting both instructors and students.

Keywords: Foundational Mathematics, Course Coordination, Professional Development

Introduction

Course coordinators are a key population within college math education because they teach and supervise instruction of thousands of undergraduate students each semester. A key element for course coordinators is streamlining their goals and expectations and communicating those to the instructors they coordinate. This paper reports on data from the first two years of a 3-year grant project (Rogers, 2022-2025) that ultimately aims to test and refine a community of practice (Wenger et al., 2002) among eight faculty members. These faculty members coordinate and teach foundational math courses (FMCs, e.g., College Algebra, Precalculus, Quantitative Reasoning, Introductory Statistics, Math for Elementary Teachers, & Calculus). A community of practice can provide a means for faculty to share/manage professional knowledge, in this case about professional development (PD) opportunities we facilitated among college math faculty.

Key elements for effective coordination include (1) fostering community and (2) maintaining uniform course components (e.g., syllabus, textbook, homework, and assessments; Martinez et al., 2022). When asked, "How do you effectively coordinate FMCs?" coordinators may focus on people-centered approaches or structural components, reflecting their course coordination orientations. These orientations demonstrate an individual's disposition toward their work. Our aim is to explore FMC coordinators' orientations and identify ways to enhance their coordination efforts through PD. We asked both FMC coordinators and graduate student instructors (GSIs) to identify the coordinators' approaches to course coordination, leading to our research questions:

1. How do GSI and FMC coordinator perceptions of course coordination relate?
2. What kinds of support structures could aid in course coordinators' PD?

Literature Review

We are particularly interested in understanding and supporting course coordinators' reasoning as they manage FMCs in ways that are responsive to the needs of the instructors they oversee. Our literature review focuses on two key areas: understanding the perspectives these

coordinators bring to their roles and exploring potential avenues for PD in this context. Over a decade ago, Speer et al. (2010) conducted a review of collegiate math teaching and found that instructors' practices—specifically their pedagogical actions and the reasoning behind them—remain largely underexplored in research. This gap persists, as research on PD in college math education, especially for instructors and tenure-track faculty, continues to be sparse (Florensa et al., 2017). Our project aims to address this gap by examining course coordinators' perspectives and providing targeted PD to enhance their coordination skills.

To investigate course coordinators' orientations, we use Martinez et al.'s (2022) framework, which distinguishes between two orientations: *Humanistic-Growth (H-G)* and *Resource-Managerial (R-M)*. An H-G orientation focuses on the individuals being coordinated, while the R-M orientation emphasizes expertise in managing others or resources (Table 1).

Table 1. Course Coordination Orientation Themes

Humanistic-Growth (H-G) Themes	Resource-Managerial (R-M) Themes
(a) Attends to Student Experience	(f) Draws on Knowledge of Course, Department, & Institution
(b) Expresses Concern about Others	(g) Demonstrates Experience Teaching & Knowledge of Teaching the Course
(c) Takes Action to Build Community	(h) Provides Materials Related to Content & Curriculum
(d) Is Attentive to Instructor Differences	(i) Communicates Clearly
(e) Provides Support for Instructors	(j) Exhibits Strong Administrative Skills

We report on two aspects of our PD for course coordinators. First, we incorporated **instructional routines for reasoning** (Kelemanik et al., 2016). These are "specific, repeatable designs for learning that support" (Lucenta & Kelemanik, 2024, para. 1) students' mathematical reasoning. Instructional routines highlight the interconnectedness of mathematical concepts and offer coordinators practical tools to guide instructors in fostering students' reasoning skills. Second, we considered the use of **dynamic calendars** (Oliver & Olkin, 2021) to organize course materials in a user-friendly, calendar-like format that coordinators can personalize to include notes and recommendations for instructors. This tool can provide a clear structure, helping coordinators communicate expectations and resources effectively to the instructors they oversee.

Method

Context: University and Course Coordinator Backgrounds

At a public, liberal arts college, participants are eight FMC coordinators whose duties include distributing course materials (syllabi, lecture notes, pacing guides, & assessments) to ensure consistency across course sections. They facilitate orientation meetings with instructors, provide observation feedback to novices (esp. GSIs), and address student concerns. They regularly teach the courses they coordinate, but GSIs teach majority of the course sections (Rogers et al., 2024). Most FMCs are taught in traditional, in-person lectures, while College Algebra is taught using a math emporium model (adaptive learning system to individualize student pathways; Cousins-Cooper et al., 2017). Patrick (Table 2) oversees FMCs at the 2-year branch of main campus.

Workshop & Professional Development Context

During the academic year, Author 1 met with the FMC coordinators twice a month to facilitate PD opportunities while also addressing course coordination, administrative, and policy needs. These meetings have expanded to include an annual summer workshop, chosen for its lighter workload for coordinators. In the latest workshop, coordinators learned about the H-G

and R-M orientations and self-identified their alignment within this framework. They also reflected on the group's alignment. We also shared data from graduate students who described their coordinators using relevant themes. The workshop concluded with a session on instructional routines and dynamic calendars, followed by individual worktime.

Data Collection and Analytical Approach

Before the FMC coordinator workshop began, all graduate students who had served as instructor of record of these coordinators' courses were invited via email to participate in a survey. The survey asked them to assess their coordinators based on the five themes of the H-G and R-M orientations. GSIs rated their coordinators using integers on a scale from 0 (theme does not describe the coordination) to 10 (theme accurately describes the coordination) and could also provide open-ended comments. During the workshop, coordinators were introduced to the two orientations and asked to identify which themes they used or did not use. Table 2 shows the count of themes self-reported by each coordinator and the mean ratings provided by GSIs.

Table 2. Participants' Coordination Details and Coordination Orientations.

Pseudonym	Course(s) coordinating	(H-G Count, R-M Count)		No. of GSI Responses
		Self-reported	GSI Reported	
Alice	College Algebra	(4, 3)	(4.80, 5.00)	1
Alexandra	College Algebra	(4, 2)	(4.77, 4.93)	3
Camille	Calc I, Calc II, Calc III	(4, 5)	(3.48, 4.57)	6
Madaline	Prospective Teacher Math	(4, 4)	No GSIs Coordinated	0
Patrick	College Algebra & Precalculus	(4, 4)	No GSIs Coordinated	0
Pricilla	Precalculus	(4, 5)	(3.66, 4.57)	7
Reema	Quantitative Reasoning & Math for Architecture	(4, 2)	(3.37, 4.10)	3
Stella	Introductory Statistics	(4, 5)	(4.73, 4.73)	3

After identifying their orientations individually and as a group, coordinators reflected on these results. To collect reflections from all, coordinators first silently answered each reflection question on a chalkboard and then added further silent reflections using sticky notes after reviewing others' answers. This activity provided qualitative insights into the coordinators' self-perceptions and perceptions of the group. Additional data came from self-reported surveys completed by each coordinator during project year 1. Given the small sample size, no statistical analyses were performed. We compared frequencies by coding open-ended responses for the H-G and R-M themes and counting the themes identified by coordinators while averaging the GSI-provided theme scores (scaling them from [0, 10] down to [0, 1]).

Findings

RQ1: Relationship Between Coordinator and GSI Perceptions

Early in the grant project, coordinators were asked to list what aspects of course coordination they enjoyed and what aspects they found challenging. Two researchers coded the items coordinators listed using the H-G and R-M orientations and reached 100% coding agreement through discussion. For aspects the coordinators enjoyed, 20% of their responses aligned with H-G and 80% with R-M. The themes most frequently mentioned as enjoyable were three R-M: (f) drawing on knowledge of course/department/institution, (g) demonstrating teaching expertise,

and (h) providing materials. Their responses about what they found most challenging were evenly split between the two orientations, exclusively mentioning challenges identified as (c) building community, (e) providing instructor support, (h) providing materials, and (j) exhibiting strong administrative skills. After later being introduced to these descriptions at the workshop, coordinators identified their orientations by recognizing themes they use or do not use (column 3, Table 2). Across the eight participants, each coordinator identified only one H-G theme they did not use and wanted to incorporate more explicitly, as follows: (b) concern about others (37.5%), (c) seeking to build community (25%), (d) attending to instructor differences (12.5%), and (e) providing instructor support (25%). These results aligned with their earlier survey responses of areas of challenge, while also adding items (b) and (d) to the list.

During the workshop, there was some noticeable agreement among the coordinators when they identified R-M themes they were and were not using. Three coordinators selected all the R-M themes as ones they were currently using. The five coordinators who indicated not using at least one R-M theme all selected (j) exhibiting strong administration skills as an aspect they were not explicitly incorporating. One or two coordinators also selected (i), (f), or (g) as R-M themes they were not currently using as well. The group discussions built upon these findings with coordinators asking how "strong" the word "strong" in item (j) was intending to be? They considered replacing that word with "some" and then they would consider selecting it. Their discussion also noted the group's strength among the R-M orientation themes. Additionally, the coordinators all agreed that there is "room to grow" in their course coordination and that they wanted to more intentionally use H-G themes (or in the case of Alice and Alexandra, who both recently were appointed as coordinators, more R-M themes).

GSIs, who were given the same descriptions of 10 themes for course coordination orientations, responded (column 4, Table 2) in ways comparable to the coordinators. Building community (c) was commonly one of the lower scored themes by GSIs, but not the lowest theme for any single coordinator. An exception is Alice and Alexandra, who see GSIs every day due to the emporium format, for whom (c) building community was the highest-scored GSI-response theme. Alice and Alexandra also self-selected (c) building community as something they currently use. In the open-ended responses, GSIs most commonly discussed the absences of (d) attentiveness to instructor differences and (e) providing instructor support. When asked to comment about the presence or absence of any R-M themes, only 16% of GSIs chose to provide open-ended responses as opposed to 44% of GSIs commenting about H-G themes.

RQ2: Coordinators' Receptivity of Professional Development Tools

On the second day of the workshop, after further debriefing some GSI-responses together, coordinators engaged with examples of dynamic calendars and instructional routines for reasoning and then broke out for an hour of practical work time. Coordinators' reception of the newly introduced dynamic calendar consisted of positive remarks, with half of the group either actively creating a dynamic calendar for an FMC during the work time or commenting on ways they plan to do so soon. For example, Camille began designing a cloud-based document with key topics, activities, and sample assessments for the first week of Calc I. She plans to continue constructing it during the semester to share next semester. Although Camille had these various documents accessible to instructors in a variety of folders, the dynamic calendar provided an

overarching structure and direct links to the most relevant files for each week. Providing such a resource could potentially address some of the H-G and R-M themes (e.g., (e) provides instructor support, (i) communicates clearly, and (j) administrative skills).

We modeled three types of instructional routines, and half of the group spent the entire work time adapting class materials to an instructional routine structure, while a fifth coordinator split their work time between the two tools. Using existing FMC activities, coordinators noted how they could adapt them to the routine structure, thereby providing opportunities for students to think deeply about the mathematics. Alice reflected on the experience saying incorporating instructional routines could help "students build some problem-solving skills and see past just the steps of a single topic." We noticed that four of these five coordinators identified at least one R-M theme they wanted to more explicitly incorporate, and their written workshop reflections suggest they saw routines to provide more effective course materials for FMCs.

Discussion and Questions for the RUME Audience

After being introduced to the course coordination orientation framework, the coordinators self-identified one H-G area that they had not previously identified that was also not apparent in the GSI responses: (b) expressing concern about others. The groups of GSIs and coordinators both noted that (c) building community and (e) providing instructor support were areas that could use more explicit attention. We posit that the integration of instructional routines or dynamic calendar could be a mechanism for addressing some of these areas needing attention. We emphasized that coordinator are not expected to address all ten themes across both orientations. But noted areas where the coordinators desired to (and GSIs requested they) give more attention.

Moreover, in analyzing these findings, we observed a parallel between the roles of instructional routines for reasoning and dynamic calendars, each serving different populations within FMCs. We compare how instructional routines can influence undergraduate students' conceptualization of mathematics and how dynamic calendars can shape FMC instructors' teaching practices. When instructors employ instructional routines for reasoning, they provide students with a framework to approach mathematical problems and engage in reasoning across various contexts. Similarly, when coordinators develop dynamic calendars for FMCs, they are constructing a framework for instructors to guide their course planning and teaching. This organizational structure can help instructors conceptualize the course, choosing which resources or teaching strategies to incorporate. We are eager to gather feedback from RUME on how these parallel structures might further support both students and instructors. We want to hear thoughts on how this relationship might inform future PD initiatives. Additionally, we plan to explore potential challenges or pushback coordinators might face when implementing these strategies, which could lead to valuable insights for refining our approach. To facilitate this discussion, we will also ask the audience the following questions during our presentation:

- What research-based, practitioner-focused resources should we consider to help coordinators to learn from course coordination efforts nationally?
- As we enter the final year of this project, what data should we prioritize gathering about coordinators' instructional and course coordination practices, and why?

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Using the Instructional Tetrahedron to Classify College Algebra Instructors' Peer Observations

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Professional learning communities (PLCs) can be a way to support college mathematics instructors in shifting their instruction to be more student-centered. This study focuses on what College Algebra instructors noticed when observing their colleagues' teaching in a video club setting within the PLC. We used the instructional tetrahedron to classify what instructors noticed according to the four vertices: Content, Students, Teacher, and Tools and Technology. Results highlight that instructors' discussions shifted away from Tools and Technology after returning to in-person classes and that instructors' discussions about Tools and Technology differed depending on the in person or online class context. We also found that instructors frequently noticed Teacher interactions, which points to their perceptions about the goals of the PLC - to provide feedback about teaching and to generate ideas about how to improve one's own instruction. Finally, what instructors notice about their colleagues' teaching might shed light on that instructors' goals and beliefs about teaching.

Keywords: Instructional triangle, Peer observations, Professional learning community

There has been increased attention on instructional practices that center students' activity and thinking during class time because of the positive student outcomes associated with these practices (Freeman et al., 2014; Laursen et al., 2014). However, instructional change at the college level is often difficult to catalyze (Apkarian et al., 2021). As such, college mathematics instructors may need additional support to shift their instructional practices to center students and their thinking. For this study, we leveraged a professional learning community (PLC) as a means of professional development. PLCs provide a long-term structure for on-going discussions and incremental improvements, which can increase instructors' interest in making changes to their teaching (Wildman et al., 2000). We also incorporated video clubs within the PLC as these have been shown to support teachers' learning and focus on student thinking and can result in shifts in instructional practices (Sherin & Han, 2004; van Es & Sherin, 2010).

Theoretical and Analytic Framework

The Instructional Triangle is a classic model for conceptualizing instruction since it consists of "interactions among teachers and students around content, in environments" (Cohen & Ball, 2001; p. 122). The Instructional Triangle represents these three components (teacher, students, and content) as vertices and includes edges that represent the interactions between each of the vertices. Recent research at the college level has used the Instructional Triangle to characterize instructor discussions during an online working group centered around inquiry-oriented instruction in Abstract Algebra (Kelley & Johnson, 2022). Kelley & Johnson (2022) found that the focus of instructors' comments shifted over the course of the working group to focus less on content and more on the relationship between content and students. In other recent research, Liu & Buchbinder (2024) proposed the Instructional Technology Tetrahedron, which is an extension of the Instructional Triangle to include a vertex for technology. Their conceptualization of the Instructional Technology Tetrahedron places the Technology node in the center of the Instructional Triangle and in the same plane as the other vertices, indicating that technology is more than an element of the learning environment (which is how prior extensions of the

Instructional Triangle have included Technology; Tall, 1986; Olive et al., 2010). Liu & Buchbinder (2024) used the Instructional Technology Tetrahedron to analyze pre-service teachers' noticing of their own online teaching, and how what they noticed changed as they gained experience.

In order to investigate the types of things the instructors in this study noticed, we leveraged the Socio-didactical Tetrahedron (Rezat & Sträßer, 2012), which is an extension of the Instructional Triangle to include a vertex for artifacts such as physical or digital manipulatives, digital technologies, and textbooks. Rezat & Sträßer's (2012) conceptualization of the Socio-didactical Tetrahedron connects a node for artifacts with the Instructional Triangle in a 3-dimensional space, such that the Instructional Triangle is the base of 3-D tetrahedron. Rezat & Sträßer (2012) argued that the mediation of artifacts is inseparably intermingled with the three vertices of the instructional triangle, and as such, perspectives of how artifacts interact with other vertices should be able to be represented by the surfaces of the Socio-didactical Tetrahedron.

Our work seeks to classify what college instructors notice when examining their colleagues' teaching in online and in-person classroom environments. Specifically, we aim to answer the following research question: What do instructors teaching the same class notice when observing a recorded lesson from one of their colleagues? This research is important because it highlights what components of instruction (content, student, teacher, and technology) instructors focus on when observing their colleagues, which could shed light on their personal goals, knowledge, and beliefs (Russ & Luna, 2013). Further, it could indicate a particular instructor's strengths or opportunities for growth with teaching.

Methods

This paper uses data gathered from a project which began in Fall 2021 and concluded in Spring 2024. From this project, we focus on two semesters, Spring 2022 and Fall 2022, in which instructors of College Algebra participated in a PLC to share and review recordings of each other's lessons in a video club setting. In the Spring semester four instructors participated in the PLC, while in the Fall semester two additional instructors joined PLC. In this study, we will focus only on observations of the four instructors for which we have two semesters of data for. Each instructor had three recorded lessons observed every semester, with the exception of Nicholas, who only had two lessons observed in Fall 2022.

Instructors were rotationally paired in such a way that they watched their partner's recorded lesson beforehand, then led a portion of the video club meeting by sharing snippets of the video-recording along with commentary of interesting aspects they noticed; after initial observations, all instructors were invited to discuss any aspects they found interesting of the lesson, too. Video-recordings of these meetings were transcribed verbatim and coded according to an adapted version of the Socio-didactical Tetrahedron, which we refer to as the Instructional Tetrahedron, composed of four vertices -Content (C), Teacher (T), Student (S), and Tools & Technology (E)- and the edges and faces between them. This resulted in fourteen codes. For this coding, we considered Tools and Technology to diversely cover computer-based software like Desmos, Zoom, and PowerPoint, but also non-digital classroom tools such as textbooks and whiteboards, since all items listed are used as support tools for learning.

Participating instructors were interviewed at the end of each semester by a facilitator of the PLC. The interviews included questions where the instructors were asked to reflect on their experiences with technology and peer observations. We use quotes of two instructors from these interviews to highlight possible reasoning for features found in results of our data.

Results

For this report, we focus on overlapping clusters of the four main vertices of the Instructional Tetrahedron -Content, Teacher, Students, and Tools & Technology- because this provides a coarse breakdown of what instructors focused on in their observations of each instructor. These are referred to as Cluster C, Cluster T, Cluster S, and Cluster E. Note that the clusters are composed of overlapping codes. For example, a Content-Teacher interaction will be reflected in both Cluster C and Cluster T. Table 1 includes the percentage of coded segments for each cluster from the instructors' discussion in the video club when observing the lessons taught by the instructor listed in the left column. Since the clusters are composed of overlapping codes the percentages of the rows do not sum to 100%.

Table 1. Frequency of Codes from each Instructor's Observations

Instructor		Cluster C	Cluster T	Cluster S	Cluster E
Alex	%	48.99	74.75	50.51	28.79
Nicholas		39.25	70.09	75.70	28.04
Ivy		44.71	66.47	44.12	32.35
Shay		42.00	71.33	43.33	37.33
Average	%	**43.74**	**70.66**	**53.41**	**31.63**

In addition to looking at what the instructors focused on when observing a given instructor, it is also important to consider the context in which the two semesters' data were collected. The semester of Spring 2022 began online, which then shifted to in-person instruction for most participants. By the Fall 2022 semester, only Nicholas was leading online classes. Table 2 highlights the percentage of segments from the observation discussions for all instructors related to each cluster from Spring 2022 to Fall 2022. In the following sections we discuss insights from Table 1 and Table 2.

Table 2. Average Frequency of Codes from Each Semester

Semester		Cluster C	Cluster T	Cluster S	Cluster E
Spring 2022	%	32.77	71.44	40.48	38.16
Fall 2022	%	62.50	69.11	62.26	21.51

Noticing of Tools and Technology-Focused Interactions (Cluster E)

Overall, the discussions focused on Tools and Technology interactions were consistently lower than other clusters, with an average of only 31.63% of the conversation revolving around this cluster (Table 1). Each instructor implemented tools like Desmos, Zoom, PowerPoint, whiteboards, etc., in a variety of ways during lessons, and many of these implementations are discussed within the video clubs. However, there is a decreased focus on Cluster E from Spring 2022 to Fall 2022. Specifically, Table 2, which breaks down the discussions revolving around each cluster by the semester these discussions occurred, shows instructor noticing of Tools and Technology decreases from 38.16% to 21.51%. We hypothesize this change of about 17% is related to the switch of online to in-person classes for the majority of instructors. Since we also

consider technology to be an environment, we speculate that when instructors are removed from this environment, their noticing of Cluster E interactions instead shift toward viewing technology as a tool. Instead of instructing through a medium of technology by necessity, after shifting to in-person classes instructors may or may not deliberately bring in technology to lessons as learning tools.

In order to shed light on the types of Tools and Technology interactions that were part of the discussions within the video club, we provide a comparison of instructor noticing during observations of Nicholas, an online instructor, and Shay, an in-person instructor. For comparison of these two instructors, we will refer to Table 1. Nicholas, who solely teaches online classes in both semesters and must constantly utilize technology for all of his lessons, only has 28.04% of instructors' noticing discussion points to Tools and Technology interactions. This could be because the technology serves as an environment that operates in the background, almost unnoticed. Interestingly, observations of Shay have the highest percentage of noticing focused on Tools and Technology, with 37.33%. Many of these observations come from Shay's use of Tools and Technology to segment lessons, such as personal whiteboards and review games created in PowerPoint. During the final video club of Fall 2022, when leading discussion of one of Shay's lessons, Alex describes the lesson as "high energy," and goes on to elaborate more during this same lesson:

> But then I kept it going because, showing the whiteboards I thought was such a good idea, bringing in those whiteboards, mixing it up. They're not on the computer. Like, you could feel that the energy in that class is changing just from something new happening.

As a complimentary point of view, in an interview in Spring 2022 with Nicholas, when asked about whether he would have implemented active learning and technology like Desmos without intervention of the project, he responded:

> We were already doing group activities in the classroom and having that active learning happening. And so…I guess the only thing that changed was the implementation method. And so now getting them to do it online, we are using the Desmos [Classroom].

When we consider the environments of Shay's and Nicholas' classrooms, they are both heavily influenced and reliant on Tools and Technology in different manners: one reliant on technology to function and the other bringing in tools as learning supports.

Finally, we notice that observations focused on content-interactions (Cluster C) increase by about 30%, and student-interactions (Cluster S) increase by about 22% from the Spring 2022 to Fall 2022 semester. We hypothesize that the instructors' discussion is shifting away from Tools and Technology (Cluster E) as they are becoming more comfortable with the technology used in the project, Desmos Classroom, which also allows them to be more focused on other aspects of instruction.

Noticing of Teacher-Focused Interactions (Cluster T)

The teacher-interactions (Cluster T) only has about a 2% shift from one semester to the next. This consistency of noticing teacher interactions may suggest that participating instructors view one of the main goals of video clubs as a time to give feedback to colleagues and reflect on their own pedagogy. An excerpt from Shay when interviewed in Spring 2024, gives clarification of her view of the valuableness of the video-club observations:

> Observations are really good for course coordination, but again, not setting it up for people to criticize other people. It's more of 'oh, I did that. And this went really well for me', 'Oh, you know, this is something that I could do in the future.' So I like observation reports because it was spun in such a positive way.

Shay's reflection of video clubs shows that she considers dialogue in these meetings as useful feedback to compare and develop with her own classroom experiences. Shay's response shows how she may be contributing to the collective noticing of teacher-interactions, due to her perception of the video clubs.

Noticing of Student-Focused Interactions (Cluster S)

During viewings of Nicholas' lessons, instructors observed more student-related interactions when compared to other instructors' teaching. While observations of Alex, Ivy, and Shay range from about 43% to 51% in noticed student-interactions (Cluster S), observations of student-interactions during Nicholas' lessons are 75.70% of discussions. We hypothesize the noticing of student interactions, during Nicholas' lessons, to Nicholas actively focusing on a student-centered classroom environment. In an interview in 2023, Nicholas reflected on his view of tools like Desmos during the previous year, mentioning that he considered Desmos to contribute to "more student interaction and independence." In a later interview from Spring 2024 with Nicholas, he is asked to reflect on his role as an instructor, which he responds to by calling himself a "facilitator." He then expands on this by explaining:

> I don't feel the need to take control of the class. And I feel like that gives [the students] back ownership for the class, that I think they take their work and their commitment to the class differently because of it. Because it's not my class.

We consider the autonomy Nicholas has afforded his students in their learning to be noticed by other instructors, as indicated by the percentage of compiled noticing of Cluster S. We also see this from comments during a Fall 2022 video club session, when reviewing a lesson from Nicholas, a comment from Alex coded as a teacher-student-technology interaction (TSE) is:

> [I] guess just for like the wider context, Nicholas spends most of the class like going through the Zoom and or the Desmos and always looking at student work while also shuffling through rooms. But if they're not having a problem in the room, he doesn't, you know, bother them or anything. He's just checking on all the rooms and also looking at all the Desmos stuff.

In an observation of himself from Fall 2022, Nicholas simply notes himself to be "a more effective instructor" in online classrooms like Zoom, which other instructors also tend to notice during video clubs.

Discussion

This work shows what aspects of the instructional tetrahedron instructors participating in a video club within a PLC noticed when observing their colleagues' teaching. It is important to highlight that what happened in class (what the instructors were doing and what the students were doing) during a video-recorded lesson impacted what might have been notable in an observation. For example, if the instructor was primarily lecturing while the students listened, then an observation would likely highlight either Teacher or Content interactions. This raises the question: Could a shift in the focus of discussion in a video club point to a shift in instructional practices? We hypothesized that the shift in discussion about Tools and Technology after the change from online to in person teaching resulted in less discussion about this cluster in the video clubs. It will be important for future work to investigate this connection between what is noticed and discussed during a video club and how that might reflect shifts in teaching practices.

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"If you are good enough, that can transcend the gender stuff"

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Women in advanced mathematics continue to be underrepresented despite representational progress in K-16 mathematics. Existing literature has investigated women's mathematical identities at the undergraduate level and revealed varying ways women organize their mathematical selves in response to the masculine context of mathematics. In this paper, I expand on the existing identity research about women in undergraduate mathematics to better understand the mathematical identities of women in doctoral mathematics. I discuss themes related to women's mathematical identities and the cultural discourse of "Women in Math."

Keywords: Gender, graduate mathematics, identity

Despite improvement in representation at the undergraduate level, women are still underrepresented in doctoral mathematics with only 28% of doctoral degrees awarded to women in 2018 (National Science Foundation, 2021). In addition to inequitable representation, women in doctoral mathematics have experienced overt prejudice, feelings of invisibility, insufficient or negative advising experiences, and a lack of belonging (Herzig, 2004). Furthermore, women's reports of success are often mitigated by expressions of struggle, bringing into question the quality of their experiences (Ataide Pinheiro, 2021).

The limited research investigating women in advanced mathematics is overwhelmingly focused on understanding the individual factors influencing persistence and attrition (Herzig, 2002a, 2002b; Miller, 2015), with few studies (e.g., Ataide Pinheiro, 2021; Lubienski & Ataide Pinheiro, 2020) focused on the school and societal contexts and their effect on women. Identity research is a potential solution in bridging broader contexts to individual experiences and allows one to reveal the taken-for-granted or status quo that privileges some and excludes others (Darragh, 2016; Gutiérrez, 2013). Identity framing affords researchers the ability to highlight how students "fit into a broader discourse about the politics of language, racism, sexism, and teaching and learning" (Gutiérrez, 2013, p. 44).

Gendered discourses and discourses that create hierarchies of mathematical ability are pervasive in mathematics (Leyva et al., 2021; Mendick, 2006). Mendick (2006) details how gendered identities and practices are produced and reinforced in mathematics classrooms and how certain discourses contribute to the association of mathematics with masculinity. The association of mathematics with masculinity, paired with the idea that mathematical ability is innate, creates an impasse for women who are generally underrepresented in mathematics. At the undergraduate level, mathematics education researchers have taken up the investigation of women's mathematical identities and their relationship to broader discourses about mathematics and gender, finding that these discourses make developing strong mathematical identities a challenge (Damarin, 2008; Hottinger, 2016; Mendick, 2006; Rodd & Bartholomew, 2006).

Women in undergraduate mathematics have responded to these seemingly incompatible identities of woman and mathematician in several ways. They may explain their mathematical talent while maintaining the feminine identity, and/or distancing themselves from a masculine identity by explaining their mathematical ability as abnormal or special. Alternatively, women may choose to downplay their ability as a result of hard work rather than natural talent, contradicting how they ascribe innate mathematical ability to men (Rodd & Bartholomew, 2006).

Solomon et al. (2016) describe another option in which women may develop hybridized mathematical and woman identities through recognition of and reflection on the contradiction posed by being a woman and mathematician. While these studies have not focused on graduate mathematics, there is no reason to believe these discourses are not shared across mathematical contexts and that women in advanced mathematics face some of the same cultural obstacles in adopting mathematical identities. However, doctoral mathematics also differs from undergraduate study and more research is needed to understand the experiences of women in advanced mathematics.

Building on and expanding the literature, I focus on women doctoral students. My goal is to understand the discursive processes by which women construct their mathematical identities. I investigate how woman doctoral students respond to the discursive label of "Woman in Math." by answering the research question: *How do women in doctoral mathematics describe being a "Woman in Math?"*

Theoretical Framework

In response to research of women in doctoral mathematics which focuses on individual factors, I adopt the figured worlds framework, in an effort to connect individual experiences to broader societal narratives about what it means to be a woman in mathematics (Holland et al., 1998). Figured worlds are defined as "socially and culturally constructed realms of interpretation in which particular characters and actors are recognized, significance is assigned to certain acts, and particular outcomes are valued over others" (p. 52). As a sociocultural theory, figured worlds assumes a reflexive relationship between social activity and individual development.

In particular, Holland et al. (1998) emphasize that identities are developed through participation in the activities organized by figured worlds, and figured worlds are produced and reproduced through the work of participants. Their definition of identity states:

> People tell others who they are, but even more important, they tell themselves and then try to act as though they are who they say they are. These self-understandings…are what we refer to as identities (p. 3).

This conception of identity includes the stories we tell about ourselves and others as well as our behaviors which are influenced by these stories.

Drawing from Vygotsky, Holland et al. (1998) emphasizes semiotic mediation with respect to human activity and identity building. This means that cultural artifacts such as objects, discourses, and even people are ascribed meaning and taken up as individuals structure their identities. Holland et al. (1998) draw connections that equate discourses and artifacts, with both being used as tools for mediation. Others have described discourses as "symbols, signs, artifacts, and other cultural representations" (Reinholz, 2022, p. 37) and a source of individuals' meaning making and knowledge (Gutiérrez, 2013). Under this perspective discourses do not describe objective truths; they structure our worlds. The parallels between artifacts and discourses help to understand the way broad social narratives occur in daily social interactions and are appropriated by individuals as they develop their identities.

In the context of doctoral mathematics many aspects of the figured world may signify meaning and value and be taken up by women as they refine their mathematical identities. For example, if a student has a prestigious advisor this might elevate their status in the department and lead to more confidence. Other factors such as grades, qualifying exams, social connections, and of course gender may also influence how they view their mathematical selves.

Gender

While the figured worlds framework does not have a particular focus on gender, I take the understanding, based on the literature (i.e., Mendick, 2006), that gender is part of the value-laden context of doctoral mathematics. Furthermore, I take gender to be a social construct that is continually performed and enforced through various cultural and institutional practices (Butler, 1990). This suggests that gender is not innately residing within individuals but is rather something they *do*. Butler (1990) argues that gender is performed through a repetition of norms, gestures, and discourses that present an image of stable identity. This conception of gender destabilizes the idea that masculinity and femininity are essential, fixed categories (as evidenced in Solomon et al. (2016)). However, despite the theoretical understanding of gender as a continuous, fluid performance, there is a societal commitment to a gender binary (Butler, 1990; Wilchins, 2011), which is paralleled in mathematical contexts and evidenced through a gendered hierarchy of mathematical ability (Hottinger, 2016; Mendick, 2006).

Woman in Math

In the process of interviewing and getting to know the participants in this study I realized that the label Woman in Math held more meaning for them beyond their individual experiences. It invoked many responses, some proud, some distancing. It became apparent that there was a larger cultural meaning associated with the label "Women in Math." Furthermore, all the participants were members of their local Association for Women in Mathematics (AWM), creating a local cultural meaning for them in their department. In retrospect this makes sense considering all the many programs and initiatives taken to increase gender diversity in mathematics at all levels. As such, I consider "Woman in Math" a societal discourse and take it up as an object of interest in this study.

Context and Methods

Southwest University (SU) is a large public university in the southwest United States and is classified as a very high research activity institution. The mathematics department at SU has approximately 200 students enrolled in their doctoral program, with about 15% of students identifying as women. There are approximately 60 faculty, a minority of whom are women. The structure of the program is typical of many mathematics PhD programs in that the first and second years of the program are dedicated to coursework in preparation for three written qualifying examinations, followed by original research.

Participants included five women enrolled, or recently enrolled, in the doctoral mathematics program at SU. The women were selected using a stratified purposeful sampling (Patton, 1990) with respect to years' experience in the program. This included one woman enrolled in coursework, two women in early research, one woman in advanced research, and one woman taking a leave of absence after completing coursework. Additional sampling criteria included participation in the local graduate chapter of the AWM. Data sources included one-hour narrative interviews (Mueller, 2019) from each of the five participants and a focus group interview with four of the five participants. The focus group and narrative interviews included a range of questions about the participants' experience in doctoral mathematics. In this report I focus my analysis on two questions – "*What does it mean to be a woman in mathematics?*" and "*How has being a woman shaped your experiences in mathematics?*" The interviews and focus group were transcribed and open coded for distinct, meaningful ideas (Braun & Clarke, 2006). Preliminary results include three themes – *anticipating doubt, proving yourself,* and *transcending gender* – and their potential connections.

Preliminary Results

There were myriad responses in response to the questions analyzed. However, most participants expressed some version of the first two themes, anticipating doubt and proving yourself. I include that last theme, transcending gender, despite only two excerpts coded from the same participant as it is closely related to the theme of prove yourself.

Anticipating Doubt

Four of the five participants expressed the idea that people might make assumptions about them because they are women in mathematical contexts. For example, one participant explained:

> I also think that a lot of people kind of like do not expect women to be good at maths. Like even in grad school that I think that maybe like in study groups like not maybe taking women seriously or like kind of ignoring what they're saying, because there is kind of an implicit assumption that it's not going to be like, useful or correct.

Here the participant describes her expectation for people to doubt her mathematical ability. She notes different contexts, such as study groups and her phrasing of "even in grad school" implies this an experience she has had in the past and present. Another participant states:

> I think that the instant assumption first off, is going to be that I'm not a mathematician at all. And secondly, I have an idea in my head that may not be true, that I'm sort of guilty until proven innocent.

Despite her brief expression of doubt this participant makes a stronger statement, that people will assume she is not a mathematician at all. The latter half of her statement also expresses the idea that people's first instinct will be to doubt her ability. Overall, this was a common experience to most of the participants. Some women also expressed responding to this perceived doubt by proving themselves to be competent mathematicians.

Proving Yourself

Like the previous theme, four of the five participants expressed the need to prove their mathematical ability. One woman said:

> I felt like to get someone's respect, you had to usually show you were smart enough. Maybe you're working on a problem set and you figure something out and then maybe people will then start listening to you and including you.

In this excerpt the participant describes having to prove she was smart enough but also that if she did this could lead to her gaining more access to working with others. In this way there appears to be consequences and benefits to proving her mathematical ability. Another participant presents her perspective as more high stakes:

> I always felt that I had to be unequivocally better than all of the men in order to be taken seriously… So I always had to have perfect scores, have the best grade in the class, have everything super, super correct because I always felt like if I missed five points on the

> exam, if I was a 95 instead of a hundred, then that would be the reason that someone would write me off.

This passage describes a need for perfection in order to be beyond reproach. She expresses that there is no room for mistakes otherwise her credibility will be questioned. Unlike the previous excerpt, this participant explicitly compares her performance to men. This aspect of her response highlights a connection to the previously discussed hierarchy of mathematical ability and the association of mathematics with masculinity. Most of the women who expressed the need to prove themselves did not describe a need for this level of perfection. However, this participant is the only woman who had responses coded as transcending gender.

Transcending Gender

The transcending gender theme captures the idea that if you are recognized as exceptionally good or successful, then you can escape any implications of being a woman in mathematics. Only one participant expressed ideas related to this theme in response to the interview questions. The first time she discusses the mathematical prestige of her advisor as well as her own accomplishments. She explains:

> …I think something that I've talked to with one of my friends a lot is that I have a certain, have a lot of good at math privilege, even though I don't have male privilege…my advisor's very impressive and it seems that the results that I have done in my PhD are very impressive. So it seems that, yeah, my advisor's a woman and I think she probably dealt with lots more things than I dealt with, but the weight of her famousness in the field and her good reputation far outweighs any sort of women in math stuff.

Here she acknowledges that her standing as someone who is successful in mathematics allows her a certain privilege that is valuable but distinct from the privilege men have. She offers evidence of her privilege through the status of her advisor and of her PhD results. She goes on to say that her advisor's reputation distances her from any implications of being a woman in math. Later in the interview she revisits the idea when she says "But I have experienced a sense that if you are good enough, that can transcend the gender stuff." In this excerpt she speaks strictly about her own experience. Taken with her previous sentiments about her PhD results and her "good at math privilege" one can get a sense of what it is required to transcend gender as a woman in mathematics. Furthermore, her statements in the previous section about needing to be perfect to be beyond reproach gain new meaning as she explains what the reward is for perfection; she can outperform her position as woman and "other" in mathematics.

Conclusion

Preliminary analysis shows that women often anticipate that their mathematical abilities will be doubted. Additionally, women may respond to this anticipation by working to prove themselves as competent mathematicians. Some women saw proving themselves as a way to gain access to participation in study groups or mathematical conversations. However, one woman expressed that if she proves herself enough, she can escape being a woman in math completely. These themes reflect an awareness and response by the participants that being a woman is a meaningful aspect of their experiences in advanced mathematics.

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Mathematics Instructors' Perspectives on the Affordances of LA Implementations in the Undergraduate Classroom

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This study explores undergraduate mathematics instructors' perspectives on the benefits of Learning Assistant (LA) programs. Interviews with seven faculty members highlight LAs' roles in providing student support, enhancing instruction, and promoting student-centered learning. Instructors also noted that LAs gain valuable teaching experience, build relationships with faculty, and grow in confidence. These findings align with the established LA program goals of improving undergraduate education and preparing future teachers. However, the LA program goals of engaging faculty with educational research and institutional change were less evident. Future research will examine how LA use aligns with or diverges from established program objectives.

Keywords: Learning Assistants, Student Support, Faculty Perspectives

Learning Assistants (LAs) are typically undergraduate students who have successfully completed a course and support students currently enrolled in the same class (Talbot et al., 2015). Since its development in 2003, the LA model has become commonplace in higher education. This paper presents a preliminary analysis of interviews with mathematics faculty who have worked with LAs within the past five years (since Fall 2019). This study aims to identify the key benefits that faculty perceive LA programs offer to students, LAs, mathematics programs, and institutions of higher education.

Literature Review

The LA model was developed at the University of Colorado Boulder (CU Boulder) and the University of Colorado Denver with four key goals: (1) transform undergraduate STEM education, (2) recruit and prepare future teachers, (3) engage faculty in discipline-based educational research, and (4) change departmental and institutional culture (Otero, 2015). Talbot, Hartley, Marzetta, and Wee (2015) exhibit similar, directly related goals of improving student learning in undergraduate science courses, supporting reform efforts, and giving high-performing students opportunities to be exposed to science teaching as a career choice.

In addition, Talbot et al. (2015) describe three "foundations" of any LA program – content, pedagogy, and practice. Firstly, content (or preparation) refers to LAs meeting regularly to discuss course content, plan upcoming classes, reflect on past activities, and provide input from students' perspectives to the instructor. Second, LAs participate in a regular pedagogy to discuss learning theory, teaching strategies, formative assessment, and student conceptions in the context of their work. Thirdly, LAs primarily serve to interact with students to guide them in students' own learning process.

Barrasso and Spilios (2021) conducted a scoping review of 39 research papers to assess the effectiveness of Learning Assistant (LA) programs in achieving the four goals outlined by Otero (2015). First, their review highlights evidence that LA programs can lead to increased student satisfaction, reduced DFW rates, greater learning gains, and enhanced student comfort in interacting with peers and faculty—factors that contribute to a stronger STEM identity. Second,

although there is limited evidence regarding improved teacher preparation, the LA program at CU Boulder has seen a rise in student enrollment in its teaching certification program since its implementation (Otero et al., 2006). Third, several studies suggest that the LA model can influence faculty teaching practices (Van Dusen et al., 2015; Caravez et al, 2017). Finally, evidence of departmental and institutional changes can be found in case studies of multi-institutional partnerships utilizing LAs (Cochran et al., 2016; De Leone et al., 2019).

Recent research by Karch, Mashhour, Koss, and Caspari-Gnann (2024) with instructors and LAs demonstrates the ways instructors implement the LA model can vary across institutions and classes and are influenced by their perspectives on learning and curricular goals. Moreover, these differences result in distinct roles for LAs, which impact how LAs learn and interact with students. Thus, the way an LA model is implemented (and its success) depends on how instructors define and implement the role of LAs in the classroom.

This study is a preliminary exploration of how instructors position LAs in their classrooms and why they do so. Specifically, it investigates the extent to which instructors from institutions of varying sizes align their use of LAs with the established goals of the LA model. The research questions guiding this study are:

1. What goals or benefits of utilizing LAs do instructors describe for both classroom environments and departmental culture?
2. In what ways do instructors' descriptions of their work with LAs embody the goals of the LA model?

Methods

Participants

Participants were recruited through a screening survey sent by email from a RUME listserv. A total of 15 participants were chosen based on the following criteria. First, all participants had experience teaching undergraduate courses with one or more LAs within the past five years. Second, participants represented, as best as possible, a range of institution sizes. For this preliminary report, we further reduced our sample to seven participants as outlined in the Analysis section below.

Table 1. Distribution of participants by institution size

Institution Size	Count
1,000 - 4,999 students	1
5,000 - 9,999 students	3
10,000 - 19,999 students	4
20,000 students or more	7
Total Participants	15

Data Collection

Each participant engaged in a one-hour semi-structured interview following the same basic sequence of questions. First, five questions explored the structure and format of course(s) taught with LAs. Second, seven questions explored the ways the student and participant (instructor)

interacted with the LA. A third sequence of questions explored which aspects of learning the LAs provided the most support. Finally, participants were asked for advice for someone training or working with LAs for the first time. Each interview was conducted online by one of the authors using video conferencing software. Interview recordings and computer-generated transcriptions were used for subsequent analysis. Additionally, a post-interview survey was sent to participants to collect additional demographic, educational, and institutional information along with any comments they thought of after the conclusion of the interview.

Analysis

An individual interview was used as the unit of analysis. For this preliminary analysis, one participant's interview was chosen for trial coding based on the clarity of the examples provided during that interview. Both authors individually coded this interview for instructor evidence of work and teaching with LAs in ways that aligned with the goals of the LA model, as described by Otero (2015), along with the subgoals developed in Barrasso and Spilios' (2021) review of the LA literature.

Both authors then met to compare their coding, agreeing on a set of codes that best captured the range of affordances of LAs described in the interviews. These codes are shown in Table 2. The resulting coding scheme closely aligns with the provisional coding method described by Saldaña (2013), as codes were based on existing research frameworks but expanded to better reflect the range of affordances of LAs as described by instructors.

For instance, under the goal of "recruiting and preparing future STEM teachers," the subcodes "giving LAs first-hand leadership experience" and "providing LAs with practice in supporting student questions" were added, as distinct from "engaging LAs in learning about learning" (which we further defined as second-hand study of learning and pedagogy).

At the time of this report, the authors selected six additional participants (for a total of 7 out of 15) to represent the range of institution sizes in the data corpus; the remaining interviews will be analyzed in the completed analysis. One author coded each participant's interview using the coding scheme mentioned earlier. Lastly the codes were aggregated for analysis as described in the next section.

Results

Table 2 displays the coding scheme used in the preliminary analysis of interviews of seven of the 15 participants in our study. Subcodes are listed under primary codes and are indented. A column is provided for each of the seven coded interviews, with the interview heading showing the size of the participant's home institution, as detailed in the key at the bottom of Table 2. Each cell is marked with a "1" if the code was present in that participant's interview, and a "0" if it was not. Additionally, the total number of occurrences of each code across the seven participants is shown on the right, indicating the relative frequency of each code representation across the seven interviews.

Primary code 1 and its associated subcodes relate to the benefits LAs provide to students. In 100% of the interviews analyzed thus far, participants made observations related to subcodes 1a, 1b, and 1c. Code 1a indicates these seven instructors have observed students receiving direct support from LAs in the form of LAs answering their questions about mathematics or institutional knowledge or LAs providing affective support or encouragement. Codes 1b and 1c are closely related; this result indicates that all seven instructors feel that having LAs in their

classroom both improves the overall instruction students receive as well as the amount of student-centered instruction students provide. One participant captured a sentiment shared by several others: "[LAs] can feel the pulse of the room much more than [I can]." Four participants also spoke about code 1d, which relates to improving student retention and filling a need for remediation.

Table 2. Coding scheme and codes present for each participant's interview

Codes and Subcodes **Institution Size***	**A**	**B**	**C**	**C**	**C**	**D**	**D**	**Total**
1. Transforming undergraduate STEM education								
1a. Direct student support	1	1	1	1	1	1	1	**7/7**
1b. Improving instruction in general	1	1	1	1	1	1	1	**7/7**
1c. Developing student-centered instruction	1	1	1	1	1	1	1	**7/7**
1d. Supporting reform efforts	0	0	1	1	1	0	1	**4/7**
2. Recruiting and preparing future STEM teachers								
2a. Building mentorships with faculty	1	1	1	0	0	1	1	**5/7**
2b. Engaging LAs in learning about learning	1	0	1	1	1	1	1	**6/7**
2c. Promoting positive STEM identities in LAs	1	0	0	0	0	1	1	**3/7**
2d. Giving LAs classroom leadership experience	0	1	0	0	0	1	0	**2/7**
3. Engaging faculty in discipline-based educational research literature	0	0	0	0	0	0	0	**0/7**
4. Changing departmental instructional culture to value evidence-based teaching	0	0	0	0	0	1	0	**1/7**

** Institution size: **A** = 1-5k students; **B** = 5-10k; **C** = 10-20k; **D** = above 20k students*

Primary Code 2 and its subcodes describe the benefits LAs receive through their participation in the LA program at their institution. While these benefits concern preparing LAs as future STEM teachers, many of the LAs described by our participants were neither pre-service teachers nor enrolled in teacher education programs. The most frequently mentioned subcode of Primary Code 2, cited by our seven participants, is 2b, which relates to teacher professional development. Most often, participants indicated that LAs met at least once per week either the instructor of the course in which they assisted, the course coordinator, and/or the LA program coordinator at their institution. Some participants also indicated that LAs were required to take a pedagogy course, where they discussed classroom challenges and topics related to metacognition.

Five participants mentioned that LAs built relationships with the instructors they assisted, providing LAs with a faculty mentor they could come to for advice or letters of recommendation. Subcode 2c was referenced by three participants, who observed LAs gaining confidence in their mathematical and teaching abilities. Subcode 2d was mentioned twice, indicating that at two of the participants' institutions, LAs had the opportunity to lead a classroom, either through leading recitations or hosting review sessions.

Lastly, the authors note that it is not surprising that the interviews analyzed contained so few instances of participants mentioning primary codes 3 or 4, as these were not directly referenced in our interview protocol. However, several open-ended questions allowed participants opportunities to mention these topics if they felt them relevant. The one reference to changing

departmental culture to value evidence-based teaching was from a senior faculty that also served as coordinator for the course.

Future Work

Based on this preliminary work, instructors' perspectives of their work with LAs appear focused and aligned on the first two goals of the LA model – transforming undergraduate STEM education and recruiting and preparing future STEM teachers. The degree to which instructors are aware of and motivated by goals three and four (engagement with discipline-based educational research and change in departmental culture to value evidence-based teaching) falls outside the scope of the methods of this current study. Thus, we plan to expand our analysis to better understand: (1) how instructors and departments position LAs in active learning classrooms, (2) the reasons instructors assign LAs to these roles, and (3) how LAs are supported or trained to develop these roles. In this way, we hope to better understand ways the productive use of LAs may align and diverge from the goals outlined in current LA program research.

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Underrepresented Students' Legitimization as Peripheral Participants in the Mathematics Research Community of Practice

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This paper presents preliminary findings on underrepresented students' experiences as they navigate the mathematics research community. Three undergraduate mathematics majors from underrepresented groups were interviewed regarding their experiences at their home institution, research experiences for undergraduates, and an undergraduate mathematics conference. Using communities of practice as a theoretical framework, we explore these students' status as legitimate peripheral participants in the mathematics research community of practice. Analysis led to bridging the communities of practice framework with Gutierrez's (2009) four dimension of equity. In this paper, we focus on the ways in which students' real or perceived legitimacy were impacted by identity and power. In particular, students experienced de-legitimization via the power of the dominant group and via less peripheral members.

Keywords: mathematics majors, community of practice, identity, equity

Mathematics majors engage with mathematics research in a variety of ways during their time as an undergraduate. These experiences go beyond the classroom and their work with their advisor at their home institution. Increasingly, STEM majors participate in activities such as research experiences for undergraduates (REUs) (National Academies of Sciences, Engineering, and Medicine, 2017) and conferences. These experiences are essential for students seeking entry into graduate school (National Academies of Sciences, Engineering, and Medicine, 2017; Peterson & Rubinstein, 2016; Russell et al., 2007). Activities such as REUs expose students to greater opportunities for research, faculty connections, and peer networking. Work examining the holistic experience of mathematics majors must take into account these critical aspects of their development as researchers.

In addition, little research exists on the experiences of underrepresented students as they navigate mathematics research, REUs, or conferences. Research on STEM undergraduates in REUs has shown increases in persistence and promotion of mathematics identity formation (National Academies of Sciences, Engineering, and Medicine, 2017). However, much of the work on the role of REUs in identity formation has not been specific to underrepresented students (Palmer et al., 2015; Seymore et al., 2004). Given importance of these activities for access to graduate school, equity work within the mathematics community must consider how underrepresented students experience not only their local home institution environment but also their holistic experience including activities such as REUs and conferences.

Using a community of practice lens, this work seeks to answer the research question: How do underrepresented mathematics majors describe their experiences in the mathematics research community?

Theoretical Framework

A community of practice is defined by Wenger (1998) as having three dimensions: a joint enterprise, mutual engagement, and shared practices. For example, the work of tailors is a community of practice (Lave & Wenger, 1991). In a community of practice, experienced members (experienced tailors) mentor newcomers (apprentices) to take-up the practices of the

community. These practices include ways of doing, knowledge, and language. To join the community, newcomers must be granted access through sponsorship by an old-timer. At this time, newcomers become legitimate peripheral participants in the community, having gained legitimacy through their sponsorship. The newcomers are participants in that they engage in the joint enterprise of the community, but their work is peripheral as they are given smaller, easier subcomponents of the work and have a great deal of oversight from a more experienced member. Overtime, newcomers are given more complex tasks and independence as they update the community's practices and become fuller members.

To better understand the experiences of underrepresented minorities as mathematics majors, we seek to bridge community of practice (Lave & Wegner, 1991) with Gutierrez's (2009) four dimensions of equity. Gutierrez outlines four dimensions of equity, lying on two axes. On the dominate axis are access and achievement and on the critical axis are identity and power. We view each of these four dimensions as tied to aspects of legitimate peripheral participation. Newcomers to a community of practice need access as granted by sponsorship by an experienced member, providing their legitimacy as a member of the community. As newcomers gain achievement, their participation becomes less peripheral as they moved towards full participation and membership. Further, as newcomers participate in the community, their identities shift based on how they perceive themselves and their participation in the community (White & Lowenthal, 2011). Finally, the full members in the community who hold power are those who are able to decide what it means to be a legitimate participant (Gutierrez, 2013). White and Lowenthal argue minority students must "adopt a form of discourse that originated in and often perpetuates oppression" (p.289) to participate in the community, as dictated by those in power.

In this work, we focus on the mathematics research community as a community of practice. Newcomers are students who are mathematics majors and experienced member are advisors and tenured faculty. Non-tenured faculty, graduate students, and more advanced peers are relative experienced members to mathematics majors These members of the community of practice have greater legitimacy and fuller participation than the mathematics majors but less legitimacy and participation than tenured faculty. Students choose to enter the mathematics research community by declaring a mathematics major and are granted access, or legitimacy in the community, by their advisors. The ability to gain legitimacy and enter the community is connected to Gutierrez's (2012) dimension of access. Students are peripheral participants in the mathematics research community as they are given smaller tasks with close oversight by advisors. Achievement in these activities and coursework, as well as uptake of community practices are required to become a fuller participant in the community and to move away from the periphery. As students negotiate their place in the community, their identities will shift. Students will possibly experience tensions between their dynamic identities and as their perception of their role within the mathematics research community changes. Each student will experience the power relations within the mathematics research community differently. Some will find the privileged mathematic research community discourse to be similar to their everyday discourse. Others will need to learn a new discourse to "fit in". Among legitimate peripheral participants, power relations may be based on identity, institution, area of research, awards won, or any number of achievements, access, and identity markers.

Methods

Data for this study was collected during a conference for undergraduate mathematicians. At the end of the application process for the conference, students were asked to complete a demographic survey, a subset of questions from the X-PIPS-M survey (Apkarian et al., 2019). Of

the 59 students accepted in the undergraduate mathematics conference, 18 students completed the survey and consented to participate in the research. The results of the demographic survey were used to select students to interview, seeking women and/or underrepresented students. This work reports on three mathematics majors with intersectional identities as underrepresented students across multiple demographics. Pseudonyms are used to protect student identities. Amira self-identified as female, black or African American, middle eastern, and white. She was also a transfer student. Cuc self-identified as non-binary and southeast Asian. They were a first-generation college student and transfer student. Ethan identified as male and Hispanic or Latinx. He was an international student and first-generation college student. Interviews were conducted on the final day of the conference and the week after. Interviews were conducted as semi-structured with students being asked questions about their experience as a mathematics major at their home institution, their experience with research experiences for undergraduates (REUs), and the undergraduate mathematics conference. Questions were aimed to get a sense of factors influencing students' trajectories and how their identities contributed to their experiences. Interviews were then transcribed. Data analysis was conducted via thematic analysis (Braun & Clarke, 2006). Interview transcripts were read multiple times before inductively coding each talking turn to capture the essence of the turn. Codes were then compared to look for themes across the data. Themes continued to evolve as frameworks were applied to make sense of them.

Results and Discussion

Our results focus on the most encompassing theme which emerged from the data: Legitimacy as a Participant in the Mathematics Research Community of Practice. In our data, we found students' perceptions of their legitimacy to be heavily influenced by their interactions with peers. Specifically, students questioned their legitimacy based on their interactions with students in the dominant group and with students perceived as having greater achievement in the mathematics research community.

We connect students' real or perceived legitimacy within the mathematics research community to Gutierrez's (2012) critical axis of identity and power. We see power as held by members in the community who are able to grant or restrict legitimacy within the community. Students belonging to the dominant group within the mathematics research community have greater power over their nondominant peers. In addition, students with greater achievement have moved further into the community, becoming fuller participants, granting them increased legitimacy and power. Students' identities as legitimate members of the mathematics research community are determined in part by the individual and in part by others (Gutierrez, 2013). That is, their peers play a role in students' mathematics research community identity formation.

Delegitimization via the Power of the Dominant Group

Several students detailed negative experiences with peers either at their home institution, an REU, and/or the undergraduate mathematics conference. For some, these included microaggressions such as scoffing, interrupting or "mansplaining". Amira described feeling "on edge" before asking a question, nervous she would justify her peers' behaviors. Through their behaviors, peers indicated Amira didn't belong in the mathematics research community, that is, she was not a legitimate participant. Davies (2005) notes that it is not enough to participate in the community, but one must be accepted as well to become a full member. Amira was concerned that her mathematics questions or contributions in class would prevent her from being accepted into the community and confirm that she was not a legitimate participant.

For similar reasons, Ethan described that as a Hispanic first-generation college student, he had developed a "math personality". Ethan described this personality as "very professional" and that he would talk with others in the community "about math…not so much on my background because there's just not a lot of people from my background." By describing his mathematics research community identity as professional, Ethan implied that his background was not professional and did not belong in the community. Ethan adopted an identity which he felt legitimized him as a participant in the mathematics research community. He focused on connecting with his peers through mathematics and their similar mathematics experiences, avoiding discussing his background. Similarly, Cuc felt the need to hide their use of she/they pronouns after a negative interaction with a few students whom they felt didn't "have good allyship". Ethan and Cuc felt the need to change or hide aspects of their identity in order to belong to and be a legitimate member of the mathematics research community. White and Lowenthal (2011) describe conflicts that occur when students' individual culture differs from the academic discourse community, here, the mathematics research community. Ethan and Cuc both describe code switching to navigate between their individual culture or identity, and what was required to belong in the mathematics research community.

De-legitimization via Perceptions of Less Peripheral Members

Amira and Ethan both described feeling intimidated by their peers. Ethan felt some students were pretentious and tried to impress others which led him to feel the mathematics research community wasn't for him. Peers used the discourse of the mathematics research community to come across as less peripheral members. This display of power resulted in Ethan feeling as though he didn't belong in the community. However, Ethan also stated that over time, he knew more about mathematics and could tell when students were trying to sound impressive. As Ethan himself became a less peripheral member, he could identify these embellishments of power. At the same time, he was still cautious, not wanting to look dumb or confirm stereotypes. Similarly, Amira described feeling nervous around "celebrity" students whom she described as high achievers who are known across institutions, and who are prestigious based on who they work with and know. "I was very nervous because…I knew that there were like, very prestigious programs that like, sent their students [to the undergraduate mathematics conference] and very high-level students and I was sort of worried that like, for example, my presentation might pale in comparison". Amira was concerned others would think her research wasn't good enough, or legitimate. Both Ethan and Amira worried about their legitimacy in the mathematics research community in relation to peers they viewed as more legitimate and less peripheral, experiencing imposter syndrome (Murray et al., 2023). Murray and colleagues found those with imposter syndrome do a large amount of emotional work to "survive and thrive in exclusionary environments" (p.754). The power the less peripheral members held led both Ethan and Amira to question the own legitimacy in the community, leading to additional emotional work as they engaged in the practices of the mathematics research community.

Conclusion

Underrepresented minority mathematics majors described experiences related to power and identity as they navigated the mathematics research community. As they interacted with peers, these students questioned their own identity as a member of community. In particular, peers who belonged to the dominant group or with greater achievement in the mathematics research community had power to grant to rescind these students' legitimacy to the community.

Future work will explore the ways in which these students described their entrance into the mathematics research community. Specifically, why they chose to join the community and how they experienced gaining access to the community. In addition, future work will examine the students' access to *mirrors* (Gutierrez, 2012) with which to see themselves in the mathematics research community.

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A Methodology for Mapping Connections Between Persistent Learning Challenges

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Much research within mathematics education is dedicated to understanding the challenges that students often face when encountering new mathematical ideas. Some challenges, called epistemological obstacles (EOs), persist despite research-based instruction designed to target those challenges. The purpose of this report is to describe a methodology for mapping connections between EOs as they emerge during mathematical discussion and problem-solving. We illustrate this methodology through a preliminary analysis of whole-class data collected in an introductory proofs course.

Keywords: Methodology; Epistemological Obstacles; Transition-to-Proof

Understanding the obstacles that students face in mathematics teaching and learning is an essential part of mathematics education research. Much research has focused on identifying and understanding students' cognitive obstacles in mathematics learning (e.g., Smith et al., 1994). Such obstacles are often identified and studied independently, but we have found that many obstacles do not occur independently. Existing research methods, however, do not allow for identifying ways that cognitive obstacles may co-occur. This report provides such a method, allowing researchers to gain greater insight into the ecology of EOs across research areas.

Brousseau (2002) defined three main types of obstacles that students may experience: ontogenetic, didactical, and epistemological. Ontogenetic obstacles occur due to limitations in one's current biological (neuro-physiological) maturation. Didactical obstacles occur when instruction overlooks or potentially exacerbates conceptual challenges in students' mathematical understanding. However, even with instruction designed to target known conceptual challenges, and with sufficient biological maturation (e.g., students who have reached Piaget's formal operational stage), some challenges may persist in students' mathematical reasoning and sense-making. Following Brousseau (2002), we refer to these persistent challenges as *epistemological obstacles* (EOs).

EOs often manifest as a pattern of repeated errors or uncertainties observed by a teacher or researcher. However, EOs are not simply mistakes or the product of ignorance; they are "the effect of a previous piece of knowledge which was interesting and successful, but which now is revealed as false or simply unadapted" (Brousseau, 2002, p. 82). EOs are part of a complex cognitive system and cannot be simply replaced or eradicated (Smith et al., 1994). Our conceptualization of EOs is consistent with what Steffe calls *necessary errors* (Steffe, 1992); they are products of a student's current overall way of operating.

Students may experience an EO when they experience cognitive dissonance, often accompanied by a perturbation, but teachers may experience such dissonance first. Within The Proofs Project, our goal is to identify known EOs in the context of logic and proof, design tasks intended to elicit students' experience of EOs, and leverage inquiry-oriented instruction (Kuster et al., 2018) to provide opportunities for students to address EOs head-on. Consistent with our Piagetian framing, we hypothesize that overcoming an EO requires a structural change in a

student's way of operating (i.e., a metamorphic accommodation; Steffe & Thompson, 2000). Understanding connections between EOs can support instructional design for promoting such structural change.

The purpose of this report is to describe a methodology for mapping co-occurrences between EOs as they emerge through mathematical discussion and problem-solving. We first provide an overview of our methodology. We then illustrate this methodology by drawing on preliminary analysis of whole-class data from an introductory proofs course.

Methodology for Mapping Connections Between Epistemological Obstacles

Our method for mapping connections between EOs is summarized in these six steps:

1. Conduct a literature search to identify known EOs related to a particular mathematical idea, generating a codebook consisting of EOs, their definitions, and potential evidence
2. Collect video recordings of students engaging in mathematical discussion or problem solving (with each other and/or with a teacher or researcher)
3. Analyze video recordings using the EO codebook, creating a spreadsheet containing timestamps, key moments in discussion, low-inference notes justifying codes, and important student quotes, adding emergent codes to the codebook that satisfy the definition of EO as needed
4. Decide upon a "chunk" size (e.g., a discussion topic or classroom task) for determining co-occurrence of EOs, and demarcate chunks in the coding spreadsheet
5. Create an adjacency matrix by counting co-occurrences of EOs within each chunk
6. Represent adjacency using a weighted vertex-edge graph (where vertices represent EOs and weighted edges represent number of co-occurrences).

Step 1 is necessary for understanding the EOs that students may face. Being as specific as possible about potential evidence (behavioral indicators) of each EO (as informed by prior research) is critical. To conduct this search, we recommend beginning with a keyword search using Google Scholar, Scopus, and library databases such as Academic Search Complete. Because researchers use disparate terms to refer to persistent challenges in students' mathematical knowing and learning, a broad range of keywords will likely need to be included using AND/OR Boolean operators and asterisks to include characters that may follow a word, such as "misconce* OR challenge* OR error* OR mistake* OR struggle*," along with keywords pertaining to the specific topic area (e.g., "implication OR conditional"). Conducting a systematic review (e.g., Borrego et al., 2014) will yield the largest set of EOs identified in prior literature and potential evidence for each EO.

In Step 2, students are engaged in contexts where EOs may emerge. Often, EOs emerge in the context of discussing or solving mathematical problems or proofs. Step 3 employs standard qualitative analytic techniques, including making inferences about data, categorizing inferences using codes, and writing low-inference notes to justify one's inferences (Maxwell, 2013). Researchers may also identify emergent codes not identified in prior research and apply these to their analysis (e.g., Corbin & Strauss, 2015). Although not reported here, we identified emergent codes from task-based interviews (Clement, 2000) with individual students and applied these to our analysis (see Norton, Arnold, Antonides, & Kokushkin, 2023).

Step 4 is critical and is context-dependent. For us, given our focus on whole-class discussion and the overall instructional design, generating chunks based on general discussion topics and tasks was most practical. However, an analysis of connections between EOs during a clinical interview setting may permit finer-grained chunks. Step 5 involves the creation of the adjacency matrix itself. To ensure consistency, we recommend only counting each distinct EO once within

each chunk. For instance, we often coded "Reu" on multiple consecutive lines in our codebook, but if they all occurred within a single chunk, then it would only be counted once. Step 6 allows researchers to visualize connections between EOs and the strength of each connection—especially those greater than 1. Notably, our methodology was inspired by prior research that used adjacency matrices to analyze student reasoning (Selinski et al., 2014). However, our approach is relatively novel within the field, given our theoretical framing and focus on EOs.

Illustrating the Methodology

We now turn to illustrating our proposed methodology by drawing on a preliminary analysis of data from whole-class discussions in an undergraduate introductory proofs course.

Step 1: Literature Review of EOs in the Context of Logic and Proof

Prior research indicates that students often assume that if a logical implication (LI) holds, then its converse holds, treating the LI as biconditional (coded LIff; see Epp, 1999, 2003; Rumain et al., 1983; Wagner-Egger, 2007; Wason & Johnson-Laird, 1972). Such meanings are understandable and common in many non-mathematical contexts (e.g., "If you do your homework, then you can play video games," which is likely intended biconditionally). Transforming LIs into a logically related form, such as its contrapositive or negation, poses additional challenges (coded LIt; see Antonini, 2004; Goetting, 1995; Hawthorne & Rasmussen, 2015; Hub & Dawkins, 2018; Knuth, 2002; Stylianides et al., 2004). In particular, students may reason that the negation of an LI should, itself, be an LI (coded NO; see Arnold et al., in press; Epp, 2003), which we call its "opposite" (e.g., the opposite of $p \to q$ is $p \to \sim q$).

Quantifying the negation of a statement poses additional challenges (coded Qneg; see Epp, 2003), particularly uncertainty about whether the negation of an LI should be an existential or universal statement. Many statements in mathematics, including LIs, are not stated with explicit quantification, such as "If a shape is a square, then it is a rectangle." Epp (1999) refers to such statements as containing "hidden" quantifiers (coded Qh), which can pose significant challenges for students. Statements containing multiple quantifiers can pose additional challenges to students (coded Qmult). Students may assume that the order in which two quantifiers are stated does not affect the logical meaning of the statement (coded Qord; for the latter two EOs, see Dawkins & Roh, 2020; Dubinsky & Yiparaki, 2000; Epp, 2003; Piatek-Jimenez, 2010).

Challenges may also emerge when generalizing an argument. To directly prove the validity of a universally-quantified statement, the most common method is to use what Copi (1954) called the Principle of Universal Generalization: a statement holds *for all* $x \in U$ (where U is a universe of discourse) if and only if that statement holds *for any arbitrary* $x \in U$. Students often struggle to make sense of and apply this principle (coded PUG; see Norton, Arnold, Kokushkin, & Tiraphatna, 2023; Selden & Selden, 2013). Additionally, students may attend to the *value* of a variable (e.g., $\delta = .01$), but they may not attend to the *role* played by this variable in a statement or proof, such as the distance between two x-values (coded Qrv; see Dawkins & Roh, 2022).

Lastly, instructors may use visual representations to help students make sense of LIs. Euler diagrams use spatial inclusion and exclusion to illustrate relationships between truth sets of predicate statements. For instance, $p \to q$ can be interpreted as the truth set of p contained within the truth set of q; using an Euler diagram, this would be (conventionally) represented as one circle (representing the truth set of p) contained within another circle (representing the truth set of q). However, students may struggle to make sense of quantification using Euler diagrams, and students may conceptualize $p \to q$ as a circle representing the hypothesis *containing* a circle representing the conclusion (coded Reu; see Antonides et al., 2024; Hub & Dawkins, 2018).

Steps 2-4: Data Collection and Analysis

Data come from an upper-level undergraduate introductory proofs course. The instructor (third author) took an inquiry-oriented approach, with students regularly engaging in small-group and whole-class discussions. All class sessions were recorded (excluding exam days). The analysis reported here focuses on discussions from the 40 class sessions throughout the semester. These data were collected from a video camera installed in the back corner of the classroom ceiling, controlled by a researcher in the neighboring room, as well as four microphones installed in the four quadrants of the classroom.

To analyze whole-class data, we used a codebook of EOs developed by (a) searching research literature related to persistent challenges in logic and proof, and (b) supplementing with emergent codes that were identified in clinical interviews with students (Norton et al., accepted). Because emergent codes did not appear in our analysis of whole-class discussion, we focus only on those codes identified in prior literature.

A pair of researchers (first and second authors) created a spreadsheet to record line-by-line descriptions of classroom activity (e.g., tasks, questions from the instructor, and student quotes) with timestamps for each line. They coded each line that corresponded with evidence for one or more EO. Lines were then "chunked" together by task or discussion topic. Two researchers (fourth and fifth authors) created an initial adjacency matrix of EO co-occurrences between chunks, counting each distinct EO once within a given chunk.

Steps 5-6: Results and Interpretation

Table 1 shows an adjacency matrix of EO co-occurrences (omitting EOs that had zero co-occurrences with other EOs). For instance, the first row, second column, shows that LIff and LIt co-occurred 1 time during whole-class discussions. Overall, of the 10 EOs suggested by prior literature, 9 EOs co-occurred with others in at least one segment of class discussion. The final column of Table 1 shows the total number of connections between each EO and other EOs, found by taking row sums. In particular, the total connections for LIt (10) and Qneg (11) indicate that, in the context of class discussions, students' experiences transforming LIs and quantifying the negation of logical statements are highly connected to their experiences of other EOs.

Table 1. Adjacency matrix of EOs experienced in an introductory proofs course

	LIff	LIt	Qh	Qmult	Qord	Qneg	NO	PUG	Reu	Total
LIff		1	1	0	0	0	0	0	3	5
LIt	1		1	0	0	4	1	0	3	10
Qh	1	1		0	0	1	0	1	1	5
Qmult	0	0	0		3	2	1	0	0	6
Qord	0	0	0	3		0	0	0	0	3
Qneg	0	4	1	2	0		3	1	0	11
NO	0	1	0	1	0	3		1	0	6
PUG	0	0	1	0	0	1	1		0	3
Reu	3	3	1	0	0	0	0	0		7

The weighted graph shown in Figure 1 illustrates connections between EOs. Figure 1 makes salient the relatively large number of paths connecting LIff, Reu, LIt, Qneg, and NO. Additionally, Figure 1 shows that Qord is isolated from other EOs except for Qmult.

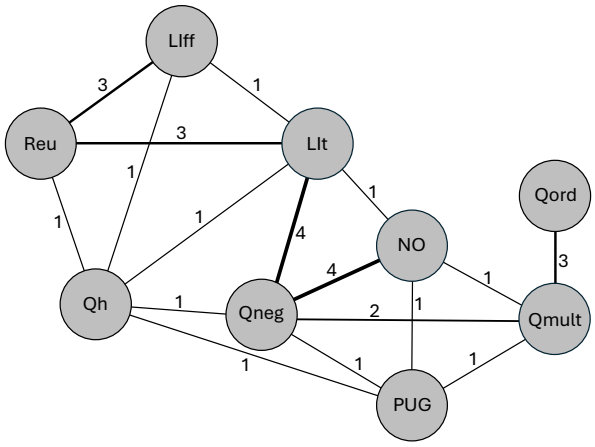


Figure 1. Vertex-edge graph illustrating adjacency among EOs from an introductory proofs course

Classroom Example of a Connection Between EOs

We provide an example of a class discussion intended to elicit students' experience of LIt, NO, and Qneg. The teacher (third author) presented students with the statement p: "For all nonnegative real numbers x, $\sqrt{x} \leq x$" (logically equivalent to the LI: if $x \geq 0$, then $\sqrt{x} \leq x$). The teacher asked students to determine "what statement is equivalent to saying p is false." Following small-group discussion, class discussion began with students mentioning the counterexample $x = ¼$. The teacher then challenged students to generalize this finding into a statement equivalent to saying p is false as follows.

Teacher: ... in general if I say "for all x, something happens," then how would you talk about the other side of the argument?

Student: Um, the false statement would be like, "for all x, something doesn't happen."

Teacher: So, do I need to show that… he said for all x we want to show that open statement doesn't happen.

Student: Oh, for some x in the same set, it doesn't.

In attending to transforming the statement (LIt), the student experienced Qneg and NO; their quantified response "for all x, something doesn't happen" is logically equivalent to the opposite LI, "if x is in the set, then something doesn't happen." The student amended their quantification to "for some x," perhaps in an effort to reconcile their quantification with their understanding that a single counterexample disproves the statement. The adjacency matrix in Table 1 cued us to looking for connections between Qneg and NO, and to proffer explanations for their relatively strong relationship. Indeed, when attempting to transform a universally quantified statement into its negation (LIt), students may initially conceptualize the negation as also needing to be a statement about all values in the universe (Qneg). If this is the case, then it makes sense that students may interpret the negation as its opposite (NO).

Conclusion

This report outlines a methodology for mapping connections between EOs in mathematical discussions and problem solving. Our preliminary analysis of data from an introductory proofs course revealed significant co-occurrences among EOs, such as LIff, Reu, LIt, Qneg, and NO. This methodology can be used broadly to highlight connections between EOs and their potential impact on students' reasoning. Future research should expand this approach across varied mathematical contexts to further refine our understanding of EOs and develop instructional tools and strategies designed to overcome them.

Acknowledgments

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Addressing Equity and Identity in a General Education Mathematics Course

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Motivated by the goal of reducing equity gaps in achievement at a predominately white, public land grant research university, a team of mathematicians designed and implemented research-based strategies for promoting equitable instruction in a coordinated multi-section general education terminal mathematics course. To assess the impact of the course on students' mathematics identity, we utilized a pre- and post-survey adapted from an instrument developed by Hazari et al. (2020) which included scales to measure the constructs of sense of belonging, interest, recognition, and utility/competence. Preliminary analysis revealed that the means of student responses changed in the desired direction from pre- to post-test, with sense of belonging showing the greatest, statistically significant change. Initial quantitative and qualitative findings suggest the instruction tasks described here, which are relatively simple and inexpensive to implement, can have a significant impact on improving mathematical identity for students in a general education mathematics course.

Keywords: mathematical identity, sense of belonging, general education mathematics

In recent years, STEM education research has focused on student identity and its impact on students in STEM courses (Tytler, 2014). Some researchers have broadly defined science identity as an individual's "understandings of self," the ways in which they "make sense of themselves" and how they perceive they are "made sense of by others" within the context of science (Kane, 2012, p. 458). This definition of identity was applied to students pursuing study in physics (Hazari, Cass, & Beattie, 2015; Hazari, Sadler, & Sonnert, 2013) using a framework developed by Carlone and Johnson (2007) that introduced three constructs of science identity. These three constructs formed the basis of a framework adapted for physics identity by Hazari et al. (2020), who also developed a fourth construct.

Additionally, efforts to promote equitable education and positive STEM identity are widespread, with stakeholders in STEM education offering workshops and programming to guide instructors wishing to implement equitable curriculum and teaching practices. Though examples are plentiful, one such program is The Inclusive STEM Teaching Project, sponsored by the National Science Foundation (NSF), Directorate for Education and Human Resources (EHR), and Division of Undergraduate Education (DUE), and developed by a team of faculty representing Boston University, Northwestern, University of Michigan – Ann Arbor, Des Moines Area Community College, The University of Utah, University of Wisconsin – Madison, and the University of Georgia. (https://www.inclusivestemteaching.org/) Having completed the program, one of the authors conducted focus groups of UNL students to analyze localized needs, and then chose and implemented changes to the curriculum and instructional tasks in the general education mathematics course, Contemporary Mathematics. The strategies involve (1) making visible the contributions of mathematicians from historically underrepresented minority groups in multiple ways (course workbook, Open Educational Resources (OER) course textbook, biography assignments, homework questions, in-class discussions), (2) connecting these contributions to content learned in class, and (3) reflecting on personal learning experiences in mathematics alongside the experiences of contemporary mathematicians via an extended writing assignment.

In this study, we utilize an adaptation of the instrument employed by Hazari et. al. (2020) to assess constructs of identity in the context of mathematics. In particular, we use a pre- and post-survey assessment of mathematical identity in a general education (terminal) undergraduate mathematics course in which concerted efforts to promote equitable instruction and positive STEM identity have been implemented.

Context and Curricular Changes at our Institution

In response to emphasis at University of Nebraska-Lincoln (UNL) on reducing equity gaps in achievement across the institution, UNL's Department of Mathematics made a commitment to implement changes designed to address these gaps. One such change was the widespread use of Open Educational Resources (OER) as textbooks and extended tutoring/support opportunities for equitable access to resources. However, curricular changes were also needed to successfully address achievement gaps. In preparation for determining appropriate changes, the department procured university funding to collect feedback from underrepresented minority (URM) students using focus groups to identify curricular and other barriers to students' success in Math 203. In addition to pedagogical and logistical changes, findings from the focus groups led to curricular revisions specifically designed to address issues of mathematical diversity and identity. Consequently, the curricular changes were intended to help all students (URM students in particular) see themselves as part of the culture of mathematics, as individuals capable of mathematical success, and as being potential contributors to a body of mathematical knowledge fundamental to modern society. This was accomplished by including new or enhanced emphasis on topics for which relevant mathematics is being developed by researchers from URM groups (e.g., mathematical modeling, epidemiology, gerrymandering) and were incorporated into the curriculum in a multi-pronged approach:

1. Relevant content was added to the OER.
2. Each unit of the OER closes with a section featuring the mathematicians from URM groups whose research relates to these topics, featuring images (for quick, visual recognition of diversity), basic biographical data and links for further information.
3. Questions about a subset of the featured mathematicians are included in homework (WeBWorK) assigned for that unit based on these entries.
4. Students are tasked with selecting one of the mathematicians and writing a brief biography following specified prompts.
5. Instructors lead a brief in-class discussion about the mathematicians featured in students' biographies.
6. An extended biography and personal reflection is assigned directing students to the MAA publication "Living Proof" (Heinrich et al., 2019).

Methods

At the beginning of the spring semester of 2024, a pre-survey was administered as part of an assignment to all students enrolled in six different sections of Contemporary Mathematics, UNL's general education mathematics course for non-STEM related majors. Of these, 118 completed the survey expressing a willingness to be part of the study. To accommodate UNL's final week policy, the post-survey was administered as an extra credit assignment and was completed by 84 students at the close of the term. For the purposes of this study, the students are categorized as 78% majority (predominantly white students) and 22% non-majority (students from underrepresented minorities).

The survey instrument is based on one developed by Hazari, et al. (2020) that measures four constructs (interest, recognition, performance/competence, sense of belonging) determined to comprise physics identity in the context of undergraduates studying physics both at the beginning and near the end of their degree programs. We utilized similar questions by replacing references to "physics" with "math" when it made sense to do so and making a few other adjustments to better reflect course content and goals. We also added questions about growth mindset structured in a manner similar to those in the instrument used by Hazari, et al. (2020). A five-point Likert scale was used for responses to the construct items; four free response items were included for justifying a few key Likert ratings and to share comments directly correlated to the new curricular tasks addressing diversity and identity as explained above.

Indicator questions related to students' sense of belonging are shown in Table 1 below, both as an example and to highlight certain significant gains.

Table 1. Belonging Indicators [scale 1=not at all, 3=neutral, 5=very much so]

With respect to other students taking [____], to what extent do you…	[Math 203]	[other math classes]
…feel different from others in the community?	Q1	Q4
…feel alone or isolated?	Q2	Q5
…feel inadequate as a member?	Q3	Q6

Means from the Likert scale responses were analyzed and are summarized in the quantitative findings below. The free response survey questions on the post-test were categorized as positive, negative, or neutral responses by the authors and summarized in the qualitative findings.

Preliminary Quantitative Findings

A two-tailed z-Test in Excel was used to compare pre- to post-means, with α=0.05. Results show means of all individual questions and the combined categories changed in the desired direction from pre- to post-test, with an increased sense of belonging showing the greatest, statistically significant change. (See Table 2.)

Table 2. Pre/post mean comparisons for construct categories

Construct category	Pre-survey Mean (n=118)	Post-survey Mean (n=84)	Difference in Means	*p*-value (*z*-Test)
Interest	2.11	2.20	0.09	0.33
Recognition	3.26	3.36	0.10	0.16
Belonging#	3.77	2.03	-1.74	0
Growth Mindset #	3.28	2.99	-0.29	0.0001
Utility/Competence	3.46	3.62	0.16	0.024
Mathematical Identity	2.88	3.05	0.17	0.057

#The desired change based on the phrasing of questions in the Belonging and Growth Mindset categories is a decrease from pre- to post-means. For all other categories, the desired change is an increase.

A closer look at some individual indicators of construct categories for which combined mean changes were not statistically significant reveals some indicators exhibited greater change than others within their respective categories. Switching to a one-tailed z-Test (based on evidence from Table 2), Table 3 shows the results for two such indicators. Using the same Likert scale explained in Table 1, the desired change is positive.

Table 3. Pre/post mean comparisons for two isolated construct indicators

Construct Category and Indicator Item	Difference in Means	*p*-value
Recognition Q3: To what extent do you believe you have the ability to understand mathematics topics/concepts well?	0.275	0.026
Identity Q3: With respect to one or more of your identities, to what extent do you believe you or people like you can excel in mathematics?	0.22	0.059

Preliminary Qualitative Findings

Here are the three of the optional free response questions with a summary:

Q32: "With respect to one or more of your identities, to what extent do you believe that people like you can excel or are excelling in mathematics?" We categorized 41 of the 58 written responses as positive, nine neutral, and four negative. Additionally, we found responses that we couldn't categorize, due to poor wording or statements like "I don't understand the question." From the 41 positive responses, many students revealed a belief that if someone works hard, they can excel independent of their identity. The positive responses included:

- I think people like me, those who struggle can either do great or not. It all depends on their mindset and work ethic. If they go and continually ask for help and make an effort to understand the material, then they can be really successful.
- I have seen examples of LGBTQ+ people and females excelling in mathematics, especially in this course. I feel that people like me can excel in mathematics if they would like to. I'm glad that there are many different types of people represented in this course and not just old dead white guys.
- I think through my own identity, as a Hispanic female, there is potential for other women like me to excel in mathematics or who are currently excelling in mathematics that influence many others to do the same.

Q33 on the post-survey asked about key takeaways from the assignments related to the biographies of URM mathematicians. 56 of 64 responses were positive, responding that they learned about a wide variety of math careers, or saw a wide variety of identities represented, or they learned that even mathematicians struggle with math. One such response was,

- A big take away from me is that these people are normal people. Some of them did struggle in math in their lives. Not only that but they come from a number of backgrounds, some very similar to mine.

Q34: "If you indicated math will be useful to you in your future career, how do you see yourself using it? If you do not believe math will be useful for your career, can you see yourself using it in daily life? If so, how?" For this question, there were 62 responses. We categorized 59 of the 62 as positive responses, two neutral and one negative. This was an encouraging result, especially since many of those who chose to respond identified themselves as future teachers.

Results from these questions suggest positive outcomes, just as we saw in the quantitative data. The written responses from students identifying themselves as future teachers suggest we

should implement instructional tasks and curriculum changes such as these in courses for future teachers.

Continued Analysis and Conclusions

Our initial statistical analysis compared means from the pooled responses for pre- and post-tests. However, the pre- and post-data can be paired, allowing additional analysis that can potentially address changes in identity at the individual level. Students can be isolated by demographics (gender, race, ethnicity, first-generation) to measure the impact on mathematical identity among these groups (the students we endeavored to reach). Another group of interest are those in the course pursuing a degree in elementary education, as most of these students take this course to fulfill a general education requirement. Improving sense of mathematical identity for future elementary teachers has significant potential to impact future learners of mathematics.

Findings from Hazari et. al. (2020) reveal that while all constructs from their study contribute to a positive physics identity, the impact of each construct is case-dependent. In particular, for undergraduate students at the beginning of their program of study, a strong sense of belonging has the strongest impact on their development of positive physics identity. While Math 203 is a general education course, it is not unreasonable to suggest results similar to those observed in this study (i.e., the strong gains in sense of belonging) have the potential to impact other undergraduate mathematics courses, particularly those at entry-level.

Implications of both quantitative and qualitative results from this study suggest tasks such as those described here, which are relatively simple and inexpensive to implement, can have a significant impact on improving students' mathematical identity for students with or without majors requiring courses in STEM.

Questions for Discussion: (a) How can we tell if we are impacting minority students when our population includes so few of them? (b) What are suggestions for analyzing the data without being affected by our own biases? (c) One goal is to analyze responses from those students going into (elementary) education. Can these curricular strategies and instructional tasks be used as recruiting tools for non-majority students to pursue teacher education? Will the positive changes in math identity among preservice teachers carry over into their practice and to their future students?

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Implementing Playful Mathematics Tasks in Calculus Sections

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This preliminary report explores how students experience playful mathematics tasks in sections of a large Calculus I class. We implemented tasks that have been established to foster playful mathematical engagement and learning in 5 sections, and examined whether students experienced playfulness across multiple dimensions. We found that students enjoyed the class, felt engaged, and considered themselves to have understood the content, which they rated as at least somewhat challenging. These findings are being used to design professional development for future Calculus I teaching assistants, in order to examine potential scalability of playful mathematics and learning to lecture-based classrooms.

Keywords: playful mathematics, calculus, undergraduate

There has been increasing interest in examining how playfulness can be incorporated into undergraduate mathematics as the productivity of play for mathematical learning across age groups is becoming more acknowledged. However, the undergraduate research thus far has focused on smaller seminar-style courses (Holton et al., 2001, Ellis et al., 2022), is often conducted with mathematics education majors who are well equipped to handle the ill-defined nature of many mathematical play tasks (Ellis, Horne, & Bloodworth, 2024; Ellis, Bloodworth, & Horne, 2024), and is usually taught by the researcher who is well conversant with supporting mathematical play. This preliminary research report addresses the first year of a multi-year study, in which the authors implemented playful calculus tasks in five Calculus I sections of a large lecture class, as the first step in understanding how to support such students in playing with and learning about mathematics. The second year will focus on conducting professional development with the section TAs (mathematics graduate students) so that they can implement playful calculus tasks in their own sections. We report on the initial results of the first year of the study, and outline our consequent plans and goals for both conducting the professional development workshops and evaluating the success of student learning from the playful mathematics tasks.

Background

Mathematical play has been well valued in early childhood, as a productive and powerful way for young children to experiment with mathematical patterns in the world (e.g., Wager & Parks, 2014). Scholars have theorized that mathematicians and young children both play with mathematics in similar ways (e.g., Williams-Pierce, 2019), and recently, Horne, Bloodworth and colleagues (2022) have empirically confirmed that mathematicians do in fact play with mathematics. They interviewed mathematicians and found evidence of mathematical play while problem solving, including markers of open exploration, enjoyment, investment, perseverance,

and flow. An open question, therefore, is how mathematical play can be supported after early childhood with learners who are not mathematicians. Researchers have begun investigating how to do so with older children, but still primarily with youth, not college aged students (e.g., Gresalfi, 2015; Williams-Pierce & Thevenow-Harrison, 2021).

A smaller but no less important community of researchers have begun investigating mathematical play in undergraduate and graduate courses. For example, Ellis and others have designed a variety of tasks specifically for undergraduates addressing covariation and function graphs, calculus topics such as derivatives and integrals, and tropical algebra, and found that playful tasks with intrinsic feedback and ample opportunities for problem posing can foster both playful engagement and learning. Instead of using the term 'mathematical play' in their work, they use the phrase 'playful mathematics,' "to describe the activities…and features of an instructional environment that can facilitate mathematical play" (Ellis, Horne, Bloodworth, & Ely, 2025, p. 2). We consider playful mathematics to be a conceptual subset of mathematical play, and one that is particularly well suited to the complex social learning contexts of undergraduate mathematics courses.

Given this considerable evidence that playful mathematics could be a powerful tool for supporting productive mathematical learning, it behooves us to understand how we could support larger-scale implementation at the undergraduate level. We focus here specifically on Calculus I, because it is commonly taught as a large lecture class, and can be a gatekeeper to STEM degrees and careers. While our long-term goal involves designing and implementing professional development for the TAs who run the smaller sections of the large lecture-based classes, and evaluating the influence of the mathematical play on students' calculus learning, the study we report on here is the foundation of that longer-term work. In particular, we wanted to better understand how students experience playful calculus tasks after being used to more traditional mathematical learning environments. Consequently, we asked: How do undergraduate students in a traditional calculus section experience playful calculus tasks? In particular, we investigated how they self-report on differing dimensions of engaging in playful calculus tasks, ranging from how much they believed they learned to how much they enjoyed (or did not enjoy) the tasks.

Methodology

This study collected data from participants in five different sections of two Calculus I courses, taught by the same professor. This 4-credit course at a large public university had a 50-minute lecture three times a week and a 50-minute section twice a week, so students engaged with the course content daily. Each course had 100 to 150 students, and each section was capped at 25 students. Researchers received permission from the professor and TAs to implement playful calculus tasks in seven different sections towards the end of the semester; in two sections, the students all opted out of participating in the study, resulting in data from five sections.

Playful Tasks

The playful tasks used in this study come from previous research (Ellis, Horne, Bloodworth, & Ely, 2025), and address the construction of function graphs as traces of covariation between area and length of swept shapes. For example, an animated growing rectangle would be shown to the students, and they would be asked to 'sketch' a graph on two axes, where y is the area and x is the length, and given no other information. There were no given labels or quantities (i.e., students could not calculate area or length numerically), and students would discuss their reasoning in small groups and then share with the whole class. Each section completed three

growing shapes tasks (rectangle, triangle, triangle-rectangle-triangle), and in only one section did we manage to the fourth task, where students would create their own growing shape.

The primary feedback the students received on their work was through their peer discussions, as the researchers limited themselves to asking questions and encouraging comparison between different student graphs. Refer to the above citation for more details on the task design, and how they foster playful mathematics. The research described in this paper is unique from the previous research in that the tasks were implemented in sections of a large lecture course with undergraduates seeking to fulfill benchmark requirements for their degree programs, whereas other research involved small-group middle school teaching experiments (e.g., Ellis et al., 2025), and undergraduate teaching experiments (Ellis, Bloodworth, & Horne, 2024; Ellis, Horne, & Bloodworth, 2024). The students were ostensibly quite familiar with the concept of covariation and the process of constructing function graphs, prior to data collection, given that they were close to finishing a Calculus I course.

Data Collection

The surveys were adapted from Riling, Dietiker, and Gates (2019), who developed a tool for evaluating students' mathematical experiences in individual lessons. These surveys are brief (one page), with 8 different items, specifically geared towards capturing a variety of student perspectives on a mathematical lesson. In Item 1, students are asked to pick up to three descriptions that 'best describe' their view of the lesson, ranging from positive valenced words such as Fun, Interesting, Surprising, to negative valenced words such as Boring, Stressful, Not Special. Item 2 asks them to rate how they felt they understood the topic - that is, evaluating their learning - with three options: Don't Understand, Somewhat Understand, Understand. Item 3 asks them to choose the best option for how they 'felt,' ranging from "Very bored: I was very bored and wish we would do something different" to "Very interested: I was very interested and the lesson was really fun and/or intriguing." Finally, Items 4-8 were 4-item Likerts ("Strongly Disagree" to "Strongly Agree") with the following prompts: Time flew by; The content of this lesson was relevant to my life; Today's math class was like most days in this class; I wish more lessons were like this one; and The content of today's lesson was challenging for me.

We only included very clean data for this analysis, and plan to incorporate the messier remains in later analyses. In particular, we only included surveys that were administered after the first playful task and at the end of class, discarding surveys administered in the middle (due to inconsistencies in data collection across sections), and surveys found after class with no identifiable temporal provenance. We also removed entire surveys when the respondent indicated more than three descriptors in Item 1. When a single item on a survey was left blank, we replaced it with the average value of other respondents; when someone chose a value between two values on the scale, we used the average value of all the respondents to determine whether to round up or down. We are in the process of incorporating the messier data that we removed for this preliminary analysis, for our next phase of analysis. We used the R base package (R Core Team, 2024) to conduct descriptive statistics on the resulting 85 surveys.

We also collected video, audio, and survey data from consenting students in those five sections (n = 50). We are still working on the transcription and thematic coding of this data due to unexpected data complexities, so the primary data source for this paper is the survey data.

Results

Item 1: Positive and Negative Valenced Words

This item asked respondents to choose three terms that best described their experience with the lesson. We enumerated the positive and negative valenced words, then summed them across students. In the 85 entries, positive valenced words accounted for 201 of 238 choices, and negative valenced words accounted for the remaining 37. On average, each survey had 2.36 positively valenced words, and the medians of 3 (positively valenced) and 0 (negatively valenced). These results indicate that students overwhelmingly enjoyed the playful tasks.

Item 2: Rating their Learning

This item asked students to rate their learning, and whether they considered themselves to have understood the lesson. We assigned values to the three possible responses: 0 (Don't understand); 1 (Somewhat understand); and 2 (Understand). We found that the median response was Somewhat understand, with an average of 1.29, above Somewhat understand. Consequently, we concluded that most students considered themselves to at least mostly understand - in other words, the students generally considered themselves to have learned during the playful task. This result was particularly intriguing when the results of Item 1 are also considered: the students positively described their experience engaging in the tasks, *even though* they only felt like they 'somewhat understood' the mathematics.

Item 3: Rating their Engagement

Item 3 asked students to choose which of four options best described how they felt about the lesson overall: 0 (Very bored); 1 (Uninterested); 2 (Interested); and 3 (Very interested). *Interested* was the most common response by far ($n = 56$), followed by *Uninterested* ($n = 20$), *Very Interested* ($n = 8$), and *Very Bored* ($n = 1$). We considered this to indicate that students tended to be interested in the playful calculus tasks.

Items 4-8: Likert Scales

These five items were developed specifically to triangulate earlier items and test other potential relationships (Riling, Dietiker, & Gates, 2019), and we examined them together. These 4-item Likert scales were forced choice, ranging from Strongly Disagree to Strongly Agree.

Table 1. Response Frequencies of Items 4 through 8. SD = Strongly Disagree; D = Disagree; A = Agree; SA = Strongly Agree.

Prompt	SD	D	A	SA
Item 4: Time flew by	4	19	48	14
Item 5: The content of this lesson was relevant to my life	14	50	21	0
Item 6: Today's math class was like most days in this class	41	38	6	0
Item 7: I wish more lessons were like this one	9	17	51	8
Item 8: The content of today's lesson was challenging for me	15	44	22	4

Students disagreed that the content of the playful calculus tasks were relevant to their lives, disagreed that the content of today's lesson was challenging for them, and tended to disagree/strongly disagree that this was a typical day in the class. On the other hand, students tended to agree that time flew by, and that they wish more classes were like this one. We unpack the complexities of these results in the discussion.

Discussion & Conclusion

In summary, we found that students enjoyed the class (Items 1 and 7) and felt engaged (Items 3 and 4), felt like they somewhat understood the content (Item 2) and agreed that it was at least somewhat challenging (Item 8). They did not consider the content particularly relevant to their lives (Item 5), and they felt that this class session was quite different from other days in section (Item 6). We conclude that our students overall had a positive experience engaging in these tasks, and likely experienced playful mathematics. As a result, we are using these preliminary results to guide the next step in our research.

We are delighted that these playful mathematics tasks appear to have worked as intended outside of their original design context of teaching experiments. In particular, the findings indicate that the students experienced several hallmarks of mathematical play, such as challenge and enjoyment (e.g., Ellis et al., 2022; Williams-Pierce & Thevenow-Harrison, 2021). However, we were unsurprised that students found this class session different from their previous experiences (Item 6). When the first and second authors were originally collecting data, we changed the chairs in each classroom so that students were grouped or facing each other, instead of facing the front of the classroom in neat lines. Several times, students walked in, looked at the chairs, and immediately walked out again to check the room number - in other words, their initial reaction to a non-lecture set up in the classroom was, *this must not be my class*. This anecdotal evidence was supported by the quantitative findings, and if we use Item 6 as a proxy for previous experiences in class, we consider it highly likely that students felt more enjoyment and engagement (as evidenced by Items 4 and 7) with these tasks than in the typical lecture-based section. However, there are numerous potential reasons why they may have experienced these tasks differently, such as the sheer novelty of a different instructor or being part of a research study. Our future research will dig deeper into these potential reasons and other limitations, as well as investigate our professional development efforts with TAs, and the resulting playful mathematics experiences of their students.

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Exploring Undergraduate Student Survey Responses About Engaging and Inclusive Experiences With ELITE PD MGTAs in Their Mathematics Courses

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We focus on equitable and inclusive teaching practices in a professional development program (PD) for mathematics graduate teaching assistants (MGTAs). In doing so, we hope that MGTAs will grow to improve the experiences of their students. We analyzed over 1200 undergraduate surveys corresponding to nine MGTAs in our PD program. We completed statistical analyses by comparing means in two ways: (1) the aggregate data for each term and (2) for the individual MGTAs for each term. In the first analysis, we found a statistically significant improvement in scores from the first term to the last term of an academic year. However, for the second analysis, when we analyzed the means of each MGTA's survey responses, we found that two of nine MGTAs had decreased scores over the course of the academic year. Ongoing research is investigating the extent to which experiences improved for different subpopulations of students within MGTAs' classrooms.

Keywords: Graduate Teaching Assistants, Professional Development, Student Experiences, Survey

Introduction

There is a growing body of research documenting undergraduate students' experiences of marginalization and exclusionary systems in mathematics classrooms (Battey et al., 2022; Leyva et al., 2021; Nieminen et al., 2024). Combating such inequities is desirable, but complicated. In recent years, scholars have recognized active learning instructional practices as capable of disrupting inequities in the mathematics classroom (Theobald et al., 2020), but also capable of perpetuating inequities without careful attention to equity and inclusion within those practices (Ernest et al., 2019; Johnson et al., 2020). As such, engaging learners through equitable and inclusive teaching practices is an important skill for instructors to develop.

Given the number of mathematics graduate teaching assistants (MGTAs) teaching undergraduate mathematics and their impact on undergraduate learners (Selinski & Milbourne, 2015), it has become increasingly common for mathematics departments to offer professional development (PD) for teaching to MGTAs (Braley et al., 2023; Ellis et al., 2016), with 'active learning' and 'equity and diversity in the classroom' becoming more frequent topics (Braley et al., 2023). As such, there is much to learn about the extent to which undergraduate students' experiences are shifting in more equitable directions when their instructors are engaged in PD with a focus on engaged learning, equity, and inclusion.

Our research team implemented a PD program for MGTAs that focuses on engaging learners through equitable and inclusive teaching practices. The goal of this preliminary report is to analyze undergraduate student survey responses from the mathematics classes of MGTAs enrolled in the PD program. Our research question is: *Do MGTAs receive improved student survey responses over time in a PD program?*

Literature Review

Many scholars and professional organizations have expressed the need for teaching practices that actively engage students in post-secondary mathematics classrooms (AMS, 2019; Freeman

et al., 2014; MAA, 2019; Saxe & Braddy, 2015). This increased attention to and creation of policies on active and engaged learning are due to years of research that have consistently illustrated that undergraduate learners experience improved outcomes when teachers engage them in mathematical thinking via open-ended questions, problem-solving activities, and opportunities to reflect on their own knowledge (Harper & Quaye, 2009; Laursen, 2013; Laursen et al., 2014; Laursen & Rasmussen, 2019; PCAST, 2012; Wieman, 2017). Active learning has "large and consistent" impacts on undergraduate learning outcomes (Freeman et al., 2014; Wieman, 2014, p. 8319). For example, Rasmussen and Ellis (2013) found that STEM-intending students who experienced more engaging pedagogy in calculus courses were more likely to continue to the next mathematics course than students in lecture based calculus courses. Freeman and colleagues (2014) posited that students in the 50th percentile of a course would improve to the 68th percentile if their instructors implemented active learning.

More recent literature has called for more attention to diversity, equity, inclusion, and access (DEIA) in post-secondary mathematics education (Adiredja & Andrews-Larson, 2017; Gutiérrez, 2018; Voigt et al., 2023). Mathematics classrooms that support DEIA are often those that are active and engage learners in meaningful mathematical activity. However, Brown's (2018) investigation of undergraduates' experiences in mathematics classrooms found that not all active-learning classrooms are equitable. Other studies have confirmed that a focus on active learning alone may be insufficient to address inequities at the classroom level and beyond (Ernest et al., 2019; Johnson et al., 2020).

Mathematics departments often rely heavily on MGTAs for teaching or supporting 100- and 200-level courses. MGTAs often lead multiple recitation or lab sections for large lecture courses or they might teach courses as a lead instructor. At some doctorate-granting institutions, MGTAs teach up to 68% of Calculus I courses (Selinski & Milbourne, 2015). This means that during their graduate programs, MGTAs likely contribute to the learning experiences of hundreds, and possibly thousands, of undergraduate students. Despite their contact with and impact on undergraduate students, there have been few studies that have investigated MGTAs' understanding of teaching practices (Miller et al., 2018; Speer et al., 2010). Thus, it is important to implement and study a PD program for MGTAs that focuses on engaging learners through equitable and inclusive teaching practices. We expect that such a program has the potential for significant impact on undergraduates' mathematics learning environments.

Data Collection and Analysis

In this report, we present initial results from one academic year during which we had four MGTA participants in their second year of the PD program and five MGTA participants in their first year of the PD program.

Context of the Study

Our PD program aims to educate and support MGTAs in enacting instructional practices that center equity and inclusion within active learning strategies. The ELITE PD program has five components that are offered to MGTAs over the course of three years. Each of the components is highly collaborative and fosters regular interactions among cohort members. These are intended to provide a model of effective teaching practices for MGTAs, highlighting facilitation practices, grouping students for active learning, group-worthy tasks, and questioning strategies. The MGTAs begin the PD program with an orientation, which takes place two to three days prior to the start of the term. The orientation includes information about the department and the graduate students' roles and expectations as MGTAs. Leaders of the orientation focus on community

building within the new cohort, and support MGTAs in developing knowledge of teaching practices focused on active group learning and dynamics, classroom logistics, as well as expanding their awareness of the diversity of experiences and access needs of undergraduate mathematics students. Second, in the term subsequent to the orientation, MGTAs take the Graduate Teaching Seminar where they continue to learn about the department, the university's teaching support resources, classroom management, and how to implement small-scale instructional practices with attention to equity and inclusivity through active engagement. The third and fourth components of ELITE PD occur in the second year of their graduate program with the Introduction to Equity and Active Learning seminar during which MGTAs continue to build upon their understanding of and facility with active learning and equitable teaching practices, and the Advanced Active Learning and Equity Seminar, which has MGTAs engaging with specific research-based pedagogical approaches as well as equity research associated with teaching. Throughout the ELITE PD program, MGTAs are observed in their classroom and given feedback about their teaching. Once the first four components of ELITE PD are completed, the fifth component has MGTAs serving as peer mentors to new MGTAs who enter the program.

Survey Items & Responses

We collected survey data from nine MGTAs' classrooms. There were two MGTAs who were assigned to a research assistantship during the year, so we do not have data for those terms for those MGTAs. We administered student surveys toward the end of each of the three academic terms. The surveys had 15 Likert-scale questions (Strongly Disagree (1), Disagree (2), Neutral (3), Agree (4), Strongly Agree (5)) that addressed their experiences with their MGTA primarily focused on students' active engagement (e.g., My TA helped facilitate group discussions that promoted my participation), inclusivity (e.g., I felt welcomed and a sense of belonging in this course), and access (e.g., My TA helped me identify and/or access resources to be successful in the course). We collected 1285 surveys total across all participants as seen in Table 1.

Table 1. Number of surveys collected from ELITE PD participant's classes.

MGTA	Cohort	Fall	Winter	Spring	*Total*
1	1	80	69	26	175
2	1	–	–	37	37
3	1	97	80	32	209
4	1	52	15	39	106
5	2	75	38	–	113
6	2	50	28	52	130
7	2	76	60	46	182
8	2	54	51	73	178

9	2	56	56	43	155
Total		540	397	348	1285

The surveys noted in the table above were collected in the same academic year. During that year, Cohort 1 participants were second year mathematics graduate students in their second year of the ELITE PD program, and Cohort 2 participants were first year graduate students in the first year of the ELITE PD program.

Analytic Methods

In our first approach to the data, we analyzed the entire data set, comparing the mean scores on the surveys Fall to Winter, Winter to Spring, and Fall to Spring. We first applied Levene's Test for Equality of Variances to the data. Then, based on the results of this test, we were able to conduct a t-test for equality of the means. In our second approach to the data, we analyzed each participant's data separately, still exploring whether their student survey responses increased over time. For this analysis, we calculated the average responses for the 15 survey items for each MGTA. Then we found the mean of these values to calculate a composite score for each term for each MGTA. Given the small sample size, we applied the Wilcoxon non-parametric statistic to compare the mean scores between Fall and Winter, Winter and Spring, Fall and Spring.

Results: Improvements in Survey Responses in MGTAs' Classes

Overall, student responses were skewed toward the positive end of the scale for all three terms, with their ratings being higher than 4 (Agree), for 14 out of 15 questions, and slightly below 4 for one question (My TA presented examples that resonated with me or related to my interests).

First Analysis

In our first analysis, where all the data across all participants were analyzed together, there was a slight improvement in means from Fall to Winter, but it was not significant (it could be just random). There was a slight improvement in means from Winter to Spring, which was marginally significant (at α =.05). From Fall to Spring, we found statistically significant improvement in means with a medium effect size (Cohen's d=.70). Table 2 shows the results of the test that compared the means from Fall to Spring. When we look for students' ratings as a whole, they gradually improve over time.

Table 2. Independent Samples Test

					95% Confidence Interval of the Difference	
t	df	Significance One-Sided p	Mean Difference	Std. Error Difference	Lower	Upper
-2.975	924	.002	-.13989	.04702	-.23216	-.04761

Second Analysis

In our second analysis, where we analyzed each participant's data separately, we observed more cases of improvement than of decline from Fall to Spring and from Winter to Spring. The negative Z values in Table 3 indicate improvement of mean responses from one term to the next, but the differences were not statistically significant.

Table 3. Wilcoxon Signed Ranks Test

	Fall to Winter	Winter to Spring	Fall to Spring
Z	-.140	-1.352	-1.521
Asymp. Sig. (2-tailed)	0.889	0.176	0.128

We noticed that seven of the nine MGTAs received improved student responses over time, whereas two of the nine MGTAs received decreases in student responses. These two MGTAs were in Cohort 2, and thus in the first year of their graduate program and the ELITE PD program. The decreases in these cases seem significant and might be skewing the data in such a way that we cannot conclude that the increases across the seven other MGTAs are statistically significant.

Discussion and Directions for Future Work

Based on the results from both analyses, we are optimistic that MGTAs are improving their practices over the course of a year, and curious about MGTAs who experienced a decrease in overall student responses. Our preliminary exploration suggests that MGTAs who participate in ELITE PD with an intentional focus on active, engaging, equitable and inclusive teaching practices have the potential to provide positive classroom experiences for undergraduate students over time. Namely, it appears that the students of such MGTAs can experience an increased sense of participation and engagement, belonging, and clarity of expectations and feedback. These preliminary findings and interpretations are substantiated with another source of data from our larger research project - video recordings of the MGTAs' classes. In our preliminary analyses of these observations (Quaisley & Beisiegel, 2024; Quaisley, Beisiegel, & 2024b), we see, for example, MGTAs learning to set norms for group work discussions and improve in their efforts to facilitate conversations through questioning intended to moderate discussion among all group members.

We will continue to analyze the data from multiple perspectives. Specifically, we aim to answer the questions: Do we see a relationship between the MGTAs' average student survey responses and: (1) the course the MGTAs are assigned to, (2) the identities of the MGTAs, (3) the identities of the students (while it was not reported in this preliminary report, the surveys also collected information about the students, such as their gender, race, age, and more), and (4) the stage of ELITE PD, i.e., before, during, and after completing ELITE PD coursework?

Questions for Further Discussion

We propose these questions to start off the conversation with conference attendees:

- What are ways we can further study students' experiences in MGTAs' classrooms?
- What other analyses should we explore with these (and other) data?
- What purpose do these results serve for the RUME community?
- In what ways do you see this study and its results as beneficial to mathematics?

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Investigating STEM Retention Program Participants' Sense of Belonging in Mathematics

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Research suggests that a sense of belonging in mathematics can influence students' decisions to persist in STEM fields. Programs such as summer bridge programs and Louis Stokes Alliances for Minority Participation (LSAMP) programs are often utilized to promote STEM retention by helping students adapt to college STEM environments. This study explores the mathematics sense of belonging and identity of students in STEM while participating in an LSAMP program after their participation in a summer bridge STEM program. Survey and interview data from the students will be analyzed to identify how participation in STEM retention initiatives may have influenced their mathematics sense of belonging and identity. After taking their first summer mathematics course, students' overall sense of belonging in mathematics appears to decline, according to initial findings. Results of this study also suggest that having interpersonal relationships allowed one student to have an increased sense of belonging in mathematics.

Keywords: sense of belonging, identity, retention, diversity, bridge programs

Students often leave science, technology, engineering, and mathematics (STEM majors) after taking mathematics courses within the first two years of college (e.g., Chen et al., 2013), and historically underrepresented college students are less likely to persist in these areas compared to their peers (e.g., Alkhasawneh & Hobson, 2011). A sense of belonging in mathematics may improve students' desire to persist in STEM (Good et al. 2012), so programs that support the development of a mathematics sense of belonging could be a critical component of STEM retention. LSAMP programs and summer bridge programs contribute to retention efforts of underrepresented STEM students through diverse representation, community, mentorship, and tutoring. However, little is known about how these programs influence mathematics sense of belonging. This project explores the development of participants' sense of belonging, confidence, and identity in mathematics after their involvement in STEM programs. In particular, I ask:

1. How does the mathematics sense of belonging of STEM retention program participants change or differ throughout their participation in these programs?
2. What aspects of bridge programs do participants value and why? How do these valued aspects relate to their sense of belonging in math?

I will answer these questions using student surveys and interviews. These research questions are part of a broader study that also includes program center observations and staff interviews.

Literature Review

Historically underrepresented students (African Americans, Hispanic Americans, Native Americans, and women) are less likely to persist in STEM areas in comparison to their counterparts at the postsecondary level (Alkasawneh & Hobson, 2011; Riegle-Crumb et al., 2019). These students are also less likely to have a sense of belonging in STEM (Rainey et al., 2018). There are many factors, such as climate, lack of representation, and students not having a sense of belonging, that impact the persistence of underrepresented students in mathematics and cause higher attrition rates in STEM (Botella et al., 2019; Chen, 2013; Meador, 2018; Strayhorn, 2012). Postsecondary institutions offer a variety of resources, such as mentoring, tutoring, and supplementary instruction, to help address attrition among underrepresented students pursuing

STEM degrees. STEM retention programs, such as summer bridge programs and academic year support programs, are one way to improve retention in STEM by helping students transition into college STEM environments by offering academic and social experiences, creating a community-oriented environment for underrepresented STEM students to thrive in. Investigating and understanding students' experiences in STEM retention programs can be important for teaching, learning, and retention in STEM.

Research shows that a low sense of belonging (Ahlqvist et al., 2013) and STEM majors leaving these disciplines when taking mathematics courses during the first two years of college (Chen et al., 2013; Seymour & Hunter, 2019) contribute to attrition in STEM. Little is known about how specific environments, such as STEM retention programs, contribute to retention in STEM through mathematics sense of belonging. Towards these ends, this study aims to explore students' mathematics sense of belonging, mathematics confidence, and mathematics identity development while participating in a STEM retention program.

Academic and Social Integration

Research has revealed that the first year of college is a critical period for attrition in STEM (Chen, 2013), particularly as the transition to college involves students negotiating both academic and personal stressors (Ruble & Seidman, 1996). Fostering a stable environment during students' transitions to undergraduate studies can be crucial to their college success.

Sense of Belonging in Mathematics

A sense of belonging can be described as the perception of social support among students on campus and the experience of mattering or feeling valued by and essential to a certain group; this is important for underrepresented students pursuing STEM (Strayhorn, 2012). Meehan et al. (2018) used Good et al.'s (2012) sense of belonging in mathematics (SBM) measure to examine the impact that the transition from secondary school to university mathematics has on students' SBM. Their study also reveals that a growth mindset, meaning that abilities can be developed, instead of mathematics ability being a fixed trait, seemed to help students navigate the mathematical transition (Meehan et al., 2018).

Mathematics Identity

Mathematics identity also impacts students' desire to pursue and persist in STEM areas (Black et al. 2009; Cribbs et al., 2021) and refers to one's perceptions of their own relationship to mathematics (Cribbs et al., 2015). To further elaborate, mathematics identity shapes how students view their abilities, experiences, and potential within the field of mathematics, ultimately influencing their confidence.

STEM Retention Programs

Goff et al. (2019) report how informal education programs can serve as a resource to improve STEM retention through impacting students' interest in mathematics. Their findings show that students who participated in informal science and mathematics environments feel more competent in STEM and report increased interest and greater engagement in STEM areas.

For students from groups historically underrepresented in STEM, two common types of models of STEM retention programs are Bridge programs and LSAMP programs. Bridge programs often focus on underrepresented students, first-generation students, low-income students, or students pursuing specific degrees (Kallison & Stader, 2012). The socially

supportive components of bridge programs include faculty mentoring, peer tutoring, peer mentoring (Bradford et al., 2021). LSAMP programs were developed in reference to Tinto's (1975) model for retention (Clewell et al., 2006) and are specifically tied to STEM retention with an emphasis on supporting historically underrepresented groups in STEM (Clewell et al., 2006).

Theoretical Framework

This work is grounded in the academic and social integration components of Tinto's Conceptual Schema for Dropout from College (1975), which identifies academic and social systems as contributors to one's dropout decision. This theoretical framework has been widely adapted for researching retention in STEM.

Methods

Data were collected using surveys and semi-structured interviews. The survey instrument included the 30-item Good et al. (2012) sense of belonging in mathematics scale, the six-item Lubienski et al. (2021) mathematics confidence scale, and short answer questions about students' identity and sense of belonging in math. A coding scheme for belonging explanations, adapted from Rainey et al. (2018), was used for initial content analysis and coding reasons such as mathematics identity, interpersonal relationships, personal interest, or competence. I used qualitative analysis methods and descriptive statistics to produce results. I describe this in further detail in the analysis section.

Studies have demonstrated that students need social integration, academic support structures, and sense of belonging to persist in STEM (Tomasko et al., 2016; Rainey et al., 2018). This study takes a closer look at sense of belonging among historically underrepresented students in STEM, specifically in relation to their mathematics experiences, as many mathematics courses are one of the first courses that undergraduate STEM majors encounter as a prerequisite for undergraduate STEM coursework. To do so, this study will examine how mathematics sense of belonging, mathematics identity, and mathematics confidence may differ over the course of one semester of participation for students who participated in two preparation programs intended to support historically underrepresented students in STEM fields**.**

Student Surveys

Bridge participants and LSMAP participants were surveyed about a variety of topics, with the goal of understanding students' mathematical sense of belonging, mathematics confidence, and mathematics identity, and the underlying factors that may contribute to these affects. They were surveyed at three points during the school year—once within the first month of semester 1 (Survey 1), once during the last two weeks of the semester (Survey 2), and lastly at the end of semester 2 (Survey 3). E-mail addresses were collected via the surveys to link the surveys and to schedule interviews with those who indicated interest. The surveys contained scales to measure students 'mathematics sense of belonging, mathematics confidence, and mathematics identity. The Good et al. (2012) 30-item Math Sense of Belonging scale was used to measure students' sense of belonging.

Student Interviews

Students were interviewed mid-semester 1, at the end of the semester 1, and at the end of semester 2 about their mathematics sense of belonging and mathematics identity for about 30 minutes each time. Questions focused on asking students to reflect on their trajectory toward pursuing their selected undergraduate degree program and to understand participants' history

with mathematics and mathematical role models. They were also asked to reflect on their bridge program or LSAMP experience to gain insight into their experiences in these STEM retention programs. Finally, participants were asked questions regarding their feelings of belonging, identity, and confidence in mathematics to gain an understanding of the students' beliefs about belonging in math and how they view themselves in mathematics. Student interviews were conducted using semi-structured format that involved me asking interviewees open-ended questions and following them up with probe questions to explore more about their experiences. (Roulston, 2013).

Planned Analysis

The descriptions of my planned analysis are written in the future tense because I have only recently started. All data, however, has been gathered prior to this proposal submission.

Research Question 1

How does the mathematics sense of belonging of STEM retention program participants change or differ throughout their participation in these programs? To analyze the survey responses to the mathematics sense of belonging scale, I will determine Likert scale total scores, frequency of each Likert scale score, and Likert scale mean scores to run dependent t-tests to calculate the mathematics sense of belonging mean scores at time point 1 and time point 2. Descriptive representations will aid in representing the comparisons between the results found at both time points by displaying how mathematics sense of belonging may change over time.

To analyze the interviews and open-ended survey questions, I will use line-by-line content analysis at the sentence or sentence cluster level (Schreier, 2014). The goal of this analysis process is to reveal themes found throughout the participants' responses about their sense of belonging in the open-ended survey questions and interviews. I will be looking for positive and negative connotations of sense of belonging in mathematics using a coding scheme adapted from Rainey et al. (2018).

Table 1. A table showing a summary of the Rainey et al. (2018) coding scheme..

Coding Scheme for Belonging Explanations
(Rainey et al., 2018)

Code	Reason for Belonging	Reason for not Belonging
Interpersonal Relationships	Feels socially connected with peers and/or faculty members.	Lacks a social connection with peers.
Math Identity	Math is a part of their identity as a person.	Lacks a personal connection to the major or material
Personal Interest	Expresses personal interest in course subject or major.	Explicit lack of interest. May find the material boring
Competence	Feels like they understand material or receives good grades	Feels like they do not understand material well

Research Question 2

What aspects of bridge programs do participants value and why? How do these valued aspects relate to their sense of belonging in math? Line-by-line content analysis will be used

to analyze each sentence or sentence cluster of the interview data collected to find what students value most while participating in these programs (Schreier, 2014). The interviews will be analyzed for most mentioned aspects of the LSAMP program that students believe impact their feelings of belonging in mathematics. I will be looking for connections between participants' responses. I will also use some of the initial coding to make sense of broader patterns and find themes of what students value most using open coding (Khandkar, 2009). Moreover, the Rainey et al. (2018) coding scheme, which will identify students' reasons for belonging, will provide insight into how these valued aspects relate to their mathematics sense of belonging.

Preliminary Results and Conclusions

Participants' feelings of membership and acceptance as measured by the Good et al. (2012) sense of belonging in mathematics scale, declined between the start and end of the summer bridge program. However, this decrease may be attributed to the intensive nature of summer courses. Despite these challenges, students valued the mentoring and program components.

Interviews with one participant revealed confidence in their sense of belonging in mathematics with the reasoning being personal interest and competence, often feeling reassured by peers and mentors in the program. Initial findings indicate a decline in students' overall sense of belonging in mathematics after their first college summer mathematics course. Yet, the positive experiences with the bridge program suggest that continued participation may lead to an improvement in their sense of belonging over time. Furthermore, the results of this study so far also suggest that having community and peer interactions allowed the student to have an increased sense of belonging in mathematics. These findings offer insight into the role that STEM retention programs play in fostering diversity within STEM through the lens of an individual students' mathematics experiences.

It will be interesting to see how the findings might evolve following their participation in a STEM retention program throughout the school year.

Next Steps and Questions

Although all data have been collected, analysis has just begun at the time of submitting this proposal. Thus, there only preliminary results to share. Next steps include completing the analysis before the conference in February, at which time I expect to provide further results and conclusions. More specifically, these steps include completing more rounds of coding to draw conclusions from interview data and open-ended survey questions. I plan to also establish connections among students' sense of belonging in mathematics, mathematics identity, and mathematics confidence.

Audience Questions

1. What role do you see institutional factors such as curriculum design, faculty support, or student programs playing in fostering a sense of belonging in mathematics?
2. Can you discuss any specific interventions or strategies you found that were particularly effective in enhancing the sense of belonging of students in your mathematics courses?
3. What are some ways educators and institutions can better assess or measure a student's sense of belonging in mathematics in real-time?
4. Can you think of other frameworks I should consider using to examine this data?

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An Analysis of Instructor Feedback on Student Work

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Feedback—the process of leaving remarks, questions, or markings on student work with the aim of enhancing student learning—is one of the more instructionally powerful but least understood features in instructional design (Cohen, 1985). Past studies offer inconsistent results on the features of feedback and little is known about how some of these factors benefit students' interactions with feedback and how deep a student engages in the revision process. This study aimed to learn more about both classroom perspectives of feedback between the instructor and the student. The main population included entry-level college mathematics instructors and students; we examined their feedback practices to gain a better understanding of what elements of feedback are being used and if they match the type of feedback that students report is most valuable. In this paper, we will only focus on the instructor study as that was the primary research question. Instructors graded and left feedback on a collection of student work samples while students completed a questionnaire that asked how they use feedback, how highly they value it, and which feedback type they find most useful. A thematic analysis was conducted on each comment and were classified into specific groupings. Results indicate that the instructor feedback was overall non-actionable but by a small margin. Students preferred actionable feedback, signifying a disconnect between the two populations.

Keywords: Instructor Feedback, Actionable Feedback, Feedback Literacy, Student Success

Introduction

Instructors across every discipline, especially those in common entry-level courses, act as one of the key players in a beginning college student's journey into their course of study, as well as possibly affecting more intrapersonal qualities such as personal identity and acceptance into the respective community they hope to join. Instructors serve as more than just educators; they play a pivotal role in students' personal and academic development. These instructors' actions and choices in the classroom can highly influence a student's path in college, even including their written communication to the student- feedback.

Researchers have defined feedback in various ways such as "information provided by an agent regarding aspects of one's performance or understanding" (Hattie & Timperley, 2007, p. 81) and also as any numerous procedures that are used to inform a student if their response is right or wrong (Kulhavy, 1977). Carless and Boud (2018) capture it with a small twist as "a process through which learners make sense of information from various sources and use it to enhance their work or learning strategies" (p.1). This definition goes beyond feedback solely being the role of teachers. It emphasizes the student component in making sense of the information and using it to revise their previously flawed understanding. We consider feedback to include teachers' comments, questions, and markings (X's, the addition of mathematical content) on students' written work.

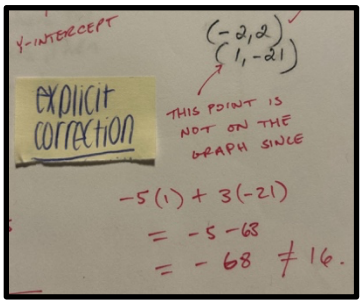
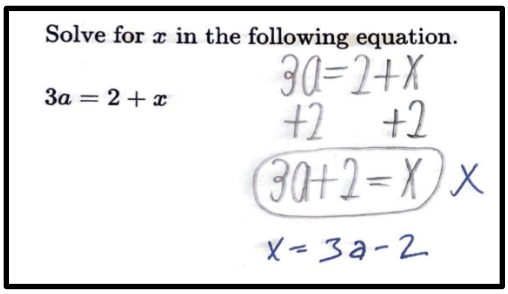
Multiple studies have identified teacher feedback as one of the most important instructional practices for promoting and advancing student learning if delivered correctly (Gentrup et.al., 2020; Hattie & Timperley, 2007; Shute, 2008; Griffiths et al., 2023). The last three words in that statement, "if delivered correctly", are the foundation of this study. The effectiveness of teacher

feedback lies not only in its delivery, but also in its specificity, timeliness, and consideration of word choice (Shute, 2008). Some goals of providing feedback to students are to challenge them to adopt new perspectives (Forsythe & Johnson 2017) and to have learners engage with feedback and use these standards as they produce new work (Sadler, 1989).

Our goal was to observe the feedback practices that are occurring within entry-level mathematics courses and answer the question: What are the feedback types commonly used by instructors within a student homework sample?

Feedback Categorizations

Instructors can increase student agency by providing students with actionable feedback- if the message suggests a "course of action that would bring the submission closer to a correct response" (Bergman & Kercher, 2022, p.48). This type of feedback allows the student the opportunity to strategize about how to correct it themselves with some direction from the instructor (Griffiths et.al, 2023; Hattie & Timperley, 2007). Actionable feedback helps promote students' independence (Lee & Hannafin, 2016) and communicates the teacher's belief in the student's ability to improve (Griffiths et.al., 2023). Non-actionable feedback does not provide the student much room to perform the correction themselves; it is either done for them or it is unclear where the student needs to continue. Below is an example of these two types of feedback in the binary system.

Actionable Feedback Example	**Non-Actionable Feedback Example**
	
This comment includes the location of the error and why it is incorrect.	This feedback sample includes the location of the error and the correct answer.

Is there a type of feedback that results in greater potential for learning? Students have different reactions to the same feedback message depending on their motivations, individual expectations, and what they hope to take out of the course so it may be difficult to satisfy each individual's needs. Researchers appear to agree that effective feedback contains both elements of feedback: both actionable and non-actionable (Shute, 2008). A short comment on the location of an error may be better in certain situations, while a more detailed explanation may help in another case. The type of student error and the severity of the total errors in their work can affect whether actionable or non-actionable notes would be better suited.

Below, Table 1 lists each of the 9 feedback classifications we used for the coding scheme extended from Lyster & Ranta (1997) and Wahyuni (2023). A description of each feedback type is included as well as whether it represents actionable or non-actionable feedback.

Table 1. Nine Main Feedback Types Categorized in the Binary System

Feedback Type		Actionable or Non-Actionable?	Description
Explicit Correction		Actionable	Feedback includes the error detection and the reason why it was incorrect.
Recast	Recast 1	Non-Actionable	A correction is only provided (Recast 1) or all of the work is provided to them, with no reason as to why it was an error (Recast 2).
	Recast 2	Non-Actionable	
Metalinguistic Feedback		Actionable	Feedback is offered about specific word choices used; correction may allow students to use better terminology in the future.
Elicitation		Actionable	Feedback is in the form of a question or contains instructions on how to continue.
Indication		Non-Actionable	No words are included in this type of feedback such as underlining or circling.
Summative Feedback	Performance	Non-Actionable	A comment about a specific general performance/ action on the assignment.
	Notice	Actionable	This feedback offers the student a comment about an action to be more aware of in the future.
Reference		Actionable	This directs a student to a certain reference, such as the instructor or a page in the textbook.
Clarification Request		Actionable/Non-Actionable	Comment suggests that the student's work is "ill-informed" and that a revision is encouraged.
Praise		Non-Actionable	Adds no mathematical comments; only offers positive words

Methods

This study involved 8 math instructors representing a mid-sized public university in the Pacific Northwest region of the United States. Data was collected during an in-person meeting involving grading and leaving feedback for 7 student work samples.

Student work samples from a College Algebra course were collected from past students of this institution. These students were not students of any of the instructors in the study; all were hypothetical students that instructors were assigned to grade and leave feedback. All identifiable information was redacted and not included in the study. Only erroneous work was included as a means to answer the call to action in a previous study that stated that "many studies fail to analyze error and correct responses separately" (Kulhavy, 1977, p. 221).

Seven student work samples were taken of the identical problem below on the topic of graphing linear functions. This topic was chosen to accommodate most instructors that teach below the Calculus level. Below is the problem that was selected.

“Given the following equation -5x+3y=16, find the slope and the y-intercept. Explain how you can use the slope and y-intercept to graph the equation. Sketch the graph of the equation.

Results

Instructor comments on student work samples were analyzed and flagged as one of the eleven detailed feedback categories. Figure 1 compares the frequency of each feedback type that was present out of all seven student work samples. In the figure, green represents the actionable feedback types, red non-actionable. 220 instances of feedback were present out of all instructors, more non-actionable comments overall.

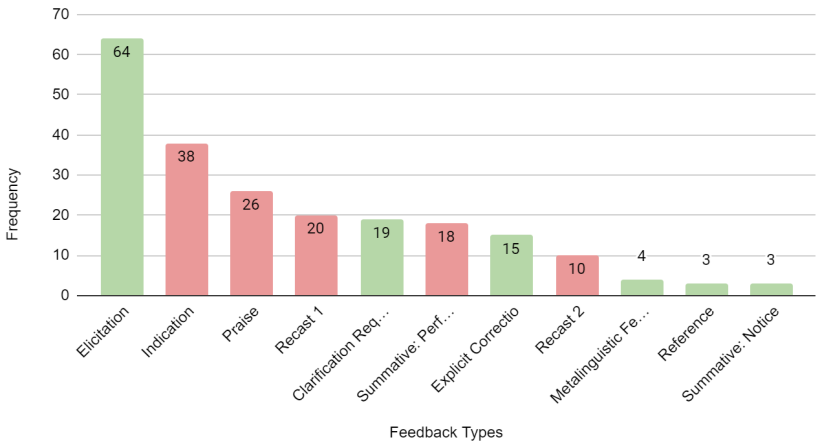


Figure 1. All Instructor Feedback Types Present

Discussion

In this study, instructors provided students with a mixture of actionable and non-actionable feedback and this relates to Shute (2008), where it was noted that effective feedback contains both elements: both actionable and non-actionable. Instructors mainly used Elicitation (actionable) out of all feedback types. Amrhein & Nassaji (2010) found that there exists a dissonance between the feedback that students prefer, actionable feedback, compared to the non-actionable feedback that teachers find more desirable for students. As previous literature found dissonance, this study complemented the Amrhein & Nassaji results and found a disconnect between the student's preferences and feedback received at this institution.

Instructors altogether used slightly more non-actionable feedback types (112 non-actionable/108 actionable comments), but there was quite an even split with non-actionable comments occurring more frequently. These non-actionable feedbacks are found to not contribute to student revision. Students in this study were found to prefer actionable feedback to better understand where their reasoning was incorrect and why.

This study presents a possible way to track and analyze the feedback provided by instructors while attending to students' preferences. The feedback categorization can be used as a professional development topic that can allow instructors to be more intentional with their feedback decisions and provide more effective feedback to students for revision purposes.

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Conceptual Chunking and Algebraic Proofs

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Research in cognitive psychology identifies chunking as one of the driving information-processing mechanisms in human cognition. Conceptual chunking, a term specific to Halford's relational complexity theory, provides theoretical insights into what constitutes chunking. Specifically, conceptual chunking explains the chunk formation as the result of temporal suppression of relations between the items within the chunk. In this article, I present the analysis of two undergraduate students' proofs from the relational complexity theoretical perspective. Preliminary results suggest that conceptual chunking can help describe students' mathematics and explain some of their difficulties related to proving algebraic conjectures.

Keywords: cognition, theory of relational complexity, chunking, proofs

Beginning in the 1950s, studies in cognitive sciences have revealed substantial evidence for a unifying information-processing mechanism known as *chunking*. In his foundational paper, Miller (1956) described chunking as combining several units of information into a single familiar unit or a chunk. Miller proposed that chunking differs from *grouping* in that the former requires the chunks to correspond to some pre-existing representations. For example, the sequence of 12 seemingly unrelated letters AMIRBRUMEBBC can be chunked into four cognitive units of varying sizes [AM][IRB][RUME][BBC] by identifying familiar words and acronyms. Similarly, Cowan (2001) defined chunks as groups of items with strong, pre-existing associations with each other but weak associations with other items and chunks.

Since Miller's work, researchers have established chunking as one of the key information-processing mechanisms in various aspects of human cognition, including mathematical reasoning (e.g., Harel & Kaput, 2002; Schoenfeld, 2001). Several influential theories in mathematics education are reminiscent of the idea of chunking, such as *encapsulation* (Ayers et al., 1988) or *reification* (Sfard, 1989). With some flexibility of interpretation, chunking can be viewed as an instance of *reflective abstraction* (Beth & Piaget, 1966), in which "a physical or mental action is reconstructed and reorganized on a higher plane of thought and so comes to be understood by the knower" (p. 247).

Despite the recognized importance of various forms of chunking for mathematical reasoning, little is known about the underlying cognitive mechanism. The construct of *conceptual chunking* provides a theoretical explanation for how chunking takes place. Proponents of the relational complexity theory (Andrews & Halford, 2002; Halford et al., 1998) suggest that conceptual chunking involves temporal suppression of the relations between the items within a chunk. Principles embodied in Halford's theory have been validated in several empirical studies (Andrews & Halford, 2002; Birney & Halford, 2002; Halford et al., 2005; Waltz et al., 1999); however, they were never explored in the context of undergraduate mathematics, and, in particular, proof production.

This preliminary study aims to investigate the psychological construct of conceptual chunking in the context of mathematical proofs. The overarching research question is *What are the aspects of conceptual chunking in proving algebraic conjectures?* In this report, I first elaborate on Halford's relational complexity theory and then apply this framework to analyze two undergraduate students' ways of producing an algebraic proof. Preliminary results suggest

that conceptual chunking can be an effective way to model students' mathematical reasoning and explain some of the difficulties related to proving algebraic conjectures.

Framework: Relational Complexity Theory and Conceptual Chunking

Halford and colleagues (e.g., Andrews & Halford, 2002; Halford et al., 1998) proposed that the cognitive complexity of a task should be defined not only in terms of the number of units of information it requires to be coordinated but also in terms of the complexity of relations between them. Consider, for example, the following problem: *Suppose five days after the day before yesterday is Friday. What day of the week is tomorrow?* (Sweller, 1993). This problem may seem to appear frustratingly difficult because of the necessity to establish relations between numerous elements presented in the first sentence, which cannot be considered meaningfully in isolation. As such, the cognitive task becomes more complex when the number of relations that need to be maintained in parallel increases.

The complexity of the task can be reduced via the mechanisms known as segmentation and conceptual chunking. Segmentation entails the decomposition of the task into smaller segments or steps that can be processed serially. Examples of segmentation include breaking the proofs into cases or implementing proof by induction as a three-step procedure (i.e., the base case, inductive hypothesis, and inductive implication).

Conceptual chunking, the construct that is central to Halford's theory and this study, entails combining units of information into a single chunk at the cost of making the relations between the units temporarily inaccessible. Halford illustrates this construct using the concept of velocity (e.g., Halford & McCredden, 1998; Halford et al., 1998). Defined as the ratio of distance over time, V=S/t, this concept involves the interaction of three variables and can be viewed as a three-dimensional representation. It is also possible to think of velocity as the number displayed on the speedometer V=60 km/h – a single-dimensional representation. The transition from the three-dimensional to the single-dimensional representations makes relations between the velocity, distance, and time temporarily inaccessible. Chunked concepts can be further combined with other chunks to construct higher-level concepts. Thus, velocity can be used to define acceleration, which can be used to define the notion of force F=ma. However, if the task requires computing the way velocity varies as the distance doubles and time does not change, the relations between the variables must be restored.

To summarize, conceptual chunking helps students to make progress on the task by temporally suppressing the relations between different cognitive units. Chunking is a matter of expertise (Chase & Simon, 1973) and can be viewed as "a form of learning that takes place over time" (Halford et al., 1998, p. 810). The three general principles of conceptual chunking are

1. A chunk functions as a single entity.
2. No relations can be represented between items within a chunk.
3. Relations between the chunk and other items, or other chunks, can be represented.

Methodology

The study draws on the data collected for a larger project investigating the role of students' gestures in offloading cognitive demands on their working memory during various proving activities, such as reading, presenting, and constructing proofs of mathematical conjectures. The participants were undergraduate students enrolled in an introductory proofs course. The students participated in several individual task-based interviews lasting approximately 45 minutes each. The participants were asked to think aloud as they worked with proofs. Because the goal of the overarching project was to study students' gestures, writing and drawing were not permitted.

In this paper, I report on two students, Ella and Sherry, proving the conjecture known as a Diophantine Equation (Figure 1).

Prove that there are no integers x and y such that
$x^2 = 4y+2$

Figure 1. Interview task.

One way to prove it is to observe that the right side of the equation is even. Therefore, the left side also has to be even. Moreover, because the left side is a perfect square, then it has to be divisible by four. However, the right side is not divisible by four, and thus, no integers x and y satisfy this equation.

At a later date, a stimulated recall interview (SRI) was conducted. During this interview, the students were shown selected video clips of themselves working on the task, and the interviewer asked questions about their mathematical reasoning. All interviews were video and audio-recorded.

Data Analysis

The analysis began with creating the transcripts of the interviews and identifying the episodes where the participants seemed to chunk mathematical objects involved in their proofs. Verbal evidence of chunking included labeling pieces of the equation as "it," "a fraction," "number," etc. Students' responses during the SRI were used to triangulate the initial findings. One of the goals of this preliminary report is to identify more rigorous data analysis procedures for future study design.

Results

In this section, I present the analysis of students' proofs from Halford's relational complexity theory perspective. Specifically, I showcase two episodes of conceptual chunking and illustrate its underlying cognitive mechanisms in constructing algebraic proofs.

Sherry's Proof

Sherry made two attempts to prove the given conjecture. In the first attempt, the student relied on the fact that the existence of rational solutions would entail the non-existence of integer solutions. After some discussion with the interviewer, she realized that her initial approach was invalid. On her second attempt, Sherry started her proof by solving for y and rearranging the equation as $y = \frac{x^2}{4} - \frac{1}{2}$. However, later on, the denominator of the fraction $\frac{x^2}{4}$ was lost, and the equation was transformed into $y = x^2 - \frac{1}{2}$:

> So, to prove that there are no integers x and y such that x squared equals to 4y plus two... Um... I think... I'll think of cases, where x and y are integers. And see if x squared equals to 4y plus two exist. And thus, I rearrange the equation... in y... so x squared minus two equals to 4y, and y equals to x squared over four minus two over four. And since the very simplified rational number of two over four, that's one over two, **we have *y* equals to *x* squared minus one over two**. And since we have assumed that x and y are both integers, ***x* squared is also going to be integers. But then integer minus one over two is just the... em... very simplified version of rational numbers** [i.e., not an integer]. And it will also give a rational

number... *x* squared equals to four... There are no integers *x* and *y*, such that *x* squared equals to 4*y* plus two.

Initially, the participant correctly solved for *y* and derived the equation $y = \frac{x^2}{4} - \frac{2}{4}$. Then, she shifted the focus of her attention to simplifying the second fraction from the right side from $\frac{2}{4}$ to $\frac{1}{2}$. In doing so, the student chunked the components of the first fraction, $\frac{x^2}{4}$, into a single cognitive unit by suppressing the relations between the numerator and the denominator of the fraction. However, the reconstruction of these relations turned out to be problematic and significantly affected Sherry's proof.

Ella's Proof

Ella's approach was based on the valid idea of factoring out 2 from the right side of the equation and observing that 2*y*+1 is odd, and therefore, 2(2*y*+1) is not divisible by 4. However, the interviewer asked her to summarize her proof, Ella lost the coefficient 2 in front of *y*, which implied that nothing can be specified regarding the parity of the quantity enclosed in parentheses:

> We're gonna try to prove that there are no integers *x* and *y* such that *x* squared equals 4*y* plus two. So first, we're going to assume that *x* and *y* are integers, in which case, *x* squared becomes a perfect square number. However, 4*y* plus two, when you pull it two out, **you can factor the greatest common factor of 4*y* plus two is two, which gives you two parentheses *y* plus one because we know that *y*... Wait... shoot... We don't know anything about *y* being even or odd... Em... Here we have cases then...** If *y* is even, then whatever's in the parentheses is an odd number, which means that there is no way that it can be a perfect square or equal to the perfect square of *x* squared. Because there's a two factor that's not matched... within the other factors... to be able to... to take the square root of... in the case that *y* is even, the whatever's in the parentheses... sorry, in the case that *y* is odd, then whatever's in the parentheses will be an even number if it's plus one.... **Oh my gosh. I forgot about the 2*y*.** Can I jump back into my previous [laughing]? Okay, that's what makes it even. That's what I was missing.

Making progress on the task, the participant chunked 2*y* and +1 into a single unit signifying an odd number. Cognitively, this could be done by suppressing the relations between 2, *y*, and 1. However, the details about why this number is odd were temporarily lost. This can be explained by Ella's difficulty in reconstructing the relations between 2 and *y.* As a result, 2*y* was replaced by *y*, transforming the expression 2*y*+1 into *y*+1, which changed Ella's proof. The observed effect of conceptual chunking was confirmed later during her SRI:

> *Researcher*: I find it interesting that, in the middle of your presentation, you decided that the parity of *y* might affect the proof. You had to consider two cases separately: *y* is even and *y* is odd.
>
> *Ella*: Right, because I was saying *y* plus one. Because in my head, it was 2*y* plus two. For some reason, I was looking at the four. Um... Because I think... when I think about, like, the structure of an odd number, right? They give us a... there's, like, 2*k* plus one, right? Where *k* is some sort of the whole number. Um... Because you know that that 2*k* is going to be even and then the other part plus one makes it odd, right? I think of the 2*k*...

together. Like I don't think of it as two times *k*, it's just 2*k*. So, I think that's where I was like **instead of 2*k* I just put *y* there, and why I forgot about that 2*y***. And so I was, like, well, then I could save the inside of that two times whatever's in the inside, *y* plus one. Cause that, they're gonna be even or odd. So, I started going into cases, randomly.

Discussion

Theories of working memory capacity (Baddeley & Hitch, 1974; Pascual-Leone, 1970) suggest that humans can operate on a limited number of chunks at the same time. According to the principle of cognitive economy (Rescher, 1989), we tend to choose the least complex representation available to make progress on a cognitively demanding task. Thus, it is the nature of human cognition to package information into chunks by reducing the complexity of relations between the units.

The study design enabled my participants to coordinate the enormous amount of information and mental representations when constructing the proof of the given algebraic conjecture. The data presented in this study indicate that Sherry and Ella implemented mental strategies that can be described as conceptual chunking. Sherry temporarily suppressed the relations between the numerator and the denominator in the fraction $\frac{x^2}{4}$. Ella temporarily suppressed the relations between 2, *y,* and 1 in the expression 2*y*+1. Although conceptual chunking did help the students to make progress on the cognitively demanding task, the reconstruction of the suppressed relations turned out to be problematic in both cases. Sherry overlooked the denominator of the fraction, and Ella lost the coefficient of 2. The described negative effect of conceptual chunking, as applied to mathematical reasoning, is a finding not previously reported in the literature. More research is needed to investigate this preliminary observation.

Questions for the Audience

I conclude my report with the following three questions for the interested audience:

1. Is conceptual chunking and the overall relational complexity theory appropriate theoretical perspective for explaining the observed phenomenon?
2. What would be an appropriate study design for future research on conceptual chunking?
3. What are the pedagogical implications of these preliminary results?

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An Analysis of one Calculus Student's Understanding of Derivative Through Gestures

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This preliminary report focuses on investigating the range of gestures students use to explain the concept of derivative in different contexts. Students were asked to explain their meaning of derivative and how they make sense of the formal limit definition of derivative. In addition to taking an embodied cognition perspective in our analysis, we also draw from Zandieh's Derivative Framework to make meaning of the range of gestures we observed. For this preliminary report, we analyzed a video recording of one student during a 30-minute assessment interview for a first semester calculus course wherein he displayed over 40 different gestures. Our findings focus on five representational gestures that were salient in the student's explanation of derivative. We discuss methodological implications about analyzing gesture alongside a cognitive framework of a mathematical concept, as well as how our findings connect with existing themes in research on mathematical gestures.

Keywords: calculus, derivative, gesture, embodied cognition

Embodiment in mathematics education research has contributed much to a more robust conceptualization of students' sensemaking of mathematics (Abrahamson et al., 2020). Introductory Calculus is often seen as a barrier to gaining access to STEM-related college majors and hence, STEM-related careers (Bressoud, Mesa, Rasmussen, 2015). This report attends to the variety of gestures used by a student, Andy, while explaining his conceptual understanding of derivative. This work may support extending current frameworks for analyzing students' understanding of derivatives (e.g., Zandieh, 2000) and promoting the value of attending to gestures by instructors when assessing and advancing student thinking.

Literature Review and Research Questions

Two common phenomena observed in studies of gesture in mathematics education are representational gestures (Alibali & Nathan, 2012) and gesture-speech "mismatches" (Alibali and Goldin-Meadow, 1993). While Alibali and Goldin-Meadow (1993) suggest mismatches indicate students' knowledge in transition or on the cusp of verbalizing it, Abrahamson et al., (2020) argue that "non-redundant gestures" may be an expression of action-based experiences.

In undergraduate mathematics education research, gesture has been studied in content areas such as complex analysis (e.g., Oehrtman et al., 2019), abstract algebra (Soto et al., 2024), and mathematical proof (Kokushkin, 2021). Prior studies of derivative have focused on experts' reasoning through gesture (Oehrtman et al., 2019) and within classroom settings (Arzarello et al., 2009). Individual analyses of Calculus students' understanding of derivatives have frequently drawn from Zandieh's (2000) Derivative Framework, which provides a deconstruction of the concept as layers of process-object pairs woven through various contexts or representations.

Yu and Uttal's (2022) analysis of pairs of Calculus students' as they solve real life rate of change problems about changes in volume of a water bottle found that some of their participants may have developed a Calculus-based way of reasoning about the problem through their gestures. For example, some used a "disk" shaped hand gesture representing a cross section of the water bottle. Alibali and Nathan (2012) refer to these as representational gestures--or those that "depict aspects of their meaning" (Alibali & Nathan, 2012, p. 252). Thus, gestures may

serve as a tool for abstracting complex ideas (such as the formal definition of derivative). The current preliminary research report examines the research question, *what are the range of gestures students use to convey their understanding of the concept of derivative in calculus?*

Theoretical Framework

Theoretically, we situate this study within embodied cognition research, which examines aspects of mathematical thinking and understanding that are "bodily-grounded, *embodied* within a shared biological and physical context" (Núnez, Edwards, and Matos, 1999, p. 46). Contributing to the larger research program of embodiment in mathematics education (Abrahamson et al., 2020), this report further broadly investigates how gestures can "ground formalisms and abstractions to the physical environment, support simulated action of mathematical ideas, and invoke conceptual blends and metaphors" (p. 3). Also supported by findings from current research on gestures, we do not assume that students' speech would always be consistent with the gestures they produce (see also "mismatching gestures" in Hostetter & Alibali, 2019). Indeed, gestures reveal information that students do not or have yet verbalized.

Drawing on Zandieh's Derivative Framework (2000) we connect our analysis of gestures to the landscape of conceptual understanding of the specific mathematical content. This framework helps us to consider gestures in multiple contexts for student understanding of derivatives (primarily graphical, verbal, physical, and symbolic), related to its different aspects (ratio, limit, and function), as well as ways that students produce the gestures as dynamic processes and/or as static objects. Zandieh (2000) argued that the concept can be represented *graphically* as the slope of the tangent line, *verbally* as the instantaneous rate of change, *physically* as speed or velocity, and *symbolically* as the limit of the difference quotient.

Methods

Data Source

The case we consider comes from an introductory Calculus course taught at a large public university in the midwestern U.S. The instructor for the course (first author) heavily embedded Desmos Activity Builder (DAB) to the design of the course to leverage students' voice in classroom discussions, and to facilitate intuition building activities focused on conceptually understanding topics in a first-year calculus course. The data collected for this report comes from a 30-minute interview assessment about derivative for students in the course. It took place approximately two months into the semester. From these meetings, video recordings and written inscriptions created by the participants were gathered as data. Three students in the course consented to the analysis of their interview data and we selected Andy, a white male student, as a case where a wide variety of gestures were used. We focused our current analysis on less than eight minutes of video data where Andy is answering the two assessment questions: (1) what is a derivative to you, and (2) how do you make sense of the formal definition of a derivative?

Analytical Methods

All three authors reviewed the video data together and stopped to discuss moments of interest. Upon observing an abundance of gestures occurring simultaneously with Andy's speech, we rewatched the video with specific attention on his gestures. We then rewatched the video with the sound muted to focus attention on potentially mathematical gestures and selected possible gestures. We watched these selected moments once again with the sound on to guide our discussions on what counts as a mathematical gesture. We considered momentary pauses as

natural separation points between dynamic gestures. When it appeared that multiple gestures occurred in a short amount of time, we watched, paused, and discussed each gesture, frame-by-frame, until we came to an agreement of the timestamps where the gestures started and ended.

We focused our analysis on gestures that demonstrate a mathematical concept (see also "representational gestures" in Alibali & Nathan (2012)), those that simultaneously occurred with speech about mathematics, or those previously identified as mathematical but occurred with different words (e.g., Gesture 3 in Findings). We noted but did not analyze gestures that are more conversational in nature (e.g., Andy's gesturing with his hand to himself while saying "that's the easiest way to explain it…at least for me."). Once agreement was reached, we placed selected gestures within Zandieh's (2000) Derivative Framework. The three authors selected the cell in the table that they feel best fit the mathematical nature of the gesture. We discussed and resolved disagreements and arrived at the placement for the five gestures we are highlighting from Andy.

Findings

During the interview, we found that Andy made over 40 gestures as he described what a derivative is and how his understanding connected with the formal definition. We highlight five gestures that we interpret as particularly salient in Andy's explanation of derivative. We organize the presentation of gestures chronologically corresponding to Andy's responses to the two questions in the Data Source section above. We present the gestures as consecutive images in each figure. We then connect them to Zandieh's (2000) Derivative Framework.

Gesture 1: "Instantaneous rate of change"

On multiple occasions, Andy makes variations of a gesture in which his hands move closer together and sometimes meet each other. The first instance of this occurs immediately after being asked to explain what a derivative is. Over the interval :52-:59, Andy states, "umm, the derivative is like the instantaneous rate of change, at any given, like on[e]--single point," and makes the following sequence of gestures (see Figure 1).



Figure 1. Andy's "instantaneous rate of change" gestures

Gesture 2: "A graph" as a context for instantaneous rate of change

After his initial response to the first question (see Gesture 1 section above), he continues "the instantaneous rate of change…of like [*gesture*]…I know it's more specific than a graph, but a graph [*gesture*]." Note that the gesture is basically the same each time, a positioning of his hand and forearm at an angle in front of himself.

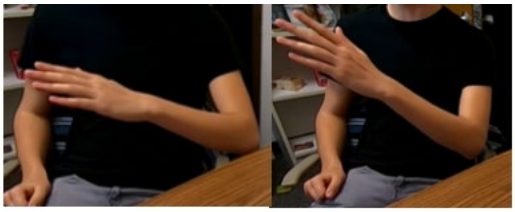


Figure 2. Andy's gesture as he says "of like" and "a graph," respectively

Gesture 3: "velocity graph" and "distance, time"

We refer the following as Andy's "velocity graph" gestures (see Figure 3 below). At 1:20, he states, "a graph that would be measuring velocity, so velocity graph." He elaborates what he means by "velocity graph," and makes two gestures that appear to trace out coordinate axes in the air in front of him (see Figure 3, second and third images) as he says, "so you got distance [*traces horizontally*], speed [*traces up*]. Or distance [*traces horizontally*] time [*traces up*]…"

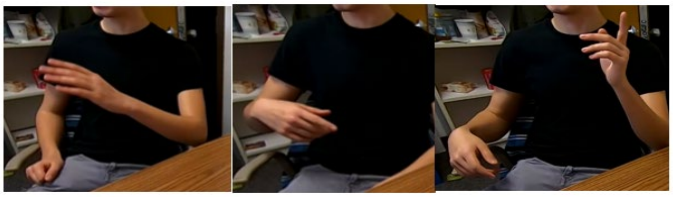


Figure 3. Andy's "velocity graph" gestures

Gesture 4: "h gets closer and closer to zero"

The next two gestures occurred after Andy's response to the second interview question about the formal definition. Prior to being asked explicitly what the h in the formula means, Andy made a similar gesture to that when initially asked what a derivative is (see Figure 1). The gesture here differs in that the emphasis is on the hands becoming closer as opposed to his hands coming to one point as is the case in Gesture 1. Over the interval 4:19-4:31, Andy states, "And then the limit as h approaches zero means the—the two points get closer and closer together, the distance between the two of them becomes less and so h gets closer and closer to zero…" while making the gestures shown in Figure 4.



Figure 4. Andy's "h gets closer and closer to zero" gesture

Gesture 5: "the h"

After being asked to explain what the h in the denominator of the formal definition means, Andy states, "uhh, the h is the distance between the first x and the second. The first input and the second." As he states this, he makes the gestures below (see Figure 5), tapping the table with each hand as he says, "first x" and "second," defining a space between his hands.

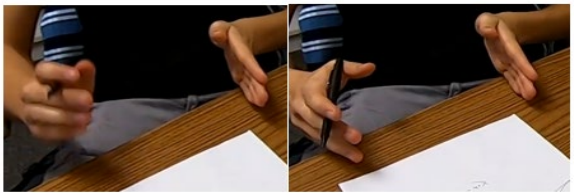


Figure 5. Andy's gesture for "the h"

Andy's gestures and the Derivative Framework

We first placed Andy's five gestures in different cells of the Derivative Framework based on the context as described by his speech when he displayed the gesture: "instantaneous rate of change" (verbal), "velocity" (physical), and limit definition (symbolic). For the process-object layer, Gesture 2 focused on the whole velocity graph so we associated it with the function layer,

whereas Gesture 3 focused on velocity as a ratio of the two "axes" of distance and time. Gesture 5 focuses on the static interval *h*, as the denominator of the difference quotient, whereas Gesture 4 was dynamically illustrating the limiting process as well as the limit. Gesture 1 emphasizes the "single point," without additional explanation. In this way, we also characterize Gesture 1 as a closed circle as the student's gesture displayed a "pseudo-object understanding" of that cell and Gesture 4 as an open circle as the gesture also displayed the limiting "process" involved (Zandieh, 2000, p. 114). There was no graphical gesture because Andy never spoke of "slope" in this explanation of derivative despite sharing gestures of graphical representations.

Table 1. Andy's Gestures within the Zandieh Derivative Framework

	Contexts				
	Graphical	Verbal	Physical	Symbolic	Other
Process-object layer	Slope	Rate	Velocity	Difference Quotient	
Ratio			Gesture 3 (_|)	Gesture 5 (| |)	
Limit		Gesture 1 (^)		Gesture 4 (|→←|)	
Function			Gesture 2 (\)		

Discussion

The Derivative Framework proved productive in interpreting the contexts represented by Andy's gestures. Moreover, it helps us interpret the gesture in terms of the process-object layers that add meaning to Andy's gestures beyond what they represent. This exercise suggests that incorporating a cognitive framework in thinking about gestures can be fruitful. In the other direction, examination of gestures students used to explain the concept has the potential to bolster the framework. The Derivative Framework has largely been used to analyze students' understanding of derivatives via students' verbal and written explanations.

We found Andy's gestures varied widely, may serve different communicative purposes, and became more precise as he attended to nuances of understanding derivative (e.g., the h in the formula). We observed non-redundancy between Andy's gestures and his verbal explanation. Given the value of non-redundant gestures to reasoning (Abrahamson et al., 2020; Alibali & Goldin-Meadow, 1993), instructors' attention to these moments can be important to gaining a deeper understanding of student thinking and might suggest the student is on the cusp of verbalizing something that relates to action-based gestures they make.

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Mathematics Graduate Teaching Assistants' Evolving Equity Conceptions: Three Profiles From Professional Development

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This study investigates mathematics graduate teaching assistants' (MGTAs) conceptions of equity based on activities from professional development (PD) focused on active learning, inclusivity, and equity. Using Gutierrez's (2008; 2009) four dimensions of equity (access, achievement, identity, and power), we analyzed survey data collected from MGTAs. These findings led us to create three MGTA profiles: Resource Advocacy, Asset-Based Collaboration, and Resilient Problem-Solving. These findings inform future PD design to enhance engagement with equity issues in mathematics education.

Keywords: Equity, Teaching Assistants, Professional Development

Background

Undergraduate mathematics courses are fundamental in shaping careers in Science, Technology, Engineering, and Mathematics (STEM), with completion of these courses often being a prerequisite for STEM professions and teaching roles (Deshler et al., 2015). However, many students initially interested in STEM do not persist, with this phenomenon disproportionately affecting minoritized students (Ellis et al., 2016; Seymour & Hunter, 2019). Research indicates that active learning strategies, such as problem-solving and interactive teaching practices (Conference Board of the Mathematical Sciences, 2016), significantly improve student outcomes (Braun et al., 2017; Freeman et al., 2014; Handelsman & Brown, 2016), they can also inadvertently exacerbate classroom inequities, particularly affecting female students who may benefit less from such methods (Reinholz et al., 2022). Hence, in addition to active learning, equitable instructional practices, such as addressing student identity and biases, are essential components of teaching mathematics (Yu et. al., 2024). Mathematics graduate teaching assistants (MGTAs) frequently teach introductory mathematics classes (Selinski & Milbourne, 2015), the courses that are commonly viewed as barriers to STEM persistence (Theobald et al., 2020), and many will go on to teach these classes later in their careers. Given their pivotal position, it is essential to ensure that MGTAs receive robust training in active and equitable teaching practices and to understand how this training influences their understanding of equity and their teaching practice.

Due to this, we endeavored to investigate the following question: *How do equity-focused activities in an MGTA professional development (PD) program prompt MGTAs to consider equity in their teaching?*

Theoretical and Analytical Framework

Gutiérrez (2008, 2009) introduced a comprehensive framework for equity in mathematics education, centered on two fundamental ideas: teaching students to "play the game" (dominant axis) and "change the game" (critical axis). The dominant axis consists of *Access* and

Achievement. *Access* involves providing students with the necessary resources, like quality teachers and curriculum, to succeed in mathematics, while *Achievement* is about meeting societal expectations through grades and test scores. The critical axis includes *Identity* and *Power*. *Identity* emphasizes the importance of students bringing their authentic selves and lived experiences into the classroom. *Power* entails student agency to challenge societal inequalities through math. Gutiérrez's equity framework is a valuable lens for analyzing facets of equity. For example, Jackson and Roberts (2017) used this framework to study preservice teachers' conceptions, demonstrating how teachers can foster access by setting high expectations and connecting mathematics to students' lives. They also highlighted the interplay between access and achievement, showing that teachers' low expectations limit students' opportunities to succeed. Similarly, we leverage this framework to analyze MGTAs' conceptions of equity related to their teaching practice.

Methodology

For the present study, we collected data from nine MGTAs participating in the ELITE PD program, which is focused on active learning, equity, and inclusivity, during the 2023-2024 academic year at a Southwestern four-year public university. The ELITE PD program is part of a multi-institutional grant-funded project aimed at developing and researching the impacts of MGTA PD that focuses on active learning, equity, and inclusion.

During the first PD session in the fall of 2023, the MGTAs participated in two activities: building hanging mobiles and the advantages checklist. For the mobile building activity (Schniedewind & Davidson, 1983), MGTAs were divided into groups of three to five and provided with bags of crafting materials (e.g., scissors, glue, colored paper, stickers, makers). They were then tasked to make a hanging mobile (a decorative structure that often hangs above baby cribs). Each bag had varying amounts and types of materials. Groups displayed their mobiles and discussed the materials used, highlighting disparities in the resources available to each group. Following this activity, the MGTAs completed an advantages checklist[1], which included questions about personal advantages such as socioeconomic status and able-bodiedness. Together, these activities help MGTAs reflect on the resources and advantages their students may or may not have access to while also considering how this compares to their own advantages and disadvantages.

After these activities, the MGTAs completed a brief survey to reflect on 1) how these activities supported equity and inclusion, 2) their main takeaways from each activity, and 3) their relevance to their roles as MGTAs. The data set for our analysis consisted of the nine survey responses to the mobile activity and advantages checklist. One MGTA was removed from the data set due to ostensibly misunderstanding the survey questions, resulting in responses that could not be analyzed. Thus, our analysis is of eight MGTA survey responses.

Data Analysis

The first two authors independently coded the eight MGTAs' survey responses using the four equity dimensions (Gutiérrez, 2008) as an a priori coding scheme while allowing patterns within that coding scheme to emerge (Miles et al., 2014). Both coders then met to reconcile coding discrepancies and discuss added codes. The data set was then coded again with the final codebook. Next, the coders looked for themes across and within individual MGTA responses,

[1] Advantage Checklist towards Educational Equity © 2024 by Jamiella Brooks is licensed under CC BY-NC 4.0)

which resulted in three prevalent profiles representing how the activities prompted MGTAs to think about equity. These themes emerged from prevalent clusters of codes (Table 1).

Results

In the coding process, three open codes emerged: *Empathy*, *Self-reflection*, and *Perseverance*. The *Empathy* code represents when an MGTA demonstrated an understanding of their students' experiences and disadvantages or advantages. The *Self-reflection* code represents instances when an MGTA reflected on their own privileges. The *Perseverance* code represents when an MGTA referred to students overcoming challenges in the classroom. Themes for MGTA profiles emerged from prevalent clusters of these open codes along with the four dimensions of equity (Table 1). Below, we detail the three profiles for MGTA responses.

Table 1. The three profiles that emerged were based on the dimensions of equity.

MGTA Profiles		
Profile	Dimensions of Equity	Open Codes
Resource Advocacy	Access	Empathy, Self-reflection
Asset-Based Collaboration	Access, Identity, Power	Perseverance
Resilient Problem-solving	Access	Perseverance

MGTA Profiles

All of our MGTAs' responses fit into one of the three profiles. These profiles reflect how the MGTAs considered the activities related to equity and how they were seen as relevant to their role as MGTAs. In this section, we describe the three profiles: Resource Advocacy, Asset-Based Collaboration, and Resilient Problem-Solving.

Resource Advocacy. Five of the eight MGTAs (MGTA3, MGTA4, MGTA6, MGTA7, MGTA8) were categorized under Resource Advocacy, with access and empathy/self-reflection being the most prevalent codes in their responses. As exemplified by the quotes in Figure 1, these MGTAs saw the PD activities as representing the importance of ensuring access to resources and cultivating empathy for students' diverse situations. Three MGTAs (MGTA3, MGTA7, MGTA8) also connected these activities to their role as MGTAs by reflecting on their own privileges and how the insights gained shaped their approach to teaching. In discussing equity, these five MGTAs primarily focused on access, viewing resources as advantages or disadvantages. Most of these MGTAs also highlighted how they could address students' varying access to resources by offering extra support.

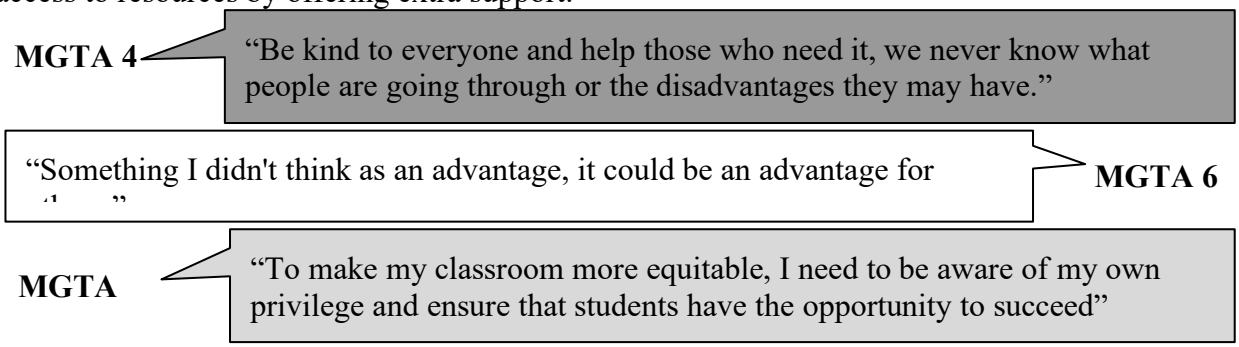


Figure 1. Example responses from the Resource Advocacy Profile.

Asset-Based Collaboration. Two of the eight MGTAs (MGTA2, MGTA5) were categorized under Asset-Based Collaboration, with their responses coded with access, identity, and power. As depicted by the quotes in Figure 2, these MGTAs saw the PD activities as representing the importance of recognizing students' diverse backgrounds and strengths as valuable assets in the classroom. They connected the activities to their role in creating an inclusive environment that leverages these differences and fosters collaboration. This differs from the Resource Advocacy profile in the sense that these MGTAs saw students' different resources as potential assets for them to help one another, rather than solely in terms of advantages or disadvantages, with one of the MGTAs (MGTA5) discussing how collaboration can help students support one another and persevere through challenges. Overall, these MGTAs' discussion of equity centered on access, identity, and power, acknowledging that while students have varying levels of access to necessary resources, their unique backgrounds could be leveraged to support each other and address classroom inequities.

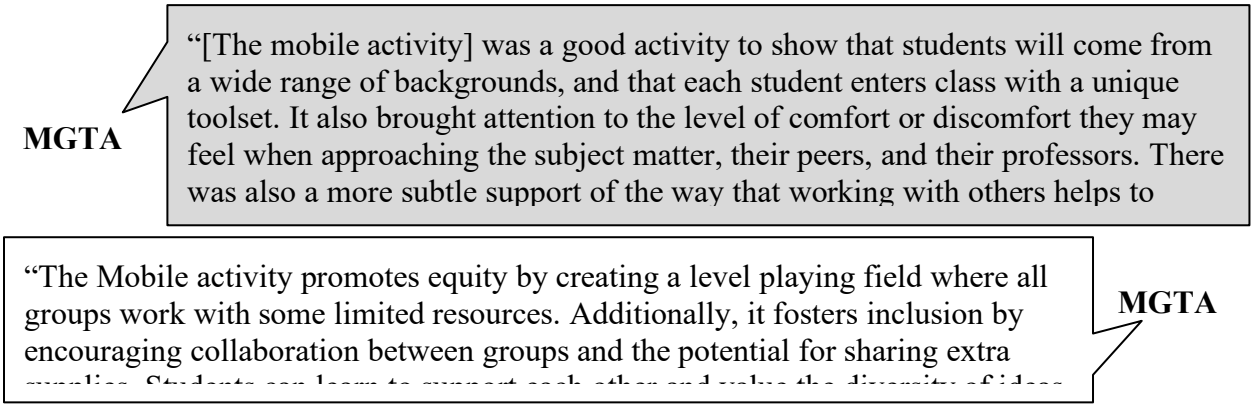


Figure 2. Example responses from the Asset-Based Collaboration Profile.

Resilient Problem-Solving. One of the MGTAs' (MGTA1) responses differed from that of the other MGTAs. As evidenced by the quote in Figure 3, this MGTA perceived equity in the activities as fostering resilience and problem-solving among their students. They linked this to their role by emphasizing the importance of creating an environment that supports students in overcoming challenges and setbacks. Their discussion of equity was focused mainly on access, emphasizing overcoming disadvantages by creating an environment where students could ask questions. This contrasts with how MGTA5 discussed identity as a tool to be leveraged to overcome challenges and support one another. Indeed, MGTA1 was focused on creating an environment where students could learn to help themselves by accessing resources rather than supporting one another by leveraging their identity in the classroom.

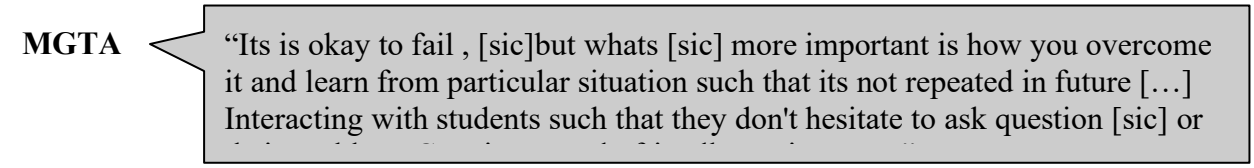


Figure 3. Example response from the Resilient Problem-Solving Profile.

Discussion

Notably, the MGTA profiles in this study reflect conceptions of equity similar to those found by Yu et al. (2024). The majority of MGTAs fit into the Resource Advocacy profile, which reflects the most prevalent conceptions of equity MGTAs have been found to hold. Yu et al. found that the most prevalent conceptions of equity included an individualized approach, awareness of diversity, and making things equal. The mobile activity and advantages checklist in this study prompted most MGTAs to think of equity regarding the resources students bring to the classroom and how these influence their ability to succeed (awareness of diversity). MGTAs related this understanding to their role in providing individualized support. They emphasized the importance of offering disadvantaged students equal opportunities, reflecting the "making things equal" conception of equity. The Asset-Based Collaboration profile seems to reflect less prevalent conceptions, such as valuing different types of thinking, in addition to the more prevalent conceptions.

These findings suggest that the mobile activity and advantages checklist are likely interpreted through the lens of an MGTAs current conception of equity. This prompted a preliminary analysis of how these MGTAs' conceptions shifted by looking at semi-structured interviews with these MGTAs after two semesters of ELITE PD. This preliminary analysis has suggested that these MGTA profiles evolve and shift, reflecting changes in their conceptions of equity. An additional finding of this preliminary analysis is that MGTAs' conceptions of equity align more with critical perspectives when discussing concrete topics like "equitable teaching" rather than the abstract idea of equity, even after a year of weekly equity-focused PD. For instance, several MGTAs mentioned practices like avoiding groups with only one female student to help them feel more comfortable and build community. This has led us to question the value of focusing on abstract equity concepts and consider whether addressing concrete issues of identity and power in the classroom, along with specific strategies, might be more effective.

Future work will expand this study to the other two institutions, allowing us to examine how different contexts influence the impact and implementation of the ELITE PD program. We plan to conduct a longitudinal analysis of exit tickets and other artifacts collected throughout the academic year to track the evolution of MGTA profiles. This analysis will help identify which activities tend to prompt shifts in thinking and in what ways, investigating the nascent shifts we see in MGTA profiles from our preliminary analysis. Additionally, we aim to explore the relationship between changes in conceptions of equity and concrete teaching practices to bridge theoretical understandings with practical application. By broadening our research scope, we hope to understand better how ELITE PD influences MGTAs' conceptions and teaching practices regarding equity.

Lastly, we pose the following questions: Based on each profile, which activities best support a shift in MGTAs' conceptions of equity, emphasizing the critical axis shift toward equity? How have other PD programs or initiatives that train MGTAs balanced equity's theoretical, abstract elements with the concrete implementation of equitable teaching practices? What are the most effective ways to ensure that theory-based activities, discussions, and readings translate into equitable teaching practices in the classroom?

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Constructing Instructors' Classroom Practices From Students' Survey Responses

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This preliminary report shares an outcome from a summer professional development (PD) activity with university instructors. Instructors participated in four PD meetings, then immediately taught a five-day summer workshop using inquiry, working primarily with first-generation minoritized students. While instructor participants' exit interviews of the project identified their experience in the summer PD as pivotal to their development, we know little of how students experienced the instructors' teaching during the workshop. Our analysis focuses on two items from student post-workshop survey wherein students shared their feedback of their instructor and their experiences more broadly. This analysis allowed us to get a good sense of the instructors' individual practices and revealed convergence in their practices. Pedagogically, instructors utilized groupwork and deemphasized direct instructions, while prioritizing students' engagement in discussions and struggling through conceptual ideas. Relationally, instructors were responsive to students' mathematical needs and created a respectful, safe, and welcoming classroom environment.

Keywords: professional development, equity, inquiry

Project Adelante was a three-year professional development (PD) project, aimed to support university instructors to develop skills in teaching with inquiry methods while simultaneously disrupting deficit narratives about minoritized students in mathematics (Adiredja et al., 2024). Broadly, the PD's goal was to develop culturally responsive mathematics teachers (Ladson-Billings, 1995; Zavala & Aguirre, 2023). The project was motivated by the ongoing educational inequities in undergraduate mathematics for minoritized students, and the need to develop skilled university instructors who are responsive to such inequities. In this way, we joined other scholars and national studies with similar foci (e.g., Leyva, 2022; Smith et al., 2021). However, the project engaged instructors and students in several activities where they explicitly discussed the role of race and gender in STEM to build empathy and solidarity. Each year's cohort of students and faculty participated in critical conversations at the university's cultural centers. Instructors conducted Funds of Knowledge-inspired (Moll et al., 1992) interview of a minoritized student from their class. This report focuses on a PD activity as part of a summer bridge program.

The year-long PD started in the summer. Our participants, instructional faculty and graduate students, participated in four meetings in July in preparation to teach a five-day inquiry-based learning (IBL) summer math workshop. The workshop served primarily first-year first-generation minoritized students who were enrolled in the courses: Pre-Calculus, Calculus I, II, Vector Calculus, and Linear Algebra in the following Fall semester. Topics of the meetings included deficit discourse in mathematics education, teaching with IBL, Funds of Knowledge, anti-deficit teaching, and IBL worksheet development. The PD used immediate implementation, and live coaching to give instructors hands-on experience with inquiry. Exit interviews with the instructors pointed to this summer activity as pivotal to their development as instructors. Here, we focus on analysis of students' feedback of their instructor and examine: *What can we learn about the instructors' teaching practices in their classroom based on the students' feedback? And to what extent did the teaching practices meet the goal of the PD?*

Conceptual Framework

We draw on Zavala and Aguirre's (2023) conception of Culturally Responsive Mathematics Teaching (CRMT) to set the landscape of desired practices with our instructors. The authors built their framework by leveraging the well-developed research in K-12 mathematics education around professional development. It is theoretically rooted in three research traditions: Math Pedagogical Content Knowledge (Grossman et al., 2005), Culturally Responsive Pedagogy (Gay, 2002), and Rehumanizing Mathematics (Gutiérrez, 2018). The authors have articulated components of CRMT and organized them into three strands and nine dimensions.

Table 1. Culturally Responsive Mathematics Teaching (CRMT) Tool.

Knowledge & Identities	Rigor & Support	Power & Participation
• Centering Cultural and Community Funds of Knowledge • (Re)Humanizing Mathematics • Honoring Student Thinking and Ideas	• Sustaining High Cognitive Demand • Scaffolding Up • Affirming Multilingualism	• Distributing Intellectual Authority • Disrupting Status and Power • Analyzing and Taking Actions

This framework articulates concrete aspects of "equity" in teaching. Fostering equity in the classroom has been articulated as one of the pillars of Inquiry Based Mathematics Education (IBME; Laursen & Rasmussen, 2019). More critically, the framework embeds active learning goals as part of culturally responsive math teaching. For example, the CRMT framework adds to "inquiring into student thinking"—a different pillar of IBME—the notion of honoring such thinking to emphasize the importance of shifting deficit narratives about URM students in mathematics (Adiredja, 2019). Power and participation issues are the focus in implementing collaboration among students and teachers. In other words, CRMT ensures that active learning strategies serve the goal of inclusive teaching.

Methods

We analyzed student feedback from the student exit survey to investigate the instructors' teaching practices in the classroom. The survey was administered to students at the end of the summer math workshop during each year of the project from 2021-2023. The workshop and the project were conducted at a research university in the Southwest of the US, which is also a Hispanic Serving Institution. In this report, we only focused on the 2023 cohort. The workshop session was led by six instructors, each teaching one course as shown in Table 2. There was a total of 71 students who attended all five days of the workshop. Sixty one of the 71 students (85.9%) responded to the exit survey. Students self-identified their race and ethnicity and might have included more than one descriptor: 38 students identified as Hispanic/Latinx/Mexican American, 26 as White, 18 as Asian/Japanese/Korean/Indian/South Asian/Pacific Islander, seven as Black/African American, three as Native American, and one as Middle Eastern.

We focused on the two questions from the survey which pertained to the instructors' teaching in the workshop. The first question asked students to give feedback to their respective workshop leader/instructor by providing at least one thing which their workshop leader/instructor did well

and one thing they could improve upon. The second question asked students to provide any additional comments they might have about this program, that were not elicited previously in the survey. Each of the students' responses were first analyzed for each instructor individually, to find common themes, and then aggregated to find similar classroom practices across instructors.

Table 2. Instructor course and demographics

Instructor	Course Taught	Title	Race	Gender
Barbara	Pre-Calculus	Career-track faculty	White	Cis-woman
Jinwoo	Calculus I	Career-track faculty	Asian	Cis-woman
Aarav	Calculus I	Graduate student	South Asian	Cis-man
Julian	Calculus II	Graduate student	Hispanic/Multi-race	Cis-man
Caspian	Vector Calculus	Graduate student	White	Non-binary
Daniel	Linear Algebra	Postdoctoral scholar	White	Cis-man

Since each student's response can contain multiple ideas, we deconstructed every response into different sentences, each consisting of a single idea. We kept track of each sentence by identifying the corresponding instructor and student. We filtered the sentence that described the pedagogical practices the instructors engaged in from the general student experiences. Starting with one instructor, we created categories by comparing and grouping the sentences with similar themes. We used these categories to group a second instructor's sentences from their students and created new categories based on the remaining unsorted sentences. We iterated this process for the remaining instructors. We then collected the data in each category for all the instructors and examined the categories shared across most or all instructors.

For example, one student responded that their instructor "[was] very good at pacing and drip-feeding concepts while we work on problems." This response was then split into two sentences: CAS_3_a: A student was comfortable with the pacing in the course, and CAS_3_b: Students were engaged with concepts while working on problems. CAS was the first three letters of the instructor's pseudonym, in this case, Caspian with 3 being the third student who responded from this instructor's class. We note that students' negative sentiment, such as they wished the instructor would lecture more was still informative. We placed that sentence into the category that says that the instructor de-emphasized the use of direct instruction.

The three authors started the analysis with survey responses for the first instructor. The second author continued the analysis of the rest of the responses for the remaining instructors with the third author. Since the second author was a past participant of the workshop, the third author led the analysis of their students' responses. Disagreements were resolved first by the second and third authors, and in some cases with the first author.

Findings

We identified ten categories, which describe classroom practices that the instructors engaged in from the student surveys. We focus on the following six categories which were commonly shared within four or more instructors' feedback: (a) Fostering a classroom environment that is safe, comfortable, and respectful; (b) Offering students opportunities to experience productive struggle; (c) Organizing the classroom around students' engagement with discussions and group work; (d) Focusing on students learning with conceptual understanding; (e) Responding to students' mathematical needs; and (f) De-emphasizing the use of direct instruction. Table 3

shows the distribution of distinct student responses within each category by instructor. Due to space constraints, we will share details about four categories in part due their richness.

Table 3. Distribution of distinct student responses within each category by instructor

Category	Barbara	Jinwoo	Aarav	Julian	Caspian	Daniel	Total
(a) Classroom environment that is safe, comfortable and respectful	2	2	4	2	1	1	12
(b) Opportunities to experience productive struggle	1	0	3	2	2	1	9
(c) Students' engagement with discussions and group work	2	1	4	4	2	1	14
(d) Students learning with conceptual understanding	0	0	1	2	3	2	8
(e) Responding to students' mathematical needs	3	2	3	3	2	7	20
(f) De-emphasizing the use of direct instruction	1	3	3	3	1	1	12

Within the category of *Fostering a classroom environment that is safe, comfortable, and respectful*, 12 total students across all six instructors' classroom identified how their respective instructor contributed to creating an environment that fostered student respect and feelings of safety for students to engage with math. Many responses spoke to how the instructor was kind and respectful, but we also found that instructors created space for exploration, mistakes, and questions. One of Daniel's students expressed that he provided less formal and open explanations that allowed the student to feel like they could learn without fear of being treated like they were dumb. Similarly, responses from Aarav's class expressed that he created an environment where he was approachable for questions and where there were "no wrong answers" or "dumb questions." The instructors fostered a classroom where students were supported and encouraged to make mistakes and ask questions.

Among nine student responses categorized as *Offering students opportunities to experience productive struggle*, we found evidence that five instructors engaged in such practices. These responses identified that students were responsible for most of the cognitive load in the classroom and were given opportunities to make sense of material on their own. In Julian's class, a student identified that he gave them freedom to learn theorems on their own. Moreover, a response from Aarav's class indicated that he did not give students explicit directions while solving problems and encouraged them to explore ideas. Instructors created classrooms where students were given space to explore, struggle, and make sense of material on their own. Some instructors also incorporated a balance between struggling on problems and the big picture ideas of the course. For example, a student expressed that Caspian gave students opportunities to struggle with solving problems on worksheets while also striking a balance with getting students back to the big picture ideas.

The category of *Organizing the classroom around students' engagement with discussions and group work* appeared in 14 students' responses across all six instructors' classrooms. This category provided evidence that these instructors engaged their students in collaborative group

work, but also gave us insight into the nature of class discussions that occurred in the classroom. A student in Barbara's class indicated that Barbara told her students to work [in groups] on vertical whiteboards. In Jinwoo's class, a student expressed they did not like the inquiry-based learning. They said they prefer being taught explicitly where they could take notes, solve practice problems and being corrected. Both these responses, despite reflecting a negative sentiment in Jinwoo's case, were placed in this current category.

Focusing on students learning with conceptual understanding was our smallest shared category with only eight students across four instructors, but it exemplified an important facet of the instructor's teaching practices. We found evidence that the instructors emphasized process and big ideas over procedures and memorization within their classrooms. In Caspian's class, a student expressed that their learning focused on conceptual ideas and problem-solving methodologies over finding computational answers. Similarly, a student in Julian's class expressed that they proved mathematical rules through problem solving and generalized these rules to solve other problems. The focus shifted from knowing a rule to getting a hands-on experience of understanding the derivation of that mathematical rule.

Discussion

Our analysis methods proved fruitful. We were able to construct our instructors' individual classrooms and the practices they engaged in during the workshop by analyzing their students' responses to the two survey items. Moreover, comparing the categories across all the instructors revealed some degree of convergence in the kinds of practices they were engaged in after the summer PD, which was an encouraging result as they are consistent with the practices we advocated during the preparation meetings. We now reflect on these findings relative to the Culturally Responsive Mathematics Teaching framework (CRMT; Zavala and Aguirre, 2023).

Considering the nine dimensions of CRMT, we found consistencies between our categories and several of the dimensions. These dimensions include, (Re)humanizing Mathematics, Honoring Students' Thinking and Ideas, Sustaining High Cognitive Demand, Scaffolding Up, Distributing Intellectual Authority, and Disrupting Status and Power. We did not find evidence for Centering Cultural and Community Funds of Knowledge, Affirming Multilingualism, and Analyzing and Taking Actions. In some ways, this result allows us to think about how much culture was centralized in the summer preparation meetings. On the other hand, we actually did not expect all nine dimensions to be reflected at this early point of the PD. It will be interesting to complement this analysis with other data point from the project where the instructors have participated in critical conversations about racial and gender stereotypes and conducted the Funds of Knowledge-inspired interview with one of their workshop students.

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Preservice Teachers and the Triangular Area Puzzle: A Look at Common Errors

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Keywords: geometry, area, conceptual understanding, teacher education

Research has demonstrated that enhancing teacher candidates' comprehension of area concepts largely emphasizes the deepening of content knowledge in geometry through the application of geometric principles to real-world problems (Litoldo & Amaral-Schio, 2021). Building on prior studies (McNeal et al., 2018; McNeal et al., 2023), this study examines the common errors identified in elementary preservice teachers' (PSTs) written responses to triangular area questions, linking these errors to PSTs' conceptual and procedural understanding.

The conceptual framework for this study is based on Hiebert and Carpenter's (1992) model, which distinguishes between two types of knowledge: procedural and conceptual understanding. Procedural knowledge encompasses the application of rules, algorithms, and symbols, whereas conceptual knowledge involves grasping the relationships among concepts and the principles underlying the procedures. This framework guided the analysis in the present study, enabling an examination of the relationship between the common errors made by preservice teachers (PSTs) in their written submissions and their conceptual and procedural understanding of the triangular area.

We collected 85 elementary preservice teachers' written work, including homework, midterm, and final assessment questions related to the calculation of triangular area. A total of 200 written submissions were gathered from four cohorts of a Geometry and Measurement content course for elementary preservice teachers over three consecutive academic years. Employing content analysis methodology (Elo et al., 2014), we first analyzed the content of the students' written submissions to identify common errors. We then examined the relationship between these errors and the preservice teachers' conceptual and procedural knowledge, in accordance with our conceptual framework. After 5 individual coding sessions and 6 group checking-in meetings, we finalized developing 6 codes of common errors in preservice teachers' written work including *using of area units, identifying height or/and base of a triangle, using of area formula, connecting the area formula with the adding and moving principle, connecting triangular area with different shapes' areas, and calculation mistake*.

Analysis revealed that among 6 identified error codes, the highest percentage of errors pertained to the use of area units. For instance, preservice teachers (PSTs) frequently confused the terms "square unit" and "unit square," and struggled to differentiate between them using examples. The second-highest percentage of errors involved difficulties in drawing or identifying the height of triangles, particularly in the case of obtuse triangles. These errors often occurred when PSTs attempted to draw the height for calculating the triangular area, struggled to extend the base and determine the appropriate point to start drawing the height, or create a non-perpendicular segment and treating it as the height. The use of incorrect units and the misidentification or incorrect drawing of the height are linked to a conceptual misunderstanding of the properties of triangles. These errors significantly contributed to the frequency of mistakes in area calculations. Although most PSTs correctly stated and applied the area formula in their written submissions, the incorrect use of units and/or height often led to incorrect answers.

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Student Mathematical Identity as a Framework for Reflective Practice

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Keywords: Student Attitudes, Reflective Practice

The 21st century has seen the rise of math education research which seeks to understand and contextualize students' affective beliefs as well as large-scale professional development and policy interventions targeting student identity (Aguirre et al., 2013). The authors of this study are teacher researchers at a large public Hispanic-Serving Institution in Southern California whose prior work investigates student attitudes. Our dual roles led us to the following research question: **How do we use knowledge of students' prior and developed mathematical attitudes in our classes to inform our future teaching?** The goal of our poster presentation is to share a methodology for how scholarship can return to teacher practice.

We use action research, specifically the dialectical action research spiral described in Mills (2018), to frame our analysis. The steps of this process – identifying an area of focus, collecting data, analyzing and interpreting the data, and developing an action plan – are linked in a loop with no fixed starting or endpoint. Based on our prior work and internal evaluation, we use the Mathematics Attitudes and Perceptions Survey (MAPS) (Code et al., 2016) to measure students' *expert-like attitudes* in the dimensions of Growth Mindset, connections to the Real World, Confidence, Interest, Persistence, Sense Making, and the nature of Answers in math; we augment this instrument with an Identity dimension taken from the High School Longitudinal Survey (HSLS:09) (Ingels et al., 2011).

Student data is drawn from the populations of the researchers' courses in Fall 2022 and Spring 2023. The MAPS+Identity survey was used as a metacognitive classroom activity in each of these courses at the beginning and end of the semester; our analysis includes only students who gave informed consent for their data to be used in research. Author 1 identifies as white and non-binary; they teach content courses for pre-service elementary and secondary teachers. Author 2 identifies as white, male, and a former low-income first generation college student; he teaches content courses for pre-service secondary teachers as well as general education courses. Author 3 identifies as white, female, a former first-generation college student, and a non-native English speaker; she teaches general education courses. All three authors are coordinators for courses in the mathematics department at our institution.

Our poster will refine and present the following preliminary results and implications for our practice, based on initial teacher researcher conversations:

- Instruments used in RUME have historically been validated primarily in the contexts of research institutions with predominantly white student populations. Our work extends the validity evidence of the MAPS to our context, which increases our confidence as teachers that these dimensions are salient for understanding our students.
- Recent prior work (Balady & Taylor, 2022) found that students' pre-course Growth Mindset scores were higher than those in earlier uses of the instrument, which the authors suggested might reflect the increasing popularity of K-12 mindset interventions. Students in our population are consistent with this trend, which suggests that such interventions may be reaching a ceiling of effectiveness.

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Improving Students' Confidence in Mathematics and Problem-Solving Skills

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Keywords: Confidence, Motivation, Attitude

Many students have negative attitudes toward mathematics. In the courses that I have taught, I frequently hear "I'm just not a math person" or "I'm bad at math." It has been proven in many different studies that attitude and achievement are highly correlated. Not only does attitude affect achievement, but achievement affects attitude. In fact, this correlation is bidirectional (Goodykoontz 5).

According to research, there are five factors that influence students' attitudes toward mathematics. Specifically, these five factors are teacher characteristics, teaching characteristics, classroom characteristics, assessments and achievement, and individual perceptions and characteristics (Goodykoontz 76). There are multiple types of student confidence: overall confidence in mathematics, topic confidence, and applications confidence (Parsons 55). My project investigated the first four external factors to determine if a single semester in a positive math classroom could increase students' attitudes and confidence in mathematics.

In Spring 2023, I was the instructor of record for MATH1005: College Algebra at the University of Iowa. The students in this course were mostly non-math majors who needed this class for their degree. On the first day of class, I handed out Survey 1. This survey asked students about their attitude, confidence, and motivation toward mathematics. Throughout the semester, I implemented the four external factors discussed above to attempt to increase student's confidence and attitude toward mathematics. For example, to implement teaching characteristics, I always wrote out every step carefully, tried to show multiple ways to approach each problem, and gave real-life examples to emphasize important concepts.

At the end of the semester, I gave the students Survey 2. This survey was nearly identical to the first survey. Survey 1 had 5 responses, which was a 21% response rate, and Survey 2 had 6 responses, which was a 25% response rate. I analyzed the data and created multiple bar graphs comparing the results from Survey 1 and Survey 2. First, I asked the students to rate their confidence for specific mathematical concepts taught in College Algebra. The data showed that students felt a higher confidence level in their abilities after learning the new concepts.

Next, I asked students to rate their level of agreement/disagreement on various statements relating to confidence and motivation. The statements with the largest amount of positive change from Survey 1 to Survey 2 were "I am good at math," "I have a mathematical mind," and "I enjoy solving challenging math problems. The statement with the largest amount of negative change was "Even if I work hard, I still struggle with math." The results show that the students had higher confidence in their mathematical abilities and a more positive attitude toward mathematics in general.

Lastly, I asked the students to identify which specific aspect of the course had the greatest impact on their feelings about math in the course. The top two answers were the instructor's personality/enthusiasm for the course and seeing each step written out carefully for problem solving. Some students also selected feedback on quizzes and exams, seeing multiple ways to approach a problem, and opportunity for one-on-one help. This shows that the teaching characteristics and teacher characteristics were the most influential factors for students' confidence.

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Mathematics for Undergraduate Nursing Students: Expansive Framing to Foster Transfer

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Keywords: expansive framing, mathematics, nursing, transfer

Mathematics plays an important role for nurses on the job yet practicing nurses do not view their work as being math-dependent, and nursing students find little value for learning mathematics beyond simple calculations (Hoyles et al., 2001). This may indicate that nursing students are struggling to successfully transfer what they learn in mathematics class into the nursing field. *Transfer* occurs when a student takes content or skills learned in one context and applies it to another. Transfer has been studied extensively but is known to be a construct that is difficult to research (Barnett & Ceci, 2002). Expansive Framing theory (Engle et al., 2006, 2011, 2012) acknowledges the transfer debate but refocuses the lens on context, positing that transfer is fostered by linking contextual elements and creating *intercontextuality*, or connections between contexts.

This research employed a teaching intervention in an online College Algebra course at a nursing school. The students in the course received instruction through a weekly synchronous webinar. Participants were placed in two groups: the first group received typical instruction using primarily bounded framing, and the second group received instruction using expansive framing across time, place, people, and topics (Lam et al., 2014). Participants were interviewed about their perceptions of value and transferability and transcripts were analyzed using reflexive thematic analysis (Braun & Clarke, 2022).

Results showed that forming intercontextuality positively influenced perceptions of value and transferability. While the webinars themselves showed mixed effectiveness, other expansively framed course activities emerged as being highly beneficial for influencing perceptions. Overall, this research indicated that broad framing of context was effective in improving perceptions of mathematics value and transferability for nursing students. Based on these results, I recommend that mathematics instructors in undergraduate programs actively seek opportunities to create intercontextuality through broad framing during instruction, but also by applying EF to classroom activities through curricular integration of content.

The accompanying poster provides examples of expansive and bounded instructional framing, quotes from student interviews, and more detailed results. This research provides a unique contribution to the field of undergraduate mathematics education, and while this research focuses on mathematics in nursing, the findings have important implications about mathematics in a variety of applied fields. Expansive Framing is a theory previously applied in the Learning Sciences field and has not yet been widely accepted in mathematics education, yet this research shows that it has the power to facilitate deeper mathematical connections across contexts through the creation of intercontextuality.

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Attributions of Mathematical Excellence from the Perspective of (Pre)Calculus Instructors.

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Keywords: attributional beliefs, deficit discourses, equity

Deficit discourses are socially constructed narratives that circulate and reinforce negative perceptions about marginalized students (Adiredia, 2019; Adiredja & Louie, 2020). Deficit discourses make claims about "students' academic and intellectual shortcomings" (Adiredia & Louis, 2020, p. 42) and attribute those shortcomings to defects of the student or their community. They claim, for example, that Black and Latinx students are responsible for their own underachievement (Nasir & Shah, 2011), that "women and Latinos are not good at mathematics" (Leyva, 2016 , p. 109), or that Mexican and Black families and cultures do not value school work (Solórzano & Yosso, 2002; Adiredia & Louie, 2020). Classroom environments that reinforce deficit discourses overlook the existing knowledge of students, denying them meaningful opportunities to engage in mathematics (Adiredja & Louis, 2020). This approach also perpetuates false hierarchies that portray "Black, Latinx and indigenous people as mathematically inferior to white and some Asians and consequently as less deserving of meaningful, prestigious and lucrative learning opportunities and careers" (Adiredja & Louis, 2020, p. 43). In a recent study, Leyva et al. (2021) found that even "seemingly neutral behaviors and comments ... reinforced whiteness and patriarchy" (p.28). They call for the future research to scrutinize "precalculus and calculus instructors' consciousness of whiteness and patriarchy embedded in undergraduate mathematics education" (p. 28). Inspired by this call, we propose investigating precalculus and calculus instructors' beliefs about the source of racial and gender differences in mathematics achievement outcomes and investigate links with "seemingly neutral behaviors and comments" (Leyva et al., 2021). We approach this work using a perspective that draws on the attributions of mathematical excellence (AME) framework (Jacobson et al., 2022) which uses theory from social psychology to characterize deficit (and anti-deficit) discourses in mathematics education as genetic, personal, social, or educational. The goal the AME framework is to examine how teachers' understanding of the origins of unequal mathematics education outcomes might be significant for instruction. Some beliefs may reinforce racist and sexist disparities, while others may help to challenge and change them. Importantly, the theory highlights how apparently identity-neutral attributional beliefs about mathematical excellence align with negative racial and gender stereotypes that correspond to deficit discourses in mathematics education. For example, the explanation that succeeding in mathematics requires sufficient effort (personal) is aligned with the discourse that Black and Latino students are underachievers. Building on the Leyva et al. (2021) work on examining racialized and gendered mechanisms in undergraduate mathematics courses, we plan to apply AME in a university context, testing its generalizability in a new environment. Specifically, we see the theory as promising to illuminate the causal factors that mathematicians use to explain students' mathematical success (and struggle), including explanations that seem to be neutral but nevertheless align with negative stereotypes. In this poster, we describe the theoretical framework and outline a study design to address the research question: How does the AME framework capture university mathematicians' attributional beliefs of mathematical excellence?

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Neurodiversity Access, Agency, and Accommodation in Mathematics Education for Undergraduate Students

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Keywords: DEIA, Neurodiversity, UDL

In recent years, there has been greater emphasis on creating diverse, equitable, inclusive, and accessible spaces for underrepresented students in mathematics. However, neurodiverse students have largely been left out of this conversation. There is evidence that students who are privileged to have access to diagnosis may not disclose their diagnosis or seek accommodations in college due to stigma or lack of information about resources (Borsotti et al., 2024; Edwards et al., 2022) and that even equity-oriented undergraduate faculty often feel under-prepared to teach neurodiverse or disabled students, or treat them only as separate groups rather than creating pedagogies that are accessible to all students (Cotán et al., 2021; Hitch et al., 2015; Moriña, 2022). There is also little research specifically on disabled student accommodations in online and hybrid formats (Edwards et al., 2022). The NAME US (Neurodiversity Access, Agency, and Accommodation in Mathematics Education for Undergraduate Students) project aims to begin this conversation and provide more research into the ways all students can be supported in a mathematics classroom.

This poster will present on the initial results from a pilot study focused on a teaching experiment that applies principles of Universal Design for Learning (UDL) (Fornauf & Erickson, 2020) to an undergraduate mathematics course at an institution that serves high number of students in demographically at-risk groups for underdiagnosis of neurodivergence and learning differences such as women, first-generation college students, and non-white students (Cunningham, 2021; Lockwood Estrin et al., 2021; Martin, 2024; Moody, 2016; Shi et al., 2021). Our specific research questions are:

1. In what ways does applying the principles of UDL to increase flexibility in learning and assessment modalities in introductory mathematics courses improve student outcomes as measured by grades and course evaluations?
2. What evidence is there of under-diagnosis of learning differences or under-disclosure of diagnosed learning differences in students in introductory mathematic course?

The teaching experiment was conducted within two in-person sections of Precalculus I and two online sections of Contemporary Mathematics. All sections of both courses were taught by the same instructor. The control sections were taught using the same materials and course structure the instructor had used previously for these courses. The experimental sections utilized the same textbook, homework assignments, and exams, but UDL inspired changes in content delivery and assessment such as greater flexibility of exam conditions (headphones with white noise or music and fidget toys explicitly allowed) and more accessible presentation of material (more accessible and larger font for textbook and other resources, explicit orientation to accessibility aids within the learning systems).

Our poster will present the initial findings from both the courses and the survey conducted to measure the potential amount of underdiagnosis of neurodiverse conditions and underutilization of campus resources for those with a formal diagnosis. We look forward to discussing the project with the RUME community as a whole and are excited to share our plans for moving forward.

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The Impact of Using the IBL Approach and the 5E Framework in Mathematics Classrooms on Students' Motivation and Engagement

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Keywords: Inquiry based learning (IBL), 5E Framework, Motivation, Engagement

Many academic fields such as engineering and economics, rely on mathematics as a fundamental discipline (D'Ambrosio, 1995; Diego-Mantecon et al., 2021). One essential goal for mathematics educators should therefore be graduating students who can address real life challenges. This requires an education system that emphasizes practical problem-solving skills. In traditional educational systems memorization is prioritized over real-world applications. As a result, students perceive mathematics as a challenging subject with no practical applications (Diego-Mantecon et al., 2021; Gravemeijer et al., 2017), causing a decline in students' motivation and engagement to learn mathematics (Grootenboer, 2013). For this reason, more and more academics are adopting a mathematics curriculum that emphasizes the relevance of the subject to society. While engagement is active participation, motivation is the willingness to engage in an activity (Fredricks et al., 2004; Singh et al., 2002). In mathematics, engagement is fostered by challenging problems and group learning, whereas motivation is the result of factors such as curiosity, perceived relevance, or outside incentives (Skinner et al., 2008).

In this study, the Inquiry-Based Learning (IBL) approach and the 5E framework were employed to teach a mathematics lesson on logarithms. The subjects of the study were grade 12 students (equivalent to the college Freshman year) in a school situated in a remote area of Lebanon. In this intervention, we aimed at investigating the impact of IBL on students' motivation and engagement. The research was guided by two questions: (1) *How does the IBL approach impact students' motivation to learn mathematics?* (2) *How does the IBL approach impact students' engagement in learning mathematics?* A variety of tools were used to examine the research questions. Those included two questionnaires and an interview. The freely available surveys were the *Mathematics Motivation questionnaire* (Fiorella et al., 2021), and the *Mathematics Learner Engagement Scale* (Rimm-Kaufman, 2010). Interviews were conducted with 6 students after the intervention. Interviews were semi-structured and included 10 questions divided into five domains: Background information, perception of Mathematics in real life, application of mathematical skills, individual needs, suggestions, and feedback. A semi-structured interview was also conducted with the class teacher, and it was employed to elicit information on the educator's perspective and experience with the application of the IBL approach.

The class teacher and the researcher collaborated to design the intervention on the topic of logarithms. To increase students' intrinsic motivation and actively engage them in the classroom, the design incorporated an activity on earthquake intensity, a timely and relevant topic since Lebanon had experienced prior to the intervention a series of rather intense earthquakes. The study findings showed that the implementation of the earthquake activity heightened students' intrinsic motivation. Results also showed that using student-centered pedagogical techniques and building a cooperative learning environment significantly impacted student engagement. Results of the teacher's interview highlighted the shortcomings of traditional teaching approaches and the importance of creative pedagogical approaches such as IBL alongside the 5E framework to pique students' interest in learning.

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Exploring Student Experiences in Calculus I: The Impact of Inquiry-Based Versus Lecture-Based Instruction on Math Anxiety

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Keywords: math anxiety, inquiry-based learning, collaboration

Math anxiety affects student learning and academic performance, causing physical, mental, and emotional symptoms (Ashcraft, 2002; Mattarella-Micke et al., 2011; Ruffins, 2007). Math-anxious individuals may dislike math and take fewer math courses, ultimately avoiding math and math-related courses (Hembree, 1990). Due to this significant impact, it is essential to diagnose the causes of math anxiety and determine potential interventions to reduce math anxiety. As such, this study investigated the effects of inquiry-based learning (IBL) vs. lecture-based instruction on Calculus I students' math anxiety through the following research questions.

1. What have been students' experiences and perceptions of taking Calculus I in an online setting via IBL or lecture-based instruction?
2. How and to what extent do these experiences and perceptions influence math anxiety?

Methodology

Sequential Explanatory Mixed Methods Design was employed to investigate quantitative and qualitative research questions. Since the data were predominantly collected through one-on-one online interviews, this study fits more into a qualitative research study.

Participants, Data Collection, and Data Analysis

The sample of this study comprised of students who were enrolled in the Calculus I course and were taught using either IBL or lecture-based instruction during Spring 2021 at a university located in the Midwestern United States. Students who received IBL instruction were in the IBL group, and those who received lectures were in a lecture-based group.

A short version of the Mathematics Anxiety Rating Scale (MARS-S) developed by Suinn and Winston (2003) was used to collect the quantitative data from 15 IBL and 20 lecture-based students. Based on the change in their anxiety scores from pre- to post-MARS-S survey, 9 students from the IBL group and 3 from the lecture-based group were selected and interviewed via Zoom. Quantitative data were analyzed using the SPSS, and qualitative data were analyzed using the NVivo software.

Findings and Implications

Results show that the activities, such as collaborative work, optional and ungraded homework, and the instructor's anytime readiness to meet with his students, decreased IBL students' anxiety, whereas the instructor's readiness to reexplain the material in class and recitation classes by the teaching assistant decreased lecture-based students' anxiety. However, the tests and exams, anticipating the instructor's call for a response, and responding to instructor's questions in front of their peers increased anxiety among both groups of students. Moreover, students with IBL experience felt more confident and empowered compared to the lecture-based learning experience

Reflecting on students' increased anxiety taking tests, it might be worth considering more group-based assignments and fewer in-class tests. Also, since IBL developed confidence and critical thinking among students, implementing an IBL in math teaching might be worth it.

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Factors Influencing Persistence for Undergraduate Women in Mathematics or Statistics

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Keywords: undergraduate women, mathematics, statistics, college experience

Few women enroll in collegiate mathematics-intensive Science, Technology, Engineering, and Mathematics (STEM) programs compared to non-STEM programs (Rivers, 2017). Meanwhile, the few women pursuing these STEM degrees in colleges are about 1.5 times more likely than men to leave their STEM programs (Ellis et al., 2016). Some studies have explored the factors that account for women's dropout from STEM programs (e.g., O'Connell & McKinnon, 2021). But for the women who continue to enroll in STEM programs, particularly those in math-intensive programs, what college factors explain their continued enrollment in their majors?

In an attempt to better understand this situation, we explored the experiences of undergraduate women in mathematics and statistics programs with a focus on their perception of academic development opportunities and social support in relation to persistence. We define persistence as students' continuous enrollment through the third year, placing them on a presumed path to STEM degree attainment from the university. Academic development refers to the formal and experiential learning opportunities provided officially within the program (e.g., courses, office hours). Likewise, social support describes the help and guidance students receive beyond the official capacity of the program/department (e.g., study groups, major-related clubs).

The participants in the study included 24 self-identified women who completed the student survey sent to all students enrolled in the mathematics and statistics programs at Pink University (pseudonym) located in the Midwestern region of the United States. The survey was intended for only women in the programs. We used an online survey to collect data on key variables such as demographics and persistence factors. The survey consisted of 31 items on a 5-point Likert scale and 5 open-ended questions. The larger study involves group interviews with select respondents but the current analysis focuses on the survey responses to address the following research question: What college factors do women in mathematics or statistics undergraduate degree programs indicate as influencing their persistence in those programs?

We found that women rated academic factors as more influential on their persistence than social factors. Specifically, 91% of the women indicated that the relevance of courses to their career goals and their performance in the courses contributes strongly to their persistence. Likewise, 69% of the women declared that the quality of instruction and internship opportunities influences their persistence in mathematics or statistics programs. However, 75% of the women revealed social support factors have a low to moderate influence on their persistence.

Our results echo similar studies (e.g., Espinosa, 2011; Brainard & Carlin, 1998), and we extend this work by looking beyond introductory calculus to hear from women in their third and fourth years about how these academic and social factors have influenced persistence. Centering undergraduate women's experiences can point to areas where interventions and support can be targeted to improve women's experiences in STEM education.

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Learning to Learn Calculus: Equipping Students with Nonacademic and Behavioral Skills

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Keywords: Calculus, Educational Interventions, College Readiness, Mathematics Anxiety

Every year, millions of students begin post-secondary education in the hopes of furthering their education. However, research shows that being college eligible is not the same as being college ready, and students struggle with the rigors of higher education. While it is typical to associate success in college with various acquired academic skills, there are numerous nonacademic factors that can determine whether a student will graduate (Capuno et al., 2019; Conley, 2008; Robbins et al., 2006; Tinto, 2017). To help students acclimate to college life and academics, universities and colleges have created student life skills (SLS) courses which focus on nonacademic skills and learning strategies. While research has shown that SLS courses provide some benefits to students, these courses often attend to general concepts and skills without considering their application. Therefore, researchers recommend combining SLS course content with discipline specific content for it to be more beneficial for students (Boaler et al., 2022; Cho & Karp, 2013; Reilly, 2014; Wingate, 2007).

One discipline that could highly benefit from incorporating SLS content into the curriculum is mathematics, as certain mathematics courses in higher education are considered gateway courses to advanced coursework. Learning to learn mathematics is multifaceted, and combining SLS content with mathematics specific content can help students succeed in their mathematics courses and complete their degree in a timely manner. In this poster session I will discuss how I created a special Learning to Learn Business Calculus workshop focused on nonacademic and mathematics strategy skills, the different components of the workshop, and the findings of the research study which measured its effectiveness and usefulness (from the student perspective) as an educational intervention. I will explain how I facilitated the intervention at a large R1 university and discuss how the different activities are connected to non-academic and mathematics strategy skills topics such as metacognition, growth mindset, calculus test-taking strategies, procrastination, mathematics anxiety, and more (I will also provide a QR code where activities can be found). The findings showed the workshop had a positive impact on student outcomes (mathematics anxiety, self-efficacy, and academic achievements) and students found the workshop useful for both personal and academic reasons. The poster will present the findings and discuss their implications for future research and educational practices.

As the integration of nonacademic and strategy skills directly into discipline-specific courses is infrequent and receives limited attention in the field of education, this presentation serves as an opportunity for the audience to become acquainted with a method that enables instructors to incorporate these skills into their curriculum. By harnessing this tool, educators could potentially enhance short- and long-term student retention rates, thereby contributing to a more successful academic experience. This, in turn, would likely lead to improved satisfaction and overall positive experiences for both teachers and students within these critical courses. By attending this session, participants can be exposed to a unique and innovative teaching strategy and get tools and advice to apply it to their own courses. Furthermore, the audience can be exposed to a mixed methods education research study which can be replicated if they wish to measure the effectiveness or usefulness of facilitating this type of intervention at their own institution.

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Exploring Students' Understanding of a Limit of a Sequence: Using ϵ-Strip Activity With Realistic Mathematics Education Framework

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Keywords: ϵ-strip activity, realistic mathematics education, defining as a mathematical activity

The concept of limit is crucial in undergraduate mathematics, serving as a foundation for rigorous mathematical reasoning and applications in calculus, real analysis, and advanced courses (Roh, 2008; Roh & Lee, 2017; Swinyard & Larsen, 2012). However, students often struggle to develop a robust understanding of limits (Oehrtman et al., 2014; Williams, 1991).

I interviewed two students to examine if the ϵ-strip activity developed by Roh (2008) could scaffold and help students leverage their intuitive understanding of the limit in a sequence to the mathematically rigorous ϵ-N definition of convergence of a sequence. In particular, I investigated how students reason about the convergence or divergence of a sequence during the ϵ-strip activity. As a theory-building effort, I analyzed the interview data by mapping it into a table figure that integrates the constructs from the Defining as a Mathematical Activity (DMA) framework and Realistic Mathematics Education (RME).

Freudenthal (1991) proposed the RME as an instructional design that aims to foster the invention of mathematical knowledge called "emergent models." It highlights the shift between how students develop *models-of* their informal mathematical activity using tools, graphs, and analytic expressions to *models-for* increasingly sophisticated and formal ways of mathematical reasoning in four phases of Situational, Referential, General, and Formal activities (Gravemeijer, 1999). Zandieh and Rasmussen (2010) further described these four phases with the DMA by integrating the notion of concept definition and concept image (Tall & Vinner, 1981). This integration adds five constructs to the four phases of RME's transition process, which are creating a concept definition, using a concept definition, creating a concept image, using a concept image, and creating a new mathematical reality.

By mapping out the interview data and identifying critical events as explicit evidence for the transition process in a figure, I observed that the ϵ-strip activity aligns well with RME's emergent model and DMA. At the beginning of the ϵ-strip activity, students created a concept image and concept definition of ϵ-strip recalling their past understanding of limit as a model-of, which is what the Situational and Referential activities in an experiential real setting attribute. As they completed more activities, students used the physical ϵ-strip as a model-for to create a concept image and concept definition of the rigorous ϵ-N definition of convergent sequences. In particular, Henry (pseudonym) almost reinvented the ϵ-N definition with De Morgan's Laws by formalizing their concept image, an occurrence of the Formal activity in the wild.

As a result, this study examined the practicability and robustness ϵ-strip learning activity grounded with DMA and RME using a figure to present how the interview data merged into the four levels of activity and five constructs of DMA. Furthermore, I envision that mathematics education researchers and practitioners not only can use the DMA framework to guide the design process of instructional sequences that scaffold complicated mathematical topics related to defining, but they can also use DMA to improve the design of existing instructional sequences like the ϵ-strip activity that has the potential to leverage students' emergent model to formalize their understanding towards rigorous mathematical statements.

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Moving Beyond the Fence: Interrogating Problematic Images of Equity and Envisioning Inspiring Metaphors

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Denison University

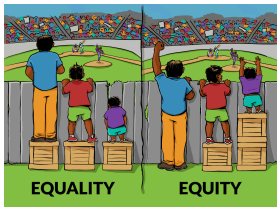
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Keywords: equity, arts-based methodology, education metaphors, departmental change

Visual imagery and metaphors are powerful. A single image can represent multitudinous and complex conceptualizations of an idea, and evoke emotions tied to a viewer's deep-rooted values, beliefs, and experiences (e.g., Adiredja & Zandieh, 2020). Unpacking this complexity is especially valuable in critical equity work (Levinson et al., 2022). Drawing on our research team's engagement with research-practice partnerships in three networked improvement communities (NICs) in mathematics departments, we are motivated by the following research questions: 1) How do NIC members respond when asked to draw or share an image representing equity work? 2) What emotions are evoked when reacting to these representations for researchers engaged in research-practice partnerships?

We compiled images from NIC members' journals (see Figure 1 for the prompt, left) and subsequently asked each research team member to describe the emotion or word that was evoked upon viewing the image. We first categorized the types of images by how each was created, content (e.g., reflecting the status quo or an aspirational future), and composition. We further categorized the researchers' emotional responses as positive (e.g., curiosity, appreciation, hopefulness) or negative (e.g., disgust, confusion, helplessness), noting how the composition and content of the images evoked our emotions. In many spaces in mathematics education, there is a common scene (or variant of it) used to distinguish equity from equality, which we refer to as *the fence* (see Figure 1, center image; Froehle, 2016). Members from all three NICs submitted images of the fence or a similar variant, e.g., picking apples, comprising 9 of the 20 images submitted. While the fence can be a useful tool for discussion, it is problematic in that it portrays the people as the problem (Equity in Education Coalition, 2018), rather than systemic structures.

"Please draw or share an image (e.g., a picture you took or an image from the internet) that represents what it means to you personally to engage in equity work. Then, write a description explaining why you chose or drew that image."



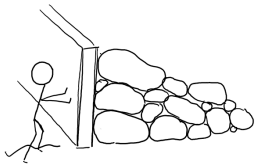


Figure 1. The journal prompt that elicited images (left), the Fence (center), and an Immovable Wall (right)

As a result, repeated images of the fence brought about emotions of irritation and frustration among the research team. We are ready to move beyond the fence. More positive emotions were evoked by other images that NIC members included, such as an immovable wall (see Figure 1, right image), a collage of kites cycling through soaring and crashing, and searching a cave with a flashlight. We found these images to more sincerely convey the tensions, joys, and challenges of doing critical equity work. For our poster, we present a collection of types of images and researcher responses in more detail, and suggest other visual representations of equity as inspiring and creative alternatives to the fence.

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Emulating Social Aspects for Argumentation in Non-STEM Classrooms

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Keywords: argumentation, non-stem students, social constraints

Studies have described accountable argumentation (Horn, 2008), collective argumentation (Brown, 2017), or dialogic argumentation (Hähkiöniemi et al., 2022) for how students can engage with each other to develop rigorous mathematical reasoning. Argumentation, justifying, and proving often come together for students when mathematical proof is not the focal point of instruction (Knuth et al., 2022). A common theme for developing proof is the idea of convincing oneself and another (Yopp et al., 2022). This often entails another person asking for further details or justification about how a mathematical claim is true. For a non-proof-oriented classroom, the idea of a convincing argument is tenable for the development of proof (Czocher & Weber, 2020). Therefore, our research purpose was to investigate how students interact with increasingly socially constrained tasks.

Methods

We developed four tasks to explore emulating the social aspect for convincing another classmate. Each task provided varying social constraints. These tasks were implemented throughout one semester on the exams of three instructors in an introductory mathematics course for non-STEM majors. The first task was "show that the sum of two even numbers is even." The second task was "convince a very stubborn classmate that when you add two even numbers, your result will be an even number. This classmate is not convinced that this is true for every number, especially very large numbers." The third task was like the second but added "This classmate has **challenged** you to show that this is true without using specific examples" at the end. The fourth task was "show that when you multiply two even numbers, the result is even."

We coded the students' responses using deductive argument (DA; akin to a mathematical proof, justifies inferences, or a teacher would accept this), example use with attention to generality (EG; example use with connections to conclusion or references shared characteristics across mathematical objects), example use only (EO; only calculations), other argument (OA; doesn't fit the other types), and non-argument (NA; unclear intentions or irrelevant).

Results and Discussion

In general, among the 46 students who participated in at least one task, EO and OA were the most common response types. For the constrained second and third tasks, the frequency of EO codes decreased, indicating it is possible these constraints interacted with argument constructions. While only one student was coded DA, many OA or EG responses contained ideas of such an argument, so the task variations may motivate deductive reasoning. Of the 42 students who completed at least two tasks, 25 students varied in responses across tasks. Students responded differently to the constraints; for example, several students who started with EO responses later constructed arguments such as OA or EG and vice versa (i.e., OA or EG to EO). Given that over half of the students changed their response type, we argue that studying what students attend to when choosing their argumentation strategies is worthwhile. In future work, we intend to conduct student interviews to focus on how they interpret these constraints.

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"Am I Missing Something?": Learning From Pre-Service Teachers' Routes Through a Mathematical Modeling Cycle

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Keywords: mathematical modeling, pre-service math teachers, modeling activity diagram

Mathematical modeling has been receiving increased focus with researchers identifying its benefits for students, such as using mathematics in real-world applications and affecting change in attitudes towards mathematics (Maaß, 2006; Stender & Kaiser, 2016). Mathematical modeling also provides the opportunity to support equitable learning by promoting cooperation among students from various mathematical backgrounds with the use of low-floor and high-ceiling tasks (Cirillo et al., 2016). With so many potential benefits, it is important to examine how groups of pre-service math teachers (PSMTs) negotiate the modeling process in order to better prepare them to include modeling activities in their future classrooms.

Modeling is a complex process, with participants needing to connect mathematical concepts with real-world contexts that are not well defined. There have been several modeling cycles published (e.g., Bliss et al., 2014; Blum & Ferri, 2009) that attempt to illustrate how individuals move through the various modeling phases. However, the modeling cycles tend to provide a simplified view of the "messy" and complex nature of these tasks (Albarracín et al., 2019). To gain insight into the modeling process, we planned and taught two mathematical modeling sessions for 24 PSMTs in a combined bachelor's and master's level mathematics teacher education course. This work enabled us to explore how small groups of PSMTs moved between phases of the Bliss et al. (2014) modeling cycle: *building the model, getting a solution, analysis and model assessment*, and *reporting results*. The research questions guiding this study are: How does PSMTs' engagement in a modeling task align with and differ from the Bliss et al. (2014) modeling cycle? What is the context that is taking place as PSMTs deviate from this cycle?

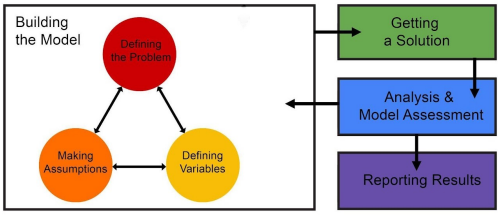


Figure 1. A Modeling Cycle Adapted From Bliss et al. (2014)

In our analysis, we used a modified version of the Modeling Activity Diagram (MAD) framework (Ärlebäck, 2009) to create a visual representation of each group's movements through the modeling cycle phases. Several different types of deviations from the Bliss et al. (2014) modeling cycle were observed. For example, students moved from *getting a solution* back to the *building the model* phase due to a need to make additional assumptions based on their model choices. We explore the causes for such deviations and discuss how this process is more complex than the cycles suggest when PSMTs engage in these tasks within the classroom. These insights can be used to help prepare PSMTs to teach mathematical modeling.

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Students' Limit Conceptions and Representational Approaches

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Keywords: Concept, limit conceptions, representational approaches

Investigating students' thinking about the limit concept in calculus through representational approaches supports the need for students to learn mathematical concepts with deep understanding and actively build new knowledge (NCTM, 2000/2014). This study takes an asset-based perspective and focuses on students' reasoning, views, and understandings of the limit concept rather than focusing on students' misconceptions and lack of understanding, as some past research has done (Bezuidenhout, 2001; Denbel, 2014; Fernández, 2004; Liang, 2016; Nagle et al., 2017). Additionally, some studies have investigated students' abilities to reinvent the conventional $\varepsilon - \delta$ limit definition (Swinyard, 2011; Swinyard & Larsen, 2012) but have not explicitly addressed how students' reasoning with representational approaches plays a role in advancing to the more formal limit definition. This poster presentation shares a portion of the results from a larger study on students' limit conceptions (Intuitive, Informal, and Formal) and their representational approaches (Graphical, Numerical, and Algebraic) to understanding the limit concept. The two research questions are:

1. *What limit conceptions do students possess?*
2. *What connections exist between students' limit conceptions and representational approaches?*

To address these research questions, pre-assessment surveys were collected from 63 students enrolled in Calculus II from a public university in the southeastern United States. The surveys have six limit-view statements (adapted from Williams, 1991/2001) that constitute the three main classifications of limit conceptions (Intuitive, Informal, Formal) and five multiple/short answer limit problems that constitute the three main representational approaches (Graphical, Numerical, Algebraic). To address research question 1, we analyzed the frequency counts for students' preferred choice of the limit-view statements that best expressed their understanding. To answer research question 2, we compared students' responses to the six limit-view statements to their choice of representational approaches to the five multiple/short answer problems.

We found that the intuitive limit-view had the most significant frequency, which could suggest that students begin to conceptualize limits from an intuitive perspective. Additionally, we found that students' *intuitive limit conceptions* are tied to graphical reasoning, and students' *informal limit conceptions* are tied to numerical reasoning. The results of this study are consistent with Adiredja et al.'s (2020) suggestion that the teaching and learning of abstract concepts in more tangible ways should be done based on students' intuitive reasoning.

Implications of this study offer both calculus instructors and curriculum reformers ways to integrate students' ways of reasoning into calculus teaching and learning. Specifically, results suggest that calculus instructors may leverage different representational approaches to provide opportunities for students to deepen their conceptual understanding of limits.

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Liberal Arts Math Through Theater

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Keywords: liberal arts, theater, interdisciplinary approach

Liberal arts mathematics courses, a crucial topic in mathematics education research, are typically designed to fulfill an academic requirement for non-STEM majors. They often focus on "real-world" mathematical applications (Penn LPS, 2024), and unlike traditional undergraduate mathematics courses that deeply explore one subject, liberal arts mathematics classes tend to highlight a variety of mathematical topics (George, 2010). Existing literature about liberal arts mathematics courses claim that these classes should highlight the "usefulness" of mathematics for those not pursuing a career in a STEM-related field (Strasser, 2011; Teixeira, 2023), and focus on the fact that "typically, students in these classes are not math enthusiasts" (Teixeira, 2023, p. 106). However, perhaps with a more intentional approach, focusing instead on the beauty of mathematics rather than the utility, these students could engage with rich mathematics in a more meaningful way.

I propose a liberal arts mathematics course that focuses on mathematics in theatrical works (liberal arts mathematics through theater, or LAMTT). Theater has the "capacity to adjoin and to assimilate every other art [the written word, music, dance, costumes, sets, etc.]" (Gertsberger, 2009, p. 70), so it is the most comprehensive medium with which to engage liberal arts students. This would be particularly appropriate for students majoring in a humanities discipline, because "meeting students at the intersection of where their interests lie along with significant mathematics allows educators to better communicate with their pupils" (Meadows & Caniglia, 2021, p. 183). The Database of Mathematical Fiction, curated and maintained by Dr. Alex Kasman is "widely recognized" as a reliable research tool in mathematical fiction, and contains a list of 71 theatrical works highlighting mathematics (Kasman, n.d.; Swallow, 2006).

I have analyzed these 71 theatrical works according to the intersection of "significant mathematics" and "student interest," as indicated by Meadows and Caniglia (2021). A variety of rich mathematical topics can be found in these theatrical works, such as: sequences, partitions, and The Functional Equation of the Riemann Zeta Function in Simon McBurney's *A Disappearing Number*; and Euclidean and fractal geometry, chaos theory, and Fermat's Last Theorem in Tom Stoppard's *Arcadia*. On the Database of Mathematical Fiction, the plays are labeled by their mathematical content (i.e. Probability/Statistics, Mathematical Physics, etc.). The 71 theatrical works also touch on interest areas that may engage liberal arts students, such as mental health in David Auburn's *Proof* and history and Lord Byron's poetry in *Arcadia*, and India and its diasporas in *A Disappearing Number.* I created a table that shows the intersection between mathematical content areas and common liberal arts content areas (i.e. *Arcadia* may fall under *(Chaos/Fractals, History)*). This analysis will be displayed in further detail on my poster, and can be used as a tool for undergraduate mathematics educators to create a LAMTT syllabus.

Ruth, a character in *A Disappearing Number,* says it best with her line: "It may be very hard to define mathematical beauty but that is true of beauty of any kind. We may not know quite what we mean by a beautiful poem, but that does not prevent us from recognising one when we read it" (McBurney, 2007, p. 42). In examining an artistic medium such as theater that both connects with student interests and focuses on the beauty rather than the utility of mathematics, liberal arts students may feel a deeper appreciation for the mathematics that they study and learn.

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Teachers' Perceptions of Artificial Intelligence in the Classroom

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We conducted an online survey to analyze how teachers view and use Artificial Intelligence (AI) tools to answer the following research questions: 1. What are teachers' perceptions of AI? 2. What are the characteristics of the teachers using AI in the classroom? 3. How often are teachers using AI in the classroom? 4. How are teachers using AI in the classroom?

Keywords: Artificial Intelligence, Teacher Perception, Curriculum Development

Artificial Intelligence (AI) is "a machine-based system that can, for a given set of human-defined objectives, make predictions, recommendations or decisions influencing real or virtual environments" (US Department of State, 2023, p. 21). According to the US Department of Education, AI is a problem because it can show discrimination and bias in its answers. It is often not helpful when giving answers because it is not human (2023). However, individuals still use it for teaching and suggest its use in the field, with some preservice teachers even considering "It's (AI) like a calculator for math resources" (Sawyer, 2024, p. 20). Yet, the field does not know how teachers perceive AI and how often they use it in their classrooms. Thus, we conducted an online survey to analyze how teachers view and use AI.

We distributed the online survey through a link on Instagram and X, using hashtags such as #eledteachers, #teachersandai, #teachersofX, #teachers, #AI, and #lessonplanning. From the survey, we gained 40 participants from 10 different US states. We analyzed the data using a grounded theory approach (Tie et al., 2019) using thematic analysis (Gibbs, 2007).

Findings

Teachers viewed AI as being able to create some activities. Still, the data stated that 84 % of participants said it needed editing, with only 16% stating it could use editing to create meaningful materials. Of the teachers who use AI, younger teachers are more likely to use AI than older teachers. The teachers identified using AI of any kind, on average, monthly. The participants identified that there are many ways that they are using AI in the classroom. For example, one teacher said they use it when creating multiple-choice tests and answer keys. In another instance in our survey, a teacher also said they use AI to give them ideas for what to write for discussion posts and/or assignments.

The data indicates that younger teachers use AI in the classroom. Thus, they must be trained in their preservice years to be critical of this new technology tool. Another important finding is that everyone believed that AI needed to be edited for the classroom; thus, individuals are already skeptical about using the tools. Mathematics Teacher Educators (MTEs) should use the young teacher's skepticism by providing examples of inaccurate mathematics and inappropriate responses by AI tools like Sawyer and Aga's (2024) paper, "Counterexample to Demonstrate Artificial Intelligence Chatbot's Lack of Knowledge in the Mathematics Classroom". If younger teachers are not educated on AI's abilities, they may end up using it in a way that could harm students. The data demonstrates that MTEs must take responsibility for supporting this vulnerable population.

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How Undergraduate STEM Students With Math Anxiety Persist Through Their Academic Careers

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Keywords: math anxiety, achievement, calculus, strategies

Background and Motivation

Math anxiety is one of the main reasons that students avoid majors that would require them to take math classes in the process (Ashcraft & Moore, 2009; Dowker et al, 2016). Lee (2009) has found that there are high achievers in math who are also highly math anxious. The antecedents of math anxiety were investigated by Luttenberger et al (2018) in which they have found that significant people in students' lives, such as teachers, can affect students' development of math anxiety. A qualitative study by Quan-Lorey (2017) found that teachers, self-imposed pressures, and bad grades are some of the roots of math anxiety. A motivation of this study is due to the observation that, despite there being numerous studies surrounding math anxiety, most of the studies are with non-STEM college students. Some participants of this study have expressed the need to address math anxiety experienced by STEM majors due to the necessity of math in their respective majors. This study highlights that math anxiety also impacts STEM majors, and what they do to navigate their college careers while persisting through their math anxiety. In this study, I seek to answer the following research questions: (1) What factors affect undergraduate STEM students' math anxiety in college? (2) What strategies are utilized by high achieving students to help them succeed despite their math anxiety? (3) What motivations do students possess to continuing in pursuing their STEM majors despite having math anxiety?

Methods

This qualitative study included 10 STEM students experiencing math anxiety that volunteered to participate. The participants are currently taking Calculus II, given most students in Calculus II are STEM majors and the target demographic. The study was conducted through semi-structured interviews where participants were asked about their earliest experience of math anxiety. From there, participants were asked how they have been navigating their academic career within their respective STEM major given their math anxiety. Furthermore, participants were asked about study and learning strategies they have implemented to help cope with their math anxiety. Aside from how they combat their math anxiety, a key factor that was investigated further was the students' motivations and why they remained in the STEM academia despite their math anxiety. The goal is in knowing how they tackle it and what motivates them to do so will allow future educators and students to be aware of and minimize their math anxiety.

Results

In this poster report, I present evidence that all of the participants have unique experiences with math and the development of their math anxiety. Most of the roots of their math anxiety are aligned with the previously mentioned findings of Quan-Lorey (2017) and Luttenberger et al (2018). As they traverse through college, they find that the levels of their math anxiety vary depending on their teachers and the difficulty of the class. When participants were asked about their motivation in pursuing/persisting through their math anxiety, a couple of themes arose. One of which is the job security that comes with the STEM major that they chose. The other is their innate passion and determination to change the future for the better. I will also discuss in this poster different strategies students implement to manage their math anxiety.

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A Cross-Disciplinary Approach to Supporting Mathematically Underprepared First-Year STEM Students

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Keywords: college student mental health, STEM, introductory mathematics courses

Many first-year students intending to earn a degree in a science, technology, engineering, and mathematics (STEM) field begin their studies with inadequate mathematical preparation (ACTA, 2019). These STEM majors are also at a high risk of developing mental health issues due to the stress-inducing nature of their coursework and are less likely than their non-STEM counterparts to seek counseling services (Kalkbrenner et al., 2022). In fact, a third of STEM majors who abandon their degree plans report having struggled "with depression, high levels of stress, chronic anxiety, overwhelm, feelings of guilt, regret, shame, and self-blame" (Holland et al., 2019, p. 330). In response to these issues, the Mathematics and Counseling departments at Montclair State University (MSU) collaborated to integrate counseling services into a unique new introductory mathematics course, Approaches to College Mathematics (ACM).

The MSU Mathematics department developed the ACM course as a prerequisite for students whose placement exam scores indicate they are not ready to take Precalculus. In the past, nearly all first-year MSU STEM majors were placed into Precalculus, which had DFW rates that were sometimes as high as 48%. The content of ACM was therefore chosen to focus on mathematical skills necessary for success in Precalculus (e.g., solving linear and quadratic equations). The ACM developers were cognizant that traditional remedial mathematics courses often have negative impacts on students (Cohen & Kelly, 2020; Larnell, 2016), so they were committed to designing a course that would engage students with course content in a new way, and would help them develop other academic skills required for success in tertiary STEM coursework. The ACM developers created an extensive repository of in-class activities designed to leverage research-based pedagogies, such as engaging students in collaborative learning (Liljedahl, 2020) and situating course concepts in meaningful real-world contexts (Bernacki & Walkington, 2018). They also developed a list of "growth objectives" for the course that were interwoven into class assignments and activities, such as *budgeting time and prioritizing*, *studying effectively*, *learning from mistakes*, and *navigating resources*. Our poster will include examples of the ACM learning activities and how the growth objectives are integrated into coursework.

The MSU Counseling Department provided the ACM program with an experienced counselor to offer counseling services to struggling ACM students, usually identified by their instructors. Students choosing to utilize the counseling resource meet with the counselor through virtual or in-person therapeutic interactions, where the counselor provides support in a safe space where students can discuss the challenges they are facing. Additionally, the counselor visits classrooms at the request of ACM instructors to give presentations and lead classroom discussions about various mental health issues students may be facing. The ACM team is in the early stages of collecting data on students' pre- and post-semester attitudes towards mathematics, their feedback on the course, and the percentage of students utilizing the counseling services. We will also be collecting longitudinal data on their successes and challenges in Precalculus. Some of this data will be available by the time of the conference and will be shared in the poster. This work aims to generate and contribute to conversations about how educators can support mathematically underprepared STEM majors experiencing anxiety and stress in math.

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Choices You Make Should Matter: Asynchronous Discovery Activities

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Keywords: Activity Theory, Asynchronous Activities, Digital Applet

In a live undergraduate classroom, the instructor and other students can provide dynamic responses to anticipated and unanticipated student actions. This study investigates the need for theoretically-sound development of math activities where a computer might also provide dynamic responses to anticipated student actions. This poster addresses the following research question: How can sociocultural and cognitive points of conflict be used to develop asynchronous discovery-based activities?

Theoretical Framework – Activity Theory

Activity Theory is a sociocultural theory that attempts to describe the coupling of consciousness and activity, where consciousness is the product of an individual's interactions with people and artifacts in the context of everyday practical activity (Kaptelinin & Nardi, 2009). In Activity Theory, points of conflicts (referred to as *contradictions*) act as opportunities for development. They mark key actions a student would need to make similar to how introducing cognitive dissonance provides an opportunity for concept development (Cooper, 2019) in constructivist theories. By blending Activity Theory with a constructivist lens for a particular mathematical concept, a discovery-based activity can be theoretically-sound by constructing a series of anticipated actions and reactions. This is similar to how a "Choose Your Own Adventure" story has points where the reader chooses an action (go in the left door) and the book provides a reaction (turn to page 105). An overview of key actions for the developed and studied dynamic activity included in a College Algebra course is presented in Figure 1.

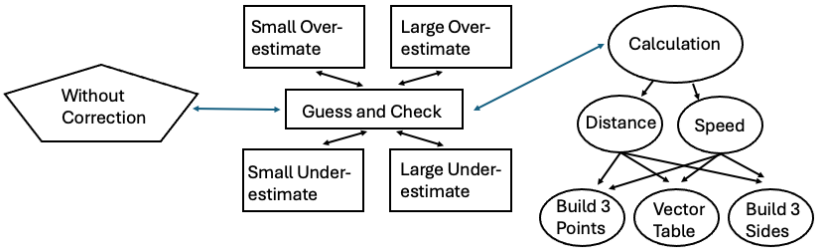


Figure 1: *Graph flow of dynamic activity, with each shape representing a series of prompts and arrows between shapes represent reactions to students' key actions.*

Methodology and Results

Undergraduate student responses to a static and dynamic version of this activity were coded for key actions they presented using either Activity Theory (sociocultural actions) or thematic analysis (cognitive actions). Three new sociocultural actions and one new series of cognitive actions were identified in student responses with the dynamic version of the activity.

Conclusions and Next Steps

Overall, the dynamic activity reflected common methods students used to solve the task. A future study will track every interaction students make with the task.

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The "Wicked" Challenge of Improving Education Outcomes

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Keywords: Wicked problems, change theory

There are certain types of challenges that are so complex, so heavily influenced by social forces and so enduring that scholars have labeled them "wicked" (Head, 2019; Ritchey, 2013). Examples of wicked challenges include addressing homelessness, reducing poverty, or mitigating climate change. It is difficult to make progress on these challenges because of the ever-changing influences they are subject to along with the differing views humans may have about their causes and about the nature of both the challenge and what a solution might look like. Such problems stand in contrast to "tame" problems where the conditions are known (and stable), where solutions to similar problems can serve as models, and where most (if not all) people would agree on criteria for knowing when a solution has been found. Most mathematics problems are "tame;" most mathematics education ones are "wicked."

Improvement of education outcomes often makes its way onto lists of wicked problems, owing to the complex, intertwined influences on teaching and learning, as well as the diverse views and priorities possessed by the large set of stakeholders involved. Members of the RUME community have generated findings that give powerful insights into how students think about and learn many ideas in the undergraduate curriculum. Findings have also revealed what instructors do when creating rich, equitable learning opportunities for students, as well as what instructors need to know and be able to do to create those opportunities. Despite these many contributions to theory and to practice, a challenge we continue to face is achieving wide-spread utilization of findings from RUME in how learning opportunities are designed and implemented. If our challenges were "tame," the history of RUME's role in mathematics education might look quite different, with clear recipes for how research directly shapes practice.

Our challenge fits the definition of a "wicked problem." Often roots or histories of wicked problems are complex and tangled (Camillus, 2008). Such problems are described differently by various stakeholders (Ritchey, 2013). As progress is made, what remains to be worked on shifts, and the wicked problem "changes with every attempt to address it" (Camillus, 2008). "Solutions are temporary, as issues keep morphing into new problems." (Manville, 2016, p. 4). And of immense importance: wicked problems are never completely solved. The very best one can hope for is continual improvement in addressing the issue (Schmida, 2020).

In the 50 years since the term "wicked" was first used to characterize problems and challenges (Rittel & Webber, 1973) a great deal of theory-focused and practice-focused work has occurred. These developments in areas such as public policy and business have helped others learn to make progress. I contend that viewing some challenges in RUME as "wicked" could enhance our capacity to both understand and to make progress on those challenges.

This poster presentation offers a tour of characteristics of wicked problems as seen in scholarship from variety of fields. These characteristics are mapped to features of challenges from the RUME community to provide insights into the nature of those challenges and to leverage what others have learned. This analysis is used to support the claim that there may be significant benefits to our community if we learn how to see some of our challenges as wicked and design our approaches to them with that view in mind.

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What Might We Learn About Leadership From Those Outside of RUME?

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Keywords: Leadership practices, wicked problems, change theory

Improving the teaching and learning of undergraduate mathematics shares many features with other substantial challenges society faces. Those other challenges include addressing climate change, reducing poverty, and others things that have been described as "wicked problems" (Head, 2019; Ritchey, 2013). These kinds of challenges are complex, subject to the ever-changing influences of society, and often viewed or defined differently by the (often numerous) groups of stakeholders. As a result, wicked problems can never be definitively "solved" and instead the goal is making progress in addressing the issues (Schmida, 2020).

Some scholars from non-education sectors have found that progress on wicked problems can be enhanced with use of particular types of leadership (Manville, 2016). These leadership practices reflect what is known about the nature of wicked problems and the approaches that supported groups to make progress on such problems. Given that one objective of the RUME community is to positively influence the teaching and learning of undergraduate mathematics, and given the wickedness of that challenge, we might want to consider what we could learn from those who have provided some insights into leadership for wicked challenges.

Recommendations include ensuring representation and authentic engagement from all stakeholder groups. It can be tempting to involve only a small group in your efforts but we are cautioned that "Those left out won't contribute their valuable knowledge, and may even sabotage the solution" (Manville, 2016, p. 5). Work in RUME includes examples of projects where collaboration between those with research expertise and those with practice-based expertise has generated powerful outcomes. For such collaborations to be of value, there must be trust among those working on the challenge. Thus building (and sustaining) trust needs to be a priority, followed by facilitating the group to some (perhaps short-term) successes to enhance trust and build capacity to take on more difficult aspects of the challenge. Those who work on challenges with long histories (e.g., enhancing equitable access to learning opportunities, designing effective professional learning for instructors) are well-aware of the ever-shifting nature of these wicked problems. Progress takes time and therefore building "on-going, adaptive learning into the process" (Manville, 2016, p. 8) is key. Data-informed processes of the sort found in design experiments and other forms of research can help build "sustainable capacity for inquiry and adaptation in light of new information and issues that inevitably arise" (Manville, 2016, p. 8).

Some of these leadership practices are evident in our change efforts. Others may not yet be part of how RUME leaders have approached our various challenges. This poster illustrates what implementation of leadership practices for wicked problems can entail in undergraduate teaching and learning change efforts. To understand the utility of these practices, the community could undertake several types of research. A meta-analysis of studies of change efforts could reveal which of the leadership practices have been utilized and to what effect. Those findings could then inform studies of new efforts to build leadership capacity and skill in key practices so those could be examined explicitly in various change efforts taken up by our community.

Participants in the poster session will be invited to share their experiences with these leadership practices and contribute to draft ideas for research studies designed to investigate the ways such practices shape change efforts in undergraduate mathematics education.

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Considering Emotions in Undergraduate Mathematics Education

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Keywords: emotional intelligence, mathematical affect, social emotional learning, mathematical intimacy

This poster aims to motivate the consideration of emotions in undergraduate mathematics education. An underexplored area of undergraduate mathematics education is the role of emotional intelligence and the related concept of social emotional learning. Emotional intelligence is a subcategory of social intelligence consisting of three competencies: recognizing and expressing emotions accurately, regulating emotions in oneself and others, and utilizing emotions to problem solve (Salovey & Mayer, 1990). This poster will review constructs such as affect, emotional intelligence, social emotional learning, and mathematical intimacy and connect these constructs to undergraduate mathematics education. In particular, these constructs can be leveraged to improve communication and reduce the impact of frustration in collaborative classrooms.

Emotions are part of mathematical affect but are understudied in undergraduate contexts as compared to other components of the affective domain, such as attitudes and beliefs (McLeod, 1992). One reason for this is that emotions are fleeting, making their study challenging (Leder & Grootenboer, 2005), although the role of emotion has been studied in other educational contexts. For example, secondary school students with emotionally intelligent skills perform better academically (Sánchez-Álvarez et al., 2020); in a high school mathematics classroom where active learning practices were implemented, Ilyas and colleagues concluded that emotional intelligence was productive for the students' learning, indicated by increased responsibility, increased focus, and skillful communication (Ilyas et al., 2018). In light of the calls for undergraduate mathematics classes to become more collaborative, I hypothesize that research focused on emotions could be beneficial to students' academic success at the undergraduate level. For example, in courses which employ inquiry oriented instruction (IOI). One of the four pillars for inquiry based mathematics education is "student collaboration for sensemaking" (Laursen & Rasmussen, 2019). The expectation of inquiry oriented classrooms is that students are sharing their thoughts to make more sense of the material (Kuster et al., 2017).

Emotional responses are also an important aspect of an individual's mathematical problem-solving process (Carlson & Bloom, 2005). Studies have documented that confronting a challenging mathematics problem can make one feel vulnerable, sometimes leading to avoidance of the subject altogether (e.g. Haase et al., 2019; DeBellis, 1998; DeBellis & Goldin, 1999). Researchers have coined terms like "math anxiety" to denote the unique, negative responses mathematics causes for some students (Ashcraft, 2002). *Mathematical intimacy* describes a relationship between internally represented mathematics and its potential connection to that student's value or sense of self. Mathematical intimacy has the potential to bring positive outcomes, but the effects of intimate betrayal can be drastic, such as if a student decides to engage and struggle through a problem but consistently gets it incorrect. There is a vulnerability associated with mathematical intimacy, so experiencing a betrayal can lead someone to develop a negative sense of self (DeBellis & Goldin, 1999). This poster will show how research exploring ways of encouraging and preserving mathematical intimacy and preventing betrayals can support increased student success and persistence in STEM.

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A Feminist Approach to Mathematical Modeling: Using Data Feminism for Task Design

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Keywords: mathematical modeling, feminism, task design

Feminist pedagogy allows for connecting "to our roots, our past, and to envision the future" (Shrewsbury, 1987, p. 6). By recognizing individuals as unique contributors within the social, cultural, and intellectual contexts, feminist pedagogy helps motivate students to learn about themselves and the diverse experiences of others (Torres, 2014). Feminism in education may involve exploration of social-justice issues (e.g. Louie et al., 2022, Gutstein, 2003) or integration of feminist theories into classroom norms (e.g. Abrahamson et al., 2020, Raven, 2014). One relevant feminist perspective is Data Feminism (D'Ignazio & Klein, 2020). The framework provides a way of thinking about data science and data ethics through considering seven core principles: examine power, challenge power, elevate emotion and embodiment, rethink binaries and hierarchies, embrace pluralism, consider context, and make labor visible (p. 17-18). Despite the invaluable nature of feminist pedagogy, the intersection of feminism and mathematics has rarely been crossed (Damarin, 2008), prompting the question: *How can feminism be incorporated into mathematics education?*

Mathematical modeling (MM) offers a viable avenue for including feminism. First, as modeling "uses mathematics to represent, analyze, make predictions" about real world phenomena (GAIMME, p. 8), students can draw on their funds of knowledge and experiences (Moll et al., 1992). This approach aligns with feminist ideal of positioning the scientist within the study (Rubel et al. 2022), enabling students to integrate new learning while modifying past understandings (Shrewsbury, 1987). Second, scholars have found that students have high interest and enjoyment with little boredom when solving real-world problems (e.g., Hartman & Schukajlow, 2021). Thus, modeling has the capacity to engage students emotionally, echoing D' I'ganzio and Klein's (2020) assertion that data is neither neutral nor objective but should instead embrace "the knowledge that comes from people as living, feeling bodies in the world" (pg. 73). Prior research has incorporated students' lived experiences (e.g., Toor & Mgombelo, 2016), emotion (e.g., Azevedo et al., 2017), and embodiment (e.g., Sinha & Malhotra, 2022) in STEM education; but more work can be done in including feminism to mathematics. Specifically, there remains a gap in understanding how Data Feminism can be integrated into MM curricula. This study will employ Data Feminism principles as a heuristic to (a) design two MM tasks and (b) discuss why the tasks are characterized as "feminist modeling." The question guiding this research is *How can feminism be incorporated into the design of mathematical modeling task?*

Our task design intentionally integrates compatible theoretical frameworks, such as the MM process diagram (e.g., Anhalt et al., 2018), Data Feminism (D' Ignazio & Klein, 2020), and Rubel et al.'s (2022) re-envisioning problem contexts. These frameworks will guide the selection of appropriate context for the tasks (e.g., personal, occupational, societal, and scientific). Tasks will be outlined and compared with "non-feminist" modeling tasks and assessed for their potential in engaging students beyond just the mathematics (e.g. personal experiences, social justice). Through highlighting Data Feminism as a design framework, we stand to help educators create an environment where students are not only equipped to "read the world with mathematics" (Gutstein, 2003, p. 44) but also to develop "mathematical power" (Gutstein, 2003, p. 45). Additionally, this study could influence how modeling tasks are designed and executed.

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How One Graduate Teaching Assistant Restructured a Course on Teaching Undergraduate Research

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Keywords: GTA development, Ungrading, Gradual release, backward design

Expectations that undergraduate students have engaged in research activities before applying to graduate school are becoming more common. Notably lacking from these expectations is the training of those undergraduate students' instructional and communication skills. The development of these skills is critical for producing a well-rounded undergraduate student who will be a successful graduate student. The Iowa Sciences Academy is part of The University of Iowa, our mission is to support the success of undergraduate students interested in research and scientific communication. In furtherance of that mission the course t*eaching your undergraduate research* has been on offer for around five years and is based on the prior work of Hoffmann and Lenoch (Hoffmann & Lenoch, 2013). The course's established goal is to provide experience in instructional design and implementation to undergraduates whose intention is to continue to graduate school. This poster describes a course redesign which focused on implementing a gradual release of responsibility model (Fisher & Frey, 2013) with ungrading assessments (Stommel, 2023) and was carried out by a Graduate Teaching Assistant (GTA).

For the redesign, the following goals were established:

1. Students can engage in teaching as a structured activity.
2. Students are confident to go rogue in their own learning.
3. Students acknowledge that failure breeds success.

Using qualitative methods, we gathered evidence for how the undergraduate students in one course section (n=7) met these three goals. Course discussions and presentations, student created artifacts, regular instructor touchpoints, and peer feedback from multiple teaching sessions all provided rich examples of students working toward these goals. Our preliminary analysis provides an overview of the ways undergraduate students met these three goals. Goal one was met in the creation of a teaching portfolio documenting student, most students selecting the instructor modeled backward design paradigm (Wiggins & McTighe, 2008). Student feedback during touchpoints indicates an increase apperception for the intricacies of instruction as well as increased confidence in students' self-identification as an instructor. Goal two seemed to be more challenging for all the students possibly because their previous success in school was contingent upon their following the rules, where going rogue was likely to have negative effects to their grades. Goal three is seen in the students own retrospective analysis. Students acknowledge failures in their teaching and improvement strategies for their materials. Ungrading, moving assessment from a rubric evaluating teaching to the conversations on the act of preparation seems to have allowed for students to decouple the failure of an aspect of their preparation from the students' failure in the course.

Overall, the gradual release and ungrading models were successful for this small group of undergraduates. Extending these ideas to other course sections would provide additional broader data that would help us disaggregate data across several subgroups of students. We will continue using these instructional strategies and look for others that will help support our three goals.

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Using Provided Guided Notes in Coordinated First-Year College Mathematics Courses

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Keywords: coordinated courses; guided notes; post-secondary mathematics

Lecture-based instruction prevails as the primary pedagogical approach used to teach undergraduate proof-based courses; thus, mathematics education researchers should investigate ways to improve the lecture rather than replace lectures (Melhuish et al., 2022). Course coordination has been successfully implemented in mathematics programs to standardize student experience (Regier et al., 2024), to meet legislative needs (Jensen-Vallin, 2013), and to shift the pedagogical approach across multiple sections of the same course (Golnabi et al., 2020; Rahman et al., 2021). Alternatively, at the individual level, instructors and students have observed positive impacts of the use of guided notes during classroom lecture (Cardetti et al., 2010; Krapf and Pfefforkorn, 2022). As the pedagogy of mathematics education evolves, universities are encountering the overlap of course coordination with the use of guided notes.

The goal of our study is to examine the use of guided notes in coordinated courses. Specifically, for the coordinated courses considered in this study, the guided notes were created by a current/former coordinator of the course and not by the instructor. This study will investigate the research question, "What is the impact of coordinated guided notes (not co-created) in a post-secondary mathematics course from both the perspective of instructor pedagogical choices and student behaviors and expectations?"

Methodology

The study will focus on first-year undergraduate mathematics courses at a public R1 university in the U.S. We will pilot the study with multiple sections of a coordinated course in Spring 2025 and plan to expand the study to gather data on additional coordinated courses in Fall 2025. The goal of the poster will be to outline the methodology and seek feedback.

To understand the coordination expected of the instructors in our study, we first will interview the course coordinators to determine what curricular materials are provided and how instructors are expected to use the provided materials. Then, we will ask the instructors about their plans to use provided guided notes while teaching their course and survey their Mathematics Teacher Self-Efficacy (Zuya et al. 2016). During the semester in which they are teaching the coordinated course, we will host group interviews of the instructors to discuss their classroom approach, focusing on their use of guided notes and their math teacher self-efficacy. Each of these components will provide insight on how and why the post-secondary math instructors use guided notes in lectures. We will investigate the degree to which instructors incorporate their mathematical insight in their lectures.

Additionally, we will observe class meetings that cover common mathematical content to assess student engagement in coordinated courses with guided notes. After the observed class meetings, the researchers will gather student note-taking artifacts to document how students record formal and informal mathematics. A questionnaire will be shared with students to obtain their perception and expectation of utilizing guided notes in the class lectures. These components will focus on the students' behaviors and expectations. Compiling the various resources will delineate the instructor-student interaction to determine how the student and instructor expectations align.

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Development and Preliminary Validation of Enrolment in Engineering and Robotic Sensing Scale (EERSS)

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This study addressed the persistent gender disparity in STEM fields, particularly in engineering and robotics, by developing a survey instrument, "Enrolment in Engineering and Robotics Sensing Scale (EERSS)." The results of the study demonstrated good psychometric properties for the EERS scale, supporting the interpretation and use of the scale for understanding and measuring factors that influence the participation of female students in STEM fields.

Keywords: Engineering, Robotics, STEM, Gender Enrolment, Psychometrics, Scale development

STEM education makes learning real by allowing students to see the connections between the content they are studying and the application of that content in relevant ways (NSTA, 2024). At the K-12 levels, the involvement of boys and girls is integral in promoting STEM education (Merayo & Ayso, 2023). Despite ongoing efforts to ensure gender inclusivity in STEM education, the enrollment of girls in engineering and robotics remains disproportionately low compared to their male counterparts (Merayo & Ayuso, 2023). Studies have highlighted various factors and barriers, including lack of exposure, students' interest, motivation, self-efficacy (Msambwa et al., 2024), perception of STEM, and role model influence (Dasgupta & Stout, 2014). Other barriers include gender stereotypes (Cheryan et al., 2017), inadequate support systems, confidence in STEM, engagement, educational policies, attitudes, and pedagogy (Ibanez, 2022; Van der Hurk et al., 2019).

Research suggests that fostering interest and confidence, ensuring a compelling and inclusive school vision, etc., in STEM subjects like engineering through early interventions can significantly increase the likelihood of pursuing engineering-related careers, especially for girls (Ibanze, 2022; Wang et al., 2023). These factors, which influence the decision of students to enroll in STEM careers, especially engineering and robotics, need to be precisely measured and understood through evidence-based strategies. Thus, the study developed and validated a survey instrument (Enrolment in Engineering and Robotic Sensing Scale-EERSS), to identify these factors that could influence females' enrolment in engineering and robotics careers. The study addressed the research question: What are the psychometric properties of EERSS?

This paper reported on several stages of survey development, such as item generation, expert validation, and pilot testing. The scale has a demographic section addressing students' grade level, gender, race, and language and engineering. The generated items were reviewed by subject experts to ensure the content and face validity of the scale. A pilot study was then conducted on a sample of 96 secondary school students to refine the instrument through exploratory factor analysis (EFA) and reliability testing. The study used the Partial Credit Model (PCM) to evaluate the psychometric properties of the EERS scale. The results all showed acceptable fit indices for the EERS scale. These findings presented empirical evidence that supports the interpretation and use of EERS scale in the measurement of students' enrolment in engineering and robotic sensing discipline.

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Exploring Impact of Task on Mathematical Sensemaking in Undergraduate Biology

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Keywords: mathematical sensemaking, biology, task

Mathematical equations are a key part of undergraduate biology instruction (Cohen, 2004). While biology is often perceived as the refuge of the math averse (Steen, 2010), students in biology classes have mathematical resources they can use to make sense of mathematical equations (Schuchardt & Schunn, 2016). Sensemaking has been defined as using pre-existing knowledge or skills to figure out a problem (Martin & Kasmer, 2009; Odden & Russ, 2019). Five types of sensemaking in mathematics have been identified (Zhao & Schuchardt, 2021). Theoretically, the tasks students engage in could affect the types of sensemaking they use (Li & Schoenfeld, 2019). The effect of different tasks on students' use of mathematical sensemaking is underexplored. **RQs:** How do different tasks on mathematical equations in biology affect the types of mathematical sensemaking, and connections between types, discussed by students?

Incoming biology majors at a large R1 university participated in a session on mathematical modeling of population growth as part of a summer orientation program. Pairs of students were recorded as they engaged in a sequence of two mathematical tasks. The first task varied in different sessions, but the second task was always the same, modifying an equation from the first task. The first task was one of the following: DM) developing a mathematical equation, CM) comparing two mathematical equations, MR) describing relationships between variables in an equation, and PS) solving problems after being shown how to do so. Four to six recorded discussions for each task were coded by two independent coders for types of mathematical sensemaking and for connections between mathematical sensemaking types (Krippendorf's alpha= .78, .76). Counts were standardized for number of groups and time on task.

Students who engaged in PS in the first task tended to focus on how to perform the steps in the problem-solving process (Math-procedure sensemaking, 67% of instances) compared to students in the other tasks (DM=29%, CM=18%, MR=44%). Students in DM and CM tasks tended to focus on equation structure (Math-structure sensemaking, DM=57%, CM=62% vs. MR=17% and PS=6%). For all groups, the second Modify tasks were more similar in their sensemaking with frequency of Math-procedure sensemaking for all tasks ranging from 23% for CM to 40% for PS, and Math-structure frequency ranging from 37% for PS to 57% for CM.

Connections between mathematical sensemaking types may enhance student problem solving and understanding (Li & Schoenfeld, 2019). Across the first tasks, connections were low, ranging from 4 to 7 average instances per group. For the second task, Modify after CM averaged only 3 instances of connections per group while the other Modify tasks averaged from 12 (after PS) to 19 instances (after MR). Across the Modify DM and CM tasks, group discussion focused on connections between Math-procedure and Math-structure sensemaking and Math-structure and Math-relation sensemaking (relationships between variables). However, for Modify MR and PS tasks, these connections occurred less often and connections between Math-procedure and Math-relation occurred frequently (49%, 66% of connections respectively).

Results suggest an impact of task on type of sensemaking that students discuss and the connections they make, suggesting that for research and instruction, it is important to align task with intended objective. Qualitative analysis is underway to provide an explanation for observed differences in patterns by determining how students are using sensemaking in the tasks.

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Connecting Mathematics and Social Justice Goals in Preservice Mathematics Teachers' Lesson Plans

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Keywords: Preservice mathematics teachers, Lesson plans, Teaching mathematics for social justice, Culturally responsive mathematics teaching

As students in K-12 classrooms in the U.S. become increasingly diverse, it is necessary to prepare preservice mathematics teachers (PSMTs) to consider their students' learning needs, understand the nature of achievement gaps, and foster the development of their mathematics identity (De Freitas, 2008; Dunn, 2005; Koestler, 2012; Register et al., 2022; White, 2002). Designing and implementing mathematics lessons based on equity-based mathematics teaching practices can help PSMTs support their culturally and linguistically diverse students (Moldavan & Gonzalez, 2023). In particular, researchers (Aguirre & Zavala, 2013; Gay, 2000; Ladson-Billings, 1995) underscore the importance of attending to students' funds of knowledge and teaching mathematics for social justice in order to promote academic prowess and equity for marginalized students. Thus, it is important to understand how PSMTs connect mathematics to social justice (SJ) goals when designing lessons. The research question that guides this study is: *How do preservice mathematics teachers connect mathematics and social justice goals when designing lesson plans?*

Aguirre and Zavala's (2013) theoretical and analytic framework – culturally responsive mathematics teaching (CRMT) – forms the basis of this study to investigate how PSMTs make connections between mathematics and SJ goals in their lesson plans. CRMT informs PSMTs by helping them understand equity-based teaching practices and how to effectively teach culturally, ethnically, linguistically, and socio-economically diverse students. Thus, CRMT is useful to analyze teacher practices in their lesson plans to see if these plans meet diverse students' learning needs.

In this study, we collected and analyzed a total of 41 lesson plans designed by PSMTs in a middle school methods course throughout five semesters (Fall 2021–Spring 2023). In the lesson plans, PSMTs were asked to include both mathematics and SJ goals. PSMTs were particularly asked to choose SJ standards in a framework for anti-bias education (Learning for Justice, 2022). By using open coding, we analyzed which mathematical topics and SJ standards are frequently used as well as the relationship between them.

As a result, we found that the three most frequent mathematical topics are graphs, rational numbers, and linear equations; and the most frequent SJ standard is Justice 12 (Students will recognize unfairness on the individual level and injustice at the systemic level). Our findings imply that graphs, rational numbers, and linear equations are useful to express and recognize social issues (e.g., unfairness and injustice). Supporting PSMTs in connecting mathematics to SJ goals can help them learn about social issues themselves and equip PSMTs to support their future students' learning of critical issues.

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Assessment Design Promotes Peer Learning in a Calculus Course for Engineering Students

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Keywords: Peer learning; Assessment; Community of Practice; Student engagement

This study reports on how students experience peer learning in a first-year engineering mathematics course with portfolio assessment. The course has recently changed the assessment form from a high stake, closed book exam at the end of the semester, to a portfolio exam. An analysis of the traditional approach to teaching mathematics, with large lectures, problem solving sessions, and a high stake, closed book exam at the end of the semester, shows that students participate passively in their own learning (Zakariya et al., 2022). Students reports that they receive little feedback, and the course suffers a high failure rate (up to 40%). Several studies, such as Freeman et al. (2014) claims that students in STEM subjects perform better in classes with active learning methods compared to traditional lectures. Biggs et al. (2007) argue that all types of students will benefit from active learning methods. It is what activities the students do that is important for their learning. The term "active learning" can describe many learning activities. Here we will define peer learning as active learning. Peer learning is when students form study groups and work together with mutual engagement and shared responsibility for learning. We describe these groups using Community of Practice (CoP) framework from Wenger (1998).

This study analyzes data collected from the course's midterm evaluation, a standardized survey administered and anonymized by the university. The survey comprises 17 Likert-scale questions (5-point scale) and 4 open-ended questions. Of the 447 students enrolled in the course, 194 (43.4%) completed the survey. The analysis incorporates mixed approaches.

The course assessment consists of 4 digital tests and a written assignment. The digital tests were closed-book exams, utilized the STACK assessment tool (*STACK*, 2024) to generate randomized questions. Students could attempt each test up to four times, with their highest score contributing to their final grade. During these tests, students solved problems manually on paper and submitted answers digitally, receiving immediate feedback on correctness.

After the test we observe that students gathered in groups to discuss their solutions, suggesting that peer learning played a significant role in their preparation for subsequent attempts. On the 5-point Likert question "Academic collaboration among students (in groups or study circles) contributes to a good learning environment" 117 (60%) of the students answered, "Strongly agree" (5) and 59 (30%) students answered "agree" (4). While the course retains traditional instructional components, such as large lectures and problem-solving sessions, we suggest that the new assessment format foster peer learning. Students report frequently using the university's mathematics support center, further indicating a strong reliance on peer and institutional support networks. Peer learning is particularly valued for its contributions to understanding complex concepts, maintaining motivation, and building a collaborative learning culture. Observations suggest that students naturally form Communities of Practice because of the digital tests, using these interactions to enhance their learning outcomes.

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"Trust me, I am an Expert!": Combinatorial Problem Solving and Thinking Processes in Researchers With High Mathematical Preparation

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Keywords: Combinatorics, Problem Solving, Experts, Combinatorial Thinking

Combinatorics plays a key role in mathematics and its teaching: on the one hand, it represents a core in discrete mathematics while, on the other hand, in the school curriculum, it plays a role in the development of topics both in mathematics and in other disciplines (Batanero et al., 1994), offering students the opportunity to develop and train problem solving skills in an environment that does not rely on specific mathematical prerequisites (Kapur, 1970). In this sense, the research literature presents several studies that focus on the problem solving processes developed by students from elementary school (e.g., English, 1991) to university (e.g., Lockwood & Gibson, 2016). We note, however, that most of the research has focused on solvers without high combinatorial expertise. With our work, we propose a further step in the development of this line of research by focusing on combinatorial problem solving processes as developed by what we call experts, who in this work are to be considered as academics whose field of research and/or expertise is or includes combinatorics (e.g., statistics, probability, etc.).

Framing our research within Schoenfeld's model of problem solving (1985), the focus of our work is on analyzing the different phases that occur during the resolution of some combinatorial tasks, paying particular attention to the heuristics involved (based on the comprehensive description presented in Lamanna et al., 2022) and the dynamics that develop across the dimensions of combinatorial reasoning (formulas/expressions, set of outcomes, counting processes - Lockwood, 2013). We proposed an elementary task (i.e., solvable with a single application of a combinatorial formula) and a compound non-standard task (i.e., a task for which a closed direct solution formula does not exist) to observe how experts approach the two tasks and if/how they change and adapt their approaches between the two tasks. In summary, our research question is: Which heuristics and dimensions of combinatorial reasoning are involved by a group of experts when solving an elementary and a non-standard combinatorial task?

To answer our research question, we involved five experts in individual interviews in which they were proposed two combinatorial tasks to solve in a think-aloud process explaining each step of their reasoning. The results showed that in the case of the elementary task, experts tend to provide a solution that includes heuristics for direct counting of all configurations, thus involving only two of the three dimensions of combinatorial thinking and often skipping the problem solving phases of exploration of validation. On the other hand, when dealing with a non-standard task, experts rely on a broader set of heuristics (in particular involving enumerative processes), thus shifting the dynamics of their thinking towards the inclusion of all dimensions of Lockwood's model – which are often involved in the problem solving phases that were skipped in the solution of the elementary task. Comparing the two solving processes, we observe that while there are cases in which the experts seem to use efficient counting techniques, there are also situations in which they demonstrate an expertise that is built on a base that includes a variety of different heuristics – including enumerative processes that are often discouraged, especially in the case of high school combinatorics (Lamanna & Branchetti, 2024). We observe that the results of this work provide some key insights on the learning objectives that one should pursue to enhance a structured development of students' combinatorial thinking.

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Qualitative Themes for Mathematics Capital: Investigations of Graduates with Dual Majors in Mathematics and Mathematics Education

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Keywords: Mathematics Capital, Science Capital, Bourdieu, Undergraduate Mathematics

We previously defined mathematics capital as "mathematical assets that actively influence a student's capability or power in the field of mathematics education and how the student utilizes these assets to exert power in the field of mathematics education" (Anderson et al., 2024, p. 494) through quantitative work. Here, we aim to operationalize mathematics capital through qualitative work via thematic descriptions, as Bourdieu did in many of his works in capital (Bourdieu, 1977; Bourdieu & Wacquant, 1992). We answer the research question: *How can mathematics capital be qualitatively described?*

Using a Bourdieusian lens (Bourdieu, 1977; Bourdieu & Wacquant, 1992), we view the components of mathematics education as intertwined. Mathematics capital acts as a bridge between other constructs like social factors, cultural capital, emotions, personal experiences to mathematics education (Anderson, 2024, cf., Bourdieu, 1984). In past work, Archer and colleagues (2014, 2015) used a Bourdieusian lens (1977, 1984) to develop a three-part model for science capital. We aim to continue to extend Bourdieu's notion of capital into the field of mathematics education (Anderson et al., 2024, cf. Archer et al., 2014, 2015, Dewitt et al., 2016).

We conducted semi-structured interviews with six recent graduates receiving dual degrees in mathematics and mathematics education. Then, we performed qualitative analyses using a combination of deductive and inductive coding (Miles et al., 2014) where we developed three overarching themes for mathematics capital (through mutual coding consensus).

We take this thematic approach to redefine mathematics capital to deepen our understanding and to clearly communicate its impact on mathematics education. Theme names and summarized definitions for each are: (a) compounded mathematical achievements *(i.e., valued achievements in mathematics (formal and informal settings) altering responses to mathematics)* (b) productive disposition towards mathematics, *(i.e., a growth mindset, a positive, intrinsic relationship with mathematics, and belief that mathematics is meaningful)* and (c) expected mathematical achievements, *(i.e., expected mathematical success and a nonchalant disposition towards mathematics success*.

We developed three primary themes that characterize and describe mathematics capital. In these, we find appropriate amendments to redefine mathematics capital as "mathematical assets and expectations that influence a student's capacity and power in the field of mathematics education, and the disposition that a student assumes to utilize these mathematical assets and expectations and exert power in the field of mathematics education." In short, these themes help capture the essence of mathematics capital, which extends throughout a person's lifetime and influences how a person *interacts* with, *reacts* to, and *responds* to mathematics and their potential mathematical outcomes. These *interactions*, *reactions,* and *responses* can be intrinsic or extrinsic (i.e., what a person believes or how a person close to another may impose their beliefs on another) and have far reaching implications for mathematics learning.

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Designing Games For Approaches To College Mathematics

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Keywords: Card games, Incoming math students, Order of operations, Functions

With mathematics being a singular obstacle to college retention and degree attainment (Douglas & Salzman, 2019), it becomes especially important that students who struggle to place into first-year mathematics courses are provided a strong foundation. We are developing a new class entitled Approaches to College Mathematics (ACM) for students who failed to place into Precalculus. The course developers recognized that the complex system that contributed to these students' struggles wasn't working. We thus spent the summer developing a student-centered ACM and deliberately hired instructors with philosophies supporting the interwoven nature of conceptual and procedural understanding (NRC, 2001). We hoped that this contrast with widespread, teacher-centered, and strictly procedural approaches (Cohen & Mehta, 2017) would provide the missing connections needed to bolster these students' foundations for Precalculus.

Of the 820 students taking ACM and Precalculus, 333 placed into ACM. This poster focuses on 57 of those students. All these students experienced a pandemic in their freshman year of high school. Preliminary surveys indicate that 42 students are interested in Biology; 16 characterize mathematics in strictly negative terms ("stressful," "hard"), while 10 invoked both positive *and* negative attributes ("difficult, complicated, useful"). In their conceptions of a "math person," 21 students relayed attributes focused on intelligence ("someone who is smart"), while 15 recognized a math person's joy in doing mathematics ("enjoys the problem-solving process"). Survey responses indicate the challenge of connecting hard work (not innate intelligence) with mathematics learning, and the promise of playful experiences in enhancing that learning.

Mathematical play is a crucial part of learning and doing mathematics. Encouraging students to play with mathematical ideas can help them to extend their horizons, make their own conjectures and connections, and build their confidence in an environment where mistakes do not feel so threatening (Holton et al., 2001). As a structured form of play, games can be designed or modified to reflect mathematical structures and then used in undergraduate mathematics classrooms to support students in playing with those ideas (Naik, 2017). For example, Herman (2014) adapted Trivial Pursuit for use in mathematics classes for business majors. Students who played the modified game participated more in class discussions, seemed less afraid of asking questions, reported enjoyment, and showed improvement on test scores. Similarly, Galarza (2017) designed a new card game called Groups as a learning tool for undergraduates studying group theory. After playing Groups, students commented on its accessible and enjoyable format, its ability to offer insight into rich mathematics, its use as a learning tool for novices, and its use as both a learning and a review tool for experts. These examples highlight the potential of two different game design strategies: (1) modifying existing games to map mathematical ideas onto their existing rules and (2) designing entirely new games that reflect mathematical ideas.

In our poster we present two games we designed for ACM, one using each of these two strategies. Functionominoes is a modified dominoes game based on categories and features of functions, while the Order of Operations (OOPS) Game is a new card game about building and evaluating expressions. The poster will feature a synopsis of the value of games to math learning, a description of two ACM classes and students, an overview of our two game designs, examples of our students' work with our games, and links to our game printables and activity guides.

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From Theory to Practice: Examining Pedagogical Development in Graduate Mathematics TA Seminars

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Keywords: Graduate Mathematics Teaching Assistants, Teaching Assistant Training

Graduate mathematics teaching assistants (GMTAs) typically have a great deal of the responsibility for the instruction of undergraduates but often have little opportunity to explore teaching methods beyond those they have personally experienced or seen demonstrated by their faculty mentors (Speer, Gutmann, & Murphy, 2005). Most students arrive to their graduate programs with little to no teaching experience (Speer et al., 2005). With little formal training and limited guidance, GMTAs are expected to help undergraduates bridge the gap between high school and college-level mathematics while managing the heavy demands of their own graduate coursework (Belnap, 2005). This unique challenge highlights a need for structured and supportive TA preparation that will help GMTAs be successful teachers and graduate students.

Through this study, we aim to add to the knowledge of GMTA preparation by examining how one training seminar prepares GMTAs for their teaching responsibilities. We conducted a case study on a teaching seminar course for beginning GMTAs at a large, public, research university in the Southern United States. Our data collection methods included observations of the seminar, semi-structured interviews with the instructor and GMTAs, and document analysis of instructor observations of GMTAs and course materials, which were triangulated to ensure reliability (Stake, 1995). This study aimed to answer the overarching research question: *How does a graduate mathematics teaching seminar shape GMTAs' development as educators, and how does it compare to established TA training frameworks?*

This study was framed using an adapted version of Justice's (2020) framework for preparing GTAs to teach statistics. This framework emphasizes community-building, challenging teaching beliefs, and developing teaching knowledge. Selected for its STEM focus and alignment with the literature, the framework was modified to fit mathematics better and emphasize the importance of cultural understanding and inclusivity in teacher training (Felton-Koestler, 2020).

Analyzing our data through this lens, we found that the GMTA seminar provided opportunities for pedagogical growth through collaboration, approximations of practice, and technology training. The most valuable aspect, according to both GMTAs and the instructor, was the gradual shift toward legitimate peripheral participation. However, a significant challenge was the persistence of pre-existing beliefs, which limited the seminar's overall impact. GMTAs also expressed a desire for more informal spaces to discuss teaching, where they could share experiences, ask questions, and exchange advice. Additionally, some felt that the seminar's structure did not fully reflect the instructional strategies it aimed to promote, leading to frustration with the disconnect between its design and its goals. Addressing these concerns could strengthen the seminar's overall effectiveness.

The goal of our poster presentation is to engage RUME conference participants in discussing how we can support GMTAs in their dual roles as educators and students. The findings from this study have the potential to inform the (re)development of professional learning opportunities for incoming GMTAs and shape future research on how these professional learning opportunities impact GMTAs' development as educators.

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Homework Resubmissions: What is the Impact on College Students' Exam Scores?

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Keywords: College Algebra, Exam, Undergraduate Students

Homework is a debated issue in educational research. Many researchers have explored aspects of homework including: effort, amount, justification, and value (Núñez et al., 2014; Cooper et al., 2006; Xu, 2023; Fan et al., 2017; Epstein, 2023). This research aims to assess the different qualities of this common practice for both students and teachers. One area of homework research that currently contains few studies is homework resubmissions. Homework resubmissions have been researched in different disciplines, countries, and age groups (e. g. Zhu & Leung, 2012); this study addresses the gap in research on homework resubmissions and exam scores in undergraduate mathematics in the United States. In this study we address the research question: how are undergraduate College Algebra students' homework resubmission patterns influencing their exam scores?

Data Collection and Analysis

We collected 141 undergraduate College Algebra students' homework scores and exam scores during one semester. Each exam had a number of associated homeworks (HWs): four HWs for Exam 1, three HWs for Exam 2, three HWs for Exam 3, and all 12 HWs for the Final. For each exam, students' HW resubmission patterns were ranked into five groups. Two examples of rankings are as follows: (1) students who resubmitted their HWs and improved their scores were ranked above students who resubmitted and did not improve their score, (2) students who submitted their homework and received perfect scores were ranked above students who resubmitted and improved their HW scores. An ANOVA was conducted to compare students' HW resubmission patterns with their exam scores.

Results and Conclusions

We report here specific results from Exam 2. Two significant results were found from our analysis of students' HW resubmission patterns. First, there was a significant difference in exam scores among the HW resubmission patterns ($F(4, 136) = 12.394$, $p<0.001$). Tukey's HSD post hoc test results revealed that the students who resubmitted HWs earned exam scores that were not significantly lower (M=95.24, SD=4.07) compared to students who received perfect HW scores (M=96.35, SD=4.6). Second, Tukey's HSD post hoc test results revealed that students who did not submit or not resubmit their HW earned exam scores significantly lower (M=61.6, SD=15.59; M=76.94, SD=17.16) compared to students who resubmitted or received perfect scores (M=95.24, SD=4.07).

These results suggest that homework resubmissions are beneficial for student learning. The positive relationship between homework resubmissions patterns and exam scores supports the inclusion of homework resubmission in more undergraduate math classes. We hope our results inspire more research into homework resubmissions at the undergraduate level and more opportunities for students to learn content through the resubmission process.

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Community College Mathematics Instructors' Conceptions of Equity

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Keywords: equity, community college instructors, professional development

Active learning has been shown to improve student success in STEM courses (e.g., Freeman et al., 2014; Theobold et al., 2020). However, research also shows how such instruction can have inequitable impacts on the student experience and subsequent success (Reinholz et al., 2022). Thus the implementation of active teaching practices needs to be grounded in equity. A goal of this project is to implement a professional development (PD) program for community college introductory mathematics instructors. As part of that work, three rounds of interviews were conducted with instructor participants at a Hispanic-Serving two-year college (11 in fall; 8 in spring). Instructors were asked about their conceptions of equity as related to teaching during these interviews. In this poster, I will share some common themes from four instructors' conceptions of equity and how those conceptions evolved over the course of a year while engaging in PD centered on active, equitable, and inclusive teaching.

Results

Table 1. Four teachers' initial views of equity, final views of equity, and equity goals.

Instructor	Initial Views	Final Views	Goals
Paul	Equity is providing the right amount of support to each person.	Inquiry can be leveraged to improve equity within a class.	Wants to transfer lessons about gender equity into racial or ethnic equity.
Stella	Considering and attending to individual needs	New Perception of student needs	None
Sandra	Providing many forms of support to increase meaningful engagement and a sense of belonging	Considers race and ethnicity as factors is participation and sees a need to be flexible	To close the achievement gap within her class and increase the passing rate
Sun	Provide students with the means to succeed based on their individual needs	Give females a stronger voice and make effort to gain insight into students	Create class environment for students to fully engage and feel welcomed

Instructors discussed various teacher moves to support these views and goals that included purposefully creating mixed demographic groups, increasing wait time, allowing for multiple forms of participation, and increasing access to resources.

Conclusion

With a better understanding of how instructors think about equity, the project team is better positioned to meet the instructors where they are in terms of equity-focused PD. We can strengthen the connection between equity and teaching practices, for example, like making the connection for Stella that wait time can be a useful tool for attending to equity. Lastly, we can celebrate small steps toward equity. Equity does not have to mean big changes at the start. Rather, progress towards achieving classroom equity can be an ongoing process.

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Mathematics and Science Sensemaking While Developing Equations Representing the North Atlantic Current

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Keywords: Mathematical Models, Integrated STEM, Environmental Science

Global warming is often believed to be synonymous with climate change, a misconception which leads individuals to think that climate change entails that every region on Earth warms in a similar fashion (Renert, 2011). However, the North Atlantic Current, for instance, can slow due to rising temperatures and freshwater influxes from melting ice, limiting the flow of warmer water traveling to the polar regions (Stommel, 1961). With less warmer water traveling to polar regions, Earth's polar regions become even colder as the equatorial regions remain warm. Such dynamics of the North Atlantic Current can be represented with mathematical equations that carry scientific meaning. Thus, working with such mathematical equations provides opportunities to deepen students' understanding of mathematics and science.

I studied how college algebra students made sense of the mathematics and science behind a two-box system of the North Atlantic Current. This model accounts for how density differences, determined by the salinity level and temperature of the ocean water, affect the rate of flow of water between the polar and equatorial boxes of the two-box model. Typically, mathematical models representing these dynamics consist of a system of ordinary differential equations, which is typically learned after taking the college calculus series (Kaper & Engler, 2011). Instead, I made these concepts accessible to students who have not taken calculus by requiring students to use recursive equations, which primarily relied on their algebra knowledge.

I analyzed students' *sensemaking*—i.e., making meaning of new situations based on prior experiences and knowledge (Odden & Russ, 2019)—as they developed these equations over the course of three 75-minute class periods. More specifically, I coded instances of students' sensemaking from transcripts of student discourse as they worked on the mathematical model, according to the *Sci-Math Sensemaking Framework* (Zhao & Schuchardt, 2021). Preliminarily, I have found that students used a variety of mathematics and science sensemaking types to justify the choices they made in the mathematical structure of their equations. Students used science mechanisms such as heat transfer to justify the structure of equations, as well as making qualitative predictions to describe what they think will occur in the system over time and how this is reflected in their equations. Students also discussed mathematical concepts related to limits and changing rates of change. Overall, students made sense of mathematical relationships and concepts, alongside justifications from science, to translate the complexities of the North Atlantic Current into mathematical equations.

Students also reflected on the broader implications of their mathematical model in the real world. When considering the impacts of a slowed North Atlantic Current, students reflected on the multitude of significant impacts this would have on humans and other living organisms. Some students also recognized that models of the real world are imperfect and require assumptions and simplifications. Thus, not only did students make sense of the mathematics and science behind the mathematical model, but they also gained insight into the limitations of mathematical models, as well as a deeper understanding of the role of ocean circulation on Earth's climate, ecosystems, and society.

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Mathematics Credit Loss Upon Transfer: The Need for Specific Research

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Keywords: community colleges, credit transfer, vertical transfer, gateway math

Introduction

Mathematics courses are often regarded as gatekeeper courses for not only STEM fields, but degree attainment in general (Kopparla, 2019). Entry-level or developmental mathematics courses in particular have high rates of repetition or failure across institutions (Fink et al., 2018; Schudde & Keisler, 2019; Zientek et al., 2020; Snead et al., 2022); in the case of students who begin at two-year institutions, this effect is more pronounced, as their mathematics courses must be transferred from the two-year institution to a four-year institution in order to earn their baccalaureate degree (Hodara et al., 2017). This type of transfer, called *vertical transfer*, has been the subject of limited study (Doyle, 2006; Bicak, 2023). We hypothesize that it is likely that losing credit for first- and second-year mathematics courses during vertical transfer is likely to have a similar effect on student outcomes as repeating a course for other reasons, and echo the call of Bicak and colleagues (2023) to urge that specific research on the vertical transfer of mathematics credit is needed.

Review

Vertical transfer of credit is the process by which a student completes work at a two-year institution and transfers to a four-year institution to complete their baccalaureate degree (Chase 2010). Among community college students, there are major inefficiencies in the credit transfer process; this is associated with delays in degree attainment (Doyle, 2006; Monaghan & Attewell, 2015), decrease in persistence to degree (Monaghan & Attewell, 2015; Hodara et al. 2017), and an increase in financial commitment to the degree (Fink et al., 2018; Sandoval-Lucero et al., 2024). Vertical transfer rate has been associated with successful completion of mathematics courses (Zhang 2019), and repetition of entry-level mathematics courses has been shown to be a contributor to attrition from STEM degrees or from higher education altogether (Kopparla, 2019; Bicak et al. 2023).

Academic study of vertical transfer tends to focus on snapshots of particular transfer programs at particular times and tend to measure success as the percentage of credits transferred for any kind of credit at the receiving institution (Doyle, 2006; Giani, 2019), due in part to limited data set availability (Giani, 2019). Credit loss is attributable to many things, including grade requirements, coursework content, program applicability, or state-by-state differences (GAO, 2017). A study done within the CUNY system found that approximately 47% of transfer students lost credits to vertical transfer, with an average of between 10 and 13 credits for each student (CUNY Office of Policy Research, 2022), while a similar study in Texas approximated that students lost approximately 12 credits upon transfer (Bicak et al., 2023). Monaghan and Attewell (2015) used national data to estimate that approximately a third of transferring students lost between 10% and 89% of credits upon transfer. Mathematics coursework, especially in entry-level courses, functions as a predictor of degree attainment and time to completion and excess credit accumulation (Fink et al., 2018; Schudde & Keisler, 2019; Bicak et al., 2023). Specific studies of the vertical transfer process for mathematics credits are necessary as a better understanding of the process will likely result in improvements of transfer student outcomes.

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Department Culture and Math Graduate Teaching Assistant Professional Development: A Tale of Two Institutions

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Keywords: graduate teaching assistants, professional development, departmental change

Mathematics Graduate Teaching Assistants (MGTAs) play a crucial role in undergraduate mathematics education, often providing instruction in smaller settings (e.g., recitation sections) where they can directly impact student learning and experiences (Selinski & Milbourne, 2015; Smith et al., 2021). While professional development (PD) is necessary to enhance MGTA teaching effectiveness, PD varies widely across institutions (Deshler et al., 2015).

As part of the larger ELITE PD program focused on MGTA PD in active learning, equity, and inclusion, we analyzed 11 interviews with department leaders (e.g., chairs, course coordinators, PD providers, etc.) from two participating institutions (Institutions Epsilon and Gamma) during the 2021-2022 academic year. Our goal was to investigate what supports or inhibits change when implementing the ELITE PD program. Using Reinholz and Apkarian's (2018) adaptation of Bolman and Deal's (2017) four frames (power, people, structures, and symbols) as a lens through which to examine each institution's culture, we identified five distinct areas (agenda setting and authority, autonomy and constraints, expertise and resources, collaboration and alliances, and institutional and departmental initiatives) that highlight the relationships between the frames within each department. For this poster, we focus on the relationships between the structural and symbolic frames, as symbols fundamentally influence how structures are implemented (Reinholz & Apkarian, 2018).

Our results revealed distinct approaches to MGTA development and involvement at the two institutions. Institution Epsilon emphasized symbols of teaching excellence and structured community, implementing hierarchical agenda setting through mandatory coordination meetings and PD focused on active learning, reflecting autonomy constraints. MGTAs participated in orientation, PD, and received support from experienced peer leaders, reflecting structured expertise and resource sharing. In contrast, Institution Gamma symbolic value of individual autonomy was reflected in informal coordination and optional coordination meeting attendance for MGTAs. While both symbolically valued active learning and equity, Institution Gamma's structures allowed more flexibility in their interpretation, positioning MGTAs as developing professionals through a required teaching and career readiness seminar. These contrasting approaches suggest different pathways for implementing the ELITE PD program. Understanding these institutional differences is crucial for effective PD implementation and fostering sustainable change. Future work will investigate how the ELITE PD program was taken up at each and its impact on MGTAs' teaching practices and institutional beliefs. A third institution in the ELITE PD program will also be included for further analysis and comparison.

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Unnecessarily Hard: How Student Emotions, Course Components and Rigor Collide in Calculus

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Keywords: mathematical rigor, student emotions, calculus, case study, focus groups

Rigor is considered an important component of successful Calculus courses (Rasmussen et al., 2014). The authors provide a definition of rigor adapted from the Dana Center at UT Austin (2019): In a rigorous math classroom, students (a) use mathematical language to communicate their work with clarity and precision; (b) demonstrate that what they have done works within the appropriate context; (c) answer the question, "How do we know?"; and (d) apply their understanding to new contexts and situations. In addition, student emotions are connected to mathematical achievement (Pekrun et al., 2017). Confidence and anxiety play important roles in student success and persistence (Ellis et al. 2016; Barroso et al., 2021).

In this poster, we present a case study of Kappa University (KU), a private institution in the Southeastern United States. Our case study focuses on students' experiences of their introductory calculus courses, which they often describe as "unnecessarily hard." There may be a disparity at KU between what instructors and students perceive as necessary for student learning. An investigation of the interactions between student emotions, course components, and instructor conceptions of rigor could help instructors better support students in rigorous courses. We thus chose to investigate the question: How do classroom practices associated with the instructor ideal of rigor align with students' emotional experiences in introductory math at KU?

To perform this investigation, a team of faculty, undergraduates, and graduate students at KU formed a Networked Improvement Community (NIC) as part of a broader multi-institutional DEI initiative. NIC members conducted 4 hour-long focus groups of 5-8 students each (for a total of 26 student perspectives). All but one participant had taken a single-variable calculus course at KU. The questions asked included, "How did you experience the rigor or challenge of your course? Do you feel like you were challenged appropriately?"

We leveraged Braun et al.'s (2023) reflexive thematic analysis process, which began with open coding. Codes were collected into three categories: course components, student emotions, and rigor. While the method of data collection makes frequency analysis limited in scope, we studied the relative frequency of coincident codes. Preliminary analysis shows that, with access to sufficient resources, students can feel supported in working on challenging novel problems. For example, one student described, "*[KU] does give you harder problems, like harder versions of the same concept… and they don't necessarily give you exactly what you need to figure it out… I think that's intentional, so that you like, learn to figure it out, learn to be able to do a problem at that level, so that you could eventually do every and any problem… But I think, yeah, the fact that they have the help room there… made it definitely not as bad as I was expecting.*"

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Digital Gestures in a Math Game

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Keywords: Fractions, Video Games, Gestures, Gestures as Simulated Actions

Objective

Autumn Hostetter and Martha Alibali (2019) framework called *Gestures as Simulated Action* (GSA) suggests that people's cognitive processes trigger gestures. It does not yet account for digital interactions and environments. In 2011, Alibali et al. conducted research examining the role of limiting physical gestures on reasoning. When playing a game to develop math knowledge, players have a digital world and a controller that all play a role in the gesturing process. The primary research question of this poster is: How does inhibition in the digital space work when a player is reasoning in a game designed to improve fraction knowledge?

The brain is the intersection of reason, perception, and movement. Nemirovsky et al. (2012) asserted that the "understanding of mathematics is expressed in and constituted by perceptuo-motor activity, and learning of mathematical ideas is shaped in non-deterministic ways" (p. 288). Math is often understood as grounded metaphor (Lakoff & Nunez, 2000) Videogame levels can be grounded metaphors for participants to experience virtually to support their fraction knowledge development.

Methodology

This study took place at a large university located on the East Coast. The participants were four soon-to-be first-year students at the university who were enrolled in a remedial summer program. Students played Mario Maker 2 levels designed to evoke the actions in fraction number schemes. Students were asked replay the levels and to make connections between game levels and mathematical content of a fraction worksheet given that day. These replay interviews were transcribed using gesture-enhanced transcription methods. This study used a thematic analysis.

Results

There were three different categories of gestures that occurred. The most common types of gestures were with one hand on the controller using the other free hand to gesture and as a pointing device, and with no hands on the controller and gesturing with both hands. There was also one player who kept two hands on the controller and used the avatar to gesture within the game, including slowing down the avatar to focus on parts of the screen (displaying the game) and making the character jump/move as a pointing device. The controller affected student play and gestures.

The act of play and the digital gestures associated with making connections can be in conflict with one another. It appears that players who see the act of play as incompatible with the digital gestures of connection-making physically gesture outside of the game, whereas players who see the act of play as compatible with such gestures digitally gesture with the avatar. We need studies to highlight what exactly is the difference in connection strategies between digital avatar gestures and physical gestures. Designers of educational games need to actively consider the player avatar movement in their design.

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Undergraduate Students' Conceptions of the Function of Mathematical Definitions in Proving

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Keywords: Proof, mathematical definitions, definition-function, meta-mathematical aspects

In mathematics education, researchers have attempted to elaborate on many meta-mathematical aspects of mathematical definitions by exploring novice and expert conceptions of the function of definitions in proving (Edwards & Ward, 2004, 2008; Gilboa et al., 2023; Parameswaran, 2010; Torkildsen et al., 2023; Zaslavsky & Shir, 2005). We define *students' conceptions of the function of mathematical definitions* in proving as the work a mathematical definition does for a student during proving (i.e. what a student uses or believes they can use a definition for during the proving process). While studies have suggested that novice provers (undergraduates) may have differing conceptions from expert provers (graduates or mathematicians) about the function of mathematical definitions in proving (Edwards & Ward, 2004; Zaslavsky & Shir, 2005), these studies are almost entirely observational, interpreted through the lens of the proof educator/researcher (Gilboa et al., 2023; Haj-Yahya et al., 2023; Tabach & Nachlieli, 2015). To date, there exists a paucity of research characterizing students' conceptions of the function of mathematical definitions in their own words: the language of the novice prover. In this project, we sought to answer the question: What conceptions do novice provers have about the function of definitions in mathematical proving?

Methods

This study was conducted in two introduction-to-proof (ITP) courses at two primarily undergraduate universities in the Eastern United States. Written reflection data were collected from a total of 26 undergraduate students enrolled in the ITP courses at two different points in the semester. Following in-class collaborative proving activities relating to logical equivalence and rational numbers, participants were prompted to write a response to, "What function did the mathematical definition for [topic] serve in your proof construction attempts?" Responses to this question were first open coded (Saldaña, 2016) and then open codes were sorted into thematic categories. Finally we deductively coded the data with these categories. All data were coded by both authors, and we discussed any disagreements until 100% consensus was met.

Findings

Through our deductive analysis we identified ten different conceptions that students held regarding the function of mathematical definitions in proving. The conception most frequently cited by participants (n=19, 73.1% of reflections) involved how definitions provided something operable within their proving. For example, Shawn wrote, "the definition was our cornerstone[...] it was about manipulating and applying them to form a coherent and sound proof." We will include a complete summary of the findings in our poster presentation, including examples of students' conceptions from each theme. Through improving the RUME community's knowledge of students' conceptions about the function of definitions in their proofs, this project can help proof instructors and researchers leverage students' conceptions into their development as authentic provers. During the poster session, we will solicit feedback for how to continue this investigation and questions about students' conceptions of the function of mathematical definitions that the RUME community would like to see answered in the future.

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Factors Affecting Students' Mathematical Affect: Layered Analysis

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Keywords: quantitative methods, mathematics affect, student experience, precalculus, calculus

Introductory mathematics courses are a critical juncture in undergraduate students' STEM trajectories. Unfortunately, these courses often erode students' *mathematical affect* – their attitudes and beliefs about their relationship to mathematics – reducing success and persistence (Ellis et al., 2016; McLeod, 1992; Rainey et al., 2018). We set out to better understand, at scale, what aspects of students' experiences influence changes in students' mathematical affect while enrolled in introductory courses by answering the following research questions: (1) What factors influence students' reported change in mathematics affect across a term? (2) What influences students' *reports of those factors*, when in the same classroom?

Data for this poster comes from a large set of precalculus and calculus student responses from 21 universities. The survey covered students' experience in their course, academic and personal identity, beliefs about themselves and mathematics, and more (Apkarian et al., 2019). Of particular relevance to this study, these included both retrospective and introspective questions about students' mathematics affect, which were compared to create the composite variable *change in math affect* (ΔMA); items about instructional practices were transformed using principle component analysis and oblimin rotation into composite factors (Jolliffe & Cadima, 2016). We leveraged *least absolute shrinkage and selection operator* (LASSO) regression analysis methods to answer both research questions. We used 10-fold cross-validation to train the model on 80% of the, reserving 20% for testing. LASSO creates a parsimonious, interpretable model by penalizing the absolute size of regression coefficients, dropping less important models out of the model by forcing their coefficients to zero. We select the model with the fewest variables following the "one-standard-error rule," a conservative approach which further reduces overfitting (Hastie et al., 2009).

To answer RQ1, we employ LASSO with the outcome variable ΔMA and 15 input variables, including demographic characteristics and reports of classroom practices (N=22,923). In this model (MAE=0.74; HL=0.45; R^2=0.21), the only variable whose coefficient does not shrink to zero was *MathENG*, a composite variable measuring students' reported engagement in meaningful mathematics. MathENG exhibits wide variance, even among students enrolled in the same course, leading us to specify the second research question as: *What influences variation in students' reports of MathENG within the same course?*

To answer RQ2, we employ LASSO regression with the outcome variable *MathENG_cen*, a recalibration of students' responses centered around the mean response of their classmates, modeling how 12 variables relate to students reporting higher/lower MathENG than their peers (N=23,201). Factors which did not disappear from this model (MAE=0.58; HL=0.22; R^2=0.03) are gender (negative impact on women vs. men); major (positive impact on STEM majors vs. non-STEM majors); and reports of how class time was spent (positive association with class discussion; negative association with listening to the instructor lecture and solve problems).

The poster will further detail the model and fit characteristics and explore conjectures, implications for research and practice, and next steps needed to further understand these phenomena.

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Student Trajectories in a College Algebra With Support Course Using Standards-Based Grading

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Keywords: College Algebra, Standards-Based Grading, Assessment

We share how the implementation of a College Algebra with Support (CAWS) course affected student success in mathematics and subsequent course taking behavior. The CAWS course added one additional credit hour to the existing College Algebra course and includes student supports such as just-in-time reviews and study skills. The course uses student-centered pedagogies (Abell et al., 2018; Bressoud et al, 2015; Deslauriers et al., 2019; Freeman et al, 2014; Theobald et al., 2020) and standards-based grading (Nilson & Stanny, 2015; Zimmerman, 2020). CAWS was first implemented in Fall 2023, with an enrollment of 610 students. In Spring 2024, 507 students enrolled in CAWS. As part of the standards-based grading structure, a significant portion of a student's grade was earned by completing learning checks (quizzes) on individual learning objectives (LOs). Students were expected to demonstrate skill twice on each LO. We sought to examine the question: *What are common student trajectories in a CAWS course that uses standards-based grading?*

Institutional data were collected from the university Systems Intelligence and Data Analytics Office. Course level data on when LOs were met during the semester were gathered from course coordinators. Although five years of data were collected, the analysis presented here focuses on Fall 2023 and Spring 2024. Data included: demographics, mathematics course enrollments, final course grades, and learning objective data from CAWS in Fall 2023 and Spring 2024.

To analyze the data, we first looked broadly at how final course grades were distributed based on student demographic data. We then conducted hierarchical clustering analysis on the Fall 2023 data based on the week when students successfully completed each LO for the first and second time. Students were grouped into six clusters by the similarity of their performance on LOs. Group 1 has 272 students (44.6%) who demonstrated consistent strong learning performance and mostly received A's. Group 2 contains 61 students (10%) who started late in LO#1 but caught up since LO#2 and consistently met most of the LOs twice. Most of them received A or B's. Group 3 has 66 students (10.8%) who had a slow start and most LOs were met after week 9. Most students in this group received B or C's. Group 4 has 34 students (5.6%) who started strong in the first three LOs but then struggled to complete later LOs. Group 5 with 99 students (16.2%) mostly started after week 7 and were inconsistent with completing LOs. Group 6 has 78 students (12.8%) who did not complete almost all the LOs. Groups 4 and 5 have high DFW rates at around 80%, while students in Group 6 have 100% DFW rate.

In addition to studying CAWS, we started examining student success in future courses. There has been a significant increase in the number of students taking Trigonometry as a result of the implementation of CAWS, particularly by students of color. This suggests that these students are persisting on STEM degree paths that require them to take calculus. Our goal is to use our analyses to improve CAWS and better support students as they continue to take mathematics courses. Future work will continue to analyze student trajectories in courses such as Trigonometry and Calculus I.

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Interactive Digital Activities in Real Analysis for In-Service Teachers

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In this poster presentation, we present pilot data describing how secondary mathematics teachers enrolled in a fully online, Real Analysis course in the Southeastern United States engaged with interactive digital activities. Further, we describe what mathematical meanings were elicited in this context during semi-structured interviews where students engaged with mathematical tasks via the activities. These pilot data will inform two future goals. First, it will inform the design of interactive digital activities in our Advanced Geometry course. Second, it will inform our future work on a larger study about overall engagement with various touchpoints (i.e., ways to participate) in fully asynchronous classes.

Keywords: Secondary Teachers, Real Analysis, Interactive Digital Activities

Mathematics teachers who seek promotion or to teach dual-credit classes at their jobs often pursue graduate education as a way to achieve the requirements for that promotion. As these teachers are full-time employed teachers, asynchronous online programs in graduate mathematics provide ways to achieve these professional goals. Our work exists in spaces like these that are very common across the United States. An important goal for us as mathematics educators is that even in asynchronous online environments, students engage in inquiry-based mathematics education (Laursen & Rasmussen, 2019). One way to achieve this is to include *interactive digital activities (IDA)* as ways to engage with our students remotely. Because there are many different kinds of IDAs we define each word within the term. An **activity** is a mathematical task where students actively build understandings related to a set of mathematical ideas. **Digital activities** can include applets (Moore, 2009), animations (Moore & Thompson, 2015), digital learning games (Mauntel et al., 2021), or activities that incorporate augmented or virtual reality (Kang et al., 2020). This list is not exhaustive; any activity that utilizes digital resources and engages students in mathematical activity could be considered a digital activity. Finally, there are **interactive digital activities**. These are activities where students are required to engage, in the moment, with some aspect (such as a slider, button, etc.) of the digital activity and get in-the-moment feedback (Zandieh et al., 2018). Therefore, the students' interaction depends on the nature of IDA and not all students will engage with the IDA in the same way.

Our research team collected pilot data to understand how in-service teachers engage with IDAs focusing on real analysis content and what mathematical meanings are elicited during this engagement. Our research questions are: *1) When students who are enrolled in an asynchronous online graduate mathematics class engage with interactive digital activities, what mathematical meanings are elicited? 2) How do students engage with interactive digital activities?* In this poster, we present preliminary results from semi-structured mathematical task interviews on polynomial asymptotes with six in-service teachers. The various ways in which the teachers interacted with the IDAs informed the possible meanings that could have been elicited. For instance, a teacher may have overlooked an aspect of the design of the IDA which then meant that the mathematics question they were answering was limited in scope. This work represents an initial attempt to understand how IDAs can be improved and used in asynchronous classes. The goal is to consider how these IDAs fit into larger engagement with mathematics, learning, other students, the instructor, etc. in online asynchronous classes for in-service teachers.

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Who Feels Like They're (Not) Getting Attention in Class?

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Keywords: Inclusion, Student Experience, Identity, Introductory Mathematics

Students' experiences in their introductory courses impact their likelihood of pursuing and completing undergraduate degrees in STEM (Holland, 2019; Seymour et al., 2019). A sense of belonging or inclusion supports their pursuit of these degrees, while feeling excluded hinders that pursuit. This is particularly pronounced for students from marginalized and/or underrepresented groups, who may enter such courses already feeling like "outsiders" (Ellis et al., 2016; Riegle-Crumb et al., 2019). We explore links between students' demographic characteristics and their reports of receiving similar (or more, or less) attention than their classmates.

The data corpus for this study consists of 32,137 survey responses (complete with respect to our variables of interest) from undergraduate students enrolled in precalculus and calculus courses at 21 universities (Apkarian et al., 2019; Street et al., 2021). Students reported demographic characteristics as well as their perceptions of class activities and their experiences with the course. Students were categorized into three groups: those who reported receiving more, the same, or less attention during class. This categorization was based on students' responses to five Likert items asking them to report whether (compared to their classmates) they had more/less/the same (a) opportunity to ask questions; (b) attention given to their questions; (c) encouragement from the instructor; (d) opportunity to participate in class discussions; and (e) praise from the instructor. The majority (26973; 83.9% of sample) reported receiving the same attention as their classmates, but we focus on those who reported that they were *not* receiving the same attention as their classmates. Testing was conducted to determine if students from different sub-populations were equally likely to report receiving more vs. less attention.

Overall, the 5,164 who were not in the "same" group were 3.29 times as likely to report receiving *less* as opposed to *more* attention than their classmates. We found little variation between men's and women's perceptions of their in-class treatment. However, transgender and gender non-conforming students were 5.4 times as likely to report that they receive less attention. This pattern within men and women held for most racial groups, apart from Alaskan Native and/or Native American students, among whom women were more likely to report more attention than men. We found additional discrepancies based on race. For example, 87.8% of white students (n=16,185) characterize their experience as average, a larger percentage than any other racial group. Within racial groups analyzed, Hispanic/Latinx (n=865) students were 5.13 times as likely to report receiving less attention than more, the highest proportion among groups analyzed. With regards to other special populations, we also found that first-generation students (n=1476) were 4.125 times as likely to report they received less attention than more, and parents (n=48) were 5.857 times as likely. Conversely, international students (n=248) were 1.88 times as likely and disabled students (n=197) were 2.71 times as likely. The poster will include full results for all special populations analyzed in this project, with additional statistical details.

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GTA Views on Their Own Roles and Student Roles in the Classroom

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Keywords: GTA, instructor obligations, student role, GTA role

Graduate Teaching Assistants teach undergraduate students for 50% or more of their time in a classroom. These GTAs may receive little to no training in teaching and for many of them it is their first teaching experience (Sohoni et al. 2013). GTAs are an important part of undergraduate student learning but unfortunately many universities do not have teaching training programs available for them (Luft et al. 2004). GTAs and professors not only vary in teaching experience but also in the power they have. In some cases, both GTAs and professors teach coordinated courses where some resources, such as exams, homework assignments, and lesson plans, may be available to them and possibly required to be used (Rasmussen and Ellis, 2015).There are professional obligations that instructors can use as resources to justify their teaching decisions being different than what is customary (Ko et al., 2023). These obligations are

(a) "An obligation to interpersonal dynamics and relationships expressed in social groups.
(b) An obligation to individual students.
(c) An obligation to schooling institutions.
(d) An obligation to the discipline of mathematics." (Chazan et al., 2016).

I will refer to them as Interpersonal, Individual, Institutional, and Disciplinary, respectively from this point on. The PRofessional Obligation Scenario Evaluation (PROSE) instrument looks at teachers' recognition of the above obligations (Ko et al., 2023). I designed discussion posts for 11 GTAs at a mid-size public northeastern university enrolled in a teaching course in order to expose them to pedagogy using the PROSE instrument as a guide. There are two scenarios for each obligation. This ensured that every GTA was seeing one prompt related to a class they are teaching and one prompt related to a class they are not teaching. The discussion asked GTAs to state whether or not they agree with the prompt and justify their level of agreement or disagreement, state any context they are imagining as they are reading the prompt and then choose 2 of their peers to respond to for each prompt. In their response to their peers the GTA was asked to state why they agree or disagree with the final decision, whether or not they agree with the justification for the decision, and the level to which they had the same considerations as their peers. In analyzing the discussion posts, using open coding (Corbin & Strauss, 1990), I observed differences in some responses being whether the GTA is responding to the specific question topic in the prompt versus for general topics. Some GTAs related their responses to the courses they are teaching. GTA also discussed how they view the student's role or their own role in a classroom which had related to how they justified their response to each scenario.This will inform what aspects of the Graduate Teaching experience have an effect on how they choose to structure their classroom and how this recognize the four obligations: "Interpersonal", "Individual", "Disciplinary", and "Institutional."

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Diversity, Equity, and Inclusion: Gathering Community College Students' Perspectives

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EHR #2000602, 2000644, 2000527, 2000566

Keywords: college algebra, community colleges, equity, diversity, inclusion

In this poster, we share our design process and protocols for gathering longitudinal data of students' perspectives on equity, diversity, and inclusion in the college algebra classrooms at community colleges with the goal of documenting their understanding of these terms in the context of college algebra. While some studies explored student experiences in college algebra (Gardner, 2020) and other studies document undergraduate students' experiences of inclusion and belonging in mathematics classes at predominantly white institutions (Leyva et al., 2021), we do not have studies that investigated community college students' understanding of equity, diversity, and inclusion nor how students come to understand these ideas.

We are engaging in an initial cycle of a grounded theory approach (Charmaz, 2006) with the goal of providing an initial understanding of these constructs from students' perspectives. Since community colleges draw students from their local communities, the details of students' experiences vary, and their understanding of equity is influenced by contextual aspects of the institution and of the community. As such, we plan to collect data from 80 students (10 from each college) enrolled in eight carefully selected community colleges, two from each of the following regions: Northwest, Southwest, Central, and Midwest. These colleges were selected for the diversity of students enrolled on their campuses. We seek to recruit students with particular identities including race, gender, ethnicity, physical diversity, and neurodiversity. As student recruitment is critical, our recruitment process will use a multipronged approach that includes reaching out to college units that support a diversity of students.

To capture students' views they will participate in a series of four interviews within a year. During the interviews, students will (1) discuss their personal and mathematical histories related to their current understanding of equity, diversity, and inclusion; (2) provide commentary on short videos of classroom instruction tagging specific characteristics that can be associated with equity, diversity, and inclusion; (3) engage in a Q-sort exercise (Rieber, 2020) of student statements from previous interviews geared to elicit their understandings of equity, diversity, and inclusion; and (4) reflect on this process over time. The time interval between interviews will allow for analysis of student data and fine-tuning of interview protocols. Data collected within an institution and across institutions will complete one cycle of the grounded theory approach which will generate initial theoretical propositions about students' understanding of equity, diversity, and inclusion.

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Perceptions of Mentors' Impact on the Success of Mentees

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Keywords: mentoring, graduate teaching assistants, post-secondary education

Graduate students in the mathematical sciences are responsible for the teaching of many undergraduate math courses in doctoral institutions (Eller, 2017). To that end, it is highly beneficial to make sure that these graduate teaching assistants (GTAs) are adequately prepared to take on the task of instructing a course. Peer mentors are a key resource in helping these less experienced GTAs to grow in both their teaching and research responsibilities associated with being in a graduate program (Flores-Scott & Nerad, 2012).

Literature Review

Research has shown that participants in graduate student peer-mentoring programs obtain benefits in many different domains including academic, social, psychological, and career development, and mentoring can attend to many of the diverse needs of graduate students (Lorenzetti et al., 2019; Liu et al., 2022). These mentoring relationships work in both directions with knowledge sharing, exploring different perspectives, and giving and receiving feedback (Lorenzetti et al., 2020). In our poster, we discuss mentees' and mentors' perceptions of the benefits mentoring can provide in a teaching-oriented peer mentorship program.

Data, Methods, and Results

As part of our multifaceted and multi-institution GTA training program, *Promoting Success in Undergraduate Mathematics through Graduate Teacher Training (PSUM-GTT)* (Harrell-Williams, et al., 2020; Manzanares, et al., 2024), GTAs participated in peer mentoring. Each mentor was at least a third-year doctoral student in a Mathematical Sciences program at one of the three institutions and was assigned one mentee (sometimes two). Each semester GTAs completed online surveys about their experiences with teaching and mentoring. This poster presents results from survey data from Fall 2019-Spring 2023 and is a follow-up to a conference proceedings paper (Funkhouser, et al., 2021) reporting preliminary first year results. Specifically, mentors were asked how significant they thought their mentoring was to their mentee's success as an educator, and mentees were asked how significant they thought their mentor was to their success as a GTA, both on a scale of 1-5. Across all three institutions and every year, mentees rated the assistance they got from their mentor higher (*mean=3.68, SD=0.34*) than the mentors rated themselves as being successful in mentoring their mentees (*mean=2.84, SD=0.30*). We will situate these results within a discussion of mentoring self-efficacy and how peer mentoring models can provide specific support for their mentors.

Acknowledgments

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Catalyzing Departmental Change: A Literature Review of Barriers, Drivers, and Framing

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Keywords: Departmental change, teaching culture, STEM education, barriers, drivers

Despite an abundance of research applauding the use of evidence-based instructional methods in STEM courses, the majority of math undergraduates are still taught using traditional lecture-based methods (Sonnert et al., 2015; Stains et al., 2018). As such, a deeper cultural shift may be required for these novel instructional methods to be embraced outside the educational research community. As academic departments play a pivotal role in influencing teaching quality, one natural entry point for enacting instructional change is to focus on departmental teaching culture (Henderson et al., 2011; Corbo et al., 2016; Couch et al., 2024). While the implementation of change efforts is an individualized process for each department, value can still be found in identifying key factors that have the potential to impact the change process. Through this exploratory literature review, we seek to answer the following question: what are potential barriers and drivers to departmental teaching culture change and what strategies could be used to encourage embracement of departmental change?

To answer our question, we conducted an exploratory analysis of literature via a keyword search of *(departmental) cultural change*. In particular, identified *barriers to change* include: lack of professional knowledge (Corbo et al., 2016), lack of time and resources (Reinholz & Apkarian, 2018), fear of the unknown (Gaubatz & Ensminger, 2017), satisfaction with the status quo (Gaubatz & Ensminger, 2017), lack of autonomy (Gaubatz & Ensminger, 2017), and resistance to external leaders (Gagliardi et al., 1986). Possible *drivers for change* are: lack of conflict between current values and the values associated with change (Gagliardi et al., 1986), celebration of positive outcomes throughout the change process (Couch et al., 2024), reinforcement of change as an ongoing iterative process (Gagliardi et al., 1986), a commitment to diversity and inclusion (Quan et al., 2019), and the leveraging of departmental leaders as agents of change (Kunst & Mesoudi, 2023). Finally, *change should be framed* as follows: change is at the departmental level (Gagliardi et al., 1986), change is student centered (Quan et al., 2019), everyone is a collaborator in change efforts (Quan et al., 2019), change will have positive outcomes (Kunst & Mesoudi, 2023), and change efforts are data driven (Quan et al., 2019).

Our poster will present a literature-informed action plan for RUME researchers and mathematics faculty who seek to promote cultural change with respect to teaching. For example, current departmental values must be identified to ensure they are not in conflict with the proposed change; any conflicts must be reconciled. Current departmental needs must also be identified in order to be leveraged as positive outcomes of change. In addition, departmental leaders should be identified to be leveraged as agents of change; this includes both department administrators such as chairs as well as peer leaders deemed as influential by their peers.

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Comparing the Strategies of Two Precalculus Students Responding to a Novel Mathematics Task

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Keywords: covariational reasoning, quantitative reasoning, continuity, concavity

This study examined the problem-solving abilities of two male Pre-Calculus students at a Southwestern University as they completed three novel mathematics tasks. The tasks prompted students to explain and represent how two quantities in a problem context vary together. My analysis revealed that students' success in constructing meaningful formulas and graphs depended on the students' first conceptualizing the quantities in the problem context and visualizing how they are related and vary together.

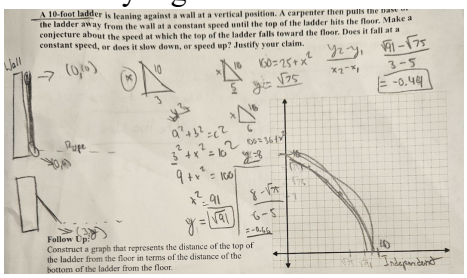


Figure 1. A picture of the task and Student A's work

Student A's mental image (Carlson, O'Bryan, 2022) of the quantitative structure of the ladder problem is illustrated (Figure 1) in the student's drawing. To address the prompt (Figure 1), he sketched the ladder's initial position (top left) and modeled its motion as the bottom of the ladder was pulled away from the wall. He constructed a sequence of triangles to represent snapshots of the ladder falling, ultimately allowing him to conceptualize and compute how much the top of the ladder dropped with each fixed change in the base's movement. His use of covariational reasoning led to him correctly recognizing that the top of the ladder would speed up as the base was moved further from the wall.

Student B immediately made a drawing of the context and concluded that he could use the triangle he constructed and the Pythagorean theorem to write the equation of a circle.

> *Student B:* … the rates of change aren't constant because [omitted] it's a circle… so the ladder is going to start slow, speed up, and then slow down again.

Student B demonstrated fluency in using geometric relationships when he represented the Pythagorean theorem as an equation of a circle and then solved for one variable. When asked to restate how the ladder would fall, he referenced the curvature of the circle on different intervals and claimed the rate of the top of the circle is the inverse rate of the bottom right-hand part. He appeared to be recalling facts he had learned rather than conceptualizing how the quantities in the problem context were related. While his approach of comparing rates of change across intervals was notable, his reliance on abstract geometric reasoning prevented him from verifying whether his answer aligned with the real-world scenario.

While there's no single correct approach to solving a novel problem, constructing a mental image of how quantities are related before modeling them mathematically helps students understand how the quantities vary together in relation to a graph's concavity. The poster will focus on this task so viewers can engage with the task and consider excerpts from each student.

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Lowering the DFW Rate in Calculus 1

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Keywords: Calculus, DFW Rate, Coordination

Educators and researchers have been working to help a wider variety of students be successful in calculus for decades (e.g., Steen, 1987). Researchers have made progress on outlining ways this can be accomplished (e.g., Johnson et al., 2022). We add to this literature by sharing our experience of implementing several small changes that considerably lowered the DFW rate in calculus 1. Specifically, the semester we implemented these changes the DFW rate was 18.8% rate compared to an average rate of 43.2% over the previous six years.

Context and Nature of the Changes

Both before and after the changes we report on were made, Calculus 1 was taught in several medium-sized coordinated sections (about 50 students each), mostly taught by graduate students. The sections met 4 days a week, with new content being introduced on three of those days and the other day being reserved for students to work on problems in class. In addition to typical homework, students also completed Mathematica assignments each week.

Increased Instructional Alignment and Strengthened Coordination

When considering how to change the course to better support students, the course coordinator began by clearly articulating learning objectives for each day of class. He then ensured that in-class activities, homework, and exam questions were aligned with these objectives. He also developed common handouts. These handouts listed the learning objectives relevant to the lesson at the top, with notes on key terms, ideas, theorems and problems below. While the handouts were intended to guide students' notetaking and problem solving in class, they also communicated a potential outline for a lesson plan to the instructors.

Additional Support and Feedback for Students

The course was also changed to include extra support for students and offer several additional opportunities for feedback. The course coordinator created short (3-6 min) videos for each class that students were supposed to watch and take a quiz on before class. The videos introduced the topics in a way that was aligned with the learning objectives. The coordinator also offered weekly reviews that covered just-in-time topics. Students who did poorly on an algebra quiz given during the first week of class were required to get additional support. One option was to attend these reviews. The course coordinator also created short daily "exit quizzes" to provide students with an opportunity to receive feedback on a per class basis. Finally, the course was modified to include three exams rather than two so students could get feedback earlier.

Discussion

The changes we reported here were relatively small in that they were not a major restructuring of the course or an overhaul of the content. Instead, these changes focused on creating strong instructional alignment and providing additional supports for students. However, they seemed to have a significant impact on the number of students who were able to be successful.

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Creating Counterspaces Through Critical Conversations About Race and Gender in STEM

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Keywords: race, gender, equity, community development

Undergraduate STEM educational spaces, and particularly, mathematics educational spaces, are uniquely racialized and cisheteronormative. (e.g., Leyva et al., 2022; Ong et al., 2011). STEM *counterspaces* are academic and social spaces that allow minoritized students to validate one another's critical knowledge from their experiences, share stories of isolation, discrimination, etc., broadly challenge deficit notions of minoritized groups, and establish positive racial climates (Ong et al., 2018, p. 209). Research has begun to uncover the benefits of counterspaces on students, such as improving students' sense of belonging, mitigating feelings of isolation, and affirming students' academic identities (McGee & Martin, 2011; Oppland-Cordel & Martin, 2015). In this poster, we explore the question: *To what extent can critical conversations about race and gender function as a counterspace for minoritized students?*

Critical Conversations (CC) about Race and Gender in STEM was an activity that we designed with the objective of supporting minoritized students and instructors to build empathy, solidarity, and community with one another through affirmations and shared stories. It was part of a larger three-year professional development project at a Hispanic Serving Institution focused on anti-deficit teaching (Adiredja et al., 2024). Three cohorts of students and instructors participated in five conversations, two during a summer math workshop and three during the following Fall semester. We used the same curriculum for each year, and each conversation was facilitated by undergraduate STEM peer mentors from minoritized backgrounds.

We focused our analysis on exit tickets after each of the CCs in the fall because they were held at the university cultural resource centers, thus involving more members of the university community. The fall CCs also discussed topics that were explicitly about race and gender in STEM: stereotype and stereotype threat (Steele, 1997), stereotype management (McGee & Martin, 2011), and community cultural wealth (Yosso, 2005) respectively, all discussed in the context of STEM education. The exit tickets were anonymous and asked for one take-away and one feedback about the conversation. Over all nine CCs, we had 90 participants (75 students and 15 instructors). From these participants, we received 89 responses.

Eleven themes emerged through a constant comparative method (Glaser, 1965), with three being especially relevant: connecting through shared experiences (26 responses), experiencing affirmation and support (20 responses), and learning how to respond to stereotypes (22 responses). These themes connect to Ong's definition of counterspaces, as they provide evidence participants are sharing stories, validating each other's knowledge and experiences, and challenging deficit narratives. Notably, we found that participants reported connecting with each other through shared experiences instead of shared identities, which may make CCs a valuable counterspace for people who feel left out of more homogeneous groups. For example, one participant reported "a good sense of shared humanity with other men," though they were initially worried their sexuality would render them invisible to straight men. They were able to connect over shared experiences "that had different sources but the same impact," and found that "really affirming." Our findings suggest that leveraging shared experiences can support the creation of counterspaces, especially in a department without critical masses of underrepresented groups. Even so, we found the collaboration with the university cultural centers to be fruitful.

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Mathematics Capital as a Latent Factor

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Keywords: Mathematics Capital, Structural Equation Modeling, Bourdieu, Mathematics Identity

Students bring their individual characteristics and identities into the classroom, which ultimately impacts their learning experiences and success in mathematics (Cribbs et al., 2015, 2021, 2022; Gutiérrez, 2009; NCTM, 2014). We seek to identify student backgrounds and characteristics that impact their success in mathematics education by analyzing data from students in undergraduate mathematics courses via structural equation modeling (SEM). Our goal is to study a novel theoretical construct, defined as *mathematics capital* (Anderson, 2023, 2024, Anderson et al., 2024, Williams & Choudry, 2016; cf, Bourdieu, 1977, Archer et al., 2014, 2015). Undergraduate students have a plethora of prior mathematical experiences. These past experiences can inform their choice of a college major and set them on a semi-fixed trajectory. Thus, we study undergraduate mathematics students because we view their course enrollment as a marker of their mathematical outcomes.

We use a Bourdieusian lens (Bourdieu, 1977; Bourdieu & Wacquant, 1992) to frame the construct of mathematics capital. Following the work of Bourdieu, we argue that mathematics capital is a unique variable that relates to and acts as a bridge between other constructs in mathematics education such as social factors, cultural capital, emotions, and personal experiences (Anderson, 2024). Like Archer and colleagues (2014, 2015) who have applied Bourdieu's ideas to study science capital, we extend this work into the field of mathematics education (Anderson et al., 2024, c.f. Dewitt et al., 2016).

We collected quantitative data in Fall 2023. A previously validated Qualtrics survey containing items related to mathematics capital (MC), mathematics mindset (MIND), self-efficacy (SE), cultural capital (CC), and mathematics teaching and learning experiences (TEAC) was distributed via email to all students enrolled in any mathematics course at an urban university. We received a total of 685 responses out of 2594 students for a response rate of 26.4%. Only participants who completed the entire survey were included in the analysis for a total of 674 participants. Variables of MIND, SE, CC, and TEAC were used with alpha values ranging from .75 to .94. To create a latent factor of the MC measure, six items were used. A single-factor confirmatory factor analysis was tested using structural equation modeling (M-Plus Ver. 7, Muthén & Muthén, 2012), resulting in a poor fit to the data. Thus, a correlation was added between item 6 and item 7) (as suggested by modification indices). The resulting model was then an acceptable fit to the data ($\chi2(8) = 32.06$, $p < .05$, CFI = .99; RMSEA = .07, SRMR = .01). The internal reliability of the latent factor of mathematics capital was acceptable ($\omega = .90$). We also tested alternative models in which either item was removed, and none were acceptable.

A full model with all other scale scores was tested to elucidate how each were related to the other, but more importantly to the latent factor of mathematics capital. The final model (visual figure on poster) remained a good fit to the data ($\chi2(31) = 103.35$, $p < .05$, CFI = .98; RMSEA = .03, SRMR = .01). This model details the positive relationships between self-efficacy, mathematics mindset, teaching and learning experiences, cultural capital to mathematics capital and reinforces mathematics capital as a separate construct as it is measurably distinct from all other variables (with existing scales) used in this study. This model supports findings from a previous pilot study and provides avenues for future research.

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Supporting Mathematical Instruction Around Definitions Through Values-Centered Collaboration

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Keywords: Norms and values of definitions, undergraduate instruction, collaboration

Extensive research shows students do not learn as much from undergraduate mathematics courses as we would like (e.g., Asiala et al., 1997; Borji & Martínez-Planell, 2020; Siyepu, 2013). Despite research demonstrating the efficacy of non-lecture-based approaches (e.g., Inquiry Oriented Abstract Algebra, Larsen, 2013; Larsen & Lockwood, 2013; Learning Abstract Algebra with ISETL; Leron & Dubinsky, 1995), most mathematics instructors lecture in lower-division courses like Calculus (Kirin et al., 2017) and upper-division courses like Abstract Algebra (Johnson et al., 2018). Dawkins and Weber (2023) suggest mathematicians choose to lecture because it meets their goals, including content coverage, preparing future mathematicians, precision and language, and efficiency; thus, they call for improving instruction by enhancing lectures in ways that feel comfortable to instructors. Our project supports this novel approach to improving undergraduate instruction by centering collaboration between mathematics education researchers and mathematics instructors. We focus on teaching the norms and values of definitions, as definitions are fundamental across different undergraduate courses and have been identified as an area where students and instructors can struggle to communicate (e.g., Martín-Molina et al., 2020).

This project integrates focus groups, class observations, surveys, and interviews with instructors and students to develop a toolkit of instructional innovations for effectively communicating mathematical norms and values around definitions in both upper-division (Abstract Algebra and Real Analysis) and lower-division (Calculus I and II) lectures. This poster presents preliminary observations in the first round of focus groups and surveys with 10 mathematics faculty from two universities. The aim is for the instructors to develop their own minimally invasive activity centered on a definition that communicates a mathematical norm or value (Rupnow & Randazzo, 2023).

Instructors reported challenges with limited time for activities in lecture, highly coordinated courses limiting freedom of instruction, as well as more content-specific challenges. For instance, some instructors explained how it was difficult to plan an activity that shared norms and values of mathematical definitions in such a way that complemented their learning goals. For example, instructors reported difficulty in communicating precision without becoming over-encumbered in fine details (e.g., tensor products). Other instructors expressed difficulty in growing students' intuition about certain definitions (e.g., Laplace Transform) or translating between students' informal and technical understandings (e.g., the informal notion of closeness with the technical nature of ϵ and δ in convergence proofs). As the project continues, the project team is continuing to collaborate with the mathematics faculty to address these challenges (e.g., by understanding which norms and values might afford which learning goals) while supporting their instruction to communicate norms and values in ways that are comfortable and non-intrusive for both the instructors and their students.

Acknowledgments

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"I'VE ALREADY DONE THIS": How Prior Exposure Affects Calculus Students' Mindsets During a Semester of Regular Reflections

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Keywords: growth mindset, mastery-oriented mindset, strategic mindset, calculus

Historically, Calculus I has been a notorious barrier to many students' completion of STEM degrees (Weston et. al, 2019). In Fall 2023, we taught a large lecture section of Calculus I (~250 students) at a large R1 public university in the Southeastern U.S., intending to increase the success rate for students by implementing a minor metacognitive intervention called "How'd it go?" prompts after problem sets. In this poster presentation, we investigate the organic manifestation of calculus student mindsets as they occurred in real-time throughout the semester by analyzing short reflective prompts that the students completed voluntarily. Specifically, we answer the research question: *How do university calculus students' mindsets manifest differently depending on the setting of their first exposure to a calculus class (high school or college)?*

In Psychological Meaning System Theory, *mindsets* are attitudes that people hold about themselves (Dweck, 2000). This study focuses on four established mindset pairings: growth/fixed (Dweck, 2016), mastery-oriented/helpless (Dweck, 2000), strategic/non-strategic (Chen et. al, 2020), and entrepreneurial/non-entrepreneurial (Bosman & Fernhaber, 2018).

The course was conducted in a 2-2 lecture-recitation format. Throughout the semester, students were provided printed packets that contained lecture notes outlines, pre-recitation prep problems, recitation problems, and (in some cases) extra practice problems. The whole semester included 21 packets (called lessons) divided into four units. In this study, we analyzed a random sampling of three packets from each unit. To analyze the data, we used a priori coding (Saldaña, 2013) with codes defined by the described mindsets: (a) growth, (b) fixed, (c) mastery-oriented, (d) helpless, (e) strategic, (f) nonstrategic, (g) curious, and (h) not curious. Note that instead of entrepreneurial and non-entrepreneurial, we only coded for curiosity or explicit non-curiosity because some qualities in entrepreneurial mindset are also qualities of a strategic mindset.

When the data was disaggregated according to students' high school calculus exposure (or not), results revealed that these groups had antithetical manifestations of mindsets. In particular, students with no high school calculus were as much as seven times more likely to display a growth mindset in Unit 2 than students with high school calculus. Results further suggest that engaging students in regular metacognitive reflection coupled with a strategic framing of calculus may significantly increase course success rates within an exam-heavy, highly coordinated grading structure.

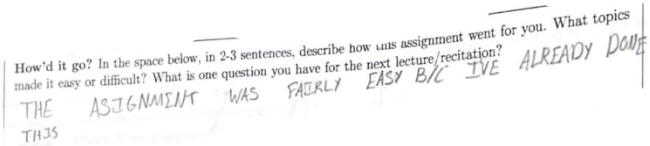

How'd it go? In the space below, in 2-3 sentences, describe how this assignment went for you. What topics made it easy or difficult? What is one question you have for the next lecture/recitation?
THE ASIGNMENT WAS FAIRLY EASY B/C IVE ALREADY DONE THIS

Figure 1. An example of a "How'd it go?" prompt.

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Exploring Student Strategies for Matrix-Vector Multiplication in a Linear Algebra Video Game

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Keywords: Design-Based Research, Inquiry-Oriented Linear Algebra, Game-Based Learning

In this work, we investigate the second level of the video game *Vector Unknown: Echelon Seas* (*VUES*, Zandieh, et al., 2018; Mauntel et al., 2019; Mauntel et al., 2020; Mauntel et al., 2021), which was designed to integrate elements of Game-Based Learning (GBL; Gee, 2003; Williams-Pierce & Thevenow-Harrison, 2021), Inquiry-Oriented Learning (IOI; Rasmussen & Kwon, 2007; Zandieh, Wawro, & Rasmussen, 2017), and Realistic Mathematics Education (RME; Gravemeijer, 1994; Freudenthal, 1991). The game, *VUES*, is set within a 3-dimensional environment, providing a unique platform for players to engage with concepts of vectors in a dynamic way. In this level, players use matrix-vector multiplication to aim and fire cannonballs at targeted ships.

Players are given a target vector that is represented by a ship. Players pick one of four matrices, represented by different cannonballs, and one of four cannon vectors, represented by different cannons. After choosing a cannonball, there is a geometric support that shows the player visualize a lattice generated by the column vectors of the matrix. The goal of the game is to choose the matrix and cannon vector that multiply together to form the ship vector. Through this process, players can explore the calculation of different matrix-vector multiplications from a transformation point of view. The game also provides a geometric visualization related to the column vectors of the matrix. By exploring these different interpretations of matrix multiplication, students can deepen their understanding of how matrix transformations influence vector directions in a practical and engaging context. Our research question is: What strategies do students use to complete the second level of *VUES*?

To investigate students' strategies in playing this level, we conducted three paired-teaching experiments (Steffe & Thompson, 2000), each lasting approximately one hour and 15 minutes. These students are currently enrolled in a linear algebra course and have prior experience with 3D games. During the interviews, students were asked to explain their strategies for choosing matrices and vectors and describe how they anticipated the cannonball's trajectory. This approach allowed us to explore their strategies for solving and completing the video game related to matrix-vector multiplication. The interviewers scaffolded information about how to use the grid lines at various levels of gameplay.

In the poster, we will present our preliminary analysis of student strategies. In each interview, the student fired at least 10 cannonballs. When firing the first few cannonballs, most students used guess and check. Some students used a numeric form of guessing and checking by choosing one cannonball (matrix) and multiplying the first row of the matrix by a vector to check whether the result matched the first element of the target vector or not. Others used a geometric form of guessing and checking that involved the geometric support from the game, which included a provided trajectory line based on their choice of cannonball and cannon, to check the multiplication results.

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Computational Thinking and Problem Solving

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Keywords: computational thinking, problem solving, mathematics curriculum

Computational thinking (CT) is a way of thinking about problems that allows their formulation and solution to be expressed using computational steps or algorithms that can be carried out by humans and/or computers (Aho, 2012; Li et al., 2020; Wing, 2006, 2008). Often complemented by mathematical thinking, CT incorporates algorithmic thinking and focuses on problem solving using computer science concepts (e.g., data structures, algorithms, various programming styles, logic, and computation).

The purpose of this paper is to illustrate problem solving using a CT approach and discuss the role of CT within the K-20 mathematics curriculum.

Two studies conducted by the author (Postelnicu, 2025; Postelnicu & Postelnicu, 2020) point out the core components of CT:

- *decomposition* - breaking the problems into manageable components or steps;
- *pattern recognition* - identifying patterns;
- *abstraction* - simplifying the problem by reducing it to a mathematical model or a visual representation;
- *algorithm design* - write a logical sequence of computational steps or an algorithm.

Participants in the studies were middle school and high school students (first study), and mathematics and computer science educators with various backgrounds, including university professors (second study, ongoing). From a mathematical perspective, the problem from the first study can be considered an optimization problem, while the problem from the second study uses recurrence relations and modular arithmetic. The solutions for the two problems highlight the interplay between CT and mathematical thinking approaches. Both approaches to problem solving may share features like *decomposition*, *pattern recognition, abstraction*, and *algorithm design*, but those features shape up differently for each approach. For example, in the CT approach, the *abstraction* may come about through a visual representation (physical model, simulation), while in in the mathematical thinking approach *abstraction* comes about through a mathematical model. The type of a*bstraction* affects the way *pattern recognition* and *algorithm design* take shape. In the CT approach, the concern for *algorithm* efficiency and computational constraints often lead to "good enough" solutions or *algorithms* that are based on heuristics and may lack formal theoretical justifications. In the mathematical thinking approach, solutions or *algorithms* are based on theory, and the aim is to construct exact solutions or *algorithms*, rigorously justified, whenever possible.

The role of CT within the K-20 mathematics curriculum will be discussed. Weintrop et al. (2016) point to the challenges faced by integrating CT and subject-matter curricula, including defining CT and its role in the context of our mathematics curriculum. The discussion is pertinent, given that CT is necessary for understanding data, algorithms, and digital technologies that permeate multiple aspects of our life (Kafai and Proctor (2022).

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Sustaining Instructional Practices in Challenging Times

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Keywords: professional development, technology, online teaching, sustainable change

Motivation and Overview

Durable change of instructional practices depends heavily on instructors' ability to sustain the use of new practices in challenging times, whether the challenges arise at the scale of society (e.g., Covid-19), institution (e.g., leadership turnover), or individual (e.g., life circumstances). The Covid-19 pandemic created unprecedented disruptions to people's routines, work, and lives. For example, in higher education, instructors had to move all teaching online on short notice, drawing on their personal teaching systems (knowledge, beliefs and practices) to shape their teaching in the online setting (Brunetto et al., 2022). Understanding how instructors adapted to pandemic teaching conditions over time provides insight into the durability of their use of evidence-based instructional practices during a challenging era.

Our mixed methods study uses larger contextual systems (Hora & Hunter, 2014) as framing to help us understand the work of undergraduate mathematics instructors' who are committed to improving their teaching and how they adapted to remote instruction. We sought to answer the following questions about how undergraduate mathematics instructors responded during the online instruction phase of the pandemic: 1) What instructional practices did instructors maintain, revert to, or attempt during this phase? 2) What technology and institutional supports did instructors find helpful for teaching online? and 3) When instructors attempted new instructional practices out of necessity, what, if any, of those practices did they hope to retain for in-person, post-pandemic teaching?

Methods, Results and Discussion

Data for this study were collected in spring, summer and early fall 2020 during the online teaching phase of the pandemic, from a sample of instructors who were already participating in professional development. We analyzed instructors' responses to multiple-choice questions about how much time they spent using a variety of instructional practices prior to the pandemic compared to during online instruction, and text responses to open-ended questions about professional learning, technological support and new instructional strategies used online.

This poster will share how instructional practices were different online compared to before the pandemic. For example, instructors reported being less likely to ask students to present problems in class, and more likely to answer student questions via email. Instructors reported posting recorded lectures more often online and they expected to continue this practice after returning to in-person classrooms. We interpret these findings in light of instructors' work within larger systems and consider how what worked for people during the pandemic may shed light on how they sustain—or not—the use of new teaching practices through challenging times.

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Redesigning Assignments and Assessments in Calculus for Life Sciences Courses

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Keywords: Summative Assessments, Hispanic Serving Institutions, Scientific Identity, Calculus for Life Sciences, Applications

Math 11A/B is the Calculus for Life Sciences sequence at UC Santa Cruz which focuses on applications of Calculus most relevant to students who are proposing to major in Biology, Chemistry, Ecology & Evolutionary Biology, Psychology, Earth Science, Global & Community Health. In the Fall of 2019, there were approximately 4,500 Latinx students enrolled at UCSC with 28% of these students declaring or proposing a major requiring Math 11A or 11B. From Fall 2015 to Fall 2019, the DFW rate overall in Math 11A/B was 24%, with a DFW rate of 32% for Latinx students and 20% for White students. This data led to the acquisition of a Graduating and Advancing New American Scholars (GANAS) grant to fund the redesign of Math 11A/B.

Methods

Starting in the Summer of 2021, a team of faculty and graduate students worked together to redesign Math 11A/B at UCSC. The team was interested in decreasing the DFW rates and equity gaps and focused on: (a) Which content from the current curriculum for Math 11A/B was being used in the proposed majors of the students and in which contexts was the material being used? (b) How can we support students to succeed in alternative forms of assessment?

We completed interviews with faculty across STEM departments to determine which mathematical concepts were most relevant to their upper division courses, how these concepts are used in their STEM fields, and which mathematical concepts were the largest barriers to student success in their major-specific courses. Following the advice of (Hutchings et al., 2014, p. 10) that assignments should "be connected to one another and aligned and integrated with a broader curricular and co-curricular set of experiences", we designed projects which closely align with the authentic use cases of mathematical concepts and could replace traditional exam(s). This involved reading current scientific papers to distill their necessary background and use of calculus in order to highlight the relevance of mathematics in science and build aspirational capital (Yosso, 2005). We also designed weekly worksheets to supplement students' understanding and to offer low-stakes assignments to practice the broader scientific analysis which they are being asked to perform on the projects.

Results and Discussion

We have data showing a decline in the DFW rates for all students in both Math 11A and 11B since implementing the redesigned materials. We are in the process of reviewing the data to determine what is causing the decline in the DFW rate. We still see significant equity gaps and hope to gather more data to begin investigating the causes.

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Navigating Obstacles to Foster Success in Learning Optimization Problems in First Semester Calculus

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Keywords: Calculus; Optimization; Problem Solving; Teaching Experiment

Calculus I serves as either a "gateway" into STEM majors or a "filter" that limits students' career intentions (Ayebo et al., 2017; Baisley & Adams, 2019; Biza, 2022; Bressoud et al., 2015; Rasmussen & Ellis, 2013; Sonnert et al., 2015). Baisley and Adams (2019) asserted that poor performance in calculus I can lead to students dropping out of STEM majors, with significant retention concerns in engineering programs. It is historically documented that many calculus students struggle with solving optimization problems due to its complexity which necessitates quantitative reasoning, modeling, and problem-solving skills (Borgen & Manu, 2002; Brijlall & Ndlovu, 2013; Dominguez, 2013; LaRue, 2016; Mkhatshwa, 2016; Swanagan, 2012; White & Mitchelmore, 1996). Speer and Kung (2016) suggested that further investigation into optimization problems is needed. This poster will highlight aspects of this investigation.

The Pilot Study

Drawing on the social constructivist perspective (Cobb et al., 1997; Confrey, 1990; Noddings, 1990), a qualitative pilot study was conducted to understand how engagement in a collaborative problem-solving activity could reveal evidence of understanding or misconceptions of optimization problems from prior calculus learning. Four students who had recently completed Calculus I in a traditional setting worked together to solve an optimization problem. Student interactions revealed a heavy reliance on procedures and formula recall, with participants exchanging ideas and exploring alternative pathways. Although they referred to calculus strategies like using derivatives and identifying extreme values, these were often applied unsuccessfully or abandoned without thorough consideration. Despite their enthusiasm, gaps in conceptual understanding led to frustration and hindered progress. In exit interviews, students voiced dissatisfaction with traditional calculus instruction and suggested ways to create a more supportive learning environment for future students. The findings of this study highlighted the need to further explore how to support calculus students in developing a deeper understanding of optimization.

Results from the pilot study informed the design of a qualitative teaching experiment aiming to improve the learning of calculus optimization by facilitating the building of knowledge representations among a diverse sample of current calculus students. Guided by Davis's "cognitive science approach to mathematics education," this study investigates: (1) what knowledge representation structures related to optimization do students develop, and (2) how the teaching intervention influences the construction of such a knowledge representation? To deepen students' understanding of calculus optimization, participants worked in pairs on five tasks designed to promote conceptual understanding while connecting optimization to prior knowledge.

Results and Significance

The poster will share brief data from the Fall 2023 pilot study that led to a more extensive teaching intervention in Fall 2024, along with some preliminary findings from the ongoing study.

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Exploratory PCA and AI for Assessment Analysis

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Keywords: AI, PCA, R, validation

Assessment plays a critical role in improving student learning outcomes (Brown, 2022; Karaman, 2021). Exploratory Principal Component Analysis (PCA) can identify latent skill structures in exams, offering educators insight into how well assessments align with their goals (Jolliffe, 2002). Simultaneously, emerging AI tools like ChatGPT-4 Turbo present opportunities for enhancing data analysis accessibility in educational research. However, as AI can hallucinate facts, it requires careful validation (Lee, 2023). Using the author's College Algebra Exam 1 as a test case, this study sought to 1) investigate the latent skill dimensions in a College Algebra exam using PCA, and 2) Evaluate ChatGPT-4 Turbo's reliability and utility in conducting PCA.

Pre-existing data were collected from a 19-question College Algebra exam administered to 60 students across two sections of 30 students each, both taught by the author. ChatGPT-4 Turbo was tasked with performing a PCA, and the author conducted a PCA in R and compared results. Interpretation of the PCA results in both ChatGPT-4 Turbo and R focused on identifying the underlying skills being tested by each component. Key ChatGPT prompts and outputs were saved and are included in the poster (via QR code) for reproducibility.

The outputs from R and ChatGPT differed slightly, though the results were similar enough that the interpretation of the underlying skills was not different. Inspection of the chat history shows that ChatGPT wrote its own code to execute the PCA, and it seems its algorithm differs from R's, given the mildly different numerical outputs. This outcome highlights the potential of ChatGPT but underscores the need for human oversight (Lee, 2023). Preliminary findings identified components reflecting distinct skill areas, such as graphical reasoning and multi-step algebraic problem-solving. The component that explained the most variance suggested a counterindication between procedural and conceptual fluency. As literature overwhelmingly suggests that conceptual fluency naturally increases procedural fluency, this result warrants further exploration (Hurrell, 2021; Rittle-Johnson, Schneider, & Star, 2015). Tools such as PCAs and factor analysis offer a potential framework for uncovering skill structures in exams, which can help educators refine and align test design with learning goals (Jolliffe, 2002). While ChatGPT offers accessible entry points for conducting data analysis, human validation remains critical to ensure reliability (Lee, 2023). The intent of this poster is to highlight the potential of this methodology, rather than focusing exclusively on the structure of a single professor's non-standardized college algebra exam.

Since this study, a new AI version, ChatGPT-4o, was released. In the future, the author will investigate this new version's capabilities in this context. If sufficiently reliable, the methodology could be extended to other datasets to enhance generalizability. Further, AI is a developing field, and best practices are still being determined, such as making prompts transparent for reproducibility. Finally, the counterindication between procedural and conceptual fluency in the author's College Algebra class merits further investigation, as these are typically expected to be positively correlated (Rittle-Johnson, Schneider, & Star, 2015). This study demonstrates the utility of PCA in assessing alignment and explores the potential of AI tools in conducting analyses. While AI offers promising avenues of analysis in educational research, human oversight remains crucial to ensure accuracy and validity.

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Preliminary Analysis of Undergraduate Student Problem Posing in a Scaffolded Calculus Intervention

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Keywords: problem posing, Calculus, mathematical representations, mathematical thinking list

Problem posing, that is, the reformatting of existing problems and the generation of new ones, has been promoted for many years by the National Council of Teachers of Mathematics (NCTM, 2000, 1991) as a valuable practice for deepening students' *mathematical thinking* and habits of mind. Problem posing has been associated with problem-solving ability (English, 1998; Silver & Cai, 1996, 1993) and mathematical creativity (Van Harpen & Sriraman, 2013; Yuan & Sriraman, 2011; Leung & Silver 1997). As a practice, problem posing has also been shown to develop student conceptual understanding (Kapur, 2015; Cai et al., 2013). Problem-posing tasks are not only more cognitively demanding than problem-solving tasks. Due to their low-floor nature, they are also more accessible to students of different mathematical backgrounds (Cai & Hwang, 2021), including undergraduate students (Silber & Cai, 2021).

Single-Variable Calculus serves as a gateway course for undergraduates of various majors and mathematical backgrounds. Yet, such 100-level courses may not provide many opportunities for deep mathematical thinking. NCTM (2000) policy has called for student use of multiple *mathematical representations* for promoting mathematical thinking. Such representations are important for developing and reflecting understandings of mathematical ideas (Huntley et al., 2000; Lloyd & Wilson, 1998). In this study, I explored what kinds of problem scenarios (i.e., analogous, somewhat original, or original) undergraduate Calculus students created during a scaffolded, representation-based intervention to analyze students' depth of mathematical thinking.

During two semesters, I, the principal investigator, led a nine-week intervention in four undergraduate Calculus courses in a small, mid-Atlantic college. Over the nine weeks, all students received instruction on average rate of change, limits, continuity, the derivative, and applications. Students were paired randomly by an outside researcher. As a preliminary analysis, problem scenarios created by six dyads with similar Calculus and problem-posing backgrounds were analyzed. For Phase 1 (five weeks), students were told to create problem scenarios for approximately 20 minutes per session based on the mathematical content taught that week. For Phase 2 (four weeks), students were instructed on the concept of problem posing, that is, the difference between analogous problems, "what if?" problems, and "original" problems (e.g., inversions), along with the incorporation of one, then two mathematical representations. Then, students created problem scenarios for approximately 20 minutes per session.

During Phase 1, almost 90% of the created scenarios were analogous to those received during instruction (not original), no matter the students' experiences with Calculus (if any) or with problem posing prior to taking the course. However, during Phase 2, most scenarios were somewhat original or original in nature. Such scenarios in Phase 2 were mostly more mathematically challenging than those in Phase 1. However, they varied in mathematical reasoning, understanding required by the solver, and in complexity. Phase 2 appeared to foster deeper mathematical thinking. Yet, as this did not occur for all student groups, more research is needed to explore ways to use the activity to foster greater originality.

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Students' Perceptions and Academic Outcomes in an Active Learning-Based Calculus I Course: Insights From Mediation Analysis

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Keywords: Active Learning, Calculus, Student Perceptions

Active learning can increase engagement, comprehension, and achievement in mathematics (Freeman et al., 2014; Fuchs & Sahmbi, 2024). Despite the widely reported benefits of active learning, students' perceptions remain mixed, with some studies noting resistance to these methods (Andrews et al., 2020; Finelli et al., 2018; Tharayil et al., 2018). This study investigates students' perceptions of the impact of active learning on their interest and enjoyment in mathematics and confidence in their mathematical abilities. Mediation analysis was used to explore whether students' perception of confidence mediates the relationships between interest in mathematics and academic achievement (grades) and interest and enjoyment. Research questions: (1) How do students perceive the impact of active learning sessions on their interest, enjoyment, and confidence in mathematics, and does this perception vary significantly depending on the teaching assistant? (2) Does confidence mediate the relationship between interest and academic outcomes (grades/enjoyment) in an active learning calculus environment?

Our analysis revealed that students' perception of the impact of active learning was positive overall. Specifically, out of the 142 students, over 40% of students reported an increase in their interest (65, 46%), enjoyment (63, 44%), and confidence (99, 70%) in mathematics after taking the calculus I course. Analysis of variance results indicated that teaching assistant (TA) was a significant factor for students' confidence, $F(3, 138)= 3.498$, $p = .017$, and enjoyment, $F(3, 138)=3.052$, $p = .031$, but not for students' interest, $F(3, 138)=2.571$, $p=.057$, at the .05 significance level. Correlation analysis shows the highest correlation (r=.86) between interest in mathematics and enjoyment in mathematics, moderate correlation (r =.60) between confidence in mathematics and the other two perception measures and weaker correlation (r=.37) between the course grade and the three perception measures. Mediation analysis was conducted to test the mediating role of confidence in the relationship between interest (independent variable) and grades and enjoyment as outcomes, respectively. Results showed that confidence partially mediates the effect of interest on grades, with a significant direct effect (b = .23, 95% CI [.05, .43], p =.02) and a significant indirect effect through confidence (b =.10, 95% CI [.01, .44], p =.04). Additionally, confidence significantly mediates the effect of interest on enjoyment, with both a significant direct effect (b = .84, CI [.74, .94], p < .01) and a significant indirect effect through confidence (b = .10, CI [.02, .18], p < .001). Pre-calculus experience, teaching assistant (TA), and small group participation score were control variables.

In line with prior research (e.g., Code et al., 2014), this study demonstrates that active learning positively impacts students' interest, enjoyment, and confidence in mathematics. Results also suggest that confidence is a crucial factor connecting interest to improved grades and enjoyment in addition to serving as a reliable predictor of success orientation and achievement (Kirikkanat & Soyer, 2018; Chue, 2020). Finally, although the TA was a significant factor on several student perception variables, further research is needed on the role of instructional support within active learning environments.

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Comparison of Graduate Teaching Assistants' Ratings of Individual Program Components in a Multi-Component Pedagogical Training Program

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Keywords: graduate teaching assistants, post-secondary education, pedagogical training components

Sadera et al's (2024) scoping review on sustainable professional development (PD) of graduate teaching assistants (GTAs), teaching assistants, and tutors, noted that continuous, discipline-specific training, both before and during teaching, was important to the development of GTA's teaching identity and self-efficacy and their attitudes towards teaching-oriented PD and research. Designed to be a continuous, multi-component pedagogical development program using a framework of community of practice (Lave, 1991; Wenger, 1988), the National Science Foundation (NSF) funded *Promoting Undergraduate Success in Mathematics through Graduate Teaching Assistant Training* (PSUM-GTT) project (Harrell-Williams et al., 2020; Manzanares et al., 2024) provided (1) practical training, grounded in relevant literature, regarding issues in undergraduate mathematical sciences instruction prior to or during their first semester of teaching; (2) opportunities to mentor and be mentored by peers; (3) K-12 outreach opportunities to gain an understanding of the student pipeline, (4) a Critical Issues in Undergraduate STEM Education/Learning (CIUSE) seminar series; and (5) instructional support through a peer TA Coach. Hence, knowledge of how beneficial each component was to the GTAs is important information for the current institutions and for other institutions considering which component(s) they might adopt.

Due to the COVID-19 pandemics' impact on institutional policy and instructional format in the early project years, assessment of early outcomes heavily relied on GTA-provided data. Specifically, this poster focuses on quantitative data from end-of-year surveys administered in each of the four academic years of program implementation (2019-2020 to 2022-2023) by summarizing and comparing GTAs' ratings of how beneficial each of the five program components were to them. Results varied across the three universities and four years, indicating that the importance of each component could be dependent on the specific cohorts of GTAs as well as other factors such as institutional policy and changes in department culture. For example, at one university, the peer mentoring program had the highest average ratings component in the first year (4.39 out of 5), which was .74 higher than the other two schools. At the same university, the K-12 outreach had the highest average rating (4.5) in the last two years. However, average ratings for outreach were low at the other two universities, with means at or less than 2. Implications for program component adaptation and adoption will be discussed.

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Implementation Supports and Outcomes of Online Professional Development Workshops

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Keywords: Professional development, online workshops, teaching

Introduction

Online professional development (PD) workshops can be more accessible for math instructors whose responsibilities, constraints, or preferences do not align with participation in in-person PD (e.g., caregiving responsibilities, under-resourced institutions, concerns for environmental impacts of travel). In response, the MAA offered a series of online workshops that covered a variety of teaching topics (e.g., content-focused PD vs. pedagogy-focused PD) in the summers of 2022 and 2023. While the workshops appeared to be effective in helping instructors make changes to their teaching (Valadez Fraire et al., 2024), we also wanted to know how participants were changing their teaching and which workshop features, if any, were impactful in supporting these changes and how these supports and implementation may differ by workshop topic.

Methods

We surveyed math instructors who participated in any of 16 online workshops in the summers of 2022 and 2023. Surveys were administered about one year after instructors took part in the workshop. Respondents were asked if they had implemented anything in their classes that they had learned in the workshop, and if so, what, and which workshop features, if any, helped them to make these teaching changes. We computed descriptive statistics for quantitative items and conducted thematic analysis for open-ended items.

Results and Discussion

Respondents reported incorporating concrete examples, group work, and pedagogy changes (e.g., classroom activities and classroom materials). Respondents identified several workshop features that aided their implementation of what they learned at the workshop, such as access to resources, concrete examples, time to plan how they would use what they learned in the workshop in their teaching, and interactions with peers and facilitators. These findings demonstrate the different ways that participants implemented what they learned during an online PD workshop and the workshop features that helped them to enact their planned implementation in their classrooms. PD providers could utilize these features to create implementation-focused PD opportunities.

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How Does the Level of Education of Students' Familial and Social Networks Influence Their Perception of Barriers to Applying to Graduate School in Mathematics?

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Keywords: Parental Education, Graduate School Applications, Social Capital

Graduate enrollment is heavily influenced by a student's parent's level of education, especially for doctoral programs (Mullen et al., 2003). One possible explanation could include the impact of students' familiar and social networks on the graduate school application process, due to the theory of social capital (Coleman, 1988). We use this framework to discuss our results. The following data was collected from 519 survey responses to a 57-question survey investigating student knowledge about applying to graduate school. The following two tables show selected results from two questions in the survey. Based on the first table, there is an association between students seeing graduate school application fees as a barrier to applying and their immediate members having less education. From the second table, one observes that students with someone in their social network with a higher degree are less likely to view obtaining letters of recommendation as a barrier to applying to graduate school. Due to the known association with educational attainment and social inequality (Blau & Duncan, 1967; Wells et al., 2011), not all students worry about the financial resources needed to apply to graduate school. Students with members of their social network who have graduate degrees are less worried about finding recommenders for their application and know what it is like to travel for graduate school. Therefore, these students who have people with higher degrees in their immediate circles have additional resources to apply to graduate school due to their social capital.

Table 1. Kruskal-Wallis test results for selected items for the question "To what extent are the following factors a potential barrier to your pursuit of graduate school?" using the level of education of immediate family member variable.

Item	Degree	*N*	Mean	Mean Rank	*H*	*df*	*p*	*r*
Graduate Application Fees	< Bachelor's	104	3.28	341.55	43.701	3	<.001	0.078
	Bachelor's	151	2.58	263.41				
	Graduate	265	2.29	232.91				
	Other	4	1.50	133.50				

Table 2. Kruskal-Wallis test results for selected items for the question "To what extent are the following factors a potential barrier to your pursuit of graduate school?" using the level of education of social network variable.

Item	Degree	*N*	Mean	Mean Rank	*H*	*df*	*p*	*r*
Obtaining letters of recommendation	< Bachelor's	44	3.07	328.99	12.625	3	.006	0.019
	Bachelor's	122	2.56	270.27				
	Graduate	346	2.36	248.62				
	Other	9	2.67	279.00				
The possibility of having to relocate	< Bachelor's	44	3.41	329.88	17.226	3	<.001	0.028
	Bachelor's	122	2.93	282.11				
	Graduate	344	2.57	243.34				
	Other	9	2.78	255.28				

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Introductory Students' Definition and Views Regarding Proof

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Keywords: introduction to proof students, student conceptions, importance of proof

In order to meet students where they are, we must first understand where they are at. As Rouhani and Zazkis (2023) noted in their RUME paper, "the existing research on conception of what a proof is and what counts as a proof is primarily focused on expert mathematicians (e.g., Inglis, & Aberdein, 2016, Weber & Czocher, 2019)". Complementary to Rouhani and Zazkis's work on students' conceptions about *proofiness*, we looked at how Introduction to Proof students defined proof and why they thought proofs were important.

Methods

The participants were undergraduate students in a public university, who were enrolled in an introduction to proofs course. Twenty students voluntarily agreed to participate and completed at least the initial reflective though prompts: 1) How would you define or describe the concept of proof? 2) Why do you think proofs are important in mathematics? We merged the students' initial and final reflective thought prompt answers to form our unit of analysis, since several students did not answer both the initial and final prompts. The authors independently analyzed and open-coded each entry before reaching an agreed upon final set of codes and themes. Independent coding was then revised before having a discussion to resolve differences.

Findings

These preliminary findings provide evidence that Introduction to Proof students reflected on four major components when defining proof and its importance: *Coherence*, *Community*, *Building Concepts*, and *Mental Processes*. Each of these four components is comprised of subcategories which will be discussed below.

In terms of *Coherence*, students primarily reflected on the importance of illustrating the validity of statements, the sequencing or logical ordering of statements, and argumentation. Although generalization, claims, and abstractions were mentioned, they were done by only a handful of students. Moreover, proof *coherence* was present for all students who reflected on communication to a *community*. However, the participants were not always explicit in the intent for communication; rather, they were vague in the notion that "You are proving…" for some purpose or would speak in terms of the collective "We", such as, "…concepts that we already know to be true". Additionally, of the students who mentioned using previous theorems, axioms, or definitions to help establish ideas, three (out of eight) did so without notions of communications. Similar to coherence, students would vaguely speak about the ideas of *Building Concepts*, which we noted as the regard of proof as a foundational component that mathematics is built upon or using known concepts to advance mathematics. Lastly, *Mental Processes* was prevalently seen as students noting the usefulness of proofs. However, several students also noted on their emotional experiences regarding proofs.

In conclusion, there is evidence to suggest that students find proofs important and useful for the foundation and expansion of the mathematical community knowledge. However, when it comes to communication and ideas of what a proof is, the notions, although present, could be more nuanced. Special attention in proof courses could be given for the importance of proof.

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Graphing as a Tool to Support Covariational Reasoning

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Keywords: Covariational Reasoning, Student Cognition, Graphing

There are limitations to using graphing tasks to assess students' covariational reasoning since students' nonnormative graphing schemes can impede their ability to graphically re-present their images of covarying quantities (Frank, 2018; Moore et al., 2024). Thus, Drimalla et al. (2020) encouraged researchers to attend to students covariational reasoning in non-graphing contexts. In this poster we document the challenges in demonstrating advanced levels of covariational reasoning without normative mathematical representations, such as graphs.

Covariational reasoning entails an *anticipation* to track how two quantities magnitudes' change together. This involves three essential constructions: (1) constructing two quantities - measurable attributes of a situation, (2) imagining each quantity to vary (variational reasoning) and (3) coupling these images of variation through a multiplicative object (Thompson and Carlson, 2017). The multiplicative object is an essential construction in **co**variational reasoning. As Saldanha and Thompson (1998) explain, "As a multiplicative object, one tracks either quantity's value with the immediate, explicit, and persistent realization that, at every moment, the other quantity also has a value" (pp. 1-2). That is, as the student imagines variation in one quantity they necessarily imagine variation in the other.

One can conceptualize a point in the Cartesian coordinate system as a multiplicative object that simultaneously entails the magnitudes of two quantities. One can then construct a graph by tracking this point in the plane as they imagine the quantities to co-vary. Through this construction the graph is a record of the multiplicative object – the point – under variation. The physical construction of tracking point in the plane allows one to present and coordinate images of simultaneous variation. In contrast, there are linguistic difficulties in expressing one's anticipation to simultaneously attend to two quantities' magnitudes as they vary together: at any moment in time the language one uses will be focused on one quantity or another. One cannot simultaneously communicate variation of the constructed multiplicative object. This is a methodological limitation to studying students' covariational reasoning outside conventional representations, such as graphs.

In this poster we use the work of Sam, an undergraduate first semester calculus student, to document how his graphing activity supported and inhibited his construction of quantities, images of variation, and a multiplicative object. We report that while Sam's nonnormative graphing schemes inhibited him in graphing the gross covariation of two quantities, the activity of try to construct a graph supported him in constructing quantities from a dynamic situation and coordinating gross images of variation. We have no evidence that Sam constructed a multiplicative object to unite these images of variation. We caution this might be evidence of a linguistic challenge, and not a limitation to Sam's covariational reasoning. Sam's work illustrates some challenges of assessing students' covariational reasoning outside the context of graphing and suggests that the action of graphing might be necessary to enact advanced levels of covariational reasoning.

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Designing With ChatGPT: A Critical Conversation

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Keywords: Task Design, Undergraduate Mathematics, Technology, Artificial Intelligence

In recent years there has been an explosion of mathematics software supported by artificial intelligence that has supported the ways students think about mathematics. With the advent of generative AI (ChatGPT, NotebookLM, Gemini, etc.), the use of AI has steadily grown among the student population. This creates the demand to both understand the use and capabilities of AI by mathematics educators and how it can be leveraged for teaching (Frieder et al., 2023; Hodge-Zickerman & York, 2024; Sapkota & Bondurant, 2024; Wardat et al., 2023). Generative AI can take on several different supporting roles (Eager & Brunton, 2023; UNESCO, 2023) including being a creative engine, personal tutor, conversational buddy, etc. We approach the use of generative AI from the perspective of task design for an undergraduate mathematics educator. In this work we will address the research question: What are the affordances and constraints of the use of AI for undergraduate mathematics task design? We will be focusing on the capabilities of AI to provide background research for task design, generate code for use in task design, provide contexts for task generation, and create entire tasks and task sequences from user prompting.

For this research, we plan on doing a self-study approach (LaBoskey, 2004) – a qualitative approach of practice, analysis, and critical conversations – to reflect upon our designing and supporting the design of tasks for undergraduate mathematics students. Both authors are experienced task designers in undergraduate mathematics education, often make use of technology in task design, and have published work on the task design. For this analysis, each of us used ChatGPT in different ways to aid in the design of tasks. We shared our dialogues with ChatGPT, read through the dialogues, and then had critical friend conversations for each dialogue about how we used ChatGPT for different stages in task design. These conversations centered on the affordances and constraints of using ChatGPT for task design, the role of the task designer in the conversation, and the possible implications/uses for undergraduates using ChatGPT in a mathematics classroom. The transcripts of these conversations were transcribed, and we will present our initial analysis of these conversations on the poster.

The four dialogues that serve as a basis for this research include topics the following topics:

- Prompting ChatGPT to explore the mathematics behind the single value decomposition and related applications to data science from a mathematical source.
- Using ChatGPT to develop coding for Python and Matlab to illustrate different calculations in linear algebra.
- Exploring the use of ChatGPT to synthesize different frameworks for task design, modify the framework according for specific considerations of the instructor, and generate tasks using the modified framework.
- Generating realistic starting points for advanced mathematical topics such as Lie groups that can be leveraged for task design.

We will build on this work by examining the intricacies of task design more broadly, and particularly how technology and AI can be effectively integrated with current task design best practices.

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Mapping Instructor Movement: Use of Classroom Space as a Communication Act

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Keywords: classroom observations, interactions, socio-cultural norms, non-verbal communication

Storylines within discourse contain the imagined roles and interactions carried by the speaker (Herbel-Eisenmann et al., 2015). Participants in a classroom act out their storylines and in doing so negotiate roles. Positioning theory describes these roles as "positioning" actors (i.e., teachers and students) with agency and authority according to their role (Anderson, 2009; Herbel-Eisenmann et al., 2015; Langer-Osuna & McKinney de Royston, 2017; Wagner & Herbel-Eisenmann, 2014). Researchers examine discourse within teacher/student interactions to understand the student experience in the classroom (Adger & Wright, 2015; Chronaki & Kollosche, 2019; Herbel-Eisenmann & Wagner, 2010; Wagner & Herbel-Eisenmann, 2008).

Discourse (capital "D") includes verbal and non-verbal communication that contributes to messages of stance, power, agency, and authority (Bishop, 2012; Herbel-Eisenmann et al., 2015; Razfar, 2012). Non-verbal communication in this context includes gestures, facial expressions, and body language. In this poster, we purport movement is also a non-verbal discursive act that communicates positioning, agency, and storyline.

Classroom observations were done in a variety of courses (mathematics education content, mathematics, and statistics) as an exploratory phase of a larger research study at a Midwestern university. The recordings were played double-speed to focus on teacher movement. Examples of mapping the instructors' movements are provided below. Each map represents the first twenty minutes of instructional time for a single observation.

Every trip the instructor made from the front and around the room was given a unique color, and stops were shown as an "x." Looking at a map, one can identify whether the instructor stops to talk with students, centering themselves within conversations or standing to the side, or if they walk, pivot, and return to the front. The proposed poster will demonstrate many of these maps with details of the positioning displayed by each.

The information gleaned from these maps provides a glimpse of movement-as-communication, which informs storylines and impacts student positioning.

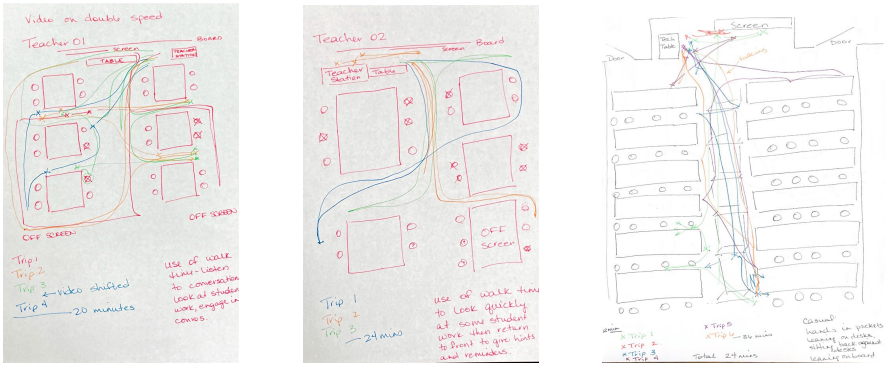


Figure 1: Examples of maps of teacher movement within the first 20 minutes of instructional time

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Towards an Awareness of Instructional Approaches in Graduate Mathematics

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Keywords: Graduate Mathematics, Classroom Pedagogies, Instructional Practices

Mathematics teaching and learning research has occurred in elementary through secondary levels for at least the past century (Perry, 1901). Researchers have debated what mathematics should be taught (Romberg, 1992; Apple, 1992; McLeod et al., 1996), how content should be taught (National Research Council, 2001; Gutstein, 2016), and even who deserves to learn math (Guttiérrez, 2018; Schoenfeld, 2002; Louie, 2017). We have learned many things from these studies such as different perceptions of mathematics learning (Nasir, 2002; Sfard, 1998), and benefits of teaching strategies that center student participation (Boaler & Greeno, 2000). Recently, there have been efforts to research undergraduate math, namely the calculus sequence, abstract algebra, and linear algebra (Kogan & Laursen, 2014; Abdulwahed et al., 2012). These efforts have had impacts on how some universities (re-)structure their math courses and have resulted in greater student success in these courses (Laursen et al., 2014). All of these results came from investigating mathematics at various levels of a learner's trajectory, but there exists a higher level of tertiary mathematics where little research exists: graduate mathematics courses.

In many PhD mathematics programs across the United States, there is a common course trajectory for students. This trajectory involves a pillar sequence of classes, typically in Algebra, Analysis, and Topology. These courses are assessed using department-created tests, which the institution of study called preliminary exams, that are designed to evaluate student understanding of the concepts related to these core courses. Different schools have different expectations for the quantity/time frame for tests and testing windows and what are considered pillar courses. Overall, the trend for many students going into a mathematics PhD programs in the United States is to take course sequences that have high stakes assessment and then the courses beyond these are evaluated by instructor discretion.

This poster shows student/instructor responses about instructional practices during different stages of a students PhD program. Research has shared very little about course sequencing in graduate programs nor how this sequencing relates to student/instructor experiences in the classroom. Through interviews with PhD students and instructors of graduate mathematics courses, as well as classroom observations in multiple courses through the program at one PhD granting midwest university, I have had the opportunity to discuss teaching strategies and observe graduate classes in action. One finding from these data shows the difference in assessment strategies used and also instructional practices. For example, one student participant explained how"the pillar courses and the after pillar courses are very different", specifically pointing to her experience in a non pillar Analysis class where the instructor would "lecture at us [students] for a couple of days, asking us for our input [and] we got to ask questions and direct the lecture, depending upon our understanding." This contrasts with her pillar experience where, "There's no active learning. There's no group work. There's no discussions really happening in class for certain instructors." This shows one example of different teaching styles that are used.

Discussing differences between classroom instruction can begin conversations about mathematical achievement in PhD programs. Who benefits/struggles from the different instructional styles? How do the different approaches to teaching impact student retention? These questions and more are very important to consider when looking at classroom practices.

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Gender, Rank, and Service: Unpacking Inequities Among Math Faculty at Research Universities

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Keywords: Service Responsibilities, Gender Identity, Math Faculty

Research consistently demonstrates that cisgender women faculty shoulder a greater service burden than cisgender men (CM), revealing a gender-based disparity in service performance and expectations (Guarino & Borden, 2017; Misra et al., 2021; O'Meara et al., 2017; Pyke, 2011). To address this issue, we aim to explore how tenure-stream faculty within mathematics departments at research-intensive institutions describe gender inequities through self-reports of time spent on service. We focused on mathematics departments and research-intensive institutions because of the absence of studies in this area and the significant underrepresentation of gender marginalized individuals (i.e., cisgender women and genderqueer (CWGQ)) (Blair et al., 2014).

Data for this analysis comes from the distribution of the Faculty Service Survey, developed and distributed via Qualtrics in the Spring 2024. We used convenience sampling through emails to reach tenure-stream math faculty at research-intensive institutions (high or very high) as defined by the Carnegie classification. Out of 218 responses, 75 participants met the eligibility criteria. We examined numeric responses on the time spent per month on various service activities. We measured these activities by a monthly timescale to capture a comprehensive view of faculty responsibilities and account for any service occurring on a once-a-month obligation. We analyzed the average hours on each type of service by academic rank (Assistant, Associate, Full), gender (CWGQ and CM), and the intersection of gender and ranks.

Preliminary findings indicate significant gender disparities in the performance of service responsibilities among tenure-stream math faculty. CWGQ faculty reported higher averages for service aligning with "feminized" service, i.e., "relational-oriented" service, (Hanasono et al., 2019) whereas CM faculty reported higher averages for service aligning with "masculinized" service, i.e., "goal-oriented" service (Hanasono et al., 2019).

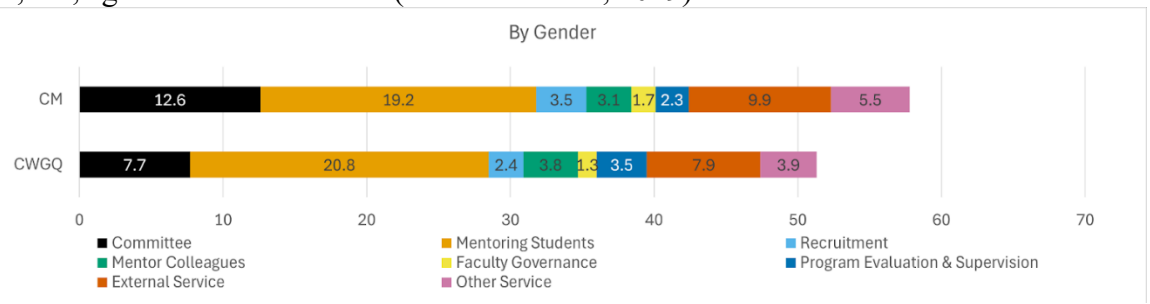


Figure 1. Stacked bar chart of monthly service hours by gender.

The data additionally revealed that Full Professors generally perform the most service hours, except in mentoring students and external service, where Associate Professors lead. Assistant Professors reported the least service overall, with notable exceptions in mentoring roles. These findings highlight the need for further analysis to understand these inequities and develop policies for more equitable distribution of service responsibilities.

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Training AI Tutors: Tutor Training of the 21st Century

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Keywords: Tutor Training, Multivariable Calculus, AI Tutor

Introduction and Theoretical Framework

AI tools like ChatGPT and other Large Language Models (LLM) are here to stay, and already in use in educational platforms like Khanmigo. These tools show a lot of potential for use in education, but in their current form appear to be most useful for semi-supervised content generation (Nye et al., 2023). While AI tutors have potential to help students learn, provide personalized guidance and expand educational support they still require training and supervision. Using a lens of undergraduate tutor training (Johns, 2024) we sought to answer the questions: How does an AI tutor respond to training and how effectively does it support students?

Methods

This study was a case study of an AI tutor developed to support students in a multivariable calculus course. Students had the option of attending an in-person problem solving session led by an undergraduate course assistant or engaging with the same problems assisted by an AI tutor. The tutor was built on the TeachGPT platform which integrates ChatGPT 4 and 4o into educational tools for personalized learning (Kestin et al., 2024). The AI tutor was trained to be a friendly and encouraging persona focused on inquiry-based learning, guiding students through real-life applications of multivariable calculus. It was directed to avoid giving direct answers, instead offering hints or guiding questions like, "What do you think would happen if...?" to foster critical thinking and reflection. For each workshop, problems and solutions with detailed instructions, including learning objectives, solution steps, common misconceptions, and strategies to address them, creating a supportive and engaging learning environment were uploaded to the platform. Data for this study were the training prompts and logs of student tutor interactions over the course of the semester.

Results

After initial training, the AI tutor was helpful to students and did not appear to hallucinate or provide incorrect information. However, it appeared to be “too helpful” in that it was likely to provide a complete solution to the problem after a short question from the student. Further training from the course instructor was needed to direct the AI tutor to provide shorter hints and guide students via questioning rather than providing full solutions initially.

Discussion and Implications

This study demonstrates one potential use of LLMs in an undergraduate mathematics classroom. The course instructor felt the fully trained model performed well as a tutor and this modality was popular with students although many students still preferred to attend in-person sessions. Further research is needed to provide more general guidance for effective training of LLMs to act as mathematics tutors for students. With the current state of this technology the role of the human instructor remains central to the success of the AI tutor.

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Supporting Playful Mathematical Problem Solving With Puzzles: Variant Sudoku

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Keywords: mathematical problem solving, mathematical puzzles, play

The development of mathematical problem solving skills has been long-recognized by a variety of stakeholders as critical for all learners (English & Sriraman, 2010; Pólya, 1945). Additionally, solving mathematical problems can be a playful, enjoyable, and beautiful process (Holton et al., 2001; Su, 2020; Williams-Pierce & Thevenow-Harrison, 2021). Yet, many students who might have become excellent mathematical problem solvers are pushed out of mathematics early on in their educational experiences and may experience mathematics anxiety or other negative emotions when they participate in undergraduate mathematics courses (Maciejewski et al., 2024; Pajares & Miller, 1994), and may struggle to experience mathematics in the playful way that mathematicians do (Bernier, 2024).

Puzzles have long been suggested as a means of introducing students to enjoying mathematics and to engaging in mathematical problem solving (Burton & Burton, 1980). In general, a puzzle can be construed as a game or fun activity with a correct solution or right answer (Kim, 2006; Salen & Zimmerman, 2004). Puzzles can also involve mathematical problem solving and playful mathematics, particularly if they invoke mathematical structures, ideas, and objects directly in their presentation. A popular example is sudoku, a genre of puzzle where the solver must use given information to fill each row, column, and 3-by-3 box within a 9-by-9 grid with each of the digits 1 through 9. Sudoku has been used to support students engagement in and learning about mathematics in a variety of contexts (Butler & Butler, 2007; Lorch & Lorch, 2008; Tengah, 2011; Wanko, 2010).

In recent years, a community of puzzle solvers and setters has developed around what is known as "variant sudoku." Variant sudoku puzzles are sudoku puzzles which use additional rules and structures on top of the standard sudoku rules. For example, the "German whisper line" restriction indicates that adjacent in cells falling along a line must differ by at least 5. Under this condition, if two cells are joined by a line and one of them contains a 4, the other must contain a 9. These restrictions, combined with clever placement, allow the crafting of puzzles that provide few or no digits initially yet are still solvable through logical and mathematical reasoning. Moreover, these additional restrictions allow for the application of additional mathematical topics beyond those permitted in standard sudoku.

Our team is working to integrate variant sudoku into undergraduate mathematics coursework and to research that integration. Currently in the works is a study into the mathematical identity development of students who participate in variant sudoku solving and setting in an undergraduate honors course. Additionally, we are developing a design-based research project to craft open-access materials for a general education course centered around solving and setting variant sudoku puzzles. These projects are in early stages; as such, our poster will present information on the projects at their stage at the time of the conference. It will also include opportunities for attendees to participate in variant sudoku solving live.

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Exploring Gender Differences in Perceptions of Undergraduate Mathematics Instruction

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Keywords: Instructional experience, gender, bold problem solving, college math

Curriculum and instructional modality are not neutral—each reflects individual choices and disciplinary norms that may be experienced differently by students depending on their background and identity (Leyva et al., 2021). The current work is preliminary work exploring ways to measure the influence of students' gendered attributes on their perceptions of their experiences in undergraduate mathematics classes. In particular, we ask: (a) What characteristics of interactions in mathematics classes result in different responses based on the student's binary sex and (b) which gendered attributes of the students relate to students' responses to surveys about mathematics classroom instruction?

Theoretical Framework

Gender is widely understood to be rooted in socialization, and we use it with this intent. Researchers have been attempting to unpack gendered attributes beyond the binary to understand the socialized tendencies associated with men and women. For example Stets et al. (2023) asked over 900 ethnically diverse undergraduate students about words in response to the phrases "Men usually are" and "Women usually are". They found that 30% of the descriptors describing men could be categorized as showing agency. In contrast, over half of the attributes found to describe women could be categorized as communal or competent. Taken together, it is evident that the concepts of personality and gender are conflated. This has implications for math experiences as Alcock et al. (2014) found that personality had predictive power above binary sex for math achievement, particularly the characteristics of agreeableness and conscientiousness.

Methods

A convenience sample of 67 students enrolled in undergraduate mathematics classes were surveyed via an electronically distributed survey link. The survey asked a battery of gender identification items and scales, including measures based on work by Stets et al. (2023). The survey also included slightly truncated versions of the "What Is Happening In the Class? (WIHIC)" questionnaire (Dorman, 2023) and the "Student Postsecondary Instructional Practices Survey in Mathematics (SPIPS-M)" (Apkarian et al., 2019; Creagar et al., 2022).

In order to answer research question 1, we used Kendall's W coefficient of concordance to determine if there are statistical differences in how males and females rate the mathematics class in the 6 subscales of the MCAQ. To answer research question 2, we looked at the correlations between the three gendered attributes and the 6 components of the MCAQ.

Results and Discussion

A variety of suggestive but small or non-significant relationships were evident that warrant further investigation and which will be elaborated on in the longer poster format. For example, Agency (from one of our gendered attribute scales) had a negative correlation with Teacher-Student-Content Interactions and positive correlations with Student-Student and Student-Student-Content Interactions. Further studies with a larger sample size may result in significance and additional qualitative analyses are also warranted to unpack observed relations.

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An Undergraduate Student's Experiences with Mathematical Proof's Values and Norms

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Keywords: Proof, Norms and Values, Poststructuralism, Unstructured Interview

I use a poststructuralist stance to interrogate some of the discourses for mathematical proofs and their relationship to their author. Discourses are anything in our world that can be thought of and brought into existence through writing or creation which also influences the way that we think about or view social entities and/or relations or even what can be said, thought, or done in particular situations (Fendler, 2010; Walshaw, 2016). Scholars have adopted a poststructuralist perspective due to its ability to question the "normalities" in different social settings (Gutiérrez, 2013). While some mathematics education studies use poststructuralist theory (Dubbs & Herbel-Eisenmann, 2021; Johnson et al., 2022; Stinson, 2008, 2013), this perspective has not yet been widely used to explore questions related to the practices of mathematical proof. I view values and norms for mathematical proof as some of the discourses that inform the mathematics community about what is "normal" in mathematical proof.

Dawkins and Weber (2017) conceptualize some values for mathematical proofs, including the aim for mathematics that mathematical truth is purely objective and should not rely on the experiences of the author or their community. Norms upholding this value include omitting how an individual may have come to understand an object or concept in a particular way in a final proof or the norm that the author leaves out how they came to build their proof as the final argument should stand on its own without the experiences of the author. Researchers have worked to support students' *access* to these and other mathematical norms and values (Rupnow & Randazzo, 2023; Vroom et al., 2024; Vroom & Ellis, 2024); here, I aim to interrogate this value and associated norms. I address the research question: What are some ways that mathematics students respond to reading about norms that adhere to the value that mathematical proofs should be independent from the author and proofs which follow or breach these norms? During an unstructured interview, I presented an undergraduate student, Sherine, who completed an introduction to proofs course, with an excerpt from Dawkins and Weber's (2017) "Values and norms of proof for mathematicians and students" along with a theorem's textbook proof and a modified version of this proof which is meant to breach the norms mentioned above within an unstructured interview session setting. After I presented each material to Sherine, we discussed how she has experienced these values and norms.

From her reading of the excerpt, Sherine questioned the convention that an ideal mathematician should conceal that they are human. Sherine said, "We know that they are human, so why would we conceal that fact?" Despite this questioning, Sherine expressed her appreciation for the textbook proof which she said was "short and clear" compared to my personalized proof that, "makes sense and it's an explanation but that's not what a proof is." Instead, Sherine felt that the personalized proof sounded as if someone were verbally sharing the proof with another person or in a lecture. Sherine's recognition of my personalized proof as something to be shared in a lecture could have implications for how proofs are presented to students. In my future research, I will continue to investigate how more students in a focus group setting respond to this excerpt and different versions of proofs.

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Undergraduate Students' Nature of Mathematics in an Introduction to Proof Course

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Keywords: Nature of Mathematics, Introduction to Proof, Student Reasoning

What is mathematics? This question, and the study of the nature of mathematics more broadly, are of interest to many mathematicians. However, many universities do not include dedicated coursework to studying the nature of mathematics. Reflecting on the nature of mathematics is important for both personal development as a mathematician (i.e., Hersh, 1998; Lakatos, 1976), as well as for the development of mathematical concepts itself (i.e., Ernest, 1985; Lakoff & Nunez, 2000).

Transition to proof courses offer a time for students to reflect on the nature of mathematics due to the shift from computational calculus courses to more abstract proof courses. Boyle and colleagues (2015) argued a transition to prof course may be the first time students are asked to become creators of mathematics rather than reproducing what they have previously seen. This course presents an opportunity to study what views these students hold of the nature of mathematics. In this study, I asked students in an introduction to proof course to respond to the prompt, "In your own words, what is math?" This question was asked multiple times through the semester after several intentional activities to force reflection were completed. Ten students responded to three prompts each over the course of the semester. Each batch of 10 responses were open coded for common responses. Nine of the ten students were math majors in their third or fourth years of the major, all having completed the calculus sequence. The remaining student was a computer science major with a mathematics minor.

As of the writing of this poster, the final writing prompt is being given to student participants. Initial results indicate an overwhelming majority (8 of 10) of students initially viewed the nature of mathematics as simply a "toolbox" to be used for other disciplines, or for problem solving. Seven of the 10 students remarked they had never formally considered the question "What is mathematics" before answering this prompt. Halfway through the semester when the question was asked again, responses started shifting slightly away from a toolbox approach to include more reasoning for validation or to answer questions mathematicians themselves ask.

Students were asked to reflect on what contributed to their evolving understanding of the nature of mathematics. Two students mentioned watching YouTube series, while the other students mentioned working with proofs in a general sense to shift their nature of mathematics. By February, I will have analyzed the third group of responses to see how this shift across the semester continues.

In this poster session, I hope to gain insight from attendees into what aspects of this research are most convincing, and any future directions such as intentional readings to include or classroom activities to include to help foster growth in the nature of mathematics.

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An Intersectional, Quantitative Analysis of Undergraduate Students' Responses to Racialized and Gendered Instructional Events in Undergraduate Calculus

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Keywords: Calculus; Equity; Black; Latin*; Women

It is widely agreed that undergraduate calculus is a gatekeeper to STEM degrees, weeding out Black and Latin* students and leading to a disproportionate loss of women (Ellis et al., 2016; Weston et al., 2019). Students' experiences in calculus classrooms can reveal how instruction reinforces inequities in mathematics learning opportunities (Leyva et al., 2021; Leyva, 2016). Our poster presents a quantitative analysis of the impact from two marginalizing events in calculus instruction on students' emotions and participation across race-gender intersections.

Previously, our team completed a qualitative study where cisgender Black students, Latin* students, and white women journaled about marginalizing instructional events in their calculus classrooms. Journaling responses were used to develop stimulus events for individual and group interviews to solicit student perceptions of these events as potentially racialized and/or gendered. We developed theoretical accounts for how seemingly neutral and commonplace events, such as ignoring a student's raised hand, can be marginalizing when accounting for broader influences of racism and misogyny (e.g., stereotypes of mathematical ability; Leyva et al., 2021). This work examines the extent to which our prior qualitative accounts generalize to a larger population.

The present quantitative study explores national survey data collected in Fall 2019 and Fall 2021 from students at 108 state colleges and universities with similar demographics to the state university where the qualitative study was conducted. Survey respondents included 322 Black, Latin*, American Indigenous, Middle Eastern and/or North African students, and multi-racial students. Our analysis focused on students' responses to two instructional events that our prior work identified as marginalizing (Leyva et al., 2021). The first event involves a student attempting to correct an instructor's mistake and getting interrupted by the instructor. The second event features an instructor laughing at a student and repeatedly not providing an answer to their question. For each event, our participants addressed if it would: (i) make them feel angry, frustrated or discouraged, (ii) impact their future participation, and (iii) make them aware of their race or gender. A factor analysis based on participants' responses to items in categories (i) and (ii) identified two scales. We report average scores on a scale (α=0.890) accounting for negative affective and participatory impacts of each event.

Results and Significance

On average, Black women (M=3.14, SD=0.60; M=3.08, SD=0.69) and white gender nonconforming students (M=3.11, SD=0.47; M=3.33, SD=0.36) were negatively affected by both events. (An average score of 3 corresponds to choosing "agree" to items indicating negative impacts on a 5-point Likert scale). Latin* women, white women, and Black men averaged above a 3 for one event. The results speak to the differential impact of instructional events by race-gender intersection and can help us create more affirming classroom spaces for historically marginalized students.

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Immediate IBL Implementation With Minoritized Students as a Professional Development Activity

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Keywords: classroom pedagogy, professional development, equity

Research has documented the benefits of Inquiry Based Learning (IBL) on student outcomes (e.g., Laursen et al., 2014). Even as new studies more carefully examine the differentiated benefits on minoritized populations (Johnson et al., 2020), professional development (PD) efforts, like the Academy of Inquiry Based Learning continue to attract many university educators nationally. The impact of such PD on participants' adoption of IBL practices have been a focus of study (e.g., Archie et al., 2022). The nature of PD usually involves workshops where people learn about teaching or how to adapt their teaching material to apply ideas from the PD. To different degrees, programs would incorporate student work or video data as a reflection material for participants. In this report we ask, *what are the impacts of immediately engaging PD participants in implementing IBL with minoritized students on the participants' teaching?*

This analysis was part of a larger project investigating instructors' development of teaching using active learning and anti-deficit perspective. Instructional faculty and graduate students at a research university applied to participate in the PD, and the year-long program went from the summer to the following spring semester. One core activity was the participants' leading a five-day IBL workshop in pre-calculus, calculus, or linear algebra. A month prior to the workshop, participants participated in four preparation meetings focusing on: 1) deficit discourse in mathematics education and teaching using IBL, 2) anti-deficit perspective and funds of knowledge, 3) writing an IBL worksheet (individual consult); 4) sharing the first worksheet and lessons about how to write worksheet; 5) sharing hopes and fears about the upcoming workshop. The five-day workshop was a part of a summer bridge program serving primarily first-year first-generation STEM students of color, and other interested minoritized students at the university. The instructors also received coaching on IBL teaching on their second day of the workshop.

Our findings come from analysis of exit interview data with a cohort of eight participants. We report on the significance of the immediate implementation of IBL with minoritized students by analyzing responses to two non-consecutive questions: "What is a major highlight of your experience participating in this program?" and "How do you view your role as a classroom teacher now?" Constant comparison (Glaser, 1965) revealed themes in the data. Four of the five common highlights pertain to implementing IBL in the workshop: *experiencing teaching using IBL, learning to design worksheets, playing a role of a guide in the classroom, broadening communication in the classroom*. Moreover, they aligned with the six themes of how participants viewed their role in the classroom eight months after the workshop, including *creating an empowering classroom community*, *shifting the thinking from instructor to students*, *actively listening to student thinking, affirming and honoring students' work, expecting students to communicate with one another, and expanding participation to include non-visible involvement.* Seven out of eight participants gave explicit examples on how they performed their roles in the classroom during their regular semesters, highlighting impacts on their practice. The emerged roles encouragingly align with many culturally responsive teaching practices (Zavala & Aguirre, 2024), which we connect to their actual experience implementing IBL with minoritized students.

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The Disconnect Between Undergraduate Standard Mathematics and Modern Applied Mathematics: A Review of Integral Transform Research

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Keywords: Applied mathematics, Information processing, Fourier analysis, Laplace Transform

Recent concerns highlight a shortage of engineering graduates to meet modern demands, in part due to the high drop-out rates in undergraduate engineering programs (Faulkner et al., 2019; Flegg et al., 2012). Various studies have attributed this to engineering students' struggles to transfer mathematical knowledge to engineering contexts (Ekici et al., 2020; Harris et al., 2015). This concern is further highlighted by Chui and Jiang (2013), who described a recent shift in modern applied mathematics from modeling through differential equations to information processing and data science, requiring a different set of mathematical concepts which have not been emphasized in the past, such as Fourier analysis and abstract linear algebra. This literature review aims to study the disconnect between the standard mathematics curriculum and modern applied mathematics. Due to the breadth of existing research, this review focuses on integral transforms (Laplace and Fourier Transforms), which provide a bridge between the standard mathematics curriculum and modern applied mathematics. The following two research questions guide the review: How are the Laplace and Fourier Transforms viewed differently between mathematicians and engineers? Is the standard mathematics curriculum effective in preparing engineers for their future studies, particularly in the field of Fourier analysis?

Methods

A literature search of mathematics education and engineering education journal articles and conference proceedings was performed with the keywords "Engineering Mathematics Education" and "Applied Mathematics Education". Articles not explicitly addressing differential equations, Fourier analysis, Fourier Transforms, or Laplace Transforms were excluded. A total of 57 articles were found, primarily in the *International Journal of Research in Undergraduate Mathematics Education*, *International Journal of Mathematics Education in Science and Technology*, and *IEEE Transactions on Education*. Open coding was performed on the 57 articles where key words and ideas along with specific excerpts were extracted. Themes and connections within the literature relating to the research questions were then identified.

Preliminary Findings

Integral transforms are viewed differently between mathematics and engineering due to the primary generating question, or question motivating integral transform use, in these contents. In mathematics, transform methods are a tool to solve differential equations. In engineering, transform methods are the key to characterizing and analyzing linear systems without finding closed-form solutions (Rønning, 2021; Rezvanifard and Radmehr, 2024). This results in key tasks, techniques, justifications, and theories involving the information processing dimension of mathematics, including complex numbers, functional analysis thinking, and the frequency domain, being underdeveloped in undergraduate mathematics courses (Ekici et al., 2020; Hochmuth & Peters, 2021). During the poster session, I plan to share findings and implications of this review. I also plan to obtain feedback regarding future directions and questions the RUME community would like explored regarding modern applied mathematics education.

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The Impact of Physics Context on Student Thinking About Vector Product Directions

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Keywords: Vector products, Student thinking

Introduction and Background

Although in mathematics curricula, vectors are typically not covered in depth until calculus 3 or linear algebra, vector quantities are used extensively in introductory physics (e.g., Knight, 2017); mathematical operations on vectors are prevalent in this context. The body of literature examining student understanding of vectors includes assessment instruments, but typically with significant focus on computational issues and/or the impact of context (Van Deventer, 2008; Mikula & Heckler, 2013; Heckler & Scaife, 2015; Barniol & Zavala, 2014; Carli et al., 2020). Other work has attended to more conceptual understanding about dot and cross products (Zavala & Barniol, 2013; Barniol, Zavala & Hinojosa, 2013). To more explicitly explore the role of context and aspects of task features on student performance, our research goals are to identify (1) how different contexts impact student ideas of vector concepts and (2) how performance on vector concepts varies with course level. We analyze how introductory physics students describe the direction of dot- and cross products in different contexts.

Methods

Multiple-choice, online surveys were administered in each semester of calculus-based introductory physics (PHY1, PHY2) after relevant instruction. Several prior tasks have had answer choices that are a mix of scalars and vectors, leaving some ambiguity about students' understanding of the vector or scalar nature of the products. To disentangle calculation errors from student understanding, we asked about the magnitude and direction of each product in separate questions. Surveys included several tasks asking about the magnitude and direction of dot and cross products, first as a pair of generic vectors in a 2-D coordinate system and second in a physical context. The physics contexts were mechanical work, magnetic flux, and the Lorentz force. We discuss the results of the direction tasks.

Results and Discussion

In the non-physics context, PHY2 students performed better than PHY1 students on both the dot- and cross product direction tasks. The most common incorrect answer, in both classes, was a direction *between* the two vectors; student explanations suggest the incorrect application of vector addition principles and ideas of combining vectors. From the non-physics to the physics context within each course level, there was no significant shift between correct and incorrect responses, for both the dot- and cross product tasks. However, there was a significant shift from *between* to other answers. The most common incorrect responses varied in the physics contexts in ways that reflect reasonable confusion with the physical quantities.

These results suggest that context does not impact overall performance but does impact which distractors students select. We are working on adopting or modifying existing analytical frameworks to explain the context dependence of student responses. Symbolic forms (Sherin, 2001), models of transfer (e.g., Lobato, 2008; Rebello et al., 2017), and dual-process theories of reasoning (Evans, 2006; Heckler, 2011; Kryjevskaia et al., 2014; Speirs et al., 2021) are three promising prospects that may provide insight on the impact of different contexts.

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The Landscape of Research on the Meta-mathematical Aspects of Definitions

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Keywords: Proof, mathematical definitions, literature review, meta-mathematical aspects

Meta-mathematical aspects of definitions (MMAD) are the general notions held about the nature, roles, functions, and characteristics of mathematical definitions (Gilboa et al., 2023). Mathematics education research has produced several rich branches of research exploring the meta-mathematical conceptions that proof learners, educators, mathematicians, and researchers hold about mathematical definitions (Edwards & Ward, 2008; Gilboa et al., 2023; Torkildsen et al., 2023; Zaslavsky & Shir, 2005). Researchers have found the MMAD play a role in students' success in producing mathematical proofs and developing their conceptions of mathematical definitions (Gilboa et al., 2023; Torkildsen et al., 2023). While the MMAD have been discussed both implicitly and explicitly in mathematics education literature, we have anecdotally noticed a trend of conflation in these aspects that may lead to confusion for novice researchers. In this project, we seek to review MMAD literature produced by the top ten mathematics education research journals to characterize the conceptions held about the MMAD by the mathematics education community. We seek to answer the questions: What MMAD have been most and least explored by mathematics education researchers? How has extant literature characterized these MMAD? What group has been most represented in research exploring conceptions of MMAD?

We conducted a systematic review of the top ten mathematics education journals, as identified by Williams and Leatham (2017). These include *Educational Studies in Mathematics*, *Journal for Research in Mathematics Education*, *Journal of Mathematical Behavior*, *For the Learning of Mathematics*, *Mathematical Thinking and Learning*, *Journal of Mathematics Teacher Education*, *ZDM: The International Journal on Mathematics Education*, *Mathematics Education Research Journal*, *International Journal of Mathematics Education in Science and Technology*, and *School Science and Mathematics*. The search was performed by identifying articles with the keywords "function of definition," "definition proof," "mathematical definitions," and "definition usage" in the body. A total of 108 articles were identified. The articles were coded separately by two researchers, searching for mentions of the meaning of mathematical definitions and concepts, with special focus on the meta-mathematical aspects (i.e. roles, functions, criteria, properties, and usage) of definitions. After disagreements were resolved, articles not implicitly or explicitly discussing the meta-mathematical aspects of definitions were excluded, resulting in 66 articles for further analysis in nine of the ten journals.

In this poster, we plan to share preliminary findings related to the implicit and explicit descriptions of MMAD as well as the meta-mathematical focus of the articles. These articles will be categorized according to the specific categories of the meta-mathematical aspects of definitions discussed. We offer a framework to clarify distinctions in the categories of MMAD, and thus aid in the streamlining of future MMAD research. Finally, we will describe the demographic (i.e. undergraduate students, pre-service teachers, mathematicians) that has been most-often featured in meta-mathematical definition research. During the poster session, we intend to gather feedback from the RUME community on future directions of the study and questions about mathematical definition research the community would like answered.

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Shifting Perceptions: Using Stories to Understand Changes in Students' Attitudes After Completing an Introductory Mathematics Course

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Keywords: Mathematical Attitudes, Mathematical Narratives

Students' attitudes toward mathematics significantly influence their mathematical learning and achievement (Turgut & Bakir, 2024). Further, studies have suggested that mathematical achievement can be a predictor in whether students graduate from high school (McKee & Caldarella, 2016). These issues persist in college, with many students underprepared for their introductory mathematics coursework (Martinez et al., 2016) and drop/failure rates over 40% in these courses (Herriot, 2006). In a previous study (Reed & Worley, 2024), we investigated students' attitudes towards mathematics by examining their mathematical autobiographies or narratives in an introductory college mathematics course, akin to College Algebra. It was found that students' stories fit into four general categories: Friendly; Familial; Antagonist; and Frenemy, with a majority of students having a negatively leaning story (Reed & Worley, 2024).

The goal of this study was to examine and categorize the reasons which students expressed had shifted their attitude towards mathematics or their mathematical narrative in a more positive direction. To investigate this, we had students in a 'Launch into the University' initiative over the summer of 2024 to complete Pre- and Post-Narratives about their relationship with mathematics. The university offered introductory English and Mathematics courses, including English 101 and College Algebra, with integrated tutoring assistance. Many participants in these classes had experienced significant educational challenges during the COVID-19 pandemic disruptions, as mentioned in the Pre-Narratives. In examining these students' Post-Narratives, we found several factors which influenced their attitudes, and helped to promote a more positive outlook about themselves as mathematics learners. Despite the prevalent negative attitudes towards mathematics documented in prior research, our analysis of students' Post-Narratives revealed several key factors that contributed to positive attitudinal shifts. These factors included: (a) supportive and effective teaching approaches that create engaging learning environments, (b) students' development of resilience and help-seeking behaviors, (c) growing confidence and sense of mathematical empowerment, (d) recognition of mathematics' real-world relevance, and (e) challenging internalized negative perceptions about mathematical ability. By identifying these transformative elements, our study provides insights into potential interventions that may help mitigate the persistent challenges of mathematical anxiety and underachievement in higher education.

In this poster session, we hope to prompt the following discussion: *how do we leverage these results to continue to shift students' attitudes in a more positive direction?*

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Sudoku: A Bridge to Proof Validation

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Keywords: proof writing, proof validation, curriculum

It is well established in the literature that undergraduate students struggle when first learning how to write proofs, writing, and validating the proofs of others (e.g. Alcock & Weber, 2005; Selden & Selden, 2003; Stylianides, et al., 2017). The literature base hypothesized that explicit instruction on proof validation would help students improve their proof validating skills, and this study attempted to measure whether that hypothesis is true.

Curriculum Components

The curriculum introduced proofs through Sudoku, inspired by Greenhill's (2011) activity. This low-barrier experience allowed all students to engage equitably in the lessons. The second lesson focused on counterexamples and encouraged collaboration. The third lesson centered on identifying assumptions and conclusions as used in proof frameworks (Selden, et al., 2017).

Lesson four focused on proof validation and was emphasized due to its application in upper-level proof courses and everyday logic. This segment of the curriculum used the language of the Toulman scheme (1958) and included a handout that walked students through two guided examples on validating proofs. The students then tried examples on their own or in small groups.

Methodology

A pre-post test model was implemented. Twenty-three participants completed the pre-test, however only five participants had paired pre- and post-tests. The pre-test and post-test were identical assessments and consisted of 10 questions. Five questions provided proofs with unknown validity and asked the participants to choose if the proof was valid or invalid and explain why. One question gave the participants both a proof and disproof of the same statement and they had to determine which one was valid. The students were also asked to write their own proof of a statement and to then discuss the validity of their proof.

Results

On the pre-test, the participants in this study performed in ways consistent with the literature (Alcock & Weber, 2005; Ko, 2010; Weber & Ramos, 2014). The one result that counters previous research findings is that these participants demonstrated a conceptual understanding of counterexample with each counterexample question validated correctly at least 60% of the time. One notable finding is that although a majority of the participants wrote a valid proof, more than 50% of the participants did not answer the question asking them to validate their own proof.

Although the sample size is insufficient to make a statistical conclusion, there is some evidence that the students effectively learned proof validation techniques from the curriculum. Across the 50 questions, the scores decreased in two instances, improved in 20 instances, and stayed the same the remaining 28 times. Even more compelling is the changes in vocabulary observed from the textual analysis. On the post-test the participants made several references to the “warrants” for claims and to the “proof framework.” This result is triangulated by the survey given at the end of the course where the majority of the respondents said they used the proof framework every time or almost every time when writing their proofs.

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Mathematical Modeling and Quantitative Reasoning: The Case of Task Design

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Keywords: Math Education, Modeling, Quantitative Reasoning, Task Design

Mathematical modeling, the process of representing a real-world situation through mathematics, has gained increasing interest across K-16 education. Engaging in modeling requires certain competencies that are unique for solving modeling tasks. Despite the prevalence of modeling across curricula, numerous studies have documented the challenges students face as they engage in modeling (e.g., Klock & Siller, 2020). One promising approach for supporting learners' modeling competencies is to have students engage in quantitative reasoning as they construct models for real-world situations (Czocher et al., 2022). Quantitative reasoning entails conceiving a situation consisting of quantities and quantitative relationships (Thompson, 2011). Scholars have argued that quantitative reasoning not only supports students' understanding of *mathematical content* (e.g., Carlson et al., 2015) but also *mathematical processes* such as proofs and modeling (e.g., Belin & Akar, 2020). However, the first step in systematically supporting students' modeling competencies via their quantitative reasoning involves developing modeling tasks that are the ideal learning ecologies for students' quantitative reasoning to emerge. The central question that guides this study is: *In what ways can quantitative reasoning be integrated into the design of modeling tasks?* We address this question by designing a model development sequence (Lesh & Doerr, 2003) that integrates modeling principles with quantitative reasoning.

The model development sequence has three stages *Model Eliciting* (phase 1), *Model Exploration* (phase 2), and *Model Adaptation* (phase 3). In phase 1, students will develop strategies for building models and constructing quantitative relationships. In phase 2, students will analyze the models they constructed through examining the relationship between quantities and their variations. In phase 3, students will apply and refine the techniques developed in phase 1 and phase 2 to model a more complex, open-ended situation with limited information. We developed four sequences each consisting of three tasks. The sequences are *optimization*, *proportionality*, *flow*, and *growth*. In this poster we highlight the optimization sequence, where students use their algebraic knowledge to construct, analyze, and interpret functions that model real-world situations. Phase one (*food truck task*) involves modeling the relationship between demands and profit of a food truck business. Phase two (*catering*) involves modeling the profit for a catering company based on seating capacity. The instructional goal for the students is to construct a piecewise profit function for varying seating capacities. Phase 3 (*honey business*) involves modeling local business' profit changes due to COVID. The instructional goal for the students is to model the relationship between price, demand, and profit through building on and expanding on the quantitative reasoning strategies developed in earlier tasks.

To test the feasibility of our tasks, we intend to implement our tasks with undergraduate students in a clinical setting. Following a design-based experiment (Brown, 1992) methodology, results from the clinical interviews will be used to modify the existing tasks or design new ones if needed. This study ultimately will provide insights into the characteristics modeling tasks that have the capacity to elicit students' quantitative reasoning.

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Not All (Former) Friends Are the Same

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Keywords: personification, math anxiety, reflection, mindset

Mathematics is often perceived to be a purely logical abstract field with no space for creativity, negatively impacting students' confidence and engagement (Boaler, 2016). Personification, attributing human characteristics to something nonhuman (Abrams & Harpham, 2012), helps students reflect on their personal relationship with mathematics and provides instructors insights into their students' mindset (Zazkis, 2015). By asking students to personify mathematics, educators encourage students to describe math as if it were a person in their lives, revealing their perceptions of the subject (Zazkis, 2015).

In this study, we analyzed students' perceptions by examining their responses to a personification prompt at the beginning and end of a Math Reasoning for the Arts course. Using Zazkis' (2015) framework of Monsters, Lovers, and Former Friends, we analyzed 80 pre- and post-personification responses across five semesters, identifying additional themes. We are still currently analyzing the data, however, preliminary results revealed categories such as Perfectionist, Mentor, Sibling, Antagonist, Current Friend, or Acquaintance. Additionally, we identified an extend theme of "Former Friends" that was combined with either Antagonist, Current Friend, or Acquaintance.

Out of 80 responses, 30 classified math as a Former Friend, defined as a friendship that has turned sour or dwindled, sometimes with the hope of mending, where an identifiable shift in or separation of the friendship is seen. Of these 30, the responses fell into one of three categories. Antagonist: (60%, n=18), where math is perceived as a bully who is hurtful, but not cruel; there is no intention of mending the relationship. Acquaintance (37%, n=11), where math is acknowledged as present in their life but is not considered a friend; there is no personal relationship. Current Friend (3%, n=1), where the friendship with math has been rekindled, or was maintained through hardships. These three extended Former Friend categories reflect distinct ways students perceive their evolving relationship with math, serving as a foundation for examining whether and how these perceptions shift over time. Building on this categorization, we investigated changes in students' views at the end of the Math Reasoning for the Arts course, after they had been exposed to mathematics in relation to art, music, and history.

Preliminary results indicate the following shifts in perception: Positive/Positive (49%), students express a new appreciation for math; Positive/Neutral (10%), students gain some respect or awareness; Neutral/Positive (23%), students show little change but develop a different perspective; Neutral (3%), students report minimal change in their feelings; Negative/Negative (1%), students feel worse and see no positive aspects; and No Change (14%), students continue to feel the same. Additionally, the content and structure of the Math Reasoning for the Arts course will be included in the presentation.

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PAC-Math Protocol: A Framework for Assessing the Professional Development of Graduate Teaching Assistants

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Keywords: Graduate Teaching Assistant, Professional Development, Program Assessment

Graduate Teaching Assistants (GTAs) play a critical role in undergraduate mathematics instruction across the U.S., and their preparedness directly impacts undergraduate student experiences (Ellis et al., 2016). Given the importance of Professional Development for Teaching (PDT) in preparing GTAs, stakeholders are increasingly interested in understanding the effectiveness of these programs (Braley et al., 2023). Key questions arise: Are these PDT programs improving GTAs' teaching performance? Are they adequately preparing GTAs for their future careers? Addressing these questions requires a robust, cyclical assessment framework that can guide departments in evaluating and improving their programs. This poster presents the PAC-Math Assessment Protocol, a new tool designed to systematically assess and improve GTA PDT programs.

The PAC-Math Protocol provides a flexible, iterative approach to assessment. It is built around six key phases: **Actions, Aims, Assessment, Analysis, Adaption, and Alignment.** These phases guide programs in defining their goals, collecting meaningful evidence, analyzing the outcomes, and making informed adaptations. The framework is intentionally adaptable, allowing programs to refine their aims and activities based on the evolving needs of their GTAs, departments, and institutions. This cyclical protocol also promotes alignment between program actions and intended outcomes, encouraging programs to sustain continuous improvement in their PDT efforts.

The PAC-Math Protocol has been piloted in a small-scale study with six mathematics department's GTA PDT programs. This initial phase asked pilot sites to focus on a specific component of their GTA PDT program (e.g. orientation, micro-teaching practice) and to use the PAC-Math protocol to guide their assessment. The pilot supported departments in initiating program assessment in a flexible way to meet their particular departmental goals. This poster will present the results of the PAC-Math pilot study, including feedback from program providers about the usefulness of the protocol. Visitors to the poster will gain insights into the effectiveness of the PAC-Math Protocol in capturing meaningful data and informing programmatic changes. By sharing these results, we aim to promote discussion about GTA PDT assessment and contribute to the broader conversation about improving the preparation of future educators in mathematics.

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Collaborative Teamwork Builds Confidence and Independence in Discrete Mathematics

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Keywords: equity, discrete mathematics, self-efficacy, agency, collaboration

For many students, discrete mathematics is a first exposure to formal logical reasoning, proofs, and proving. Undergraduates entering discrete mathematics have a long history with procedural problem solving and often attribute the authority for sense-making to textbook, instructor, or software (Herbel-Eisenmann, 2007). Like calculus, discrete mathematics shares a history of enormous disparities in outcomes correlated to demographic variables (Campisi et al., 2024). Research in undergraduate mathematics education, notably in calculus, has indicated that disparities can be addressed by linked changes in student materials and instructional methods (Bressoud & Rasmussen, 2015). Groupworthy learning opportunities for students combined with use of socio-culturally informed instructional approaches can build positive relationships among students, between students and instructor (and institution), and foster engagement in and outside of class (Archie et al., 2022; Laursen et al., 2014; Leyva et al., 2022; Yoshinobu et al., 2023).

As part of a project developing and researching the use of team-worthy lessons in discrete mathematics, the poster reports on a small study of 3 instructors (M, A, and P) and their 105 students. Classroom observations indicated that the use of team-worthy lessons changed the structure of classroom activity from primarily didactic lecture to at least 75% student-centered activity (e.g., group work, student presentation; Weston et al., 2023). However, a change in observed structure is only evidence of an affordance, not of students' experience of it (was it equitable? productive for learning?). After each team-worthy lesson students were surveyed about what went well (and not). Analysis of student responses used a framework based on social learning theory (Bandura, 1997) and dimensions of equity (Guitierrez, 2009).

The poster presents results about students' experience of equity, agency, authority, creativity, self-efficacy, and collaboration. The majority of students indicated collaboration as an aspect they deemed positive. Yet, collaboration was also the area they felt needed the most improvement! For example, two of the survey prompts were: "Share at least one aspect of your experience today with math teamwork that NEEDS IMPROVEMENT" and "Share at least one aspect of your experience today with math teamwork that was POSITIVE." For the first question, students in M's class felt mostly satisfied (60%), while A's and P's students highlighted needs for clearer instructions, improved participation, and better collaboration. For the "positive" question, students from all three instructors indicated collaboration most frequently, with at least 30% for each instructor (Carmody, 2024). As will be illustrated on the poster, these distributions reflect students' reports of their needs, both met and unmet, for establishing and sustaining a more equitable learning environment.

Poster-side conversations will explore next steps in research and practice. In particular, the project is developing a short-course for instructors who use the lessons. The short-course focuses on student-centered methods, particularly noticing student thinking, and the idea of how to balance the use of new and old instructional skills. A likely next research area is: What (re)balancing of instructor skills is needed to realize the potential of the (re)balancing of activity structures afforded by the lessons? Future work can examine how instructional skills, combined with the affordances of lesson content, contribute to the development of agency, self-efficacy, belonging, and engagement in mathematical sense-making among discrete mathematics students.

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Evidence of Equity Practices in Teaching-focused Professional Development

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Keywords: equity, novice instructor, professional development

Entry-level college mathematics classes are often taught by graduate students (about 35%) and most graduate students (more than 90%) will teach at some point (Blair et al., 2018). Research indicates that college mathematics instructors in effective undergraduate programs, particularly novice instructors such as graduate students, benefit from well-structured teaching-focused professional development (TPD) (Connolly et al., 2016; Friedlaender et al., 2014; Gehrtz et al., 2022). Those responsible for offering TPD, including those who lead workshops, courses, and seminars as well as those who facilitate TPD as course coordinators, have come to be known as *Providers* of TPD (Bookman & Murphy, 2019; Braley et al., 2023). Providers have a critical role, yet little is known about what they provide in TPD for novice instructors (Yee at al., 2023, 2024). At the same time, institutions vary in structure and resources in ways that are consequential for TPD. This leads to the questions: What are local Provider practices in the TPD they design and implement? In particular, what is the evidence that TPD is offering opportunities to learn equitable and effective practices?

This poster reports on a national effort to identify equitable practices promoted by and used in novice instructor TPD. Poster attendees will participate in refining the research design for a new extension to the project.

The two driving research questions for the new work:

1. TPD Content: What opportunities to learn about effective use of equity practices for teaching undergraduates are novice college mathematics instructors offered in TPD?
2. TPD Processes: In what ways do current Providers of TPD attend to equity practices themselves in the design and delivery of activities, formats, and topics in TPD?

Project research is driven by social cognitive theory (Bandura, 1997) and aligned with the five perspectives on instructional practices in the Mathematical Association of America's *Instructional Practices Guide* (IPG, 2018): *classroom, design, assessment, equity, and technology*. As published, the topic of equity practices was an appendix in the IPG. Subsequent research and development have refined and specified equity practices in detail for college teaching and learning in sciences, technology, engineering, and mathematics (National Academies, forthcoming).

The focus of the poster is designing research for gathering evidence of equity practices for research question 1 on TPD Content. The first step in the extension project is the development of a short (15-minute) survey for Providers. The research goal is collecting evidence about the current opportunities novice instructors have in local TPD programs to develop knowledge, skills, and productive dispositions related to equity practices (e.g., student-centered and instructor-decentered types and uses of formative assessment, identifying and leveraging "productive struggle," uses of roles in group activities). Given the time-constraint on the survey, it is especially important that items are written to gather information about Provider actions and actual in-use TPD formats, activities, and topics (rather than Provider or program intentions). Attendees will give feedback on potential survey items in support of design that will increase the feasibility of the survey to elicit information to address the two research questions.

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Using Theory to Design Professional Development for Calculus I and II Instructors

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Keywords: Professional Development, Calculus, Active Learning, Equity, Teaching Practices

Calculus courses are a gateway for students entering STEM fields, but they sometimes hinder students, especially those from historically marginalized groups, from pursuing STEM (Bressoud et al., 2015; Ellis et al., 2016; Kogan & Laursen, 2014; Seymour & Hewitt, 1997). Thus, there is a need for instructional innovation in Calculus and the creation of professional development (PD) programs to better equip instructors to support students' success. Researchers have identified characteristics of successful college calculus programs, some of which include the "use of student-centered pedagogies and active-learning strategies" (p. 145) and diversity, equity, and inclusion practices (Hagman, 2021). There is a need for researchers to design PD programs that train calculus instructors to implement active learning and equitable teaching practices. Our project contributes to this need by designing such a PD program for Calculus I and II instructors at two Hispanic Serving Institutions, a research university and a community college. For this poster, we report on how varied theoretical frameworks inform this PD's design.

The literature on active learning and equity in mathematics is vast. There are myriad theoretical frameworks and recommendations for teaching practices that instructors can use to equitably implement active learning (e.g., Freeman et al., 2014; Mathematical Association of America, 2018). Thus, it can be a challenge for PD designers to identify the main characteristics of active learning and equitable teaching practices that are essential for instructors to learn to implement. To address this, we employed a qualitative analysis on several articles and books (i.e., Bartell et al., 2017; Freeman et al., 2014; Gutiérrez, 2009; Kuster et al., 2018; Laursen & Rasmussen, 2019; Reinholz, 2023), as well as the National Council of Teachers of Mathematics' (2014) list of effective teaching practices and the Mathematical Association of America's (2018) Instructional Practices Guide, that outlined active learning or equitable teaching practices. We inductively coded (Miles et al., 2013) each manuscript to identify the active learning or equitable teaching practices listed in frameworks, examples, or recommendations for instructors. We then performed axial coding (Miles et al., 2013) to group the coded practices into broader categories, including: *(i) use meaningful engaging math tasks, (ii) facilitate student collaboration, (iii) elicit, notice, and use evidence of students' mathematical thinking, (iv) orchestrate class discussion on students' mathematical ideas, (v) attend to participatory equity and access in the classroom, (vi) foster an inclusive class community, and (vii) equitably assess student work*.

We used these categories to inform the design of our PD program for Calculus I and II instructors. The PD program consists of a workshop series over two days before the beginning of the course with follow-up biweekly meetings for the instructors in a professional learning community (Stoll et al., 2006) to reflect and receive feedback on their implementation of the equitable active learning teaching practices they learned. Each workshop was designed to train instructors to implement techniques pertaining to one of the broader categories of equitable active learning teaching practices. The literature-based categories we identified were productive for informing the design of our PD workshop series, so we recommend for researchers and practitioners to use this theoretical framework to inform the design of their PD programs for instructors. Future research can further refine this framework and address the effectiveness of PDs designed with this framework of equitable active learning teaching practices.

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Distractor-Based Distractors: Adventures in Assessment Item Analysis

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Keywords: Assessment, Item-Level Analysis, Distractors

Motivation and Context

This poster presents a methodology for capturing underlying student conceptions based on their answers to a series of multiple-choice questions. We quantitatively and qualitatively examine publisher-supplied assessment items across four exams in College Algebra.

Item-Level Assessment Analysis

To appropriately capture students' mathematical conceptions, we perform an item-level analysis utilizing the following statistical measures: Item Difficulty (ID), Rasch Logistic Model (RLM) Estimated Item Difficulty (EID), and Point-Biserial Correlation (PBC). Item Difficulty (ID) refers to the proportion of students that correctly answered a given item. The PBC coefficient measures the correlation between students' performance on a single item and their performance on the exam as a whole. The RLM is a mathematical log function that relates the probability of a (correct) response on an item to the person's mathematical ability and to the item's difficulty. In this model, students have the highest probability of answering items with EID near their estimated ability (Krabbe, 2017). Unlike PBC, RLM takes into account a student's performance on each item in relation to the rest of the class' performance on the item. Example statistics are presented in Table 1 for two items on the exam.

Table 1. Item Response Analysis for two assessment items. Both items have similar difficulty, that is, Item Difficulty and RLM Estimated Item Difficulty are both consistent, but 4C22 is predictive of student performance on the exam (PBC close to 0.5), while 4C10 is not (PBC close to zero).

IRT Analysis			
Item	RLM EID	PBC	ID
4C10	2.91	0.03	0.36
4C22	2.75	0.38	0.39

Distractor Identification and Analysis

To investigate differences in item-level statistics, we investigate items' distractors – plausible but incorrect solutions (Chamberlain Jr. & Jeter, 2024). This leads to a classification of distractor types that can capture students' content misconceptions and provides evidence for distractor type impact on item-level statistics.

Discussion and Conclusions

These analyses and the items of interest they point to raise the following questions: What do these distractors tell us about student thinking? Why do high performing students get these questions right and low performing students get them wrong? To extend a multiple-choice assessment to fill a more formative role, it is important to align choices of distractors with the ways students might be thinking. Right/wrong dichotomy of questions in the analysis limits the amount of predictive power when it comes to student thinking.

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Undergraduate Students' Conceptualizations of Four Big Ideas of Mathematical Modeling

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Keywords: Mathematical modeling, Thematic analysis, Big ideas

Mathematical modeling situates mathematical problem solving and critical thinking squarely in students' day-to-day experiences, broadening students' understanding of mathematics in everyday decision-making. Mathematical modeling is the *process* of creating mathematical representations to understand and make decisions about the world. When students model, they bring their individual experiences to understand a problem and inform their solution strategy. Thus, it is likely that the *modeling process* will unfold in different ways for different problems. Rather than focusing on modeling as a series of steps in a cyclic process, Arnold et al. (2021) described mathematical modeling in terms of four big ideas that characterize it.

1. Modeling begins and ends outside the mathematical world.
2. Modeling deals with situations that are open and complex.
3. Modelers exercise judgment when investigating problems.
4. Modelers decide when a solution is good enough.

We investigated how undergraduate students conceptualized these four big ideas during their study of and engagement with mathematical modeling. We designed and taught an upper-division, semester-long course in mathematical modeling at two different public land-grant universities. We created and implemented four modeling projects throughout the semester in our respective courses, with each project illuminating the four big ideas of modeling. We collected student work on these projects, along with survey data, and facilitated a written correspondence between undergraduates at the two universities. In those communications, our undergraduates reflected on their overall understanding of mathematical modeling and described their conceptualization of these four big ideas.

We conducted a thematic analysis (Braun & Clarke, 2006) on these data using an open and axial coding scheme (Strauss & Corbin, 1998). In our initial read of the data, we noted which of the four big ideas of mathematical modeling were prevalent. Then, using descriptions of the big ideas, we developed and used codes to describe the undergraduates' conceptualization of the four big ideas. Overall findings indicate that the big ideas were a tool to help undergraduates organize their own understanding of the process of mathematical modeling. We found that within undergraduates' conceptualizations, the four big ideas were connected and interdependent; when discussing one big idea, undergraduates also included connections to at least one other big idea. Tension also was evident between undergraduates' conceptualizations of Big Idea 2 and Big Idea 4; while they expressed excitement about the openness of a modeling problem, they also struggled with the notion of having to decide for themselves when their model and its solution were good enough. Lastly, we detected evidence of empathetic critical thinking skills in undergraduates' conceptualizations as they described how the modeling projects that focused on addressing justice in their own communities fostered their sense of mathematics and community.

In this poster, we (a) summarize the four big ideas of mathematical modeling, (b) describe the modeling projects we developed and implemented, (c) describe undergraduates' conceptualizations of the four big ideas, (d) summarize our findings, and (e) provide recommendations on how to facilitate undergraduates' learning of and engagement with the four big ideas of mathematical modeling.

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Towards a Unified Theory of Abstraction: Abstracting and Reducing Abstraction as Reciprocal Categories of Mathematical Practices

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Keywords: Abstraction, Reducing Abstraction, Unified Theory, Abstract Algebra, Disciplinary Practices

In the undergraduate classroom, abstract algebra concepts can be presented at varying levels of abstraction. Traditional lecture-based instruction presents concepts abstractly, and students often rely on their own cognitive strategies to make sense of them (Hazzan, 1999). Instructors using inquiry-based teaching methods often begin at a lower level of abstraction via concrete examples, guiding students to a higher level of abstraction through activities by which they discover and formalize the abstract properties of algebraic structures (for example see Ernst, 2016; Larsen et al., 2013). Many courses use a combination of teaching methods, requiring students to navigate between higher and lower levels of abstraction in both directions.

Existing theories of abstraction usually address activities and/or cognitive practices in one direction. Hazzan's (1999) reducing abstraction framework, expanded by Melhuish et al. (2018), classifies cognitive strategies that abstract algebra students use to *reduce* abstraction level. Other frameworks in the literature, including the generalization framework (Ellis et al., 2017) and Advancing Mathematical Activity (Rasmussen et al., 2005), describe ways in which students' understanding of mathematical topics *increases* in abstraction level through activities (referred to as disciplinary practices) such as generalizing, defining, and symbolizing.

Drawing on the existing literature and informed by my observations, I propose a unified theory of abstraction that connects multiple theories of learning through their approaches to abstraction. This unified theory comes out of my own need to make sense of observations of students' abstracting activities within the context of my dissertation study investigating abstraction in a groups first versus a rings first abstract algebra course, in which I observed the students and instructor moving fluidly between more and less abstract representations of abstract algebra ideas. Thus, this theory positions *abstracting* and *reducing abstraction* as reciprocal sets of linked disciplinary practices. This positioning reflects the bidirectional nature of navigating between levels of abstraction and conjectures that fluency in both directions is an indicator of student learning.

Abstracting encompasses the practices by which students move from less abstract to more abstract, including practices such as decontextualizing, relating, extending, generalizing, and symbolizing. *Reducing abstraction* encompasses the practices by which students move from more abstract to less abstract, and includes practices such as contextualizing, exemplifying, computing, and embodying. I define these practices within five categories of abstraction. Three of the categories (relationship between the object of thought and the thinking person; reflection of the process-object duality; complexity of concept of thought) were defined in Hazzan's (1999) original framework and the fourth (degree of precision/formality) added in the expanded framework of Melhuish, et al. (2018). I add a fifth category, interpreting abstraction as the degree of concrete embodiment of the concept of thought, based on observations and the literature on concreteness fading (Fyfe et. al, 2014). I define some of the practices by which students abstract and reduce abstraction within each category and provide examples of how students use these practices from my observations of an abstract algebra class.

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Argument-Mirroring Proof-Texts: A Tool for Leveraging Students' Incoherent Thinking to Effect Greater Coherence

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Keywords: calculus, chain rule, coherence, clinical interview, teaching experiment

This report details a teaching experiment (Steffe & Thompson, 2000) which consisted of clinical interviews (Clement, 2000) and exploratory teaching interviews (Sellers, 2020) designed to perturb selected students' incoherent thinking. In short, this study addressed the following research questions:

1) In what ways are students' understandings of the chain rule coherent with one another?
2) How do students respond to proof-texts which mirror their own incoherent thinking?

Theoretical Perspective

Glasersfeld (1988) describes schemes as quanta of knowledge held by an individual. Per Thompson (2008), incoherence is a lack of consistency between schemes. I used Toulmin's (1958) method to identify within students' arguments the claim – an assertion to be proven, data – evidence used to support the claim, and warrant – why the data is relevant to the claim.

Data Collection and Analysis

Ten students from a public research university in the American southwest participated in the initial clinical interviews, designed to examine the subjects' cognition regarding the chain rule of differentiation and its fundamental ideas e.g., continuity, differentiability, composition of functions, subjects were each presented with three functions and asked where each function was defined, continuous, and differentiable. The student was encouraged to think aloud and make annotations as appropriate.

Analysis focused on partitioning each student's transcript into arguments and identifying relevant claims, data, and warrants. These facets of a student's argument were then compared to one another and where appropriate (in my estimation), categorized as incoherent in one of four ways, including *incoherent claims* – two or more claims result in an impossibility and *justifications not broadly applicable* – the warrant from one argument would cause the claim from another argument to be altered. From each source of incoherence, *argument-mirroring proof-texts*, conditional statements and justifications built from the participant's reasoning were generated and presented to them to examine how they responded to their incoherent thinking and whether they take measures to resolve it.

Results and Discussion

The bulk of this report focuses on the four subjects who participated in both the clinical interviews and the argument-mirroring proof-text phase of the study and the effectiveness of the argument-mirroring proof-texts in particular. The latter method showed great effectiveness in a pilot study (Ruiz, in press). Furthermore, argument-mirroring proof-texts are highly individualizable, have the potential to be applied to a wide variety of contexts inside and outside of mathematics, and as such, the method represents a novel contribution to the field.

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Exploring the Intersection of Social Justice and Quantitative Reasoning: Engaging Non-Stem Undergraduates With Payday Loans

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Keywords: Financial inequality, Social Justice, Compound Interest, Quantitative Reasoning

Social Justice Mathematics (SJM) represents a broad and evolving space within mathematics education. SJM integrates mathematical concepts with social justice issues and aims to achieve dual goals: enhancing students' mathematical understanding while fostering critical awareness of social injustices (Gutstein, 2003; Frankenstein, 1983). Much of the existing work on social justice mathematics has focused on middle and high school students (Gutstein, 2003; Stinson, Bidwell, & Powell, 2012), as well as preservice and in-service teachers (Gonzalez, 2009, 2003; Bartell, 2013). Although there are a few studies with undergraduate students in disciplines such as calculus (e.g., Erksine et al., 2022), computer science (Padiyath et al., 2024), and statistics (Conway & Nguyen, 2020)), there is limited research on how non-STEM undergraduate students engage with SJM, leaving a gap in understanding how these students respond to lessons designed to integrate mathematical content with social justice themes. Thus, we ask the question: "In what ways does integrating social justice themes, such as payday loans, impact non-STEM students' understanding of financial inequality and/or concepts of simple and compound interest?"

We draw from the work of Branson and George (2024), who explored the connection between payday loans and mathematical concepts, providing a framework for examining how undergraduate students relate to these concepts through the lens of financial inequality. This study was conducted at a university in the North Eastern U.S. Participants in the study were enrolled in a quantitative reasoning course for non-STEM undergraduate students, primarily from a business administration program. The aim of the course is to equip students with essential mathematical skills and apply them to real-world problems.

In this poster, we present a case study examining students' pre-existing perspectives on payday loans and their disproportionate impact on marginalized communities. We addressed our research questions through pre- and post-surveys assessing students' attitudes toward both mathematics and social justice, followed by interviews before and after the lesson. The data were coded, revealing key themes, such as students' views on financial systems and how these systems disproportionately affect marginalized groups. This poster will present emerging themes from our findings, illustrating how the integration of social justice issues into the quantitative reasoning course influenced students' understanding of both mathematical concepts and financial inequality.

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Student Perceptions of Inequities and Benefits of Active Assessment Strategies in the Math Classroom

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Keywords: active assessment, active learning, student experience

This study investigated student responses to active assessments in place of timed in-class exams in mathematics courses. *Active assessment* refers to assessment approaches that align with active learning paradigms (e.g., group presentations, portfolios; Katz, 2022). This research explored student perceptions of the benefits and the potential inequities of active assessment strategies. The poster will highlight both the student-identified advantages and areas of concern related to fairness, relevance, and resource availability. Findings contribute to ongoing discussions on equitable assessment practices in mathematics education for meeting the needs of all learners (Katz, 2022; Black and William, 2009; Laursen and Rasmussen, 2019).

Methods

Data for the study was collected through a two-stage process during the Fall 2023 and Spring 2024 semesters. Across the two terms, a total of 278 students from 13 different lower and upper division undergraduate mathematics courses completed a survey (during class time) which gathered their perceptions and experiences regarding the benefits and inequities of the particular active assessment strategies implemented in the course in which they were enrolled. The courses were taught by 8 participating instructors who implemented active assessment strategies. Following the survey, a subset of voluntary participants were selected for follow-up small-group interviews, for more in-depth discussions about their experiences. The qualitative data from these surveys and interviews was transcribed and analyzed to identify key patterns and insights related to the benefits and inequities of active assessments.

Results

Preliminary findings indicated the most common themes arising for perceived benefits included reduced anxiety, development of social skills, and connection to real-world applications. Common themes that emerged for potential inequities included uneven group dynamics (e.g., relative amount of work done by different group members) and format accessibility (e.g., assessments calling for advanced writing or linguistic skills). More than two-thirds of students responded that they preferred the active assessment strategy over a timed in-class exam. Overall, results suggest that, for students, active assessment benefits outweighed potential inequities.

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Developing CALC: Developing Conceptual Assessments of Learning in Calculus

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Keywords: Calculus, Assessment, Productive Understandings, Integrals

Introductory calculus is a gateway to study in STEM disciplines. It introduces concepts about rate of change, accumulation, and co-variation, which are foundational concepts for modeling across STEM fields. Thus, assessing student understanding of calculus concepts is important to researchers and practitioners. Currently, no curriculum-independent concept inventory for introductory calculus has been validated and found to conform to accepted educational testing standards (American Educational Research Association, 2014; Gleason, White, Thomas, Bagley, & Rice, 2015). In 2007, Epstein and colleagues presented the first Calculus Concept Inventory (CCI) to evaluate the effectiveness of teaching methods on conceptual learning in calculus. However, the inventory was not research-based, was narrowly designed as a single multiple-choice summative exam (Epstein, 2007, 2013), and Gleason et al. (2015) found that the CCI failed a thorough psychometric analysis. Our purpose is to develop a reliable, well-validated curriculum-independent concept assessment to test productive understandings of calculus.

Instead of developing one assessment for all of calculus, we decided smaller, modular assessments for the various calculus topics would be more helpful. Initially, we focused on the topic of definite integrals and accumulation. We created a taxonomy of concepts related to definite integrals and further narrowed our initial focus to definite integrals involving area or other applications. Literature on student understanding of definite integrals shows two main themes. First, students who view integrals as adding up pieces and understand the pieces quantitatively are more successful in problem solving (e.g. Sealey, 2014; Simmons & Oehrtman, 2023). Second, many students over-rely on integrals as area without being able to unpack the meaning of the quantities that form the area (e.g. Sealey, 2006; Jones, 2015).

Productive understandings involve procedural and conceptual understandings. We utilize Hiebert and Carpenters' (1992) distinction between *procedural understanding*, where students solve problems using a predetermined sequence of actions, and *conceptual understanding*, where students solve problems by relying on their networks and relationships of ideas within the problem. Since many seemingly conceptual calculus problems are familiar to students and can be solved with only a procedural understanding, we aimed to create conceptual questions which, in particular, test students' ability to reverse-engineer why their problem-solving methods work.

We developed and refined seven initial assessment items which contained both standard tasks and slight variations, which were intended to disrupt application of memorized procedures and required application of conceptual knowledge. Based on clinical interviews with 15 students at 3 institutions, we observe that students' responses to these variations reveal insight that may further distinguish productive conceptual understanding from memorized procedures. Future research will involve designing additional small suites of conceptual questions on derivatives and limits. We will eventually invite other researchers to contribute to the development of these assessments in other areas of calculus.

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Interdisciplinary Aspects of Linear Algebra: Student-identified Applications in STEM

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Keywords: linear algebra, Interdisciplinary Applications, STEM Education

Motivation and Research Questions. Linear algebra bridges abstract mathematical principles with advanced applications across diverse STEM fields (Stewart et al., 2022). Students often perceive its concepts as disconnected from their fields of study to their disciplines (Stewart et al., 2019). This study investigates: What is the nature of discipline-relevant examples identified by students at the beginning of an applied linear algebra course?

Literature and Theory. Existing literature highlights the importance of linear algebra in solving real-world problems across STEM and beyond. Stewart et al. (2019) emphasizes the need to connect abstract concepts like matrices and eigenvalues to practical applications, while Muhammad & Saidi (2024) demonstrate its role in data science through tasks such as dimensionality reduction and predictive modeling. Wang (2019) and Arangala (2023) emphasize the relevance of conceptual understanding in fields like machine learning, engineering optimization, and cryptography. Beyond STEM, linear algebra's applications in areas such as digital art and forensic psychology showcase its versatility (Stewart et al., 2022). This study aims to enhance teaching strategies that bridge theory and practice.

Data Sources and Methods. We analyze online discussion board responses from 66 U.S. undergraduate linear algebra students, grouped into five major-area categories: Applied Mathematics, Biomathematics, and Mathematics Education; Computer and Computational Science; Mechanical, Electrical, and Biomedical Engineering; Statistics, Actuarial Science, and Economics; and Non-STEM majors. Early in the term, students were asked to identify an example problem from their area of study that uses linear algebra. Data-driven codes capture the range of examples students provided (deCuir-Gunby et al., 2011). Examples provided by students were classified as Q&A (when a specific question and sample solution was provided, illustrative examples (when a demonstration of a particular idea was provided), or for descriptions of broad ideas (general principle in their discipline was provided). Examples were further analyzed for dimensional context (low or high) and core linear algebra concepts.

Results and Discussion. The representation of majors in our sample of 66 student responses was: Applied Mathematics (~16.67%), Computer Science (~24.24%), Industrial Engineering (~19.70%), Statistics and Actuarial Science (~34.85%), and non-STEM majors (~4.55%). The majority of responses were coded as Q&A (~39.29%) or illustrative examples (~44.64%), with about (~16.07%) coded as broad ideas. High-dimensional examples accounted for 62.5% of responses, prevalent in Computer Science and Statistics, while low-dimensional examples were more frequent in non-STEM fields and Engineering. The most frequently cited topics included Matrix (35 mentions, ~20%), matrix operations (29 mentions, ~16.5%), and eigenvalues (23 mentions, ~13%), with significant overlaps observed between disciplines. For instance, optimization techniques connected Engineering and Computer Science, while transformations bridged art and non-STEM applications. These patterns varied within and across disciplinary clusters, reflecting both shared and unique applications of linear algebra concepts. In our poster, we will feature a selection of examples across majors and linear algebra topics to highlight trends in dimensionality issues and overlap of discipline specific knowledge and linear algebra in endeavors focused on student selected topics.

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Coding- and Spreadsheet-Integrated Tasks to Enhance Students' Quantitative Reasoning and Covariational Reasoning for Exponential and Logarithmic Functions

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Keywords: Coding-integrated tasks, Quantitative reasoning, covariational reasoning, logarithmic and exponential functions

Quantitative Reasoning (QR) is defined as a conceptualization of quantities as measurable attributes of an object(s), reasoning about and operating on quantities, and quantitative relationships in learning functions (Smith & Thompson, 2008). QR sets a foundation for covariational reasoning (CR); CR is the cognitive process involved in coordinating two quantities that covary while paying attention to how they change with respect to each other (Carlson et al., 2002). Studies have examined students' QR and CR skills across various mathematical domains, including exponential relationships (Castillo-Garsow, 2010; Confrey & Smith, 1995; Ellis, Ozgur, Kulow, Williams, & Amidon, 2012), trigonometry (Moore, 2010), rate of change (Johnson, 2012, 2015a, 2015b), functions (Carlson et al., 2002; Johnson, 2012, 2015a, 2015b), logarithmic-exponential functions (Ferrari-Escolá et al., 2016), the fundamental theorem of calculus (Thompson, 1994), graphs of functions (Carlson et al., 2002; Moore, Paoletti, & Musgrave, 2013), logistic function (Altindis et al., 2024), and differential equations (Castillo-Garsow, 2010). While research highlights the importance of QR and CR across various domains, these skills are complex due to the need to process multiple quantitative relationships simultaneously. Thus, we developed Python-integrated tasks to teach logarithmic and exponential functions.

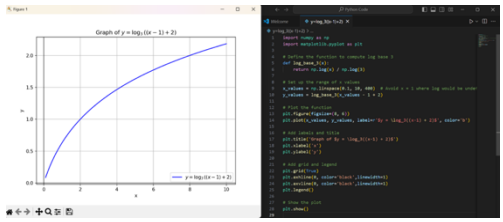


Figure 1. A Logarithmic Function Graph Created with Python Code

This poster will report on the affordances and challenges of coding and spreadsheet-integrated tasks and how these tasks allow students to model and visualize quantities and quantitative relationships. Using multiple representations (symbolic, graphical, numerical) enhances their understanding of function concepts. We will also report on preliminary testing of these tasks with high school students and their open survey responses. Additionally, we will discuss how students engage in multiple representations of mathematical phenomena, using Python codes to simulate and visualize the relationships between varying quantities. Furthermore, we will report that integrating coding into math lessons significantly increased students' engagement with the material, making abstract concepts more tangible and interactive. These findings highlight how Python coding can serve as a tool for enhancing students' quantitative reasoning and covariational reasoning skills, ultimately contributing to a meaningful understanding of logarithmic and exponential functions.

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Undergraduate Students' Positions and Interactions during Group Work in a Calculus Class

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Keywords: group work, student interaction, positioning, calculus

Group learning in the mathematics classroom aims to encourage deeper individual learning through creating a sense of autonomy and making space to negotiate new ideas. When a student proposes a new idea to the group, they reposition themselves with increased authority and autonomy; furthermore, the acceptance or rejection of new ideas may impact the success or failure of a group in reaching their desired goal (Barron, 2003; Cobb, et. al., 2009; DeJarnette & González, 2015). From this perspective, examining the positions of group members during collaboration may provide insight into what makes a group is successful. The central goal of this research is to evaluate how the interactions within groups of students impact individual learning. In this study, we examine undergraduate students' group interactions while learning to express distances within the Cartesian coordinate system.

The primary data set includes 31 videos that were taken in an undergraduate Calculus II course at a small, private college from 2022-2024. The recordings show groups of 2-3 students working on an "Interpreting Graphs for Calculus Activity" during a 50-minute class period. The activity begins with expressing distances on number lines and then advances to representing distances on the Cartesian plane. Data was collected from the same group of students in an individual pre-test, two questions on their third exam, and one question on their final exam. From these videos, we narrowed the dataset to 27 episodes involving pairs of students, roughly one to five minutes in length, which contained what we considered to be significant interactions. Significant interactions included discussions around the validity and certainty of a proposed solution, instances in which students shared different solution paths, and those in which students provided new ideas, or the students or instructor gave feedback on a proposed solution.

Currently, we are applying a coding scheme for interactional positions (Parr et al., 2023; in preparation) to identify the roles that students were taking on during these episodes of interaction, expanded from coding schemes developed by Lo & Ruef (2020) and González & DeJarnette (2015). The central codes that we have found to be relevant to this dataset are codes of primary knower (K1), secondary knower (K2), and primary uncertain knower (K1-U). Based on the application of these codes, we created categories to represent themes of interaction we found within the groups: Independently Working (each student displayed K1 and K2), Shared Understanding (each student displayed K1), Shared Uncertainty (each student displayed K1-U and K2), and Single Understanding (one student displayed K1 and the other displayed K1-U and K2). These categories capture both individual student roles and the nature of the students' interaction. Notably, of the first 16 coded episodes, we have identified 10 (62.5%) as Single Understanding, 3 (18.8%) as Shared Understanding, and 2 (12.5%) as Shared Uncertainty.

These differences in collaborative interaction between groups indicate the complexities of mathematical learning within a group. Analysis is ongoing to examine the relationship between these categories and positional codes with students' individual performance on related tasks following the group interaction.

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An Examination of the Self-Regulation Strategies of Students in an Inquiry-Based Learning With ALEKS Support Course

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Keywords: self-regulated learning, inquiry-based learning, ALEKS, developmental

Extensive research has shown that a vital component of student success in mathematics classes is the ability to self-regulate (e.g., Bol et al., 2016; Isaacson & Fujita, 2006; Kramarski & Gutman, 2006; Ley & Young, 1998; Mevarech & Fridkin, 2006; Throndsen, 2011). Self-regulated learning (SRL) is the ability of a student to take an active part in their learning, by setting goals and motivating themselves, using different strategies to complete their learning task, and by reflecting on their learning and strategy usage (Zimmerman, 2002). A prominent model of SRL and the one being used for this project is Zimmerman's (2002) model, which has three phases: the forethought phase, the performance phase, and the self-reflection phase (Zimmerman, 2002). In addition, there has been research into different classroom practices to combat the issues of low academic achievement, such as more active-learning style classes (e.g., Freeman et al., 2014) and the use of technology (Cosyn et al., 2021). A specific type of active-learning model is known as Inquiry-based learning (IBL), where students are given investigative tasks that allow them to be engaged in the creation of important mathematics (Laursen & Rasmussen, 2019), and has shown promising results in university courses (e.g. Kogan & Laursen, 2014; Laursen et al., 2014). The intelligent-tutoring software ALEKS has also been adopted in many classrooms (Cosyn et al., 2021), where research has seen positive results when the software was used as a supplement to a regular classroom (e.g., Boyce & O'Halloran 2020; S. Sun et al., 2021). My project focuses on how students self-regulate their learning in classrooms that utilize inquiry-based learning (IBL) and ALEKS software.

This project integrates student questionnaires and classroom observations to determine how students self-regulate their learning in this novel classroom environment. The surveys include a numerical measure of students' SRL skills using a subset of the Motivated Strategies for Learning Questionnaire (MSLQ; Pintrich & De Groot, 1990). The classroom observations use the Toolkit for Assessing Mathematics Instruction – Observation Protocol (TAMI-OP; Hayward et al., 2018), while the intensity of IBL is measured using the Inquiry-Oriented Instructional Measure (IOIM; Kuster et al., 2019). I examine the self-regulated learning scores of students and the intensity of inquiry-based learning to look for relationships. This poster will present some of the observational data collected, along with the numerical values of the SRL skills, to answer the questions: (1) Is there a relationship between the intensity of IBL and students' self-regulated learning skills? (2) Are there differences in the intensity of IBL across sections? By analyzing how classroom practices may relate to self-regulated learning skills, we can then begin adjusting these practices to focus on improving the SRL skills of students in future courses.

For five different sections of this coordinated class with all different instructors, interesting differences were observed through the TAMI-OP, as well as differences in the implementation and intensity of IBL via the IOIM. For example, some sections had higher scores on the TAMI-OP, reflecting a more active-based classroom, but lower IOIM scores. In addition, some sections had higher IOIM scores but had lower self-regulated learning skills in students.

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Developmental Math: Is A Traditional Approach Serving Underprepared Students?

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Keywords: developmental, undergraduate, higher education, math education, co-requisite

This poster presents emerging results from a data-focused investigation of the effects Developmental Mathematics courses on student success at a large, comprehensive university. A growing research base about Developmental Mathematics suggests that effective Developmental Mathematics teaching requires a multifaceted overhaul from traditional models requiring multi-semester pre-requisite course sequencing, high stakes test placement, and front facing lectures by the professor. In particular, the literature suggests that co-requisite models, improving extracurricular supports, providing holistic placement methods, accelerating the developmental pathway, and enhancing professional development are all methods to support students progressing through the developmental pathway (Childers & Shi, 2021; Emblom-Callahan et al., 2019; S. S. Jaggars et al., 2014; S. Jaggars & Stacey, 2014; Miller, 2022; Park et al., 2018; Parker, 2019). Before implementing any changes, assessments need to be made on a case-by-case basis to determine which models are appropriate.

In the current study, I aimed to conduct a preliminary investigation of student success at a single university based on enrollment in a traditional algebra track developmental pathway. This institution's developmental math pathway is characterized as traditional because it is: a pre-requisite path from beginner to intermediate to college algebra; co-requisite offerings are only available to a specific subset of students; minimal active learning is utilized. My research question was: To what extent do students successfully proceed through a traditional developmental mathematics pathway? This research will add to the knowledge base of the field by contributing to the literature regarding failures of traditional pathways.

Methods

In order to examine student success rates, I accessed data from the Institutional Research board at the institution. I compiled demographic data, math course history and grades, and term GPA for all students enrolled in the algebra pathway (two sequential non-credit preparatory courses and College Algebra), during Fall 2021, Spring 2022, or Fall 2022. These data were analyzed to compare pass rates across demographics and various entry points to the pathway.

Results

Data analysis revealed that student success in this traditional developmental mathematics pathway is low. Students who enrolled in the lowest offered course only succeeded in passing college math in the minimum number of semesters seventeen percent of the time. The data also indicated that students that were failing the developmental courses had GPAs about a full letter grade lower than students that were passing the courses. These findings add to the research consensus that traditional developmental pathways do not well serve the aspirations of underprepared students. Future work will examine pathway interventions (such as course acceleration, implementation of active learning, providing more holistic placement methods, and enhancing professional development) to improve these outcomes and enable students to succeed.

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Relationship Among Classroom Climate, Self-Concept, and Interest in Undergraduate Mathematics Courses: What Can We Learn From a Literature Synthesis?

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Keywords: classroom climate, classroom environment, math self-concept, math interest

Mathematics courses have become barriers for undergraduate students, including those pursuing STEM-related degrees (Olson & Riordan, 2012; Leyva et al., 2022). Some possible explanations for the difficulties that students have in mathematics coursework are the lack of belief in their mathematical abilities (Adlawon et al., 2022) and the lack of perceived relevance, as students wonder, "When am I ever going to use this?" (Chazan, 1999). The two aforementioned psychological processes, self-concept, and interest in mathematics, that influence learning are affected by several factors, including the multidimensional construct of the classroom environment.

Much research has focused on specific pedagogical practices teachers adopt, students' views of those practices, and the effect of particular practices on learning outcomes. Some studies explored the relationship between specific pedagogical or instructional practices and student learning outcomes in terms of achievement-related outcomes and affective outcomes (e.g., Kerrigan et al., 2020; Downing, 2020). For instance, culturally relevant pedagogy in college algebra positively affected students' attitudes toward mathematics and increased confidence in mathematics (Downing, 2020). However, a few studies have collected students' perceptions of the classroom environment more generally using established surveys (e.g., University Mathematics Classroom Environment Questionnaire) and evaluated the relationship between those perceptions and students' cognitive and affective outcomes.

In the present review, I synthesized quantitative studies examining how students' perceptions of their classroom environment are related to their interest and self-concept in math. In the poster, I include a table of the statistical evidence that the included papers showed of how different dimensions of the classroom environment are associated with constructs of students' self-concept and interest in college mathematics.

Results indicate that most studies used correlational, multiple regression, and structural equation modeling techniques on data collected using various measurement scales of classroom climate, self-concept, and interest in math. Findings demonstrated positive correlations (values ranging from 0.2 to 0.58) or absolute standardized path (values ranging from absolute 0.11 to 0.2) characterizing the relationship between students' perceptions of the components of classroom climate and their self-concept and interest in mathematics (e.g., Wang et al., 2017; Yin et al., 2020).

From the literature results on this poster, RUME researchers could identify research gaps of classroom conditions influencing affective outcomes and consider their classroom environment's impact on students' self-concept and interest in mathematics. This is an important avenue for future research as improving students' learning experience will positively impact persistence in pursuing STEM degrees. I discuss how future research could build and test a theoretical structural model to investigate the causal relationships between the various dimensions of the classroom climate and students' self-concept and interest in mathematics grounded in sociocultural theories, Krapp's (2007) person-oriented theory of interest, and Bronfenbrenner's (1977) ecological systems framework.

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Mathematical Wellbeing as an Inclusive Framework for Equity: A Systematic Review

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Keywords: mathematical wellbeing, literature review, culture, organizational change

For many undergraduate students, navigating the demands of mathematics education can feel like navigating a pressure-cooker system. The term "mathematical wellbeing," coined in 2010, describes how individuals relate to mathematics in ways that are positive, functional, and sustainable throughout their lives (Clarkson et al., 2010; Part, 2012; Hill et al., 2021; Almora Rios, 2024; Hunter & Hill, 2024). Although relatively new, research on mathematical wellbeing has grown rapidly. This poster presents initial findings from a large-scale systematic review of the emerging literature on mathematical wellbeing, focusing on the rationales authors use to motivate its study. Specifically, this study highlights how centering mathematical wellbeing in organizational change efforts can promote student persistence in introductory math course sequences and foster positive, lifelong relationships with mathematical practice.

Methods

The Preferred Reporting Items for Systematic Reviews and Meta-Analysis (PRISMA) guidelines were used as a framework for data collection (Page et al., 2021). Following prior systematic review methods in mathematics education research (e.g., Darragh, 2016; Radovic et al., 2018; DiMartino et al., 2022; Cevikbas et al., 2022), the article corpus for review was constructed using databases in psychology and education research (ERIC, APA PsycInfo, ProQuest Dissertations & Theses Global, Web of Science Core Collection, Academic Search Complete, Education Source), international conference proceedings (2021-2024 International Group for the Psychology of Mathematics Education (PME) conference), citation searching, and gray literature found on Google Scholar (within first 300 results; see Haddaway et al., 2015). The search criteria used in this review was: (Keyword:Education AND Title:math*) AND Abstract/Title:(wellbeing OR flourishing OR happiness OR thriving OR satisfaction OR eudaemonia OR "mental health" OR "quality of life") after considering the usage and frequency of various synonymous search terms to 'wellbeing' in academic journals (see Iasiello et al., 2024; Butler & Kern, 2016; Veenhoven, 2015; Hill et al., 2021). Eligible studies were published in English and centered (not just utilized) the mathematics education context at any level. Articles passed screening by focusing on the conceptualization, measurement, or theorization of concepts related to the persistence of math learning or practice. Articles related to job satisfaction, in-service teachers, educators, the faculty workplace, or distance, online, computer-based, or e-learning, were excluded, as well as book reviews and oral communications from conference proceedings.

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Native and Non-Native English Speakers in an Introductory Proof-Based College Math Course

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Keywords: Non-Native English Speakers, Introduction to Proof, English Learners, Language

Research on the influence of language on mathematics learning at the K-12 level (see de Araujo et al., 2018, for a review of this literature) suggests that in primarily English-speaking classrooms, perceptions of non-native English speakers as "deficient" can harm their learning of mathematics, but with appropriate supports, non-native English speakers can flourish. This study compares native and non-native English speakers' perceptions of the effects of their language status with their performance on proof-based assessments at the university level in a course taught and assessed in English. We sought to answer two questions: (1) to what extent do non-native English speakers perceive their language background as an advantage or a disadvantage in an introduction to proof course taught in English? And (2) to what extent do assessment outcomes between native and non-native English speakers align with students' perceptions?

Methods

Data were collected from a discrete math course focused on proofs at University of Manitoba where English was the language of instruction. Just before the end of the term, a questionnaire was sent to all students enrolled in all sections of the course. This questionnaire asked each student to indicate (Q1) whether they considered themselves a native speaker of English or a non-native English speaker and (Q2) whether they believed their English-speaker status contributed positively or negatively to their learning in their discrete math course. Respondents' itemized final exam scores were also obtained. The final exam in the course contained 12 questions with two questions consisting of two sub-parts; we treated these sub-parts as independent questions, yielding a total of 14 questions for analysis.

Respondents were sorted into two groups: native English speakers and non-native English speakers, as indicated in their response to Q1. Each respondent was then assigned a Perception Score (PS) of -1, 0, or +1 based on their response to Q2. A respondent was assigned a PS of -1 if they indicated a belief that their language status negatively impacted their learning, a PS of +1 if they indicated a positive impact, or a PS of 0 if their response was neutral.

A *t*-test was performed comparing the PS of native English speakers against the PS of non-native English speakers. Additionally, a *t*-test was performed on each final exam question, comparing the numerical scores earned by native English speakers against those earned by non-native English speakers.

Findings

The native English speakers ($n_1 = 40,\ M_1 = 0.675,\ SD_1 = 0.616$) had a significantly higher PS than the non-native English speakers ($n_2 = 52,\ M_2 = -0.0769,\ SD_2 = 0.788,\ p < 0.001$). There was a significant difference in scores on only two questions from the final exam: Q6(a) ($M_1 = 0.363,\ SD_1 = 0.660;\ M_2 = 1.010,\ SD_2 = 1.374,\ p = 0.00386$) and Q7 ($M_1 = 6.475,\ SD_1 = 3.772;\ M_2 = 8.298,\ SD_2 = 3.365,\ p = 0.0184$); in both cases, non-native English speakers outperformed native English speakers. These preliminary results suggest that, despite perceptions to the contrary, non-native English speakers perform at least as well as native English speakers on proof-based assessments in university math courses taught in English.

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Exploring Students' Understanding of Belonging and Self-Efficacy in a Transformed Introductory Mathematics Course

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Keywords: sense of belonging, self-efficacy, diversity, inclusion, culturally responsive teaching

Achieving diversity, equity, inclusion, and justice (DEIJ) remains a major challenge within STEM fields, especially at the higher education level. Particularly, at The Ohio State University, students from minoritized groups in STEM have felt othered and isolated within the STEM environments. To address disparities in participation and retention of minoritized students in STEM, our HHMI grant provides both financial and structural support to foster racial, ethnic, and gender justice. The HHMI-supported courses offer culturally responsive pedagogies (CRP), active learning opportunities, and smaller classroom sizes. As a part of the project, this poster focuses on accessing the pedagogical changes in one of the four foundational STEM courses, Math 1151, which has undergone integrated DEIJ-driven adjustments.

This poster investigates how students' experiences in a transformative mathematics course shape their sense of belonging (SB) and self-efficacy (SE). Research shows that higher SE and SB are key to student success and retention (Bandura, 1977, 1993; Hausmann et al., 2007; Zimmerman, 2000). Using the mixed-method approach, we compare pre- and post-class surveys results focused on these constructs, based on the Leibowitz et al. (2020) survey of SB, students' perceptions of CRP, and SE. Additionally, we analyzed open-ended questions to offer insights into how their course experiences impact their SB and SE.

The Fall 2024 study included a sample of 766 undergraduates who completed both pre- and post-class surveys, which included five-point-Likert-scale and open-ended questions. The sample was 82% first-year students, with 6% enrolled in HHMI sections. Analysis showed modest changes: students' SE slightly decreased, while perceptions of CRP increased. Group comparisons between HHMI and non-HHMI sections showed higher mean scores in HHMI, but not statistically significant pre-post changes. Further investigation of class standing revealed that first-year students reported significantly greater increases in CRP perceptions compared to upper-class students. Several themes emerged from the open-ended questions on how instructors have supported and could better support students' learning, including providing clear explanations and engaging environments which were cited for both improvement and appreciation. The most common response for improvement was that students already felt well-supported by their instructors suggesting that factors outside of this course (e.g. prior academic experiences, other course engagements, peer relationships) may have influenced the quantitative results. Although no statistically significant increases were observed in HHMI courses, the mean scores for SB and SE were already relatively high at baseline—4.02 and 4.14, respectively—aligning with participants' satisfaction with their instructors in open-ended responses. As the project progresses, we anticipate observing further improvements.

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Exploring Computer Science and Mathematics Students' Cognitive Processes in Discrete Mathematics

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Keywords: Discrete Mathematics, Undergraduate Student Reasoning, Statistics

Introduction

Discrete mathematics lies at the intersection of mathematics and computer science and provides a landscape for mathematics education researchers to analyze the interaction between mathematical reasoning and computational thinking. In this poster, we provide preliminary findings from our analysis of undergraduate mathematics and computer science (CS) students' problem-solving strategies on a set of discrete mathematics questions.

Methods

Twenty-two students were interviewed in the Fall 2024 semester through think-aloud protocols (TAPs) to elicit their mathematical activity (e.g., Reinhart et al., 2022). In these TAPs, students were asked to verbalize their thinking as they solved mathematics problems. Mathematics and CS undergraduate students from four cohorts were interviewed. The first and second cohorts consist of 4 first- and second-year students in CS and 2 first- and second-year students in mathematics, respectively. The third and fourth cohorts consisted of 8 third- and fourth-year students in CS and mathematics, respectively. Students were presented with an assessment protocol consisting of 13 discrete mathematics problems. The problems included the following concepts: functions, combinatorics, game theory, recursion, set theory, logic, recursion, decision trees, and graph theory. As a preliminary pass through the data, all questions students answered were graded as either correct (1) or incorrect (0). A more detailed analysis will be conducted as we move forward with the project.

Results

For this poster, the analysis compares and contrasts how students from the four cohorts performed on the protocol questions. On average, mathematics and CS majors answered 7 out of the 13 questions on the protocol. Of the 7 answered, the math cohorts had a median score of 73%, while students in the CS cohorts scored 70%. A Wilcoxon rank sum test was conducted to evaluate differences in scores between math majors (n=10) and CS majors (n=12). The results indicated no significant difference in scores between the two groups ($W=59.5$, $p=1.0$). The CS cohorts exhibited larger variability in their assessment scores compared to the math cohorts. For a more in-depth analysis, a cluster analysis to inform the identification of natural groupings of students will be provided on the final poster.

Discussion

Preliminary analysis indicates that students from both majors performed about the same on the assessment protocol. However, interviews with the different cohorts of students revealed many different solution strategies and varying levels of sophistication. Our goal with this analysis is to yield fruitful insights into student reasoning in discrete mathematics and contribute to more effective mathematics education at the undergraduate level.

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Examining Series Convergence via Script Writing: The Case of Jordan

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Keywords: calculus, script writing, series convergence

Undergraduate students in the United States typically encounter the topic of infinite series in second-semester calculus courses (Kung & Speer, 2013). Students study techniques for determining convergence of infinite series which also rely upon leveraging prior knowledge about limits and convergence for both functions and sequences. As such, McDonald et al. (2000) assert the necessity for students to have "a rich sequence schema … to fully understand series as the limit of a sequence of partial sums" (p. 98). Similarly, Oehrtman et al. (2014) found students engaging in guided reinvention for sequence convergence capable of giving a definition for series convergence that aligned with its concept definition. Other research on student understanding of series convergence focuses on understandings of series and meanings for series-related concepts including ways students conceptualize and think about series convergence (e.g., Eckman & Roh, 2022; Martínez-Planell et al., 2012). In addition, Eckman and Roh (2024a, 2024b) and Larsen et al. (2024) explore how students interpret summation notation or develop notation describing partial sums of a series while Alcock and Simpson (2004) and Martin (2013) examine the type of imagery students rely upon when examining convergence of power and Taylor series. Researchers acknowledge that students face difficulties learning infinite series and that meaningful learning activities should facilitate the development of deeper understandings of series and series convergence (e.g., Lindaman & Gay, 2012; Martínez-Planell et al., 2012).

This exploratory case study aims to examine second-semester calculus students' understanding of series convergence and identify ways that script writing elicits aspects of their understanding. Script writing (Zazkis & Zazkis, 2014), used primarily with prospective teachers in mathematics courses for engaging in an imagined role-play through the process of writing a script, can also be used to explore or assess mathematical understandings (Zazkis, 2018). We examine student understanding through the theoretical lens of concept image and definition (Tall & Vinner, 1981). This poster presents findings related to the following research questions: (1) What aspects of student understanding of convergence of infinite series arise when working a scripting task versus a non-scripting-based task? and (2) In what ways do scripting task responses convey a student's concept image and definition regarding series convergence?

Jordan (pseudonym) participated in two interview sessions spaced two weeks apart, taking place in weeks 11 and 13 of a 15-week semester. Jordan completed the course midterm exam on series two weeks prior to the first session. The first session included a 75-minute task-based interview that involved general questions and tasks on sequences and series along with a 30-minute scripting training activity. The second session involved Jordan creating a script on series convergence within 45 minutes followed by a 45-minute interview focused on Jordan's script.

During the first task-based interview, Jordan confounds the sequence of partial sums of a series with the terms of the series. Jordan also claims that a series converges given that the terms of the series converge to zero. In the task-based script writing interview, Jordan's partial understandings persisted, but their script exposes these partial understandings in a more robust manner and provides distinct opportunities to convey their understandings of the concept definition. This exploratory work provides some evidence for using script writing as a viable alternative assessment of student understanding of series convergence.

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Research-based Curriculum for Developing Mathematical Tools for Upper-Division Physics

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Keywords: Curriculum, Physics, Calculus, Differential Equations

As part of a larger project that investigates the gap between mathematics that is taught in math courses and the mathematics used in upper-division physics courses, we present curriculum on Separation of Variables and Taylor Series. While students are likely to be first introduced to these mathematical concepts in a math course, there exist specific instances where and how physics uses these tools in teaching content. Both mathematical tools are incorporated into the courses of Mechanics, Electricity & Magnetism as well as Quantum Mechanics.

Separation of Variables and the Taylor Series expansion, as used in a physics classroom, often have different emphasis and utility compared to when they are taught in math classrooms. For instance, the Taylor Series is used to expand a complicated equation about a small quantity within the equation in the limit as another quantity becomes very large, thereby simplifying the expression. The infinite Taylor Series is typically truncated to allow for further insight into an approximate expression that's useful for analyzing complex systems.

Methodology

The tutorials were developed to allow students work through the relevant mathematics using a sequence of physics examples from different content areas (the aforementioned Quantum Mechanics, E&M and Upper-division Mechanics). Despite using course-relevant symbols and phrasing, the tutorials attempt to avoid the use of sophisticated physics content. This is intended to alleviate the need to get caught up on physics concepts so that students can gain experience and focus on the complexities of the mathematics at play. Additionally, multiple versions of the tutorials have been created, such that the first topic encountered would be most relevant to the course in which it was given, with additional, less scaffolded examples are intended to boost students' familiarity and skill with each mathematical tool. The tutorials were vetted by experts in mathematics and physics at several points in their development. Furthermore, one-on-one interviews with undergraduate and graduate students were conducted with numerous modifications made to address areas of confusion. Finally, both tutorials have been used in classroom settings where independent student groups working through the tutorials were recorded, transcribed and analyzed.

Results

While initial results are positive–both tutorials were successfully deployed in a classroom environment and generated desired discussion among students–the level of student activity and interaction varied. Engagement in the Separation of Variables Tutorial was limited compared to the Taylor Series Tutorial; however, it is unclear if this was a result of existing classroom dynamics or an indication of needed modifications in the tutorials.[1]

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Investigating Outcomes for Students Repeating Calculus I

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Keywords: calculus, repeated course

In the undergraduate STEM research community, research has focused on interventions to decrease DFW (students earning grades of D, F, or withdrawing from a course) rates in introductory courses (Norton et al., 2018) or has delved further into why students leave STEM pathways in college. (Seymour & Hewitt, 1997). Johns Hopkins University (JHU), like many universities, has had a number of initiatives focused on supporting students taking introductory STEM courses, including the Gateway Sciences Initiative, and the To, Through, and Beyond initiative focused on students who are first generation and low-income students. Both initiatives focused on interventions to increase student success in introductory STEM courses including calculus.

One set of students we are specifically considering is students retaking calculus because they don't pass the first time. Work has been done studying students retaking developmental math at community colleges (Canfield, 2013; Bickerstaff et al., 2022) and those retaking calculus at a mid-sized public university (Dibbs, 2019). We hope to add to this growing literature by considering students retaking calculus at a small private and research-focused university.

At JHU, students can drop a course without a transcript mark until the seventh week of the fifteen-week semester, likely encompassing students facing challenges similar to those with DFW outcomes. To include this broader group, we will track these students as well, referring to them collectively as DDFW students. This approach represents a key divergence from existing literature, which typically focuses solely on DFW outcomes.

Through the examination of placement data, surveys, and interviews with DDFW students, we aim to describe the what, when and why of the DDFW students' experiences and gain insight into what helps these students succeed.

By documenting commonalities among students who successfully pass calculus on the second try, and identifying factors that contributed to failing, we hope to propose interventions to help increase the pass rate of calculus for both first time takers and repeating students. This poster will present an overview of the literature to date and research questions posed to investigate the outcomes for students who retake calculus.

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Planning Professional Development to Support Mathematical Play

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Keywords: playful mathematics, calculus, teaching assistants, professional development

Our team has been conducting research on mathematical play and learning for some time. Recently, we began examining how Calculus I undergraduate students enrolled in a large lecture class can be supported in experiencing mathematical play during their sections. The first author implemented playful calculus tasks focusing on covariation in five different sections, and we have been analyzing that data to identify what teacher moves supported or discouraged playful mathematical engagement. In our discussions, we have identified a key tension between *establishing a playful environment* and *supporting mathematical content goals.* Particularly, the low-stakes nature of mathematical play (Williams-Pierce & Thevenow-Harrison, 2021) may sometimes be in conflict with the explicit mathematics learning goals, and vice versa. In this poster, we share some preliminary findings from this pilot study, with a particular focus on teacher moves. Our future plans involve conducting professional development (PD) with teaching assistants (TAs) so they can implement mathematical play opportunities in their sections. We consider a key part of that PD to be teacher moves that carefully thread that tension between establishing playful environments and facilitating productive mathematical learning.

For example, a key part of mathematical play is establishing an environment students are comfortably *playing* in. Play often involves exploration, agency, and learning from low-stakes failure ((Ellis et al., 2024a; Ellis et al., 2024; Williams-Pierce, 2019), which can be quite the contrast with the more traditional format of large lecture classes. Even the much smaller class sizes of the sections in this study served as more intimate extensions of the lecture, with desks arranged to face the TA at the chalkboard. In fact, in multiple of our sections, students walked into the classroom, saw that the desks had been rearranged into collaborative groups, and walked out again to double check the room number. In each section, we saw evidence that the first author engaged in ***nonjudgmental teacher moves*** that supported students in experiencing a more playful environment by the end of class. For example, a common teacher move was nonjudgmental contrasting of different student responses, such as following up a student sharing their graphed line by choosing the next student to be one who graphed a curve. By the end of class, students were more likely to be working loudly together, laughing, and taking ***public mathematical risks*** by volunteering to share their work even when they were unsure of the work or themselves. Fostering a newly playful environment is a sensitive process that works in tension with more explicit mathematical guidance – nonjudgmental teacher moves supported playfulness, but may not have efficiently improved covariational reasoning. We are continuing to explore this highly productive tension between play and content learning, and will discuss this in more detail in our poster.

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Developing a Coding Scheme to Analyze Instructor Questioning in Mathematical Discussions

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Keywords: instructor questioning, large group discussion, active learning

Mathematical discussion is a central feature of mathematics instruction, which is shaped by the types of questions an instructor poses and the ways that students respond. Existing literature has introduced frameworks delineating types of questions that an instructor can pose, such as open and closed questions (Sullivan & Lilburn, 2002), higher-order and lower-order questions (Brahier, 2020), and open questions not calling for reasoning (Vacc, 1993). Additional work has also looked at the types of questions that could be asked that connect to relational practices (Wood et al., 2015), e.g., "How are we feeling about this?". Yet, we do not know how an instructor's questions impact the way students begin to develop rich, self-prompted responses.

We aim to investigate the following research question: How can an instructor use questioning and evaluation to motivate students to autonomously contribute toward mathematical discussions in more meaningful ways? In this poster, we outline the development of our coding scheme, present preliminary findings on mathematics classroom instruction, and discuss the nuances of the coding scheme. As a theoretical framing, we adopt one of Anthony's (1996) constructivist formulations for active learning, which characterizes students as mentally engaged in active thinking which promotes intellectual involvement (Kyriacou & Marshall, 1989). Incorporating this perspective into instructor-facilitated mathematical discussions, we assume that if a student provides a high-level response to questions posed by the instructor (i.e., by supporting an answer with details about their process or reasoning), then they are demonstrating an internalized understanding of the mathematical material.

The dataset used for this study consists of two 30-minute class recordings of an undergraduate mathematics content course for future elementary teachers taught by Author 2 taken from the middle and end of the semester. From the transcripts, each solicitation of input from Author 2 and each student's contribution was logged using the developing coding scheme. The coding scheme was then developed through multiple iterations of data analysis. The EQUIP tool (Reinholz & Shah, 2018) served as an initial coding scheme, including codes for Solicitation Type (*What*/*How*/*Why*), Length of Talk, and Student Talk Type (*What*/*How*/*Why*/*Other*). After a first iteration of coding, we created codes that accounted for non-mathematical types of questions, such as personal questions fostering relationships with students and logistical questions. After a second round of coding, we created codes that accounted for whether a *What* question was *Open-Ended* or *Closed-Ended* as well as codes that allowed us to identify non-verbal ways of participation.

Preliminary findings show that students demonstrated richer (e.g., more contributions coded as *How* or *What* codes) and more extensive mathematical responses later in the semester, which featured more varied types of instructor solicitations. There were multiple instances in both segments where students provided higher-level responses than the solicitation type. Through this data, we aim to stimulate conversations about the multiple ways instructors can facilitate high-level mathematical discussion and further refine the coding scheme. Subsequent work will incorporate this revised scheme into a larger analysis of an entire semester of the course.

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Critical Consciousness Activation Through Undergraduate Mathematics Curriculum

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Keywords: critical consciousness, curriculum, teacher education

Critical consciousness (CC) has been advocated as a necessary part of mathematics learning through culturally relevant pedagogy (Ladson-Billings, 1995). The ways in which CC is conceptualized varies greatly (Freire, 1970; Ladson-Billings, 1995; Jemel, 2017; Watts, Diemer, & Voight, 2011). Furthermore, the idea of *critical* mathematics is also approached from various ways in literature (e.g., Gutiérrez, 2009; Skovsmose, 2013). Thus, operationalizing CC in mathematics education research can occur from a variety of lenses leading to challenges for researchers. In this poster we present our approach to identifying evidence of prospective mathematics teachers' (PTs) CC through a construct that we call *critical consciousness resources* (CCRs; Witt, Lozano, & Anhalt, 2024) as they engage with a culturally responsive precalculus curriculum. Our prior work (Witt, Lozano, & Anhalt, 2024) presented CCRs as, "ideas that elevate a person's awareness or sense of agency to 'engage the world and others critically'" (Ladson-Billings, 1995, p.162). In this poster, we present our expanded methodology for analyzing CCRs.

Methods

Prior work on CC in mathematics, and critical mathematics education have been primarily theoretical in nature. This study can be situated alongside others who have worked on critical mathematics curriculum (e.g., Brantlinger, 2013; Gutstein, 2006; Gutiérrez, 2013). However, we aim to contribute a conceptualization of what makes an emergent idea about contextually-rich mathematics *critical* while engaging with such curricula. Our poster will present our approach to analyzing how two cohorts of prospective mathematics teachers activated CCRs in various ways through engaging with a precalculus task *grounded in place* (Lozano, 2023). The task we feature comes from a place-based, mathematically authentic and rigorous curriculum that centers the identities and cultures of students from the Southwestern United States. Through PTs' "I-statements" (e.g., "I will try to inspire [my students] by talking about the possibility of change, and how they can better the world."), we coded whether the statement was *critical* or *dominant* (cf. Gutiérrez, 2009), centered mathematics or the context, and determined a level of agentic disposition (e.g., awareness or agency). Through a thematic analysis (Clark & Baun, 2017), we describe broader salient ideas represented in the CCRs across multiple PTs.

Findings and Discussion

We offer this conceptualization as an analytic tool to help mathematics education researchers consider the kinds of CCRs that may be activated while teaching with asset-based, contextually and culturally rich tasks such as those featured in our poster. Our findings indicate that several levels of agentic dispositions were uncovered as PTs engaged with the task. We argue that teaching to activate CCRs contributes to equitable and affirming learning environments in undergraduate mathematics classrooms. These are important research considerations for gaining insights into how mathematics curriculum can critically impact the students who engage with it.

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GTA's Categories of Perception of Precalculus Students' Covariational Reasoning

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Keywords: graduate teaching assistants, covariational reasoning, student thinking

Herbst et al. (2023) published four categories of perception teachers use when considering student contributions: normativity, serviceability, responsiveness, and framing. The authors argue that though attending to the mathematical sophistication of a students' work is important, the task of teaching requires teachers to attend to more, stating that these categories provide additional relevant insights teachers can and should make. Though described with a geometric problem, the authors argue their relevance to other topics. In this study, I used this framework to identify potential categories of perception when interacting with students engaged in tasks offering opportunities to engage in constructing and interpreting graphs via covariational reasoning—reasoning about two quantities changing together. Thompson and Carlson (2017) argued for the importance of this reasoning in supporting students' conceptions of rate of change in precalculus courses and beyond. The Bottle Problem (Carlson et al., 2002), in which students are given a bottle and asked to construct a corresponding volume-height graph, has been widely used to support this end. In this study, I analyzed how graduate teaching assistants (GTAs) reason about various bottle problems and then assess video of precalculus students working on the same problems to understand how these GTAs perceive these students' reasoning.

The group of 11 graduate students were enrolled in a one-credit hour seminar on teaching undergraduate mathematics. In a course assignment, the GTAs first read an introduction to procedural vs. conceptual understanding, which includes an introduction to Carlson's Precalculus Concept Assessment (Carlson, Oehrtman, and Engelke, 2010) and potential student responses to a bottle problem (Abell et al., 2024). Then, the GTAs completed 8 total bottle-graph pairs and submitted a 3-5 minute video of their reasoning. Then, the GTAs watched two precalculus students explain their reasoning on the same problems and submitted their reflections on those videos, including sharing how they would respond to those students. The student videos were strategically selected; one student exhibited amounts of change reasoning, and another student used phenomenological language about "speed".

To attend to the GTAs' responses to students' covariational reasoning, the following analysis occurred. First, GTA responses were coded for correctness (final graph shape), framing (color coding, labeling), normativity (production of reasonable responses), and serviceability (attention to covariational reasoning/mathematical relationships). Second, GTA responses to the precalculus student videos were analyzed according to the same categories of perception.

Results show that the GTAs do attend to various categories of perception, though at varying degrees. The poster will detail overall patterns in the categories of perception as well as consistency between what the GTA attended to in their own work vs. the students' work. Overall, this study points to a continued need to learn about how to support students' covariational reasoning at the undergraduate level by considering what current categories of perception instructors attend to, and particularly how we can support GTAs in intentionally considering a broad range of categories when analyzing student work.

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Exploring How Educational Math Videos Affect Student Motivation

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Keywords: online learning, student motivation, epistemic messaging, YouTube videos

Finding ways to generate and sustain student motivation for learning mathematics is a beneficial endeavor. Researchers have begun to examine how to measure student motivation over time in an undergraduate course (Alibek, 2024), as well as how to motivate students through problem sets aimed at promoting intellectual need (Weinberg et al., 2022). Current work to both measure and promote student motivation is primarily focused on classroom settings. However, research shows that students increasingly consult external online resources, including videos on YouTube, to support them in their math courses (e.g., Muir, 2014). Although studies have looked at how such videos may benefit students' learning of mathematical content (e.g, Esparza Puga & Aguilar, 2023; Radmehr, 2024), researchers have yet to examine how online math videos can influence motivation. This project aims to begin to address this gap by asking the question: How do popular online math videos affect undergraduate student motivation?

Three popular mathematics YouTube channels include Khan Academy, Professor Dave Explains, and 3Blue1Brown. We view such popular math videos as existing on a spectrum from conceptual to procedural, based on the goals and approaches taken by these different content creators. From our perspective, a highly conceptual video may focus more on broad math concepts and developing a general mathematical intuition (i.e. many 3Blue1Brown videos), whereas a largely procedural video may focus more on how to solve specific math problems that would be relevant to assignments in a math course (i.e. Khan Academy, Professor Dave Explains). For this study, we have selected one video from each end of this spectrum. We have begun conducting interviews with undergraduate students and ask questions to gauge their interest in and view of mathematics before, during, and after watching each of these videos.

To frame our study and support our data analysis, we apply the lens of epistemic messaging, which describes the ways in which teachers convey to students what ideas are valued in a classroom and what ways of thinking are important (Russ, 2018). By applying this concept to videos, we are able to observe how the specific epistemic messaging within each video affects student interest in mathematics. With this lens, we are additionally able to observe trends in epistemic messaging between the videos at either end of the conceptual-procedural spectrum.

This poster will describe the structure of the interviews and report findings from our grounded analysis. Although data collection and analysis are ongoing, initial findings suggest the potential for a conceptual video to influence a student's perception of mathematics. However, we have not found this across the board for students from a variety of backgrounds. Thus far, we have found that a student's mathematical background and STEM-related interests may prime some students to be more receptive to the messages contained in the videos. Additionally, a student's inherent disposition towards learning may also impact the extent to which a video may impact their perceived motivation. These initial findings suggest that students may have very different take-aways from the videos, meaning that video recommendations may need to be tailored for individual students in order to affect student motivation.

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Analyzing Student-Advisor Interactions During First-Year Undergraduate Mathematics Advising

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Keywords: Gender, Advising, Women, Identity, Embodiment

This poster reports on findings from a larger research project focused on a mathematics department's advising practices for incoming undergraduate students. This analysis explores how talk and gesture in student-advisor interactions play a role in how students are given particular mathematics course recommendations and explores the mediating role of gender in the interaction.

Recent trends show men and women are earning bachelor's degrees at near equal rates, however women only account for about one-third of undergraduates earning bachelor's degrees in a STEM discipline. Furthermore, women account for an even smaller fraction when compared to men among those earning bachelor's degrees in math-intensive STEM majors (Canaan & Mouganie, 2023; Jiang, 2021; Louten, 2022; Suárez & Beatty, 2022). While examinations of the gender gap in math-intensive STEM majors is not a novel research area, existing research has largely focused on in-classroom, faculty-level, and curriculum-specific interventions while minimizing the role of one-on-one interactions and outside-of-classroom contexts. One such overlooked site for intervention is undergraduate, pre-course-enrollment mathematics advising.

The mathematics advising explored in this study takes place at a large, public, midwestern university. Incoming undergraduate students opt in to meeting one-on-one with a member of the mathematics department (i.e., tenured faculty, postdoctoral researchers, graduate students) while they consider which math class to take. The sessions are short, lasting just 5-10 minutes on average and conclude with the student and advisor reaching a non-binding math course recommendation. Advisors and students have had no prior interactions. Advisors receive minimal to no training prior to engaging in the work of advising (Gholson et al., 2021; Margolis et al., 2023).

In a very short amount of time, students must provide information about their math background, academic desires, and future goals, and advisors must absorb this information and leverage their understanding of course details and major requirements to reach a course recommendation. Initial analyses of the advising sessions uncovered that students with similar mathematics backgrounds are sometimes given quite distinct math course recommendations that differ widely in terms of their ambitiousness (e.g., Statistics versus Honors Calculus I). Why is this happening? The analysis reported on in this poster examines paired sessions where students with similar mathematics backgrounds are given distinct course recommendations and is guided by the following questions:

1. How is student mathematical competence co-constructed through student-advisor interaction in undergraduate mathematics advising sessions?
2. How, if at all, does gender mediate the co-construction of a student's mathematical competence and the final course recommendation a student receives?

Student and advisor talk are examined in tandem with an embodied analysis based on video recordings of the advising sessions and corresponding transcripts to answer these questions. The layering of discourse analysis and embodied analysis provides a fuller picture of the advising interaction compared to either method in isolation and offers multiple avenues for exploring how gender might mediate interaction.

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The Effect of an Activity-Based Quantitative Reasoning Course on Students' Perspectives About Mathematics

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Keywords: quantitative reasoning, non-STEM, activity-based, anxiety, relevance

Previous studies identify that student-centered approaches to teaching and learning result in academic outcomes that are as good or better than more traditional lecture-based approaches and that the positive effect is more evident for previously low-achieving students (Freeman et al., 2014; Kogan & Laursen et al., 2014). These results have driven change efforts toward course design and content delivery in introductory mathematics courses nationwide (Smith et al., 2021). In light of such results, we redesigned our non-STEM quantitative reasoning course to be activity-based and interactive. Studies also investigate students' attitudes in a variety of active learning courses including calculus and linear algebra (Duran et al., 2022; Murphy et al., 2015). We aim to know whether students in our quantitative reasoning course, who often hold a negative view and prior experience with math, have changed perspectives after completing the course. In particular, has this course impacted students' views about mathematics, learning, and the relevancy of mathematics to their lives?

The redesigned course, *Mathematical Investigations*, is required for all students at our institution seeking a 4-year degree that is not in a Bachelor of Science or Bachelor of Business Administration program. Approximately 200-250 students enroll in the course each semester. The sections are capped at 25 students and are primarily taught in person by full-time, non-tenure-track faculty with some sections offered on a satellite campus and others offered asynchronously online. In Fall 2020, we introduced the course redesign which transformed our traditional, computational, lecture-based course to a course that emphasizes relevant, real-world applications through classroom activities and course projects. We collected anonymous responses to identical pre-course and post-course surveys to informally gauge the effectiveness of the redesigned course. After early indicators of success, we shifted to distributing the surveys through Qualtrics, using personalized 5-digit codes to link the anonymous pre-course survey responses to the individual's anonymous post-course survey responses.

The complete survey includes three open-response questions and seven Likert scale questions. Across four semesters beginning with Spring 2023, we have approximately 795 pre-course and 490 post-course respondents including 392 paired survey sets. The current analysis focuses on the paired responses to a subset of the Likert scale questions. We see favorable results in response to the statement "I feel anxiety about math" (n=392). While 48 students disagreed or strongly disagreed with the statement in both surveys, 166 students who did not initially disagree with the statement indicated improvement, shifting in the direction of "strongly disagree" in the post-course survey when compared to their pre-course response. In response to the statement "Mathematics has no relevance to my life" (n=391), 191 students disagreed or strongly disagreed with the statement in both the pre- and post-course surveys. An additional 111 students indicated improvement, shifting in the direction of "strongly disagree" in the post-course survey.

Our results indicate that students' perspectives were improved after completing the course. Our future work is to conduct a qualitative analysis on the paired responses to the open-ended survey questions which are tied to students' feelings about math and making mistakes in math.

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Building Metaphors That Describe Mathematics Identity and Experiences among Mathematics GTAs: Locks to Change

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Keywords: Mathematics; Graduate Teaching Assistants; Reflection; Identity; Professional Development

Graduate teaching assistants (GTAs) hold unique positions in higher education as they take on roles of both learner and instructor. Kress (2020) found that GTA identity includes not only their sense as a teacher and student, but also their own relationship with mathematics and how they may - or may not - align with the field. In this poster, we share data from a GTA professional development (PD) activity focused on race. We analyzed work from seven mathematics GTAs who responded reflectively to short biographies of mathematicians of color and to at least one other GTA's writing. Reflections focused on their own racial identities and lived experiences. While this study was not driven by a predetermined research question, the responses from the GTAs on the reflective writing assignment were more complex than expected, which prompted further investigation. In particular, we wondered: *How do biographies of mathematicians of color prompt GTAs to reflect upon their own lived experiences in mathematics?*

Women and students of color are underrepresented in STEM fields (Fry et al., 2021) and thus their mathematics experiences are often viewed through windows of white others, with notably few opportunities for mirrors (Leyva, Quea et al., 2021; Guiterrez, 2018). Motivated by this discrepancy, we leveraged Style's work (1996/1988), which focuses on the concepts of a window and a mirror, where windows allow for one "to see the realities of others" and mirrors provide opportunities for one to "see her/his own reality reflected" (p. 1). Abo-zena et al. (2019) expand on the windows and mirrors metaphor by including doors: a way to foster engagement between diverse individuals and provide opportunities for sharing of ideas and perspectives among those with different backgrounds and lived experiences. Aligning with the house metaphor, we added our own construct, *locks*, to capture statements that highlighted obstacles, challenges, or adversity that the GTAs described.

Our findings show the application of the a priori codes windows, mirrors, doors, and locks. The mirror code occurred in all seven discussion board threads, many of which contain posts by more than one GTA. In relation to the mirror codes, every time a lock code was identified it was linked to a mirror. Five of the six locks occurred where GTAs were looking for a connection to autobiographers through personal challenges. In these cases, they saw the obstacles that they themselves had faced in their mathematics career reflected back by successful mathematicians who looked like them. Often locks were followed up by doors in discussion threads.

Our exploratory study, which emerged from a pedagogical experiment, suggests that GTAs, who are both students and emerging educators, may be positively affected by the reading of biographies of successful mathematicians from underrepresented populations, especially when accompanied by self-reflexive writing. Exploring the reflective practice of this PD may create more inclusive and equitable mathematics classrooms by supporting GTAs in considering personal and systemic experiences of marginalization.

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Designing Courseware for Equitable Access, Engagement and Learning in Gateway Mathematics: Centering the Experiences and Perspectives of Students and Faculty

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Keywords: Gateway mathematics, learning technology, user-centered design

Postsecondary math faculty are facing several related challenges that existed prior to but were accelerated by the COVID-19 pandemic (Lopez et al., 2021). At the institutional level, concerns over declining enrollment and budget cuts have led to staffing declines and increased workloads. Faculty have had to teach in a wider array of classroom scenarios, including fully online synchronous or asynchronous, hybrid/hyflex, and in-person, with some teaching courses in all of these scenarios. Furthermore, faculty are finding that, compared to prior to the pandemic, students are entering gateway math courses with greater challenges in mathematical knowledge and skills, surprising gaps in fundamental technology skills, and under-preparation in college-ready organizational skills. Students are also experiencing anxiety and trauma related to mathematics. As a core component of many gateway mathematics courses, digital courseware has the potential to help faculty address these challenges but also presents barriers and challenges to faculty and students (Ganimian et al., 2020; Viberg et al., 2020).

This poster reports on a nationwide study of the perspectives, experiences, and needs of math faculty and students regarding courseware technology (i.e., digital content and tools/platforms designed to support learning) in gateway mathematics courses. The goal of the study, funded by the Gates Foundation, is to inform the design and implementation of mathematics courseware technologies that can improve the mathematics learning experiences and outcomes of Black, Latinx, Indigenous, and low-income students. The study focuses on Quantitative Reasoning, College Algebra, and Calculus and draws upon equity-focused and user-centered design research methods. Data include interviews, surveys, and diary studies, with intentional sampling of the priority student populations (Black/Latinx/ Indigenous/Low-Income) across institution types (2-year and 4-year, particularly TCUs, HSIs, and HBCUs), as well as faculty (teaching, research, adjunct). The analysis draws upon approximately 75 hour long semi-structured interviews of instructors of the selected courses and 75 interviews of students. Seven faculty participated in diary studies, providing daily logs of activities involving mathematics and uses of courseware.

The poster reports on findings from the faculty analysis related to three key themes: (1) Equity: pertaining to issues of access to and proficiency with technology as well as the capacities of courseware to promote inclusion; (2) Engagement: barriers to and features supporting student engagement with courseware content; and (3) Effectiveness: insights related to how courseware can support meaningful mathematics learning and identity development. The poster also presents products to inform design and development of courseware derived from the study analysis, including faculty personas, journey maps, and a map of the ecosystem of resources, supports, barriers, and obstacles in faculty's use of courseware in gateway mathematics.

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