PROCEEDINGS OF THE 14TH ANNUAL CONFERENCE ON RESEARCH IN UNDERGRADUATE MATHEMATICS EDUCATION

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FOREWORD

The research reports and proceedings papers in these volumes were presented at the 14^{th} Annual Conference on Research in Undergraduate Mathematics Education, which took place in Portland, Oregon from February 24 to February 27, 2011.

Volumes 1 and 2, the RUME Conference Proceedings, include conference papers that underwent a rigorous review by two or more reviewers. These papers represent current important work in the field of undergraduate mathematics education and are elaborations of the RUME conference reports.

Volume 1 begins with the winner of the best paper award, an honor bestowed upon papers that make a substantial contribution to the field in terms of raising new questions or gaining insights into existing research programs.

Volume 3, the RUME Conference Reports, includes the Contributed Research Reports that were presented at the conference and that underwent a rigorous review by at least three reviewers prior to the conference. Contributed Research Reports discuss completed research studies on undergraduate mathematics education and address findings from these studies, contemporary theoretical perspectives, and research paradigms.

Volume 4, the RUME Conference Reports, includes the Preliminary Research Reports that were presented at the conference and that underwent a rigorous review by at least three reviewers prior to the conference. Preliminary Research Reports discuss ongoing and exploratory research studies of undergraduate mathematics education. To foster growth in our community, during the conference significant discussion time followed each presentation to allow for feedback and suggestions for future directions for the research.

We wish to acknowledge the conference program committee and reviewers, for their substantial contributions and our institutions, for their support.

Sincerely,

Stacy Brown,
RUME Organizational Director & Conference Chairperson

Sean Larsen,
RUME Program Chair

Karen Marrongelle
RUME Co-coordinator & Conference Local Organizer

Michael Oehrtman
RUME Coordinator Elect

VOLUME 2

CONFERENCE PROCEEDINGS PAPERS

VOLUME 2
TABLE OF CONTENTS

Promoting Success in College Algebra by Using Worked Examples in Weekly Active Group Sessions

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Abstract: At a research university near the east coast, researchers have restructured a College Algebra course by formatting the course into two large lectures a week, an active recitation size laboratory class once a week, and an extra day devoted to active group work called Supplemental Practice (SP). SP was added as an extra day of class where the SP leader has students to work in groups on a worksheet of examples and problems, based off of worked example research, that were covered in the previous week's class material. Two sections of the course were randomly chosen to be the experimental group and the other section was the control group. The experimental group was given the SP worksheets and the control group was given a question and answer session. The experimental group significantly outperformed the control on a variety of components in the course, particularly when prior knowledge is factored in.

Keywords: College Algebra, Cognitive Science, Worked Examples, Large Lecture, Supplemental Sessions

Introduction

A Commitment to America's Future: Responding to the Crisis in Mathematics and Science Education states that "nationally 22% of all college freshman fail to meet the performance levels required for entry level mathematics courses and must begin their college experience in remedial courses" (2005, p. 6). The enrollment in College Algebra has grown recently to the point that nationally there are estimated 650,000 to 750,000 students per year (Haver, 2007) and has surpassed the enrollment in Calculus. Although there are almost three fourths of 1 million students enrolling in College Algebra, it is estimated conservatively that 45% of these students fail to receive a grade of A, B, or C and can reach percentages in the sixties at some colleges. To address this non-success of students at a large research university in the eastern part of the United States, faculty members teaching College Algebra have implemented a new structure in the course that emphasizes active learning through a day called Supplemental Practice.

Theoretical Framework

The Interactive, Compensatory Model of Learning (ICML) provides the framework for understanding and improving classroom learning (see figure 1). Schraw and Brooks (1999) refer to a wide range of literature that reinforces ICML. There are five main components of ICML: cognitive ability, knowledge, metacognition, strategies, and motivation, which affect learning. The following are brief definitions of the five main components of ICML found in (Schraw and Brooks, 1999) and readers should reference their work for more detailed information. *Cognitive Ability* refers to one's general capacity to learn and *knowledge* refers to organized domain-specific and general knowledge in long-term memory. *Metacognition* includes knowledge about oneself as a learner and how to regulate one's learning. *Strategies* refer to procedures that enable learners to solve specific problems and *motivation* refers to beliefs about one's ability to successfully perform a task, as well as one's goals for performing a task.

Figure 1 shows that knowledge, metacognition, and strategies are so closely connected that they are combined together into one area in the figure. We will refer to this one area as knowledge-regulation component. The ICML captures the interactions between these four components that affect learning and describes how one component can compensate for deficiencies in others.

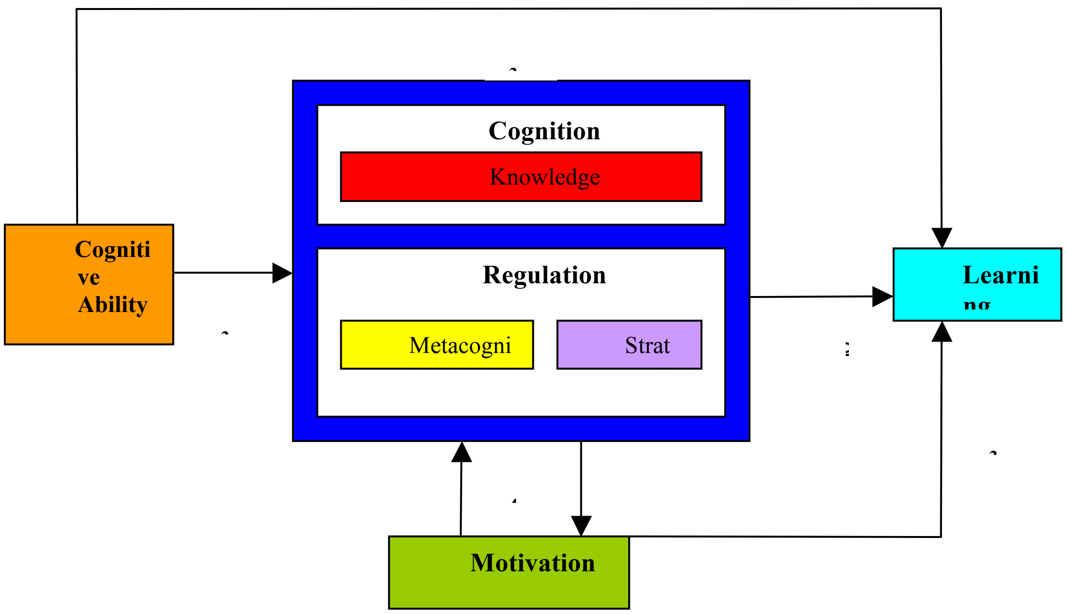

Figure 1: Interactive, Compensatory Model of Learning

Each component can affect learning either directly or indirectly. For example, cognitive ability is related to learning directly, but also indirectly through knowledge-regulation. From Figure 1, one can see that each component directly affect learning (the arrows from each component to learning) while only some components affects learning indirectly through another component. For example, motivation indirectly affects learning through knowledge-regulation, but not through cognitive ability. The numbers in the figure refer to the estimated correlation coefficient between two components. Each correlation coefficient is the estimated value of what has been measured in a number of empirical studies. Cognitive ability is correlated to learning with correlation coefficients ranging from 0.3 to 0.4 (Brody, 1992) and hence, the correlation coefficient of 0.3 between these components. The other correlation coefficients are shown on Figure 1. Schraw and Brooks (1999) state that "most experts agree that knowledge and regulation exert a strong direct effect on learning that is greater than the effects of either ability or motivational beliefs" (p. 9).

The compensatory part of the model refers to how students can compensate for a weakness in one component with a stronger component. For example, students who have weaker cognitive abilities (literature refers to this as intelligence) can compensate by a stronger knowledge-regulation component. That is, students can regulate their learning while they work diligently to increase their knowledge about a particular topic. Through this iterative process, as they go from one topic to another topic in the course to gain knowledge, they successful compensate for their lower cognitive ability than other students. The notion of compensatory processes is supported by many different theories (Gardner, 1983; Perkins, 1987: Sternberg, 1994). Schraw and Brooks (1999) state the following compensation can occur:

1) ability compensates in part for knowledge and regulation
2) knowledge and regulation compensate for cognitive ability and motivation, and
3) motivation compensates for ability, knowledge and regulation.

The next section discusses the history of supplemental practice and briefly covers literature on worked examples.

Background and Literature Review

Supplemental Practice Structure

The idea of Supplemental Practice, denoted SP, was implemented during the Fall 2004 and was originally modeled after Supplemental Instruction (Arendale, 1994; SI Staff, 1997). The normal structure of the algebra class that consisted of three lectures a week morphed into a structure of two lectures a week in a large lecture room, and an active laboratory class once a week in computer classrooms where students meet in smaller groups. The lab class was held on Tuesdays while the lecture class was held on Mondays and Fridays. The SP days on Wednesdays were originally added to the schedule to help lower-achieving students. This was done by requiring students that scored lower than an 80 on a placement exam, or scored lower than a 70 on any regular exam, to attend the SP sessions. Starting in the Fall 2006 semester, the SP sessions have since morphed into active problem-session days modeled after the cognitive science "worked-out example" research. The worked-out example research, henceforth denoted worked examples, asks students to study a worked example for a particular topic, ask questions about anything in the example that they do not understand, and finally work a similar example without reference to the worked example, nor other outside sources (Cooper and Sweller, 1985; Ward and Sweller, 1990; Zhu and Simon, 1987; Carroll, 1994, Tarmizi and Sweller, 1988). The SP sessions and worksheets have been developed based off of the worked example research for the following reasons: 1) helps students to be actively engaged with the material in a setting where they can get feedback and assistance as they solve problems, 2) assists students in transferring information from working (short-term) memory to long-term memory, 3) helps students to regulate their learning and build confidence that they can work problems, 4) allows the instructor to work in the large lecture class to assist students as they learn the material, and 5) helps students as they work on homework, quizzes, and while they study for tests outside of class.

Worked Example Research

The discipline of cognitive science deals with the mental processes of learning, memory, and problem solving. Worked example research was developed based from Seller's cognitive load theory (1988). The total load on working memory at any moment in time is referred as the cognitive load. Most people can retain about seven "chunks" of information in their working memory and when they exceed that limit at any moment in time, there will be a loss of information in the working memory. In other words, there is an overflow of information in the working memory and cognitive overload. Cognitive overload can be thwarted if one limits information so that it does not exceed the students' working memory. One way this can be done is to transfer information from working memory to long-term memory as information is being processed (or soon after). According to Sweller (1988), optimum learning occurs in humans when one minimizes the load on working memory, which in turn facilitates changes in long-term memory.

Worked example research (Cooper and Sweller, 1985; Ward and Sweller, 1990; Zhu and Simon, 1987; Carroll, 1994; Tarmizi and Sweller, 1988) present students with a worked example on paper and tells them to study the example. Once the students are done studying the worked example and have asked any questions, the instructor asks the student to solve a similar problem without any help from the worked example. It has been suggested that worked examples reduce the cognitive load on a student and might optimize schema acquisition (Sweller and Owen, 1989**;** Sweller and Cooper, 1985). In addition, worked examples have been researched (and used) in a variety of subjects: mathematics (Cooper and Sweller, 1985; Zhu and Simon, 1987), engineering (Chi et al., 1989), physics (Ward and Sweller, 1990), computer science (Catrambone and Yuasa, 2006), chemistry (Crippen and Boyd, 2007), and education (Hilbert, Schworm, and Renkl, 2004**).** Readers can reference these studies to gain more information and insight. We will mention highlights of a few of these to end this section.

Sweller and Cooper (1985) conducted one of the first studies on worked examples in high school-level Algebra. Through five experiments, they examined the use of worked examples as a substitute for problem solving and consistently found that the worked example group outperformed the conventional group (worked conventional homework problems instead of studying worked examples) and had less time on task. Zhu and Simon (1987) demonstrated the feasibility and effectiveness of teaching mathematical skills through chosen sequences of worked examples and problems in a Chinese middle school's Algebra and Geometry curriculum without lectures or other direct instruction. Chi et. al. (1989) showed that while students studied worked examples, "good" students generally monitored their own understanding and misunderstanding through self-explanations. Compare this to "poor" students who did not generate sufficient self-explanations or monitored their learning inaccurately. Ward and Sweller (1990) established that students who used worked examples formatted to reduce the need for students to mentally integrate multiple sources of information achieved test performances superior to either those exposed to conventional problems or to those shown worked examples that required students to split their attention.

Past research on worked examples in mathematics has been conducted in a laboratory setting. This research is conducted in a large lecture classroom setting and concentrates on determining if worked examples helped promote success in the course. In addition, past worked example research in mathematics has not dealt with college mathematics courses, classes in a large lecture setting, or implementing an extra day of class to focus on working with students to master material. The research could be valuable to other researchers that are working to promote student success in large lecture classes. The research question that will be addressed in this study is "Do students in the experimental group earn significantly different course grades/exam scores/quiz scores/etc… than students in the control group?"

Prior Data

Data has been generated for all students in College Algebra, including SP sessions attended, since Spring 2007. Students historically perform better in the class as the number of SP days they attended increases and the success rates – the number of students that receive a specific grade, usually a D or C, divided by the total number of students – reach the 60 – 80% range for students that attended 9 or more SP days in a given semester. Figure 1 shows success rates versus number of SP days attended in the Spring 2007.

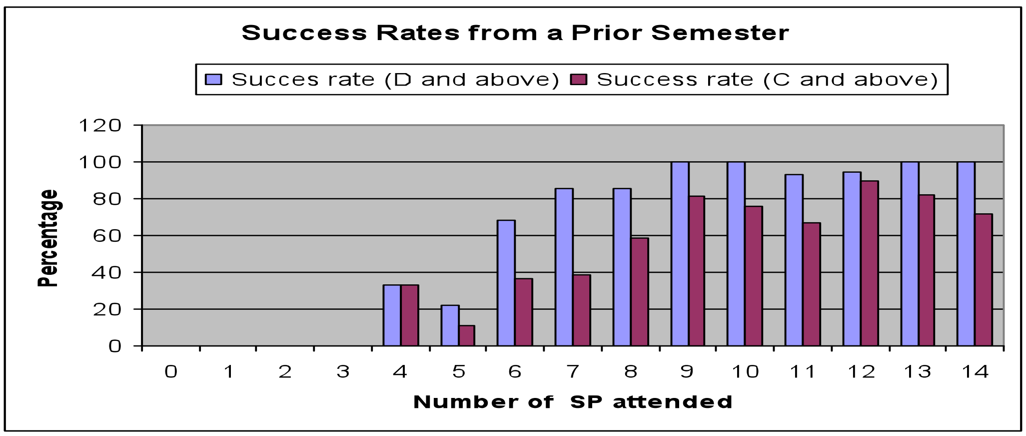

Figure 2

The overall course averages in the spring 2007 College Algebra course is shown in Table 1 below. We see that students that attended 8 or more SP days earned overall course averages in the 70 percentile range with a maximum course average of 77.8% for students that attended 12 SP days. Also all, but the students who attend 8 and 11 SP days earned course averages of 75% and above. The data looks very promising in showing that SP help students in the course. However, the data could be suspect because students voluntarily attended the supplement sessions and perhaps only the motivated students attended the majority of the sessions. To be able to prove that SP sessions were beneficial to the students, the author designed an experiment prior to the Fall 2009 semester to give the experimental group the treatment (worked example worksheets) and the control group an alternative session.

# of Supplemental days attended	Course Average	# of students
0	10.615	2
1	16.185	10

2	22.615	4
3	20.893	3
4	35.630	3
5	45.078	9
6	63.376	22
7	67.040	54
8	71.746	34
9	75.354	27
10	74.963	29
11	73.771	15
12	77.838	19
13	76.625	11
14	75.579	7

Table 1

Methodology

Course

The setting for the research was a large lecture 4-day College Algebra course with an annual enrollment of around 1000 students. This course is one of three different types of College Algebra courses at the university. One type of College Algebra is called a 3-day large lecture College Algebra course that comprises of two lectures a week in a large lecture setting and one day a week in the lab where students actively work in smaller-group math labs. The second type is the 4-day College Algebra course which has the same format as the 3-day College Algebra course, except the 4th day is spent in SP. The final type is a 5-day College Algebra course that is comprised of 5 lectures a week in a class size of approximately 40 students. The 5-day College Algebra class takes all their quizzes and exams by pencil and paper and does not have a laboratory component. All College Algebra courses require specific placement exams scores. The 3-day College Algebra course requires the highest placement score and the 5-day College Algebra course requiring the lowest placement score. In addition, students in the 3-day and 4-day large lecture College Algebra courses take their computerized exams during lab and their quizzes at home on a computer. The pencil and paper labs are completed during lab time one day a week with the help of a java-based applet grapher.

Worked Example Worksheets

During the SP days, worked example worksheets were handed out to the students to work on in groups. SP days were moved from Wednesdays to Mondays in Fall 2008 so that students would have SP days before the exams that occurred periodically on Tuesdays. Since the class was still in the large lecture classroom setting with theatre-seating structure, students formed groups with other students near them as they see fit. Usually students worked with 1 to 3 other students seated close to them. The worked example worksheets consisted of an expert solution of a College Algebra problem followed by a problem for the students to work out. For example, the following worked examples (see figure 3) were given on worksheet 3 (left column) and 11 (right column) during the 5th and 13th SP sessions (the first SP occurs on the third Monday of the semester). These two problems are stated exactly like they are on the worksheet. The worksheet is always given to the students as one sheet (front and back) in a two column format with headings on all worked examples, followed by the section in the textbook (Sullivan and Sullivan, 2006) that can be referenced later outside of class. There are approximately 8 to 10 worked examples and problems on each worksheet. The material on the worksheets consisted of some of the material covered during the previous week in lecture (too much material to

Worked-out Example (2.2):

Find the slope and y-intercept for the equation $2y + 3x = 1$.

To find the slope and y-intercept we need to rewrite the equation into the form $y = mx + b$. That is, solve for y. Thus rewriting $2y + 3x = 1$ we have $2y = -3x + 1$ which simplifies to . So the slope is $-\frac{3}{2}$ and y-intercept is $\left(0, \frac{1}{2}\right)$.

2(b) Find the slope and y-intercept for $\frac{1}{2}y + 5x = -3$.

Worked-out Example (6.6):

Solve for x.

Using the property $\log_b x + \log_b y = \log_b(xy)$, we have . Changing this from logarithm to exponential form, we have $4^1 = x(x-3)$. So

$$4^1 = x^2 - 3x$$
$$0 = x^2 - 3x - 4$$
$$0 = (x-4)(x+1).$$

So or . Therefore $x = 4$ or $x = -1$. But we can not include $x = -1$ as a solution because when we substitute it back into the original equation it yields $\log(-1)$ and $\log(-4)$ which are undefined since we can not take a logarithm of a negative number. Solution is $x = 4$.

1. Solve $\ln(x+1) - \ln x = 2$ for x.

Figure 3: Sample Worked-out Examples from a Worked-out Example Worksheet

cover it all). No new material was ever covered and the worksheets comprised of problems directly from or derived from the problems in the textbook. The worksheets were never developed while referencing material from exams, quizzes, or labs. However, most of the questions from the exams and quizzes are similar to the homework in the book. Finally, the worksheets are modeled after worked example research since it presents an expert's solution to a problem followed by a problem for the student to work out. The only difference is that it is not plausible to ask the students to not reference the worked example while working another problem and so this was never done. Furthermore, most studies on worked examples state that the student should be given a similar problem (very similar in some cases), but in SP the problems students were asked to do, vary from very similar to somewhat different problems.

Experiment

The researcher randomly designated one of the course sections as the control group (n = 177) and the other two sections as the experimental group (n = 320). In the experimental group, the students were given a “worked-out example” worksheet at the beginning of each of the 13 SP days and asked to work in groups to complete the worksheet. Three to four class assistants circulated around the room to answer any student questions about the worksheet. In the control group, a graduate student organized a question-and-answer session during the extra day instead of giving a worksheet to the students. Students were able to get any question answered, but the graduate student only answered student questions and did not generate questions themselves. For the most part, the graduate student spent most of the class time answering student-generated questions. Quantitative data (course scores on exams and quizzes, supplemental days attended, class attendance, total points,…) was collected for each student in both the control and experimental groups and analyzed at the end of the semester. There were similar demographics in both the control and experimental groups.

Data

Data from the experimental and control groups were compared on a variety of levels by using t-test with equal and unequal variances depending on the data. The experimental and control group had similar levels of retention (number of students that completed the course) at 80.5% and 84%, respectively. At the beginning of the semester, all students were given an old ACT math test that consisted of 60 questions. Students were given extra credit points for the ACT test on a sliding scale. This ensured that the better a student performed, the more extra credit, up to 10 points, they earned. The ACT exam gave a good measure of students’ prior mathematical knowledge. Figure 4 shows the control and experimental groups’ mean scores of 28.40 and 26.91 with standard deviations of 6.41 and 6.86, respectively. The control group significantly outperformed the experimental group (p=0.01) on the (pre) ACT test.

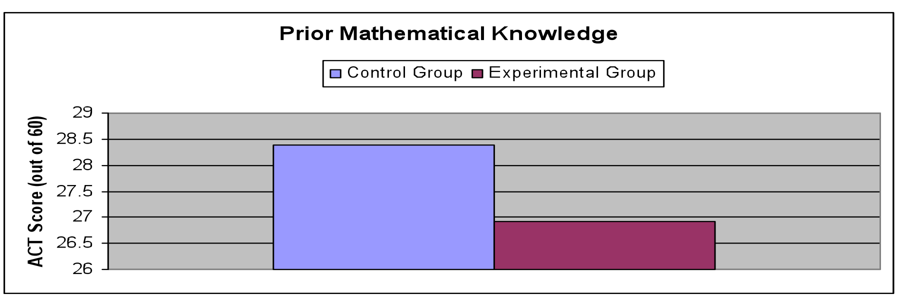

Figure 4

At the end of the semester, students were given the same old ACT exam to measure their post mathematical knowledge. Figure 5 shows the control and experimental groups earned a mean ACT score of 32.81 and 32.16, with standard deviation of 6.46 and 7.22, respectively. There was no significant difference between the mean ACT scores of the two groups.

The data for total points in the course (Current Points), total points without attendance (CP w/o Attend), total points without attendance or labs (CP w/o Attend, Labs), and Current points for just exams (CP Tests Only), were compared between the three groups.

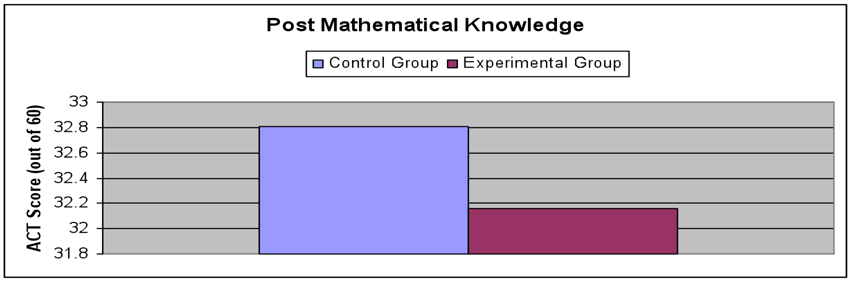

Figure 5

Figure 6 shows the current points for the two groups and Table 2 shows the mean current points with standard deviations.

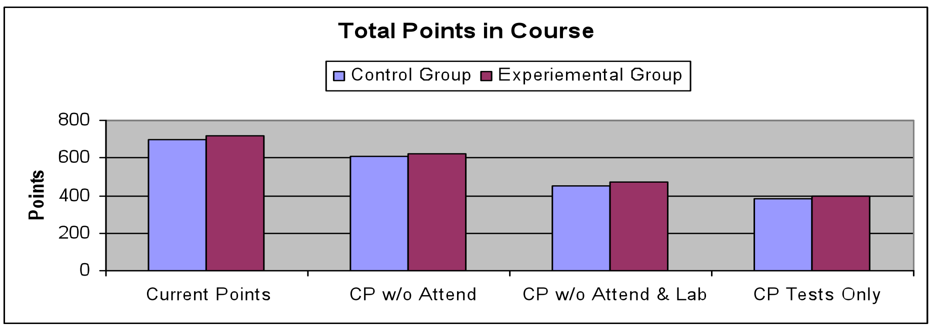

Figure 6

	Current Points	CP w/o Attend	CP w/o Attend, Lab	CP Tests Only
Control Group (n=177)	698.81 (150.51)	605.88 (140.25)	453.24 (120.04)	381.72 (105.48)
Experimental Group (n=320)	718.60 (151.60)	625.37 (141.29)	470.82 (119.49)	396.25 (103.98)

Table 2: Means and standard deviations

There were almost significant differences between the mean scores of the control and experimental groups with respect to Current Points ($p = 0.082$), CP w/o Attend ($p = 0.070$), and CP Tests Only ($p = 0.069$). The experimental group had significantly better current points without including the attendance and laboratory points than the control group ($p = 0.058$). Recall that current points did not include any extra credit (i.e. pre/post ACT exam).

The two groups were compared with respect to each exam, the final, and quizzes. Figure 7 shows the two groups mean scores on each exam, the final, and quizzes and Table 3 shows the exact scores and standard deviation in parentheses.

The experimental group significantly outperformed the control group on test 3 ($p = 0.031$), quizzes ($p =0.048$), and final exam ($p = 0.029$), and almost significantly outperformed the control group on test 2 ($p = 0.083$). There was no difference between the control and experimental groups with respect to test 1 or test 4.

Course grade point average was calculated to compare the two groups on the average course grade earned. This was accomplished by assigned a quantitative score for the final grade that each student earned in the course (A = 4, B = 3, C = 2, D = 1, and F = 0). Figure 8 shows the course grade point average for control group (1.97) and experimental group (2.13) with standard deviations 1.17 and 1.27, respectively. The experimental group had an almost significantly different course grade point average than the control group (p = 0.075).

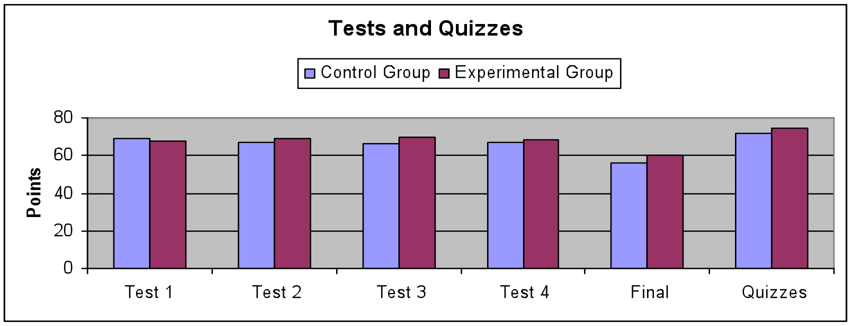

Figure 7

	Test 1	Test 2	Test 3	Test 4	Final	Quizzes
Control Group (n=177)	68.84 (16.21)	66.86 (19.93)	66.58 (19.61)	67.23 (22.67)	112.20 ()	71.52 (18.89)
Experimental Group (n=320)	67.52 (16.18)	69.38 (17.99)	70.05 (19.76)	68.66 (23.88)	120.66 ()	74.57 (20.65)

Table 3

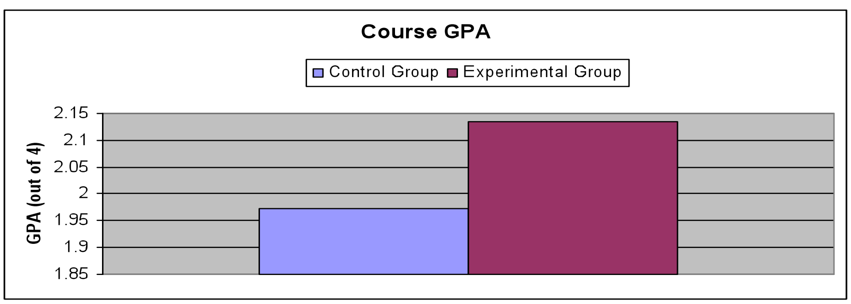

Figure 8

Experimental Group versus Control Group Based on Levels of Prior Knowledge

The ICML states that students can compensate for weaker prior knowledge by having a stronger knowledge-regulation and motivation component. To investigate this, the researcher examined the difference (significant or not) between the control and experimental groups based on levels of prior knowledge. The high prior mathematical knowledge group was defined to be all students with a 31 or higher score (out of 60) on the old ACT math exam. The high prior mathematical knowledge control (n = 54) and experimental group (n = 88) was denoted as high control and high experimental. The medium prior mathematical knowledge group was defined to be all students with an old ACT math exam score of 26 to 30. The middle prior mathematical knowledge control (n = 59) and experimental group (n = 89) was denoted as middle control and middle experimental. The low prior mathematical knowledge group was defined to be all students with an old ACT math exam score of 25 or below. The low prior mathematical knowledge control (n = 49) and experimental group (n = 119) was denoted as low control and low experimental.

Figure 9 shows the prior/post mathematical knowledge for the control and experimental group based on the level of prior knowledge groups. For prior and post, there was no significant difference in mathematical knowledge for the high control (pre = 35.26, post = 37.21) and high experimental (34.84, 37.36), nor the low control (21.14, 28.74) and low experimental (20.45, 27.82) with standard deviations (4.02,5.41), (3.74, 5.24), (3.25, 5.47), and (4.07, 6.72). However for the prior, the middle control (28.14) significantly outperformed the middle experimental (27.71) (p = 0.041) with standard deviations of (1.48, 5.30) and (1.43, 5.71), respectively. For the post, there was no significant difference in the middle groups.

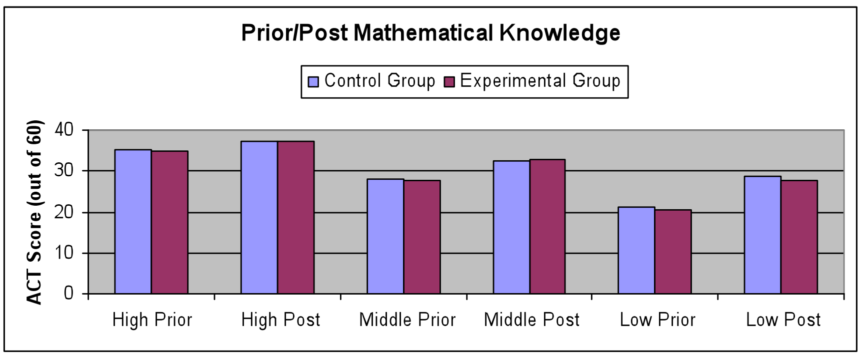

Figure 9

Furthermore, there is no significant difference between the performance of the high control group and the high experimental except on test 3. The high experimental (78.13) almost significantly outperformed the high control (74.17) on test 3 (p = 0.065) with standard deviations 15.34 and 14.59, respectively. Similarly, there is no significant difference between the performance of the low control and low experimental except on test 1 and test 2. The low control (64.80) almost significantly outperformed the low experimental (61.30) on test 1 (p = 0.085) with standard deviations of 14.25 and 16.42, respectively. In addition, the low experimental (63.91) significantly outperformed the low control (59.08) on test 2 (p = 0.051) with standard deviations of 19.71 and 16.06, respectively. The middle prior mathematical knowledge level is where the significant data emerged. The middle experimental significantly or almost significantly (two cases) outperformed the middle control on every course component except test 1. See Table 4, Figure 10, and Figure 11 for more details.

	Mean Control (n=59)	Mean Experimental (n=89)	Standard Deviation Control	Standard Deviation Experimental	P – Value
Current Points	669.18	726.58	175.70	140.70	0.019
CP w/o	578.58	632.09	160.41	130.56	0.017

Attend					
CP w/o Attend, Lab	431.33	474.89	131.46	110.62	0.019
CP Tests Only	362.12	398.65	114.91	95.93	0.023
Test 1	67.88	69.66	15.18	13.87	0.235
Test 2	64.41	69.49	22.32	15.45	0.065
Test 3	66.19	72.53	19.94	17.44	0.025
Test 4	63.47	69.33	24.48	24.54	0.079
Final	100.17	117.64	54.63	45.90	0.023
Quizzes	69.21	76.24	20.09	20.43	0.020
Course GPA	1.85	2.19	1.26	1.22	0.050

Table 4: Middle Prior Knowledge Mathematical Level

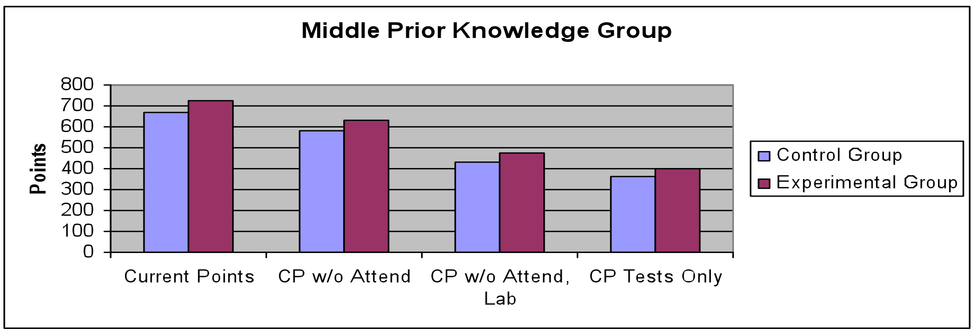

Figure 10

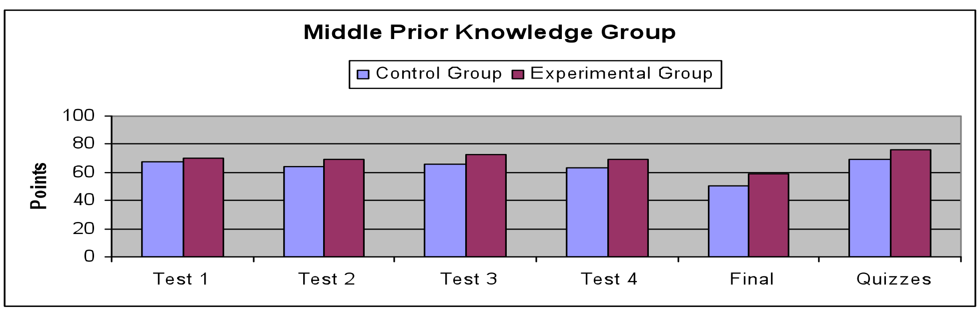

Figure 11

Results

For the most part, prior data has shown that the more days of supplemental practice that a student attends, the better the student performs overall in College Algebra. Specifically, students who attend 9 or more SP days have been very successful in earning a grade of C or better in the class. In this past data that has been collected, students voluntarily attended the SP sessions on Mondays. One could argue that the students that were motivated came to the SP sessions 9 or more times and hence that is why they were successful. This research study was developed to investigate whether the SP sessions help students to be more successful on course components and overall.

We see that the control group started with a significantly higher prior mathematical knowledge than the experimental group. However, the control and experimental groups ended the semester with no significant difference in post mathematical knowledge. The students in the experimental group added to their prior knowledge throughout the class so that they had similar post mathematical knowledge, assessed by an old ACT exam. On average students in the experimental group increased their ACT exam score by 5.15 points compared to an increase of 4.32 points for the control group. This is very important since ICML states that students can compensate a weaker knowledge component with a stronger motivation component. Examining things a little further, we see that the high and low level prior mathematical knowledge groups started off with similar prior knowledge but the middle prior mathematical knowledge control group was significantly higher than the middle prior mathematical knowledge experimental group. However, all three experimental groups (high, middle, and low) ended the semester with similar post mathematical knowledge as the three control groups. The middle group started with significantly lower prior knowledge, but ended with similar post knowledge in the course. We can conclude that it was the middle group's mathematical knowledge that was affected the most by the worked examples. The worked examples worksheets helped the students in the middle experimental group to accumulate the knowledge needed to bring their mathematical knowledge to a similar level of the middle control group.

In terms of total points in the course, although the experimental group did not always significantly outperform the control group, the experimental group almost significantly outperformed the control group. When attendance and laboratory components of the grade were removed from the current points, the experimental group significantly outperformed the control group. Investigating things further, we find that, for the most part, the middle prior mathematical knowledge experimental group significantly outperformed the middle prior mathematical knowledge control group. In fact, there were only three components where no significant difference or almost significance occurred. The non-significance on exam 1 for the middle groups is understandable since there were only two SP days before exam 1 and students had to get comfortable with the worked example worksheets and the structure of the SP sessions. Both exam 2 and exam 4 performances showed that the middle experimental group outperformed the middle control by a half letter grade and was very close to being significant.

In conclusion, we find that the SP sessions benefited the middle experimental the most out of all other groups. Using the ICML framework, SP sessions helped the students in the middle experimental gain a stronger motivation component to compensate for a weaker knowledge component. In addition, as their stronger motivation compensated for weaker knowledge, they increased (strengthened) their knowledge through learning in an iterative process throughout the semester. This led to more learning directly from their knowledge-regulation component and indirectly through motivation. We propose that the high experimental was not motivated any differently than the high control. That is, the high prior mathematical knowledge group would learn the material no matter what intervention was given in the class. According to ICML, there are two components that can compensate for the knowledge-regulation component: ability and motivation. It is not completely clear why there was no significant difference in the lower prior mathematical knowledge groups. The research by Chi (1989) might shed some light, perhaps the high prior mathematical knowledge group is comprised of the "good" students that have sufficient self-regulation skills and the low prior mathematical knowledge group is comprised of the "poor" students that do not have sufficient self-regulation skills. We propose as a whole that the lower groups start the course with such a low prior knowledge-base that they are not able to compensate for this lack of knowledge through cognitive ability or motivation. It would be interesting to see if the higher cognitive ability students in the low prior mathematical knowledge group are able to compensate for the weak knowledge-regulation component. This would illuminate whether cognitive ability compensates for knowledge-regulation for the lower prior mathematical knowledge group. Future research will attempt to investigate this and why the middle group seems to benefit the most from motivation. It is important to note that data has been analyzed more for the experimental group based on their attendance of SP days. It turns out that the students in the experimental group that attend 8 or more SP days outperform the control group significantly on every comparison (p-values less than 0.0091 expect a p-value of 0.047 for test 4). The results are even more significant when we compare the students in the experimental group that attended 13 SP days to the control group. In fact, this group significantly outperforms any other group of students with a specific number of SP days attended. For example, the group of students in the experimental group that attended 13 days of SP significantly outperform the group of students in the experimental group that attend 12 days of SP. Details of this research will be presented in future papers.

Implications to Teaching

Many college instructors teach large lecture sections of introductory mathematics classes and struggle with high percentages of students that earn grades of D or F, or simply withdraw (DFW rate) from the class. It takes resources to offer recitation sessions, out-of-class sessions, or tutoring. These are familiar modes of intervention that colleges use to lower the DFW rate and, in general, help students be more successful in learning material in a course. This study shows that carefully designed worksheets modeled after worked examples coupled with active group sessions can be very beneficial in helping students become more successful despite lower prior knowledge in a course. The SP day each week allow extra active session where students can work on comprehending material in groups. These SP days are like an extra day in class, however, they only emphasize material that has already been covered during the lecture days. The obligation of the instructor is to have a group of class assistants ready to help students with the material. This extra day of class per week is very important because, for the most part, students will show up for the extra day of class to actively participate compared to an out-of-class session. Moreover, students are less likely to visit the instructor in his office during office hours, nor go to a mathematical tutoring center. For the instructor, SP days is the most efficient way to help many students at the same time. There would be no way for the instructor to help this many students during office visits and reduces the task of explaining material many different times to different students during office hours. The worked example worksheets act like a tutor by presenting students with a number of examples and problems to practice. Since it was shown that the high and low prior mathematical knowledge group did not benefit, other modes might be designed to help these students. For example, maybe smaller group session using the worked example worksheets with focus on helping the students would benefit the low prior mathematical knowledge group more.

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MATHEMATICIANS' PEDAGOGICAL THOUGHTS AND PRACTICES IN PROOF PRESENTATION

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This descriptive study investigates the ways in which proofs are presented in upper division proof-based undergraduate mathematics courses at a large comprehensive research university in the Midwest. To pursue this inquiry, four faculty members who were teaching such courses were interviewed and three of the four participated in video-taped observations periodically throughout the course of the semester. Interview data were used to construct a framework with which to analyze the observation data. The observation data were analyzed to determine the level of engagement that the professor seemed to expect of the students, and to identify some proof presentation strategies that were used.

Keywords: proof presentation, teaching proof, expected engagement, traditional lecture

Introduction and Literature Review

The teaching and learning of mathematical proof has proven to be problematic for decades (Grassl & Mingus, 2007; Larsen, 2009; Larsen & Zandieh, 2008; Selden & Selden, 2003; Tall 1997). Research in this area usually comes in two flavors: investigating student thinking (Knuth, 2002; Larsen, 2009; Simpson & Stehlikova, 2006; Healey & Hoyles, 2000; Selden & Selden, 2003) and developmental research projects that focus on developing innovative ways to teach proof (Leron and Dubinski,1992; Larsen, 2009; Weber, 2006). These studies shed light on teaching and learning in the context of mathematical proof, but it is often difficult to apply these findings directly. A foundation of knowledge about the range of current teaching practices is needed before we can adequately understand how to improve student learning.

It is generally acknowledged that lecture is the norm in most university mathematics classrooms. The traditional lecture style has been criticized by many, especially by those who propose alternative, more interactive teaching methods such as computer activities or guided reinvention of the concepts (Leron & Dubinski, 1995; Larsen, 2009). These types of reform curriculum require a dramatic shift in the delivery of the material from teacher-centered to student-centered. Leron (1985) merely called for a divergence from a linear proof presentation method in favor of "heuristic" presentations, which give the audience a better idea of how the ideas were constructed. In the traditional lecture format, students are expected to learn how to prove theorems by attending to the presentations of proofs given by their instructor in class and by working the homework problems that are assigned. Even though the presentation of proof in class is a primary tool for teaching proof, there are few research studies addressing this topic (Weber, 2010).

We believe that the traditional lecture style of teaching mathematical proof should be investigated more closely in order to catalog the strategies that professors are using. In a recent article, Speer, Smith, & Horvath (2010) claim that "very little empirical research has yet described and analyzed the practices of teachers of mathematics" (p. 99). They mention that although the "effects of instructional activities have been examined… the actions of the teachers using those activities have not" (p. 101). They conducted an extensive literature search for published articles addressing collegiate teaching practice, only to find five articles that fit their criteria. All of the five articles were case studies with one faculty member as a participant, and all used observation and interview data to analyze the instructor's teaching practices. Only one of the articles mentioned was an analysis of a proof-based course (Weber, 2004).

Weber's (2004) study analyzed a professor's use of the 'definition-theorem-proof' model. Although this course was taught in the traditional lecture style, the professor didn't seem to present proofs in a linear style. He made an effort to reveal the reasoning behind the proof construction so that students could learn to construct original proofs themselves. Another recent study examines the pedagogical choices of a faculty member teaching a traditional, proof-based course. Fukawa-Connelly (2010) observed a mathematics faculty member over the course of a semester in an abstract algebra course. This study gives an existence proof that university mathematics professors do not always use a "pure telling" method of proof presentation. He analyzed classroom dialogue through the lens of pedagogical content tools, looking for instances in which the faculty member 'modeled mathematical behaviors.' One of the mathematical behaviors he identified was *proof writing,* which he described as a collection of proof creation heuristics. This study will explore the mathematical behavior of *proof writing* further to identify some strategies that mathematicians use when presenting proofs in class.

There have been several studies that investigate mathematics faculty members' pedagogy in regard to proof presentation (Weber, 2010; Alcock, 2010; Yopp, 2011; Hemmi, 2010). These studies have used interviews of faculty members to investigate why and how they teach proof, and have described how these mathematicians talk about their intentions and pedagogical perspectives.

Two of the studies investigated the reasons that the participants presented proofs to their students. Yopp (2011) interviewed 14 mathematicians and statisticians, and all of the participants mentioned that they present proofs to develop their students' ability to write proofs on their own. No other reason was mentioned so frequently by the subjects. Of the nine mathematicians interviewed by Weber (2010), six of them mentioned that they would present a proof when it illustrates a new proving technique. This may be because their goal is also to equip students to write proofs on their own. Other reasons for presenting proofs included cultural reasons, to expose students to proof, or to illustrate the truth of a theorem.

Alcock (2010) identified four modes of thinking that the faculty members aim to teach their students by presenting proofs. The mathematicians considered these ways of thinking natural for them, but not for students. *Instantiation* is used to "understand a mathematical statement by thinking about its referent objects," (p. 69) and includes thinking about examples or images. Mathematicians often use this mode of thinking when presented with a new definition. *Structural thinking* is a way of reducing abstraction and making use of the logical structures to drive the construction of the proof. *Creative thinking* includes experimenting with examples with the hope to generalize or attempting to construct a counterexample. The goal of *critical thinking* is to check for the correctness of each line of a proof.

Hemmi (2010) outlined three dimensions of presenting proof that seem to encapsulate the different opinions of faculty members. Induction/deduction refers to whether the presentation generalizes from examples or starts with a generalization. Visibility/invisibility refers to the degree to which the logical structures are made explicit. Intuition/formality refers to the amount of heuristic and visual arguments that are used versus purely formal, rigorous proof methods.

Though these studies help to describe how faculty members talk about their teaching practice, they make no effort to describe what is actually happening in the classroom. Hemmi (2010) acknowledges that "the relationship between talk and reality is complex. The mathematicians' talk about proof and the teaching and learning of proof is considered as shedding light on the practice, but not as an objective description of it" (p. 272). This study will address this gap in the literature by collecting and analyzing both interview and observation data from faculty members who are teaching mathematical proof.

Research Methods

In this two-phase study, we first conducted interviews with mathematics faculty members who were teaching a proof-based course to develop a backdrop and framework. By "proof-based," we mean a course in which students are expected to construct original proofs on a regular basis. Secondly, we conducted video-taped observations of the same faculty members throughout the course of the semester. We then analyzed the proof presentations in the video data using the framework that we developed from the interview data.

We seek neither to criticize nor to commend the presentation methods of these mathematicians. Our goal is merely to describe the practices of mathematics professors as they present proofs in a traditional style in upper division, undergraduate level mathematics courses. In our description, we will highlight some of the tools that are being used to present mathematical proof, and catalog the level of engagement that the instructor seems to expect from the students during proof presentations. Our research questions are the following:

1. What do mathematics faculty members contemplate as they plan lectures that include proof presentations?
2. What pedagogical moves do mathematics faculty members make when presenting proofs in a traditional undergraduate classroom?
3. To what degree and in what ways do faculty members engage students when presenting proofs?

This paper will concentrate on the second and third research questions.

Participants: The four participants in this study were tenured faculty members at a large comprehensive research university in the Midwest. All were experienced teachers and researchers, and all were scheduled to teach an upper-division proof-based course. The four participants were interviewed, and three of the participants agreed to allow video-taping of their lectures. The participant who declined to be video-taped was observed, and field notes were collected, however, the researcher did not feel that enough information was gathered to include that data in the results, because the coding scheme had not yet been developed. All of the instructors who participated in the observations identified their teaching style as traditional lecture. The courses in which the observations took place were Introduction to Abstract Algebra, Geometry, and Number Theory.

Because the data were collected at one university, we must take extra care to protect the identities of the participants in this study. Although the mathematicians' gender, age, ethnicity, and area of research may shed some light on data analysis, we cannot reveal this information. We will therefore use pseudonyms, and will use the pronoun "he" to refer to all of the participants throughout the discussion, regardless of their gender.

Phase 1: The first phase consisted of semi-structured interviews addressing what the instructors do when they present proofs in class, why they make those choices, and what they do to help students understand their presentation of proofs in class. The interview data were transcribed and broken into chunks which were sorted into groups to identify themes. Several themes emerged, but two overarching themes were selected to use for analysis of the observations. These two themes were *expected engagement* and *proof presentation strategies*. Both of these themes also appeared in the literature. In the area of geo-science education, Markley, Miller, Kneeshaw, and Herbert (2009) conducted faculty interviews and classroom observations to determine the level of student engagement. Weber (2010) cataloged several strategies that faculty members claimed to use to help students understand proof. Our presentation strategies overlapped with some of Alcock's (2010) modes of thinking and with Mejia-Ramos, Weber, Fuller, Samkoff, Search, & Rhoads' (2010) dimensions of proof comprehension. The interview data were used to construct levels of expected engagement and indicators for each proof presentation strategy.

Phase 2: Throughout the semester, observations of three of the instructors were conducted and analyzed in detail. Each participant was video-taped 6 to 7 times throughout the semester, approximately once every two weeks. The participants communicated exam dates so that those could be avoided, and also occasionally communicated when a particular important proof would be presented. So, the dates for observation were selected by convenience sampling.

Transcriptions were made of each instance of proof presentation, beginning and ending at natural breaks in the dialogue. The researcher chose instances where the professor wrote a formal proof on the board, provided justification for an argument, or worked problems on the level of students' homework. Because the homework problems required proof or some kind of formal justification, we also counted homework problems that were worked or discussed in class as proof presentations.

Once the transcriptions were complete, the researcher read through each observation again, and took note of the amount of time spent on each proof. The times were analyzed to determine the amount of class time spent on proof, the number of proofs per 50 minute class period, and the average amount of time spent on each proof. The researcher then coded each proof according to the *proof presentation strategies* that appeared and the level of *expected engagement*, which will be discussed further in the next section.

Results

As this is a preliminary report, the results presented in this paper represent the framework that we constructed from the interview data and our initial pass through the observation data. This paper will describe the framework that we constructed, but will not contain a detailed analysis of the interview data. Our initial analysis of the observation data determined the amount of class time each professor spent on proof presentations, and these results are shown in Table 1. Next, the observation data were analyzed using the framework we constructed from the interview data. We identified the *proof presentation strategies* used in the proofs, and the level of *expected engagement*.

We identified four *proof presentation strategies* from comments made by the professors in the interview data. We gave them the names *outline, instantiation, logical structure,* and *context.* As we worked to identify these strategies, we echo Yopp's lament that "it is challenging to define the categories in a way that distinctions between them are clear. This issue seems to be heightened when authors discuss instruction" (Yopp, 2011, p. 3). We make no claim that these strategies are mutually exclusive or exhaustive, they merely represent some of the strategies mentioned by our participants in the interviews. We will describe each of these strategies and discuss the indicators that we used in our coding. Although it would be helpful to include instances of each strategy from the observation data, we will not do so in this paper for lack of space.

Table 1: Analysis of class time spent on proof

	Dr. Grey	Dr.	Dr.

		Nelson	Adams
Percentage of class time spent on proof	70.1%	36.1%	42.5%
Average number of proofs per 50 minute class period	3.7	1.3	2
Average amount of time spent on each proof in minutes	7:58	11:33	11:27

When asked what they do to help students understand a proof presentation, several of the participants described a process of outlining or talking informally about the key ideas of the proof *before* they began to write the proof on the board. For example, Dr. Nelson said that before he begins to present a proof in class, he tries to "*read* the… theorem, and I say, 'What's important here? What do you think of when you read this?' And I ask for ideas, what relevant theorems might be true…" We called this strategy *outline.* It sometimes took a form similar to Leron's (1985) top-down approach in which he begins a proof by giving an overview, but other times it was just an informal discussion about how to begin the proof. When coding, we looked for instances where the faculty member discussed with students about the ideas of the proof before starting the proof, and we also looked for when the professor was very clear about what was just informal talk and what was a formal proof.

The mathematicians also mentioned drawing pictures or using specific examples to help students understand the meanings of terms or statements, or to motivate the proof. Dr. Nelson said that "…if it's a proof of a pattern, then I certainly… emphasize computation. First, you have to compute a lot to try to figure out what the pattern is." Although we first identified *drawing pictures* and *using examples* as separate proof presentation strategies, it seemed to be that the amount of pictures or numerical examples used seemed to rely on the mathematical content. In Alcock's (2010) study, she used the term *instantiation* to mean "understand[ing] a mathematical statement by thinking about its referent objects." This terminology seemed much less content specific. In the observation data, we were looking for proofs in which the professor presented algebraic, numerical, or pictorial examples to explain the mathematical statements or the proof strategy.

Though the mathematicians who participated had differing views about whether the logic of proof writing should be taught along with content or separately, they all mentioned that they spend time talking about *logical structure*. When Dr. Adams begins a proof presentation, he says, "I make the statement on the board, and I point out if it's an if-then statement, or if it's a, both, if-and-only-if statement, and I talk about that, what we have to prove then." When analyzing the data, proofs in which the professor used the *outline* strategy often also emphasized *logical structure*, but not always. Indicators of *logical structure* included pointing out when hypotheses are used, explicitly discussing the structure of the statement to be proved, or summarizing the logic of the proof after the proof was completed.

The final strategy that emerged from the interview data was *context.* We used this to describe instances when the professor placed the ideas of the proof in historical context, pointed out standard arguments as they appeared in presentations, or highlighted the key ideas and how they fit within a larger context of the mathematical content area. For example, when describing how he presented the proof that trisecting an angle is impossible, Dr. Grey said that "…the whole idea that you can prove that it's impossible, no matter what you do… that's kind of a big thing to understand right there." When presenting the proof, he emphasized the standard arguments required to show that a construction is impossible.

Each proof was analyzed and coded with the presentation strategies that occurred in the proof. There were very few of the proofs that used none of the strategies, and some that used all four. Although we have not completed a detailed analysis, it is our impression that professors are more likely to employ multiple strategies when the theorem is more important.

Table 2: Percentages of proofs using each presentation strategy

	Dr. Grey	Dr. Nelson	Dr. Adams
Total Number of Proofs in Observation Data	22	8	12
Outline	54%	63%	42%
Instantiation	27%	50%	67%
Logical Structure	36%	50%	25%
Context	27%	38%	17%

By *expected engagement*, we mean the extent to which the professor seemed to expect students to contribute to the proof presentation. We used each individual proof as the unit of analysis, and assigned a code of 1-5 representing different levels of expected engagement. Proofs coded 1 or 2 represented instances where the instructor did not seem to expect the students to be actively contributing to the proof presentation. The code 3 was assigned to proofs in which the instructor seemed to expect students to contribute factual information, and 4 or 5 was assigned to proofs in which the professor expected students to contribute both factual information and key ideas for the proof presentation. To determine the level of expected engagement, we looked for both the types of questions that the instructor posed as well as the amount of time that he waited for a response. For example, if the instructor posed a question and immediately answered his own question, we assumed that he did not expect the students to respond. The coding scheme for expected engagement is summarized in Table 3.

Of the 42 proofs that were video-taped and transcribed, 16.6% of them were coded 1 or 2, 50% of them were coded with a 3, and 33.3% were coded with a 4 or 5. So, this means that in well over half of the proof presentations, the faculty members expected the students to actively contribute to the presentation, whether by providing some factual information or actually helping to contribute key ideas for the proof. The data for each professor is shown in Table 4. Since this was a study of faculty members and not students, there was no data collected about the actual number of students who participated in class, or the other ways in which students were engaging. We were merely documenting the apparent expectation of the professor in regard to student input in the proof presentations.

Table 3: Coding scheme for expected engagement

	Description
1	Professor does not request active contribution during the proof presentation, and does not appear to engage students.
2	Professor does not request active contribution during the proof presentation, but seems to adapt the presentation based on monitoring, non-verbal communication with students, or feedback from closed-ended questions.
3	Professor expects students to contribute some factual information during the proof presentation.
4	Professor expects students to contribute some factual information and some key ideas of the proof during proof presentation
5	Professor expects students to generate the majority of the ideas for the proof during the proof presentation.

We will now give examples of different instances of proof presentations from the observation data and how they were coded for *expected engagement*. Because the video camera was directed at the instructor, any comments from the students are simply labeled 'Student.' It is not necessarily the case that the same student spoke each time. Also, single quotes are used to identify times when the instructor was writing on the chalk board while talking aloud.

The expected engagement code 1 or 2 represents proof presentations where the faculty member did not appear to expect the students to actively contribute to the proof presentation. This does not imply that there was no interaction at all, or that the students were not engaged in other ways such as note-taking or non-verbal communication. In most of these instances, the instructor used monitoring to adapt his presentation. For example, he would ask if the students understand, watch their reactions, and modify the presentation if he deemed necessary. One such example is taken from Dr. Adams' class, where he presented a proof that the composition of homomorphisms is a homomorphism.

Dr. A: What does that mean? '*G*, *H*, um, *P* groups. $\theta : G \to H$,' and uh, let's see… theta... let's say psi, '$\psi : H \to P$, homomorphisms, then $\psi \circ \theta$ is a homomorphism.' So, we know that composition is an operation on mappings. So, when I say theta and psi are homomorphisms, first of all, they are mappings of the underlying point-sets. So these are mappings. Hence, their composition makes sense. So, their composition is a mapping. So, the question here is… we know this is a mapping *(circles $\psi \circ \theta$).* Is it a homomorphism?

Dr. A: So, 'Proof:' What do you need to do to prove this? Um, well, we have to show that multiplication in *G*, if you take multiplication and take it under the composition, it is the multiplication of the factors in *P*, uh, of the images in *P*. So, we 'Let $g_1, g_2 \in G$,' and we 'Consider $\psi \circ \theta(g_1 g_2)$ ' Ok? What is

	that equal to? Well, first off, it is equal to ' ' that is… that's what the definition is of a composition. Ok? Theta, though, is a homomorphism. So this is '$\psi(\theta(g_1)\theta(g_2))$' Now, all these guys have big symbols here… lots of symbols. This is a single element in H, *(circles $\theta(g_1)$ with his finger)* this is a single element in H *(circles with his finger)*. Now, would it be better if I put those stars and diamonds and dots in there for everybody, or are you ok with this? Ok? And so we have this. But this now, and we totally forget G for a minute, this is an h_1 *(circles $\theta(g_1)$ with his finger)*, this is an h_2 *(circles with his finger)*, these are just elements in H. The image of the product is the product of the image, because psi is a homomorphism.
Dr. A:	So, this is, this uh, theta a homomorphism, and now we do this as '$(\psi\theta(g_1))(\psi\theta(g_2))$' and this is because psi is a homomorphism. But now, again, we go back to the definition, this is '$((\psi\circ\theta)(g_1))((\psi\circ\theta)(g_2))$' and so, if you take this guy on the product, you end up with here, the product of the images. So, here, the image of the product is the product of the images. So, it is *(writes '$\psi\circ\theta$ is a homomorphism')*. Ok?

So, we can see that Dr. Adams did ask some questions, however, he did not wait for a reply from the students. The questions posed were merely part of his presentation style. He may have been modeling the questions that he would ask himself as he wrote the proof, but there did not appear to be an expectation that the students would actively contribute to the proof presentation.

Table 4: Percentages of proofs coded for expected engagement

	Dr. G	Dr. N	Dr. A
Number of Proofs in Observation Data	22	8	12
Coded 1 or 2	4.5%	0%	50%
Coded 3	54.5%	50%	41.6%
Coded 4 or 5	40.9%	50%	8.3%

Proofs coded 3 represent an instance where the professor did expect the students to actively engage in the presentation, but only to contribute factual information. There did not appear to be an expectation that the students would create any of the big ideas for the structure of the proof. An example of this level of expected engagement comes from Dr. Grey's presentation about the summit angles of a half-rectangle.

Dr. G:	A half rectangle is a convex quadrilateral with two right angles at the base. And, what I want to prove is that in a half-rectangle, I want to look at these two angles… this is going to look like a test problem before we're through…

If I look at those two angles, I claim that the greater side is across from the greater angle. So, 'In a half-rectangle, the greater side is across from the greater summit angle.' Those are the summit angles. So, what I want to do, what's the theorem? The theorem is angle 1, well, let's do it this way. 'Suppose BC is bigger than AD,' and I want to prove that angle 1 is, uh, bigger than angle 2. Suppose this is true. Ok, Well, let's see, let's 'choose a point *E* so that *B*E*C* and *BE* congruent to *AD*.' So, pick a point so that those two things are congruent. What kind of an animal is this? *DABE*? It has a name.

Student: Saccheri rectangle?

Dr. G: It's not a Saccheri.. don't say rectangle.

Student: Saccheri Quadrilateral?

Dr. G: Saccheri quadrilateral. Ok? 'So, *ABED* is a Saccheri quadrilateral.' What do I know about summit angles of Saccheri quadrilaterals?

Student: They're congruent?

Dr. G: Yeah. We proved that on a test, I think, or on a previous test, or a homework. 'Angle 3 congruent to angle 4.' Oh, now it's gonna look like the test we just did, because angle 1, degree measure, is bigger than angle 3, which is the same in degree measure as angle 4, and this angle is… how does that compare with angle 2? How does angle 4 compare with angle 2?

Student: It's exterior.

Dr. G: It's exterior, isn't it? It's exterior to this triangle. *(re-draws part of the picture to emphasize that angle 4 is exterior)* Angle 4 is exterior to the triangle, so it's, um, bigger than angle 2. Ok.

In this proof, we can see that Dr. Grey expected the students to respond to most of his questions. When he is outlining the structure of the proof, he seems like he is going to ask the students how to begin the proof, but he doesn't wait for their response. He then lays out the beginning of the proof, and begins to ask the students factual questions along the way. We believe that he expects the students to respond to these questions because he asks the same question multiple times, and waits for a student response. When Dr. Grey hears the desired response, he takes that information and immediately proceeds to the next step in the proof. The students do dialogue with Dr. Grey, and they do contribute to the proof, however, Dr. Grey is clearly in control of the construction of the proof.

Since the expected engagement level of 4 or 5 may differ slightly from the way we conceptualize lecture style proof presentations, we included three examples, one from each faculty member. This excerpt from Dr. Nelson's class was coded 4, because the students give the key ideas for how to construct a proof of uniqueness.

Dr. N: Um, all right, well, unfortunately after all that success, you still have uniqueness to prove. And, um, you'd like to think that maybe uniqueness was part of what you were doing, but it's really hard to justify that. So, the best way to prove uniqueness is to… I mean, what's the standard way that you prove that there's only one formula? There's sort of a standard strategy that you do. When you've done this already… we did this for the division algorithm. You remember what I said? At the end there's a uniqueness part of the division algorithm, and… how did we prove that uniqueness was actually true? Yeah, so let me write up here, 'proof of uniqueness' And there's always sort of a standard way. So, you've proved that there is a formula, and you want to show that that formula is unique, so what's the strategy for doing it?

Student: Assume there's two, and show it isn't possible.

Dr. N: Exactly! All right, that's all I want you to say is just assume that there's two. Assume there's two and then show that they are actually the same. So, 'Suppose there are two such formulas for the same n.'

Student: So are we just proving contradiction?

Dr. N: Yeah, this would be an example of proof by contradiction. What's the opposite of uniqueness? The opposite of uniqueness is that there's more than one. So, I'm going to assume that there are… this is kind of a specialized version of proof by contradiction. It always works for uniqueness. You always assume there are two, and then you try to prove that they are actually the same. So, suppose there are two such formulas, um, so the first one we got was that over there, let me just summarize it, ' ' and for the second one, we have to use a different letter for the coefficients, b is the same, but I'm going to change the a to a c, and the k to an l, because it's possible that if I did it in some other crazy way, maybe I got more, uh, more digits or something. '$n = c_l b^l + ... + c_1 b + c_0$' And now the strategy is to prove that something is wrong with this. Suppose there are two different ones. Suppose they are actually different. We have two different formulas. Now, how would I know that these are really different? What specifically would.. what would happen that would tell me that these are really different? Now, you sort of have to hone in on what's the difference between the two formulas. You know, what's the specific difference. Now, if they are really different, something has to happen.

Student: *(mumbles)*

Dr. N: Now, you're saying *a*, but there are a lot of *a*'s, you have to be specific.

Student: a_j is not equal to c_i, or c_j.

Dr. N: No, that's not correct, or, what did you say at the end?

Student: c_j

Dr. N: c_j, ok, right, ok? So, it doesn't matter that this digit is not equal to that one, it's the corresponding ones. So, what Nathan is saying here is that '$a_j \neq c_j$ for some j.' Not for all of them, but for one of them, they have to be different. Ok, now this is really hard to motivate until you've seen this kind of proof, but, one of them, I know that in one of them they are different, but I want to pick, I want to sort of be real specific about j. I want to say, 'take j to be' Now, j could be zero, it could be 1, it could be anything in those ranges there, but I want to work specifically with j having a real specific property here. You know, until you get into this, it's really hard to motivate. Do you know what I want j to be? I want the two digits to be different, but I want to know something more about j. *(mumbles)* You have to learn this…

Student: The smallest one?

Dr. N: The smallest one, there we go. You got it. You know, when you're trying to hone in, you always want to go to an extreme case…

We can see that Dr. Nelson continues to ask the students questions about how to structure the proof, and asks the same question in different ways until the students give an answer. Although the students are contributing significant ideas to the proof, this presentation is still teacher-centered. When the students give the expected answer, Dr. Nelson spends a little bit of time explaining why the answer that the student gave is correct, and then carries on with the proof.

One common approach in Dr. Grey's class was to have the students identify the contradiction at the end of an indirect proof. The students didn't structure the proof, but they did contribute a key idea to the proof by identifying the contradiction. Here is an example of this type of engagement.

Dr. G: So, I was going to say, we did this proof before, but I was just going to remind you how it goes. How do I show that if Euclid 5 is true, then the converse to alternate interior angles is true? This is what I want to prove *(underlines 'converse to AIA' writes 'prove this' underneath)* But, since I'm assuming Euclid 5 is true, I get to use this. *(underlines 'Euclid 5' and writes 'Use this in my proof' underneath.)* So, let's see, I want to show the converse to alternate interior angles is true, so I say, 'suppose that k is parallel to l,' and I need to 'prove that angle 1 is congruent to angle 2.' So how do I do that? Well, suppose it's not. 'Suppose angle 1 is not congruent to angle 2.' Well, then what are we going to do, is use my protractor axiom to reproduce angle 1 right here. I'm running out of letters. n. So, let's choose n through Q so that angle 3 is congruent to angle 1.

Student: So, you're saying that angle 1 is bigger than angle 2?

Dr. G: This picture makes it look greater, It would, it's gonna be the… it's gonna be the same argument if it were smaller. So, I'm just assuming they're different. Ok, I claim that we're going to get a contradiction here all ready. I claim I'm done. Where is my contradiction? Well, what do I… let me… what do I know about n and k?

Student: They're lines.

Dr. G: They're lines, I do know that. That's correct, but I would hope to know more.

Student: Parallel.

Dr. G: Why?

Student: Alternate interior.

Dr. G: Alternate interior angles? Ok, so 'k is parallel to n by alternate interior angles.' It's not the converse, but the theorem itself.

Student: And then Euclid 5, there's a contradiction because it says there's a unique parallel.

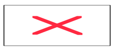
Dr. G: Right. I get to use Euclid 5 in my proof, and now I've just built two lines parallel to k through the point Q. Now, 'both l and n are parallel to k and through Q.' That's a contradiction. So, that didn't happen. Those angles were the same.

Dr. Adams frequently answered questions about homework problems at the beginning of class. On one such occasion, we can see that the professor expected students to be involved in both example generation and a discussion about the necessity of the hypotheses in this particular proof.

Dr. A: Uh, '*a* equivalent to *b* if and only if  for some . A)

Prove that this is an equivalence relation,' and 'B) give a complete set of equivalence class representatives.' So, you had a problem with showing that it's an equivalence relation, right? And, so *a* is equal to *b* to the *10 k* for some *k*, so, um, we, uh, have to have, then, so let's see. So what's the equivalence class, say, of… what are *a* and *b*?

Student: Natural numbers

Dr. A: *(writes*  *at the top)* So, let's see. What's equivalent to 1? Yeah, we want a complete set of equivalence classes.

Student: 1.

Dr. A: Well, 1 is. That's a good start. What else?

Student: 10.

Dr. A: Right. So, 1, 10, uh, and then we go 100, … and then we go 10^k. *(writes {1,10,100,…* 10^k *…})* '*k* belonging to *Z*.' So, I should maybe, uh, do it the other direction too, so since we have k negative, um, let's see…

Student: Can *k* be negative, though? I don't think *k* can be negative, because then it makes a… not a natural number.

Student: It can be negative if *b* is greater, like a multiple of 10.

Dr. A: So, did it say some *k* belonging to *Z*?

Student: Yes.

Dr. A: So, *k* has to be greater than, *k*, and that, uh, greater than zero. Ok, so that's…

Student: So that's…

Student: It can be zero.

Student: It doesn't have to be greater than zero. If *a* is a multiple of 10 then *b* can be, like 4, and then, whatever, and then it just decreases… Does anybody else see what I'm saying?

Student: If *k* is not equal to zero, we don't have the reflexive property of 1 being equivalent to itself.

Dr. A: Ok. *(erases k>0)* So, at least, for a natural numbers, things that are equivalent to 1 should be that. Do you agree with that? And now, 2, we get, what? 20, 200, so on and so forth. Right? *(writes {2,20,200,…2* $\cdot 10^k$ *…})*. Now, for general *n*, what would we get?

In this instance, Dr. Adams was posing questions and actively listening to the students' responses. The entire class was exploring the mathematics together, and Dr. Adams was merely facilitating the discussion. This may be because the student asked a question about a homework problem with which Dr. Adams was unfamiliar, and so he himself was working to make sense of the problem. This discussion continues until the class comes to an agreement about the possible values of *k* and how they are used in constructing a proof that this relationship is an equivalence relation, but they do not formally write down the proof because it was a homework problem.

We can see that in all of these instances the mathematician engaged the students, and seemed to expect that the students contribute key ideas to the proof. So, we have begun to examine the strategies that mathematicians mentioned in the interviews, and how they manifest themselves in the classroom.

Limitations of Study

This study analyzed several observations of each faculty member periodically throughout the course of a semester. The mathematicians in this study were teaching different courses in different content areas, so naturally the content could influence presentation strategies. We are not attempting to compare the instructors to each other across different content areas, but are seeking to describe the presentation strategies that each used in their classroom. Although we did make an effort to choose proof presentation strategies that were content independent, it may not be possible to completely divorce the content from the presentation style.

The observation data reflects the manner in which the material was presented during the semester in which the observations were made, but may not reflect the past or future behavior of the faculty member involved. The proof presentation strategies used and expected engagement level could depend on several factors, such as the ability levels and experience of the students in the classroom, or the classroom dynamic created by each particular group of students. The interactions between the instructor and his students may also reflect the instructor's current mathematical content knowledge and pedagogical content knowledge, which both change over time.

The subjects of this study were faculty members, and therefore data was not explicitly collected from the students. We have no data on the number of students who interacted with the faculty member or ways in which they engaged.

The results presented here were coded by one researcher independently. It would increase the validity of this study if more than one researcher participated in the coding. Also, the coding was based only on the instances of proof presentation, so it is possible that other strategies were used by the instructor when not explicitly presenting proofs.

Conclusions

As we work to describe how faculty members present proofs in class, our goal is to contribute to the a foundation of knowledge about how proofs are presented in the undergraduate mathematics classroom. Although much more work needs to be done, we can draw a few conclusions from the preliminary analysis of our data.

First of all, we can conclude that faculty members in a traditional lecture classroom expect engagement from their student, and that the level of engagement during proof presentations varies. Even in the early stages of analysis, we can already see that these instructors do expect students to contribute the proof presentation. All three professors presented proofs with the expectation that the students would provide factual information during the proof presentation in approximately half of the presentations, and they also expected students to contribute key ideas the proofs in many of the presentations that were in our video data.

Another conclusion that we can draw from this data is that the faculty members are not only talking about strategies for proof presentations, but are actually using the strategies that they talk about. The proof presentation strategies that we identified from the interview data were not entirely unique to this study, but bore similarities to the strategies mentioned in the literature (Weber, 2010; Alcock, 2010; Mejia-Ramos, et. al., 2010). We have documented that these instructors not only think about the pedagogy of proof presentation, but that they use specific strategies with the goal of enhancing student learning.

We have seen that the lecture style of proof presentation is richer and more diverse than is currently documented, and can include significant levels of student engagement. Future directions for this research will include follow-up interviews with the faculty members. This data will strengthen the validity of our study and may give us ideas for different ways to look at our video data. It may also be interesting to collect data from the students who are attending to and participating in the proof presentations. We will continue to analyze the data from different perspectives to draw out other facets of proof presentations in the classroom.

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RELATIONSHIPS BETWEEN QUANTITATIVE REASONING AND STUDENTS' PROBLEM SOLVING BEHAVIORS

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Over the past half-century, one strand of problem solving research has investigated problem solvers' behaviors during various phases of the problem solving process. A second strand of problem solving research has explored the mental actions, reasoning abilities, and understandings involved in using specific concepts to solve novel problems. These two strands of research provide valuable contributions to mathematics education, but they have remained as separate strands of inquiry within the research literature. This paper reports on the results of a study that investigated three students' problem solving behaviors in the context of their confronting novel problems when learning central ideas of trigonometry. The study's findings illustrate specific mental processes that enabled or hindered the students in engaging in productive problem solving behaviors. Particularly, the study describes how a student's ability to engage in quantitative reasoning shaped their movement through the problem solving phases.

Key Words: Precalculus, Problem Solving, Student Reasoning, Quantitative Reasoning

Introduction

Problem solving has been a focus of mathematicians and mathematics educators for the past half-century, with several studies characterizing students' and mathematicians' problem solving processes and performance (e.g., M. Carlson, 1999; M. P. Carlson & Bloom, 2005; DeFranco, 1996; Lester Jr., 1994; Pólya, 1957; Schoenfeld, 2007). These authors describe problem solving as a complex process of interrelated factors (e.g., affect, monitoring, and conceptual knowledge) and phases (e.g., planning and checking) with recent studies (M. Carlson, 1999; M. P. Carlson & Bloom, 2005; Schoenfeld, 2007) revealing the interaction of various factors in the problem solving process.

Another strand of mathematics education research investigates student thinking and reasoning in the context of students acquiring an understanding of a mathematical concept or idea as students encounter novel problems (e.g., Ellis, 2007; Hackenberg, 2010; Moore, 2010; Oehrtman, Carlson, & Thompson, 2008; Smith III & Thompson, 2008; Thompson, 1994a, 1994b). These studies' findings provide insights into the critical reasoning processes that support productive problem solving abilities. This collection of studies particularly identifies quantitative reasoning, as defined by Smith III and Thompson (2008), as critical for developing flexible and coherent understandings of mathematical topics.

Quantitative reasoning (Smith III & Thompson, 2008) refers to the mental actions involved in conceptualizing measurable attributes (quantities) of objects and relationships between these measurable attributes. In addition to the studies that highlight the role of quantitative reasoning in students acquiring understandings of a mathematical idea, Moore, Carlson, and Oehrtman (submitted) identified that quantitative reasoning is an essential reasoning pattern that leads to students building the mental imagery needed to construct meaningful formulas and graphs.

This study builds on the current body of problem solving research by investigating the reasoning students employ when moving through the problem solving phases. In order to accomplish this goal, I explored three precalculus students' thinking as they solved novel problems that required reasoning about angle measure and trigonometric functions. The students participated in a 5-week teaching experiment (Steffe & Thompson, 2000) that included a series of group teaching sessions. Each student also participated in a series of exploratory teaching interviews.

In order to describe the students' thinking during the problem solving process, I initially examined individual student written solutions and interview data. I then classified their mental actions in terms of the problem solving phases of orienting, planning, executing, and checking and drew relationships between their mental actions and their behaviors during the problem solving phases. The results of this analysis illustrate that the students' ability to engage in quantitative reasoning and build rich quantitative structures of a problem context influenced their actions and the products that they constructed during each of the problem solving phases. The study's results offer insights into the mental actions that either hinder or support students' engagement in productive problem solving behaviors.

Background

In starting any discussion on problem solving, it is natural to begin by referencing Pólya's (1957) initial work in describing four problem solving phases: a) understanding the problem, b) developing a plan, c) carrying out this developed plan, and d) looking back at the solution. Since Pólya's initial work, contributions to the body of literature on problem solving (see Lester Jr. (1994) and Schoenfeld (2007) for exhaustive reviews) provide many insights into the characteristics of successful problem solvers, including comparisons of the behaviors exhibited by successful and unsuccessful problem solvers. These studies have revealed many factors that contribute to an individual's problem solving ability, including an individual's knowledge and her/his attitudes and beliefs about mathematics and problem solving. Several authors illustrate (M. Carlson, 1999; DeBellis & Goldin, 1999; Hannula, 1999) the influence of affective variables (e.g., beliefs, attitudes, and emotions) on an individual's problem solving activity, while other studies highlight the critical role of metacognition in problem solving (Schoenfeld, 1992; DeFranco, 1996, Carlson, 1999). As a particular example, Schoenfeld (1992) identified that successful problem solvers often monitored and regulated their own actions while unsuccessful problem solvers did this on a less frequent basis.

In the early 1990s, the main focus of research on problem solving turned away from exploring problem solvers' behaviors (Schoenfeld, 2007). Rather than focusing on the mental actions of problem solvers and attempting to determine how and why problem solvers make their choices, the emphasis of the community turned to exploring learning environments for problem solving. These studies explored sense-making classrooms, discourse communities, participant accountability, and productive classroom cultures. Although these studies provide valuable insights into social environments, I note that the main focus of problem solving transitioned away from investigating students' cognitive actions to looking at social settings, though recent studies (M. P. Carlson & Bloom, 2005; Schoenfeld, 2007) suggest that much is to be learned by continuing to examine the cognitive actions involved in problem solving. These insights are needed to better understand how to foster the development of flexible and productive problem solving abilities in students.

In an attempt to provide a finer characterization of problem solvers' cognitive processes, Carlson and Bloom (2005) investigated the problem solving activity of 12 mathematicians. Drawing from analysis of interviews with the mathematicians, as well as previous research on problem solving (DeBellis & Goldin, 1999; DeFranco, 1996; Hannula, 1999; Lester Jr., 1994; Pólya, 1957; Schoenfeld, 2007), the authors created the *Multidimensional Problem-Solving Framework*. This framework expands on Póyla's problem solving phases by characterizing how various attributes (e.g., affect and the use of heuristics) influence a problem solver's behaviors during the problem solving phases. The authors also identified multiple problem solving cycles within the four problem solving phases (e.g., orientation, planning, executing, and checking). For instance, the authors noted that the mathematicians frequently engaged in a *conjecture-imagine-evaluate* sub-cycle of the planning phase to mentally play out a solution. The authors also identified that the mathematicians exhibited problem solving behaviors indicating a more formal and deliberate *plan-execute-check* problem solving cycle.

Carlson and Bloom's (2005) study of mathematicians' problem solving behaviors emphasized that further explorations are needed to identify the role of various reasoning patterns in problem solving. Particularly, the authors claimed, "[the mathematicians'] ability to play out possible solution paths to explore the viability of different approaches appears to have contributed significantly to their…problem solving success," while suggesting that subsequent studies explore such connections between problem solvers' thinking and their problem solving behaviors (M. P. Carlson & Bloom, 2005, p. 69). In response to the authors' call for studies that explore the mental actions involved in problem solving, myself and two colleagues (Moore, Carlson, & Oehrtman, submitted) examined the role of quantitative reasoning (Smith III & Thompson, 2008) when precalculus students orient to novel problems. We were curious to know why precalculus students were exhibiting difficulties that were not reported in studies of expert problem solvers, and leveraged Smith III and Thompson's (2008) model of quantitative reasoning to explain these difficulties.

In short, quantitative reasoning (Smith III & Thompson, 2008) refers to the mental actions involved in conceiving of situation such that it is composed of measurable attributes, called quantities, and relationships between these quantities. The quantitative structure that results from these mental actions can support students in developing meaningful formulas, calculations, and graphs. In analyzing precalculus students' problem solving behaviors, we (Moore, Carlson, & Oehrtman, submitted) identified the critical role of quantitative reasoning during the orientating problem solving phase. The students' solutions to the problems relied on the images they constructed of the quantities and their relationships when initially making sense of the words in the problem. Students who developed robust images of a problem's context constructed quantitative structures that supported their determining meaningful formulas and graphs. In light of this finding, the authors concluded that problem solving research could benefit from exploring further connections between quantitative reasoning and students' problem solving behaviors.

The argument for investigating connections between quantitative reasoning and problem solving is further supported by findings (Ellis, 2007; Hackenberg, 2010; Moore, 2010; Oehrtman et al., 2008; Smith III & Thompson, 2008; Thompson, 1994a, 1994b) that describe quantitative reasoning as an important aspect of learning mathematics. These studies explored students' thinking as they solved novel problems with the intentions of characterizing the students' thinking relative to a particular mathematical topic. Though these studies did not explicitly explore the students' solutions in the context of literature on problem solving (e.g., the problem solving phases), the studies' findings imply that quantitative reasoning is a critical component of solving novel problems. In light of the collection of research findings on quantitative reasoning, I conjectured that quantitative reasoning provides a useful lens for exploring students' problem solving behaviors during the various problem solving phases.

Methods and Subjects

Three students from an undergraduate precalculus course at a large public university in the southwest United States participated in the study. The researcher (myself) was the instructor of the course. I chose the participants on a voluntary basis and they were monetarily compensated for their involvement. I drew the participants from a precalculus classroom that was part of a design research study. Theory on the processes of covariational reasoning and select literature about mathematical discourse and problem solving (M. Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; M. P. Carlson & Bloom, 2005; Clark, Moore, & Carlson, 2008) informed the design of the course. All three participants (Judy, Zac, and Amy) were full-time students at the time of the study. Judy was a female in her mid-twenties and a biochemistry student. Zac was a male in his early twenties and an ethnomusicology and audio technology major. Amy was a female in her late teens and an undeclared major planning on pursuing nursing.

The design of the study followed a teaching experiment methodology, as described by Steffe and Thompson (2000). The teaching experiment consisted of eight 75-minute teaching sessions within a span of five weeks. Each teaching session included all three participants, the instructor/researcher (myself), and an observer. Immediately after each session, I debriefed with the observer to discuss various observations during the teaching sessions and the design of future teaching sessions and interviews. I also conducted exploratory teaching interviews with each participant. I used the exploratory teaching interviews to gain additional insights into the participants' thinking. These one-on-one interviews followed the teaching experiment principles, and helped me to identify the students' thinking and the influence of their thinking on their behaviors during the problem solving phases.

I analyzed the data using an open and axial coding approach (Strauss & Corbin, 1998). I first analyzed the students' behaviors in an attempt to determine the mental actions that contributed to their solutions. I then characterized the mental actions inferred from the students' behaviors in terms of the problem solving phases identified by Carlson and Bloom (2005). This phase of the data analysis involved identifying how the students' mental actions influenced their behaviors during the four problem solving phases. This approach to analyzing the data enabled me to classify how various reasoning patterns related to the students' problem solving behaviors. Lastly, I compared and contrasted the students' mental actions and problem solving behaviors. In this stage of the analysis, I drew connections between the students' propensity to engage in quantitative reasoning and their activity during the problem solving phases.

Results

This section describes the students' solutions to novel problems by characterizing their mental actions in the context of the problem solving phases of *orienting, planning, checking, and executing*. As the study progressed, Judy and Zac revealed similar approaches to solving novel problems, while Amy's problem solving behaviors differed considerably from Judy's and Zac's behaviors. Paralleling this observation, Judy's and Zac's mental actions were in stark contrast to Amy's mental actions when confronting the problems. In order to illustrate relationships between the students' thinking and their behaviors during the problem solving phases, I compare and contrast Zac's and Amy's solutions.

After an introduction to trigonometric functions in a circular motion context, the students attempted a problem set within a right triangle context (Table 1). Zac *oriented* to the problem by constructing a diagram of the situation and labeling the given values (Figure 1).

Table 1
The Empire State Building Problem

While site seeing in New York City, Bob stopped 1000 feet from the Empire State Building and looked up to see the top of the Building. Given that the angle of Bob's site from the ground was 56 degrees, determine the height of the Empire State Building.

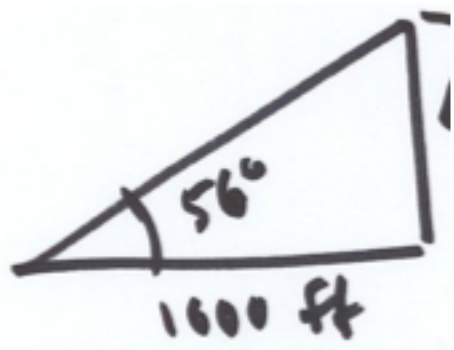

Figure 1. Zac's initial diagram on The Empire State Building Problem.

After drawing a diagram of the situation, Zac explained, "From the circle, or triangle, we can determine that cosine of fifty six degrees is equal to one thousand feet (*writing corresponding equation*)…one thousand feet is equal to [a number of radius lengths], because cosine fifty six degrees is determined in radius lengths." After converting the given angle measure to a number of radians, Zac continued to explain his solution (Excerpt 1).

Excerpt 1

Zac: Ok, alright, so. Scratch that, point nine eight. This (*referring to cosine expression*) is equal to a thousand feet…then a thousand feet is equal to the radians times the radius length (*writing '(rad)(r)'*), or *r*.

KM: Ok.

Zac: Ok, 'cause the radians is just a percentage of the radius length. Oook. So, now what I want to do is figure out what cosine point nine eight is actually equal to, and using that I can find out what the radius length is (*pointing to r*). So then when I do sine of point nine eight, I already know what the radius length is, so when I get the answer to that (*referring to sine*) all I have to do is multiply by the radius length and I'll get that part (*identifying the vertical segment on his triangle*).

KM: Ok, so if you wanna go ahead and do that.

Zac: Ya. Ok, so, (*using calculator*) cosine point nine eight is equal to, (*writing*) equals point five six radians. And so, all I have to do is (*using calculator*) divide one-thousand by point five six, as shown in this equation right here (*referring to '1000=(rad)(r)'*), to isolate the radius all I have to do is divide it by the radians. (*using calculator*) And I get a big number. So that means (*writing*) *r* is equal to one seven eight five point seven one. So then I do (*writing*) sine point nine eight, (*using calculator*) and I'm given a radius length, or a percentage of a radius length, (*writing*) equal to point eight three. Now all I have to do is multiply that (*writing*) by *r* and I'll get the length of that side (*pointing to vertical segment on his triangle*), so, times (*using calculator*) one seven eight five point seven one. So that means the length of that side is equal to (*writing*) one four eight three point O three feet. Figure definitely not drawn to scale.

Though Zac constructed (Figure 2) a mathematically incorrect equation ($\cos(0.98) = 1000$), he *planned* his solution by mentally playing out a sequence of calculations. Specifically, Zac anticipated "what cosine point nine eight is actually equal to," and explained that $\cos(0.98)$ represented a fraction of the radius without *executing* a calculation to determine this value (lines 5-11). Instead, he imagined determining a measure in radii, using this measure to determine the radius length, and then determining a value that represented a multiplicative relationship between a length and the radius (e.g., the output of the sine function).

Zac's actions reveal him *planning* a sequence of calculations without immediately *executing* the calculations (lines 8-11). Then, as Zac *executed* the calculations, he continually referred to the quantities of the situation and the meaning of the various measurements he obtained to monitor and *check* his solution (lines 14-27).

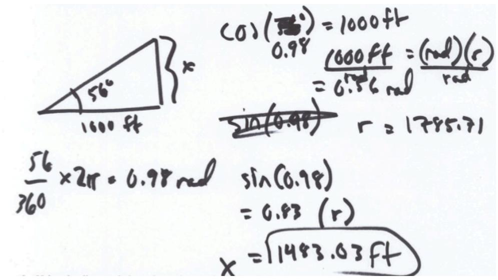

Figure 2. Zac's work on The Empire State Building Problem.

A right triangle context was not introduced during a teaching experiment sessions previous to this problem, yet Zac reasoned about multiplicative relationships between a radius and various lengths to construct a quantitative structure that enabled him to *plan* his solution before *executing* calculations. After providing his solution, Zac also explained that he *oriented* to the problem by conceiving of the hypotenuse of the right triangle as the radius of a circle (Figure 3). This *orienting* act, which was unobservable to myself previous to his drawing Figure 3, aided him in constructing a robust image of the problem's context (e.g., a well-developed quantitative structure) that supported him in *planning* his solution.

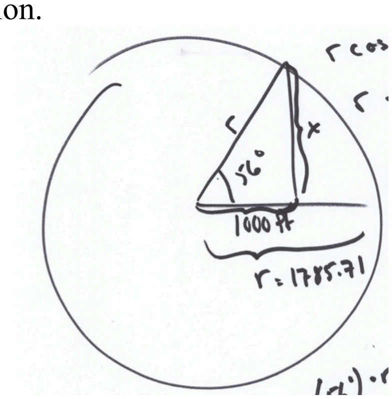

Figure 3. Zac's refined diagram on The Empire State Building Problem.

As an example of another student's solution, Amy read the problem statement, drew a diagram, and then claimed, "I know how to do this, I did this last night…I just did this…give me the first step and maybe I can figure it out from there…and we just learned this." Her claim, "I know how to do this…give me the first step," implies that she expected to recall a post procedure when solving the problem.

Amy subsequently suggested converting the angle measure to a number of degrees (the problem differed from that in Table 1 and included a given height of 670 feet and a given angle measure of 1.1 radians), but expressed that she unsure how the measure related to the goal of the problem. Amy then attempted to determine the hypotenuse of the right triangle (Excerpt 2).

Excerpt 2

Amy: Um (*pause*). Doesn't six hundred and seventy divided by this side (*pointing to the side adjacent the given angle*) give us the radius? Right?
KM: What do you think?
Amy: Well I know it does, but.
KM: Six hundred and seventy divided by what side?
Amy: Right here (*pointing to the side adjacent the given angle*).
KM: So why do you think that gives us the radius?
Amy: I know it does.
KM: So why do you know it does?
Amy: 'Cause that's what is in my notes.

Amy attempted to recall a calculation (lines 1-2), looked to the researcher for approval, and claimed that her suggested calculation was in her notes (lines 8-10). In this case, Amy's conjectures focused on calculations not rooted in quantitative relationships, which likely limited her ability to *check* the correctness of her conjectures beyond referencing her notes. Amy was then unable to move forward on the problem.

Over the course of the study, Zac and Amy exhibited actions similar to those described in Excerpts 1 and 2. After reading the problem statement and drawing a diagram, Amy frequently *oriented* and *planned* her solution by attempting to recall past problems, calculations, and procedures. Amy's explanations also revealed that the recalled calculations and procedures did not stem from quantitative relationships. For instance, Amy often immediately *executed* (or emphasized a need to *execute*) recalled calculations (and formulas), but she was unable to justify or *check* these calculations beyond claiming the she used the same procedure or calculation to solve a previous problem. As a result, she frequently had difficulty obtaining solutions to the posed problems.

Zac (and Judy), on the other hand, frequently *oriented* to a novel problem by drawing a diagram and they then *planned* their solution by discussing quantitative relationships and calculations that stemmed from these relationships. Such actions enabled Zac and Judy to anticipate *executing* a sequence of calculations without needing to perform the calculations. As Zac and Judy *executed* calculations, they also maintained a focus on the context of the problem in order to monitor and *check* their progress.

Due to Amy experiencing difficulty in providing a solution to the problems, I implemented tasks that were more conducive to recalling calculations or procedures (e.g., tasks that required only one or two calculations to correctly solve the problem). I compared and contrasted the students' actions during these problems and I also compared the students' solutions to the solutions they provided during more complex problems (e.g., Table 1).

The Arc Length Problem (Table 2) prompted the students to determine various arc lengths when given an angle measure in degrees.

Table 2
The Arc Length Problem

Given that the following angle measurement θ is 35 degrees, determine the length of each arc cut off by the angle. Consider the circles to have radius lengths of 2 inches, 2.4 inches, and 2.9 inches. (*Task includes diagram with an angle and three concentric circles centered at the vertex of the angle*)

After reading the problem statement and identifying each given measurement on a diagram (*orienting*) Zac *planned* a sequence of calculations previous to *executing* the calculations (Excerpt 3).

Excerpt 3

1	Zac:	Um, ok. So what I plan on doing for this one is converting thirty-five
2		degrees into radians. And a very easy way of doing that is putting thirty-
3		five over three sixty is equal to *x* over two pi (*writing corresponding*
4		*equation*)…And then with that all I have do is just multiply the answer
5		(*pointing to x*) by two inches, two point four inches, and two point nine
6		inches (*pointing to each value in the problem statement*) to get the
7		different arc lengths (*identifying each arc length with his pen tip*) right
8		there, because radians is just a percentage of a radius.

Zac first described converting the angle measure from degrees to radians without *executing* the conversion (lines 1-2). He then reasoned about a multiplicative relationship between a subtended arc and the radius (a percentage of the radius) in order to anticipate *executing* a calculation that determines each arc length (lines 4-8). These actions illustrate Zac reasoning about a measurement in radians as a multiplicative comparison (e.g., a quantitative relationship) between an arc length and the radius length to *plan* his solution, which supported him in mentally playing out a sequence of calculations without experiencing a need to *execute* each calculation previous to considering the subsequent calculation.

After Excerpt 3 and in order to describe his reasoning when converting the give angle measure to a number of radians, Zac explained, “Well what you're doing is just technically finding a percentage. Like thirty-five over three sixty is (*using calculator*), is nine point seven percent of the full circumference.” Similar to Zac’s ability to reason about a multiplicative relationship between a subtended arc and the radius to anticipate determining an arc length, Zac’s explanation reveals that his ability to reason about angle measure as conveying a fraction of a circle’s circumference subtended by the angle supported him in reasoning about a converted angle measure without needing to *execute* the calculation. Thus, Zac’s constructed quantitative structure enabled him to *plan* his solution by mentally playing out a sequence of calculations.

As an example of Amy’s solution on The Arc Length Problem (Table 2), consider her initial attempt to solve the problem (Excerpt 4).

Excerpt 4

1	Amy:	I remember doing this, I'm just trying to remember how. Ok. Radius.
2		Hmm, and the angle measure's thirty-five degrees. And I'm trying to
3		determine the length (*tracing arc length*), like in inches?
4	KM:	Ya.
5	Amy:	Ok. Alright, let's see. Um. Two inches. And it's thirty-five degrees out of
6		three hundred sixty degrees (*writing ratio of 35 to 360*). Hmmm. If two
7		inches (*pause*), no. Hmm, I swear I know how to do this, I just can't
8		remember. (*long pause*) (*sigh*) Ok. (*long pause*) Hmm. I'm trying to
9		determine the length of the arc. Oh I promise I know how to do this. I
10		can't remember though (*long pause*).

After reading the problem statement, Amy attempted to “remember” a solution to the problem (line 1), though she did not solve a similar problem during a previous session. Amy continually emphasized that she knew how to complete the problem, but that she was unable to recall a solution for the problem (lines 5-10). Amy’s utterances suggest that she equated solving the problem to implementing a previously determined procedure, and though she did write a ratio, she did not provide a meaning for this ratio (lines 5-6).

Amy subsequently suggested using the equation $\frac{35}{360}=\frac{x}{2\pi}$, but explained, “I don’t know if, 'cause then why would I, I don't know. I think I need to incorporate the radius somehow." I then asked Amy to explain her equation and she suggested an uncertainty about the result of solving the equation. After *executing* a cross-multiplication procedure to solve for *x*, she explained that the value was a number of radians and admitted to memorizing the equation during a past session. After determining the angle measure in radians, I prompted her to continue her attempt to solve the problem (Excerpt 5).

Excerpt 5

1	Amy:	Ya, wait. Do I take the percentage? No. (*looking at the researcher*) Is it
2		the percentage divided by the radius or opposite? No, that, no. That's not
3		right (*pause*). That's sixty one percent of one radian. And (*pause*). My
4		radius is two. Would using cosine and sine be (*pause*) helpful?

Similar to Excerpt 2, Amy exhibited *planning* actions in the form of suggesting calculations and "using cosine and sine," which were introduced in a previous session. She also looked to the researcher for approval of her suggestions and she subsequently claimed that she was unable to proceed.

Amy often exhibited actions consistent with those in Excerpts 2, 4, and 5. She predominately attempted to recall calculations and procedures after reading a problem statement and regularly *executed* calculations or expressed a need to *execute* a calculation without first considering subsequent steps in her solution or explaining a meaning for these calculations. As another example of Amy's actions, consider her response when determining a circle's circumference given that an angle subtended 0.3 inches, or 22%, of a circle's circumference. Amy obtained a correct solution to the problem by determining an equation that enabled her to *execute* a cross-multiplying procedure (Excerpt 6).

Excerpt 6

Amy: Given that an arc-length of point zero three inches is twenty two percent of a circle's circumference, what is the circle's circumference? Alright, so I've got an arc length that is point zero three inches. A circle's circumference is pi times diameter. Isn't that pi times the diameter?

KM: Pi times the diameter.

Amy: (*writing πd*) And then, ok, all we have it point three. So we've got point three and the whole circle is pi times the diameter (*writing .03 above πd*). Of the circle's circumference. Hmmm. This would be better if I had a radius. Ok, it's twenty-two percent of the circumference. Twenty-two is the result of one hundred (*writing the ratio of 22 to 100*). (*calculating 100 times .03*) I don't know if this is right, I'm just gonna give it a shot. I get three. Oh, I need an x somewhere. Hmm.

KM: So what'd you do there? You did...

Amy: Ya, well I was gonna try and like cross-multiply and everything. But I...

KM: So what do you mean by you need an x? What are you referring to?

Amy: I need something that I'm gonna solve. Which would be the rest, which would be the whole circumference. So, ya, it would be like point three over x if we wanted to find it in inches (*replacing πd with x*). So it would be equals 3 (*writing $22x=3$*) and that doesn't make sense.

KM: So what doesn't...

Amy: The whole circle's point one three inches. I mean, the circumference. I don't know if that makes sense.

KM: So what do you think? How long was your arc length?

Amy: My arc length was point zero three, I just don't like uneven stuff.

After reading the problem statement, she recalled a formula for the circumference of the circle and desired to use this formula and the length of the radius to *execute* a calculation (lines 3-4 and 8-9). Amy subsequently constructed an equation between two ratios using a part to whole correspondence (lines 6-10), and though she was unsure of her equation she immediately attempted to *execute* a sequence of calculations (e.g., cross-multiplication) (lines 11-12). However, she was uncomfortable because the equation did not include "an x…something I'm gonna solve." After altering her equation to appease her discomfort and solving the equation, Amy encountered difficulty *checking* her solution beyond considering the aesthetics of the result (e.g., "uneven stuff").

Beyond using a part to whole correspondence when constructing an equation, Amy's actions did not suggest that her solution and calculations stemmed from a constructed quantitative structure. Rather, her solution stemmed from her deeming the problem to be appropriate for applying a cross-multiplication procedure. Her *planning* actions consisted of setting up an equation that supported *executing* cross-multiplication. Furthermore, without a well-developed quantitative structure to support her solution (e.g., the ratios Amy determined did not reflect a multiplicative relationship), Amy was unable to *check* her solution beyond trusting the memorized procedure and judging the aesthetics of the solution.

After this interaction, Amy was unable to explain her solution beyond claiming that cross-multiplication was a prescribed procedure for such a problem. She also added, "I like doing cross-multiplying," conveying that she experienced comfort in *executing* this procedure. On similar problems, Amy attempted to determine equations that supported *executing* cross-multiplication. When asked to consider an alternate solution in such cases, Amy often provided responses along the lines of, "I know there's an easier way, I just, never bothered to learn it. I like my method of cross-multiplying."

Discussion

The students' solutions imply a strong correlation between the nature of their mental actions and their corresponding problem solving behaviors during the various problem solving phases. Judy and Zac made sense of a problem's context by developing rich quantitative structures that they leveraged during the problem solving phases. Contrary to this, Amy focused on executing procedures and calculations and rarely conceptualized a problem's quantities, making little progress in advancing her solutions.

When orienting to a problem, Judy and Zac regularly drew and labeled a diagram of the problem's context with known and unknown measurements. They also discussed various relationships between quantities when drawing these diagrams. During the planning phase, they reasoned about relationships between quantities in order to *anticipate* executing a sequence of calculations (Excerpts 1 and 3). The students' ability to reason about relationships between quantities without performing numerical calculations enabled the students to engage in the *conjecture-imagine-evaluate* cycle identified by Carlson and Bloom (2005) in order to mentally play out their solutions. Also, in the case that Judy and Zac recalled formulas during the planning phase, they described these formulas in terms of the quantities of the situation and used the formulas to represent values without first evaluating the formulas.

When Judy and Zac executed their planned calculations, they continued to describe the calculations in terms of the quantities of the situation and consistently used a diagram to illustrate the quantity referenced by a determined value (Excerpt 1). Also, stemming from their robust quantitative structures that they constructed during the planning phases, the students established a quantitative meaning for the result of a calculation previous to performing the calculation. When executing calculations, Judy's and Zac's images of the problem contexts also supported their monitoring and checking the appropriateness of the calculated values. When they obtained values that were not consistent with their image of a problem's context, they returned to the context to further orient to the problem, check their solution, and modify their solution. Thus, Judy's and Zac's actions of constructing a robust image of the problems' contexts supported their productively engaging in the *plan-execute-check* cycle identified by Carlson and Bloom (2005).

Contrary to Judy's and Zac's behaviors, Amy exhibited an inclination to reason about calculations and procedures. When orienting to a problem, she often drew a diagram, but she infrequently labeled unknown values on the diagram and spent limited time verbally discussing a problem's context. Instead, she regularly referred to previously completed problems deemed similar to the current problem and attempted to recall previous procedures and calculations (Excerpts 2, 4, and 5). In the cases that she recalled previous procedures, she immediately executed the procedures without explaining or analyzing the procedure in terms of the problem's context. In the cases that she did not recall a previous solution, she often suggested calculations and encountered difficulty in determining a correct solution (Excerpts 2, 4, and 5). Furthermore, when I asked Amy to explain a meaning of her suggested calculations, she first executed the calculation and experienced difficulty determining how the obtained value related to the problem's context and the goal of the problem.

Amy's inclination to use calculations devoid of quantitative meaning also presented her difficulties when she attempted to check her solutions or calculations. Amy often relied on the aesthetic quality of her answers (e.g., values not "too big," "too small," or "uneven") and her comfort with the applied procedure (Excerpt 6). For instance, Amy felt confident when she encountered a problem that she could apply a cross-multiplication procedure. In the cases that the she believed her solution was incorrect or that she was unable to determine a solution, she looked to me for assistance. Amy's actions suggest that she did not construct quantitative structures that supported her solutions or engaging in the problem solving sub-cycles identified by Carlson and Bloom (2005).

Table 3 provides a summary of the students' problem solving behaviors while comparing Judy's and Zac's actions with Amy's actions during each problem solving phase. Overall, Judy's and Zac's actions reveal a continued focus on the context of the problem. Namely, Judy and Zac constructed quantitative structures that enabled them to engaged in connected and productive problem solving phases during both types of problems (e.g., the multi-step problems and the one- or two-step problems). Amy's actions during each phase, regardless of the problem type, centered on recalling and performing procedures and calculations, which did not support productive and connected problem solving phases. For instance, Judy and Zac frequently engaged in the *conjecture-imagine-verify* sub-cycle during the planning phase, which also supported their subsequent actions during the executing and checking phases. However, Amy did not exhibit actions that indicated she engaged in the *conjecture-imagine-verify* sub-cycle and a majority of her efforts were given to recalling a procedure to execute. These contrasting approaches to solving novel problems provide insights into the influence of a student's reasoning on her/his problem solving behaviors.

Table 3
Problem Solving Activity and Orientations

Phase	**Actions**	
	Judy and Zac	Amy
Orientation	- Spending a significant amount of time describing the context of the problem and developing an image of the problem's context (e.g., drawing a diagram of the situation). - Returning to the diagram of the situation to label determined values during their solution.	- Spending minimal time describing the context of the situation and drawing a diagram of the situation. - Rarely returning to the diagram of the situation after the initial orientation process - Recalling a previously completed problem deemed similar to the current problem.

Planning	- Reasoning about relationships between quantities to anticipate a series of calculations and obtaining unknown values previous to executing the calculations. - Recalling a formula and describing it in terms of quantitative relationships. - Continuing to reference the context of the problem and making alterations to the diagram of the context. - Engaging in the *conjecture-imagine-verify* sub-cycle to mentally play out a sequence of calculations.	- Recalling a formula and expressing a need to evaluate the formula. - Attempting to remember the steps and calculations of a previous procedure. - Expressing the need to execute a calculation previous to explaining a meaning of a calculation or considering subsequent calculations. - Looking to the researcher for suggestions.
Executing	- Describing calculations in terms of the quantities of the situation and relationships between these quantities. - Describing a meaning of a value previous to obtaining the value.	- Describing calculations in terms of numerical operations and numbers. - Executing calculations without referencing the context of the problem. - Encountering the need to determine a meaning of a calculated value. - Experiencing difficulty in determining a meaning of a calculated value and how the value related to the goal of the problem.
Checking	- Returning to the context of the situation when obtaining an unexpected value (e.g., returning to the orientation and planning phases). - Limited checking of calculations, as calculations were planned in terms of quantitative relationships. - Exhibiting confidence in their solutions.	- Checking numerical operations relative to a recalled procedure. - Focusing on the aesthetic quality of the solution (e.g., values not "too big" or "too small"). - Limited referencing to the context of the situation. - Looking for outside approval (e.g., interviewer or teacher) of calculations and results.

Conclusions and Implications

The students' problem solving approaches reveal relationships between students' reasoning and their problem solving behaviors. The students' solutions also reveal that their approaches to problem solving can either support or hinder their ability to solve novel problems. The study's findings suggest that students who construct robust images of a problem's context (e.g., a cognitive structure of quantities and quantitative relationships) are better able to engage in connected and productive problem solving behaviors that support their constructing logical arguments and solutions. Furthermore, the students' actions suggest that students can more flexibly recall and leverage understandings rooted in quantitative relationships to solve novel problems, as opposed to understandings that are rooted in memorized procedures and calculations. The students' actions imply that this claim holds for one- or two-step problems, in addition to more complex problems (e.g., problems that require fluent function conceptions).

The students' solutions to the problems also suggest that a student's ability and inclination to engage in quantitative reasoning influences their ability to exercise monitoring behaviors during the problem solving process. Judy and Zac were better able to monitor their actions, while Amy was often unable to judge the effectiveness and appropriateness of her solutions. These findings imply that understandings and reasoning rooted in quantitative relationships support students in not only developing resources of greater utility, but also improving their ability to monitor and analyze their thought process when solving novel problems. Several authors (Carlson and Bloom, 2005; Schoenfeld, 1992) highlight the critical role of these metacognition activities, and the findings of this study shed further light on the mental actions that engender metacognition behaviors.

In addition to contributing to the body of research on problem solving, this study's findings should inform curriculum designers and teachers about the reasoning abilities and problem solving behaviors that support students in solving novel problems. Instruction that focuses on performing calculations and procedures devoid of quantitative meaning may not support students in developing powerful problem solving abilities or knowledge structures that support solving novel problems. Rather, students are better prepared to solve novel problems when they engage in reasoning patterns and construct understandings that are rooted in quantities and quantitative relationships. For instance, Judy and Zac *did* suggest calculations and attempt to recall procedures at various (though rare) times during the study. In the case of executing a calculation that they could not anticipate, they often executed the calculations and then immediately attempted to interpret the resulting value in terms of a quantitative relationship with the values used to perform the calculation. By maintaining this quantitative focus, Judy and Zac were able to use such calculations to inform their solution. However, in the case that they attempted to recall a procedure or calculation without considering the quantities of the situation, they frequently failed to progress on the problem until they abandoned such a solution attempt.

Future Research

This study consisted of a teaching experiment in trigonometry and the students' problem solving behaviors were only investigated in the context of trigonometry problems. Still, the findings from this study (e.g., Table 3) can provide a foundation for future problem solving research that explores students' solutions to non-trigonometry problems. For instance, future studies that investigate students' problem solving behaviors within other mathematical topics can provide data to compare to and extend the results of this study.

The findings I present above suggest that students benefit from a quantitative approach to problem solving. Future research should investigate this claim and attempt to determine ways to support students in developing meaningful approaches to solving novel problems. During this study, I observed students alternating problem solving approaches between problems, as well as within a single problem. Studies should explore reasons for such transitions, and the instruction necessary to promote students developing problem solving approaches and reasoning patterns that support their constructing meaningful and correct solutions to novel problems. For instance, the findings from this study imply that understanding rooted in quantitative relationships support a student's ability to solve novel problems. This implication provides a path of exploration for future studies.

Judy and Zac also appeared to be more reflective (e.g., engage in monitoring and checking actions) during their problem solving activity. This may have been a result of their constructing quantitative structures that created foundations for reflective actions. Future research should investigate this phenomenon, and its implications for using problem solving to support students in learning mathematics.

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THE PHYSICALITY OF SYMBOL USE

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We propose viewing the ways in which people use symbols and drawings as having an intrinsic physicality. When perceived as an extension of gesture-making, symbol use can give us insight into how symbol users experience the mathematics they're considering. To illustrate this, we apply this lens to examine selections from a video-recorded interview of a mathematician regarding one of his published papers. This results in several phenomenological characterizations of the mathematician's embodied symbol use that, in turn, offer potential insight into many instances of symbol use beyond this single case study.

Key words: Mathematicians, gesture, embodiment, phenomenology, symbols

Our main purpose here is to advance a perspective on symbol use informed by embodied cognition that is currently emerging in the field of mathematics education (Arzarello et al., 2009; Radford, 2009; Roth, 2004; Roth & Thom, 2009). In short, we choose to view symbol use as having an intrinsic physicality, which is to say that the meaning each of us perceives in symbols emerges from our actual and sensible-to-us interactions with them such as through drawing, erasing, pointing, changing body posture, and so on. We offer this perspective in contrast to a more commonly held one that symbols represent ideas via some kind of mental link that can be, but needn't be, demonstrated through bodily action.

Rather than detail the theory in full at the start, though, we think it may be easier to convey what we intend by showing you how a few particular instances of data appear when seen through this theoretical lens. In doing so we hope to share with you a vision of how symbol use is actually experienced in some very abstract domains.

Overview of Perspective

The data we'll use in this paper come from a video-recorded interview with a research mathematician, whom we'll refer to as "J". J is a topologist at a large doctorate-granting university in the southwestern United States. We interviewed J for an hour and fifteen minutes about a topology paper J had published ten years prior. The paper offered a geometric interpretation of a device known as the "6*j* symbol", which is involved in the addition of angular momentum ("spin") of quantum particles. A great deal of the interview involved J drawing analogies between quantum spin and its classical analogs, which J frequently depicted via rotating spheres.

The following screenshot was taken near the beginning of the interview. The video from which this screenshot was taken is a juxtaposition of two cameras' video data that have been time-synchronized. Thus they show the same scene from two different angles.

Figure 1 – Animating a Sphere

The transcript surrounding this moment is as follows: *"If you had a spin-, classical spinning particle like a ball, I mean its, its angular momentum is described by a vector…."* Mathematically, what J is describing here is the fact that any classical rotation can be described by a vector that is perpendicular to the plane of rotation and whose length indicates the speed of rotation. For instance, a top that's spinning counterclockwise could have its rotation represented by a vector that points in the direction of the top's handle (usually straight upward). In Figure 1, J attempts to convey this mathematical idea by drawing a sphere that we're to see as rotating, and he draws the rotation vector as an arrow coming out of the top of the sphere.

However, we'd like to draw your attention to his hand movement directly in front of the blackboard. After drawing the arrow coming out of the "north pole" of the sphere, J withdrew his right hand slightly from the board, held the piece of chalk he had been using so that it pointed roughly upward, and then twisted his hand about that piece of chalk counterclockwise as viewed from above. (The orange arrow in the right-hand image indicates this twisting movement.) In other words, he held the chalk as though it were the rotation vector for the sphere and then moved his hand in a way that conveyed the rotational motion of the sphere.

It seems prudent to ask why J did this. We aren't likely to come up with a definitive answer with such a small snapshot of data, but we can still develop something useful by exploring the question.

One way to attempt an answer would be to ask how J represents classical rotation in his mind. We could try to describe this in terms of some kind of mental construct; for instance, we might guess that J has a concept image (Tall & Vinner, 1980) for classical angular momentum that makes salient a mental picture that looks very much like the one he drew on the blackboard. We could test this hypothesis by asking J questions in a follow-up interview that should help to confirm or deny our guess, such as inviting J to share how he thinks about angular momentum when dealing with flat objects like spinning disks. At this tentative stage, this general approach would make us inclined to suggest that the reason J gestured as shown in Figure 1 is that he's trying to communicate a concept and is using his hands to help convey his concept image.

This is certainly a valid way to approach this situation, and there's probably something to be gained from it, but we'd like to suggest a perspective that's different from this and offers a different kind of insight. We posit that what J is doing here is actually *part of what he means* in his depiction of the sphere. It's not that his twisting of his hand expresses an idea he wants to convey, but that instead his hand rotation partially *constitutes* that idea by being a simultaneous practicing and conveying of the diagram's meaning as an embodied experience for him. For instance, the sense he has of the diagram depicting a three-dimensional sphere rather than a two-dimensional circle emerged from the spatiality of his hand movement. His proprioceptive sense of his hand's position around the imagined sphere is tied to the blackboard inscription by timing with respect to his spoken words, by proximity to the diagram, and by a repeated back-and-forth between indicating the drawing and doing hand movements similar to those in Figure 1. In doing so, he binds to the diagram the sense that he can reach out, grab, and rotate the sphere, giving the diagram its sense of implicit depth.

In fact, we suggest that the meaning J attributes to this drawing of a sphere is made up entirely of such perceptuo-motor activity. In other words, there is no “idea” of angular momentum that J “has” in a way that can be made independent of his embodied behavior. We can’t separate J’s gesturing from the meaning he perceives in the same way that we can separate a monitor from a computer; instead, it would be more like trying to separate a computer from its hardware.

We certainly acknowledge that there’s more to J’s understanding of the sphere he drew than this one particular gesture shown in Figure 1. However, we suggest that the impression that there are “hidden ideas” associated with this diagram comes from the existence of other perceptuo-motor activity that J perceives as pertinent to the diagram *but chooses not to act on.* For instance, it’s likely that J understands full well that were the sphere to be tilted so that the angular momentum vector pointed to the right, the sphere would instead be spinning vertically (i.e. counterclockwise as viewed from someone standing next to the blackboard to J’s right where the vector would be pointing). This understanding could be constituted in part by, say, a hand movement where J might “grab” a visualized sphere in front of the board with both hands and tilt it manually like one might tip a globe. Yet we don’t see J having actually performing any physical action like this, presumably because he did not see any point in enacting that part of his understanding of angular momentum at that particular moment. This “hidden” aspect of his understanding was therefore not acted upon because J inhibited that possibility even though he presumably recognized it on some level.

By reframing J’s movements this way, we no longer have to create formal models of what’s "really" going on in J’s mind. The mind is no longer viewed as a black box whose contents we need to model. Instead, we see J’s mind as distributed (Hutchins, 1999) across his bodily action and environment, both as acted upon (as in the case of the twisting movement in Figure 1) and as perceived as possible but inhibited (as in the case of tilting the sphere). This means we can gain insight into what shapes J’s phenomenological experience of his symbol use by carefully observing his bodily activity as he relates to the diagrams on the board (Erickson, 2004; Gallagher & Zahavi, 2008; Stivers & Sidnell, 2008). For instance, the moment depicted in Figure 1 shows a process of binding a physical movement to a diagram on the board in order to animate the diagram, so to speak, and part of that process in at least this particular instance requires that the animation occur very physically close to the diagram. This isn’t the only way J can bind movement to diagrams, and indeed at other times in this interview he shows a number of variations on how the movement can be linked to whatever is on the blackboard, but this one instance still provides us with some fair degree of insight about at least one way of animating and giving depth to drawings involved in higher mathematics.

Mathematical Regions

Another structuring pattern to J’s symbol use is with what we call *mathematical regions*. These are areas that occupy physical space in which J’s mathematical activity seems to occur. These regions appear both on the blackboard and sometimes in J’s peripersonal space when he turns or steps away from the board. Consider the following example (Figures 2 and 3):

Figure 2 – Defining a Mathematical Region

Figure 3 – Engaging in Another Region

The transcript is as follows: *"Maybe, you know, maybe you're just thinking about a system of interacting particles…. You need to do this kind of calculation…."* Here J is describing the need for the kind of spin addition he has written on the board when considering a system of interacting particles. Figure 2 shows J dropping both hands in synchrony (right before he says "system"). Figure 3 shows him stepping back, looking up at the board, and pointing with his left hand while he brings his right hand with the paper more toward his center line.

Although it's not depicted here, after the moment shown in Figure 2 J goes on to perform a number of gestures in the space between where his hands are in Figure 2. It's as though this initial gesture describes for J the boundaries of some kind of space in which he intends us to see this imaginary system of particles. His forward-leaning posture together with his gaze so obviously intent on something invisible between his hands seems to convey a vivid sense that something significant to him is going on in that portion of space. Having done so, he has created one instance of what we refer to as a mathematical region.

Just a few seconds later, though, J disengages from this space and engages with the blackboard. Figure 3 shows the result of this transition: J has stepped away from the space he had been using, straightens his posture, turns his gaze to the notation on the blackboard, and even uses his right arm and the paper it's holding to act as a kind of barrier between him and the area in which he had been working just moments before. Although his association with the blackboard material is at a distance (keeping the inscriptions in his extrapersonal space), the direction in which he has turned his body and gaze, as well as the shift in his spoken words, make it very clear that his perceptual and motor activity is engaged with the mathematical region on the blackboard now.

Notice how J's interaction with each of these two regions differ both from one another and from his style of interacting with the sphere in Figure 1. With the sphere, he stays close, keeping the drawing in his peripersonal space and alternating between gestures in front of the drawing and modifications of the drawing. With the region in space shown in Figure 2, J doesn't refer to specific written diagrams at all, instead outlining or representing objects or collections of objects and their actions via gestures. When J engages with the region of focus in Figure 3, he indicates it as though he's outside of it despite his attention being on it. These three examples help to illustrate the variety of ways in which someone can engage with mathematical regions to produce different effects. There are numerous other ways of engaging with such regions, too, and we'll detail some of them later in this paper.

At the same time, there are some remarkably stable patterns of interaction that seem to hold between these and other styles of regional engagement. One example is the defining of bounds so vividly depicted in Figure 2. This is far from an isolated case. Every time J creates a region, he does something to define its boundary, usually in one of three ways: create an embodied indication of its edges (as in Figure 2), use physical distance to distinguish between regions (as in J's having stepped away from the board to gesture as he did in Figure 2), and in the case of regions on the blackboard, he'll sometimes draw lines to indicate boundaries as in Figure 4:

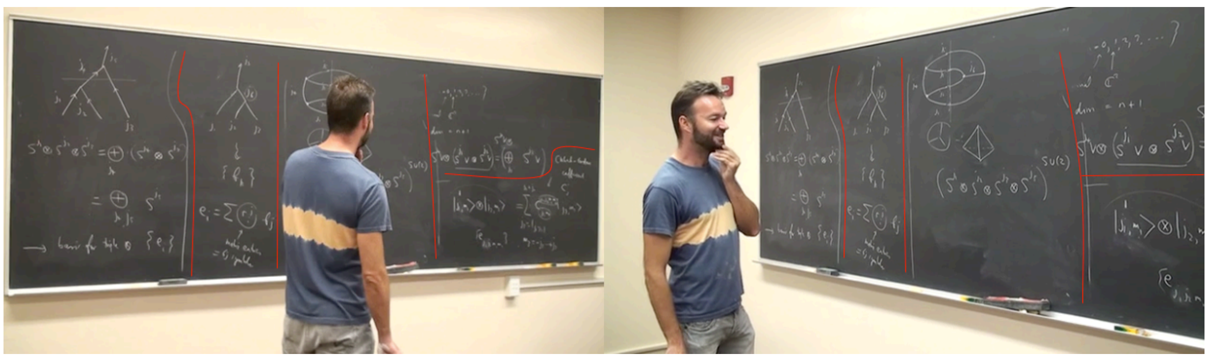

Figure 4 – Boundaries of Blackboard Regions

In Figure 4, we've outlined roughly where the boundaries between different regions on the blackboard are. You can see how the vertical boundary lines are actually drawn on the blackboard for the most part. Notice, though, that these drawn boundaries aren't firm: the "S" immediately above the juncture between the right vertical and horizontal boundaries is actually drawn across the vertical line. This is actually fairly typical, as there are many times where the gestures and diagrams J uses go beyond his predefined boundaries this way.

The horizontal boundary line from Figure 4 raises an issue that's worth commenting on. You might notice that there's just a tiny dash that represents that boundary between the rightmost regions. It's not drawn all the way across the same way the vertical lines are drawn all the way down. This indicates that although the drawn lines help to inform where J intends regions to be differentiated from one another, it's the *perception* of that separation rather than the physical division that's important to J. This is reflected again in the fact that J didn't draw any line at all to indicate the "kink" at the far right of the board. Instead, the awareness that the algebra in the top region don't relate to the words written at the far right comes from when J created each of these inscriptions and the manner in which he associated them with other regions. We'll discuss this method of defining boundaries in more depth a little later in this paper.

We find a reflection of this idea of mathematical regions in Husserl's (1913/1983) phenomenological description of a *horizon*:

> That which is actually perceived, that which is more or less clearly co-present and determined (or at least to some extent determined) is partly permeated, partly surrounded, by a *dimly given horizon of indeterminate reality*…. [Such] indeterminateness is populated with intuitive possibilities or likelihoods…. [A] never fully determinable horizon is

> necessarily there. (Husserl, 1913/1983, p. 52. Italics in the original; the translation from German has been partially adjusted on the basis of Smith & McIntyre, 1982, p. 236)

In other words, a horizon is a sort of realm of possibilities that encompasses all the ways of perceiving and interacting with a given scenario that seem available to the individual in question. For instance, we can reach out and move a coffee mug immediately in front of us, changing how it looks to us as we do so. Alternatively, we can move ourselves, also changing the angle of perception. All the different ways we could move the mug or ourselves, together with the ways in which we anticipate this will change how we experience it, are included in the horizon of that situation.

Husserl emphasized how horizons can never be completely determined, which is to say that we cannot specify *all* the possibilities any one horizon contains. There are countless ways that we could move a coffee mug, for instance, and our interactions with other objects around the mug can also affect how we perceive the mug as well. Our perceptions can change even without explicit physical interaction: remembering how one got this particular coffee mug while on one's honeymoon can transform the way the mug appears by making its personal history and significance more salient.

Mathematical regions appear to be horizons of a sort. J creates a region in order to have a place to work with mathematical objects, which for him elicit a host of possible ways of their being perceived and manipulated. Most of these possibilities never get expressed or even reach consciousness, such as the matter of tilting the sphere as discussed in connection with Figure 1 earlier. So in this sense, these regions are areas within which J can choose to engage and navigate, but he cannot fully describe or define. It's therefore understandable for us to find that sometimes J's symbols cross the lines that describe the boundaries of the regions: those lines don't actually define the regions in a strict and precise way.

Diagrams Within Mathematical Regions

In keeping with our theoretical perspective, we need to describe the "mathematical objects" that populate these mathematical regions in a way that's grounded in perceptuo-motor activity. To keep the discussion focused, we'll restrict ourselves to the objects populating regions that are projected onto the blackboard – i.e. what most of us would like to say are the symbols J used in the course of the interview. To do so, we'd like to employ Charles Pierce's notion of a *diagram* (Stjernfelt, 2000, 2007) to describe these objects.

In short, a diagram in Pierce's sense is a symbolic representation that's amenable to transformations and experimentations that can reveal previously unnoticed properties of the referent.[1] This definition ends up capturing most inscriptions one usually thinks of as being involved in mathematics because it includes algebraic symbols: we can manipulate a line's equation given in point-slope form so that it's in slope-intercept form, which reveals something about the line that we might not have noticed from the initial presentation. We contrast this with symbolic representations like words: the written word "mathematics" isn't likely to reveal anything new to us about math no matter how we might rearrange the letters.

We'd like to highlight two phenomena we noticed about the ways in which J used diagrams as a result of our framework. First, there was a way in which J would engage with his diagrams a sort of imaginative "play". Think of how we treat photographs of people: it's perfectly normal to point at one and say "That's Aunt Norma" instead of "That's a picture of Aunt Norma." Both are fine to say, and we'd never actually confuse a picture of Aunt Norma with the person, but in the moment of use we see right through the image straight to what the image represents. In a very similar way, J would frequently "see through" his diagrams, treating them as though they were actually the objects that they represented. We saw this earlier when, in the situation shown in

Figure 1, J would refer to the collection of curves he had drawn on the blackboard as "a ball" even though they didn't literally constitute a ball. Notice how this is naturally compatible with the perspective of J's interactions with his diagrams as *constituting* the diagrams' meanings rather than indicating an unseen but ontologically real referent.

Second, J frequently employed a kind of ambiguity as to whether he was dealing with a particular case or with a whole category of cases. We saw this too in the example of the sphere from Figure 1: even though J clearly drew a particular sphere with a particular rotation vector, it was just as clear that he was talking about rotation *in general*. Yet even though he was talking about general rotation, his gestural animation of the diagram made it clear he was engaging with the particular example of a sphere with a rotation vector pointing upward and spinning at a rate of at least two or three revolutions per second. This is reminiscent of when someone uses a particular quadratic equation as a stand-in for *all* quadratic equations as differentiated from, say, cubic or linear equations.

Embodied Paths

Armed with a description of the objects within the mathematical regions attached to the blackboard, we would now like to describe the main method by which J would connect these objects in order to provide context. We refer to this method as the travel along or indication of *paths* within and between regions. Consider the following example (Figure 5):

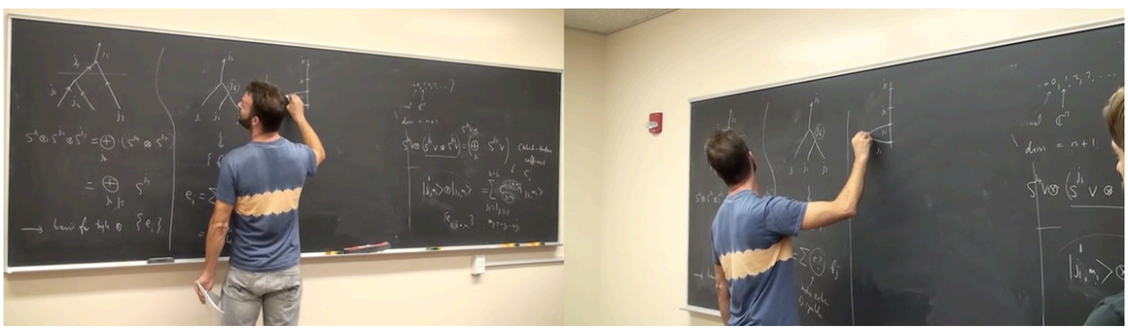

Figure 5 – Connecting Regions with Gaze

In the screenshot shown in Figure 5, J is in the process of creating a diagram that combines the tree diagrams directly to the new drawing's left. He's in the process of drawing the leftmost tree on the left side of the vertical line. You can see here how he vividly embodies that connection by keeping his writing hand engaged with the diagram currently being constructed but fixating his gaze on the leftmost tree diagram. His gaze switches back and forth between the new diagram and the older tree diagram as he draws this side. He then spends some time using gestures and words to highlight (Goodwin, 1994) the parts of the rightmost tree diagram that he wants us to attend to (see Figure 6).

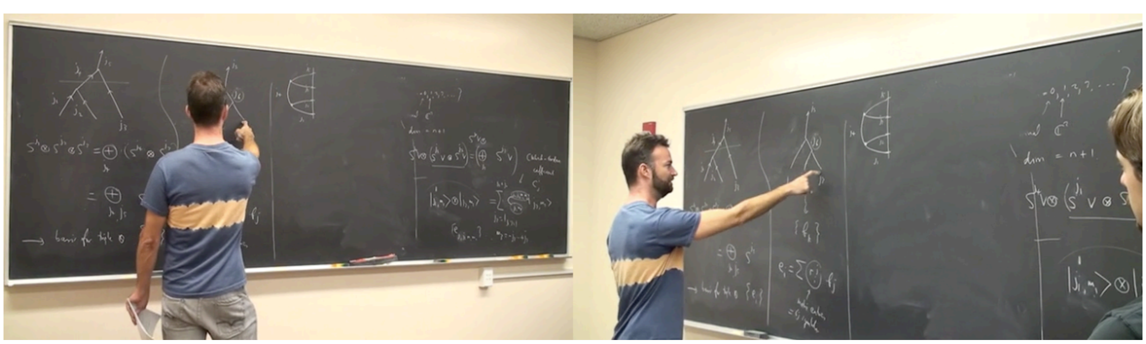

Figure 6 – Highlighting the Path's Contours

He finally repeats the gaze-connection process to bind the second tree diagram to the newest diagram, bringing the latter to completion (see Figure 7).

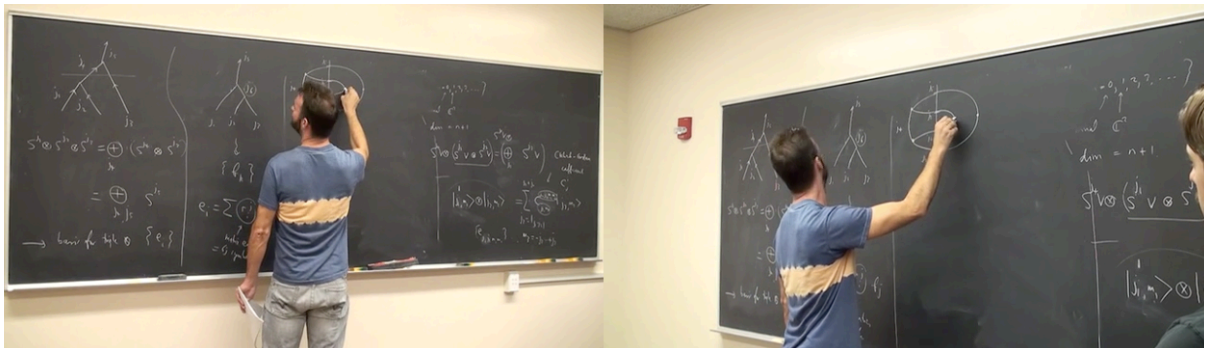

Figure 7 – Connecting More Regions

What we see emerging from this is that the meaning of that circular diagram in the third column is made up not only of J's interactions with that particular diagram but also emerges from the way he connects it to these other two diagrams in two other regions. J's intent appears to be for us to see the circular diagram as built up from aspects of these tree diagrams, and in order to convey that J puts a lot of effort into explicitly connecting all three regions. Yet he does not merge them. Instead, he describes via embodied enactment how one should perceptually travel through and between these regions in order to make the intended meaning of the newest one salient.

There are many different ways in which paths appear, and their various forms change the way meaning is intended and, presumably, experienced. In the above example, J went back and forth between the newest diagram and the older ones repeatedly while constructing the new drawing. This differs significantly from the way he later redrew the circular diagram in what he referred to as a "Mercedes badge" (see Figure 8).

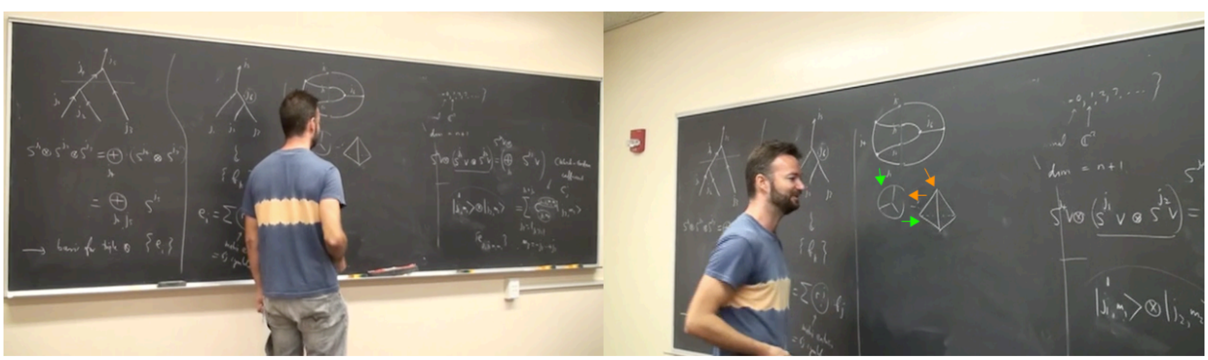

Figure 8 – Paths Within a Region

In Figure 8, J has drawn three different diagrams, namely the one he was working on in Figures 5-7, a drawing of a tetrahedron, and a Mercedes badge consisting of a circle with three equidistant radial lines within it. J seemed to intend the Mercedes badge to be a redrawing of the large circular diagram and indicated the latter via gestures only once right before drawing the badge. In this sense the meaning flows from the large diagram to the badge *but not the other way around*. In the course of the story J is telling here, the badge *replaces* the original large diagram. This has a very different quality than the relationship between the large drawing and the two tree diagrams described earlier: the back-and-forth embodied travel between them created an association of meaning between them that allows us to understand them as *different yet related* mathematical objects.

Embodied paths aren't necessarily formed in the order that the symbol-user intends them to be followed. The orange arrows in Figure 8 indicate the order in which J drew the diagrams in that central blackboard region. However, the green arrows show the path of meaning that connects them: the Mercedes badge is a rewriting of the large diagram, and one gets the tetrahedron from that badge by straightening the badge's edges and lifting its center out of the blackboard. One can certainly understand that by understanding the mathematical context, but in this case the (green) path was actually traced out via J's gestures and gaze. This is akin to how someone might draw a map and then point to indicate the route within that map one should take.

Furthermore, paths needn't be tied to the board. J demonstrated this in the time surrounding Figure 2 when he disengaged from the region on the blackboard, created and worked in a region made up of gestural objects, and then (in Figure 3) stepped back to return attention to the blackboard region. The context in which to understand his intent requires traveling off the board into a region of invisible objects and then returning to the board with a slightly different perspective for having gone on this trip. This, again, emphasizes the point that the meaning of a diagram emerges from an incredibly wide array of embodied activity.

Juxtaposition of Regions

One last embodied phenomenon we'd like to point out from this episode is that of juxtaposition of mathematical regions. At times, J would make a point by comparing and contrasting different mathematical regions *from the outside* instead of engaging with them and animating the diagrams within.

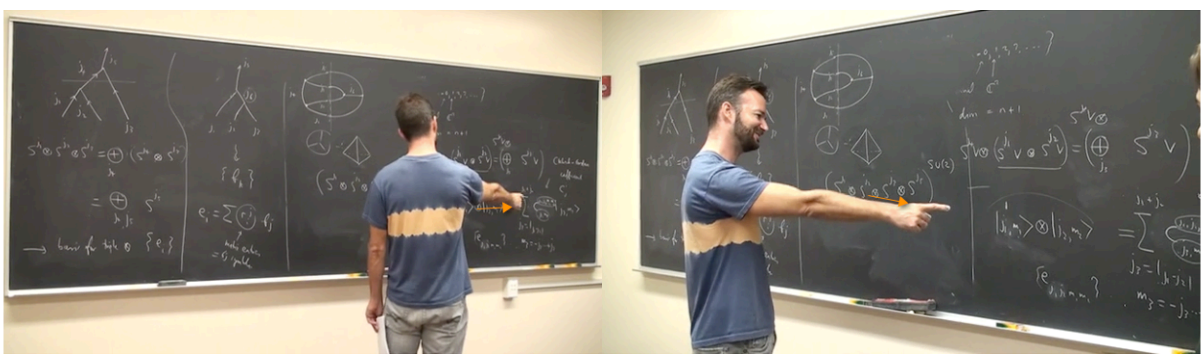

Figure 9a – Juxtaposing ("matrix-based calculations")

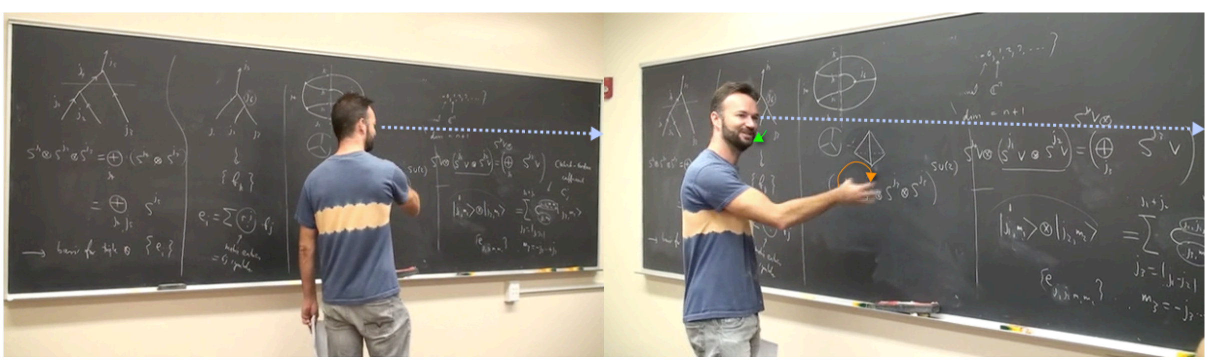

Figure 9b – Juxtaposing ("conceptual idea")

Figure 9 shows J juxtaposing two regions. Here he says, *"…I hate matrix-based calculations like this. I think it's just ugly! And, uh, you know, having a kind of* *conceptual idea of what you're, you're doing is much, much more useful to a mathematician."* He points at the "matrix-based calculations" in Figure 9a, sweeping his hand side to side to indicate the whole region. He then backs up away from the board for a moment and then moves closer to it to wave his hand around circularly as shown in Figure 9b.

The main point we'd like to highlight here is how J seems to place himself *outside of* the regions he's indicating. Rather than traveling back and forth between specific diagrams in the two regions, he seems to want us to take each region as a whole object to be considered. He still focuses his attention on specific diagrams within those regions that seem to be representative of them in the context he's describing (namely the bra-ket notation in Figure 9a and the tensor product in Figure 9b), but he clearly doesn't want our attention to lock onto those diagrams to the exclusion of the rest of the region. We see this in the imprecision of his gestures: the waving back and forth of his pointing in Figure 9a and the open-handed circular movement in Figure 9b offer a kind of ambiguity that bespeaks Husserl's (1913/1983) horizons of ephemeral boundary.

Concluding Remarks

Throughout this paper we have attempted to share an alternative to viewing symbol use as references to unseen conceptual referents. This alternative is based on perceiving the physical ways in which symbol-users interact with the symbols in question as actually constituting the meanings of those inscriptions. This theoretical perspective allows us to get insight into how the subject navigates symbolic meaning by observing how he bodily engages with his environment.

We would like to emphasize that we intend this not as a critique of reference/referent approaches, but instead as a powerful alternative that provides a different kind of insight. In particular, we usually find that this kind of analysis helps us to understand by what means a subject structures his or her experience of and approach with the mathematics at hand. For instance, the fact that embodied path travel appears to be different from external juxtaposition of regions suggests a particular difference in how thinking involving each of these approaches is experienced by the subject, which would be a very challenging observation to make if we were trying to discern, say, J's schemas of the various mathematical ideas involved.

As with more cognitivist approaches, the value of a case study of using this theoretical lens is in the insight it can provide us on beyond the case in question. We've elaborated on the phenomenological constructs that we have – namely mathematical regions, blackboard inscriptions as diagrams (Stjernfelt, 2000, 2007), embodied paths, and juxtaposition of regions – because these appear in a wide variety of cases. Having noticed these phenomena, we can now see it in the data we've collected on many other mathematicians and also in the symbol use of non-mathematicians. Our hope, in part, is that you too will find your vision of symbol use affected by noticing these phenomena when and where they occur.

We also hope that you recognize that the *kind* of insight that emerges from our theoretical perspective is very different than one of, say, exemplifying a particular student's understanding of a derivative in terms of mental schemes. The latter most certainly can give us some insight into matters such as why some students make the reasoning mistakes that they do in calculus classes, and we think there's most definitely value in such approaches. But the embodied approach we're attempting to share here can help us to see more general phenomena about how mathematical experience is structured and created, which can in turn help us to understand mathematical teaching and learning in new and potentially powerful ways.

Endnote

(1) Technically, the description of a diagram that we're using here is actually Pierce's definition of an *icon*. Diagrams are one of three sorts of icons, distinguished from

the other two by the fact that it captures “a skeleton-like sketch of relations” (Stjernfelt, 2000, p. 358). There is potential value in emphasizing the diagrammatic rather than just iconic nature of J’s inscriptions, but in the interest of brevity we’ve omitted that analysis.

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FROM INTUITION TO RIGOR: CALCULUS STUDENTS' REINVENTION OF THE DEFINITION OF SEQUENCE CONVERGENCE

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Little research exists on the ways in which students may develop an understanding of formal limit definitions. We conducted a study to i) generate insights into how students might leverage their intuitive understandings of sequence convergence to construct a formal definition and ii) assess the extent to which a previously established approximation scheme may support students in constructing their definition. Our research is rooted in the theory of Realistic Mathematics Education and employed the methodology of guided reinvention in a teaching experiment. In three 90-minute sessions, two students, neither of whom had previously seen a formal definition of sequence convergence, constructed a rigorous definition using formal mathematical notation and quantification equivalent to the conventional definition. The students' use of an approximation scheme and concrete examples were both central to their progress, and each portion of their definition emerged in response to overcoming specific cognitive challenges.

Keywords: Limits, Definition, Guided Reinvention, Approximation, Examples

Introduction and Research Questions

A robust understanding of formal limit definitions is foundational for undergraduate mathematics students proceeding to upper-division analysis-based courses. Definitions of limits often serve as a starting point for developing facility with formal proof techniques, making sense of rigorous, formally-quantified mathematical statements, and transitioning to abstract thinking. The majority of the literature on students' understanding of limits (e.g., Bezuidenhout, 2001; Cornu, 1991; Davis & Vinner, 1986; Monaghan, 1991; Tall, 1992; Williams, 1991) describes informal student reasoning about limits, with particular attention given to a myriad of student misconceptions. However, there is a paucity of research on student reasoning about formal definitions of limits. The general consensus among the few studies in this area seems clear – calculus students have great difficulty reasoning coherently about the formal definition (Artigue, 2000; Bezuidenhout, 2001; Cornu, 1991; Tall, 1992; Williams, 1991). What is less clear, however, is *how students do come to understand the formal definition*. Indeed, this is an open question with few empirical insights from research to inform it (Cottrill et al., 1996; Roh, 2008; Swinyard, in press). Oehrtman (2008) proposed an approach to developing the concepts in calculus through a conceptually accessible framework for limits in terms of approximation and error analysis. The design criteria of this approach are to maximize coherence across topics and representations and to establish a potential foundation for abstraction and formalization.

Students were recruited to participate in our study from a course that employed Oehrtman's approach. This report addresses the following research questions:

1. What are the cognitive challenges that students encounter during a process of guided reinvention of the formal definition for sequence convergence?
2. How do students' resolve these cognitive challenges, and how does that process help advance their mathematical thinking about limits of sequences?
3. To what extent and in what ways do these students use aspects of approximation and error analysis in their reinvention of the formal definition?
4. What implications can be drawn about the nature of students' and researchers' co-productive activity in the process of a guided reinvention?

Theoretical Perspective

We adopted a *developmental research* design. The purpose of developmental research is "to design instructional activities that (a) link up with the informal situated knowledge of the students, and (b) enable them to develop more sophisticated, abstract, formal knowledge, while (c) complying with the basic principle of intellectual autonomy" (Gravemeijer, 1998, p. 279). Task design was supported by the *guided reinvention* heuristic, rooted in the theory of Realistic Mathematics Education (Freudenthal, 1973). Guided reinvention is described by Gravemeijer, K., Cobb, P., Bowers, J., and Whitenack, J. (2000) as "a process by which students formalize their informal understandings and intuitions" (p. 237). Faced with the alternative of having the students reason about a formal definition provided by us (i.e, a conventional definition from a textbook), we felt that engaging the students in the articulation of a personal concept definition (Tall & Vinner, 1981) would better position us to identify *authentic* challenges students experience as they formalize their intuitive understandings of sequence convergence. Such an approach has provided rich data in other studies about students' understandings of formal definitions in advanced calculus (Swinyard, in press). The design of the teaching experiment activities described in the methods below was inspired by the *proofs and refutations* design heuristic adapted by Larsen and Zandieh (2007) based on Lakatos' (1976) framework for historical mathematical discovery.

We employed Oehrtman's (2008) approximation framework for instruction of calculus to provide the rich informal foundations on which such formalization might be based. Oehrtman (2009) framed the necessary systematization of students' spontaneous concepts about approximations in terms of the dialectic interplay between everyday and scientific concepts in Vygotsky's (1987) zone of proximal development:

> Having already traveled the long path of development from below to above, everyday concepts have blazed the trail for the continued downward growth of scientific concepts; they have created the structures required for the emergence of the lower or more elementary characteristics of the scientific concept. In the same way, having covered a certain portion of the path from above to below, scientific concepts have blazed the trail for the development of everyday concepts. They have prepared the structural formations necessary for the mastery of the higher characteristics of the everyday concept. (p. 219).

In particular, Oehrtman (2008) engages students in activities that repeatedly develop the concepts in calculus defined in terms of limits through the consistent structure represented by viewing the limit (L) as an unknown value to be approximated, the argument of the limit (say a_n in the case of sequences) as the approximations, the magnitude of difference between the two ($|a_n - L|$) as the error, ε as the error bound ($| a_n - L | < \varepsilon$), and the domain process as allowing one to control the degree of accuracy ($n \to \infty$). Our expectation was that the class activities in developing such everyday concepts about approximation to limit structures would provide students with a significant benefit for the formal definition.

Methods

The authors conducted a six-day teaching experiment with two students at a large, southwest, urban university. Five students volunteered for the study and two average to slightly above average performing students, Megan and Belinda (pseudonyms), were selected for the teaching experiment based on our estimation of their ability to effectively work as a pair. Evidence of this ability was based on instructor observations of their communication skills, engagement in class material, and their familiarity with one another from in-class group work. The full teaching experiment was comprised of six 90-120 minute sessions with a pair of students currently taking a second-semester calculus course whose topics included sequences, series, and Taylor series. The central objective of the teaching experiment was for the students to generate rigorous definitions of *sequence*, *series*, and *pointwise convergence*. Megan and Belinda had not received instruction on these formal limit definitions, as confirmed by their current and prior instructors. The research reported here focuses on the evolution of the two students' definition of sequence convergence over the course of the first three sessions of the teaching experiment.

Activities commenced with students generating prototypical examples of sequences that converge to 5 and sequences that do not converge to 5. The majority of each session then consisted of the students' iterative refinement of a definition, with the aim of precisely characterizing sequence convergence. The students were to evaluate their own progress by determining whether their definition included all of the examples of convergent sequences and excluded all of the non-examples. We then guided them in making explicit the nature of any conflicts, hypothesizing several ways the conflicts might be resolved, then employing a chosen approach. This established a cyclic process of definition refinement (Figure 1), that resulted in 26 distinct written definitions over the first three sessions of the teaching experiment.

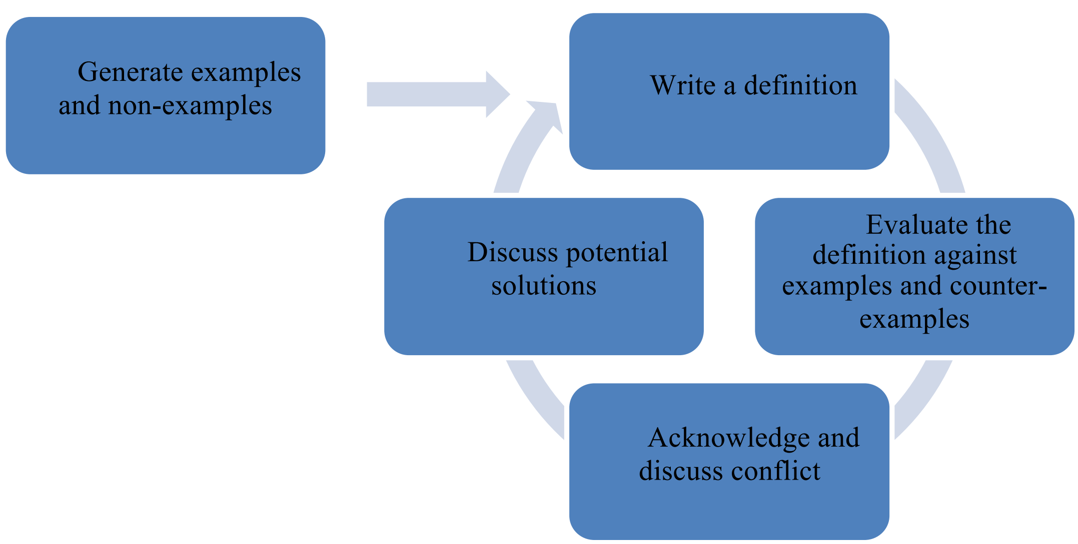

Figure 1. Iterative refinement in the process of guided reinvention of a formal definition.

Of the five-member research team, two researchers served as facilitators of the reinvention process, while the other three observed a live audio and video feed from a separate room. A live text chat between the two sets of researchers allowed the three observers to suggest to the two facilitators areas to probe the students or changes in direction for the protocols. The two researchers facilitating the guided reinvention always had discretion of where to focus the discussion. The entire research team debriefed immediately after each session, and then reviewed the video tapes to develop hypotheses about the progress of the guided reinvention, unforeseen challenges, and potential resolutions. The team then made adjustments to the protocols for the following day's session.

Starting the week after the teaching experiment sessions, the research team quickly transcribed all video tapes and created detailed content logs of each session. The logs consisted of time-stamped descriptions of the activity of the students and researchers with separate theoretical notes about how that activity was progressing toward the formal definition. We then coded the video timeline for what problem(s) the students were explicitly addressing. For each problem, we attempted to determine both the origin and the students' resolution if one occurred.

Results

We found it beneficial to distinguish between *problems*, which were those issues explicitly raised by the students most commonly as conflicts between their evoked concept image and their stated definition, and *problematic issues*, those that arose unbeknownst to the students as conflicts between the students' stated definition and the standard formal definition. Since problematic issues were not salient for the students, we often found it necessary for the facilitating researchers to prompt reflection on them. When successful, we note that the problematic issue became a problem for the students. In some cases this occurred without our intervention, as the students uncovered the problematic issues on their own. Whether prompted by the facilitating researchers or initiated by the students, we observed that resolution of problems always required explicit conversation focused intentionally and specifically on the problem. On the other hand, we also observed several instances in which problematic issues were resolved implicitly, in the background, while the focus of conversation was on resolving another problem (sometimes related while other times unrelated). In the following, we provide several examples of problematic issues, problems, and implicit and explicit resolutions while noting the interplay between the facilitating researchers and the students.

The examples and non-examples created during the preceding class session were prominently displayed on large sheets of paper on the wall near the table where Megan and Belinda were seated. These graphs were then constantly available for the facilitators to reference in questions and for the students to use in explanations and in testing their definition. The facilitators began by asking the students to work together to complete the statement "A sequence converges to 5 as $n\to\infty$ provided…" in a way that included all of the examples but excluded all of the non-examples. The first definition created by Megan and Belinda reflected a standard conception of always getting closer.

> Definition 1: A sequence converges to 5 as $n\to\infty$ provided n+1 is closer to 5 than n for any n value.

We note that their metonymic use of n+1 and n for a_{n+1} and a_n was clarified throughout their discussion and did not create any discernible confusion or miscommunication between them. They quickly reframed their definition in dynamic terms.

> Definition 2: A sequence converges to 5 as $n\to\infty$ provided that the number approaches or is 5 and no other number.

For both of these definitions, Megan and Belinda pointed out examples of sequences converging to 5 posted on the wall that were not monotonic early in the sequence and commented that their definitions did not apply to those. Their attention to this issue and expressed desire to change their definition to include sequences that might have "bad early behavior" constituted the first explicit problem to which the students attended. Six minutes into the session, Megan suggested, "What if we were to say after some point n it...We have random stuff going on [*pointing at all of the convergent graphs*], but at some point it does start doing the closer and closer and closer thing." Belinda immediately agreed and they incorporated the phrase "some point n" into their definition. Some version of this idea appeared in almost all subsequent definitions (exceptions appeared to be out of oversight and were usually changed to include the phrase). For example their third definition read,

Definition 3: As $n \to \infty$, at some n, a_{n+1} becomes closer to (or is) 5.

Although the students were happy with this resolution and it is reminiscent of the N in a standard ε-N definition, the researchers noted four problematic issues with their implementation of it. Specifically, i) their description of the "some point n" referred to where the sequence began exhibiting "convergent behavior" and as such ii) was a fixed location for a given sequence, iii) they used n to refer both to this point where the behavior changed and as the index on terms of the sequence, and iv) they did not capture precisely which values of n should have the property that a_{n+1} is closer to 5 than a_n. Schematically, this problem, solution, and emergence of new problematic issues is represented in Figure 2.

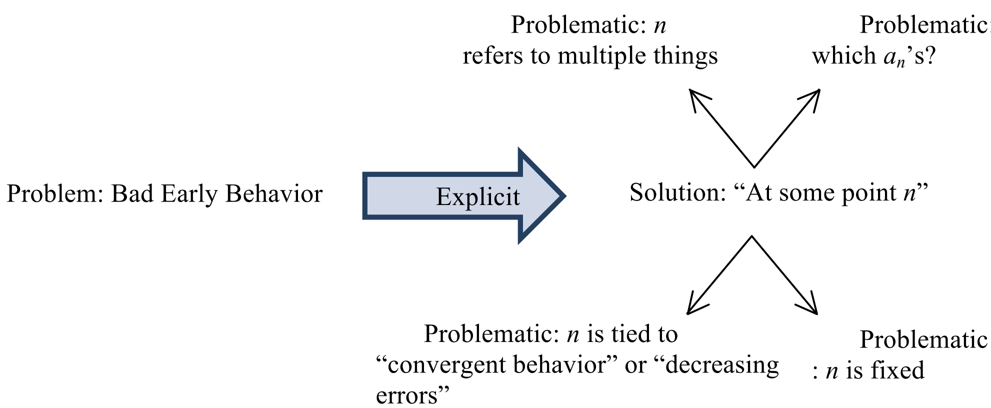

Figure 2. Megan and Belinda's first problem resolution created new problematic issues of which they remained unaware for a significant duration.

We now discuss how the first two of these problematic issues, shown in the lower region of Figure 2 were both resolved implicitly by Megan and Belinda. This means they focused on different explicit problems whose resolution also solved these problematic issues. In particular, when they started using language involving infinitesimals, Craig asked what they meant by "infinitely close."

Craig: Now both of you have mentioned this "infinitely close thing." Can you say a little bit more about what you mean by infinitely close? Or maybe is there a way that you could encompass that into your definition at all?
Megan: Negligibly small.
Belinda: Yeah, where an error exists, but it's…
Megan: Um… We were doing that stuff before where our approximation had an error. Remember when we had to find it within .0001, or whatever?

Belinda: Yeah.

Megan: I guess we can say “within a designated margin of error” or “designated error bound” because it would vary by what our error bound was for that particular problem or situation.

Thus, while trying to interpret the meaning of phrases like “infinitely close” and “negligibly small,” they recalled ideas about bounding errors from their work in class. Later they were wrestling with the same issue and Megan explained,

> Well if an alternating [graph] never actually becomes 5 but we ***accept*** it as a convergent graph, um, that must mean at some point it gets within a range of 5 [*holds hands apart horizontally*] that we accept it as converging to 5 whereas say if we’re looking at the sine graph [*moves hands farther apart*], it doesn’t get within that, that distance. It stays out at –1 to 1…. The regular sine graph. It stays within –1 to 1. That distance is too great to call it convergent. But if [the alternating graph] never actually becomes 5 [*moves hands back together*], but it gets that small little range. There’s gotta be a little acceptable range at some point that you say it’s convergent [*holds two fingers close together horizontally*] even though it never actually becomes [*bounces fingers off to her right*].

At this point we see that Megan seamlessly transferred her use of “at some point” to refer to a location where the sequence enters an “acceptable range” of the limit, and the two produced Definition 12 shortly afterward.

Definition 12: A sequence converges to 5 as $n \to \infty$ i.f.f. $\exists\, n$ such that for all a_n $|5 - a_n| \leq .01$

From this point on during the teaching experiment (with one brief exception) both students used the phrase “at some point” consistent with this interpretation. Consequently they dropped the previous usage tied to the beginning of “convergent behavior” or “decreasing errors,” implicitly resolving that problematic issue. This is represented in the lower left of Figure 3.

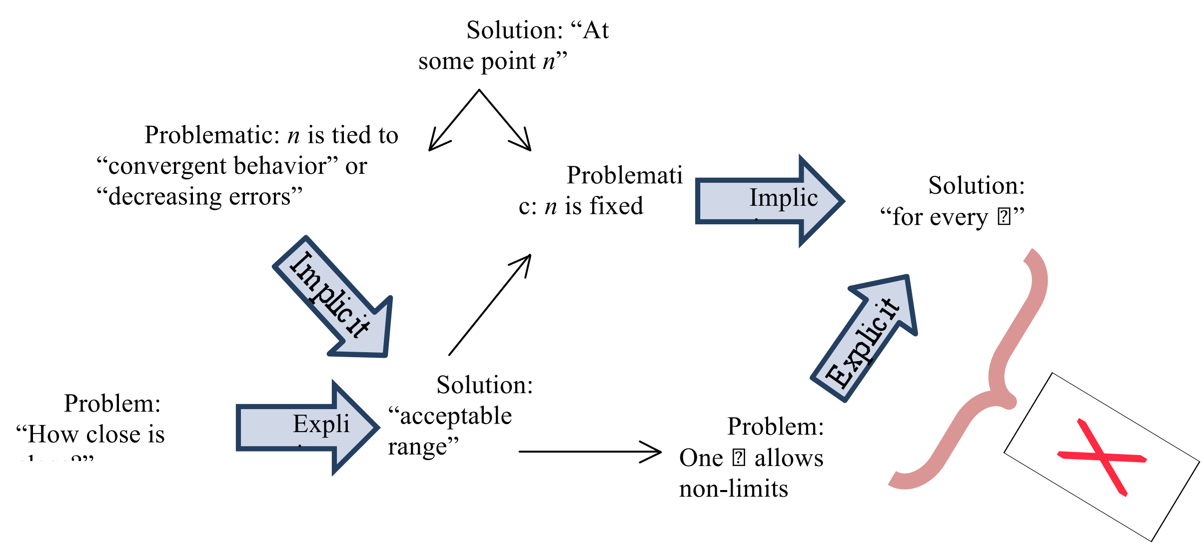

Figure 3. Implicit resolutions to two of Megan and Belinda’s problematic issues.

We also note that the problematic issue of n being a fixed location remained in this new formulation. Specifically all four references to the “acceptable range” in Megan’s quote above make clear that she viewed it as having a fixed size (not consistent with the universal quantification of ε in formal definition). Both Megan and Belinda spent the next hour of discussion clearly referring to the "acceptable range" in this way. In doing so they incorporated the idea of an "acceptable range" into their definition by either choosing some small number, as seen in Definition 12 or by replacing the number with an expression of an unknown quantity such as a question mark. The fixed nature of the "acceptable range" thus reinforced the problematic issue of "some point *n*" also being fixed. From this point on, some version of their "acceptable range" remained a part of their definition.

Megan and Belinda quickly realized that such a definition excluded some non-limits that were problems for their "approaching" formulation. Megan noted, "If we could say within this error bound of .0000ooo1 or something, you know, if it's within this range of this number [*holds hands horizontally close together*], then that way it won't be within that range of 6. It'll be within that range of 5." On the other hand, they also realized that no matter how small their "acceptable range" might be, it would still allow some non-examples to fit the definition, for example a sequence converging to 4.99999. For instance,

> Megan: Instead of 01 we'd have like 001. Or like 10 zeros and a 1. Or something that would bring it closer and closer and closer to 5, but we're not talking about an infinite number of decimals possible…. So the only thing we'd have to figure out is the possibility that it would still be converging to some number within there.
>
> Belinda: The "whatever range you want" part would exclude [4 as a limit] IF you made your range small enough.

Belinda's comment on the exclusion of a non-limit was emphatically conditioned on a sufficiently small choice of "your range." They argued that using specific numbers like .01 was "arbitrary" in the sense of not being intentionally or well-chosen. As Belinda expressed, "You could start changing the definition of convergence deciding on however, however close you want it to be at that day."

During this extended period of wrestling with the problem of a fixed "acceptable range," Megan and Belinda were also confused about the convergence of a damped oscillating sequence, as detailed by Hart-Weber, Oehrtman, Martin, Swinyard & Roh (2011). Specifically, despite introducing the idea of an "acceptable range," they also clung to their notion of requiring "decreasing errors." For example, their definition midway through the second session read,

> Definition 14: A sequence converges to 5 as $n\to\infty$ only if there exists a point n after which the error does not exceed $|5-a_n|\leq.001$ and $|5-a_{n+1}|<|5-a_n|$.

Since, no matter how far out you go, the damped oscillation always has places violating the requirement of $|a_{n+1}-5|<|a_n-5|$, they felt it must diverge. While discussing this example, Megan described one property of the damped oscillation as, "it will eventually get it within whatever range we want to put it in though, whether it's point 25 zeros and a 1 or a thousand zeros and a 1, it will get within that range and it will stay there." Here Megan verbalized a nearly perfect informal statement of the limit definition including the universal quantification on ε that would resolve their problem of the "acceptable range" being fixed. Unfortunately she did so in the context of describing what she felt was a "divergent sequence," not in the context of addressing their problem of a fixed "acceptable range" allowing non-examples. Thus, while the students were at times explicitly focusing on the right problem and at other times expressing the needed solution, they did so in separate segments of conversation.

After wrestling with the problem of a fixed "acceptable range" for over an hour, the facilitating and observing researchers all agreed that the students had extensively explored the nature of the problem and had even expressed and understood an idea that would solve their problem. We felt it would be an appropriate time to intervene and suggest bringing their solution and potential problem into common relief. As Megan and Belinda were discussing how to resolve the issue that any acceptable range would necessarily allow extraneous limit candidates, Jason intervened with a suggestion.

Jason: What about "for ***every*** error?"
Belinda: So then that would work because ***every*** error, 'cause then it has to work within ***every*** error or it's not convergent. If it doesn't work within ***every*** error, then it's not convergent.
Megan: Yeah, that works.

At this point, Megan and Belinda immediately, although somewhat awkwardly, incorporated the idea into their definition.

Definition 21: A sequence converges to 5 as $n \to \infty$ only if there exists a value N after which $|5 - a_n|$ does not exceed every error after N.

From this point on, Megan and Belinda persisted in using a universal quantification for the ε quantity in their definitions and discussions. This intervention by a facilitating researcher to bring together a problem and solution, and the resulting explicit resolution of the problem is represented in the lower right of Figure 3. Furthermore, as soon as their "acceptable range" was universally quantified, Megan and Belinda accordingly also referred to the "some point *n*" as changing depending on the particular size of the "acceptable range." This consisted of an implicit resolution to the problematic issue of the "some point *n*" being fixed, and is also represented in Figure 3.

We now return to the moment in the teaching experiment represented in Figure 1 and trace how the other two problematic issues first became explicit problems and how Megan and Belinda resolved them. The reader will recall that the students' third definition introduced the notion "at some point *n*".

Definition 3: As $n \to \infty$, at some n, a_{n+1} becomes closer to (or is) 5.

Unknowingly, the students had introduced ambiguity in a couple of important ways – first, in their definition "*n*" refers *both* to the static point on the *n*-axis beyond which the sequence behaves in a particular way and as the index for all of the individual terms of the sequence. Second, the phrase "a_{n+1} becomes closer" represented their attempt to articulate behavior of terms in the sequence after the "some *n*." Unfortunately, if n is viewed as fixed, so is $n+1$, and if n is viewed as general, then so must $n+1$. As the students' reinvention progressed during the second day, both of these ambiguities remained problematic issues, as can be seen in the following formulation.

Definition 20: A sequence converges to 5 as $n \to \infty$ only if there exists a value n after which the $|5 - a_n|$ does not exceed your chosen error and $|5 - a_{n+1}| < |5 - a_n|$.

In an effort to turn problematic issues into actual problems for the students, the facilitating researchers at times served as *conflict producers*, asking questions designed to invoke cognitive conflict for the students. We illustrate such an instance in the following excerpt, wherein Craig directed Megan and Belinda's attention to the dual use of *n* in their definition.

Craig (aka, a Conflict Producer): My question was this $5 - a_n$, what does a_n refer to?…The reason I'm asking is because it occurs to me that we're using n to mean, given some error, a static place here, right?
Megan & Belinda: Mmm-hmm
Craig: And then this n refers to a whole bunch of points.
Megan: Yeah.

Craig: Um, could it clear up confusion to call one of them something else maybe?
Belinda: Maybe a big N for this?

As she spoke, Belinda erased the n representing the static place on the n-axis, and replaced it with N (see Figure 4). Megan and Belinda then refined their definition to reflect this change.

Definition 21: A sequence converges to 5 as $n\to\infty$ only if there exists a value N after which the $|5-a_n|$ does not exceed your chosen error after N and $|5-a_{n+1}|<|5-a_n|$.

For the remainder of the teaching experiment, Megan and Belinda consistently used N to denote the point on the n-axis beyond which the sequence behaved in a particular way.

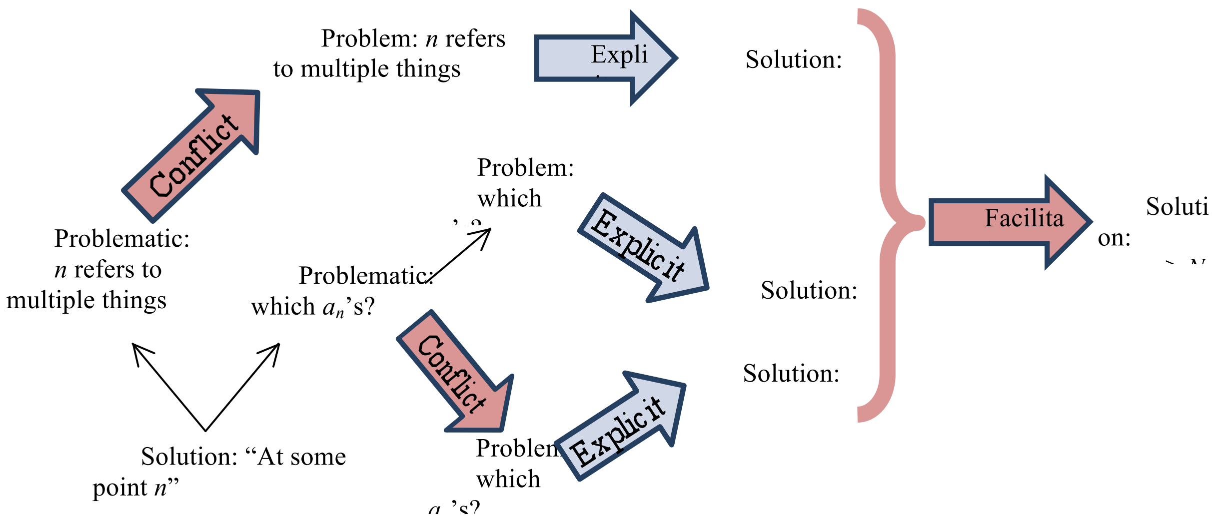

Figure 4. Students' and facilitators' activity in resolving problematic issues related to the index n.

The other ambiguity was resolved in a similar manner. In fact, each of the students (on two separate occasions) voiced concern about how to refer to the terms of the sequence after the "some point n." In this sense, the problematic issue had already become an actual problem for both Megan and Belinda (albeit on separate occasions). For instance, toward the end of the first session Megan refined Definition 12 by adding a subscript x to denote the terms of the sequence that were of interest to her, after which she verbalized the motivation for her refinement.

Definition 13: "A sequence converges to 5 as $n\to\infty$ i.f.f. $\exists$ n_x such that for all □□□ terms after $|5\text{-}a_n|\leq.01$"

After writing this, Megan explained, "I don't know. Just trying to think of a way to designate that point n forward and neglecting what's going on before that."

Similarly, during the second session, Belinda's attempt to capture the distinction between the terms of the sequence before and after the "some point n" suggests that this problematic issue had become an actual problem for her.

Craig (aka, a Conflict Producer): What does a_n refer to?
Megan: That's the approximation.
Belinda: That's whatever the term is. The approximation. All to the right of n.
Craig: Oh but not these guys back [here], because these are a_n's too aren't they?
Megan & Belinda: Yes.
Craig: But you're just saying the a_n's over here.

Megan: Mmm-hmm
Belinda: So maybe if we put like a little plus sign on that n to show that it's like, to the right.

As the third session progressed, Megan and Belinda still had not adopted a consistent means of denoting the terms of the sequence after N. To focus their attention on this issue, the facilitating researchers again acted as conflict producers.

Craig: [*pointing to* $|U–a_n|$ *in their definition*] You're saying that distance is within every error bound epsilon?
Belinda: Yeah.
Jason (aka, Conflict Producer): Is that true up there [*on the board*]?
Belinda: No, cause it's not true until you get to the N.
Megan: Yeah.
Belinda: So after some value N.

Next, Craig directed the students' attention to the graph seen in Figure 5.

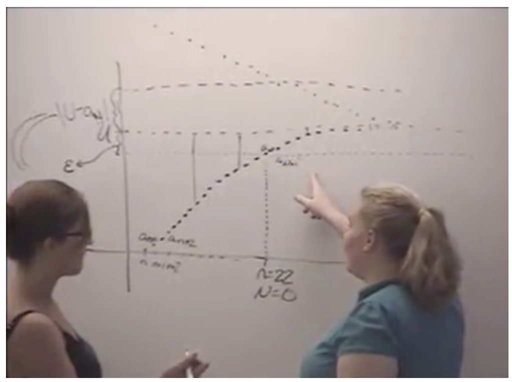

Figure 5. Determining how to refer to sequence terms after N.

For the epsilon they had chosen, Craig asked the students to imagine that the corresponding N would occur when n=22. He then asked them to describe the relationship between the indices of the terms after that point and 22. They noted that the subsequent indices would all be larger than N, at which point Craig asked them to capture that relationship in writing. After some discussion, the following final definition emerged:

Definition 23: A sequence converges to U when $\forall \varepsilon > 0$ there exists some N, $\forall\ n \geq N$, $|U - a_n| < \varepsilon$.

Conclusions

In this paper, we reported results that support three general findings:

1. Ideas developed during the guided reinvention became useful to the students and persistent in their reasoning to the extent that they were generated as solutions to problems explicitly engaged and resolved by the students.
2. Aspects of approximation and error analysis introduced crucial conceptual entities and ways of reasoning for the students.
3. The researchers facilitating the guided reinvention process played important roles i) adhering to the cyclic process of definition refinement, ii) focusing students on

problems that needed to be resolved, and iii) at appropriate times, suggesting potential solutions to problems that remained intractable to the students.

The resolution of problems

Our distinction between problems and problematic issues emphasizes that it was the explicit solutions to problems consciously addressed problems that became generally useful to the students and persisted throughout their activity. Ideas that were raised in other contexts had neither of these properties. For example, students' initial universal quantification of epsilon occurred in a discussion separate from the problem of any single value of epsilon allowing non-examples. In this instance, although they fully understood the idea, they were not able to use it to advance their definition. Only when the universal quantification of epsilon was recognized as solving an explicit problem was it used and adopted. In general, powerful use of logical quantifiers and mathematical expressions emerged only after the students had i) fully developed the underlying conceptual structure of convergence in informal terms, ii) wrestled with the problem of how to rigorously express those ideas, and iii) seen the quantifiers and expressions as viable solutions to these problems.

The students' recognition and resolution of problems in their reinvention efforts were aided considerably by the presence of the examples they constructed at the start of the experiment. These examples served as sources of cognitive conflict when their definition failed to fully capture the necessary and sufficient conditions under which sequences converge. For example, the students' initial definitions were predictably couched in language that was vague, intuitive, and dynamic. The students immediately identified weaknesses in these definitions as they applied them to their examples that increase monotonically to 4, alternate around 5 or behave erratically before eventually looking like a standard example of a convergent sequence. Having identified these weaknesses, they also looked to their examples to provide direction for their revisions. This pattern of evaluating and refining their definitions against the examples repeated over the 23 cycles during the first three days of the teaching experiment.

Since the students' final definition was constructed through several iterations of a problem-solving process, they developed strong ownership over each piece and the overall organization. When shown a formal definition in a textbook, Megan and Belinda laughed and were delighted to see "their" definition in a book. The students expressed strong appreciation for the power of the quantifiers and mathematical notation in their definition, citing multiple problems that each part efficiently resolved and extolling its benefits over their earlier definitions stated in vague, informal language.

The Effect of an Approximation Scheme for Limits

The two students in this teaching experiment had only experienced instruction aimed at developing a systematic approximation scheme for reasoning about limits for a portion of one calculus course. Consequently, it is not surprising that they did not immediately invoke this scheme as they began to wrestle with generating a definition of sequence convergence and that the scheme emerged in pieces. Nevertheless, it did not take them long to turn to approximation ideas, and each portion of their evoked scheme emerged in response to particular problems for which it was well-suited to address. We note that these students progressed much more quickly towards a formal definition and through resolving several cognitive challenges than students not introduced to the approximation framework (Swinyard, in press). Once evoked, the students' ideas about approximation remained consistent. Further, their images and application of their scheme were sufficiently strong to provide them considerable guidance and conceptual support for reasoning about the formal definition.

The students' familiarity with a previously established approximation scheme provided significant leverage for i) focusing on relevant quantities in the formal definition, ii) fluently working with the relationships between these quantities, and iii) making the necessary but difficult cognitive shift to focus on N as a function of ε (Roh, 2008; Swinyard, in press). For example, during the first 12 minutes of the teaching experiment, the students did not invoke language about approximations to describe aspects of a sequence $\{a_n\}$. During this time they did not discuss or represent the quantity $|a_n - 5|$ in any form and all descriptions of convergence involved informal dynamic language. Twelve minutes into the interview, Megan rearticulated her requirement that the terms be getting closer to the limit in approximation language, "the error gets smaller and smaller at some point, after some value n [*holding her hand up and pinching her fingers together as she swept her hand to her right*]." When Craig asked what she meant by error, both students described the limit as the value being approximated, the terms a_n as the approximations, and the distance between the limit and the approximations as the error, which they represented as $|a_n - 5|$. These ideas and representations immediately appeared upon their invoking of approximation language and became an integral part of their arguments and written definitions. In the middle of the first session, they reformulated their idea that "each subsequent term is closer to 5 than the previous term" as the inequality "$|a_{n+1} - 5| < |a_n - 5|$" which they then used consistently throughout the remaining sessions.

After first introducing "approximations" and "errors," Megan and Belinda shifted to discussing how close the terms needed to get to 5 to consider the sequence convergent to 5. Twenty-six minutes into the session, as discussed in our results on the development of Definition 12, the students invoked the idea of an error bound (corresponding to ε in the formal definition) to address this question and focused on how to make the error smaller than this bound. At this point they seamlessly transferred their prior use of "some point n" (corresponding to N in the formal definition) to indicate the point at which approximations entered the desired error bound. Afterward, they consistently reasoned that this "point n depends [on] what the acceptable error is." For the remainder of Day 1 and throughout Days 2 and 3 of the teaching experiment, the students continued to rely on this approximation scheme to describe the relevant quantities and to keep track of the relationships among them.

The Nature of Co-Productive Activity of Students and Researchers in Guided Reinvention

We conclude our discussion with some observations of the important roles played by the researchers in guiding the students defining activity. The observing and facilitating researchers constantly monitored the students' progress, assessing the potential value of wrestling with the problems in which they were currently engaged, and being vigilant of the emergence of problematic issues not noticed by the students. Based on these assessments the facilitating researchers had three essential mechanisms for steering the discourse in a productive manner while complying with Gravemeijer's (1998) "basic principle of intellectual autonomy."

Most frequently and preferentially, the facilitators simply asked the students to return to or move to the logically next phase in the cyclic refinement process illustrated in Figure 1. For example, after the students had written a definition, we would typically simply read it back to the students and wait for any comments or clarification. If the students did not apply it to the examples on their own, we would ask them if it kept all examples in and non-examples out. This regularly led to recognition of conflicts between their stated definition and concept images, which they would begin to attempt to resolve. When they had generated potentially useful ideas, we would ask them to try to incorporate them into their definition, thus beginning a new cycle.

When guiding the general process was not sufficient to maintain productive activity, a slightly stronger intervention was required, but we strove to be careful in these moments to maintain the students' intellectual autonomy. As illustrated in the results, several problematic issues never surfaced explicitly through the students' own reflection, requiring our action as *conflict producers*. We only engaged in this role when we became convinced that the students were unlikely to recognize a particular issue that needed resolution. Often asking them to consider a specific portion of their definition in light of a specific example was sufficient to initiate the recognition of a conflict. On other occasions, students wrestled with a problem for a significant amount of time, and we became convinced that they would not resolve it on their own in a reasonable amount of time. In these instances, we played the role of *solution providers*. In order to maintain intellectual autonomy, we attempted to only a provide solution when we felt the students had a sufficient understanding of its elements that they were prepared to adopt immediate ownership of it and directly apply it to the problem at hand. An optimal example was bringing the explicit problem of any one epsilon value allowing non-examples into the same conversation with their previously expressed universal quantification of epsilon represents. That the students immediately agreed with the idea, applied it to several examples, incorporated it into their definition, and persisted with its use throughout the remainder of the sessions supports our decision that this would be a productive intervention.

Limitations and future work

This study drew from data collected in a teaching experiment with only two students and we acknowledge that each individual will follow unique paths. Further, orchestrating this type of discussion for an entire class would certainly involve significant differences from what was possible with focused attention on two students. Nevertheless, these students' reinvention of the definition serves not only as an existence proof that students can construct a coherent definition of sequence convergence, but also as an illustration of *how* students might reason as they do so. Our findings shed light on several relevant cognitive challenges engaged by the students, how they resolved these difficulties, and the resulting conceptual power derived from their solutions. These results are guiding our future work to develop, evaluate and refine classroom activities for introductory analysis courses. A further limitation of the study is that students' refinements of the definition were entirely based on checking against their concept image, thus was very descriptive in nature. As a result, we expect their definition to be useful for the students in powerfully and precisely describing their new image of convergence, but we do not expect them to be able to immediately use their definition to prove theorems. We are currently pursuing refinements to the teaching experiment protocols to incorporate assessments and justifications of statements about sequences to strengthen this aspect of students' reasoning.

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How Intuition and Language Use Relate to Students' Understanding of Span and Linear Independence

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This report describes a case study in an undergraduate elementary linear algebra class about the relationship between students' understanding of span and linear independence and their intuition and language use. The study participants were seven students with a range of understanding levels. Findings indicated that students were more likely to successfully solve problems and explain concepts involving span and linear independence if they had low levels of self-evident intuitions and were better able to communicate their thinking in writing. Examples of students' interfering intuitions regarding span and linear independence are described. The report also includes possible instructional implications.

Keywords: Intuition, Language use, Linear algebra, Linear independence, Span

In an essay about his experiences teaching linear algebra, David Carlson (1997) posed a question that has become emblematic of students' learning in linear algebra: Must the fog always roll in? This question, he writes,

> refers to something that seems to happen whenever I teach linear algebra. My students first learn how to solve systems of linear equations, and how to calculate products of matrices. These are easy for them. But when we get to subspaces, spanning, and linear independence, my students become confused and disoriented. It is as if a heavy fog rolled over them, and they cannot see where they are or where they are going. (p. 39)

Research into the teaching and learning of linear algebra has spanned several decades, but the issue of how to clear the fog for students is still outstanding. In this report, I describe a research study designed to contribute to the understanding of how students learn concepts in linear algebra.

In his epilogue of *Advanced Mathematical Thinking*, Tall (1991) noted that many of the book's contributors believed students' difficulties in learning advanced mathematics could be explained by the discrepancies between the way students viewed mathematics and classroom instruction, which is often based on the formal structure of mathematics. More recently, in their discussion of advanced mathematical thinking, Mamona-Downs and Downs (2002) suggested traditional teaching of mathematics does not "connect with the students' need to develop their own intuitions and ways of thinking" (p. 170). An impediment to developing instructional theory based on students' intuitions is an incomplete understanding of how people develop abstract mathematical knowledge. Pegg and Tall (2005) compared several theories of concept development and derived a fundamental cycle of concept construction underlying each of the theories. However, there is no consensus on the mechanism of how this concept development occurs.

Some evidence exists to suggest language may play a role in concept development (Dehaene, Spelke, Pinel, Stanescu, & Tsivkin, 1999; Devlin, 2000). Pugalee (2007) contends "language and competence in mathematics are not separable" (p. 1). MacGregor and Price (1999) and Boero, Douek, and Ferrari (2002) believe that metalinguistic awareness is necessary for students to coordinate the various notation systems in mathematics. Yet, little research exists that explores the relationship between students' language abilities and mathematics learning (Barwell, 2005; Huang & Normandia, 2007; MacGregor & Price). Interestingly, though, just as mathematics education researchers have found a contrast between intuitive thinking and formal mathematics, language researchers have found this same contrast between everyday language use and the demands of formal school language (Schleppegrell, 2001, 2007). It is possible, then, that language plays an important role in how students move from intuitive, everyday thinking to understanding formal mathematical concepts and theory.

The purpose of this study was to address two outstanding issues in the learning of advanced mathematics. The first issue is a theoretical difference between the ways in which students learn "naturally" and the formal structure of mathematics, and how this difference may or may not influence students' mathematical understanding. The second issue is the relationship between students' language use and their mathematical understanding. In particular, I elected to study these issues in the context of linear algebra. My research question was:

> How do students' intuition and language use relate to the nature of their understanding of span and linear independence in an elementary linear algebra class?

Theoretical Perspective

The theoretical perspective of learning for this research was the *emergent perspective* described by Cobb and Yackel (Cobb, 1995; Cobb & Yackel, 1996; Yackel & Cobb, 1996). The emergent perspective is a type of social constructivism that coordinates the social and psychological (individual) views (Cobb & Yackel). The *interactionist* view of classroom processes (Bauersfeld, Krummheuer, & Voigt, 1988) represents the social perspective, while a constructivist view of individuals' (both students and teacher) activity (von Glasersfeld, 1984, 1987) represents the psychological perspective. The study's methodology reflected this theoretical perspective in that the social environment of the classroom was used as a lens through which to analyze individual student's work.

I drew from the literature for theories about intuition and language use. Fischbein (1987), Torff and Sternberg (2001), and Wilson (2002) all provide definitions of intuition (or the "adaptive unconscious" in the case of Wilson's work) that characterize intuition as an unconscious process that influences thought and behavior. Wilson and Fischbein postulate slightly different reasons for the existence of intuition in people, but common to both their reasons is the idea that intuition helps make sense of and organize people's incoming perceptual information so they can operate meaningfully in the world. Wilson documents some general characteristics of intuitions. People tend to regard their intuitions as self-evident. This is related to the tendency of early experiences to dominate later experiences in the formation of intuitions. Once formed, intuitions tend to be durable and resistant to change.

A key role of intuition is implicit learning, which results in tacit knowledge (Frade, da Rocha, & Falcão, 2008; Torff & Sternberg; Wilson). Ernest (1998a, b), in his model of mathematical knowledge, suggests tacit mathematical competences include knowledge-use of mathematical language, procedures, and strategies, meta-mathematical views, and personal beliefs about mathematics. There is little doubt intuition plays a role in students' classroom learning – both positively and negatively – although the exact nature of this role is still unknown (Torff & Sternberg, 2001).

As with intuition theory, the theoretical foundations explaining the role of language in learning are not well established. However, existing theory does suggest linkages between language, learning, and intuition. At a minimum, language may facilitate learning, and it may be that some in some cases, as with advanced mathematics, it is necessary for learning (Ferrari, 2004). The role of language in learning plausibly fits within the emergent perspective. In the psychological perspective, language may mediate students' explicit and tacit knowledge construction (Leron & Hazzan, 2006) and in the social perspective, language is an important tool for developing mathematical meaning (Sfard, 2001).

Setting, Participants, and Data Collection

This research was a case study with cases drawn from one elementary linear algebra class with 28 students. Broadly, the unit of analysis for this study was individual students. However, in alignment with my research question, I focused my analysis on students' understanding of span and linear independence and on their intuition and language use related to these understandings. Using homework sets and the first exam, I evaluated the overall level of the 22 students who volunteered to participate in the study. Then, from the pool of students who consistently attended class and turned in their homework, I selected seven students to represent individual cases. The seven case students represented maximal variation in understanding levels among the students in the class.

I collected two types of data: instruction environment data and individual student data. The instruction environment data provided the contextual information. These data consisted of classroom field observations, classroom video tapes, instructional artifacts, and the course textbook, which was Lay's (2003) *Linear Algebra and Its Applications*. I attended each class in order to observe and record the classroom interactions. This provided me with an understanding of the instruction context of students' learning. From these data I determined instruction students received about specific problems as well as the classroom norms.

The individual student data provided information for analyzing the case students' understanding, language use, and intuition. The three types of individual student work were: written work consisting of homework and exams, journals, and interviews. There were 10 homework assignments, three mid-term exams, and the final exam. The homework problems were primarily from the course textbook, but occasionally the instructor assigned supplemental problems. Over the semester, the instructor assigned six journals in which students were asked to provide an overview of the course content since the previous journal and explain what difficulties, if any, they were having.

I conducted two interviews with each of the case students. The first interview was around the fifth week of the semester, which was a week after the first exam, and the second interview was one to two weeks prior to the final exam. The interviews were semi-structured (Merriam, 1998) as some interview questions were prepared in advance, but others were based on students' responses. I focused the first interview on students' initial understanding of span and linear independence. For the second interview, I focused on students' understanding of span and linear independence in the context of vector spaces and bases. Each interview lasted about an hour.

Data Analysis

In order to analyze the data with respect to the research question, I needed to operationalize the theoretical constructs of understanding, language use, and intuition. There were no generally accepted data analysis methods for these constructs, so I developed methods based on my literature review and the nature of my data. Following is a discussion of the methods I used to analyze and measure students' understanding, intuition, and language use.

Understanding Data Analysis

My goal in analyzing students' understanding was to be able to compare the nature and quality of understanding between students. I concluded there were two essential elements of understanding: definitional understanding and problem solving ability. I drew from Zandieh's (2000) understanding framework to develop definitional concepts. These are concepts embedded in the definitions of span and linear independence, so are necessary to understand these definitions. For instance, linear combination is a concept embedded in both definitions. One cannot conceptually understand the span of a set of vectors without knowing the role of linear combinations in constructing a span. Similarly, one cannot conceptually understand linear independence unless he or she understands the role of linear combinations of vectors is establishing linear dependence. Solution is another concept embedded in the definitions of span and linear independence. Determining linear independence is based on the solution of the homogeneous equation. For span, the existence of a solution to a linear system determines if a vector is in the span of a given set of vectors.

I expanded the notion of definitional concepts beyond concepts embedded in the definitions. I also included how well students understood the nature of the objects associated with the definitions. For example, students were not always clear what objects were linearly independent, sometimes referring to a matrix as being linearly independent, rather than the columns vectors of the matrix. With span, students sometimes mischaracterized the objects by specifying the vector in the span of a given set of vectors as the object creating the span. The other notion I included in definitional concepts was being able to understand how matrix operations related to the definitions. This included whether students associated pivot rows with the span of a set of vectors and pivot columns with the linear independence of a set of vectors.

I decomposed the second component of understanding, problem solving, into several dimensions that I derived from existing literature (Kannemeyer, 2005; National Resource Council, 2001) and the data. These were strategic competence, procedural fluency, work interpretation, and conceptual understanding. Strategic competence indicates the students' ability to select an appropriate strategy or heuristic to solve the problem. Procedural fluency indicates how successfully a student completes the procedures he or she selected to solve the problem. Work interpretation indicates how well the student interpreted the results of their computations with respect to what the problem is asking. Conceptual understanding indicates the degree to which the student appeared to understand the mathematical conceptions embedded in the problem or question. See Table 1 for a summary of the dimensions of understanding.

The data analysis for understanding consisted of coding the problems involving span or linear independence on the students' homework and exams. I used a four-level rubric to represent each dimension of definitional understanding and problem solving. A level 4 meant the data contained evidence that the student had competent, error-free knowledge with respect to a dimension. A level 3 meant the student work contained no evidence of problematic understanding or knowledge, but neither was there sufficient evidence to conclude competent, error-free knowledge. A level 2 meant there was evidence of both correct and incorrect understandings, and a level 1 meant there were significant problems with respect to the dimension.

I coded each student's work based on how he or she solved the problem therefore for a given problem I did not always code the same dimensions of understanding for all students. Table 2 contains the coding for a problem in which students were given two vectors from R^3 and asked to determine by inspection if they were linearly independent. (The vectors were multiples of each other, so were not linearly independent.) The dimensions that were relevant to the students' work were coded with one of the four rubric levels. The other dimensions were coded with an 'x'. For most procedural problems, such as the one in Table 2, conceptual understanding (CU) was not coded. In contrast, true/false problems that required an explanation typically were coded only for conceptual understanding and not coded for strategic competence (SC), procedural fluency (PF), or work interpretation (WI).

Table 1. Understanding Dimensions for Coding

Problem Solving Dimensions	Definitional Understanding Dimensions
SC – Strategic competence	M – Matrix operations
PF – Procedural fluency	S – Solution
WI – Work Interpretation	L – Linear combination
CU – Conceptual Understanding	O – Object reference

Table 2. Sample Coding for Understanding

Problem: Given 2 vectors from R^3, determine by inspection if they are linearly independent.									
Student	Student Response	Problem Solving				Definitional Understanding			
		SC	PF	WI	CU	M	S	L	O
1	The vectors are dependent because the[y] are multiples of each other.	4	4	4	x	x	x	4	4
2	*Does row reduction with augmented matrix, but interprets the augmented column as x_3.* No, by definition the LD, with x_2 being a free variable there are more than one possible solutions. When referring to LI there is only one unique solution.	1	4	2	x	4	3	x	x

Language Use Data Analysis

I found only one commonly used tool for analyzing the quality of students' writing: the 6-trait assessment rubric (Spandel, 2009). The purpose of the 6-trait rubric is to assess writing quality that consists of one or more paragraphs. The rubric contains six dimensions: ideas, organization, word choice, voice, sentence structure, and conventions. The student writing in my data typically consisted of at most a few sentences making it difficult to ascertain multiple quality dimensions in the writing. Therefore, I used a holistic score to measure the quality of students' written explanations. This score is a composition of three aspects of the 6-trait rubric: ideas, word choice, and sentence structure. I refer to this score as the *understandability* of the explanation. Note that this score was not related to the mathematical correctness of a students' explanation; it was related only to how well the student communicated their thinking.

To be consistent with the understanding rubrics, I used a four-level rubric to code the understandability of students' written explanations on homework and exams. The four levels were:

4 – The explanation uses math vocabulary appropriately, is clear and complete, and clearly relates to the problem.

3 – The explanation is not a 4, but the problems do not significantly interfere with the reader's ability to interpret the meaning of the explanation.

2 – The explanation has more significant problems in terms of understandability, completeness, and/or vocabulary use.

1 – The explanation is very difficult or impossible to understand.

If students wrote computational or other mathematical work, I interpreted the written explanations in light of this work.

Besides coding for understandability, I also recorded other characteristics of the students' writing based on systemic functional linguistic (SFL). The SFL literature contains descriptions of the lexical, grammatical, and content differences between informal and formal writing (Schleppegrell, 2001, 2007). Because the issues involved in helping students move from informal writing to formal writing seemed to mirror the issues involved in helping students build formal mathematical understanding from their intuitions, I decided to use characteristics of informal writing as indicators of the quality of students' language use. From the characteristics described in the literature, I selected those that commonly occurred in my data. These were mathematics vocabulary use, sentence grammar, and logical completeness. In addition, because lexical density (a measure of the frequency of content-specific vocabulary use) is often a differentiating feature between informal and formal writing (Schleppegrell), I decided to measure lexical density as well. I coded for the existence or non-existence of problems within each characteristic and use qualitative analysis to describe the nature of the problems. Mathematics vocabulary problems included referencing an incorrect object type for the specified process, using an incorrect form of the vocabulary word, and using the word in an incorrect context. Sentence grammar problems included incomplete sentences, run-on sentences, using pronouns with unclear referents, and atypical or awkward phrasing. Completeness problems included expressing ideas that were not clearly linked and not relating the explanation to the problem being solved.

Table 3 contains a sample of coding for language use. The 'Ex' column contains the code for understandability of the students' explanation based on the 4-level rubric. The 'V' (vocabulary use), 'S' (sentence grammar), and 'C' (completeness) columns contain check marks if there are problems with these characteristics in the students' writing.

Table 3. Sample Coding for Language Use

Problem: Given 2 vectors from R^3, determine by inspection if they are linearly independent.					
Student	Student Response	Ex	V	S	C
1	The vectors are dependent because the[y] are multiples of each other.	4			
2	No, by definition the LD, with x2 being a free variable there are more than one possible solutions. When referring to LI there is only one unique solution.	3		✓	
3	These vectors can be LD because there [?] 3/2*v1 is a weight of v2.	2	✓		✓

Understanding and Language Use Descriptive Coding

Once the data coding of the homework and exam problems was complete for the dimension of understanding and the understandability of language, I averaged the numerical assignments across problems for each student. Then I assigned descriptive labels to number ranges as shown in Table 4. For example, I assigned a 'Strong' label to averages that were 3.7 or higher.

Table 4. Category Labels for Definitional Concept Score Ranges

Category	Score Range
Strong	3.7 – 4.0
Good	3.3 – 3.6
Fair	2.7 – 3.2
Weak	2.3 – 2.6
Poor	< 2.3

Intuition Data Analysis

The students' homework and exam problems were the primary sources for the understanding and language use analysis while interviews served as the primary data for the intuition analysis. (The journals were used for both the understanding and intuition analysis.) For ascertaining intuition, I began by developing a list of characteristics of intuitive thinking from my literature review (see Table 5).

Table 5. Intuition Indicators

Intuition Indicators
1. The conception makes sense from an "everyday" perspective (Stavy & Tirosh, 2000).
2. The conception relates to initial learning experiences (Fischbein, 1987; Wilson, 2002).
3. The conception persists in the face of opportunities to change these conceptions (Fischbein; Vosnaidou, 2008; Wilson).
4. The conception builds upon on a prototypical example, rather than definitions (Sierpinska, 2000).
5. The conception is an overgeneralization of an example or concept (Ben-Zeev & Star, 2001; Fischbein).
6. The conception derives from a surface feature of a problem or representation, rather than the problem structure (Fischbein; Harel, 1999; Stavy & Tirsoh).

These characteristics were guideposts in my initial analysis, but I refined them based on my data. First, I elected to focus on intuitive thinking that appeared to interfere with students' developing mathematically correct understandings. Second, I classified students' intuition into two main categories, each of which had three sub-categories (see Table 6). These categories were self-evident indicators and surface indicators. The self-evident indicators align with the tendency of intuition to place more confidence in what "feels" right over pursuing more careful, rational analysis. The surface indicators align with the intuition's ability to quickly assess patterns and connections in the environment (Wilson, 2002). This ability, while important for being able to interpret one's environment in a timely manner, can also lead to eschewing the need to more purposefully engage in rational analysis.

Table 6. Intuition Indicators Used in Data Analysis

Self-evident Indicators	Surface Indicators
• Belief-intrusion	• Fuzzy-object
• Initial-learning	• Pseudo-definition
• Persistent-ideas	• Over-generalization

I found three types of self-evident indicators, which I labeled *belief-intrusion*, *initial-learning*, and *persistent-ideas*. Belief-intrusion occurred when the student held beliefs about the nature of mathematics that impaired their ability to comprehend mathematical concepts that felt contradictory to those beliefs. Initial-learning occurred when students' understanding appeared to be tied to initial learning experiences that subsequently interfered with correct conceptual development of ideas related to the initial learning. Persistent-ideas occurred when a students' incorrect understanding persisted even in the face of repeated exposures to contrary evidence.

I also found three types of surface indicators: *fuzzy-object*, *pseudo-definition*, and *over-generalization*. The fuzzy-object indicator represented when students did not understand or were not clear about what objects were related to definitions and procedures. This was characterized by students either incorrectly or inconsistently associating objects with definitions, such as thinking matrices were linearly independent rather than a set of vectors. The pseudo-definition indicator marked when students used procedures, abbreviated formal definitions, properties, or other ideas in lieu of formal definitions. The over-generalization indicator represented cases in which students' applied a concept or procedure in an inappropriate context. Pseudo-definitions could, in some cases, be considered a form of over-generalization. However, because of the importance of definitions in learning advanced mathematics, I distinguished between cases of over-generalization relating to definition and other cases of over-generalization. An example of a non-definition over-generalization was a student thinking that the variable m always represents the number of columns in a matrix even in the notation A_{nxm}.

This categorization of intuition was helpful in understanding the different ways in which intuitive thinking operated. However, not all patterns of thought were easily cast into a single category and sometimes indicators appeared to influence each other. Overall, though, this classification system was productive in providing a sense of how the students' intuitive thinking was similar and different.

I rated the level of each of these intuition indicators for each student as low, medium, or high based on how frequently the intuition indicators appeared in the students' data. A student with a low rating for an intuition indicator had no or few instances of that indicator interfering with the student developing mathematically correct understanding. In contrast, a student with a high rating had frequent instances of that indicator interfering with their developing mathematically correct understanding.

Findings

Table 7 contains a summary of the ratings of each student's understanding, language use, and intuition. The understanding and language use labels ranged from weak to strong. The intuition labels were low, medium, and high, with low indicating a low occurrence of intuition indicators in a student's thinking. The intuition labels consist of two parts, which reflect the two aspects of intuition: self-evident intuition and surface intuition. Based on this summary, the short answer to this study's research question is that there appeared to be an association between the quality of students' language use and the quality of their understanding. That is, students who were better able to effectively communicate their thinking in writing generally exhibited better understanding of span and linear independence. There was also an association between the degree to which a student's cognition had intuitive indicators and the quality of his/her understanding. The more a student's thinking had intuitive characteristics, the less likely he or she was to successfully solve problems and explain concepts involving span and linear independence.

Table 7. Summary of Student Understanding, Language Use, and Intuition Levels

Student	Understanding	Language Use	Intuition[1]
Anna	Strong	Strong	Low/Low
Brad	Good	Good	Low/Med-Low
Brian	Good	Good	Low/Med-Low
Carl	Good/Fair	Fair	Low/Med-High
Delia	Fair	Fair	Medium/High
Diane	Fair	Fair	Medium/Med-High
Elaine	Weak	Weak	High/High

[1]Self-evident intuitions rating / Surface intuitions rating

Understanding, Language Use, and Intuition Findings

Definitional understanding and problem solving had differing influences on the variations in the students' overall understanding ratings. Students with good or strong understanding had competent problem solving skills, while those with fair or weak understanding were less competent with respect to the work interpretation and conceptual understanding dimensions of problem solving. However, students' definitional understanding more closely aligned with their overall understanding. The students' definitional understanding of span and linear independence was not significantly different. Students' understanding of matrix operations and object reference tended to be stronger than their understanding of solution and linear combination, although this pattern was more evident for linear independence than for span.

Students' writing patterns were relatively consistent over the semester. The quality of explanations was slightly better for linear independence problems than for span problems, which might be a reflection of students' generally higher confidence in their linear independence understanding as compared to their understanding of span. The explanation characteristics (vocabulary use, sentence structure, and completeness) were elements of consideration in evaluating the quality (understandability) of an explanation, but they were not determining factors. That is, the presence or absence of these characteristics did not dictate the quality level. However, completeness appeared to be associated with the explanation quality. For the good and strong explanations, the percent of explanations tagged as incomplete was no more than about 30%. For lesser equality explanations (fair, weak), the percent of incomplete explanations was about 40% or more.

Vocabulary use was weakly associated with explanation quality as it differentiated the strongest and weakest explanations, but was more constant in the middle levels (good, fair). Sentence structure did not appear to be associated with explanation quality. This characteristic related to grammar, such as the presence of run-on sentences. However, such issues did not always significantly interfere with the understandability of the explanation. Lexical density was not associated with explanation quality. In the literature, formal school writing tends to have a higher lexical density than informal writing. In this study, lexical density might not have differentiated explanation quality because 1) the explanations were relatively short (typically fewer than 30 words) and 2) mathematical words in students' explanations contributed to the lexical density of the explanation regardless of whether the students used them correctly.

The two types of intuition indicators were differently associated with students' quality of understanding. Students with good or strong understanding had low levels of self-evident intuitions, while students with fair or weak understanding had medium or high levels of self-evident intuitions. In contrast, the surface intuitions tended to more directly vary with understanding level. While medium or high levels of self-evident intuitions were associated with medium or high levels of surface intuitions, the converse was not true. Thus, it is possible that self-evident intuition causes or influences surface intuition, but that surface intuition can develop without strong self-evident intuition.

Examples of belief-intrusions were a philosophy of mathematics that conflicted with the philosophy of mathematics underlying the course and a belief that mathematical definitions should align with everyday definitions of words. Several of the students exhibited an initial-learning indicator when their conception of span was dominated by the first span theorem in the textbook, even though this only represented one aspect of span. Elaine's learning was impaired by her attachment to learning she acquired prior to the class. Overall, though, students did not exhibit significant levels of initial-learning intuition. In examining the persistent-ideas, I believe these ideas are more likely to be associated with learning that occurred prior to the class. For example, notions about R^n and variables tended to be persistent and were certainly encountered prior to the linear algebra course.

Pseudo-definitions were the most common type of surface indicator. This might have been related to three factors. First, students avoided the textbook definitions because they felt the definitions were too difficult to interpret. In particular, they found the notation difficult to follow. Second, some students did not believe that knowing definitions was important to the class, so focused their learning on the properties and procedures associated with span and linear independence and used these in lieu of formal definitions. Third, students were able to use their pseudo-definitions successfully in their written explanations. For example, Delia commonly wrote "one unique solution" as a justification for vectors being linearly independent even though she had no real understanding of what that phrase meant with respect to the formal definition of linear independence. Another common example was students equating trivial solution with a unique solution. When the students said the homogeneous equation has a trivial solution, it is true that it is both trivial and unique, so their understanding was never found wanting.

The most common fuzzy-object thinking was that linear independence related to a matrix rather than a set of vectors. This most likely stemmed from students working primarily with matrix operations in determining linear independence rather than the formal definition of linear independence. Students tended to be less clear about the objects associated with span than they were about the objects associated with linear independence. This might be because it is necessary to specify two objects when discussing span (e.g., a set of vectors spans a vector space, a vector is in the span of a set of vectors) and the order of the objects is important.

The types of over-generalizations students made varied. However, a couple of students mentioned it was difficult for them to distinguish between the definitions of terms and/or what different problems were asking. This might be an indication that their tendency to over-generalize interfered with their being able to differentiate definitions and problems. The students' abilities to interpret mathematical language might also be a mediating factor. Mathematical notation was sometimes a cause of over-generalization as was the case when Carl thought that span was in the definition of linear independence because he saw a vector equation that triggered his thinking about linear combinations, which he associated with span.

Span and Linear Independence Findings

Students' quality of understanding of span was the same or slightly poorer than their understanding of linear independence. However, in the course, students had more practice on homework problems with linear independence than with span. Even with equal practice opportunities, the findings suggest span might be a more difficult concept to learn than linear independence. Span has a more complex object structure, involving a set of vectors and a vector space, while linear independence involves only a set of vectors. Since object referencing was difficult for some students, this may interfere with the learning of span. Also, span can be viewed in reference to spanning an entire space or a subspace, a difference some students never mastered. Linguistically, *span* can be used in various forms, such as *spanned by* and *spans.* Students with poorer language skills may struggle to interpret these variations correctly.

Although there were differences in how well individual students understood span as compared to linear independence, these differences were never large. That is, a student never had a strong understanding of linear independence and a weak understanding of span. In addition, there was no significant difference between the understandability of the students' language use involving span as compared to their language use involving linear independence. It is possible this is due to the intuitive habits students brought to the learning of both concepts. If these habits were productive, they supported the learning of both concepts, while, in contrast, if these habits were not productive, they inhibited the learning of both concepts. Another explanation is that span and linear independence both have similar definitional concepts (solution and linear combination). How well students understood these definitional concepts likely influenced the quality of understanding for span and linear independence in a similar way.

For both span and linear independence, students tended to rely on procedural learning rather than conceptual learning. This appeared to be a function of two factors. First, some students preferred to learn mathematics procedurally and even felt uncomfortable approaching these concepts conceptually. This stemmed from beliefs students held about the nature of mathematics. Second, based on homework and test questions, most students believed that procedural skills were mostly sufficient for them to be successful in the class. The textbook played a role in this perception. Many problems in the textbook (Lay, 2003) could be done without much conceptual understanding. Thus, students' interpretation of the instruction environment led them to downplay conceptual learning. Sierpinska, Nnadozie, and Okta (2002) found the linear algebra courses they studied emphasized procedures over concepts as well. In their research they concluded, "The theory remained the frolic of the university professor" (p. 164).

Students could rarely state a formal definition of span or linear independence and, when given the definition, sometimes struggled to interpret it correctly. As with their pursuit of procedural learning, students concluded they did not need to know definitions. Another reason students did not learn the definitions was they found them to be too difficult to understand, mostly because they were not able to interpret the notation. This finding mirrors other research about students' learning of definitions in advanced mathematics courses (Edwards & Ward, 2004;

Vinner, 1991).

In this study students were not always clear on what object was linearly independent, sometimes thinking it was a matrix instead of a set of vectors. In concept development theory, students need to reify their procedural understanding in order to develop structural understanding. In the case of linear independence, students need to move from knowing linear independence as a process involving matrix operations to knowing linear independence as a characteristic of a set of vectors. It may be important that students reify foundational concepts in order to learn new concepts. According to Semadeni (2008), reification can lead to “deep intuition,” which can then facilitate deductive thinking.

It seems possible that not knowing the objects associated with processes could interfere with reification of the process and thus impede students’ developing structural understanding. For example, students who routinely ignore or misunderstand the objects associated with processes, may have a weak sense of what structural understanding in mathematics is because they focus their energies on procedures and algorithms and do not seek broader generalities and connections. If linear independence is simply determining if there is a pivot in every column of a matrix and this understanding is functional for the student, then the student does not consider linear independence as a characteristic of a set of vectors and so never reifies the concept of linear independence. Conversely, it may be that helping students understand the objects associated with processes could facilitate reification of those processes.

Discussion

This study generated preliminary evidence to suggest that students whose thinking habits are characterized by self-evident intuition indicators as defined in this research are less likely to develop mathematically correct understandings in linear algebra. Surface intuitions in the absence of self-evident intuitions were less detrimental to students’ success in learning, although higher levels of these intuitions still hindered students’ conceptual development. A more robust theory about what these intuitions are, how they influence learning, and how they can be remediated might help lead to improvements in advanced mathematics instruction.

Another finding in this study is that students’ ability to write understandable explanations of their thinking, regardless of the mathematical correctness of those explanations, was associated with the quality of students’ understanding. This suggests a link between students’ language use and their mathematical learning. An implication of this finding may be that students’ mathematical writing has some validity in assessing students’ mathematical understanding. In addition, it may be that instruction that supports students’ effective language use in mathematical explanations may help support students’ mathematical learning.

The study has several limitations. Because it was conducted in a single class, the findings may have limited transferability. Also, the nature of the data sources (student work and student interviews) may have limited the validity of the findings. Future research may refine or extend this study’s findings in other linear algebra classes. It may also be fruitful to explore this research question in other advanced mathematics classes, such as abstract algebra and analysis.

The findings suggest possible classroom implications. While none of the instructional methods are new, this research may underscore their validity in supporting students’ learning of mathematics by reducing the role of interfering intuitions. Instructional recommendations include helping students develop metacognitive awareness (Fischbein, 1987) and implementing compare and contrast activities (Marzano, Pickering, & Pollock, 2001). Several researchers have outlined more elaborate instructional tools. These include the instructional practices developed by researchers studying the role of beliefs in mathematics (Muis, 2004), conceptual change in science and mathematics (Vosniadou & Vamvakoussi, 2006), and in reducing misconceptions in mathematics (Stavy & Tirosh, 2000). In order to help students develop their language skills, which in turn may support their mathematical learning, it may be helpful to provide opportunities for students to engage oral and written language practice.

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THE INTERNAL DISCIPLINARIAN: WHO IS IN CONTROL?

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A group of mathematicians and mathematics educators are collaborating in the fine-grained examination of selected 'slices' of video recordings of lectures drawing on Schoenfeld's ROG framework of teaching-in-context. We seek to examine ways in which this model can be extended to examine university lecturing. In the process we have identified a number of lecturer behaviours. There are times when, in what appears to be an internal dialogue, lecturing decisions are driven by the mathematician within the lecturer despite the pre-stated intentions of the lecturer to be a teacher. We analyse three scenarios: in the first the teacher prevails, in the second the mathematician, and in the third the mathematician appears to prompt the teacher to be more rigorous. We analyse these behaviours against the lecturers' ROGs, both stated and inferred. The value of the approach used as a professional development model is considered.

Introduction

Delivering mathematics content to large numbers of students via the medium of lectures presents a number of pedagogical difficulties that are seldom explicitly addressed (Speer, Smith, & Horvath, 2010). This paper presents a small part of a two-year project examining the feasibility of a professional development model, with two key aims. The first is to investigate whether Schoenfeld's Resources, Orientations and Goals (ROG) theoretical framework (see details below) can be adapted to analyse university mathematics lecturing. The second broad question is whether an effective lecturing professional development strategy can be built around a community of practice focussed on an examination of lecture practice. Our primary hypothesis related to these aims is that a personal awareness of our resources, orientations and goals, and a willingness to make changes to them, are major catalysts for professional development (PD) as a lecturer (and indeed as any teacher).

Theoretical Framework

Schoenfeld (1998, 2002, 2007, 2008, 2011) and his Teacher Model Group (TMG) at the University of California at Berkeley have developed a theoretical framework of teaching-in-context, with a goal of answering how and why teachers make the in-the-moment choices they do while they are engaged in the act of teaching. The current version of the framework (Schoenfeld, 2011) is based on Resources, Orientations and Goals (ROG) that teachers bring to their practice. One formulation of the process is that "what a teacher decides to do while engaged in teaching is a function of the teacher's goals (some of which are determined prior to the instruction and some of which emerge as the lesson unfolds), beliefs (which serve to re-prioritize goals as some goals are satisfied or new goals emerge), and knowledge (including various routines the teacher has for achieving various goals)" (2008, p. 9). In the current discussion of the framework beliefs have been replaced by orientations, including dispositions, beliefs, values, tastes and preferences, and knowledge by resources (although still with an emphasis on knowledge). Thus when a teacher enters a classroom they use their orientations to adjust to the situation. Then goals are established based on the orientation, and relevant knowledge (R) is activated. Decisions consistent with the goals are made, consciously or unconsciously, about the directions to pursue and the resources to use (Schoenfeld, 2011). In an earlier paper he argues that:

> Teaching, …… depends on a large skill and knowledge base … its practice involves a significant amount of routine activity punctuated by occasional and at times unplanned but critically important decision making – decision making that can determine the success or failure of the effort. (Schoenfeld, 2007, p. 33)

These decisions made in the classroom are crucial. Since they are often made in-the-moment, yet "The quality of people's decision making…affects how successfully people attain the goals they set for themselves." (Schoenfeld, 2011, p. 36), analysing these decisions should be a vital part of a professional development programme. However, rather than being a single coherent set, an individual's ROG may likely, in any teaching situation, contain competing goals each inspired by a teacher's various orientations. It is this latter situation that is the subject of this paper, as we sought to analyse how the conflict of competing goals arising from an internal dialogue between mathematician and teacher were resolved.

Professional Development of Mathematics Lecturers

While the effectiveness of various approaches to professional development in the K-12 domain has been extensively examined, comparatively little is known at the collegiate level (Speer et al., 2010). Addressing this need, in this project we structured community interactions to prioritise three practices identified as effective. Firstly we focused on "small, but meaningful, aspects of practice" (Speer, 2008, p. 219), "at the very level of detail when development and change appear to occur—the moment-to-moment decisions and practices of teachers." (Speer, 2008, p. 263). Such a fine-detailed examination, called microteaching, has been successfully used in teacher education and has been shown to influence student teacher achievement, with an effect size of 0.88, which is remarkably high compared to the "typical" effect-size of 0.40 (Hattie, 2009). Secondly, all discussions within the group followed the protocol that these should develop from concerns identified by the lecturers themselves. This aligns well with Robinson (1989), who argues for an empowerment paradigm in professional development that recognises the teacher as a professional and works from the 'bottom-up', providing teachers with opportunities to make meaningful choices, and Barton and Paterson (2009), who showed that teachers evidenced positive changes when working on self-identified areas of concern in their practice. Finally the development of a community of practice was actively fostered (Buckley & du Toit, 2010). In this forum common tools and language enable the development of shared meanings and sharing of tacit knowledge. In a manner similar to Jaworski, Treffert, and Bartsch (2009) we sought to "establish a collaborative process in which insider and outsider researchers both study the practices … in mathematics teaching-learning and use the research as a basis for understanding and reconsidering the practices involved" (Jaworski et al., p. 250), and argue that classroom-based enquiry can contribute to the "development of an inquiry community where research into teaching becomes a regular part of teaching practice and the community becomes more knowledgeable about its teaching" (Jaworski et al., p. 254).

This paper reports on an aspect of a project that explores how Schoenfeld's ROG framework may be used to direct lecturers' attention to aspects of their decision-making in the lecture theatre as a professional development activity. The project is informed by research concerning how a teacher's knowledge, orientations, or beliefs, and goals impact on their teaching practice (Schoenfeld, 2007; Ball, Bass & Hill, 2004; Shulman, 1986; Speer, Smith & Horvath, 2010; Törner, Tolka, Rosken & Sriraman, 2010). However, these are studies of teacher practice in primary and secondary schools and similar work at the college level is 'virtually non-existent' according to Speer et al. (2010). The project is also designed to build on the effectiveness of communities of practice (Lave & Wenger, 1991) and a culture of enquiring conversation (Rowland, 2000) for professional development. The project is part of a larger research study, which is described in more detail in Thomas, Kensington-Miller, Bartholomew, Barton, Paterson, and Yoon (under review).

The project aims to examine ways in which Schoenfeld's model can be used, and extended to examine university lecturing and to support the professional development of lecturers (Van Ort, Woodtli, & Hazard, 1991). In the process we have identified a number of lecturer behaviours, one of which is discussed in this paper. We have observed a number of instances of what appears to be an inner argument, or regulation by an inner voice, in the lecturer's communication with the class. In subsequent group discussion it has become clear that many lecturers are aware of this. In this paper we present three ways in which this tension plays out in the decision making process.

Method

In this study a group of four mathematicians and four mathematics educators, all from Auckland University, have formed a community of practice to re-examine lecturing practice by collaborating in the fine-grained examination and discussion of lecturer actions in video recordings of lectures (Kazemi, Franke, & Lampert, 2009; Prushiek, McCarty, & Mcintyre, 2001). Lectures were video-recorded and the lecturer then chose a small section of less than five minutes that the whole group watched and discussed together in a fully supportive manner. This focussed discussion was audio-recorded and later transcribed, along with the brief sections of the lecture. We found that the discussion often started with the video content but then moved on to examine other relevant, related issues of learning, practice and mathematics. The video and audio transcription data was supplemented by an observer record, and a written lecturer-ROG (a statement by the lecturer of the knowledge used, orientation held, and goals, both specific goals intended for the lecture and more general educational goals) and interviews. The data was analysed by focussing on the relationship between decision points and the ROGs.

Descriptions of the Three Scenarios

The three situations we describe involved three of the mathematicians, who were experienced lecturers, two male who we call Sandy and Simon and a female called Abi. Sandy presented an applied mathematics lecture to first year students, Simon a post-graduate lecture in number theory, and Abi a general education first year lecture on the Fibonacci sequence.

Sandy – The teacher takes control

The primary purpose of Sandy's lecture was to consider solutions, for various values of the parameter q, of a difference equation that reduced to the form $x_m = qx_{m-1}\left(1-x_{m-1}\right)$. Sandy revealed a number of orientations that are relevant to this lecture. Firstly, as a teacher he values demonstrating results, and doing so without prior knowledge of precisely what may occur:

O1: "It is good to demonstrate things to the students rather than just tell them."

O2: "I'm pretty happy to experiment in this course and things will occasionally go wrong when you do that."

While he is doing so he wants the students to follow closely, without distractions:

O3: "I hope that my slides, being based on the course-book, allow students to follow what is going on without the distraction of extensive note taking”

He also has a pedagogical belief that, since he is part of a teaching team:

O4: It is important to stick to the course book and cover all the material.

With regard to the use of Matlab, his orientations were:

O5: It has value for exploration “I just wanted to explore, show them the graphs on the screen using Matlab and change the Matlab a little bit to get a closer look at the graphs”

These were some of the orientations that led to the establishment of a number of goals, some of which Sandy explicitly wrote down, and some we may infer. These include:

G1: To show students that interesting, unexpected things happen to the solution as the parameter changes.

G2: Students understanding, from the demonstration, that the solution of the difference equation approaches a periodic function.

G3: To show how easy it is to discover these things by using Matlab.

G4: To have students appreciate the value of Matlab as a mathematician’s research tool. “But basically the use of computers to um.. explore mathematics that’s something that I see as a mathematician and that I’d like to impart that.. I’d like the students to see that, the fact that they can learn a lot about mathematics using a computer.” “The goal is to show students how we can solve problems with this technology and sometimes discover interesting mathematics in the process. We often try things that are not in the course book while not straying too far from it.”

G5: To keep students interested. “…we wanted to keep them interested and so this was an extra lecture showing some more advanced features of the logistic equation that’s usually taught at graduate level.”

G6: To explain clearly the mathematical basis of the construction and solutions of the difference equation.

G7: To stick to the course book and cover all the assigned material.

To achieve these goals he called on resources, including his mathematical knowledge (R1) and the computer program Matlab (R2), which was used to plot solutions of the equation for various values of the parameter. The solutions were then displayed using an overhead projector (R3). At one point Sandy made the decision to move to the projected graph and show the students the periodicity of the function (see Figure 1a).

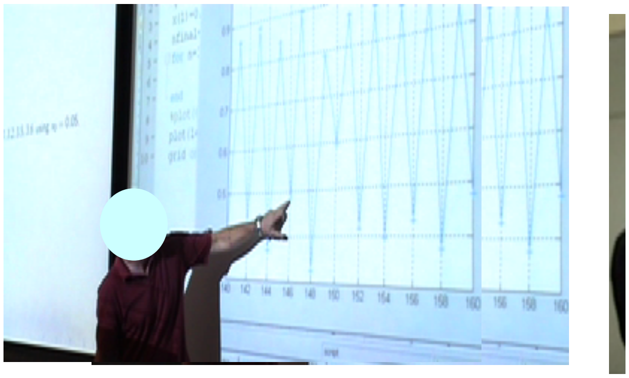

Figure 1: Stills from the video showing Sandy and Simon at critical decision points.

Why did he decide to do this? In line with his orientations O1, 2 and 5 it was part of his attainment of the goals G2, 3 and 6. O1 was crucial here, the belief that demonstrating and not just telling leads to understanding better, as he said "In fact the solutions are periodic and it was a bit hard to look and see that straight off that the solution's periodic so that's why I wanted to do the counting to count from one time step up until the time step when the solution repeats itself." However, it was during this process of counting the function local minima that a crucial decision point arose. The lecture transcription follows.

1. What's happening here it looks even more complicated, 3, 6…[*3 to 4 secs*] yeh so you can see that if you look at it closely…[*walks to screen*]
2 Suppose you start by looking at this value here [*pointing at the graph on the projection*] then there's going to be 1, 2, 3, 4, 5, 6, 7, you can count to 8 I think maybe.. do I ever get back to where I started, maybe not [*he realises that there is a problem*]
3 9, 10, 11, 12 ,13, 14 um.. how many values? So it looks like there's a period of um.. let's see 1, 2, 3, 4, 5, 6, 7, 8.. [*starts to count again*] it looks like there's a period of 14. Whether that's the case or not I'm not sure.
4 We might not have got to the limiting value yet. But it looks like we've settled down to a period of 14. By a period of 14 I mean that it takes 14 um.. we need *n* to change by 14 to get back to where you started from. It looks like that.. something like that is happening anyway. So it seems to be settling down to some complicated periodic um.. solution.
5 All this.. all these things happen just by changing *q*. In fact if you do analyse this a bit further you can you can look at.. There are critical values of *q* where you do get a change. And this allows you to draw what is called a bifurcation diagram. Bifurcation diagrams are diagrams that show how the solution splits up into different solutions as you change the parameter.

We see here that the count of the period arrives at 14, but Sandy knows that the true value is 16, as he later explained "…because the period doubles each time, so it goes from 2 to 4 to 8 to 16, so.. and so on, so there's a theory that actually says the period has to double." This discrepancy between 14 and 16 was unexpected, "I guess the thing that I was probably concerned about was um.. observing something that I didn't expect and not being about to explain it immediately". Hence, in-the-moment, he has to decide whether to address what is, for him, a mathematical discrepancy. How did he make the decision?

Analysing the decision – Sandy

Arriving at the decision involved an internal dialogue between the lecturer as a mathematician and as a teacher. This dialogue had as its aim the resolution of conflict between the competing pedagogical goals, G1, 2, 3 and 5, and the mathematician's goals of G4 and G6. We see this from Sandy's comments about this.

> Yeah, in fact my decision was based on the fact that I'd already spoken far longer than I'd planned to [G7, teacher] on the existing equation and it was time to actually go and do some problems [G7, teacher] which was supposed to be the rest of the lecture so I got onto that [G7, teacher].

> Actually I would have liked to have pursued it a bit [G6, mathematician] but we had already spent more than the allotted amount of time on this demonstration [G7, teacher] and I had shown them periodicity for shorter periods already [G2, teacher] so I think they had grasped the concept quite well so the fact that I didn't actually get a period of 16 bugged me a bit [G6, mathematician] but not enough to ruin the rest of the lecture [G7, teacher].

> I couldn't figure out at that particular time what the problem was [mathematician], uh.. I think I know now but even then it's not obvious… I think I actually probably could have shown them that it was periodic with period 16. [teacher–he has to be able to explain in detail]

> I certainly made a decision not to continue with an unexpected outcome on a graph in the first part of the lecture. Part of me wanted to address this at the time [G4, 6 the mathematician] but I had already gone over time with this part of the lecture and had achieved the goals I desired [G1, 2, 3 the teacher].

We see that, in *this* situation, with *these* students, the teacher wins out over the mathematician. The reason seems to be that the predominant goal was G2, to demonstrate that 'that the solution of the difference equation is a periodic function' and this had been accomplished, and the pedagogical goals were considered met, releasing him from the need to explain the mathematical anomaly, as Sandy said "Actually I didn't get any reaction from the students…I never did tell them it was really 16." He confirmed that, as a teacher, he was happy with the outcome of the lecture, including this decision:

> I'm pretty happy with the way the lecture went. Students seemed interested [G5, teacher], in our exploration of the logistic equation [O1, 2, 5; G1, 3, teacher] and participated in the exploration. In response to questions, we changed the Matlab code to zoom in on graphs of solutions, which allowed us to clearly see the periodicity [G2, teacher]. This part of the lecture occupied more time that I had anticipated [G7, teacher].

> The actual lecture itself went pretty well I was pleased with the demonstration [O1, 2, 5; G1, 2, 3 teacher] um.. I think it was clear enough. The students were able to see what was going on [G2, teacher].

Simon – The mathematician takes control

The primary purpose of Simon's lecture was to introduce the students to continued fractions. He too revealed a number of orientations that are relevant to this lecture and the decision we examine in detail.

O1 To emphasise to students that the right theoretical tools and proof techniques can tame a mathematical problem.

O2 Some proofs are more interesting and important than others. “The real reason I think it’s a cool proof is the fact that you prove a more general result. It’s one of these things that happens a lot in mathematics but you don’t see it so much at the junior level.”

O3 Mathematics needs to be correct. “Oh, this is not really right, I don’t like it not to be right.”

O4 Mathematical notation needs to be consistent and accurate. “Right so the symbols h_j over k_j will from now on will mean precisely one of these things for the specific numbers I am interested in.”

O5 Lecture notes need to be correct. “My response after this exercise was to go away and fix the notes because they were broken” and “it’s a tricky one because the notes clearly weren’t ideal because this whole thing was sort of hidden.”

O6 Lecture notes are useful and provide much of the detail. “I hand them out lecture notes at the beginning of each lecture with pretty much everything on them, they just have to fill in a few gaps.”

O7 Students who are talented at mathematics can cope when a lecturer dwells on the finer points in mathematics. “There is also a confidence that the students can cope with – if I go off on my own little journey the students will have the tools to deal with that … Whereas in another class you would be worried that if I’d lost them after 15 minutes then that’s it.”

O8 Some (but not all) all students at this level are ready to be inducted into mathematics. “Last year’s class I didn’t feel like they were being inducted into mathematics so it wasn’t necessary to dwell on this particular issue of the more general result.”

He also states the following beliefs about the class:

O9 The students are for the most part mathematically ‘strong’ and serious about learning mathematics: “They have a strong background in algebra and are able to work out details and read proofs in their own time.”

O10 There are 1 or 2 students who are not as quick as the others (but are still good and able students), and it is important for me not to go too quickly for them.

He discusses his role as a lecturer.

O11 Traditional lecturing by dictating notes is error prone

O12 My role as a lecturer is to provide a reason for coming to the lecture rather than reading it in a book. “I have something to add over the notes so it’s very much saying ‘Here’s the crucial bit. Here’s the important bit. Here’s the nice bit. Here’s the beautiful bit. Here’s the horrible dirty bit. This bit is really important, I should make sure I really know this bit. Or, yes this is meant to be a kind of tedious bit I shouldn’t feel bad about myself if I find this kind of boring. It’s to give them some sort of framework in their experience.”

In the ROG he writes before the lecture he states the following goals for the course:

G1 To help students to understand the theory and do proofs

G2 To provide good general preparation for post-graduate study in number theory.

G3 To increase the students’ mathematical maturity (is this about inducting them into mathematics?)

G4 To give exposure to different proof techniques.

In particular in this lecture he wants the students to learn a particular proof.

G5 The most important theoretical part is to state and prove correctness of the recurrence formulae for computing the convergents.

G6 "I hope to give a flavour for just how neat this proof is."

Some of Simon's goals were not stated in his written ROG but became apparent during discussion.

G7 To engage with the mathematics for its own sake, 'it's fun.'

G8 To ensure that the mathematics he does is 'right'; it's part of his role as a lecturer

G9 To use notation that is consistent.

G10 To induct (some) students into thinking and behaving like mathematicians.

G11 To present mathematics to his students in a way that gives them a "framework for their experience."

In order to satisfy his goals he draws on a number of resources, including:

R1 His knowledge of mathematics in general and number theory, both theoretical and computational, in particular

R2 His assessment of the students' mathematical ability and interest.

The decision we will discuss here is one he made when he suddenly realised that he was going to encounter a 'notational conundrum' while proving the correctness of the recurrence formulae for computing the convergents. [G5, teacher] in which he is aiming to 'show them a very cool proof' [G6, teacher] which would satisfy the course goals [G1, 2, 3 and 4, teacher]. When re-viewing the lecture he said, "It's like I am labouring that point. This is the point [Transcript 2 below] where I have made a decision. I have somehow made a decision to somehow do it right."[G8, mathematician] Which goals and orientations determine the outcome of the in-the-moment decision?

The lecture transcription follows.

1. This is the sum of a_0 plus, this thing is □□+*1*□□+*1* plus … [*indicates a succession of terms on board*] plus □□+*1*□□+*1*.
   So when you consider that separate extra term and we bundle that whole thing into one piece [*circles it on the board*] They are the same object.
   [*Stands back from the board and looks at class and then back at the board*]
2. So, by the inductive hypothesis, [*starts to write*] I know what this [*gestures in swirl over previous line*] It is some h_i over k_i. [*looks at board*][*looks as if he is thinking*]
3. I'm going to call it … Did I give it a name? [*Looks at paper*]
   I didn't give it a name. It's just some h_i over k_i. But whatever that h_i over k_i is it apparently satisfies the recurrence formula [*points to paper he is holding and looks at class*] [*Stands back and looks at the board*] [*Pauses a moment*] [*Moves*] [*Appears to come to decision*]
4. Yeah I mean this is *an* h_i over k_i but it's not *the* h_i over k_i that I am really thinking of [*gestures back to previous expression*] This is a very subtle point

[*Said very quietly - thinking aloud?*]

5. [*Come back to board and writes above previous 2 expressions*]
   Let's define h_i over … Let's define h_j over k_j to be these things up to a_j where I have worked these out already all the way up to *i* [*writes down and puts in rectangular box above previous 2 expressions*] *Right* so the symbols h_i over k_i will from now on mean precisely one of these things for the specific numbers I am interested in. This thing I have written down here [*gestures to bottom expression*] is *not* the h_i over k_i in that notation because this end term is wrong.

Analysing the decision – Simon

When we asked Simon why he thought he had made the decision to 'labour the point' and disentangle a problem of which the students were not (yet) aware and which "could only create confusion, because, dwelling on a fairly fine detail to such an extent might have obscured the bigger picture." He said it was really the mathematician within going "Oh this is not really right" [G8, mathematician] and added "at that point the whole world disappeared and it's just me and the mathematics." [G7, mathematician]

As he says in the interview, "This is exactly the point where I suddenly realise that it is sort of not quite fitting how I was using those symbols previously. I was thinking ahead to where the proof was going and suddenly it becomes clear to me that there is a problem ahead but it's not clear to anyone else yet." [R1] Visualising himself as the driver of a group of interested, but unworried, tourists (the students) who has just, to his surprise, noticed a sign saying 'Danger Ahead" on the roadside while the others "are all still happily chatting at the back of the bus" he says "I could put my foot down and hope." In that case he could have "swept it under the carpet," perhaps telling the students "It's in the notes … it would be a good exercise for you to do carefully." But he chose to sort it out, to labour the point, and take the bus down the bumpy road. [G7, mathematician and G2 and 10, teacher].

When he told the research group why he had chosen this section to re-view he spoke about ROG dissonance, a mismatch between what he did and the ROG that he had written before the lecture. Discussion uncovered higher order goals that drove the decision - his unwritten orientation and goals as a mathematician [G7,8,9, mathematician] and his desire to induct talented students into mathematics [G2,10,11, teacher]. Simon is not alone in needing to resolve this tension; Jaworski et al. (2009) also speak of the tensions experienced by lecturers arising from a desire to satisfy both student needs and mathematical values.

It was his estimation of students' ability [O9] that allowed him to take the 'detour'. His original assumption that because he saw them as good students he would say 'Just do it' proved incorrect in-the-moment. As he says 'last year there was not the same ability and I had a lower expectation of what they would understand and how they would work and so on so I must have gone through the proof much more lightly with them. Maybe it comes back to this inducting into being mathematicians. Last year's class I didn't feel like they were being inducted into mathematics so it wasn't necessary to dwell on this particular issue of the more general result.' [G10] The need to 'get it right' [G8,9] and to induct them [G10] won the day.

Abi – The mathematician interrupts

The lecture was an introduction to the Fibonacci sequence and the golden ratio for a general education first year course. In her personal ROG, written before the lecture, Abi stated the following, some of which we have classified as R, O or G:

R1 Knowledge of the mathematical subject material.

R2 Knowledge of the level of the students – and the mix of levels in the class: "A similarly difficult part of the class for me is and trying to keep everyone engaged and interested."

O1 It is important to engage and intrigue students; "I have the class bring in pineapples, pinecones and sunflowers themselves – announcing in the previous lecture that they should do so with no hint of why – the idea being to make them think about why they would have to bring in a pineapple and hence create intrigue".

O2 I see this whole course partially as an exercise in "public understanding of mathematics", and so try to treat the lectures as such – rarely going into much depth mathematically.

O3 Mathematical enquiry begins with observation of the real world and we value that which we discover for themselves: "I give the class 5 or 10 minutes to try and count the spirals in each object. We then write down all the numbers we have seen in order and try to spot the pattern."

O4 The physical counting is hard to do: "In the past this has caused some difficulties because it is actually quite hard to do, so this time I am going to bring electric tape to be used as a guide/marker for the spirals."

O5 Prior knowledge of the students may be a problem: "Many of the students already know about Fibonacci and so announce the answer straight away, so I think there isn't time for the students who don't already know the answer to think about it."

G1 The main aim is to give the students some idea of mathematical patterns that appear in nature, specifically patterns involving Fibonacci numbers.

G2 To keep the attention of the bright maths students whilst explaining the idea of a recurrence relation for the non-maths students.

G3 To get the students to spot the pattern(s).

G4 Rarely "going into much depth mathematically."

G5 To help the students to calculate the value of the golden ratio.

From earlier discussions and her behaviour we were also able to infer the following belief:

O6 The mathematics we put in front of students needs to be correct.

Abi satisfies many of her goals by allowing time in class for the students to count the whorls of the pineapples and pinecones and providing the data to generate a table of the early Fibonacci numbers [O1, 3, 4 G1 teacher]. The students then provide a rule for the generation of the series based on their observations [G1, 3 teacher]. The lecture moves on to an examination of the ratio between successive terms. At this point we see an instance of the lecturer's need for rigour. Note that nowhere in her written ROG did she mention rigour, on the contrary she says that she has a goal of "rarely going into much depth mathematically" [R1, G4 teacher/mathematician].

She is trying to establish in the students' thinking that the limiting value for f_{n+1} divided by f_n is the golden ratio, ψ [R1 mathematician]. There is some literal hand-waving as it is established by calculation that the value of □□+1□□ oscillates about 1.6 something [O2 teacher], and then she says:

1. OK suppose you want to compute what this number actually is.
   And it seems to be converging – **and it does actually converge** [*who is she reassuring*?]
2. So you know that f_{n+1} is bigger than f_n so this is going to be a number that is bigger than 1.
   Right? [*sounds as if she hears herself and adds this*] **Or equal to 1**.
3. So ..If I am thinking about what this ratio becomes as n gets really, really big.
   So, for any specific n these 2 things are going to be different.
   Right?
4. Because for one thing it was 1.6 and for the next one it was 1.625.
   So for any specific n it's going to be different.
   But as n gets bigger and bigger and bigger these 2 things are going to get closer and closer together.
5. As long as n is big enough. So we will assume that we are in a place where n is big enough then **we can make this approximation.**

In this interlude we see and hear her spending a lot of time emphasising that for particular values of n the values of f_n and f_{n+1} are different [R1 mathematician] and under what circumstances they are justified in making the approximation [O6, mathematician]. However, there is no need to establish the limit ψ rigorously in this class [G4 teacher]. The bold highlighted language in this excerpt shows her need to be mathematically explicit in her terminology; hand waving will not do even in a class that is 'an exercise in public understanding of mathematics.' As the observer notes show (see Figure 2), later in the lecture Abi simply wanted to use the limit to calculate ψ from the equation
□□+1□□= 1+ □□□□−1 without introducing the unnecessary concept of limit, other than to say "if n is big enough".

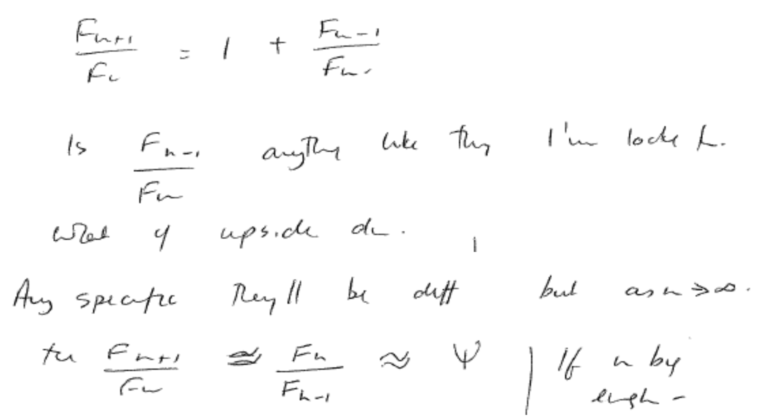

Figure 2. Observer notes from Abi's lecture showing the argument without use of limit.

In this instance it seems to us that the mathematician in the 'inner' discussion interrupts rather that taking over. Abi satisfies her need to be correct by adding qualifiers and conditions to the more general statements to ensure their mathematical correctness. This aligns with her remark during the earlier discussion but is at odds with her stated ROG prior to the lecture.

DISCUSSION

We have described above three different scenarios and the resulting decision paths that arose, based on the inner discussion between the teacher and the mathematician. Group discussions have confirmed that such a dialogue is recognised as a reality, one that we have found is familiar to other lecturers as well. In our case, on one occasion Simon noted that:

> I do this kind of thing all the time, I think it's really distracting because you've gone out and tried to make your big point and then you get all flustered over some detail [of mathematics] and you say oh sorry you know, you have to get it right and the students go "what the hell is going on and now I'm completely confused because it sounded really simple."

To which Abi responded "So should you just ignore that corner [of important mathematics] and just hope that it's not noticed? But then is that bad because you've somehow told them something incorrect?" Our contention in this is that an awareness of this inner tension is an important part of reflecting on our role as tertiary teachers, which in turn will enable incremental growth in professional development, which Speer (2008) claims is an effective approach.

Is there any other value in discussing these examples? Do we have any evidence that this kind of discussion and analysis can aid lecturer professional development? Is a community of practice analysing in depth small parts of a lecture, against a framework of ROGs, in a manner similar to microteaching in schools, of pedagogical value? We believe that the feedback from all involved suggests that the answer to these questions is yes, and there are a number of important aspects contributing to this.

One is that having a 'mixed' community of practice of mathematics educators and mathematicians is important, enabling cross-fertilisation of ideas. For example, Sandy commented that "I'm happy with the discussion that the DATUM group had about the video clip. The analysis of the clip is encouraging in that I gained a mathematics education perspective of the clip, which clarified in my own mind what I do when I teach." and "…you come in with your mathematics education theory from time to time explaining some of the things that we all do and that's very useful as well. It's good to have some of the theory behind it." One aspect of the community that should not be underestimated was the opportunity to see others' teaching. Sandy thought that "And also seeing other people teaching, that's wonderful," and all agreed with a comment from Tim (another member of the group) that "not only is it useful for me and for feedback to mine but I actually find watching your lectures amazing." So much so that there was no shortage of volunteers to handle the video camera in others' lectures! Towards the end of one discussion session it was agreed that it would be a good thing if the practice of watching others became 'business as usual' in the department. This was confirmed in a comment from Abi that "When this project is finished it might be nice to somehow have some sort of setup where lecturers are paired up or in threes or whatever and regularly go to each others' lectures." Thus she proposed continuing some form of supportive departmental community after the conclusion of the research. While we agree that this would be useful, we contend that the subsequent, focussed discussion is also extremely important.

As second aspect is that the community was supportive. Sandy's thoughts on this were that "It was quite reassuring that nobody thought that [he looked silly]. Yes, so that was good." and that the group was "Oh yes, very supportive, very supportive." Simon's comment, "It's pretty revealing watching yourself being videoed isn't it?" shows that lecturers were sensitive about exposing their practice. However he also stated "I would like to sort of record that I have been very happy with the supportive atmosphere, it's been very.. I haven't felt nervous about having people in my lecture or watching the video." It is worth noting that none of the participants was reluctant to repeat the process. From a practical, teaching perspective it was thought that feedback and discussion was important. Sandy's view was "So that's one of the nice things about having these um.. these discussions and um.. about the videos is getting feedback from people." He also mentioned "So it's good to get that feedback from other people and in some cases people identify things that I do that I wasn't even aware of." and "It's good to have feedback on that. Because like other people I guess I have my usual techniques for teaching and um.. but it's good to get some opinions on these techniques from other people." The discussions called Simon's attention to how his decision was predicated on the assumption that this was a 'better' class – prompting him to say "anything that encourages me personally, to put more thought into who the audience really are, what actually they know. That's extremely useful."

The development of a community of practice was further evidenced by the fact that when we were viewing Simon's lecture we observed that "you looked how [Sandy] looked when he was worried about the thing." No-one needed elaboration of 'the thing,' we all knew it referred to his dilemma and subsequent decision. As we work the repertoire of shared decision moments is growing and enables new ones to feed off them.

Apart from the discussions, the process of thinking about, and then writing, one's ROG was another feature commented on by several community members. Sandy said how "It just clarified what I planned to do. I think it may have affected what I did, what I was going to do…It was good to have a clearer idea of what the goals were before going in. It probably helped to make the lecture, well to improve the lecture. I'm sure it did." In Simon's case the 'dissonance' he perceived between his stated ROG and his decision, led to a discussion of his higher order goals. This discussion about his decision increased his awareness of how his decision was predicated on the assumption that this was a 'better' class – and led him to say that he would like to think more about the audience. "Before a lecture you should sit down and remind yourself who is in the class, what they know, and it's astonishing to think that that is not automatically done." Tim said that "I actually found the writing of the ROG quite a useful thing to do…It made me.. in some ways that made me think more than I ever have about what I'm doing in the lecture." Simon agreed with this:

> I agree with [Tim] that writing the ROG was a very.. I think a very useful thing to do and I wish I could hold on to that better. I think.. It forced me to think much more clearly about what I.. what my expectations of the students really are…I think I probably should write a ROG every time five minutes before my lecture just for the pure reason to remind myself that there are students out there and they have a position.

The ROG structure also provides a framework for discussing the 'objects of practice' the lecturers chose – similar to the manner in which Simon says he want to provide a mathematical framework for his maths students. Engaging mathematicians (even more than teachers) in a conversations about pedagogy – particularly their own – is enabled by linguistic and theoretical support that is grounded in their practice.

It appears that the fine-grained analysis, against the framework of the ROGs activates an awareness of the basis on which we make teaching decisions and prompts examination of these decisions leading to development of practice. In addition the analysis adds to what we know about professional development at collegiate level and the influence of beliefs on teaching practices "at the very level of detail where it appears development most *productively occurs*" (Speer, 2008, p. 219, italics original) and informs practice. Along with Jaworski et al. (2009) we argue that knowing more about the practices we, and our colleagues, engage in will enable us to develop them in informed ways. There is a nice parallel with the practice in mathematics of using and interrogating the specific, special case to begin to build an understanding of the general.

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THE IMPACT OF TECHNOLOGY ON A GRADUATE MATHEMATICS EDUCATION COURSE CONTRIBUTED RESEARCH REPORT

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Given the rise in distance delivered graduate programs, educators continue to seek ways to improve teaching and learning in online environments. In particular, the need for high quality K-12 teachers requires superior teacher education programs that model good instructional practice, especially in mathematics. This paper explores the challenges and opportunities presented to the instructor of an online mathematics education course designed for inservice mathematics teachers. The mixed-methods study utilized data from class observations and survey data from participants to capture perceptions of the course. Results of these data are presented and used with the instructor's reflections to make specific recommendations for improving the course and to offer insight to others using distance learning technology to teach graduate mathematics education courses.

Key words: online professional development, mathematics teacher education, teaching geometry

The advances in online technology continue to transform how university faculty can provide teacher professional development (Hramiak, 2010). Advocates of online teacher education maintain that it "holds the possibility of developing not only vibrant explorations of knowledge and practice in the content area, but also communities of learners and practice, and lifelong learning perspectives and skills" (King, 2002, p. 224). Concurrently, problems in the design and implementation of online courses may hinder learners in these environments. Given the demand for high-quality teachers, online courses appear to be an increasingly popular way to provide teacher professional development (Signer, 2008). However, there is a clear need for continuing research in online teacher professional development to ensure that it is meeting the professional needs of teachers (Dede, Ketelhut, Whitehouse, Breit, & McCloskey, 2009).

The purpose of this paper is to present results of an investigation into the design and implementation of an online mathematics teacher education course for secondary inservice teachers as part of the Mathematics Teacher Leadership Center (Math TLC). The Math TLC is a collaboration among the University of Northern Colorado, the University of Wyoming, and partner school districts in Colorado and Wyoming in the United States and is funded by a National Science Foundation Mathematics and Science Partnership[1]. One goal is to help develop culturally competent master teachers to work locally, regionally, and nationally to improve teacher practice and student achievement. Designers of the course relied on recommendations from the literature including purposeful attention to instructor roles and community. With this goal in mind, researchers aimed to determine the attitudes students had on technology within an online teaching geometry course. Specific questions investigated were, what insights can be gleaned from learners‟ perspectives in order to improve future online courses, how are these perspectives related to the instructor roles, and how can instructor reflections on the course offer additional information to overcome challenges in online instruction?

Literature Review

The framework for the design and implementation of an online course for inservice mathematics teacher for this study is *community of practice*. Wenger (2001) defines a community of practice as "a group of people who share an interest in a domain of human endeavor and engage in a process of collective learning that creates bonds between them" (p. 2). According to Wenger (1998), in addition to learning, there are four processes that need to be in place for a successful community of practice. These four processes are (1) a practice to be learned, (2) a community within which to learn it, (3) meaning developed as part of learning the practice with a group of individuals, and (4) an identity formed as part of membership in that community. To foster these processes, designers of the course relied on an examination of instructor roles from the literature.

Maor (2003) and Johnson and Green (2007) categorize the instructor roles of distance education teachers as pedagogical, managerial, social, and technical. The *pedagogical role* entails all of the abilities involved in delivery of content, included the ability to make instructional decisions, develop appropriate learning tasks, facilitate learning, and assess for understanding. The *managerial role* comprises the abilities to administer the course, including the skills to plan the scope and sequence of the online course, monitor the teaching and learning processes, and manage the constraints of the course, including the timeline. The *social role* includes the ability to provide one-on-one, emotional support and advising to learners. The *technical role* encompasses the abilities involved in the decision-making process of selecting technology, the aptitude to use technology, and the ability to trouble-shoot problems with the technology quickly so that participants may remain focused on learning the material. Designers of the course maintained that these roles are key elements in developing a community of practice. Specifically, the managerial role helps administer the practice to be learned; the technological role helps create a virtual community within which to learn that practice; the pedagogical role helps develop meaning as part of the learning of a practice with a group of individuals; and the social role helps form an identity as part of membership in that learning group.

Research Methods

The teaching geometry course consisted of 22 inservice secondary (grades 7-12) mathematics teachers working towards a master‟s degree in the Math TLC program. Ten (45%) were men and 12 (55%) were women, all Caucasian and from predominantly middle-class backgrounds. Because of the relatively sparsely populated nature of northern Colorado and Wyoming, the participants were spread out geographically over the two states, though all learners were within 250 miles of one other. The participants met in person during a six-week session the previous summer, about seven months prior to the start of the course.

The teaching geometry course took place during a 15-week semester in the spring of 2010 and focused on current research and practices of teaching, learning, and assessing geometry in secondary schools. The instructor facilitated the course completely online with both asynchronous and synchronous components. Participants used the course management system Blackboard to access course materials and submit assignments as well as to post occasionally on assigned discussion board topics. The participants of the course used the online collaboration system Elluminate to meet virtually every Monday night in a webinar, where they used live audio and video conferencing to conduct real-time class discussions, small group work, and lecture. The instructor held virtual office hours using Elluminate and communicated with learners regularly by email.

The study was a mixed-methods design involving survey and observations. Near the end of the course, the grant research team administered a 55-item survey. The electronic survey contained quantitative questions to obtain feedback from teacher-participants about their attitudes about the role of the instructor and the impact of technology on their learning. Thirteen of the 22 participants (59%) completed the survey: nine (69%) were women, and three (23%) were men (one did not respond to this item). Additionally, throughout the semester a graduate assistant acted as both teaching assistant and research observer who took field notes capturing synchronous classroom activity. The instructor and graduate assistant also met regularly during the semester and a few times after the course to consult on the effectiveness of the course, examining instructor roles to identify areas of strength and areas in need of improvement. Information shared during these meeting became part of the instructor's reflection on the course.

Results

Overall findings from the survey and observational data indicated participants had successful educational and learning experiences with the class (Table 1). Most participants

Table 1. Overall participant experiences ($N = 13$)

Survey Item	Responses				
	poor	fair	satisfactory	good	excellent
My overall educational experience in taking this course, so far, is:	0	0	3	6	4
	successful			not successful	
My learning experiences to date with this course have been:	13			0	

indicated they were satisfied with aspects of technology associated with all four instructor roles; specifically participants rated survey items highest on those related to technological and pedagogical roles, with social and managerial roles receiving positive but more widely distributed responses. Results organized by the four instructor roles are summarized below, including relevant survey questions, observational notes, and instructor reflections.

Technical Role

One of the most important tasks the instructor performed in his technical role was selecting an appropriate online collaboration system for the synchronous portion of the course. Prior to the semester, the instructor considered two options. After experimenting with each platform and considering the educational objectives of the course, the instructor chose to use Elluminate. One of the deciding factors was that Elluminate allowed for up to six participants to simultaneously share audio and video during synchronous sessions, promoting increased in-class dialogue and providing for greater community interaction. Also, the online collaboration system allowed for breakout rooms to facilitate small group work.

Overall results of the study indicate that Elluminate was technologically successful in facilitating the course. While specific learning and evaluation tasks the program allowed are detailed in the pedagogical section below, the technology had relatively few problems in facilitating the synchronous classroom session according to survey responders and the instructor. As indicated in Table 2, most students had sufficient familiarity with using computers and had experience using online course systems. Observations of the course revealed that although one participant had consistent troubles with technology when speaking and displaying video, most students had little trouble with the technology and indicated they were satisfied with the selection of Elluminate as an online collaboration system.

Table 2. Student self-evaluation of ability to use technology.

Survey Item	Responses			
I would rate my level of computer expertise as:	novice: 1	intermediate: 9	expert: 3	
Using technology to participate in this course seems to be:	getting easier: 11	about the same as day 1: 2	getting harder: 0	
Before taking this class I already had experience using online course delivery methods:	yes: 9	no: 3		
For me to use computer technology to participate in this course is:	easy: 8	some what easy: 4	somewhat difficult: 1	difficult: 0

The technical role of the instructor also encompassed the instructor's aptitude and trouble-shooting abilities. A substantial portion of the survey questions focused on these duties of the instructor. Course participants generally thought the technology was reliable, clear, and easy to use (see Table 3). Additionally, a majority of participants indicated that technology concerns did not often interfere with their understanding of classroom material. According to the instructor, the few major problems, such as a delay in the start of a session, and the relatively minor problems, such as failing to set a timer during breakout sessions, subsided as his skills using the Elluminate grew as the course progressed. Increase familiarity with technology also made him a better technical advisor. When issues would come up, on both his computer or on the students" computers, he dealt with them quickly and fittingly to continue educational instruction.

Table 3. Student responses to quality of technology.

Survey	Ne	Someti	Of	Alwa	No basis for an
The video image from other sites is clear.	0	0	3	1	0
What others are saying at distant sites	0	4	9	0	0
The technology is reliable.	0	4	8	1	0
I can see enough of the speaker's body	0	2	4	7	0
The online communication software (e.g., Elluminate) is easy to use.	0	1	5	7	0

Observations indicated that student attitudes toward technology were positive, validating the survey findings and instructor reflections. While occasional issues would arise, both students and instructor seemed satisfied in the technological role he played. One student"s response to an open-ended survey question asking overall comments about technology emulated this view: "The design of the course, with the integration of the technology as a medium is outstanding - provided the technology is functioning."

Managerial Role

Balancing the administration roles of the course required planning and monitoring various aspects of the course. Before the semester began, the instructor outlined the scope and sequence of the course, including goals, main projects, and a week-by-week thematic schedule. Reflecting on the role, the instructor was generally satisfied with the pace and structure of the course. He believed the constraints of the course (i.e., limited time and contact with students) were handled appropriately by incorporating Elluminate, Blackboard, and email to overcome these challenges. Student opinions on the pace of the course seemed generally positive; when asked if technology was "being used to support instruction at an appropriate pace", five students responded "always", four "often", three "sometimes", zero "never", with one student indicating he or she had no basis for an opinion. One student chose to elaborate on this view in an open- ended question by stating, "The course involved too much time for a 2-hour class. I was overwhelmed as never before."

The instructor used technological features to monitor the teacher and learning process within the course. Elluminate allowed a participants list to be printed, ensuring the instructor was aware of who was present at each synchronous meeting. The Elluminate display also allowed the instructor to see which participants spoke within the break-out groups, helping to determine which groups were active in their discussions. The asynchronous discussion board on who had submitted assignments on time. All of these tools were used to monitor the course. Student perceptions of the processes were mixed; students seemed satisfied with the synchronous portion of the course (Table 4). Although no survey items directly inquired about the asynchronous portion of the course for confirmability, comments from observations both in and out of class indicated students‟ dissatisfaction with the use of Blackboard for asynchronous discussion. The instructor reflected that the discussion board was used infrequently because the majority of discussion was taking place during the weekly meetings; while Blackboard provided the tools to monitor learning, the instructor did not glean much information from this feature.

Table 4. Student perceptions of the synchronous teaching and learning process

Survey Item	Never	Sometimes	Often	Always	No basis for an opinion
I am comfortable with the way I can be present in class:					

a. through audio	0	2	4	7	0
b. through video	0	2	2	8	1
c. in breakout or group activities	0	1	3	9	0

Pedagogical Role

The pedagogical role of the instructor included his ability to deliver course content, develop appropriate learning tasks, and assess students for understanding. Regarding the delivery of the course, the instructor integrated a few novel approaches to facilitate instruction. The typical class consisted of an interactive lecture using the whiteboard feature, where a PowerPoint presentation displayed content material and breakout discussion and whole-group questions. At least one breakout room was implemented each class to facilitate small group discussion. The instructor on occasion had group reporters write on the whiteboard to provide responses from the breakout sessions for discussion. Dialogue between the instructor and students was frequent partially due to the online protocols the instructor established early in the course. The instructor emphasized the importance of students raising their virtual hand to indicate they had a question or a comment to contribute, and to use the video camera when speaking so the entire group could clearly identify the speaker.

As previously reported (see Table 1) students seemed generally satisfied with their learning experience. Table 5 provides specific perceptions of students on the use of technology to facilitate learning in the teaching geometry class. Respondents generally felt that technology concerns did not interfere with their understanding of the material and that the instructor used technology in clear ways that supported understanding. They were mixed in their responses to the items, “The technology used does not interfere with my understanding of the content” and “My classroom experience in this course is as valuable as my experience in face-to-face only instruction.”

The pedagogical role of the instructor also included his ability to assess students‟ understanding. The instructor designed assessments involving thoughtful application of material covered in the course. For instance, designing a lesson involving geometry concepts that applied to the 21st century was a major course project. Observational data indicated that the instructor at least partially achieved his goal of assigning tasks that the teacher-participants found useful in their classrooms. Most teachers implemented, or tried to incorporate, parts of several course assignments in their classrooms. One participant commented on the survey about the assessments by saying, “I was able to use and apply most of the material I created for the course assignments in my own classroom.”

Table 5. Student perceptions of the use of technology to facilitate learning.

Survey Item	Never	Sometimes	Often	Always

No basis for an opinion					
Technology concerns interfere with understanding.	5	8	0	0	0
I have to strain to understand the speaker‟s message.	6	7	0	0	0
The graphical material and presentations are clear.	0	0	3	10	0
The transition between video of speakers and visual displays is	0	0	6	7	0
The technology used does not With my understanding of the	3	2	3	5	0
The use of technology is appropriate for the course content.	0	1	5	7	0
My classroom experience in this course is as valuable as my in face-to-face only instruction.	3	3	5	2	0
I feel I am not being heard in class.	11	1	0	0	1

Some of the major projects in the class required students to work in sustained groups, called Professional Learning Communities (PLCs), for several weeks. The PLCs consisted of three to five students working together. Observations indicated that the PLCs functioned appropriately and reflections by the instructor indicated that his satisfaction with how most of the final projects turned out. However, on one survey item, “The arrangement for communicating with my PLC is as effective as traditional classroom communication,” responses were mixed: three (23%) responded never, three (23%) responded sometimes, two (15%) responded often, two (15%) responded always, and three (23%) said they had no basis for an opinion. Open-ended responses that related to PLCs ranged from “I am not sure what PLC is referring to in the context of this course” to a respondent who said the aspect of the course that most contributed to the successful evaluation was “working in groups and discussing projects with the instructor in small groups.”

Social Role

The social role of the instructor included having the ability to provide one-on-one advising and emotional support to participants, as well as assisting in the development and maintenance of feelings of community within the course. Participants favorably viewed the instructor‟s social role. Observations of the social role included the use email and virtual office hours that provided individual communication with the instructor and the use of breakout sessions that seemed to foster a sense of community with their peers. While one survey question revealed some students felt it was harder to ask questions in the online environment in comparison to a face-to-face classroom, most responses regarding the instructor-student communication were positive (see Table 6). One student commented in an open-ended question on his or her dissatisfaction with online office hours: “The professor needs to be available more than 10 minutes once a week to meet „face to face‟ on Elluminate.”

Table 6. Student perceptions of the instructor-student contact.

Survey Item	Never	Sometimes	Often	Always	No basis for an opinion
I am satisfied with the type of contact I have with the instructor during:					
a. class sessions	0	0	4	9	0
b. online office hours	0	4	2	5	2
It is harder to ask a question than it is in a traditional (face-to-face only) class.	3	8	2	0	0
I am satisfied with the amount of contact I have with the instructor/TA through email.	0	1	4	8	0
The use of the whiteboard to communicate is working well.	0	1	6	5	1

Participants viewed a sense of community to be quite strong. As the first cohort in the Math TLC master‟s program all participants worked with one another for the previous two semesters, including three hybrid learning classes (i.e., a combination of face-to-face and site-to-site instruction) and one online class, before enrolling in the teaching geometry course. Survey data validated these findings; respondents indicated a strong sense of community through high amounts communication among participants, feelings of rapport with each other, and low feelings of isolation (see Table 7). The respondent who indicated that he or she often felt feelings isolation also indicated in the corresponding open response item that, “I only felt isolated when my audio would stop working and I could see people talking but could not let them know that I was having computer problems.” Another student attributed the sense of community to the group work: “I really like to see break out groups throughout the online session. This really helps take the online class and make it feel like we are in one classroom.”

Table 7. Feelings of Community

Survey Items	Never	Sometimes	Often	Always	
I am comfortable with the way I can be present in class:					
a. through chatting	0	1	4	7	1
b. through talking to those	0	1	1	5	6
I feel a rapport with other class members who are in the distant classroom.	1	2	5	4	1
I feel isolated from the rest of the class.	8	4	1	0	0

No basis for an opinion

Reflecting upon his social role, the instructor thought he did an effective job in maintaining a sense of community through structured group work and the creation of a positive classroom atmosphere. He recognizes the benefit of teaching participants who came into the course already feelings a sense of community; that is, having a sense of community already developed eased his job greatly, especially given the research indicating online instructors must work harder than face-to-face instructors to establish rapport and open lines of communication with learners (Rovai 2001, 2002a, 2002b; Shea et al., 2006).

Suggested Improvements

From the learners‟ perspective, two survey items inquired about what improvements they saw fit for the online course they had taken, and for future mathematics and mathematics education courses (Figure 1). Additionally, when asked to reflect on ways the instructor would improve the course, two themes emerged. First, the instructor felt the workload for the course was greater than the two credits not only for the learners but also for him. Thus, one improvement he considered afterwards was to reduce the workload of both the students and instructor to a more manageable rate, given the number of credit hours associated with the course. This recommendation appeared to be shared by participants. For example, one survey respondent wrote, "Remember that we are all full time educators. A heads up about assignments would be really helpful. Assigning readings and work on a Friday afternoon to be completed by Monday evening before class was difficult." Second, based on feedback from participants (e.g., "Post assignments ahead of time so that questions can be asked when the class meets online"),

Figure 1. Responses of ways to improve the course

I have the following suggestions for improving the course:

- Help us with doing websites so we can upload the project to the internet
- I feel like everything is ok for now. More experience is key.

Given that some future Math TLC mathematics education courses will be offered completely online, what kind of supports would you like to see (e.g., ideas for course format, instruction, curriculum, technology, etc.) to promote your success in these online mathematics education courses?

- I have felt successful in all the online math education courses. My suggestion for improvement would be to remember that we are all full-time teachers during the school year in addition to taking these courses.
- Technology that never fails. Perhaps a technology department that can check computer, appropriate software (for the course), peripherals to ensure they are functioning properly.
- Links to assignments at the end of each lecture.

Given that some future Math TLC mathematics courses will be offered completely online, what kind of supports would you like to see (e.g., ideas for course format, instruction, curriculum, technology, etc.) to promote your success in these online mathematics courses?

- I have only taken one math course online previously and I would not do it again. I am thankful that I will have met my MATH requirements without having to take an online math course – I signed up for the program at UNC with the understanding that the math courses would all be face-to-face and I do not think this was a good change to the this program at UNC.

- Better technology for quickly writing equations
- I have not attempted a pure math class, so I do not know what to suggest

the instructor recommended that assignments be prepared at least a week in advance. According

to the instructor, "in an online class a teacher cannot simply go into the classroom and improvise instruction, rather instruction must be purposeful and well prepared ahead of time." Ideally, the course would be complete prepared and available from the onset, but may be difficult to accomplish is most circumstances.

From the data gathered over the duration of the semester, the instructor compiled recommendations for future instructors of online mathematics education courses in this program. First, both instructor and participants thought the use of break-out sessions and polling in the course was important not only as tools for learning but also for community building. Second, participants considered a sense of community an important factor in their learning, a finding supported by literature (Rovai 2001, 2002a). Third, the study results indicated most learners were satisfied with the amount of community they felt, though a few were only slightly satisfied. Now aware of both the importance and the challenge of building community in online courses, the instructor suggests that this aspect of the course be a focus in the future.

Overall, the instructor and the participants thought that the course was educationally successful. In the future, the instructor plans to continue incorporating technology that fosters knowledge and community building. Evaluating his own teaching using the four instructor roles was helpful in identifying strengths and areas for improvement in the online course, and he recommended this approach for other educators.

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MATHEMATICAL KNOWLEDGE FOR TEACHING: EXEMPLARY HIGH SCHOOL TEACHERS' VIEWS

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Eleven exemplary high school mathematics teachers were interviewed to investigate their views on mathematical knowledge for teaching. Teachers took part in a one-hour interview and discussed a written lesson plan. Teachers believed the following aspects of mathematical knowledge for teaching to be important: (a) building mathematical ideas using prior mathematics, (b) a range of examples that illustrate a mathematical concept, (c) appropriate applications of a concept, (d) connections between mathematical ideas within and beyond the high school curriculum, and (e) various approaches to problem solving. Teachers also discussed the development of their mathematical knowledge for teaching, which they believed came from their teaching experience and personal experiences in addition to coursework.

Key words: Mathematical knowledge for teaching, high school mathematics teachers, mathematics teacher education

Introduction

Researchers and practitioners generally agree that a robust knowledge of mathematics is needed in order to teach mathematics successfully. Some researchers have used the term *mathematical knowledge for teaching* (MKT) to describe the specific knowledge that mathematics teachers require (e.g., Ball, Thames, & Phelps, 2008; Silverman & Thompson, 2008). Although there has been a great deal of research on MKT, most literature has focused on elementary or middle school teachers (e.g., Ball, et al., 2008; Hill, Ball, & Schilling, 2008; Ma, 1999). Less research has been devoted specifically to secondary teachers. Hence, the mathematical knowledge that secondary teachers require in order to teach effectively remains an open question.

Much of the research on MKT has investigated to what extent teachers have aspects of MKT that researchers believe to be important (e.g., Hill, et al., 2008; Kahn, Cooper, & Bethea, 2003; Kiorala, Davis, & Johnson, 2008). Teachers' voices about what they believe are important components of MKT are largely absent from the literature.

Although any secondary math teacher could give insight into MKT that is important to their teaching, exemplary teachers may be especially reflective and hence informative in this regard (Collinson, 1994). In addition, since exemplary teachers are quite successful, they likely have sufficient MKT to teach effectively. Hence, studying exemplary teachers' MKT may help to articulate the necessary components of MKT for effective math teaching.

How teachers develop MKT is also an open question. Many secondary math teacher preparation programs require extensive work in pure mathematics. For example, in New Jersey, secondary math teachers must have 30 credits of undergraduate mathematics, 12 at the junior level or above (State of New Jersey Department of Education [NJDOE], 2010). However, it is unclear whether taking these courses helps to develop MKT (Moreira & David, 2008; Zazkis & Leikin, 2010). Other researchers have argued that MKT develops through teaching itself (e.g., Leikin & Zazkis, 2010). Exemplary teachers may also be particularly aware of the ways in which their MKT developed (Collinson, 1994). The purpose of this study is to explore exemplary teachers' views on the MKT that they believe to be important to their practice and the ways in which they believe this MKT developed.

This study aims to investigate the *mathematical* knowledge used in teaching rather than the *pedagogical* knowledge specific to mathematics. That is, this study looks at MKT with a mathematical lens, seeking subject-matter components of secondary teachers' MKT. As a result, for the purposes of this paper, the term *MKT* refers to this mathematical point of view, unless otherwise specified. However, many researchers agree that MKT is a type of specialized knowledge that combines both subject-matter knowledge and pedagogical knowledge (e.g., Ball, et al., 2008; Leikin & Zazkis, 2010). Hence, viewing MKT from a mathematical perspective does not eliminate pedagogical aspects, and elements of pedagogy are evident both in the literature reviewed for this study as well as the results of this study.

Related Literature

Mathematical knowledge for teaching. A great deal of recent research devoted to mathematics teacher knowledge has emerged in the last 25 years, but only a portion of this research has helped to define the subject-specific components of MKT. Highlights of such research will be described below.

One approach to identifying subject-matter components of MKT has been to synthesize prior research. Kennedy (1998) systematically searched reform documents and research literature to determine the mathematics teachers need to know to teach well. She found that mathematics teaching requires a conceptual understanding of subject matter, which includes (a) understanding the relevant size of numbers, (b) recognizing the central ideas within the subject, (c) understanding the relationships between central ideas, (d) an elaborated and detailed knowledge of the subject, and (e) a profound mathematical reasoning ability.

A second approach used by researchers has been to build frameworks based on prior research and then illustrate how these frameworks can be applied to study teacher knowledge. For example, Even (1990) synthesized related research to develop a framework for evaluating teachers' knowledge of a particular concept. This framework proposed seven components of teachers' knowledge of a particular concept: (a) essential features of the concept, (b) different representations of the concept, (c) alternate ways of approaching the concept, (d) unique characteristics of the concept, (e) a basic repertiore of examples illustrating the concept, (f) procedural and conceptual understanding of a concept, and (g) the nature of mathematics as a discipline. Even then surveyed 162 preservice secondary teachers to investigate their knowledge of the concept of *function.* Follow-up interviews were conducted with 10 of these teachers, and Even's framework was applied to analyze preservice teachers' understanding. A similar approach was used by Chinnappan and Lawson (2005) in the domain of geometry.

A third approach has been to use grounded theory to identify subject-specific components of MKT. Ma's (1999) seminal work described features of the elementary mathematics knowledge of successful Chinese math teachers. These included (a) providing rationales for algorithms, (b) justifying explanations with symbolic derivations, (c) using multiple approaches to computational procedures, and (d) recognizing relationships between mathematical ideas. At the secondary level, Zazkis and Leikin (2010) surveyed 52 math teachers about their use of advanced mathematical knowledge in their teaching. Teachers claimed that because of their advanced mathematical knowledge, they were able to, among other things, (a) make connections within and beyond the high school curriculum, (b) provide alternative solutions problems and alternative representations of mathematical ideas, (c) provide applications of mathematics outside of the classroom, (d) consider why mathematical ideas are true, and (e) use problem-solving skills.

More empirical research that aims to identify the subject-specific components of MKT and provide clear examples of these components is sorely needed, especially at the secondary level. This study will qualitatively explore the aspects of MKT that successful teachers have in order to add to the research base on MKT.

The development of mathematical knowledge for teaching. In order to improve teachers' MKT, it is necessary to understand how this MKT develops. Some research has aimed to describe this development.

As part of her 1999 study, Ma investigated when and how a teacher may develop their MKT. She interviewed Chinese teachers whom she considered to have strong MKT, exploring how these teachers believed they acquired this MKT. Teachers indicated that they developed MKT by intensely studying teaching materials, working with colleagues, working with students, and solving mathematical problems in several ways. Potari, Zachariades, Christou, Kyriazis, and Pitta-Pantazi (2007) observed and interviewed nine calculus teachers to investigate their MKT as well as their beliefs on how their knowledge developed. These teachers generally agreed that their mathematics coursework was not helpful to their development of MKT, but they did find integrated courses, which focused on both mathematics and students' thinking in mathematics, to be helpful.

While both of the studies mentioned here provide insight into the development of MKT, more research is needed in this area. It is unclear how the Chinese teachers' development in Ma's (1999) study might extend to secondary teachers in the United States or if the beliefs discussed in Potari, et al.'s (2007) study hold true for exemplary teachers. The present study will contribute to this research by seeking exemplary secondary teachers' beliefs regarding the development of their MKT.

Research Questions

1. What subject matter components of MKT do exemplary high school teachers believe are important for their practice?
2. When and how do these teachers believe that their MKT developed?

Methods

Participants

Selection process. Participants for this study were drawn from 26 teachers who taught high school (9th through 12th grade) mathematics and had been recognized for their teaching in New Jersey between 1999 and 2010 in at least one of three ways[1]:

1. Ten teachers were state and/ or national finalists for the Presidential Award for Mathematics and Science Teaching (PAMST; National Science Foundation, 2009).
2. Eleven teachers were named County Teacher of the Year (CTOY; NJDOE, 2006).
3. Nine teachers were National Board Certified Teachers (NBCTs; National Board for Professional Teaching Standards, 2010) in adolescence and early adulthood mathematics.

Teachers' knowledge of mathematics and mathematics pedagogy are significant criteria for the three awards listed above.

Six of these 26 teachers could not be located because they had either retired or changed employers since they received their award. As a result, 20 teachers were invited to participate in this study by email or phone. Two declined to participate, and seven did not respond to repeated invitations. The remaining teachers agreed to participate.

Participant characteristics. Eleven teachers agreed to participate in this study: four received the PAMST, three were CTOTY, and seven were NBCTs. (Note that some teachers met more than one criterion. The specific awards for each teacher are not listed in order to maintain confidentiality.) Participants were eight females and three males with teaching experience ranging from 10 to 32 years. The median number of years of experience was 17. All but one of the participants had obtained Master's degrees. One participant was a Doctor of Education, and one participant was a candidate for Doctor of Education.

At the time of the interview, two participants taught at private schools, and eight participants taught at public schools. Teachers were employed by a wide range of districts in terms of socioeconomic status[2]. The New Jersey Monthly magazine 2010 rankings[3] of the public high schools where participants taught ranged from the top 10% in the state to the bottom 20% in the state, with the mean rank being 166 out of 322, and the median ranking being 174 out of 316 (New Jersey Monthly, 2010).

Procedure

Two sources of data were collected to answer the research questions: lesson plans and interviews. Data was collected March through June of 2010.

Lesson plans. Each participant was asked to submit one lesson plan from a high school course (i.e., not Calculus or AP Statistics[4]). Lesson plans were obtained anywhere from one day to one two weeks before the scheduled interviews. Courses in which lessons were taught included Algebra I (one teacher), Geometry (three teachers), Algebra II (five teachers), Precalculus (one teacher), and International Baccalaureate Math (one teacher).

[1] Note that some teachers met more than one criterion.

[2] New Jersey public school districts are assigned district factor groups (DFGs) as approximate measures socioeconomic status. Groups range from A to J: A indicates low socioeconomic status, and J indicates high socioeconomic status. Public school teachers in this sample were employed by districts with DFGs ranging from B to I at the time of the study, with at least one teacher representing every group in between (NJDOE, 2004).

[3] New Jersey Monthly magazine ranks public high schools every two years according to factors such as average class size, dropout rate, percentage of teachers with advanced degrees, SAT scores, state test scores, and so on.

[4] Examples from calculus and AP statistics naturally came out in some interview discussions.

Prior to interviews, mathematical or pedagogical points of the lesson that needed further explanation as well as points of the lesson that had the potential to yield valuable information about teachers' use of their content knowledge were noted, and interview questions were added according to these points. Lesson plans were used during the interviews in order to help prompt concrete examples of how MKT was used in practice.

Interviews. Interviews were semi-structured and lasted approximately one hour. All interviews were audio-recorded, and detailed written notes were taken to record teachers' responses and researcher observations.

The interview protocol included general questions about the teacher's background in math education, both general and specific questions about the lesson plan the teacher submitted, general questions about the teacher's use of mathematical knowledge in their practice, and questions regarding how teachers believed their MKT developed. Most of the interview questions focused on the lesson plan, as it was expected that in-depth discussion of the lesson plans would reveal elements of MKT that teachers may not be able to answer in more general questions about their mathematical knowledge (Meade & McMeniman, 1992).

Analysis

All interviews were fully transcribed for analysis. A grounded theory approach to analysis was used in the style of Strauss and Corbin (1990).

Components of mathematical knowledge for teaching. After listening to the interviews and reading through the transcripts, initial codes were assigned to teachers' answers to two explicit questions about their MKT: "How did you apply your mathematics knowledge in this lesson?", and "Describe how your knowledge of mathematics has influenced your teaching." Next, full transcripts were coded for episodes that pointed to teachers' MKT, using the initial codes created from the responses to the two interview questions mentioned above as a non-exclusive framework. Like codes were organized to form categories using the constant-comparative method (Strauss & Corbin, 1990). Categories were refined and new codes and categories were formed when appropriate.

Next, lesson plans were revisited. Elements of the lesson plan which teachers discussed in their interviews were coded independently according to the categories developed from interview analysis. In most cases, the analysis of lesson plans supported the analysis of interview data. In cases where the lesson plans provided disconfirming evidence, codes and categories were revised to accommodate the data from the lesson plans or led to proposed explanations for why the disconfirming evidence existed (Creswell, 2007).

How knowledge developed. In a similar way to the coding of teachers' MKT, codes were assigned to passages in the transcripts that referred to the development of teachers' knowledge. All of these passages were explicit discussions of this development. As with the components of MKT, like codes were organized to form categories, and after revisiting transcripts, categories and codes were revised and refined as necessary. Lesson plans were not used as data for this part of the analysis.

Components of Mathematical Knowledge for Teaching

Teachers' MKT was categorized into five components. The teachers in this study valued: (a) building mathematical ideas using prior mathematics, (b) a range of examples of a concept, (c) appropriate applications of concepts, (d) connections between mathematical ideas, and (e) approaches to problem solving. Each of these components is defined in this section, and supporting examples are provided. Table 1 summarizes the results for this section. Note that it is not reported when teachers discussed a procedural knowledge of mathematics. That is, it is assumed that all math teachers, particularly exemplary teachers, have knowledge which includes knowing correct formulas for mathematical concepts, making calculations accurately, and representing functions as graphs and equations. In this section, more substantial components of MKT are reported.

Table 1
Components of Mathematical Knowledge for Teaching

Component of MKT	**Teachers**
Building mathematical ideas using prior mathematics *e.g., Using "completing the square" to derive the quadratic formula*	**11**
Range of examples of concepts	**8**
• Including "all" cases *e.g., Continuous and non-continuous periodic functions*	2
• Emphasizing various numerical values *e.g., Non-integer solutions to problems involving length*	4
• Providing counterexamples *e.g., Find numbers for x and y such that xy = 6, but ln(6) ≠ ln(x) + ln(y)*	3
• Creating intriguing examples *e.g., The graph of f(x) = x + sin(x)*	3
Applications of concepts *e.g., Using logarithms for the Richter scale*	**11**
Connections between mathematical ideas	**8**
• Between ideas in high school curriculum *e.g., Connecting combinatorics and the binomial theorem*	6
• Between high school and undergraduate mathematics *e.g., Knowledge of hyperbolic geometry to frame Euclidean geometry*	3
Approaches to problem solving *e.g., Using Heron's formula for area of a triangle when appropriate*	**8**

Building Mathematical Ideas using Prior Mathematics

All 11 teachers indicated that they thought it was important to build mathematical ideas in the classroom by grounding them in mathematics that students already understood. For example, Teacher 3 and Teacher 11 both drew on properties of exponents that students had already established in order to build understanding of properties of logarithms. Teacher 11 explicitly asked her students to justify properties of logarithms, such as $\ln(xy) = \ln(x) + \ln(y)$, in terms of properties of exponents (in this case, $e^a e^b = e^{(a+b)}$). She described the importance of using students' prior mathematics to deepen understanding:

> [Students have] seen logs before, but ninety percent of them have no idea why they do it. They have no idea what it means. They have no idea where it comes from. It's just something

> where someone taught them a process, they do the process, and they move on. They don't ever think about why it makes sense. And I feel like if I can show them why it makes sense and show them the math behind it, like that's why I like math.

In a similar way, Teacher 8 described how she used students' knowledge of "completing the square" to solve a quadratic equation in order to derive the quadratic formula:

> I really think that the most important thing is in the explanation of why . . . just where it comes from. . . . [When we study] the quadratic formula, [the students ask], "Well who invented that?" [Like] it just came out of the sky and someone just put these letters down. . . Completing the square is one way to factor, and it's not really that popular to teach it so much anymore because you can always use the quadratic formula. . . . You take that *ax* squared plus *bx* plus *c* equals zero, and you use your algebra on it and it becomes the quadratic formula. I always show the kids. This just didn't magically appear one day under the square roots. It came from something very simple that you know.

When discussing this piece of their knowledge, teachers discussed the fact that it was important for students to make sense of and remember mathematical ideas. Teacher 9 elaborated, "Don't make it something totally foreign. Make it something connecting back to what [the students have] already done. Because the more you can do that, the stronger it makes it, easier it makes it, the more likely they'll remember it."

A Range of Examples of Concepts

Eight teachers indicated that they know a range of examples that exemplify the extent of a concept and as well as counterexamples that show the limits of a concept. Teachers discussed four aspects of these examples.

"All" cases. Two teachers described how they wanted to include "all" cases of a particular concept. For example, Teacher 8 explained how she wanted students to see "all the different possibilities" of parabolas, including those with two real roots, those with one real root, and those with no real roots. She also wanted students to see relationships between the real roots, the complex roots, and the discriminant for each of these cases. Teacher 4 mentioned "cases" that were less common in the high school curriculum. When teaching periodic functions, Teacher 4 made sure to include examples that were not continuous, such as piecewise periodic functions, in order to stretch students' ideas of periodic functions. She discussed this planning process during the interview: "When I created the graphs, I wanted to make sure that I had representatives of all of the different cases that were going to come up."

Various numerical values. Four teachers used examples to emphasize the numerical values that were appropriate for a particular concept. One way that teachers did so was by trying to consistently include a range of real numbers in their examples (i.e., integers, rational numbers, and irrational numbers) as they were appropriate for the concept. Teacher 1 explained how, without encouraging students to consider several numerical values, they tend to focus on integers: "If I don't specify integers or real numbers, the students tend to focus on integers for some reason, and they don't think 'Oh yeah, a length can be 2.4.'" Similarly, Teacher 2 described how students often consider only integers when using non-calculus based methods to find the maximum value of a function. As a result, he often includes examples where the maximum value is not an integer in order to expand students' thinking.

Counterexamples. In order to exemplify the limits of particular concepts, three teachers discussed the importance of including counterexamples. For instance, Teacher 11 challenged her students to find numbers for x and y such that $xy = 6$, but $\ln(6) \neq \ln(x) + \ln(y)$. When discussing the triangle inequality in class, Teacher 1 asked students whether a triangle could have side lengths of 2, 3, and 5, for example. She explained that she included this because, “The [example] that students have the most trouble with is the one where the two shorter sides sum to exactly equal to the third side.” She also asked students to find the possible values of x for a triangle whose side lengths are x, $2x + 1$, and $x – 1$. In this case, there are no such possible values for x.

Creating intriguing examples. Three teachers discussed how they created examples that that conveyed a particular idea by modifying other examples with which they were familiar. Teacher 6 described how she used parametric equations to create a graph that looked like a flower. Similarly, Teacher 4 discussed how she took the sum of the functions of $f(x) = x$ and $g(x) = \sin(x)$ in order to create a graph that students may not have seen before. Teacher 2 recognized that he could use a cubic function in a maximization application problem, as long as the application restricted the domain so that a maximum existed.

Applications of Concepts

All 11 teachers indicated that they know a range of appropriate applications for the mathematics they teach, and they found these applications useful for teaching. For example, Teacher 3 described the meaning of the Richter scale for measuring earthquakes and how the Richter scale relates to logarithms. Teacher 9 described how graphs with horizontal asymptotes can model the temperature of liquids cooling to room temperature, and Teacher 8 described how complex numbers are used to represent circuit elements in calculating net impedances. Teacher 11 also described how knowledge of applications of concepts was an important part of her MKT: “Because I have such a strong background in math and physics, I understand how the math is going to be used, and I can bring that into the classroom and help the kids see that as often as possible.”

Teachers also explicitly discussed the importance of using applications with students. Teacher 10 described this idea:

> I try to find something that's relevant to what we're learning and try to put a realistic spin on it to just give [the students] understanding that it is used in the real world, and there is a reason that we're learning it.

It should be noted that, although teachers felt that knowing applications was an important part of their MKT, two teachers discussed how they did not believe that it was needed in every case. Teacher 5 discussed both the value of mathematical applications, as well as the fact that applications are not always appropriate. In fact, she cited a “negative impact of her knowledge” as knowing that students do not always apply the math they are learning; they just “use it to learn more math”. Teacher 11 had a very similar sentiment:

> That's one of the hard things about teaching precalc. I feel like there's not a whole lot of quality applications that I can show the kids because the applications are either way too simple or they're not going to be able to do them until they're in calculus. And so, I try and explain that to them, and then eventually I say, “You just do it because you like math.”

Connections between Mathematical Ideas

Eight teachers indicated they were able to see connections between different concepts and bring these connections into their teaching. This knowledge manifested itself in two ways: teachers made connections across the high school curriculum, or teachers made connections to advanced mathematics. It should be noted that this component of MKT is viewed as distinct from that of building mathematical ideas using prior mathematics. Teachers who demonstrated knowledge of the latter recognized a natural progression of mathematical ideas that helped students to build new ideas from previous ones. Teachers who demonstrated knowledge of connections between mathematical ideas were able to see relationships between mathematical concepts from different domains of mathematics, and they claimed to "bring in" connections when opportunities arose with the intention of broadening students' knowledge rather than deepening it.

Connections across high school. Six teachers discussed the importance of making connections across the high school curriculum. Teacher 2 described how he connected the binomial theorem to combinatorics in one of his lessons:

> Where you're expanding $(x + y)$ to the fifth. And you go through all the theory, . . . and then you can say, "Alright, well let me just show you a little separate problem. If you have five books and you take three at a time, . . . well that's the same kind of problem as getting the coefficients for the binomial expansion." . . . The more knowledge the math teacher has, the more that they can make connections, they can bring several different ideas in.

Teacher 5 was teaching the definition of a function in Algebra I and discussing the "vertical line test". When a student asked if there was a horizontal line test, she was able to direct the lesson towards a discussion of one-to-one functions because of her knowledge of Algebra II. Teacher 5 called this a "serendipitous moment" in the classroom. In the following passage, she described drawing on her content knowledge in order to make this extension:

> Because I've got a pretty broad knowledge of mathematics from having taught it at so many different levels, I'm able to pull things out of the air, like pulling in the horizontal line test and one-to-one functions in this last lesson, without thinking too hard about it.

Teacher 1 connected geometry and probability by giving students the following task: "Randomly break a stick into three pieces. What is the probability that the three pieces form a triangle?" Teacher 10 connected regression and integration by asking students to plot data points to represent the perimeter of a two dimensional figure, find regression equations for this set of data points, and then integrate to find the area of the figure.

Connections to advanced mathematics. Three teachers discussed connecting high school concepts to undergraduate or graduate-level mathematics in their teaching. Teacher 8 described how she applied her graduate work in mathematics to high school teaching:

> I think that the more math knowledge you have, the more connections you see. And I found that even getting my Master's in math, the courses were ridiculous. . . . But then some weird thing would come up in class where he would explain it a certain way, and I'd be like, "How did I never think of it that way before?" And then I was able to take whatever that was and apply it to what I was teaching, though obviously it wasn't the same.

Teacher 5 claimed that she drew on her knowledge of college statistics when teaching high school statistics in order to "bridge some of the gaps between my students' knowledge and what they needed to know and further study." In a similar way, Teacher 9 claimed that his knowledge of hyperbolic geometry helped him to frame high school geometry and leave students open to the possibility of looking at geometry from a non-Euclidean perspective.

Approaches to Problem Solving

Eight teachers indicated that they are aware of several approaches to problem solving, and they draw on particular approaches that are appropriate for the context or situation. For example, Teacher 1 gave students two sides of a triangle and asked students to find the length of the third side that would yield the maximum area. For this problem, the teacher recognized that the use of Heron's formula[5] for the area of a triangle was appropriate because students were working with side lengths of a triangle. When Teacher 5 was discussing the restriction of $f(x) = x^2$ that was invertible and the domain of the inverse of this restricted function, she use lists of points on the graph generated by a graphing calculator to help students quickly discuss the domain. When using optimization problems in their (non-calculus based) lessons, Teacher 2 and Teacher 10 used technology to help students determine maximum and minimum values. Teacher 2 described how he felt this was an essential part of his MKT: "Getting the exact decimal, and that's where the technology comes in. So, knowing how to use the graphing calculator is really where the knowledge comes in."

Teachers also discussed the need to develop several approaches to problem solving in order to be responsive to students in the classroom. Teacher 10 described this need:

> It's now coming up with a second approach to get this formula or third approach. . . . I guess that's where our expertise comes in. I've had kids come up with methods that just don't seem kosher at first. And then you just sit there and you think about it, and sometimes you really need to reach, but they've come up with something else that you just haven't seen before. And to be able to give credibility to what they're doing, . . . Where it may not be our comfort zone. We need to be able to reach outside the box ourselves and help them keep going in the direction that they feel comfortable going.

How Knowledge Developed

The ways in which teachers discussed the development of their MKT were organized into three broad categories. Teachers discussed the influence of (a) math and math education courses, (b) professional experiences, and (c) personal experiences. Teachers were specifically prompted during interviews to discuss their undergraduate math courses but were free to talk about any other perceived influences. Each of these is described below. Table 2 summarizes the results for this section.

Table 2
The Development of Teachers' Mathematical Knowledge for Teaching

Positive Influence for Development of MKT	**Teachers**
Math and Math Education Courses	**11**

[5] Heron's formula gives the area of a triangle, A, with side lengths a, b, and c in terms of its semi-perimeter, s: $A = (s\,(s-a)(s-b)(s-c))^{1/2}$

• Content in advanced mathematics	6
• Math/ math education teacher or professor	6
• Math education courses	4
<u>Professional Experiences</u>	<u>**9**</u>
• Teaching a variety of courses	6
• Math-specific professional development while teaching	3
• Working with students and learning how students think	2
• Discussions with and ideas from colleagues	4
• Taking a leadership role in their schools	2
• Applying for National Board Certification	4
<u>Personal Experiences</u>	<u>**9**</u>
• Research and self teaching	5
• Personal/ family interest in mathematics	4
• Struggle with mathematics	5

Math and Math Education Courses

There were three elements of math and math education courses that teachers discussed when they described how they developed MKT: (a) the content of advanced mathematics, (b) a particular math teacher/ professor or math education professor, and (c) the content of math education courses. It is important to note that teachers did not discuss all of these elements in positive ways, as described below.

Content in advanced mathematics. Five teachers specifically indicated that they did *not* believe their MKT came from advanced math content. Teacher 4 described how the math that she learned as an undergraduate did not help her to teach high school:

> [In college], you start with calculus, and you go beyond, and then you become a teacher. And I didn't teach calculus until my tenth year of teaching. I'm teaching and nobody taught me how to teach geometry. I went to non-Euclidean geometry, but I'm teaching Euclidean geometry. I did four years of math in college that I'm not even touching. Ok, is it good to have that and know what that's about? Great. But, man, I could have really used some in-depth study of what I'm teaching here.

Similarly, when I asked Teacher 7 to discuss ideas from her undergraduate math courses that helped her to teach, she could not name anything:

> Interviewer: "What specific ideas from college math have helped you to teach high school?"
>
> Teacher 7: "Nothing."
>
> Interviewer: "Really?"
>
> Teacher 7: "Um, college math. Oh my God."

Teacher 3 described the influence of her undergraduate math courses, saying, "Honestly, I would love to say my college work [in mathematics] prepared me for [teaching], but I really don't think that it did."

Six teachers did believe that the content of advanced mathematics helped them to develop MKT. Teacher 9 discussed how his undergraduate degree in math helped him understand high school math more deeply:

> It's kind of like making pizza. You have to stretch it out and when you let go it shrinks back. I think needed to see where things are going, even though I've come back and I'm doing algebra, but now I understand the algebra more. . . . I definitely think having the mathematics degree helps with those math skills.

Three teachers discussed how undergraduate mathematics helped them to understand what mathematics their students would study after high school so that they could frame their lessons appropriately. (See quotes from Teachers 5, 9, and 8 in Connections between Mathematical Ideas section.) Teacher 5 also valued the fact that she could answer students' questions about the types of courses they would take after calculus.

Other teachers mentioned a single, specific undergraduate course or idea that was helpful. For example, Teacher 1 mentioned the History of Math as the one valuable math course she took as an undergraduate. Teacher 11 said that studying vectors as an undergraduate helped her to understand how to teach matrices and vectors in the IB math course she was teaching.

Math/ math education teacher or professor. Six teachers described a math or math education teacher or professor that helped them develop MKT. These influential teachers ranged from middle school teachers to graduate-level professors. For example, Teacher 10 described one of his undergraduate instructors:

> I had a professor in college that really challenged me. … He was the hardest professor I've ever had. … I knew I was going to learn something in there, and he was going to challenge me. He developed me mathematically. … I think that has really helped me to be able to think about the methods that these kids are using, and kind of process what they're doing very quickly, and to say, 'Yes, you can do this,' or 'No, you can't do it.'

Interestingly, three teachers were graduates of the same Master's program and described the same math education professor as helping them to develop MKT. Teacher 2 described how this professor had elegant ways of motivating math concepts so they made sense. Teacher 2 also claimed that this professor influenced his belief that "The math teacher must know their stuff, they must know who they're stuffing, and they must know how to stuff them elegantly."

Math education courses. Five teachers felt that undergraduate math education courses were not helpful for their development of MKT. Three teachers believed that they needed teaching experience in order to benefit from methods courses. Teacher 9 described this idea:

> You're better off teaching basics and doing the pedagogy after you taught a year. … [Math education courses are] trying to teach refining skills, which you have no skills. So how do you refine it? … I think the education courses are wasted on the [young]. … Until you've done any of it, how do you keep refining when you're doing those classes?

Three teachers felt that the undergraduate math methods courses were not helpful to the development of their MKT in its current state because the methods learned in those courses were outdated.

Four teachers did feel that math education courses or math courses specific to teachers helped them develop MKT, and these were courses taken at the graduate level, after teachers had experience in the classroom. For example, Teacher 7 described the nature of the course coupled with her teaching experience that helped her to develop MKT: “In all my classes, which were all math classes, they're like, ‘The teachers in here, can you take this back to the classroom? Can you work with this?’” Similarly, Teacher 3 described how her graduate courses were helpful to her teaching:

> I do feel like for the most part, that was related more to the classroom, as far as the math involved in my Master's degree. Because I took problem solving in mathematics, which was great. . . . I took teaching geometry and teaching algebra in the middle school, so it was things that I could use.

Teacher 10 described how, in a graduate math course for teachers, he noticed ways in which the professor was explaining new ideas that he thought he could use in the high school classroom.

Professional Experiences

Nine teachers believed that their MKT developed through teaching mathematics as well as other professional experiences and opportunities, including (a) teaching a variety of courses (six teachers), (b) math-specific professional development while teaching (three teachers), (c) working with students and learning how students think (two teachers), (d) discussions with and ideas from colleagues (four teachers), (e) taking a leadership role in their schools (two teachers), and (f) applying for National Board Certification (four teachers).

Teaching a variety of courses. Six teachers specifically discussed how teaching a variety of courses helped them to develop MKT. Teacher 1 described this learning:

> One of the things that I think has been really helpful for me as a math teacher is having taught a variety of courses. I understand the connections between, the Geometry and the Algebra II, and where is this leading into Precalculus. I don't think I would have necessarily gotten that from my math degree in and of itself. I think the fact that I had taught a variety of courses allows me to see.

Teacher 8 also described how her knowledge developed from teaching a variety of courses:

> This was an Algebra II class. I found that teaching Algebra I and then teaching Precalculus really gave insight when you see what comes before, what comes after, what's the really important part. . . . The fact of always learning yourself and seeing. Again, I really think the more broad type classes that you teach, the better teacher it makes you.

Math-specific professional development. Three teachers believed that their MKT developed through math-specific professional development programs. For example, when asked where she gained the mathematical knowledge needed to teach, Teacher 1 replied:

> [I] go to a lot of workshops and things like that, professional development. Not so much courses that are designed for education in general, but specifically for math teachers. That has been very helpful. I definitely have learned things there that I would not have thought about otherwise.

Similarly, when asked how her knowledge of mathematics influenced her teaching, Teacher 3 explained how the math-specific professional development that she received made an impact:

> I really got some really great training. . . . I felt like at those workshops that I went to, . . . they were able to focus in on the different content areas. And I think getting some math-specific help for me—I mean at that point, I had been teaching five or six years, so I had my feet on the ground. I knew what I was doing, but it was the time for me to really kind of up my ballgame, I guess. . . . It was probably where I was in my teaching career, combined with everything else; it was just what I needed to kind of help me.

Colleagues. Four teachers discussed how they learned MKT from their colleagues. Teacher 2 described how he learned high school mathematics in a deep way from visiting the classrooms of other teachers: “We've had some great teachers, master teachers, here. You just say, ‘Look, I'm free period 3, you're teaching calculus period 3, I'm there. Every day.’ … I've done that more than anybody here.” Teacher 4 felt like her school was particularly collegial so that teachers often discussed the mathematics they were teaching in depth.

Personal Experiences

Nine teachers discussed personal experiences as being influential to their development of MKT. Teachers described three general ways in which this occurred: through self-directed learning and research, through personality characteristics and personal background factors, and through a personal struggle with learning mathematics.

Self-directed research and learning. Five teachers sought out resources and researched new ways to think about math, or taught themselves mathematics. For example, Teacher 3 described, “I do a lot of research as far as how I'm going to teach a lesson, looking at different examples and different ways of how information is presented. So I do try to find some interesting ways or what worked.” Teacher 5 described how she developed MKT by teaching herself mathematics:

> I actually got a book at the public library and taught myself college algebra and trig, and CLEPed[6] out, and did the same thing with Calc I and Calc II. So some of the math and the approaches to math that I use with my students comes from having to teach it to myself and do that independent sense making. Some of the approaches I take are kind of non-traditional, because sitting at my dining room table, I had to make sense of the formula, or the limit definition of the derivative. So I kind of had to step through it myself, and that's influenced a lot of my teaching.

Personal interest and background. Four teachers mentioned personal interests and background as helping them to develop MKT. For example, Teacher 9 described that his family encouraged mathematical and logical thinking when he was growing up, so he became naturally interested in math. Teacher 10 described that he had a passion for mathematics so that he in interesting in learning it more deeply.

[6] CLEP (College Level Examination Program) offers assessments that measure basic college-level knowledge. Many U.S. tertiary institutions offer course credit for passing these exams.

Struggle with mathematics. Personal experiences that led to development of MKT were not always positive ones for teachers. Five teachers believed that their own struggles with mathematics helped them to develop MKT. Some teachers struggled with mathematics as early as middle school, while others did not struggle until taking graduate math courses. Teacher 7 described how her struggle with undergraduate mathematics developed her MKT:

> When I was first studying mathematics, . . . I didn't get every single proof. I'd chew on it forever and I always didn't get it. . . . So, I guess as I studied mathematics, and I knew where the pitfalls were and how I overcame them, I passed that along to my students.

Discussion

The exemplary teachers in this study identified five components of mathematical knowledge that they believed were necessary for effective teaching: (a) building mathematical ideas using prior mathematics, (b) a range of examples of concepts, (c) connections between mathematical ideas, (d) applications, and (e) approaches to problem solving. Although all of these components are ones that many mathematics teacher education programs strive to develop in prospective teachers, the exemplary teachers in this study cited university coursework as only one source of their mathematical knowledge. Teachers in this study attributed much of their development of MKT to professional experiences as well as personal experiences. These results are similar to those of Ma (1999). Some researchers have argued that extensive undergraduate coursework in pure mathematics may not be the best way to prepare secondary math teachers, but these arguments are often accompanied by claims that preservice teachers could benefit instead from studying mathematics from a teacher's perspective (e.g., Even, 1990; Moreira & David, 2008; Potari, et al., 2007). However, 5 of the 11 teachers in this study did not believe that math education courses taken at the undergraduate level were helpful to their development on MKT. Teachers felt that these courses quickly became outdated or that they needed classroom experience in order to benefit from mathematics education courses. In his chapter of the *Second International Handbook of Mathematics Education* (2003), Stephens echoes this sentiment:

> It is unrealistic to expect that "deep understanding" or "pedagogical content knowledge" can be achieved by the conclusion of initial training. That kind of knowledge is sometimes referred to as case knowledge or situated knowledge. It is the sort of knowledge that cannot be learned from textbooks or at best is poorly learned from textbooks. It is a knowledge that grows out of reflective practice by a professional over a number of years. (p. 773)

Like Stephens, teachers believed that their MKT developed throughout their professional career and as a result of teaching itself as well as their personal motivation to learn more about the mathematics they were teaching.

Limitations

Although this study offers intriguing insight into the MKT of exemplary high school teachers, some limitations should be noted.

First, the participants for this study were chosen because they had received at least one of three major awards for their teaching. Mathematicians and mathematics educators are among the selection committee members for each of these three awards. As a result, the selection committees may be seeking teachers with components of MKT that the committee values, so the fact that the exemplary teachers in this study value many of the same components of MKT as mathematics educators should not be too surprising.

Second, this study seeks only the *perspectives* of exemplary teachers regarding their MKT. There is a possibility other subject-matter components of MKT are central to the practice of the teachers in this study, but they were not aware of them. Indeed, many researchers have argued that elements of MKT may be tacit, even for the most reflective teachers (e.g., Kennedy, 1998; Zazkis & Leikin, 2010). By the same token, there may be some components of MKT reported by teachers that are not actually useful for their practice.

Third, the components reported in this study are certainly not exhaustive. Teachers in this study cited other, more pedagogically-focused components of MKT, such as knowledge about mathematical topics that are difficult for students and methods that are engaging for students. These components were not reported here, since the aim of this study was to explore subject-matter components. Looking at the data from a pedagogical standpoint could yield different components that the ones developed here.

Fourth, although this study focused on MKT, teachers sometimes discussed general pedagogical knowledge that they found essential to their practice, including having strong relationships with parents or appropriate levels of discipline in the classroom. Hence, neither this study nor the teachers' interviews suggest that strong MKT is sufficient for successful classroom teaching. Indeed, exemplary teaching involves the intersection of many types of knowledge and expertise (Collinson, 1994; Potari, et al., 2007).

Significance and Future Directions

Research on subject-specific components of MKT has often taken a researcher's perspective and/ or been focused at the elementary level. Since the participants in this study are drawn from an exemplary pool of teachers, this study adds to the empirical research base on teachers' MKT and gives valuable insight into the components of MKT that are necessary for successful math teaching at the secondary level. Future research is needed to see if and how these components are actually carried out in exemplary teaching. This could be explored through observation and qualitative analysis of teachers' classroom practices. Indeed, Teacher 2 suggested that I observe his teaching in order to see his expertise at work.

This study also gives insight into how exemplary teachers believe their MKT was developed. Teachers felt that their MKT was largely developed through professional experiences. If these teachers' beliefs about this development are correct, then these results may imply that teachers need extensive opportunities for subject-specific professional development throughout their career. Preservice teacher education could help teachers to prepare for career-long learning by helping teachers to develop habits of reflection on the mathematics they are teaching. More research is needed to determine specific aspects of preservice education and undergraduate-level mathematics could be helpful for teachers.

Finally, in their interviews, teachers gave hints of which types of experiences may have helped to develop particular components of their MKT. For example, Teacher 1, Teacher 5, and Teacher 8 believed that teaching a variety of courses specifically helped them to make connections between different topics in high school mathematics. Teacher 5 believed that some of her "non-traditional approaches" came from teaching herself mathematics, and Teacher 11 believed that her knowledge of applications for mathematical concepts came from her coursework as an undergraduate. Future research could investigate teachers' opinions as to which types of experiences led to the development of particular types of knowledge.

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STUDENT TEACHER AND COOPERATING TEACHER TENSIONS IN A HIGH SCHOOL MATHEMATICS TEACHER INTERNSHIP: THE CASE OF LUIS AND SHERI

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In this paper, we investigate interpersonal difficulties that student teachers and cooperating teachers experience during the teaching internship by exploring the tension between one high school mathematics student teacher and his cooperating teacher. We identified seven causes of this tension, which included different ideas about what mathematics should be taught and how it should be taught and a strained personal relationship. We compare these findings to results from interviews with 6 other student teachers and 8 of their mentors. These results suggest that (a) cooperating teachers may offer less freedom than they realize, (b) mathematics educators and cooperating teachers may have very different goals for student teaching, (c) cooperating teachers may hold unrealistic expectations about the student teachers prior to their student-teaching experience, and (d) personal relationships can greatly impact the overall student-teaching experience.

Key words: Student teaching, cooperating teacher, mathematics teacher education, teaching internship, cooperating teacher relationship

Introduction

When asked to name the most valuable part of their teacher education, both beginning and experienced teachers commonly cite field experiences, including student teaching (e.g., Beck & Kosnik, 2002; Byrd & McIntyre, 1996; Wilson, Floden, & Ferrini-Mundy, 2002). The teaching internship offers student teachers important opportunities to develop general pedagogy as well as content-specific knowledge (Feiman-Nemser & Buchmann, 1987; Wang, 2001). Although there has been notable progress in understanding effective components of mathematics field experiences (e.g., Davis & Brown, 2009; Jaworski & Gellert, 2003), much of this research focuses on elementary or middle school teachers. Many of the details of the internship[7] experiences of secondary mathematics teachers remain under-researched (Mtetwa & Thompson, 2000; Rhoads, Radu, & Weber, 2011).

More research in this area is sorely needed. Researchers in teacher education stress that teachers of mathematics need a robust pedagogical knowledge of the mathematics that they are teaching (e.g., Graeber & Tirosh, 2008; Hill, Rowan, & Ball, 2005; Ma, 1999). Some researchers have hypothesized that this pedagogical content knowledge can *only* be obtained by working with students in the context of the classroom (e.g., Fennema et al., 1996; Noddings, 1992). Teacher educators have also emphasized the importance of prospective teachers' mathematical

[7] In this paper, we use the terms *teacher internship* and *student teaching* interchangeably. These terms refer to an extended field placement during initial teacher training where student teachers assume teaching responsibilities in a cooperating teacher's classroom.

beliefs (e.g., Davis & Brown, 2009; Grant, Hiebert, & Wearne, 1998; Phillip, 2007; Thompson, 1984). Mathematics educators call for teachers to teach in reform-oriented ways that focus on student learning (e.g., Fennema et al., 1996; National Council of Teachers of Mathematics, 2000), and the student-teaching experience can be a place where teachers have opportunities to experiment with novel methods of instruction in a classroom setting.

Although the teaching internship sometimes provides prospective teachers with the opportunity to develop the content knowledge and beliefs they will need to become effective teachers, these gains are not always realized. Some student teachers report having negative internship experiences in which they learned little (e.g., Rhoads et al., 2011). The cooperating teacher with whom the student teacher is placed may have considerable influence on the prospective teacher's development. Mathematics cooperating teachers do not always challenge student teachers to develop their pedagogical content knowledge (e.g., Borko & Mayfield, 1995; Peterson & Williams, 2008), and mathematics teacher interns may emulate the mathematical beliefs and practices of their cooperating teachers, even in cases when these practices may hinder student learning (Eisenhart et al., 1993; Peterson & Williams, 2008).

More in-depth research is needed in order explore some of the issues behind mathematics student teacher and mentor interactions. In this paper, we present a case study that describes and interprets one secondary mathematics mentoring relationship. The following exploratory questions guided our research: What causes a strained relationship between a student teacher and cooperating teacher? How do the student teacher's and cooperating teacher's beliefs about the internship experience influence their behavior?

The case study in this paper focuses on one student teacher of secondary mathematics, Luis[8], and his cooperating teacher, Sheri. This study is of interest because Luis was an especially bright student who showed promise as a preservice teacher, and Sheri was regarded as a good teacher who had had positive experiences working with two student teachers in the past. However, Luis and Sheri each reported having a difficult experience working with one another. In section 0, we discuss what issues caused tensions in their working relationship from the perspectives of Luis and Sheri. Although Sheri and Luis largely agreed about the events that transpired when working with one another (i.e., we did not observe instances where Luis claimed he did something and Sheri claimed he did not or vice versa), their interpretations of these events were, at times, dramatically different. In section 0, we describe how Luis and Sheri's case related to other secondary mathematics teacher internships we investigated and discuss what can be learned from our data.

The Current Study

Overarching Design and Goals of the Study

This study took place at a large public university in the northeastern United States. This university had a five-year program for undergraduate students to obtain certification to teach secondary mathematics. In the first four years of the program, the students completed the coursework for an undergraduate degree in mathematics while taking electives in education courses. These electives included two content courses to deepen students' understanding of high school mathematics, two methods courses that taught teaching techniques, and four general education courses (e.g., classroom management, individual and cultural diversity). These

[8] Pseudonyms are used for all participants.

students graduated after their fourth year with an undergraduate degree in mathematics and began their teacher internship in the first semester of their fifth year. These teacher internships are semester-long experiences where student teachers (STs) teach a limited number of classes in a cooperating teacher's (CT's) classroom. Once a school district is selected for these students (based on negotiation of where these students would like to teach and the district's propensity to take on STs from the university where this study took place), the district assigns a CT for the student. The CTs receive little to no training on how to work with a ST, which is common in the United States (Giebelhaus & Bowman, 2002). In the year when this study occurred, there were seven STs who had teacher internships.

Near the end of the teacher internships, the STs, their CTs, and their university supervisors (USs), who observed and critiqued the STs eight times during their internship, were invited to meet individually with the first author of the paper for an interview. All seven STs agreed to participate, although some of the CTs and USs did not. The goal of these interviews was to investigate issues that we found important from previous studies, including the flexibility and freedom that STs were allowed in their student teaching, the feedback they received, and the relationships that they had with their mentors.

The Case Study

The case discussed in this paper occurred in a suburban school district, where Luis taught three classes. He taught two precalculus courses with Sheri, one of his CTs, and one algebra course with Anya, a second CT. His US was Rhonda, a retired teacher with over 30 years of experience.

Luis. *Luis was regarded as an excellent student by most who had contact with him. He was the only ST in mathematics from his university to be nominated for a prestigious statewide award for outstanding student teaching. His undergraduate GPA was 3.9 out of 4.0, which was unusually high for a prospective mathematics teacher. In her interview, Rhonda emphasized that Luis would become "a great teacher," and in her evaluations of Luis, she claimed he had the skills to develop into "a master teacher."*[9] *Anya declined to be interviewed for this study, but her written evaluations of Luis were overwhelmingly positive. Although Sheri had a difficult relationship with Luis, she acknowledged that he was extremely smart and worked well with individual students. She was also impressed with the way he made himself available to students outside of class.*

Sheri. *Sheri was a mathematics teacher with 11 years of experience. She used the "alternate route" to obtain her certification, meaning that she received on-the-job training and did not complete an education program from a university. Sheri was asked to work with Luis, in part, because her supervisor regarded her as a good teacher. She worked with two STs in the past and reported that she had very positive experiences with them. Although Luis expressed frustration*

[9] Sheri was aware of, and somewhat frustrated, by the positive opinion that Rhonda held of Luis. She felt Luis taught very well on the days in which he was observed by Rhonda, but this was not representative of the way he usually taught.

with Sheri, he claimed she was a "good teacher" and was impressed with her classroom management skills. Rhonda found Sheri to be an ideal CT with whom to work[10].

The Data

The data came from multiple sources. The primary data came from individual interviews with Luis and Sheri about their experiences in Luis' teacher internship. These interviews were semi-structured, in the sense that they were organized around particular questions, but the interviewer explored other issues as they arose in the interviews. In making sense of the experience, we also considered Rhonda's interview, the evaluations of Luis' teaching that were provided separately by Sheri, Rhonda, and Anya, and 20 pages of hand-written notes that Sheri provided for Luis during the beginning part of his student-teaching experience.

Interviews were fully transcribed for analysis. An initial pass through the transcripts indicated tension between Luis and Sheri, which was reported from both interviews[11]. We used the participants' reports to hypothesize potential causes of this tension. The first author then met with Luis for a follow-up interview, in which she shared these hypotheses with Luis to see if he felt they were accurate. She also asked Luis to expand on issues that he discussed during his previous interview that we thought were unclear. This follow-up interview was used to confirm, reject, or refine our previous hypotheses. In the following section, we present the case of Luis and Sheri organized around the causes of tension in their experience.

Causes of Tension

We located seven causes of tension between Luis and Sheri. Although Luis and Sheri generally agreed on the facts of the events that transpired, their interpretations of these facts were sometimes very different. We discuss each issue below. In section 0, we discuss the extent that these issues arose in the interviews that we had with other STs and CTs.

Freedom of Teaching Methods

One cause of tension between Luis and Sheri was Luis's perceived lack of freedom to teach in the ways that he desired. Sheri claimed to allow Luis the freedom to try out new ideas, but Luis felt he was coerced into using Sheri's teaching methods and was constrained by her demands.

Sheri gives a glimpse of some of her teaching methods in the comments below, in which she discusses the freedom that she allowed Luis and her past STs[12]:

> Sheri: I said [to Luis], "You have lots of options. . . . We have computers and television sets hooked up to all the computers in every classroom, so you can use PowerPoints in every classroom. You can use the overhead. I have an overhead graphing calculator, I have all this stuff. You can use whatever you want."

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[10] We should note that Rhonda was discriminating in her comments; she was critical of the other ST she supervised and CTs that she worked with in the past.

[11] Interestingly, Rhonda did not discuss the tensions between Luis and Sheri in her interview, which may indicate that she was unaware of the situation.

[12] Quotations were lightly edited to increase their readability. Stutters, interjections (e.g., um, er), and repeated words were deleted. Ellipses are used to indicate the removal of short passages. We do not believe our edits changed the meaning of the text.

Later in the interview, Sheri reinforced the freedom that her STs had:

> Sheri: Some of [the past STs] did PowerPoint, some of them did stuff on overheads. I don't have a preference. I say to them all the time, "Whatever you're preference is, you do. I'm not going to force you to do one thing or another. Do what you're comfortable with."

These excerpts, along with other comments made by both Luis and Sheri, suggested that Sheri's instruction relied heavily on the display of PowerPoint slides prepared prior to class. She felt that PowerPoint presentations or overhead slides were valuable aids for time management, as she did not have to spend time writing out the solutions to the problems that she covered. Luis had a different philosophy on using prepared solutions to teach, as he discussed in the following passage:

> Luis: [Sheri] definitely loved to use PowerPoint to teach, and she taught everything from PowerPoint, which is good and bad. But I always felt like, in math, PowerPoint's good for organizing, but if you're trying to show a problem, you have to, like, show it.

The fact that Luis and Sheri's philosophies differed in this way was not necessarily a cause of tension. Rhoads et al. (2011) discussed how STs did not mind working with CTs whose teaching philosophies and methods differed from theirs, so long as the STs had sufficient freedom to try out their own ideas in the classroom. Consider the following excerpt, where Sheri discussed the types of freedom that she allowed Luis:

> Interviewer: In what specific areas do you think you allowed Luis to have flexibility?
> Sheri: Pretty much everything. I pretty much let him do what he wanted to do, and only when I saw this is definitely not working would I say, "You have to change that." I gave him lots of options, lots of suggestions, as far as using the overhead or writing on the board or using the computer or PowerPoint.

Sheri believed that she was allowing Luis flexibility in his teaching because he did not use PowerPoint like she did. However, Luis felt that Sheri was not flexible about the fact that he was required to prepare written documents before a lesson, such as homework solutions that could be displayed to students (e.g., by transparency or power point). As the following excerpt illustrates, Luis felt constrained by this method of teaching:

> Luis: I started using the PowerPoint, but just for solutions for homework, which I think is good. But again, it's not that much freedom. Because if it were up to me, I feel like I waste class time going over answers. I'd rather assign the odds, which [the answers] are in the back [of the book]. And just . . . do like maybe two problems [in class].

Luis remarked that he did not mind a lack of freedom in student teaching per se; he claimed his other CT was equally strict with how he taught and he had a good relationship with her. Rather, he felt the cause of tension was that Sheri claimed to allow him the freedom to teach his own way, but was then upset if he taught using a different style than she did. This is illustrated in the excerpt below:

> Luis: Sheri was like, "You can have freedom," but later on when she's like trying to take away the freedom, that's what caused the problem.
> Interviewer: I see, OK . . . With Sheri it was kind of confusing, it was like, "I thought I could do this, but I guess I won't"?
> Luis: Yeah.

Luis and Sheri seemed to have very different ideas about what constituted “freedom.” In addition, Sheri admitted that she took away some of Luis’s freedom because of his issues with time management and the topics that he chose to emphasize, as we describe in the following sections.

What Mathematical Topics should receive Emphasis

Sheri described how she guided Luis on which mathematical topics to emphasize before he taught a lesson:

> Sheri: We would go through briefly in the book and I would say, “Ok this topic they need to know. This topic you can touch on, but it's not extremely important. This topic, you have to go really into depth.”

However, Luis and Sheri did not agree on what mathematics should be emphasized during a lesson. Sheri felt that Luis would sometimes spend too much time on higher-level ideas that would be useful in calculus, even though students in the class were struggling to understand the basics of precalculus.

> Sheri: And he said, “Well they really need this for calculus.” And I said, “Ok but we need to make sure they understand precalculus before they get to calculus because most of them are never going to take calculus. We're focusing on this year and these topics.” . . . He was always thinking higher. Which is great. I must have said a million times, he would be a great college-level teacher.

Luis also acknowledged that he and Sheri disagreed on the topics that he should emphasize. He described his philosophy of focusing on ideas that would be useful in students’ subsequent mathematics courses:

> Interviewer: So, how did you personally decide what to emphasize in a lesson or in a unit?
> Luis: I just kind of felt like what was more important, what was more applicable to things that students were going to use again in the future.
> Interviewer: What do you mean? In math class, or in the real world?
> Luis: Yeah, in math class. Like, if they're going to continue studying math, what's more important to concentrate on? For example, I always tried to give them more open-ended questions on quizzes and stuff, especially for extra credit. But then, Sheri would be like, “No, that's a waste of time.” And some of the kids liked the questions and they answered them fairly well.

The last excerpt also illustrated Luis’s philosophy on the presentation of the mathematics that he was teaching. Not only did Luis want to emphasize different topics than Sheri, but he wanted to provide students with more opportunities for critical thinking about those topics through open-ended questioning. He reinforced both of these ideas with an example of when he taught the addition of functions:

> Luis: I remember I spent like maybe like five to ten minutes the first time I taught about the domain of when you add two different functions, the intersection, why it's not the domain of the one with the function with the bigger domain. And all of a sudden she just looks at me, and she's just like, "You're spending way too much time on this.” She's like, "You don't need to focus on domain of the composition of functions." And I was like "I should" [laughing].
> Interviewer: So why did you think it was important?

Luis: Just because like conceptually what students get from it.
Interviewer: You mean just thinking about, like you just said, why it's the intersection as opposed to just the bigger one?
Luis: Yeah, exactly. For example, they have to add like f of x is x and g of x is root x and they have to add it together. Why the domain is just the positive reals and zero.
Interviewer: Why do you think she thought it wasn't as important?
Luis: I have no idea. I mean maybe she thought that it wasn't appropriate for the level.

As illustrated in the previous excerpts, Sheri felt that some ideas that Luis presented were not appropriate for precalculus students. However, there were other topics that Sheri felt were important and Luis did not, as Luis described in an example of synthetic division with complex numbers:

Luis: I remember one time she wanted to show synthetic division with complex numbers, and I didn't because I thought it was too long. And I just thought it was extra time and unnecessary to all the stuff I had to do.

Luis discussed the constraints of time in the previous excerpt. Time management was another cause of tension, as we describe in the following section.

Time Management

Both Luis and Sheri indicated that Luis had difficulty with time management when teaching. Sheri indicated that she was continually frustrated by Luis' poor time management.

Sheri: There was never time. He had a very difficult time with time management as well. It was always the bell's ringing and they're running out the door, and he's still talking. And I tried to tell him, "Luis, you've got to keep track of the time, you've got to look at the clock, you've got to kind of wind down."

To a large extent, Sheri believed that Luis' time management difficulties stemmed from Luis's failure to prepare written material for class and his decisions of which topics to emphasize, as we discussed in sections 0and 0. She noted that if Luis prepared his solutions to the homework problems ahead of time, he would not waste time figuring out how to do the problem in front of the class. In her words, covering the homework problems "would take you two minutes instead of ten." She also complained that Luis would spend too long covering things that the students should be able to do on their own (e.g., simple arithmetic) or on higher level skills that the students did not need, but Luis did not spend enough time on important ideas that were new to students.

Luis candidly admitted that he had difficulties with time management. In fact, when asked if his time management skills differed from Sheri's, Luis replied, "I would say probably not because I didn't have any time management skills." Although Luis recognized that he was not managing time appropriately, he wanted his students to have a deep understanding of the material he was teaching, and he sometimes felt like these two goals were in conflict with each other. In section 0, we described how Luis recognized the need to go over homework quickly, but he also wanted to explain homework solutions in more depth for the students. In the following excerpt, Luis described a similar conflict:

Luis: Maybe that's because I'm too strict in the way I think of a teacher.
Interviewer: You mean like you kind of have this idea of what you want to do?

> Luis: Yeah, an idea of what I want to do, what I want to explain, what I want them to understand, and I kind of feel like if I spent too much time putting together calculators and I don't get it through to them, basically. But, I have to. Classroom management is more important.. . . So, I kind of felt like, alright, I might sacrifice student understanding, but that's enough for them.

Towards the end of his experience, Luis believed he had made progress with this conflict and learned strategies for balancing time management and student understanding. His other CT complimented Luis on his progress in her evaluations. From our interviews, it seemed that Sheri did not recognize Luis's internal struggle with time management. Rather she was frustrated that Luis did not take her advice, as we discuss in section 0.

Understanding Students' Mathematical Knowledge and Difficulties

Sheri commented that Luis did not have a good grasp of students' mathematical knowledge. This manifested itself in three ways. First, Luis did not have an accurate expectation of what these students' background knowledge was. Luis acknowledged that he sometimes had difficulty "understanding where the kids were coming from" and sometimes assumed they knew more than they did. Second, Luis sometimes could not understand what students were asking. Third, he was unable to present the material in a way that students would understand it. As the excerpt below illustrates, Sheri felt that Luis' intelligence worked against him in this aspect of teaching, as the gap between what he understood and what the students understood was too wide.

> Sheri: Luis was very smart. And I think that kind of took away from him being able to present material to the kids in a way that they would be able to understand it.

Jumping In

With her previous STs, Sheri would stop attending STs' lessons halfway through the semester, providing them with more autonomy in the classroom. However, Sheri claimed that early in the semester when working with Luis, she received complaints from the parents of the students that Luis was teaching. This did not happen with the previous STs, and Sheri felt that she needed to exert more control in her classroom. Sheri noted that after Luis finished his student teaching, she would have to teach the students for the rest of the semester, and she was accountable for their mathematical learning.

> Sheri: My thought was, you know, they're my students, they're the ones that need to be learning. He needs to understand what's going on, and I'm the one who's getting them back, so we have to make sure that everything's in place.

Sheri and her supervisor discussed a modified pace of curriculum in order to remedy the situation. In addition, she felt obligated to attend Luis' lectures and "jump in" to provide Luis guidance when she felt he was teaching poorly. Sheri struggled with interrupting Luis's classes, as she discussed here:

> Sheri: So, [my supervisor and I] said, you know what, let's try [the modified pace] first, and if it doesn't work, or if it doesn't help, then we're going to have to have me kind of jump in. But I didn't want to undermine anything that [Luis's] saying or have the students kind of look at him like, "Well, you don't know what you're doing and you're not really our teacher anyway." I wanted them to really say, "This is our teacher, we have to pay attention to him, we have to listen to him, we have to ask him questions." That's why I shouldn't have been in

> the room, and I haven't been in the past. But there were just too many issues, too many, sometimes mistakes. But he tended to focus on things that they don't really need and not so much things that they did need.

In this excerpt, we see that Sheri was reluctant to "jump in" because it defeated the purpose of Luis' student teaching. However, she felt she had to, in part because Luis was not emphasizing what Sheri believed to be the appropriate mathematical ideas (see section 0). Later in the interview, Sheri again described her struggle with "jumping in," but felt she had to because Luis was not understanding the students' questions (see section 0).

> Sheri: He had a very difficult time understanding what the questions were that the kids were asking. They would say, "Can I do this?" or "Does this mean that?", and he wasn't comprehending what their question was. So he would answer a question that they weren't asking and in turn just confuse them more. And, they would just say, "Ok," and they would just drop it. And then I'm like, "Ok, do I jump in and say, 'Ok this is what they're asking and here's the answer to their question'?"

Luis also acknowledged that Sheri would interject during his teaching, and he found it frustrating, as the following excerpt illustrates:

> Luis: I mean there was like one day, I'm in the middle of something, and she's like, "Don't spend too much on that, they don't need to know that." So it's kind of annoying when you're in the middle of a lesson and you have to change it.

Luis realized that Sheri tended to interrupt his lessons when he was not emphasizing the topics that she deemed appropriate. He found this particularly frustrating because he was confused by what Sheri thought should be emphasized and because Sheri did not understand his reasons for emphasizing the topics that he did (as discussed in section 0).

The Role of Feedback

Luis and Sheri had different ideas about the role of feedback in the student-teaching experience. First, Sheri expected Luis to implement her suggestions, but Luis thought that Sheri's suggestions were optional. He eventually implemented them only to avoid complaints from her. Second, Luis expected Sheri to give consistent feedback throughout the student-teaching experience, but Sheri felt that it was appropriate to give less feedback towards the end of the experience.

In the beginning of the semester, Sheri provided Luis with detailed verbal and written feedback. As the excerpt below illustrates, Sheri found that Luis always took feedback well; however, he often did not implement Sheri's suggestions.

> Sheri: He was definitely harder on himself than I was on him. But I feel like he didn't do anything to correct it. Or, if he did, it took a really long time to get there.

When Luis reflected on his student-teaching experience, he acknowledged what he labeled as "being stubborn" and not looking for other resources to find new ways to teach.

> Luis: I kind of felt like I could have tried different things. I was kind of stubborn in a way, let's put it that way.
> Interviewer: What do you mean by stubborn?

> Luis: I kind of wanted to do things my way. I didn't look for any outside resources to help me teach any specific things.

Over time, Sheri became frustrated with Luis's failure to implement the feedback she gave him. She described the feedback she was giving as "a lot of pushing." Sheri felt that Luis's difficulties with time management in particular could be solved with her suggestions about preparing PowerPoint slides prior to class, so she began to be more forceful with the feedback she provided, as she described here:

> Sheri: STs have to learn on their own, they can't, you can't tell them, "Ok, do this, do that." It wasn't until it definitely wasn't working that I was like, "Ok you *have* to do this" [italics were Sheri's emphasis].

In the excerpt below, Sheri describes an instance when Luis followed her advice (about using PowerPoint to display homework solutions).

> Sheri: He finally did it. He said, "This helps a lot. It's much easier." And I said, "See, I know what I'm talking about. I'm telling you for a reason, to help you." I kind of felt like he didn't want to go above and beyond to do the extra work for himself. Which, in the end, created more work for him without realizing, and it wasn't until the end where he did all of his own things and he finally took my advice and did what I suggested, that he said, "Yeah, you're right. This does make it easier." Where the [past STs], when I'd make a suggestion, they took it, and they said, "Yeah, this is great."

The excerpt above indicates that Sheri felt somewhat vindicated with Luis's decision to implement her feedback.

From Luis's perspective, he seemed to think that he found a solution that minimized Sheri's complaints. Specifically, he began to review his lesson plans with Sheri before he taught, as Luis described here:

> Luis: We had really a lot of trouble in the beginning. It was more so we were going back and saying, "OK, you should have done this, you should have done this," after I taught it, as opposed to, "OK, why don't I just go to her first?". . . That way we don't have to fight about it afterwards.

As the semester progressed, Sheri offered less feedback to Luis. As the excerpt below illustrates, Luis seemed disappointed with Sheri's lack of feedback.

> Luis: She [provided positive feedback] more so in the beginning, but now it's like barely anything. I feel like she wants me out of there quickly.

Later in the interview, Luis also said that he did not know what to interpret from Sheri's lack of feedback. Sheri's perception about her lack of feedback was different. She viewed her role as providing strong guidance to her STs in the beginning of the semester and then letting them teach on their own for the remainder of the semester.

> Sheri: For the last month or so that [Luis] was here, even though I was in the room, I let him just do on his own. Every once in a while I would look over and say, "Ok this is what we need to fix." But my other STs were able to just go with it on their own.

Indeed, as discussed in section 0, the only reason she attended Luis' classes was because she felt he needed extra help.

A Difficult Personal Relationship

Both Luis and Sheri acknowledged that they had a difficult personal relationship, although each claimed not to dislike the other personally. In the excerpts below, we see that Sheri had closer relationships with her other two STs.

> Sheri: When you have more of that personal relationship or that friendship, more vulnerability, and they don't have as much of a fear opening up and showing that. And I think it's almost easier to help them when they do open up like that. It might be more of that male/ female relationship where he's afraid to show me that he had faults or that he was afraid of something or that he didn't understand something. I felt like he wanted to show that he was really smart and he knew what he was doing and he could do it. Where the [past STs], they kind of came in, in the beginning with, "I have no idea what I'm doing, I'm terrified, help me."

Sheri felt that the relationships she had with her other STs were due, in part, to their "vulnerability," which she believed that Luis lacked. However, Luis believed that the difficult relationship between him and Sheri was simply due to "clashing personalities." Several times during the interview, Luis mentioned that Sheri seemed tired of working with him. Luis may have gotten this idea by Sheri's lack of feedback towards the end of the experience, as discussed in section 0. By contrast, Luis reported having a great professional and personal relationship with his other CT.

General Discussion

In the previous section, we described seven sources of tension between Luis and Sheri. In this section, we discuss issues raised by Sheri and Luis that were present in other intern/ mentor relationships and use these issues to identify general aspects of CT/ ST interactions.

Freedom and Flexibility

To develop an independent style of teaching, STs need the freedom to try out their own pedagogical ideas in a classroom (Eisenhart et al., 1993). Researchers have noted that STs valued this freedom, and STs who lacked this freedom had negative internship experiences (Beck & Kosnik, 2002; Rhoads, et al., 2011). For Luis and other STs that we interviewed, a lack of freedom in the classroom sometimes led to tensions in their internship experience.

We believe two factors may contribute to STs' perceived lack of freedom in the classroom. First, CTs may allow STs considerably less freedom than they realize. Sheri believed that she allowed Luis freedom in "pretty much everything," yet she was particular about what mathematical topics that he emphasized (see 0) and insistent that he use slides prepared ahead of time to present his material (see 0). A second issue concerns accountability constraints that may limit the teaching flexibility that CTs can pragmatically allow. Several CTs in our study mentioned that they were ultimately responsible for students' learning and so could not allow their STs complete freedom in the classroom, and similar views are reported in the general teacher education literature (e.g., Rajuan, Beijaard, & Verloop, 2007). Luis' supervisor, Rhonda, discussed why some CTs do not allow their STs freedom in their teaching:

> Rhonda: They're looking for someone to clone their style. And don't forget they're going to get these students back again, and they want those students in the form that they want them to be, with the behaviors and the habits of the mind that they want.

Rhonda noted that CTs are ultimately responsible for the learning of their students, and CTs want their students to be conditioned to learning when they return as the classroom teacher. In section 0, we noted that Sheri was stricter with Luis after she received complaints from parents about his teaching, in part because she was responsible for these students' learning.

Time Management, Understanding Students' Mathematical Thinking, and Room to Fail

The abilities to manage time effectively and understand students' mathematical thinking are widely acknowledged to be critical for successful teaching. Luis had difficulty with both of these issues, as indicated in sections 0and 0. However, in this sense, Luis was hardly unique. Indeed, all seven STs reported having difficulty with these things, with many citing learning how to manage time and better understand students' mathematical thinking as the most valuable benefits they obtained from their student-teaching experience. Other researchers have also documented that these are common problems for beginning teachers (e.g., Ponticell & Zepeda, 1996).

What appears different about the case study reported here was how Sheri interpreted Luis' deficiencies in these regards. Other CTs and supervisors recognized that their teacher interns had difficulty with time management and understanding students' mathematical thinking, but most did not think this was problematic. Instead they thought these types of issues were the norm and something that preservice teachers were expected to develop during their internship. Sheri viewed these as serious faults of Luis.

Discrepancy between Goals of Mathematics Educators and Cooperating Teachers

There were several points were Luis was engaging in pedagogical behaviors that we believe mathematics educators would characterize as good teaching practice, but Sheri found to be inappropriate and unnecessary. This was most prevalent in section 0. In teaching a precalculus course, Luis would emphasize ideas that would be useful in calculus. Researchers such as Ma (1999) have indicated that teaching with an eye toward how the mathematics being taught will connect to the future mathematics that students will learn may help students to be more successful in mathematics. Sheri, however, found this to be a waste of time, essentially arguing that students can learn calculus if and when they take calculus. Luis also mentioned that he would like to engage students in problem-solving activities, but lamented that he did not feel like he was able to do so.

We should note, however, that other ST/ CT pairs did not seem to have conflicts over which mathematical topics to emphasize. All STs that we interviewed mentioned that their CTs would help them decide which mathematical topics to emphasize when planning a lesson or a unit. Most STs appreciated this guidance because they felt inexperienced in this regard. Luis seemed to be a special case. Perhaps this was because he was, in his words, "stubborn" and had "strict ideas about what a teacher should be." Other STs seemed to respect and go along with their CT's wishes, but Luis tried to stick to his convictions.

As described in section 0, Luis felt an internal struggle with balancing the elements of a high school classroom with his ideas about teaching and learning. Although it is important that Luis learn how to both manage a classroom and facilitate students' learning, CTs may see learning how to run a classroom as a primary goal, but mathematics educators see facilitating students' learning as more important (Leatham & Peterson, 2010).

Personal Relationship

One of the most robust findings that we found in our interviews with STs was how important their personal relationships with their CTs were to the overall quality of their internship experience. Luis and Sheri's case study is consistent with our findings. Sheri believed having a friendly personal relationship was important and, indeed, she had such relationships with her other STs in the past. Similarly Luis had a closer relationship with his other CT.

It is difficult to say whether the strained relationship between Luis and Sheri was a *cause* of the other tensions that they experienced or merely a *consequence* of these tensions. We suspect it was both, but we do not have the data to make a judgment.

Although we cannot be sure exactly what caused the difficulties in Luis and Sheri's relationship, we can offer a hypothesis. In our (non-research) based experience in working with STs and CTs, we have noticed that when CTs have problems with STs, they usually complain that the STs do not follow their advice and often attribute this to arrogance on the part of the ST. They feel insulted that the ST was not respectful of their experience and expertise. We do not have the data to conclude that this occurred in Luis' interactions with Sheri. However, when discussing the role of feedback in section 4.6, Luis commented that he could be stubborn and not seek outside counsel for his lessons, and Sheri expressed frustration that Luis would not take her advice, telling him, "See. I know what I'm talking about. I'm telling you for a reason." One implication of this hypothesis is that for some CTs, a failure to heed their advice might not be interpreted as a difference of teaching styles, personalities, or opinions, but rather as a lack of respect. This interpretation could sour the relationship between the two, and STs should be aware that some CTs are apt to make this interpretation.

Implications

Many researchers have indicated the need for quality preparation and training for STs, CTs, and USs (e.g., Feiman-Nemser, 2001; Giebelhaus & Bowman, 2002). Luis and Sheri's case and supporting evidence from the other interviews we have conducted point to some suggestions for this training. First, preparation could include discussions about the goals and purposes of student teaching according to the STs, CTs, and USs. Such conversations may not lead to consensus, but could lead to a mutual understanding and help to avoid some of the tension that we saw with Luis and Sheri. (For example, Luis seemed to expect consistent feedback throughout the semester, but Sheri believed it was her role to taper feedback.) STs and their mentors should be encouraged to continue critical dialogue about their intentions and goals throughout the student-teaching experience so that each has the opportunity to learn from the other (Graham, 1999; Feiman-Nemser, 2001). Second, common issues and sources of difficulty for STs (e.g., time management and understanding students' mathematical thinking) could be discussed during training. This would provide mentors with an idea of what to expect of their STs and make STs aware of some of the challenges that they may face. Third, the US can be encouraged to serve as a mediator between the ST and CT if necessary. USs can help to create an open and communicative atmosphere at the beginning of the student-teaching experience and encourage STs and CTs to approach them with issues. In the case of Luis and Sheri, although both respected and appreciated the US Rhonda, neither seemed to consider the option of discussing their relationship tensions with her.

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PROMOTING STUDENTS' REFLECTIVE THINKING OF MULTIPLE QUANTIFICATIONS VIA THE MAYAN ACTIVITY

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This research explored principles in designing an instructional intervention to promote students' reflective thinking about multiple quantifications. We suggested the Mayan activity and examined to what extent it promoted students' reflective thinking about the independence of ε from N in the ε-N definition of the limit of a sequence. After the Mayan activity, students recognized the problem caused by describing ε as dependent on N and hence understood the significance of the order between them. Students also developed proper reasoning and tested their hypotheses by using the Mayan stonecutter story. These results indicate that the Mayan activity provides an example logically compatible with describing ε as dependent on N, tractable, and transferable.

Introduction

The aim of this study is to explore design principles to promote students' reflective thinking about the independence of ε from N in the ε-N definition of the limit of a sequence. Although the ε-N definition is a fundamental idea in studying advanced mathematics, many students encounter difficulty in understanding the ε-N definition (e.g., Cornu, 1991; Davis & Vinner, 1986; Edwards, 1997). Students' difficulty is partially caused by the lack of understanding of multiple quantifications in general. Dubinsky and Yiparaki (2000) referred to statements of the form "for all $x \in X$, there exists $y \in Y$ such that $P(x, y)$," where $P(x, y)$ is a statement about variables x and y, as AE statements; EA statements are those of the form "there exists $y \in Y$ such that for all $x \in X$, $P(x, y)$." Students in general tend to perceive EA statements as AE statements. For instance, for a statement "there is a flower that every bug likes," undergraduate students in introductory proof courses tended to interpret it as "for every bug, there is a flower that the bug likes" (Roh, 2011). When an EA statement includes mathematical contents, even more students tend to perceive the EA statement as an AE statement. In Dawkins and Roh's (2011) study, when given an EA statement "there exists a real number x such that for any $\varepsilon > 0$, $|x| < \varepsilon$," seventy-five percent of real analysis students initially perceived the EA statement as the AE statement "for any $\varepsilon > 0$, there exists a real number x such that $|x| < \varepsilon$."

Noticing the order of variables is important in understanding the logical structure of the ε-N definition. However, understanding *why* the order of variables matters is even more important. Literature about the teaching and learning of the ε-N definition (e.g., Burn, 2005; Cory & Giarofalo, 2011; Durand-Guerrier & Arsac, 2005; Roh, 2010a) pointed out the importance of "the dependence rule;" in the AE statement, the value of y may depend on the value of x. By the dependence rule, we may choose the value of N depending on the value of ε when proving the convergence of a sequence using "*for any* $\varepsilon > 0$, *there exists* $N \in \mathbb{N}$ *such that* for all $n > N$, $|a_n - L| < \varepsilon$." On the other hand, Roh (2009) highlighted the importance of "the independence rule;" in both AE and EA statements: the value of x must be independent of the value of y. For instance, when proving the convergence of a sequence, the value of ε *must* be independent of the value of N. Similarly, when disproving the convergence of a sequence using "*there exists* $\varepsilon > 0$ *such that for any* $N \in \mathbb{N}$, there exists $n > N$ such that $|a_n - L| \geq \varepsilon$," the value of ε *must* be independent of the value of N. It is because ε is specified prior to N in these statements. The independence rule seems much harder than the dependence rule for students to grasp in the ε-N definition. When encountering invalid arguments in which ε is selected dependently on N, students experienced difficulty in figuring out what was wrong (Roh, 2009, 2010b; Roh & Lee, in press). Therefore, in order to help students understand the ε-N definition, an instructional intervention is needed to promote students' perception of the independence of ε from N.

This study suggests the Mayan activity as an instructional intervention and gives an account of its effects on students' reflective thinking in the aforementioned context. The Mayan activity consists of three steps: the first is to evaluate an argument which, by selecting ε as dependent on N, leads to the conclusion that a well-known convergent sequence is divergent; the second is to evaluate the Mayan stonecutter story (see Figure 1), which is compatible with, but more tractable than the argument in the first step; and the third is to evaluate arguments, which are described by selecting ε as dependent on N but are more complicated than the argument in the first step.

The Mayan Stonecutter Story

One of the famous Mayan architectural techniques is to build a structure with stones. These stones were ground so smoothly that there was almost no gap between two stones. It was even hard to put a razor blade between them. One day a priest came to a craftsman to request smooth stones.

Craftsman: *No matter how small of a gap you request, I can make stones as flat as you request if you give me some time.*

Priest: *I do not believe you can do it. If I ask you to flatten stones within 0.01 mm, you won't be able to do it.*

Craftsman: *Give me 10 days, and you will receive stones as flat as within 0.01 mm.*

Ten days later, the craftsman made two stones so flat that the gap between them was within 0.01 *mm*. On the 11th day, the priest came to see the stones and argued that,

Priest: *These stones are not flat within 0.001 mm. What I actually need are stones as flat as within 0.001 mm.*

Craftsman: *Okay, if you give me 5 more days, I can make the stones as flat as within 0.001 mm.*

Five days later, the craftsman made the two stones so flat that the gap between them was within 0.001 *mm*. On the 16th day, the priest came to see the stones and argued that,

Priest: *But these stones are not flat within 0.0001 mm and I meant 0.0001 mm. You don't have that kind of skill, do you?*

If the priest keeps arguing this way, is the priest really fair showing that the craftsman does not have the ability to flatten stones within any margin of error?

Figure 1 **The Mayan Stonecutter Story**

In order to evaluate the validity of the priest's argument, it is necessary to examine the relations between the margin of error and the day in the priest's argument and in the craftsman's claim, respectively. On one hand, the margin of error in the craftsman's claim is independently chosen from the day. On the other hand, the priest attempted to argue the negation of the craftsman's claim to reveal the fallacy of the craftsman's claim. However, the priest committed a mistake choosing, actually requesting, a new margin of error *dependently* on the day in his argument even if the margin of error must have been chosen *independently* from the day in the negation of the craftsman's claim. We examined how the Mayan activity leveraged students' reflective thinking about the independence rule not only during the activity but also afterwards. Through this examination, we propose design principles for instructional interventions to promote students' reflective thinking.

Theoretical Perspective: Reflective Thinking

Reflective thinking has long been of interest and broadly studied in the area of the teaching and learning of mathematics. For instance, Dubinsky (1991), Piaget (1985), and Simon (2004) examined the role of reflective thinking in the abstraction of mathematical concepts, focusing on individuals' cognitive development. Cobb, Boufi, McClain, and Whitenack (1997), and Hershkowitz and Schwarz (1999) focused on how social interactions in classrooms influence reflective thinking. Also, mathematics teachers' ability to reflect on their own teaching was explored by Artzt (1999), Lee (2005), and Silverman and Thompson (2008).

In order to build an instructional intervention for the independence rule in the context of convergence, we adopted Dewey's theory of reflective thinking. According to Dewey (1933), when an individual encounters a situation contradictory to his or her knowledge or belief, he or she experiences perplexity or frustration, then resolves it through reflective thinking. Dewey regards reflective thinking to be a continuum of the following three situations: the *pre-reflective* situation, a situation experiencing perplexity, confusion, or doubts; the *post-reflective* situation, a situation in which such perplexity, confusion, or doubts are dispelled; and the *reflective* situation, a transitionary situation from the pre-reflective situation to the post-reflective situation. In addition, Dewey characterizes the reflective situation in terms of five phases, aspects, or functions as follows: suggestions, an intellectualization, hypotheses, reasoning, and testing by (mental) actions. *Suggestions* refer to the function of suggesting merely some primitive idea(s) inhibiting a direct action when one experiences perplexity, confusion, or doubts. *Intellectualization* means the phase that the perplexity or the difficulty is defined or rises to the surface of thought. In the intellectualization phase, one is not merely emotionally annoyed or embarrassed by the difficulty, but figures out what the difficulty is as an intellectual problem. Through intellectualization, the ideas proposed in the suggestion phase can be changed or modified into more appropriate ones to resolve the difficulty. Such enhanced ideas are called *guiding* suggestions or *hypotheses*. Whereas the phases of suggestions or hypotheses examine ideas via observations, the *reasoning* phase is the phase to postulate meaningful results without observations. *Testing by (mental) actions* is the phase to verify or to confirm the hypotheses.

Research Methodology

This research was classified as a one-on-one design experiment (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003) comprised of a series of teaching sessions. By the exploratory nature of the teaching sessions (Steffe & Thompson, 2000), this research attempted to address claims about how students develop their reflective thinking related to the contexts of the convergence of sequences. The design experiment was iterated four times between the fall semester of 2006 and the spring semester of 2010. The iterative nature of the design experiment allowed us frequent cycles of design issues, (re)design of instructional interventions, prediction of student learning, implementation of the instructional interventions, and analysis of actual student learning.

In this paper, we report two studies. Each study was conducted in an advanced calculus class at a public university in the USA in the fall semester of 2006 (Study 1) and in the spring semester of 2010 (Study 2). The subjects of the studies were undergraduate students who majored in either mathematics or education with a concentration in mathematics. Calculus and introductory proof courses were prerequisite for the advanced calculus course. The first author of this paper served as an instructor in both studies. In Study 1, the advanced calculus class consisted of a series of 50-minute sessions, three times per week for 15 weeks. In Study 2, the advanced calculus class consisted of a series of two regular sessions (75 minutes in length) and a recitation (50 minutes in length, mainly for cooperative proof writing) every week for 15 weeks.

Instruction in the classes mainly followed an inquiry approach, in which students often made and justified conjectures, or evaluated given arguments. In this manner, for the first four weeks, the students explored the real number system and its related properties, and reviewed elementary logic and proof structures. For the subsequent two weeks, the students studied the limit of a sequence and some relevant properties. In particular, they dealt with the ε-N definition of the convergence of sequences, and proofs of the convergence or divergence of given sequences by applying the ε-N definition. Also, discussions in the same manner followed dealing with the definition of Cauchy sequences and their relevant properties.

Throughout both studies, students' activities were video-recorded and were then transcribed. Also, we scanned students' paper-and-pencil work including homework and exams. Dewey's theory of reflective thinking (1933) was employed as a framework for our design experiment, in particular, in the steps of raising design issues, predicting and analyzing student learning, and (re)designing instructional interventions. After abstracting the students' language expressions, gestures, and diagrams from the transcripts of the video recordings, we first coded students' evaluations into one of the following: the pre-reflective situation, reflective situation, or post-reflective situation. Next, we coded the data into one of the aspects of the reflective situation, i.e., suggestions, an intellectualization, hypotheses, reasoning, or testing by (mental) action. Through such a coding process, we analyzed whether the students experienced or resolved perplexity and to what extent reflective thinking was enhanced.

Results from Study 1

Design Issue 1 with Statement 1 and Ben's Argument

In designing an instructional intervention to promote students' reflective thinking about the independence of ε from N, the first design issue centered around how to make students perceive that selecting ε dependently on N causes a problem. Accordingly, we designed Statement 1 and Ben's argument to Statement 1 (see Figure 2).

Statement 1: If a sequence $\{a_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ is a Cauchy sequence, then for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$, .

Ben's argument: Consider $a_n = 1/n$ for any $n \in \mathbb{N}$. Since the sequence $\{a_n\}_{n=1}^{\infty}$ is convergent to 0, it is a Cauchy sequence in $\mathbb{R}$. Let for all $N \in \mathbb{N}$. Let . Then $n > N$. But

$|a_n - a_{n+1}| = |a_{N+1} - a_{N+2}| = \left|\frac{1}{N+1} - \frac{1}{N+2}\right| = \frac{1}{(N+1)(N+2)} = \varepsilon \geq \varepsilon$. Therefore, Statement 1 is false. □

Figure 2 **Statement 1 and Ben's argument**

We implemented the evaluation of Statement 1 and Ben's argument after six weeks of the class in Study 1. Six students volunteered to participate in the study (Study 1) and worked in two small groups, each of which had three members. In this paper, we use pseudonyms Ellen, Simon, and Susan for the students in Group 1 and Max, Mimi, and Scott for the students in Group 2. In fact, Statement 1 is true, but Ben's argument is an invalid argument in which ε is determined dependently on N. We expected that if students do not recognize the problem of the dependence of ε on N in Ben's argument, they would be perplexed and initiate their reflective thinking to resolve the perplexity.

Reflective thinking about the independence of ε from N in Study 1. All students in Study 1 determined that Statement 1 should be true and as the reason for their determination, they pointed out that $n+1$ can be substituted for *m*. After the students' determination of the truth of Statement 1, the instructor introduced Ben's argument. All students seemed to recognize that Ben's argument was contradictory to Statement 1 and became perplexed, which indicates that they were in the pre-reflective situation. We further analyzed the extent of students' reflective thinking about the independence of ε from *N*.

In Group 1, Susan raised a question whether choosing $\varepsilon = 1/[(N+1)(N+2)]$ might be a problem in Ben's argument because "it [ε] can change […] as […] *N* changes." However, Ellen insisted, and Susan agreed, that ε can be chosen in such a manner. Later, Susan and Ellen accepted Ben's argument, and hence they reversed themselves over their determination of Statement 1. In fact, these two students suggested Statement 1 was false, and considered Ben's argument to be valid. Since they resolved their perplexity in the wrong direction, they could not perform any proper aspects of reflective thinking about the independence of ε from *N*. On the other hand, Simon was perplexed after reading Ben's argument since he believed Statement 1 to be true but failed to find any fallacy in Ben's argument. Simon said "I don't know. I'm stuck" after Susan said "I think it [Ben's argument] makes sense." Simon would not change his determination of the truth of Statement 1, but he failed to find any fallacy in Ben's argument. Hence, he just remained in the pre-reflective situation without performing any other aspects of reflective thinking about the independence of ε from *N* in evaluating Ben's argument.

All students in Group 2 also determined Statement 1 to be true and seemed to recognize that Ben's argument was contradictory to Statement 1. Max and Mimi thought that it was not a problem to determine ε dependently on *N* in Ben's argument, and consequently they accepted Ben's argument to be valid. Only Scott did not agree that ε should depend on *N*. Max and Mimi accepted Ben's argument, saying that they "can't find anything wrong with it." Furthermore, although Scott pointed out that choosing ε as dependent on *N* is a problem in Ben's argument, Max and Mimi had no doubt about the validity of Ben's argument, saying "Why not?" (Max) and "I'm not sure that's a problem" (Mimi). By accepting Ben's argument, they reversed their determination of Statement 1. Since Max and Mimi resolved their perplexity in the wrong direction, they could not perform any proper aspects of reflective thinking about the independence of ε from *N*. On the other hand, Scott was uncomfortable about Ben's argument, saying "I don't like it." He seemed to consider that Ben's argument was not valid, and developed such a primitive idea into a hypothesis that the dependence of ε on *N* in Ben's argument may cause a problem. However, as an intellectualization in the process of deriving the hypothesis, Scott just recalled that "I made that mistake 3 or 4 times on my homework." Also, he could not perform any other proper reasoning aspect of the reflective thinking about the independence of ε from *N*, and consequently he failed to resolve his perplexity.

Results from Study 2

Design Issue 2

We can expect that when two conflicting arguments are suggested, students can recognize that at least one of the arguments is false. However, it is not assured that they will select the true statement between the two conflicting arguments. In Study 1, many students accepted Ben's argument contradictory to Statement 1 even though they determined Statement 1 to be true in the previous step. Although their reasoning was proper in deriving the truth of Statement 1, they reversed themselves about the truth of Statement 1 once they could not find that the fallacy of Ben's argument is induced by describing ε dependently on *N*. Therefore, in order to properly promote students' reflective thinking about the independence of ε from N, a new design issue centered around a way to exclude the possibility that students might regard a faulty argument described by choosing ε dependent on N to be true. Toward this end, we designed the Mayan activity. After five weeks of the class in Study 2, the instructor implemented the Mayan activity as an instructional intervention. Eleven students volunteered to participate, working in three groups, each of which had three or four members. In this paper, we focus on one group of four students: Amy, Elise, John, and Matt (all student names are pseudonyms).

The Mayan activity – Step 1: Evaluation of Sam and Bill's Arguments. In Step 1 of the Mayan activity, the students evaluated a pair of conflicting arguments about the convergence of a well-known sequence $\{1/n\}_{n=1}^{\infty}$ shown in Figure 3:

Sam's argument. For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $N\varepsilon > 1$. Then for all $n > N$,

Therefore, $\{1/n\}_{n=1}^{\infty}$ converges to 0. □

Bill's argument. For all $N \in \mathbb{N}$, choose . Then $\varepsilon > 0$ since $N > 0$. Let .

Then $n > N$. Also, $\left|\frac{1}{n} - 0\right| = \frac{1}{n} = \frac{1}{N+1} > \frac{1}{N+2} = \varepsilon$. Therefore, $\{1/n\}_{n=1}^{\infty}$ does not converge to 0. □

Figure 3 **The Mayan activity – Step 1: Sam's and Bill's arguments**

It was observed that all students in this group instantly perceived Bill's argument to be invalid. Amy and John were certain of the validity of Sam's argument and recognized that Bill's argument was contradictory to Sam's argument. In determining the validity of Sam's argument, they reflected on their previous experience that they had proved the convergence of $\{1/n\}_{n=1}^{\infty}$. Elise discerned that Bill's argument attempted to negate, but not properly, the definition of the convergence of sequences. However, she was unable to articulate what was amiss with Bill's argument. On the other hand, Matt perceived that a negation was attempted in Bill's argument, and pointed out that the problem with Bill's argument was that "the quantifiers were all negated […] but out of order." In addition, Matt understood that such an incorrect negation, by reversing the order of variables, allows ε to be described dependently on N. Through the conversation with Matt, John agreed that a negation was attempted in Bill's argument. However, John was still questioning "why does the order matter?" even though Matt pointed out that a problem bears on the order of the variables.

Matt: Is the main error the fact that you know ε is dependent on N? […] So if we have to pick one major flaw in logic, would that be it in this one? […]

Elise: […] He [Bill] is trying to negate this one, and he just does it incorrectly. He just moves it around. … He's trying to negate it. … Just that's what I see it … because the negation just made me sit there and go what? […]

Matt: So then how did he [Bill] not correctly negate it? I guess that's part of the question here.

John: Yeah, that's what I kind of don't understand here. Why does the order matter?

Matt pointed out that the logical problem in Bill's argument was to choose ε dependently on N. However, other students in this group did not seem to understand his assertion. In particular, Amy did not participate in the group discussion except arguing that Bill's argument is incorrect. Elise and John perceived that a negation was attempted in Bill's argument, but they did not understand why the reversal of the order between ε and N would be a problem in Bill's argument.

Reflective thinking about the independence of ε from N in Step 1. We could observe that the students in Study 2 immediately noticed that Sam's argument was valid and Bill's argument was contradictory to Sam's argument; thereafter, they suggested Bill's argument was invalid. From that moment, the students' reflective thinking about the independence of ε from N seemed beyond the pre-reflective situation, and was shifted to the reflective situation. On the other hand, after Amy suggested that Bill's argument was wrong, she did not perform any other aspects of the reflective situation. Unlike the case of Amy, Matt intellectualized the problem of Bill's argument, pointing out that a negation was attempted. Further, he suggested a hypothesis that the problem with Bill's argument bore on the order of quantifiers in the attempted negation, which was in fact reversed from that in the correct negation of the ε-N definition. Also, he reasoned that reversing the order of quantifiers in such a way allows ε to be described dependently on N and that Bill's argument could be unfolded due to the dependence of ε on N. Through such reasoning, Matt was able to resolve his perplexity caused by Bill's argument. Also, Elise and John explored intellectually the problem of Bill's argument, taking note that a negation was attempted. However, when Matt asked, "How did he [Bill] not correctly negate it? I guess that's part of the question here," John responded, "That's what I kind of don't understand here," and asked back to the group, "Why does the order matter?" Unlike the case of Matt, Elise and John failed to develop proper hypotheses and reasoning with regard to the independence of ε from N. Consequently, Amy, Elise, and John could not resolve their perplexity, which was caused by Bill's argument.

Design Issue 3

Results from Step 1 of the Mayan activity indicate that the evaluation step of Sam's and Bill's arguments provides a way to exclude the possibility that students might accept a faulty argument described by selecting ε dependently on N. Considering students' change in their determination of the validity of Statement 1 in Study 1, the design issue raised from Study 1 was successfully resolved through Step 1 of the Mayan activity. Still, only the task of Step 1 of the Mayan activity did not enable students, except in the case of Matt, to resolve their perplexity caused by Bill's argument. In order to resolve students' perplexity, a new design issue centered around a way to promote proper hypotheses and reasoning, hence eventually to develop students' reflective thinking about the independence of ε from N. Thus, we designed and implemented the Mayan stonecutter story (see Figure 1) in Step 2. The students presented their individual evaluation and debated whether the priest was fair or not in group discussion.

The Mayan Activity – Step 2: The Priest's Argument in the Stonecutter Story. During group discussion, the students immediately insisted that the priest was unfair. However, in explaining why the priest was unfair, they presented different bases to each other. John insisted that the priest did not give the craftsman enough time to flatten stones as requested, and hence the priest was unfair. Amy misunderstood the craftsman's claim to be "if the craftsman has some time, then he can make *some* very flat stone." In fact, on the basis that the craftsman made stones flat within *some* margin of error, she decided that the craftsman had already fulfilled his requirement. For this reason, Amy concluded that the priest's argument was not fair. Elise paid attention to the difference between the logical structures of the craftsman's claim and the priest's argument. Pointing out that the craftsman's claim is structured by "for all, there exists" whereas the priest switched it into "there exists, for all," she concluded that the priest was not fair to the craftsman. It was noted that John, Amy, and Elise compared the priest's argument with the craftsman's claim to explain the priest's fallacy. On the other hand, Matt tried to compare a negation of the craftsman's claim with the priest's argument in his attempt to explain the priest's fallacy. Matt seemed to consider that in order to show the unfairness of the priest, it is sufficient to prove that the priest's argument is equal to the negation of the craftsman's claim. He first represented the craftsman's claim as a quantified one in such a way that "for all distances, there exists a time for which the craftsman can get it done." Next, he attempted to describe the negation of his representation of the craftsman's claim and to compare it with the quantified one of the priest's argument. However, Matt fell into a logical mistake of stating the negation of the craftsman's claim to be "in a certain time, there exists a distance that the craftsman can't get to."

Through the subsequent group discussion, Matt and Elise focused on the logical structure of the priest's argument compared to that of the craftsman's claim. When Matt tried to figure out the logical meaning of the priest's argument, Elise asked him to repeat his logical interpretation of the craftsman's claim and suggested a negation of his logical interpretation in terms of a quantified statement.

Elise: What was your original one? The original like regular quantifiers, not negated? Or, what was the statement you had?

Matt: It was for all – um distances or – you know – gaps, there exists an amount of time for which the craftsman can get that distance.

Elise: Yeah, […] I think […] your negation is that there exists a distance between them such that for all time you give them – for all there – you can't make it or that.

Matt: Yeah. […] Well, and maybe he [the priest] is flipping them. He is saying for all time, there is – the existence of – a distance that you can't get to.

Elise: Yeah.

Matt: Well, flipping those is a big deal! (laughs)

Elise: (laughs)

Matt: Because if you're saying – you know – there exists a distance that no matter how much time he has, he can't get to. That's way different from saying – you know – for any times, there – you know, at this time, there exists – you know – one thing that doesn't work. Well, that doesn't tell us a whole lot about the negation of the original statement – you know –

Elise: Yeah.

Matt: but tells us what happens at that one time. And then – but then the distance is dependent on whatever time. If we're looking at any time, then it [distance] becomes dependent on that time. That distance doesn't work, so at a different time, other distance doesn't work.

Elise: Yeah.

Matt: So that becomes – it seems a big deal at that point!

Elise could suggest the correct negation of the craftsman's claim. Taking Elise's suggestion into account, Matt could recognize that the order of quantified variables, distance and time, in the priest's argument is reversed from that in the negation of the craftsman's claim. Furthermore, Elise and Matt reasoned that the reversed order of quantified variables permitted the dependence of 'distance' on 'time,' hence caused the logical problem in the priest's argument. Later, Matt pointed out that the order of quantifiers was flipped in the priest's argument, and hence the priest's argument did not give a disproof of the craftsman's claim. After listening to Matt's explanation, John recognized that he had not taken into account the dependent relationship between the variables, 'time' and 'distance.' He tried to articulate the priest's argument in terms of the distance requested and the time allowed to the craftsman. In particular, John pointed out that "some time" means the time the craftsman suggested for the previously requested distance, but the priest insisted that the craftsman did not make the newly requested distance at that time. Elise suggested further a logical approach to the problem of the priest's argument. She pointed out that for the priest to negate the craftsman's claim, the priest would have to show that "there exists a distance that no matter how much time the craftsman has, he [the craftsman] cannot make that distance," but instead the priest treated the situation as "no matter how much time the craftsman has, there was a gap that he hadn't reached yet." Other students in the group agreed upon Elise's assertion. Later, the students even compared Bill's argument with the priest's argument in terms of the logical structure. In particular, they responded that the two arguments contain a common logical problem, reversing the order of quantifiers from the desired one to draw each conclusion.

Reflective thinking about the independence of ε from N in Step 2. In Step 2, the students suggested immediately the unfairness of the priest. Also, they perceived that the priest attempted to disprove the craftsman's claim. In particular, Elise and Matt intellectualized that in order to disprove the craftsman's claim, the priest must have proven the negation of the craftsman's claim. They then compared the negation of the craftsman's claim with the priest's argument in terms of quantified statements. Afterwards, they came to recognize that the logical order between the margin of error and time in the priest's argument was reversed from that in the negation of the craftsman's claim. Elise and Matt hypothesized that the reversal of the variables in the priest's argument from that in the craftsman's claim was the crucial problem. They further reasoned that in attempting to disprove the craftsman's claim, the reversed order of the quantified variables in the priest's argument permitted the dependence of 'distance' on 'time', hence the priest's argument was irrelevant to the negation of the craftsman's claim. Through the group discussion, John revised his primitive idea and reasoned that the problem of the priest's argument was not to give sufficient time to the craftsman. In particular, John pointed out that the priest just gave the craftsman the time corresponding to the previously requested distance, but requested a new distance without giving the time corresponding to the newly requested one. Like this, Elise, John, and Matt resolved their perplexity due to the priest's argument by developing proper hypotheses and through proper reasoning. On the other hand, Amy's suggestion of the priests' unfairness was based on her misunderstanding of the craftsman's claim as "if the craftsman has some time, then he can make *some* very flat stone." In her misunderstanding of the craftsman's claim, Amy improperly intellectualized the problem of the priest's argument by regarding *any* margin of error as *some* margin of error in the craftsman's claim. Nonetheless, considering her lack of participation in Step 1, Amy's reflective thinking was substantially enhanced through the activity with the Mayan stonecutter story in Step 2.

By comparing the priest's argument with Bill's argument, the students came to hypothesize the problem with Bill's argument properly, that is, the order between ε and N in Bill's argument is reversed from that in the negation of the ε-N definition. The students also reasoned that such a reversal of the order between the variables ε and N leads to an erroneous conclusion from Bill's argument. Consequently, reflecting on their thinking process about the priest's argument, the students could resolve their perplexity caused by the problem with Bill's argument.

Design Issue 4

The results from Step 2 indicate that through the activity with the Mayan stonecutter story, students could enhance their reflective thinking about the independence of ε from N and hence resolve their perplexity due to the priest's argument in the story. In fact, they could promote proper hypotheses and reasoning through the activity with the Mayan stonecutter story; consequently, the design issue newly raised from Step 1 seemed to be achieved in Step 2.
We now consider a fourth design issue: how students utilize their reflective thinking

when evaluating different arguments with the same fallacy of determining ε dependently on N as in the priest's argument.

The Mayan Activity – Step 3: Evaluation of Jane's Argument. Immediately after Step 2 of the Mayan activity, the class dealt with some concepts and results relative to sequences including Cauchy sequences. In the subsequent week, we asked the students to evaluate Jane's argument for Question 6 (see Figure 4). Jane's argument was individually evaluated by a paper-and-pencil test and then by follow-up interviews outside of class, where the interviwer was the first author of this paper.

Question 6. Consider the sequence $\{a_n\}_{n=1}^{\infty}$ defined by $a_n = 1/2^n$ for any $N \in \mathbb{N}$. Jane was asked to prove or disprove that the sequence $\{a_n\}_{n=1}^{\infty}$ is convergent to 0. Evaluate if Jane's argument below can be regarded as a legitimate proof. Explain how you can tell.

Jane's argument. For all $N \in \mathbb{N}$, choose $\varepsilon = \frac{1}{2^{N+2}}$. Let . Then $n > N$. Also $|a_n - 0| = \left|\frac{1}{2^n} - 0\right| = \frac{1}{2^n} = \frac{1}{2^{N+1}} > \frac{1}{2^{N+2}} = \varepsilon$. Therefore, the sequence $\{a_n\}_{n=1}^{\infty}$ does not converge to 0. □

Figure 4 **The Mayan activity – Step 3: Question 6 and Jane's argument**

Jane's argument draws an erroneous conclusion by selecting ε depending on N as in Bill's argument in Step 1 of the Mayan activity. Bill's argument, the priest's argument, and Jane's argument all include a logically identical error in the sense that ε (the margin of error) is selected dependently on N (the day). Comparing students' evaluation of Jane's argument with their evaluation of Bill's argument and the priest's argument, we explored how students' reflective thinking about the independence of ε from N has been promoted after the implementation of the Mayan stonecutter story.

In the paper-and-pencil test, all students determined Jane's argument to be invalid. In the follow-up interview, Amy responded that Jane's argument attemped to prove the negation of the definition of convergence. Amy was also aware that the order between variables, ε and N, in Jane's argument was reversed from that in the negation of the definition of convergence. She pointed out that the problem in Jane's argument was to select ε dependently on N, although ε must have been chosen independently from N.

Amy: She [Jane] wants to prove $\{a_n\}_{n=1}^{\infty}$ is not convergent to 0. So she uses the negation – negation of the definition of convergence. She had to do there exists ε – which is greater than 0 – for all $N \in \mathbb{N}$, there exists n – which is greater than N – such that […] $|a_n - L| \geq \varepsilon$. But what she actually did was that she changed the order of the negation of definition. […] she [Jane] has to choose ε first and that condition [$|a_n - 0| \geq \varepsilon$] is satisfied for all $N \in \mathbb{N}$. But in here she – she gave the condition of the N first and then choose the ε. So – um now, the ε is dependent on the N. But actually she had to choose ε first independently from the N. She had to pick for ε whatever the N is in the natural numbers. But what she actually did is she gave the condition first about the N, and picked for the ε regarding to the N. So, we have the ε which is dependent on the N.

John compared Jane's argument with the negation of the definition of convergence. He noted that unlike the negation of the ε-N definition, Jane's argument selected ε dependently on N in such a way that $\varepsilon = 1/2^{N+2}$. For this reason, John regarded Jane's argument not to be equivalent to the negation of the ε-N definition.

John: The main thing is to prove that, you know, something is convergent. We need to prove that for all $\varepsilon > 0$, there exists N ↗ ☠ such that for all $n > N$, um $|a_n - L| < \varepsilon$. […] Now, she is trying to, you know, to negate this statement here, but she didn't. The correct negation of that is saying that there exists an $\varepsilon > 0$ where for all N ↗ ☠, there exists an $n > N$ such that $a_n - 0 \geq \varepsilon$, all right? This is really her intent. But instead she proved for all N ↗ ☠ N∈N, there exists an $\varepsilon > 0$. These two statements are not equivalent statements.

Roh: Why not?

John: […] It's all based on the dependence of it. So in this— in Jane's argument, you know – she's putting – making ε – you know – dependent on – dependent on N.

Roh: How can you tell?

John: Because she's setting $\varepsilon = 1/2^{N+2}$, so ε is based off of – you know – the N. But in this situation here, ε can be anything. […] ε isn't dependent on anything in particular. […]

Elise and Matt believed that Jane's argument attempted to prove the negation of the convergence but was not properly developed in negating the ε-N definition. Elise pointed out that considering the order of variables, ε depends on N in Jane's argument because ε followed N. For this reason, Elise decided that such a reversal of the order between ε and N caused the problem with Jane's argument. On the other hand, Matt noted that ε in Jane's argument was chosen dependently on N even though Jane's argument must have started off by choosing an ε before N. Consequently, he reasoned that the problem with Jane's argument followed from such a dependence of ε on N. Recalling the priest's argument of the Mayan stonecutter story and comparing it with Jane's argument, Elise pointed out that the order between quantifiers in the priest's argument was reversed in the same way as in Jane's argument. For this reason, she regarded the logical problem with Jane's argument to be the same as that of the priest's argument. Matt also compared Jane's argument with the priest's argument. In his interview, he tried to construe Jane's argument in terms of components in the priest's argument. Identifying ε and N in Jane's argument to the gap in the stones and the time, respectively, Matt pointed out that Jane must have shown the existence of the gap that cannot be achieved no matter how much time had passed; however, Jane just gave a gap that depends on time. Considering his account, we can find that Matt regarded the logical structure of Jane's argument to be the same as that of the priest's argument.

Reflective thinking about the independence of ε from N in Step 3. In Step 3, Amy intellectualized that in Jane's argument, Jane attempted to negate the ε-N definition, and Amy gave a hypothesis that the order of variables ε and N is reversed in the argument. From this hypothesis, Amy reasoned that the reversal of the order of variables ε and N allowed the dependence of ε on N. John also intellectualized that Jane's argument attempted to negate the ε-N definition. Furthermore, he pointed out the dependence of ε on N as a reason why Jane's argument was incorrect as the negation of the ε-N definition. Like Amy and John, Elise and Matt also intellectualized that Jane considered the negation of the ε-N definition. They hypothesized that the order of variables ε and N was reversed by choosing ε after N. From this hypothesis, they reasoned that the dependence of ε on N could be allowed from the reversed order between ε and N. In particular, Elise and Matt could be confident of their reasoning by testing (by mental action) that to reverse the order of the variables in Jane's argument was the same logical fallacy committed in the priest's argument. Consequently, the students all could resolve their perplexity from Jane's argument.

One of the remarkable results from Step 3 was that the students were recalling the

Mayan stonecutter story when evaluating Jane's argument. It should be noted that in Step 3 the interviewer did not guide the students to recall the Mayan stonecutter story, and did not even mention it. Nevertheless, the students compared the logical structures of Jane's argument with that of the priest's argument in the Mayan stonecutter story. Also, through the comparison, students pointed out that describing ε dependently on N is a problem in Jane's argument, and reasoned that such a problem draws an erroneous conclusion as in the priest's argument. By reflecting on the Mayan stonecutter story in their evaluation of Jane's argument, the students could be confident of their evaluation.

Discussions

In this study, we designed an instructional intervention, the Mayan activity, with the intention of enhancing students' reflective thinking about the independence of ε from N. The Mayan activity includes two pivotal instructional components: first, the Mayan activity has a means to cause students' perplexity related to the independence of ε from N. To be precise, it makes students experience perplexity through evaluating some arguments containing erroneous conclusions by describing ε dependently on N. Second, the Mayan activity has a device that enables students to experience first-hand the meaning of the independence of ε from N. In fact, it introduces the Mayan stonecutter story from which students concretely realize the problem of describing ε dependently on N. In the following, we will discuss in detail how the Mayan activity plays a role in enhancing students' reflective thinking about the independence of ε from N.

Necessity of a Convincing Argument

The Mayan activity asks in Step 1 to evaluate Bill's argument, which draws an erroneous conclusion that the sequence $\{1/n\}_{n=1}^{\infty}$ does not converge to 0, by describing ε dependently on N. The rationale for adopting Bill's argument is that almost all students in advanced calculus are confident that the sequence $\{1/n\}_{n=1}^{\infty}$ converges to 0, hence they have no doubt that the conclusion of Bill's argument is not true. The necessity of such an argument was raised from Study 1. In Study 1, Ben's argument was suggested to students instead of Bill's argument. Similar to Bill's argument, Ben's argument was also developed by describing ε dependently on N and drew an erroneous conclusion that Statement 1 was false. However, students' response in evaluating Ben's argument in Study 1 showed a different aspect from that of students in evaluating Bill's argument in Study 2. Just before evaluating Ben's argument in Study 1, students concluded that Statement 1 was true through proper reasoning. However, once failing to figure out the problem in Ben's argument, they changed their determination of the truth of Statement 1. This change of the students' decision about Statement 1 seemed to occur because they failed to perceive that describing ε dependently on N causes a problem in Ben's argument and they could not be confident of their decision about Statement 1. Therefore, in order that students perceive the problem of describing ε dependently on N and develop reflective thinking about the independence of ε from N, it was necessary to suggest some statements, instead of Statement 1, of which students could be confident. For this reason, arguments about the convergence of the sequence $\{1/n\}_{n=1}^{\infty}$ were given to students in Step 1 of the Mayan activity.

On the other hand, when being asked to evaluate two arguments that conflict with each other, students will enter the pre-reflective situation as long as they notice such a conflict. However, whether or not students progress in their reflective thinking in the direction of resolving their perplexity depends on their confidence about the truth of given arguments. Students' activities of evaluating Ben's argument in Study 1 and Bill's argument in Study 2 are a good example: in the former, students were not confident of the truth of Statement

1; hence, they could not develop properly their reflective thinking, whereas in the latter case they could. Therefore, when designing an instructional intervention to enhance students' reflective thinking, it is necessary to consider statements that the students can be confident of their truth, such as the convergence of the sequence $\{1/n\}_{n=1}^{\infty}$ in Step 1 of the Mayan activity. Through such an instructional intervention, we can help students properly develop their reflective thinking.

Necessity of Logically Compatible, Tractable, and Transferable Examples

The Mayan activity, in Step 2, provides the Mayan stonecutter story and asks students to evaluate the priest's argument in the story. The Mayan stonecutter story consists of the craftsman's claim and the priest's argument. Such a structure of the story is *logically compatible* with that of Step 1 consisting of Sam's and Bill's arguments, being contradictory to each other (see Table 1). The priest's argument also draws a logically erroneous conclusion in a similar way by describing ε dependently on N in Bill's argument. The priest's argument just differs from Bill's argument in the sense that it uses gaps between stones and days instead of the variables ε and N.

Sam's argument	*Craftsman's claim*
For any $\varepsilon > 0$	No matter how small of a gap (you request)
there exists $N \in \mathbb{N}$ (such that)	some time (I can make stones as flat as you request)
Bill's argument	*Priest's argument*
For all $N \in \mathbb{N}$	Ten days later… five more days later (this means any time)
choose (exist) $\varepsilon = 1/(N+2)$ (such that $1/(N+1) > \varepsilon$)	the stones that craftsman made are not flat within 0.001 *mm*… not flat within 0.0001 *mm* (this means the existence of some gap)

Table 8 **Logical compatibility between statements in the Mayan activity**

The priest's argument is *logically compatible* with but is more *tractable* than Bill's argument so that students easily understand the logical structure and perceive the logical fallacy in the argument. The necessity of such a tractable example was shown by results from Step 1 of the Mayan activity. In Step 1, the students could perceive that Bill's argument induced an erroneous conclusion; nonetheless, they encountered a difficulty in explaining the reason why such an erroneous conclusion could be derived. Their response to Bill's argument is due to their lack of experience with problematic situations in which an argument described by choosing ε dependently on N implies an illogical conclusion. Therefore, in order to enable students to compensate for their lack of experience, hence to overcome their difficulty, it is necessary to force them to perceive that an argument with ε selected dependently on N leads to an unreasonable conclusion, and to confirm that the illogicality of the argument comes from the dependence of ε on N. Thus, the Mayan stonecutter story was designed to provide a tractable situation related to the problem of the dependence of ε on N.

Furthermore, the stonecutter story is a *transferable* example in the sense that students can properly link the variables (gaps between stones and days) in the priest's argument to the variables (ε and N) in Bill's argument. NCTM (2000) and NRC (2001) suggest that mathematics teachers use concrete or colloquial examples to help students understand an abstract mathematical statement. A reason to use such examples is because these examples are logically compatible with the abstract statement and are more understandable. However, for students to properly understand the meaning of the abstract statement through given examples, the examples must be designed to be *transferable*, that is, students by themselves can make a correspondence between key components in the abstract statement and those in the examples, and can perceive the problem of the abstract statement by projecting the problem of the examples onto the abstract statement. In line with this viewpoint, the Mayan stonecutter story is designed to be *transferable*. In fact, the students in Step 2 matched the gaps between the stones and days to ε and N, respectively. Also the students perceived that the problem resulting from the dependence of ε on N in Bill's argument is the same as the logical problem in the priest's argument.

Design Principles

Based on the results from Study 1 and Study 2, we formulated two design principles as testable predictions for a new instructional intervention aimed at enhancing reflective thinking:

(1) *Conflicting but convincing arguments*: Students will notice when two arguments given in the task are in conflict and will be confident of the truth or falsehood of the arguments.

(2) *Logically compatible, tractable, and transferable examples*: The logical structures of the examples are compatible with those of the arguments in the first design principle. Also, the examples are tractable in the sense that students will recognize a logical fallacy and reason that erroneous conclusions might be drawn from the fallacy. Furthermore, the students themselves will identify key logical components in the examples with those in the original arguments and reflect the logical problem of the original argument onto that of the examples.

This paper reports the Mayan activity as an example of a successful instructional intervention to promote students' reflective thinking about the independence of ε from N. As in the case of the Mayan activity, an instructional intervention needs to comprise two steps: the first step is to cause perplexity, confusion, or doubt accompanied with confidence of the truth or falsehood of the given arguments, and the succeeding step is to provide an example, which is not only logically compatible and tractable but also transferable. An example designed in such a way can help students understand the meaning of the logical structure in the targeted context, and retain their understanding. Consequently, the students can utilize their understanding to comprehend the logical meaning of different arguments, which have the same logical structure as the example. Therefore, an instructional intervention designed to enhance reflective thinking should deploy conflicting but convincing arguments and develop examples to be logically compatible, tractable, and transferable.

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HOW MATHEMATICIANS USE DIAGRAMS TO CONSTRUCT PROOFS

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Although some researchers argue that diagrams can aid undergraduates' proof constructions, most undergraduates have difficulty translating a visual argument to a formal one. The processes by which undergraduates construct proofs based on visual arguments are poorly understood. We investigate this issue by presenting eight mathematicians with a mathematical task that invites the construction of a diagram and examine how they used this diagram to produce a formal proof. The main findings from the paper were that it was not trivial for mathematicians to translate an intuitive argument into a formal proof and mathematicians used diagrams for multiple purposes, including noticing mathematical properties, verifying logical deductions, representing ideas or assertions, and suggesting proof approaches.

Key words: Mathematicians, informal diagrams, proof construction

1. Introduction

1.1. Student difficulty with proof

Proving is central to mathematical practice. Hence, a primary goal of advanced undergraduate mathematics courses is to improve students' abilities to construct proofs. Yet numerous studies have documented undergraduates' frequent inability to construct mathematical proofs (e.g., Alcock & Weber, 2010a; Hart, 1994; Moore, 1994; Recio & Godino, 2001; Weber, 2001; Weber & Alcock, 2004). Research in this area has generally focused on particular difficulties that undergraduates face when writing proofs, such as having a poor understanding of advanced mathematical concepts (Hart, 1994; Moore, 1994), lack of proving strategies (Weber, 2001), and not knowing where to begin when asked to write a proof (Moore, 1994). However, research on how students can or should engage successfully in proof construction tasks has been sparse. In this paper, we examine in detail one specific suggestion for proof construction from the mathematics education literature—using a diagram as a basis for constructing a formal proof. We discuss the literature on this in the next section.

1.2 Diagrams in mathematics

Diagrams are viewed by mathematicians and mathematics educators alike as an integral component of doing and understanding mathematics (e.g., Hadamard, 1945; Stylianou, 2002). A substantial benefit that diagrams afford is they allow the problem solver access to view, compare, and integrate simultaneous pieces of information with little cognitive effort, something that is difficult when the same information is presented symbolically and sequentially (Eisenberg & Dreyfus, 1991; Larkin & Simon, 1987) – while making certain properties of mathematical concepts transparently obvious that would ordinarily be difficult to discern with non-visual representations of this concept (e.g., Piez & Voxman, 1996). Further, diagrams are often used to provide novel and more accessible explanations for mathematical phenomena or highlight aesthetics that are less accessible through symbols and logic (e.g., Hersh, 1993) as is illustrated by Nelsen's (1993) *Proofs Without Words*. Mathematicians' self-reports reveal that diagrams play a significant role in their mathematical work (e.g., Hadamard, 1945). Stylianou's (2002) investigation into mathematicians' problem solving revealed four purposes for diagrams: inferring consequences, elaborating upon mathematical ideas, creating sub-goals, and encouraging metacognitive reasoning.

Drawing diagrams is commonly cited as a heuristic for mathematical problem solving and reasoning that students should engage in (e.g., Polya, 1957; NCTM, 2000; Schoenfeld, 1985). However, despite the promise of this recommendation, the literature suggests that at least several pedagogical issues that must be addressed. First, researchers have noted that students are often reluctant to use diagrams, even for problems where their use might be highly productive (e.g., Eisenberg & Dreyfus, 1991; Piez & Voxman, 1996; Stylianou & Silver, 2004). Second, Presmeg (1986) found little correlation between high school students' propensity to visualize and their mathematical performance. Indeed, the strongest students in Presmeg's sample rarely used diagrams. Finally, Schoenfeld (1985) demonstrated that the process of implementing this type of heuristic is surprisingly complex; he argued that more explicit instruction on using diagrams (e.g., what to read from diagrams) might be necessary for students to employ this heuristic effectively.

1.3. Diagrams and formal proofs

Instructors of advanced undergraduate courses often ask their students to produce formal proofs. These proofs are expected to begin with definitions, axioms, and appropriate assumptions and proceed deductively to reach a desired conclusion, often while employing logical syntax. While informal representations of the involved concepts, including diagrams, may be included with the presentation of the proof, their role is expected to be ancillary, serving as a comprehension aid for the formal argument. The inferences within the proof are expected to be based on deductive logic, not the appearance of the diagram.

Both mathematicians and researchers in mathematics education emphasize that although formal proof results from proving, the process of proving can be, and frequently is, based on informal argumentation – often involving visual reasoning (e.g., Alcock, 2010; Burton, 2004; Boero, 2007; Dreyfus, 1991; Thurston, 1994; Raman, 2003; Schoenfeld, 1991; Weber & Alcock, 2004). Consequently, some researchers have suggested that undergraduates be encouraged to base the proofs that they construct on informal reasoning (Boero, 2007)—in particular by utilizing diagrammatic reasoning (Alcock 2010; Gibson, 1998; Raman, 2003; Weber & Alcock, 2004).

The literature supports this suggestion theoretically and empirically. Theoretically, researchers have noted benefits offered by diagrammatic reasoning in formal mathematical settings. Diagrams can, for instance, reify formal mathematical concepts to make them meaningful for students (Alcock & Simpson, 2004; Gibson, 1998), allow students to quickly draw useful inferences (Alcock & Simpson, 2004; Gibson, 1998), and help students overcome the impasses they reach when writing a proof (Gibson, 1998; Weber & Alcock, 2004). Proofs whose construction is based on diagrams may be more meaningful for students (Raman, 2003) and provide them with more learning opportunities (Weber, 2005) than proofs based on formal deduction alone. Empirically, there are case studies documenting undergraduates' success in producing proofs based on diagrams (Alcock & Weber, 2010a; Gibson, 1998), demonstrating that a diagrammatic approach to writing proofs can be useful for some students in some situations. Furthermore, there is ample empirical evidence that mathematicians frequently base their proofs on diagrams (e.g., Burton, 2004; Hadamard, 1945; Thurston, 1994). If students could engage in similar processes when they write proofs, their proving and reasoning processes might resemble more closely those used by mathematicians.

1. 4. Limitations to the effectiveness of basing proofs on diagrams

Although there is good reason to believe that undergraduates should base some proofs on diagrams, there are several factors that may limit the effectiveness of this pedagogical suggestion.

Many researchers have delineated theoretical difficulties that might prevent students from successfully basing a proof on a diagram. For instance, Duval (2007) notes that the structure of an visual argument (or any informal argument) differs significantly from the structure of a formal proof; he cautions researchers not to underestimate the cognitive complexity of the task of determining the statuses of assertions within a proof (i.e., is the assertion an assumption, an axiom, an accepted fact, or a deduction?) and organizing the chain of deductions appropriately. This can be particularly difficult when basing a proof on a diagram as each assumption may appear equally obvious, making it hard for students to distinguish what can be assumed and what needs to be shown (cf. Alcock & Simpson, 2004).

There are theoretical difficulties beyond cognitive complexity with using diagrams effectively in proof construction. Alcock and Simpson (2004) note that diagrams sometimes provide students with unwarranted confidence in the conjectures that they make, leading them to believe things that are not true or feeling that a proof of a true statement is superfluous. Alcock and Weber (2010a, 2010b) argue that students often have inaccurate representations of mathematical concepts or fail to see connections between their visual representations of a mathematical concept and the concept's formal definition, making it difficult or impossible to base formal proofs of those diagrams.

Moreover, although some published case studies in mathematics education illustrate how some undergraduates are able to use insight gleaned from diagrams to construct proofs, other case studies illustrate how undergraduates often are not able to do so. For instance, Alcock and Weber (2010b) asked eleven mathematics majors to prove that an increasing function does not have a global maximum. The four students who drew diagrams of a generic increasing function each instantly gained conviction that the statement to be proven was true but none of these students could not see how to begin writing a proof of this claim. Finally, several small-scale studies have found little to no correlation between undergraduates' propensity to use diagrams and their success in proof-writing in advanced mathematics (e.g., Alcock & Simpson, 2004, 2005). These findings call to question the claim that the unqualified suggestion to increase diagram usage will improve students' abilities to construct proofs. If we would like the benefits of diagrams that mathematicians exploit to be accessed by students, then further work must be done to determine the nature of how and when mathematicians use diagrams.

1. 5. Research questions

We find promise in the suggestion that students learn to use diagrams. However, we contend that students will need guidance in order to implement this suggestion effectively. Reflecting on her own teaching, Alcock (2010) writes:

> "Diagrams can provide insight, but it is not always easy for students to make detailed links between what is in the diagram and what is in a formal proof. This means that the step between seeing that a result must be true and proving it can seem insurmountable. Through my small-class teaching, I have also learned that students often find it difficult to draw a diagram based on verbal and algebraic information. As a result, I now spend more time walking students through the process of drawing diagrams" (p. 232-233).

We share Alcock's goal of trying to make explicit the process that connects the construction of formal proofs and the gaining of conviction from diagrams. The study reported in this paper contributes to this goal by examining how eight mathematicians use diagrams in their own proof-writing. Three questions drove our analysis:

1. To what extent did these mathematicians base their proofs on the diagrams?
2. How, and for what purposes, did these mathematicians use diagrams to assist their proof writing?

2. Methods

2. 1. Participants

Eight mathematicians participated in this study. All mathematicians currently work either in post-doctoral positions or in tenure-track positions at a research institution in universities with strong research mathematics programs in the United States. The participants' areas of research included algebra, analysis, geometric topology, and combinatorics. The range of experience of the participants also varied, ranging from 1 year to 30 years. In this report, we use pseudonym initials to refer to the mathematicians and refer to all participants with the masculine pronoun to further protect their identities.

2. 2. Materials

Participants were asked to prove a claim about the sine function:

Show that restrictions to the sine function of intervals of length greater than π cannot be injective.

This task, which we refer to as the Sine Task, features the following characteristics. First the claim is based on relatively elementary mathematics—the sine function and injectivity are part of most secondary mathematics curricula. Thus the meaning of this claim would be accessible to sophomore and junior undergraduate mathematics majors. Second, proving this claim invites the use of a diagram. We believed that upon reading the claim, most participants would sketch the graph of a sine function. In Section 3.1, we report that seven of the eight participants did construct this graph shortly after reading the claim. Third, we believed that it would be fairly easy to accept the claim as true after viewing the graph of a sine function, based on pilot data that was collected. Hence the participants would not need to spend a long time verifying the claim is correct; rather most of the work in proving this claim would involve creating an argument that established the claim. As we will show in Section 3.1, six participants indicated that they were convinced that the claim was true early on in the interview session. Fourth, despite the relative simplicity of the mathematical concepts involved and the apparent truth of the claim, we believed that writing a full proof of the claim would not be trivial. We will show in Section 3.2 that many of the participants in this study did indeed have some difficulty constructing a complete proof.

We believe that, as a consequence of the above characteristics, observing mathematicians' performance on this task has the potential to shed insight on the processes that mathematicians use to construct a proof based on a diagram in a mathematical domain accessible to undergraduate mathematics majors.

2. 3. Procedure

Participants met individually with the second author for a one-hour interview. All interviews were videotaped. Participants were handed the task described above. Participants were told to write a proof of this claim appropriate for an undergraduate textbook for mathematics majors. After producing a proof, participants were invited by the interviewer to revise the proof. Participants were not given any time limit on the task, and were allowed the opportunity to revise their final proofs if they thought it necessary.

2. 4. Analysis

The participating mathematicians collectively made a total of 321 mathematical statements. The coding of this data was conducted in the style advocated by Weber and Mejia-Ramos (2009) to analyze the contributions that different types of reasoning played in the construction of a proof. A statement was coded as an *assertion* if it was a mathematical statement that could be validated as true or false (either in an absolute sense, or within the context of previous assumptions made by mathematicians). An example of an assertion is "$\sin(x + \pi) = -\sin x$". A statement was coded as a *proof approach* if it suggested a way to proceed with the proof considered by the participant, for example, a suggestion to use a proof by cases or to employ techniques from calculus. A statement was coded as an *evaluation* if the participant judged the validity of a previously stated assertion (e.g., "$\sin(x + \pi) = \sin x$ is false"), the utility of a previously stated assertion, or the viability of a previously stated proof approach. We were able to code each mathematical statement into one of these three categories.

We next considered the source of each of the statements that participants made. We attributed statements to three sources. A *deduction* was a statement made by the participant that was a logical consequence of previous statements. Deductions included instances when a participant specifically referred to previous statements that he or she had made, indicated they were making a new deduction by using a connector such as "hence" or "consequently", or uttered a statement that was an immediate mathematical consequence of the statement that preceded it. A statement was coded as an *inference from a diagram* if the participant cited, either verbally or with gesture, a graph of the sine function or another diagram that he or she produced as a basis for making that assertion. A statement was coded as *recall or unknown* if the participant neither referenced a graph or diagram that he or she constructed nor cited previous assertions as a basis for making the new assertion.

The participants collectively drew 32 inferences from the diagrams that they constructed. To classify the types of inferences that the participants drew, we used an open coding scheme in the style of Strauss and Corbin (1990). We first wrote a short description of each inference. Similar inferences were grouped together and given preliminary category names and definitions. New inferences were placed into existing categories when appropriate, but also used to create new categories or modify the names or definitions of existing categories. This process continued until a set of categories was formed that were grounded to fit the available data. Descriptions and illustrations of each category will be provided in Section 3.3.

Finally, we coded the correctness of the participants' final proofs using the coding scheme of Malone et al (1980) that was used by Hart (1994) and Weber (2006) in their studies evaluating the correctness of students' proofs. Each proof was coded as completely correct, correct except for minor and insignificant errors, an incorrect proof with significant errors but substantial progress, or an incorrect proof with significant errors where no significant progress was made. Typed versions of two examples of proofs that were coded as completely correct are presented in the Appendix.

For each participant, the result of this analysis was a table containing every mathematical statement that the participant made and the type of statement that was made (inference from a diagram, deduction, or recall/unknown). For inferences from diagrams, the purpose and nature of this inference was described. For deductions, we could highlight the previous statements on which that deduction was based. These tables allowed us to infer how, and to what extent, diagrams played a role in the participants' proof construction process.

3. Results

A summary of the participants' behavior is presented in Table 1.

Mathematician	Drew a diagram?	Quality of proof	Time on task (minutes)	# of inferences from the diagram	Types of inferences made
TL	Y	Mostly correct	30	10	NP, RI, SPA
CY	Y	Mostly correct	11	2	NP
PV	Y	Correct	9	7	NP, ET
FR	Y	Incorrect with substantial progress	18	2	NP, ET
IR	Y	Correct	11	3	NP, ET
BT	Y	Needed substantial hints	21	2	RI, SPA
KZ	Y	Correct	33	6	NP, RI, ET, SPA
KT	N	Correct	14	0	

Table 1. A summary of mathematicians' proofs. (Key: NP=Noticing Properties/Generating Conjectures, RI=Representing/Instantiating Ideas or Assertions, ET=Estimating the Truth of an Assertion, SPA=Suggesting a Proof Approach)

3. 1. The extent of participants' diagram usage

We predicted that the Sine Task would invite the use of a diagram and this diagram would provide participants with confidence that claim in this task was correct. The participants' behavior verified these assumptions. Seven of the eight participants drew a graph of the sine function shortly after reading the problem statement. Six participants indicated they were convinced that the claim was true either prior to or immediately after drawing the graph.

Three participants drew diagrams aside from the graph of the sine function. KZ drew a right triangle while attempting to verify that $\sin(\theta) = \cos(\pi/2 - \theta)$; later, he drew a number line to help him justify that every interval $[a, b]$ of length strictly greater than π contained a point of the form $k\pi/2$ for some odd integer k. TL drew a portion of the graph of an arbitrary function containing a local maximum to prove functions cannot be injective on intervals containing local maxima. When BT reached an impasse in writing the proof and did not know how to proceed, he drew a unit circle and made two inferences from this diagram.

The diagrams played a substantial role in four of the participants' proof constructions. TL and KZ repeatedly drew inferences from the diagrams, both to suggest the proof techniques they ultimately based their proofs upon and to recognize properties of the sine function that were used in their proofs. IR and PV used their diagrams to draw inferences that were used in their final proofs, as well as to gain conviction in these deductive inferences.

In contrast, three participants did not make extensive use of their diagrams in their proof constructions. KT did not draw a single inference from a diagram. FR used his diagram solely to identify the period of the sine function, an idea that did not play a pivotal role in his final proof. As mentioned above, BT did draw two inferences from his diagram, but these were ultimately not connected to his final proof.

Finally, CY exhibited behavior between these two extremes, using the diagram to notice relevant properties of the sine function. Differential usage of diagrams in proof construction tasks has been reported in other small-scale studies, such as Raman's (2003) study on how mathematicians used graphs to prove that the derivative of an even function was an odd function.

3. 2. Participants' difficulties with the Sine Task

We predicted that constructing the required proof would not be trivial. The data displayed in Table 1 suggests this expectation was correct.

The task was not straightforward for the participants as a whole. Participants spent a mean time of 18 minutes working on this task with two participants spending more than 30 minutes on the task. While working on the proof construction, seven of the eight participants made assertions that were either false or assertions that were true but irrelevant to the proof that they produced. Only KT's proof construction proceeded in a direct and linear fashion.

Even given our predictions about task complexity, we were surprised by the extent to which the task posed more difficulties for these participants than we anticipated. It is of course hard to perform mathematics, while thinking aloud, and make only absolutely correct statements. Yet despite an invitation to revise the proof, and despite the length of time spent on the task, only four of the eight participants produced a completely correct proof. Two participants had what we judged to be minor errors in their proofs. TL argued that the arbitrary interval of length greater than π could be translated back using the periodicity of the sine function to contain the point $x = \pi/2$. However, this is not necessarily true. For instance, the interval $(-\pi, \pi/4)$ has length greater than π but cannot be shifted by a multiple of 2π, the period of sin(x), to contain the point $x = \pi/2$. (TL's argument could be salvaged by noting that the interval can be translated so it either contains the point $x = \pi/2$ or $x = -\pi/2$.) CY noted correctly that any closed interval of length greater than π would have to contain a local maximum or minimum for the sine function, but neglected to specify that one could find a maximum or minimum in the interior of this interval, a fact that would be necessary for his subsequent argument to work.

BT and FR had more serious difficulties with constructing their proofs. BT reached an impasse after trying unsuccessfully for over 11 minutes to construct a proof. At this point, he asked for hints on how to proceed and used substantial hints to produce the proof. FR produced the proof in which he uses Rolle's Theorem to deduce that sin(x) cannot be injective on an interval where its derivative is zero. However, Rolle's Theorem asserts that given real numbers a, b with $a<b$, if a continuous differentiable function f satisfies $f(a) = f(b)$, then there exists a real number c such that $a<c<b$ and $f'(c) = 0$. FR was actually applying the false converse of Rolle's Theorem, assuming that if there were a c such that $f'(c) = 0$, and f were defined on an interval around c, there must be distinct points a and b such that $f(a) = f(b)$. (Note for instance, this claim is false for $f(x) = x^3$, although it is true in the case that FR considered)

There were also cases in which participants had the essential idea for how their proof would proceed yet still had difficulty in finalizing the proof. One case of this is TL's proof construction. After working for some time, TL verbally articulated a rough sketch of a proof of the claim. In this sketch, he claimed that a continuous function cannot be injective on a neighborhood of a local maximum, which "follows immediately from the definition of what it means to be a local maximum." Yet, when TL began to write down the proof of the original claim, he stopped to pursue a proof of this sub-claim, drawing the top diagram in Figure 1, and uttered the following:

TL: Let me think for a minute. [draws middle diagram in Figure 1] OK, so let me see. So this is going to be something like Rolle's Theorem or the Intermediate Value Theorem...is basically the tool that I have to use, but I just have to figure out the right way of saying it. I don't think I can just appeal to just a theorem to do it...

In this passage, TL drew diagrams of what appeared to be generic continuous functions, and although he seems confident of the main tool that will prove this sub-claim, appears to be unsure of exactly how to proceed.

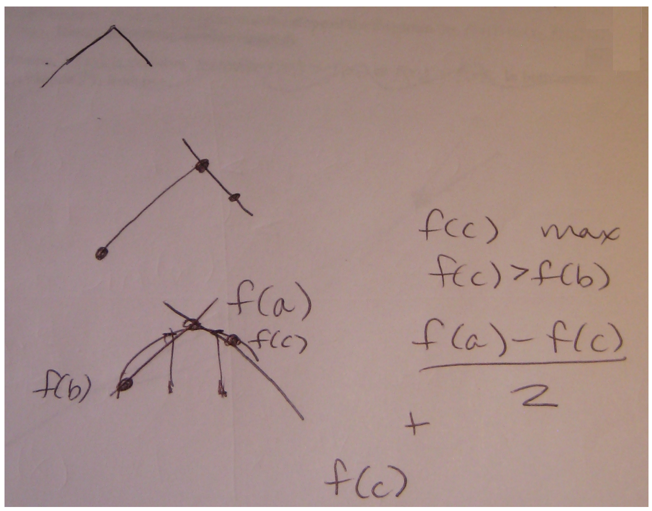

Figure 1. TL's generic continuous functions with local maxima.

After reasoning with these diagrams for nearly four minutes, TL eventually develops an argument for this sub-claim by picking points, *b,* and *c*, to the left and right, respectively, of *a,* the location of the local maximum, and applying the Intermediate Value Theorem to show that the function fails to be injective. However, TL's argument could have been more straightforward —instead of simply using the maximum of f(b) and f(c) as the value which is mapped onto twice in the interval, TL constructs a quantity which is half the distance from f(a) to max{f(b), f(c)} which he then uses to obtain the value which is mapped onto twice.

The above transcript, the time spent on developing this sub-argument, and the extraneous construction in the final solution all indicate that although TL had suggestive diagrams and an informal argument of the original claim, substantial work remained to establish the details required for a formal proof.

3. 3. Purposes of diagrams in participants' proof constructions

The purposes of participants' diagrams are summarized in Table 2.

Purpose served by diagram	Total # of instances	Participants who used the diagram for this purpose

Noticing properties and generating conjectures (NP)	11	TL, CY, PV, FR, IR, KZ
Representing/Instantiating an idea or assertion (RI)	6	TL, KZ, BT
Estimating the truth of an assertion (ET)	9	PV, FR, IR, KZ
Suggesting a proof approach (SPA)	6	TL, KZ, BT

Table 2. A summary of mathematicians' purposes of using diagrams.

3. 3. 1. Noticing properties and generating conjectures

There were 11 instances when a participant used a diagram to notice a mathematical property or make a conjecture that might plausibly be useful for proving the claim. In some cases, the participant conjectured that a property that might be true but would have to be proven, as is illustrated in the excerpt below:

> CY: I'm sort of drawing the graph. I'm trying to figure out is there some property which two points which differ by π have. Can you write down some expression for that in terms of the sine curve? So I drew a picture of the curve. … Let's see. It looks a bit like they're negative of one another. Is that true? Do I know how to prove that? I should write down something. Does sin(x) equal –sin(x+π)?

In the excerpt above, CY notices that sin(x) appears to be equal to –sin(x + π) but is not certain that this is correct and begins thinking about how to prove that. In other cases, the participant gained a high level of conviction that the property they noticed was correct after inferring it from the diagram.

> TL: [while gesturing to graph in Figure 2]. OK, so clearly, let's see. If the interval strictly contains this point, π/2 or minus π/2, then you can see from these areas... So we take this interval and we translate it back, so if that interval contains either one of these, then it's not going to be injective, sort of obviously.

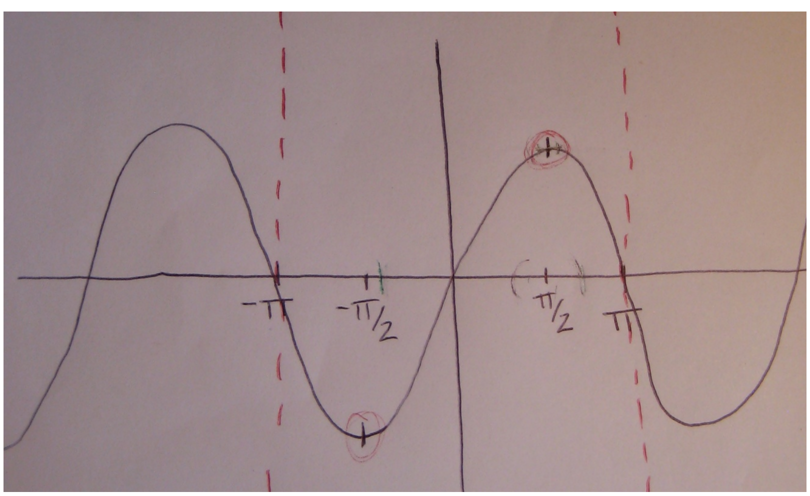

Figure 2. TL's graph of the sine function.

In this excerpt, TL uses what he drew in Figure 2 to see that the interval [a, b] could be translated to contain an interval with $\pi/2$ or $-\pi/2$ and the sine function would "sort of obviously" not be injective on this interval.

3. 3. 2. Estimating the truth of an assertion

In 9 instances, participants referred to a diagram to see whether a statement they made was correct. In some cases, these statements were conjectures; by inspecting the diagram, the participant could either gain conviction to accept the conjecture as true or find sufficient reason to reject the conjecture as false. We illustrate the latter with the excerpt below.

> PV: The sine function looks like this [draws a graph of the sine function] and this length is π [draws length] … Yeah, so if you have an interval of length greater than π, you can see the top and you can see the bottom…That's not true…

In the excerpt above, PV initially asserts that any interval of length greater than π would contain both a local maximum and a minimum for the sine function (or, in his words, "you can see the top and you can see the bottom"), but upon inspecting the diagram further, he realizes that this is not true. In a similar instance, FR initially asserted that $\sin x = \sin(x + \pi)$, but after drawing a graph quickly realized the period of sine was actually 2π and thus rejected this assertion.

In other cases, participants would look at the graph to verify inferences that they had reached previously via deductive reasoning in case they made an error at some point. In the strongest illustration of this, KZ initially recalled that $\sin \theta = \cos(\pi/2 - \theta)$ but was not sure if this formula was correct. To verify that this was correct, he said, "so this can be seen if you think of sine from the triangle" and drew a right triangle with one acute angle labeled θ and the other, $\pi/2 - \theta$. After drawing this triangle, KZ averred, "so this should be true in general". However, when he tried to prove the claim, he misremembered the cosine addition and subtraction identities and incorrectly deduced that $\cos(\pi/2 - \theta) = -\sin(\theta)$. This deduction troubled him and he was not certain whether $\cos(\pi/2 - \theta)$ was equal to $\sin \theta$ or $-\sin \theta$. To decide which was correct, KZ sketched a graph of the sine and cosine functions and found support for his original claim. Then, returning to his incorrect identity, he checked specific values of θ. Only after realizing that the identity did not hold for some values did he reject the incorrect identity and decide to retain the original claim.

This illustrates an interesting aspect of how mathematicians seek and gain conviction. It is sometimes claimed in the mathematics education literature that while empirical and visual evidence can provide support for mathematical proofs, it is deductive reasoning that provides complete certainty in a claim. For instance, Harel and Sowder (1998) state that mathematicians have deductive proof schemes, meaning that they obtain complete conviction from the deductive proofs that they produce. This is in contrast to students who hold empirical or perceptual proof schemes and believe claims are true via the inspection of diagrams or examples. As KZ's behavior illustrates, the situation is not always so straightforward. He initially gains confidence that $\cos(\pi/2 - \theta)$ equals $\sin \theta$ using the right triangle he drew. However, he doubts the veracity of this assertion after deducing a contradictory identity. What is interesting is that his (invalid) deduction did not have the last word. Rather he sought other diagrammatic evidence to resolve this contradiction and only obtained full conviction after checking his original identity with specific numbers. This suggests that while the goal of mathematicians is often to produce rigorous proofs of mathematical theorems, they might obtain conviction in these theorems via a combination of empirical, diagrammatic, and deductive reasoning; and deductive reasoning is not always the sole or dominant way of resolving mathematical questions, at least not in short-term episodes in their problem-solving work.

3. 3. 3. Suggesting proof approaches

In six instances, KZ and TL used the diagram to suggest ways that their proofs might proceed. We illustrate this with an episode of TL's work:

> TL: So I already have the picture of it [referring to Figure 2, presented earlier in this section], well let's see, so it's a periodic function. So, whatever interval that you were looking at – if you take some interval of length strictly greater than π, you can always translate it backwards so that it's around the origin, because sine is periodic. OK, so here's a period of the sine function from here [draws the two dashed lines on the sine graph in Figure 2]. OK, and we want to know about intervals of length strictly greater than π. So if we take any interval at all, I can move it back to this region, and it's necessarily going to contain the origin because length of the interval is strictly greater than π. I guess this is an easier proof than taking derivatives, but maybe it's not quite a proof yet. I have to think about it a little more carefully. OK. So we take any interval of length strictly greater than π and move it back into this fundamental region here ... So the question is I have to make a precise way of saying what it is that I'm talking about, but I can see how the argument is forming.

In this excerpt, TL lays out how his proof will proceed. He will use the periodicity of the sine function (a property he notices from his inspection of the graph of sine) to map the interval he is considering into the region indicated in Figure 2.

3. 3. 4. Instantiating or representing an idea or assertion on the graph

There were six instances in which TL or KZ represented an idea or assertion graphically. Consider the excerpt below:

> KZ: So this interval [a,b] must contain an element of the form kπ/2 because [begins drawing a number line] we're talking about π/2, 3π/2, 5π/2 and so on, the distance between those is π [labels these points on the number line as he states them]. So if you take any interval of length strictly bigger than π, it must contain at least one of them. [draws the interval of length greater than π on the number line] And moreover, it must contain some interval close to them too, both on the left and on the right, and you can choose a θ close enough there.

Throughout this excerpt, KZ is representing the ideas that he says aloud on his diagram of the number line. In particular, at the end of this excerpt, KZ chooses and draws an interval of length strictly greater than π. We believe one purpose of instantiating these ideas is to aid in the generation of new conjectures and potential proving approaches.

4. Discussion

The fact that we only looked at eight mathematicians proving a single claim limits the extent that we can generalize our findings. Nonetheless the data do reveal several interesting themes about mathematicians' use of diagrams in proof-writing.

The first theme we observed is that the participants had surprising amount of difficulty in producing complete proofs of the claim. As discussed earlier in this paper, difficulties occurred even in cases in which participants had constructed what might be called the crux or key idea of the proof. In Section 3.1, we describe how TL struggled to write a proof of a claim in his argument even after both establishing an informal argument of the original claim and drawing suggestive diagrams of the situation. In Section 3.2, we described how KZ had significant difficulties completing a symmetry-based argument because of trouble he had justifying an important trigonometric identity.

That these mathematicians did not find it trivial to translate an informal argument to a formal one illustrates an important point. Researchers such as Hanna (1991) have claimed that the essence of a proof is contained in the intuitive ideas used to try to develop the formal proof. While Hanna acknowledges that having a logically correct formal argument is important, she refers to the full rigor as a "hygiene factor", implying that translating the intuitive idea to a formal proof is a relatively uninteresting and routine part of the proof writing process for mathematicians. Although this is probably true, the data from this study remind us that this translation still introduces difficulties. If mathematicians experience these difficulties, it stands to reason that students will as well. Hence this study serves as a reminder that, although the generation of informal visual arguments to serve as a basis for formal proofs is a laudable goal in the advanced mathematics classroom, care still must be taken in describing the complex process of translating these arguments into a proof (e.g., Alcock, 2010; Duval, 2007).

The participants used diagrams for four purposes: (a) noticing properties and generating conjectures, (b) suggesting a proof approach, (c) estimating the truth of an assertion, and (d) instantiating an idea or assertion. In our previous studies on students' proof constructions using diagrams, we noticed that participants used diagrams primarily to try to understand a mathematical claim and sometimes to generate an explanation of why a theorem is true (see Alcock & Weber, 2010a, 2010b). However these students generally did not use diagrams for some of the reasons that the mathematicians did, such as verifying that their logical deductions were true, uncovering errors in their logic, or identifying false mathematical assertions. Further, in mathematics lectures that we have observed, these aspects of proof construction were often absent (see, for instance, Weber, 2004). For instance, the professors we have observed rarely show the wrong turns that they make in a proof attempt or how a diagram can reveal an error in their logical reasoning. These observations suggest a gap between mathematicians' use of diagrams in their work, and their use of diagrams in teaching. If the goal of the pedagogical suggestion of basing proofs on diagrams is to help students benefit from diagrams similarly to how mathematicians benefit, then we must find ways to raise students' awareness of and access to the variety of potential uses of diagrams in proof construction.

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Appendix.

In what follows, we give two correct proofs from our participants PV and KT. Both proofs were lightly edited for the purposes of readability and brevity but the central ideas employed in the proofs were unchanged.

Proof 1—symmetries of the sine function:

Every interval of length greater than π contains a point, x_0, of the form $\pi/2 + \pi k$ for some integer k, and a neighborhood, I, containing x_0 . Now, sine is symmetric about $\pi/2 + \pi k$ in the sense that $\sin(\pi/2 + \pi n + x) = \sin(\pi/2 + \pi n - x)$ for all real x and integral n, as can be checked using the sine angle addition formula. Choose ε such that $x_0 - \varepsilon$ and $x_0 + \varepsilon$ are elements of I. Then $\sin(x_0 + \varepsilon) = \sin(\pi/2 + \pi k + \varepsilon) = \sin(\pi/2 + \pi k - \varepsilon) = \sin(x_0 - \varepsilon)$, so sine is not injective on I, and hence not injective on the original interval.

Proof 2—Intermediate Value Theorem:

Every interval of length greater than π contains a point, x_0 of the form $\pi/2 + k\pi$ for some integer k, and a neighborhood, I, containing x_0. If $f(x) = \sin(x)$, $f'(x) = \cos x$ and $f''(x) = -\sin x$. Note that $f'(x_o) = 0$ and $f''(x_o) \neq 0$. Hence f attains a relative maximum or minimum at x_0. Assume f attains a relative maximum at x_0 (a similar argument works if x_0 is a relative minimum). Let a, b be in I such that $a < x_0 < b$. Let $y_0 = \max(f(a), f(b))$. Then, since y_0 is a relative maximum $y_0 <= f(x_0)$. Also, $f(a) <= y_0$ and $f(b) <= y_0$ by construction. Since f is continuous, we may apply the Intermediate Value Theorem: there exists a' in $[a, x_0]$ and b' in $[x_0, b]$ such that $f(a') = y_0 = f(b')$. Hence, $f(x) = \sin(x)$ is not injective on I, and hence not injective on the original interval.

Where is the Logic in Student-Constructed Proofs?

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Often university mathematics departments teach some formal logic early in a transition-to-proof course in preparation for teaching undergraduate students to construct proofs. Logic, in some form, does seem to play a crucial role in constructing proofs. Yet, this study of forty-two student-constructed proofs of theorems about sets, functions, real analysis, abstract algebra, and topology, found that only a very small part of those proofs involved logic beyond common sense reasoning. Where is the logic? How much of it is just common sense? Does proving involve forms of deductive reasoning that are logic-like, but are not immediately derivable from predicate or propositional calculus? Also, can the needed logic be taught in context while teaching proof-construction instead of first teaching it in an abstract, disembodied way? Through a theoretical framework emerging from a chunk-by-chunk analysis of student-constructed proofs and from task-based interviews with students, I try to shed light on these questions.

Keywords: Proofs, logic, transition-to-proof courses, analysis of proofs, task-based interviews

Where is the Logic in Student-Constructed Proofs? Why is this an interesting question? To obtain a Masters or Ph.D. in mathematics, or even to succeed in proof-based courses in an undergraduate mathematics major, one must often be able to construct original proofs, a common difficulty for students (Moore, 1994; Weber, 2001). This process of proof construction is usually explicitly taught, if at all, to undergraduates as a small part of a course, such as linear algebra, whose stated goal is something else, or in a transition-to-proof or "bridge" course. When universities do offer a transition-to-proof course, professors often teach some formal logic (predicate and propositional calculus) as a background for proving. But how much logic actually occurs in student-constructed proofs? In this paper, I begin to answer this question by first searching for uses of logic in a "chunk-by-chunk" analysis of student-constructed proofs from a graduate "proofs course," then by coding student-constructed proofs from a graduate homological algebra course, and lastly by examining the actions in the proving process of three graduate students and searching for additional uses of logic therein. If formal logic occurs a substantial amount, then teaching a unit on predicate and propositional calculus might be a good idea; however, if formal logic occurs infrequently, then teaching it in context, while teaching proving, may be more effective.

Background Literature

Currently, at the beginning of transition-to-proof courses, professors often include some formal logic, but how it should be taught is not so clear. Epp (2003) stated that, "I believe in presenting logic in a manner that continually links it to language and to both real world and mathematical subject matter" (p. 895). However, some mathematics education researchers maintain that there is a danger in relating logic too closely to the real world: "The example of 'mother and sweets' episode, for instance, which is 'logically wrong' but, on the other hand, compatible with norms of argumentation in everyday discourse, expresses the sizeable discrepancy between formal thinking and natural thinking" (Ayalon & Even, 2008b). In the mother and sweets scenario, the mother says to the child, "If you don't eat, you won't get any sweets" and the child responds by saying, "I ate, so I deserve some sweets." Other authors have noticed that the way logic is taught in transition-to-proof courses is at variance with how it is actually used in proving: "Beginning logic courses often seem to present logic very abstractly, in essence as a form of algebra, with examples becoming a kind of applied mathematics" (Selden & Selden, 1999, p. 8).

There are also those who think that logic does not need to be explicitly introduced at all. For example, Hanna and de Villiers (2008) stated, "It remains unclear what benefit comes from teaching formal logic to students or to prospective teachers, particularly because mathematicians have readily admitted that they seldom use formal logic in their research" (p. 311). Selden and Selden (2009) claimed that "logic does not occur within proofs as often as one might expect … [but] [w]here logic does occur within proofs, it plays an important role" (p. 347). Taken together, these differing views suggest that it would be useful for mathematics education researchers to further examine the role of logic and logic-like reasoning within proofs in order to inform professors on the ways they might best include logic in transition-to-proof courses. However, to date, only a little such research has been conducted (Baker, 2001).

Another interesting idea that has been expressed about proofs in general is that deduction occurs in proofs in a "systematic, step-by-step manner" (Ayalon & Even, 2008a). In fact, one professor quoted by Ayalon and Even (2008a) expressed the view that a student "thinks about something, he draws a conclusion, which brings him to the next thing…Logic is the procedural, algorithmic structure of things." Others tend to agree: "From most mathematical textbooks we can simply see the process of a mathematical proof [sic] as the development of a sequence of statements using only definitions and preceding results, such as deductions, axioms, or theorems" (Chin & Tall, 2002, p. 213). Rips (1994) looks at proof in a slightly more sophisticated way: "At the most general level, a formal proof is a finite sequence of sentences $(s_1, \dots, s_k)$ in which each sentence is either a premise, an axiom of the logical system, or a sentence that follows from the preceding sentences by one of the system's rules" (p. 34). Instead of sentences, I partition student-constructed proofs into usually smaller "chunks" to begin to answer the title question on logic.

Research Settings

This research was done in three separate phases: (a) By examining all of the 42 student-constructed proofs from a beginning graduate level "proofs course," (b) by examining 10 student-constructed proofs from a more advanced graduate homological algebra course, and (c) by conducting and analyzing task-based interviews with three graduate students a year after the "proofs course." The "proofs course," *Understanding and Constructing Proofs*, was offered at a large Southwestern state university, giving Masters and Ph.D.'s in mathematics. Students in the course were first-year mathematics graduate students along with a few advanced undergraduate mathematics majors. For this course, the students were given professor-created notes with a sequence of definitions, questions, and statements of theorems dealing with topics such as sets, functions, real analysis, algebra, and topology. For example, three theorems that were proved by the students were: "The product of two continuous [real] functions is continuous"; "Every semigroup has at most one minimal ideal"; and "Every compact, Hausdorff topological space is regular." The topics in the course were of less importance than its focus on the construction of differing kinds of proofs.

Students were asked to prove theorems in the notes at home, and came to the class to present their proofs on the chalkboard. After receiving critiques on content and style, one student was selected to modify his/her proof to turn in. The professors then verified all of these proofs as correct and photocopies were made for the students. There were no lectures, just discussions of student work. The class met for one hour and fifteen minutes twice a week for a total of 30 class meetings. The course was taught like a modified Moore Method course (Mahavier, 1999) by two professors, using a constructivist approach that was also somewhat Vygotskian because the professors gave their opinions on how mathematicians write proofs. All class meetings were videotaped, and field notes were taken by a graduate research assistant. The debriefing for each class meeting and preparation for the next class were also videotaped, as well as were all tutoring sessions with students.

The homological algebra course was taught lecture-style, with two theorems to prove assigned per week as homework. The professor then graded the proofs and handed them back to the students. The course covered such topics as module theory, category theory, homology and cohomology, with emphasis on the Tor and Ext functors. All students in the course had passed the graduate abstract algebra courses required before taking the Ph.D. algebra comprehensive examination. The ten proofs considered from this course were constructed by a single student who received perfect scores on all homework and actively conducts research in algebra.

The task-based interviews were conducted in a university seminar room one year after the "proofs course." The three graduate students, who participated, had all received A's in the "proofs course" and were willing to volunteer for the one hour interview. Each interviewed student received a one-page subset of the "proofs course" notes. This subset was self-contained and began with the definition of a semigroup and ended with the theorem to be proved, "Every semigroup has at most one minimal ideal." All other theorems listed therein were to be considered as proved and thus could be used by the students in the proving process. Each student was encouraged to think aloud while attempting to construct a proof on the chalkboard. Because of time constraints, after 45 minutes, each student was asked to stop proving and answer some questions about the proving process. The questions were not predetermined. Most were intended to clarify what had been said during the think-aloud process or what had been written on the chalkboard. The interviews were videotaped, and field notes were taken.

Research Methodology

The 42 proofs from the "proofs course" were first subdivided into "chunks" for coding. The "chunks" are similar to those in Miller's (1956) article, in which he stated that chunks are a "meaningful unit" in thinking. In the analysis described here, a chunk can refer to a sentence, a group of words, or even a single word, but always refers to a unit in a proof. During several iterations of the coding process, 13 categories, such as "Informal inference" and "Assumption," emerged. During one iteration, I asked two mathematics professors to observe my coding of two of the student-constructed proofs. After this training, four additional proofs were coded independently by the professors, and over 80% of the chunks were agreed upon by all three of us. It appears that once a person understands this chunking process, he/she can do the chunking with relative ease. This iteration included full agreement on which chunks to associate with specific categories, which helped solidify the coding process.

The categories and the chunks sometimes co-emerged, that is, the categories sometimes influenced the chunking. For example, "Then $x \in A$ and $x \in B$" might have been treated as a single chunk because it arose from $x \in A \cap B$ and the definition of intersection. However, it could have been split into "Then $x \in A$" and "and $x \in B$" because the two chunks seemed to follow from separate warrants. This coding process, which resulted in a total of 673 chunks in the student-constructed proofs from the "proofs course," is further described below.

In this paper, I discuss in detail just 5 of the 13 categories. The first two of these deal with the question posed at the beginning of this paper, "Where is the logic in student-constructed proofs?" The remaining three categories are those that occurred most often. Proofs from both the "proofs course" and the homology course were coded using the same categories. All 13 categories are briefly described in the Appendix.

The Categories

Informal inference (**II**) is the category that refers to a chunk of a proof that depends on common sense reasoning. While I view informal inference as being logic-like, it seems that when one uses common sense, one does so automatically and does not consciously bring to mind any formal logic. For example, given $a \in A$, one can conclude $a \in A \cup B$ by common sense reasoning, without needing to call on formal logic.

By *Formal logic* (**FL**) in this paper I mean the conscious use of predicate or propositional calculus going beyond common sense. The distinction is that formal logic is the logic a student does not normally possess before entering a transition-to-proof course. Modus Tollens and DeMorgan's Laws are two examples of formal logic that are usually not common sense for most students (Anderson, 1980; Austin, 1984). For example, given $x \notin B \cup C$, one can conclude $x \notin B$ and $x \notin C$, a typical use of DeMorgan's Laws that students often do not perform automatically, or do perform automatically, but incorrectly.

Definition of (**DEF**) refers to a chunk in a proof that calls on the definition of a mathematical term. For example, consider the line "Since $x \in A$ or $x \in B$, then $x \in A \cup B$." The conclusion "then $x \in A \cup B$" implicitly calls on the definition of union.

Assumption (**A**) is the code for a chunk that creates a mathematical object or asserts a property of an object in the proof. The category is further divided into two sub-categories: "Choice" and "Hypothesis." *Assumption (Choice)* refers to the introduction of a symbol to represent an object (often fixed, but arbitrary) about which something will be proved – but not the assumption of additional properties given in a hypothesis. In contrast, *Assumption (Hypothesis)* refers to the assumption of the hypothesis of a theorem or argument (often asserting properties of an object in the proof). An example to demonstrate the difference between the two is provided by the theorem "For all $n \in \mathbb{N}$, if $n > 5$ then $n^2 > 25$." The chunk "Let $n \in \mathbb{N}$" would be coded Assumption (Choice), and the chunk "Suppose $n > 5$" would be coded Assumption (Hypothesis).

Interior reference (**IR**) is the category for a chunk in a proof that uses a previous chunk as a warrant for a conclusion. For example, if there were a line indicating $x \in A$ earlier in the proof, then a subsequent line stating "Since $x \in A$…" later in the proof would be an interior reference.

Examples of the Coding

To fully understand the coding and categorizing process, some examples might be helpful:

Theorem 2: For sets $A, B,$ and C, if $A \subseteq B$ and $A \subseteq C$, then $A \subseteq B \cap C$.

Proof: Let $A, B,$ and C be sets. Suppose $A \subseteq B$ and $A \subseteq C$. Suppose $x \in A$. Then $x \in B$ and $x \in C$. That means by the def of intersection $x \in B \cap C$. Therefore, $A \subseteq B \cap C$.

Let $A, B,$ and C be sets.	Assumption (Choice)
Suppose $A \subseteq B$ and $A \subseteq C$.	Assumption (Hypothesis)
Suppose $x \in A$.	Assumption (Hypothesis)
Then $x \in B$	Informal inference
and $x \in C$.	Informal inference
That means by the def of intersection $x \in B \cap C$.	Definition of intersection
Therefore, $A \subseteq B \cap C$.	Conclusion statement/Definition of subset

For the above proof, the first chunk was "Let $A, B,$ and C be sets." This was coded "Assumption (Choice)" because the sets were chosen. The next chunk, "Suppose $A \subseteq B$ and $A \subseteq C$" was coded "Assumption (Hypothesis)" since those suppositions were stated in the theorem as the hypotheses. "Suppose $x \in A$" is a chunk that is the hypothesis of a subproof for proving a subset inclusion, hence it was coded "Assumption (Hypothesis)." Both conclusions "Then $x \in B$" and "and $x \in C$" were coded "Informal Inference" because they can both be argued by common sense. The chunk "That means by the def of intersection $x \in B \cap C$" takes the two previous chunks and uses the definition of intersection. Finally, the chunk "Therefore, $A \subseteq B \cap C$" uses the definition of subset and is also the conclusion of the proof. Most coded chunks were counted as one unit each, however this final chunk was assigned a half unit to each category "Conclusion statement" and "Definition of subset."

Theorem 38: If X is a Hausdorff space and $x \in X$, then $\{x\}$ is closed.

Proof: Let X be a Hausdorff space. Let $x \in X$. Note $\{x\} = X - (X - \{x\})$. Suppose $y \in X$ and $y \neq x$. Because X is Hausdorff, there is an open set P_y for which $y \in P_y$. There is also an open set R_y such that $x \in R_y$ and $P_y \cap R_y = \emptyset$. Suppose $P_y \nsubseteq X - \{x\}$, then $x \in P_y$, but $x \in R_y$. Therefore $x \in P_y \cap R_y$, which is a contradiction. Therefore, $P_y \subseteq X - \{x\}$. Thus for every $y \neq x$ there is an open set P_y where $y \in P_y$ and $P_y \subseteq X - \{x\}$. The union of all P_y is equal to $X - \{x\}$, which is thus an open set. Therefore $\{x\}$ is closed, being the complement of an open set.

Let X be a Hausdorff space.	Assumption (Hypothesis)
Let $x \in X$.	Assumption (Hypothesis)
Note $\{x\} = X - (X - \{x\})$.	Formal logic
Suppose $y \in X$ and $y \neq x$.	Assumption (Choice)
Because X is Hausdorff,	Interior reference
there is an open set P_y for which $y \in P_y$. There is also an open set R_y such that $x \in R_y$ and $P_y \cap R_y = \emptyset$.	Definition of Hausdorff
Suppose $P_y \nsubseteq X - \{x\}$,	Assumption (Hypothesis)
then $x \in P_y$,	Formal logic

but $x \in R_y$.	Interior reference
Therefore $x \in P_y \cap R_y$,	Definition of intersection
which is a contradiction.	Contradiction statement
Therefore, $P_y \subseteq X - \{x\}$.	Formal logic
Thus for every $y \neq x$ there is an open set P_y where $y \in P_y$ and $P_y \subseteq X - \{x\}$.	Conclusion statement
The union of all P_y is equal to $X - \{x\}$,	Informal inference
which is thus an open set.	Definition of topology
Therefore $\{x\}$ is closed, being the complement of an open set.	Conclusion statement/Definition of closed

The first chunk in the above proof that I would like to discuss is the third chunk "Note $\{x\} = X - (X - \{x\})$." This was coded as "Formal logic" because this inference would normally not be automatic for most students, as they would have to formulate the negations first, before concluding the set equality. The fifth chunk, "Because X is Hausdorff," restates the first chunk in the proof, so it was coded as "Interior reference." The eleventh chunk, "which is a contradiction," is just a statement to the reader that the prover has arrived at a contradiction; hence the code "Contradiction statement."

The Interviews

The three interviewed graduate students took three different approaches to the proof, including voicing different concept images for several concept definitions (Tall & Vinner, 1981). For example, for the definition of a "minimal ideal of a semigroup," one student considered Venn diagrams when reflecting on the definition, while the other two students stated in a subsequent debriefing that they had not thought of using a diagram. While all three students' proving approaches were different, none of them proved the theorem correctly. This might have been a result of time constraints.

Another result was that the actions that I had previously hypothesized for the proof construction did not match the actual actions of any of the interviewed students. For example, I had hypothesized that the students would write the assumptions as their first line, leave a space, and then write what was to be proved as their last line (as they had been encouraged to do in the earlier "proofs course"). This is a proving technique (Downs & Mamona-Downs, 2005) that is not often taught. Although all three students wrote "Let S be a semigroup" almost immediately at the beginning of their proofs, thereby introducing the letter S, only one student left a space and wrote the conclusion at the end, after some algebraic manipulations. Another student wrote definitions on scratch work before attempting the proof. She then assumed A and B were minimal ideals, and looked up the definition of a minimal ideal. She subsequently claimed (without justification) that either $A = B$ or $A \cap B = \emptyset$. After correctly using a theorem listed in the provided notes, she concluded $A = B$. She then said, "Ok, I'm done." But in fact, the above lack of justification can be seen as a gap in her proof.

The Results

In the chunk-by-chunk analysis of the proofs in the "proofs course," just 6.5% (44 chunks) of the 673 chunks were Informal inference, and just 1.9% (13 chunks) were Formal logic. However, I found that 30% (203 chunks) were Definition of, 25% (166 chunks) were

Assumption, and 16% (108 chunks) were Interior reference. Thus, Definition of, Assumption, and Interior reference accounted for nearly 71% of the chunks in the analyzed proofs. These large percentages brought up the question: Was this due to the somewhat unusual nature of the "proofs course" with its wide spread of topics that entailed the introduction of many definitions? Also, how much difference might there be in the amount of logic used if I were to do a chunk-by-chunk analysis of proofs in a graduate homology course that concentrated on a single topic?

In the subsequent chunk-by-chunk analysis of 10 proofs from the homology course, only 10% (17 chunks) of the 170 chunks were Informal inference, and just 0.6% (1 chunk) was Formal logic. Indeed, I found that 21% (36 chunks) were Definition of, 18% (31 chunks) were Assumption, and 18% (30 chunks) were Interior reference, for a total of almost 57%. This lends some support for the hypothesis that the above large percentage (71%) of Definition of, Assumption, and Interior reference was due to the nature of the "proofs course". However, there is not much difference in the percentages of both formal and informal logic used in the two courses (8.4% vs. 10.6%). The table below shows the chunk categories, complete with the rounded percentages. A brief description of all of the categories, their abbreviations (e.g., **A**, **ALG**, etc.), and examples thereof is given in the Appendix.

	# Chunks	A	ALG	C	CONT	D	DEF	ER	FL	II	IR	
"Proofs class"	673	166	23	56	4	32	203	17	13	44	108	
% of chunks		24.7	3.3	8.3	0.6	4.7	30.2	2.5	1.9	6.5	16	
Homology	170	31	4	14	0	5	36	21	1	17	30	
% of chunks		18.2	2.4	8.2	0	2.9	21.2	12.4	0.6	10	17.7	

Discussion

At first glance, these results may seem surprising. While the chunk-by-chunk coding is a convenient tool for a surface analysis of a finished written proof, there are underlying structures to, and within, proofs, such as proof by contradiction. I see these as "logic-like structures" that are not often explained in the predicate and propositional calculus discussed in most transition-to-proof courses. For example, if one wishes to prove "For all $x \in A$, $P(x)$," one starts with "Let $x \in A$" and reasons towards "$P(x)$."

"Logic-like structures" are structures that preserve truth value in an argument, yet are not easily expressible in the language of predicate or propositional calculus. Beginning a proof of a theorem whose conclusion is of the form P *or* Q by supposing *not* P and arriving at Q is another example of a logic-like structure. Structuring a proof in this way has the effect of using logic. Also, if one wishes to prove that there is a unique x with the property $P(x)$, one normally begins "Suppose $P(a)$ and $P(b)$" and reasons towards the conclusion "$a = b$." This logic-like structure appears in the proof of the theorem, "Every semigroup has at most one minimal ideal," the same theorem used in the interviews. In fact, all three interviewed students immediately assumed a semigroup S and further assumed that S contained two, or n, minimal ideals $-$ both actions which had been discussed in the "proofs course."

The fact that from the "proofs course," 30% of the chunks were definitions and 25% were assumptions suggests that there is a need to teach undergraduates how to introduce mathematical objects into proofs and how to read and use definitions. Indeed, there have been documented instances of students' struggle with definitions (Edwards & Ward, 2004). What this and the results of the chunk-by-chunk coding suggest is that there needs to be a focus on how to unpack and interpret concept definitions in order to create usable concept images (Tall & Vinner, 1981) with components that can be used directly in proofs. For example, by definition, $A \cup B$ is $\{x | x \in A \text{ or } x \in B\}$, but one usage is if one has "$x \in A$ or $x \in B$" then one can conclude "$x \in A \cup B$." Some students do not immediately connect this definition, with its set notation, to this usage. Bills and Tall (1998) introduced a similar notion saying that a definition is *formally operable* for a student if that student "is able to use it in creating or (meaningfully) reproducing a formal argument [proof]" (p. 104).

Another implication for teaching that stems from this research is that because formal logic occurs fairly rarely, one might be able to teach it in context as the need arises. Also there is a possibility that doing so might be more effective.

Future Research

It would be interesting to examine whether the kinds of chunks used in proofs varies by mathematical subject area. For example, would topology have a different distribution of categories of chunks than abstract algebra? Indeed, several mathematics professors have suggested that I code chapters of various textbooks to see how much formal logic occurs in them. Also, it may be that the kind of formal logic taught explicitly at the beginning of many transition-to-proof courses is actually psychologically, and practically, disconnected from the process of proving for students. This disconnect might lead to future difficulties in many of the proof-based courses in students' subsequent undergraduate and graduate programs. An additional interesting question that arises from the "proofs course" itself is: How many beginning graduate students need a course specifically devoted to improving their proving skills?

In future research, one might also look for instances of logic-like structures and techniques in student-constructed proofs. Solow (1982) and Velleman (1994) both discuss logic-like structures and techniques for proving, but many other transition-to-proof books touch on this very briefly, if at all. Can one identify a range of logic-like structures that students most often need in constructing proofs? Further, one might investigate the degree of a prover's automated behavioral knowledge of logic-like structures that could help to reduce the burden on his or her working memory. This might free resources to devote to the problem-solving aspects of proofs. That this might be the case was suggested by Selden, McKee and Selden (2010).

Finally, there may be additional logic that does not appear in a final written proof, but that might occur in the actions of the proving process. This would be interesting to investigate. For example, consider the first theorem in the "proofs course" notes, "For sets A and B, if $A \cap B = A$, then $A \cup B = B$." The hypothesized actions for one student-constructed proof of this theorem might be re-stating the hypothesis, "Let A and B be sets, and suppose $A \cap B = A$" at the beginning of the proof, leaving a space, and then writing the conclusion, "Then $A \cup B = B$," at the bottom of the paper or chalkboard. The next action might be to unpack the conclusion and realize that a set equality requires each set to be a subset of the other. This way of showing set equality is a technique that may not be emphasized at the transition-to-proof level, but is very useful, because something similar occurs in many branches of mathematics. In number theory, for example, proofs showing the equality of two numbers sometimes demonstrate that one number divides the other and vice versa.

The next proving action might be to unpack the definition of subset. In this hypothetical proof construction, one might first show that $A \cup B \subseteq B$ by supposing $x \in A \cup B$ and arguing towards $x \in B$. This is subtle because x could be in $A - B$, or in B. Because of this, one might take cases that cover the two possibilities. Using cases produces another logic-like structure, because logically one must exhaust all possibilities. "Suppose $x \in A$" might be the first case, and since one has the hypothesis, "$A = A \cap B$," one can conclude "$x \in B$." This was written into the student-constructed proof as "Let $x \in A$. Since $A = A \cap B$, then $x \in B$." [This can be warranted by the following argument: From $x \in A$ and $A = A \cap B$, one has $x \in A \cap B$. Then $x \in A$ and $x \in B$, so $x \in B$.] Notice that the implicit step "$x \in A \cap B$" was omitted, leaving the reader to supply a warrant. The other case is "Suppose $x \notin A$," and in this case, since $x \in A \cup B$, by common sense logic one obtains $x \in B$. Since both possibilities have thus been exhausted, one can conclude $x \in B$, and the first subset inclusion has been proved. One would then go on to prove the other subset inclusion.

Sub-arguments, such as the one in brackets above, are sometimes omitted from proofs when readers are assumed to have sufficient background knowledge of the mathematics. The above implicit sub-argument is very simple. Implicit arguments, including those from the graduate homology course proofs, also seemed to be relatively simple. I conjecture that in contrast, implicit arguments in mathematics journals are often much more difficult for a reader to verify.

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Appendix

The following is a list of all 13 categories used in the chunk-by-chunk analysis. For each category, I first give its abbreviation in parentheses (e.g., (**A**)), followed by its designation (e.g., Assumption), a definition, and an example of how it can be used in a chunk-by-chunk analysis.

(**A**) Assumption: We separate assumption into two sub-categories:

- Choice: The choice of a symbol to represent an object (often fixed, but arbitrary) about which something will be proved – but not the assumption of additional properties given in a hypothesis
- Hypothesis: The assumption of the hypothesis of a theorem or argument (often

stating properties of an object in the proof).

Example 1: For the theorem "For all $n \in \mathbb{N}$, if $n > 5$ then $n^2 > 25$."
A (Choice): "Let $n \in \mathbb{N}$" (fixed, but arbitrary)
A (Hypothesis): "Suppose $n > 5$"

Example 2: For the theorem "Let π be the ratio of the circumference of a circle to its diameter. Then $\pi > 3$."
A (Choice): "Let π be the ratio of the circumference of a circle to its diameter"
A (Hypothesis): None

(**ALG**) Algebra: Any high school or computational algebra done in the proof.
Example: "$\ldots |x+4| - 4 \leq |x| + |4| - 4 = |x| \ldots$"
ALG: "$|x+4| - 4 \leq |x| + |4| - 4$"
ALG: "$|x| + |4| - 4 = |x|$"

(**C**) Conclusion statement: A statement that summarizes the conclusion of a theorem or an argument.
Example: "…So $x \in B$. Therefore $A \subseteq B$."
C: "Therefore $A \subseteq B$."

(**CONT**) Contradiction statement: The conclusion of a proof or argument by contradiction.
Example: "…We found $x \in A$, which is a contradiction."
CONT: "which is a contradiction".

(**D**) Delimiter: A word or group of words signifying the beginning or end of a sub-argument. Common delimiters include "now," "next," "firstly," "lastly," "case 1," "in both cases," "part," "($\Rightarrow$)," "($\subseteq$)," "base case" (in an induction proof), and "by induction" (in an induction proof).
Example: "…In both cases, we conclude that $x \in B$."
D: "In both cases".

(**DEF**) Definition of: The use of a definition of a mathematical object.
Example: "…so $x \in A$… also $x \in B$… Then $x \in A \cap B$…"
DEF: "Then $x \in A \cap B$".

(**ER**) Exterior reference: A reference to a theorem proved previously.
Example: "…Now, by Theorem 6, $x \in A$…"
ER: "by Theorem 6"

(**FL**) Formal logic: Any logic that is not common sense.
Example: "…If $x \in A$ and $x \notin B$, and $A, B \subseteq X$, then $x \notin (X - A) \cup B$…"
FL: "then $x \notin (X - A) \cup B$"

(**II**) Informal inference: An inference depending on common sense logic.
Example: "…If $a \in A$ … $A \subseteq B$… then $a \in B$…"
II: "then $a \in B$"

(**IR**) Interior reference: A chunk of a proof that calls on anything stated earlier in the proof.
Example: "…Let $x \in A$ … $A \subseteq B$… Since $x \in A$, $x \in B$…"
IR: "Since $x \in A$"

(**REL**) Relabeling: Giving an object a new (usually shorter) label.
Example: "…Thus $e_a = e_b$ is the identity. Set $e = e_a = e_b$…"
REL: "Set $e = e_a = e_b$"

(**SI**) Statement of intent: A small statement in a proof that indicates what is intended in the rest of the argument.
Example: "…We want to show that $x \in A$…"
SI: "We want to show that $x \in A$"

(**SIM**) Similarity in Proof: An indication that a section of a proof can be repeated with the same arguments previously given for another part of the proof.
Example: "…Therefore A is a left ideal. Similarly, A is a right ideal…"
SIM: "Similarly, A is a right ideal"

READING ONLINE MATHEMATICS TEXTBOOKS

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Abstract:

This study explores how students read from an online mathematics textbook. The particular textbook that we are exploring is *Precalculus: Pathways to Calculus,* which was developed at Arizona State University as part of a redesigned precalculus course that focuses on developing students' ability to reason conceptually about functions and quantity. We are interested in understanding the way students read their mathematical textbooks so that research informed activities can be developed and incorporated into online textbooks to increase comprehension and retention. In order to investigate authentic student reading habits as closely as possible, we used nonintrusive screen capture software to measure activities such as scrolling, latency, and browsing, as students completed their regular reading assignments in a study hall setting. Other data sources include brief surveys, assessments and interviews. Interventions include reading instruction and embedded activities with feedback and sequences of hints that are intended to promote deeper engagement with the text.

Introduction

Many would agree that reading is critical for gaining understanding within a discipline, and that students will not reap the full benefits of their studies if they skim through (or worse yet, ignore) their reading assignments. Even in quantitative disciplines such as mathematics, teachers may assign readings from the textbook with the intent of having students come to class more prepared and giving them exposure to more material than can be taught in the time allotted to class meetings. However, few teachers would be so naïve as to believe that students actually read the text, and often complain about the unpreparedness of the students for instruction. On their part, students complain about how hard it is to read mathematics textbooks, perhaps because they lack appropriate reading strategies that might remedy the situation. Indeed, even first-year undergraduate who are good general readers do not read mathematics textbooks well (Shepherd, Selden & Selden, 2009).

One solution to the problem of getting students to read mathematics texts effectively, despite their deeply instilled poor reading habits, is to harness technology. Online mathematics textbooks are a fairly recent (and increasingly popular) addition to the available set of instructional resources. In contrast to physical textbooks, online texts have affordances for interactive and responsive engagement. In particular, online texts can include activities that foster effective reading through embedded tasks that provide feedback and hints. The purpose of this project is to begin to understand how readers interact with an online mathematics textbook in a quasi-authentic setting, and to study the effects of some scaffolded online activities intended to help students monitor their comprehension of what is read.

Literature & Theoretical Perspective

Reading involves both decoding and comprehension. On the comprehension side of the coin, research has identified several strategies that good readers employ as they engage with a

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text (Flood & Lapp, 1990; Palincsar & Brown, 1984; Pressley & Afflerbach, 1995). Of course, these strategies depend on the individual reader, the reader's goals, and the material being read. Mathematics textbooks, in particular, are "closed texts" in the sense that they seek to elicit a well-defined, "precise" response that is not open to differing interpretations from readers (Weinberg & Wiesner, 2011). Yet, many students have not been taught how to read their mathematics textbooks, and do not read them as intended. For instance, authors of mathematics texts include expository material to help students develop a deeper understanding of the mathematical concepts. Yet, despite the fact that an overwhelming percentage of students claim to read their mathematics textbooks for understanding, few students report attempts at reading the expository sections (Weinberg, in press). Our research addresses how students who *are* making an attempt to read their textbooks engage in this process, and how they might be better supported in their endeavors.

Our theoretical perspective is aligned with the view that reading is an active process of meaning-making in which knowledge of language and the world are used to construct and negotiate interpretations of texts (Flood & Lapp, 1990; Palincsar & Brown, 1984; Rosenblatt, 1994). In helping students navigate mathematics texts, we advocate reading strategies that stem from the Constructively Responsive Reading framework (CRR) that was developed in reading comprehension research (Pressley & Afflerbach, 1995). These strategies are intended to help students maximize their construction of knowledge from texts. In addition, we place an emphasis on *cautious reading* (Shepherd, Selden & Selden, under review) that helps students minimize inappropriate interpretations of their mathematics texts by detecting and correcting errors, misunderstandings, and confusions. Taken together, CRR-based strategies and cautious reading advocate encouraging students to carefully read expository text and check the correspondence between the inferences they have drawn and the author's intent, and discouraging students from forging ahead without carrying out and evaluating their performance on tasks provided by the authors.

Motivation, Framework and Research Questions

Standard "online" textbooks from the major textbook publishers look like the printed book. The pages are the same as in the printed version and have active links to videos, demonstrations, and similar problems to examples. The first author has had some experience with this type of online textbook in both a traditional precalculus and college algebra course. When the students were given reading instruction it appeared students were not likely to stop and monitor comprehension by "doing" any of the interactive activities.

The course from which volunteers were drawn uses an online text, *Precalculus: Pathways to Calculus,* which was developed at Arizona State University and was designed to foster students' ability to reason conceptually about functions and quantity (Carlson & Oehrtman, 2009). *Precalculus: Pathways to Calculus* textbook is only online. It does not look like a traditional book put into an online text with pages identical to the printed textbook and added activities. Instead, interspersed with the reading are questions to the students. The questions are asked and then the student is invited to interact with the text, signaled with the following line:

Think about it for a moment and then <u>access this link to see our answers.</u> (See Figure 1)

Figure 7-26

In the case of the Ferris wheel, when asked to evaluate $\cos\left(\frac{\pi}{2}\right)$, we are asking how far, in radii, to the right of the vertical diameter is the individual in the bucket when the Ferris wheel has rotated $\frac{\pi}{2}$ radians from the 3 o'clock position.

What does it mean to evaluate $\cos(1.4)$ in the context of the Ferris wheel problem? That is, what does the input 1.4 represent and what is the meaning of the value $\cos(1.4)$?

Think about it for a moment and then access this link to see our answers.

What does $\cos(3.6) = -0.8968$ refer to in this situation?

Think about it for a moment and then access this link to see our answers.

What is the maximum distance that an individual can be to the right of the vertical diameter?

Think about it for a moment and then access this link to see our answers.

Figure 1: Example of questions posed to reader in Pathways online precalculus book.

Readers are invited to think about an answer, then open the link that gives a "drop down" box with an explanation. (See Figure 2)

What does it mean to evaluate $\cos(1.4)$ in the context of the Ferris wheel problem? That is, what does the input 1.4 represent and what is the meaning of the value $\cos(1.4)$?

Think about it for a moment and then access this link to see our answers.

> The input of angle measure, 1.4 radians, represents 1.4 radius lengths of rotation of the individual in the bucket from the starting position. The value of $\cos(1.4)$ is the horizontal distance of the individual to the right of the vertical diameter in radians when the angle measure is 1.4 radians.

What does $\cos(3.6) = -0.8968$ refer to in this situation?

Think about it for a moment and then access this link to see our answers.

> As the individual has traveled 3.6 radians from the starting position, she is 0.8968 radii *to the left* of the vertical diameter. After the individual has traveled 3.6 radians (223.2 feet for a Ferris wheel with a radius of 62 feet) along the arc swept out by the bucket, she is -0.8968 times the length of a radius (equal to -89.68% of a radius length or -55.6016 feet) to the right of the vertical diameter. A negative value indicates the individual is actually to the left of the vertical diameter.

Figure 2: Example of explanations available to students.

These interruptions in reading this particular online textbook lead us to ask some questions.

1. How do students interact with this type of request to engage with the material?
2. Do students interact more with this type of activity request?
3. Does how students engage with these activities help them with self monitoring of understanding?

The questions with drop-down box answers are essentially passive activities—students do not have to do anything to get the answer besides just click to open it. We suspected that students might not engage with these more passive activities as fully as they should. What if we could create active "interruptions" of a similar type? We created several activities of this second type (which we affectionately called "flashlettes") using authoring tools that were developed by the Open Learning Initiative at Carnegie Mellon University. We then were able to insert these activities and created a modified text that was the same except some of the questions with drop down answers were replaced by flashlettes. (See Figure 3 and 4)

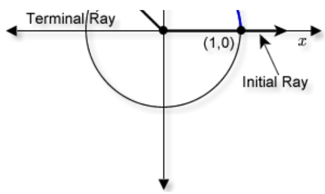


Figure 7-26

In the case of the Ferris wheel, when asked to evaluate $\cos\left(\frac{\pi}{2}\right)$, we are asking how far, in radii, to the right of the vertical diameter is the individual in the bucket when the Ferris wheel has rotated $\frac{\pi}{2}$ radians from the 3 o'clock position.

THE COSINE OF AN ANGLE MEASURED IN RADIANS — Hint

Now, let's look at a different angle. What does cos(3.6) = -0.8968 refer to in this situation?

a) As the individual has traveled 3.6 radians from the starting position, she is 0.8968 radii *above the horizontal diameter.*

b) As the individual has traveled 3.6 radians from the starting position, she is 0.8968 radii *below the horizontal diameter.*

c) As the individual has traveled 3.6 radians from the starting position, she is 0.8968 radii *to the right of the vertical diameter.*

d) As the individual has traveled 3.6 radians from the starting position, she is 0.8968 radii *to the left of the vertical diameter.*

oli — step 2 of 2

Figure 3: Example of a flashlette as it appears in the modified text—matches the second question in Figure 1.

What is the maximum distance that an individual can be to the right of the vertical diameter?

Think about it for a moment and then access this link to see our answers.

EXAMPLE 8. THE FERRIS WHEEL AND THE COSINE FUNCTION

Given that the input of the function $l = g(\theta) = \cos(\theta)$ defines the distance, in radians, the individual has traveled counterclockwise around the Ferris wheel from the 3 o'clock position, how many revolutions has the individual traveled if θ (the angle of rotation of the individual) varies from 0 radians to 5π radians?

Think about it for a moment and then access this link to see our answers.

MEANING OF THE COSINE GRAPH — Hint

When graphed, what does a point on the graph of $l = g(theta) = cos(theta)$ represent in the context of the Ferris wheel problem above?

a) It represents the **horizontal distance** of the individual **to the right of the vertical diameter** (in radii) as a function of the number of radians the individual has traveled counterclockwise around the circumference of the Ferris wheel from the 3 o'clock position.

b) It represents the **vertical distance** of the individual **above the horizontal diameter** (in radii) as a function of the number of radians the individual has traveled counterclockwise around the circumference of the Ferris wheel from the 3 o'clock position.

oli — part 1 of 2 — next

Figure 4: Example of a flashlette that shows not all passive activities replaced.

These flashlettes had in addition, scaffolded tasks that provided students not only with right/wrong feedback but also a sequence of hints, progressively leading to a correct response. For instance, the hint sequence, accessed from the green Hint corner, for the activity shown in Figure 3 was:

1. As the individual has traveled 3.6 radians, she will be a certain distance from the

vertical diameter.
2. If the entire circumference of the circle is 2pi or approximately 6.28 radians, then will she be to the left or right of the vertical diameter after traveling 3.6 radians?
3. If she is to the right of the vertical diameter, the value of cos(3.6) will be positive. If she is to the left of the vertical diameter, it will be negative.
4. As the individual has traveled 3.6 radians from the starting position, she is 0.8968 radii to the left of the vertical diameter.

Because we were able to present both the regular online text and a text modified with these more active flashlette interruptions we were able to consider some additional questions:

4. Would students interact differently with the flashlettes?
5. Would the interaction with the flashlettes be more effective in student self monitoring of understanding?

The theoretical framework that we are using for this research is the Constructively Responsive Reading (CRR) framework of Pressley and Afflerback. It is essentially a constuctivist approach to reading for comprehension that has good readers actively engaging with the material read and includes monitoring comprehension and integrating what is read with prior knowledge among other strategies and activities. Observations of students reading their standard mathematics textbooks (Shepherd, Selden & Selden, 2009) leads us to believe most students don't know how to engage effectively as they read mathematics textbooks, when they read them at all. In addition, it does appear that mathematicians do learn how to read and engage with mathematical text—we might call this *cautious reading* and this seems to go beyond the CRR framework.

Our major research questions, thus, are:

- How are students interacting with the Pathways online precalculus text?
- How effective are their apparent reading strategies to the learning of students from this online textbook?

Research Methods

The participants were 36 students enrolled in sections of a redesigned precalculus course at a large southwestern university. The course used the online text, *Precalculus: Pathways to Calculus* (Carlson & Oehrtman, 2009) discussed above. Students were recruited to volunteer for participation in seven Study Hall sessions once weekly of approximately 1.5 hours each. To encourage participation the teachers of the 10 sections using the Pathways curriculum were asked to give participants a small amount of extra credit. Not all 36 students completed all 7 weeks. (See Table1)

Number of students	Number of weeks completed
12	7
7	6
5	5
1	4
3	3

4	2
4	1

Table 1: Number of weeks of participation/number of students

During the Study Hall sessions, students were asked to complete their current reading assignment on the computers provided. Paper for scratch work was provided and collected. Copies were provided to the students if they wanted their scratch work. The students generally read the section(s) to be covered during their next class period. Some teachers were a section or so ahead of other teachers so four of the weeks, students read from different sections based on where they were in their class. (See Table 2.)

Week	Group	Module/Section	Title
1	Both	Mod 4 Sec 6	Logarithms and Solving for Exponents
2	Both	Mod 5 Sec 1 & 2	Example of Polynomials & Exploring Co-varying Quantities Qualitatively
3	A (ahead)	Mod 5 Sec 8 & 9	Exploring the Growth of Power Functions & Quadratic Functions
3	B (behind)	Mod 5 Sec 6 & 7	Evaluating and Interpreting Values of Polynomial Functions & Roots, Maximum Values, Minimum Values and Graphing Polynomial Functions
4	A	Mod 6 Sec 5 & 6	Strategy for Determining Horizontal Asymptotes & Strategy for Determining Vertical Asymptotes
4	B	Mod 6 Sec 3 & 4	Long-run Behavior: Horizontal Asymptotes & Vertical Asymptotes
5	A	Mod 7 Sec 3	The Unit Circle and Trigonometric Functions
5	B	Mod 7 Sec 2	Angles and Their Measures
6	A	Mod 7 Sec 7	Evaluating Trigonometric Functions and Solving Trigonometric Equations
6	B	Mod 7 Sec 5	The Cosine Function
7	Both	Mod 7 Sec 6	Period, Amplitude, and Translations of the Sine and Cosine Functions

Table 2: Material covered by reading each week.

In order to investigate authentic student reading habits as closely as possible, nonintrusive screen capture software was used to measure activities such as scrolling, latency, and browsing. In addition, prior to and following their reading of the text at each Study Hall session, students completed short mathematical assessments (a pre- and a post-test) based on the relevant text material. Other data sources included brief surveys addressing reading habits, and, for most students, admissions testing scores (SAT/ACT) as a control for mathematical and reading preparedness (See Table 3).

	Average Math	Avg Read/Verbal	N

ACT	24.1	24.8	15
SAT	510	535	22

Table 3: SAT/ACT average scores (Seven students had both scores reported.)

Also, half of the participants were randomly assigned during the final four Study Hall sessions to a version of the text in which some of the questions with pop-down solutions (e.g., hidden answers) were replaced with the flashlette activities. Finally, during the last three weeks, a total of 13 students were interviewed as to their reading activities and asked to step the interviewer through what they had done as the read the section of the textbook they read that day.

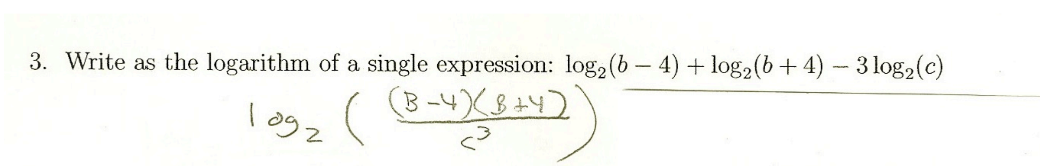
Analysis Methods

We will discuss the analysis started (some complete, some only partially complete) so far related to (1) the pre and post-tests, (2) the screen capture videos, and (3) interviews. Analysis has not yet begun on the initial survey on reading habits or the scratch work.

Pre/Post Tests. The Pre and post tests had from 4 to 12 questions, matched with only numbers or expressions slightly changed. Each question was scored on a 0—4 point scale where:

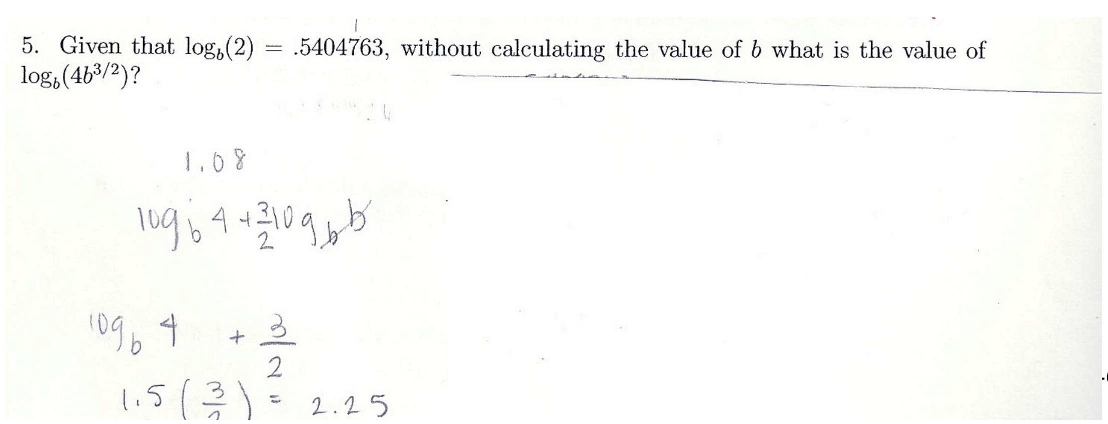
- 0 = no work (or useless work)
- 1 = some work shown, but little understanding (of what was read) demonstrated, incorrect application of material read
- 2 = some work shown and some understanding demonstrated, incorrect/incomplete application
- 3 = work demonstrates minor mistakes, correct understanding of most of what was read.
- 4 = correct work, correct understanding demonstrated

Here are two examples of scoring.


3. Write as the logarithm of a single expression: $\log_2(b-4) + \log_2(b+4) - 3\log_2(c)$

$\log_2\left(\frac{(B-4)(B+4)}{c^3}\right)$

The above work received a score of 4.


5. Given that $\log_b(2) = .5404763$, without calculating the value of b what is the value of $\log_b(4b^{3/2})$?

1.08

$\log_b 4 + \frac{3}{2}\log_b b$

$\log_b 4 + \frac{3}{2}$

$1.5\left(\frac{3}{2}\right) = 2.25$

The above work received a score of 2.

Screen Capture Video. The goal of the screen capture video was to attempt to record how students actually scrolled through and read the text. In the videos where there were no flashlette activities, each type of object (text, boxed text, figure, question, drop down box for answer) was coded according to the apparent amount of time the student spent on it.

0 = skipped
1 = scrolled quickly through/scanned
2 = scrolled as though reading
3 = long pauses in the scrolling

In addition, if there was cursor movement a "C" was added to the number code. Arrows were used to indicate where students scrolled to in the text. Brackets were used to indicate text on a page when it was difficult to tell where the student's attention was. An example coding sheet is shown in Figure 4.

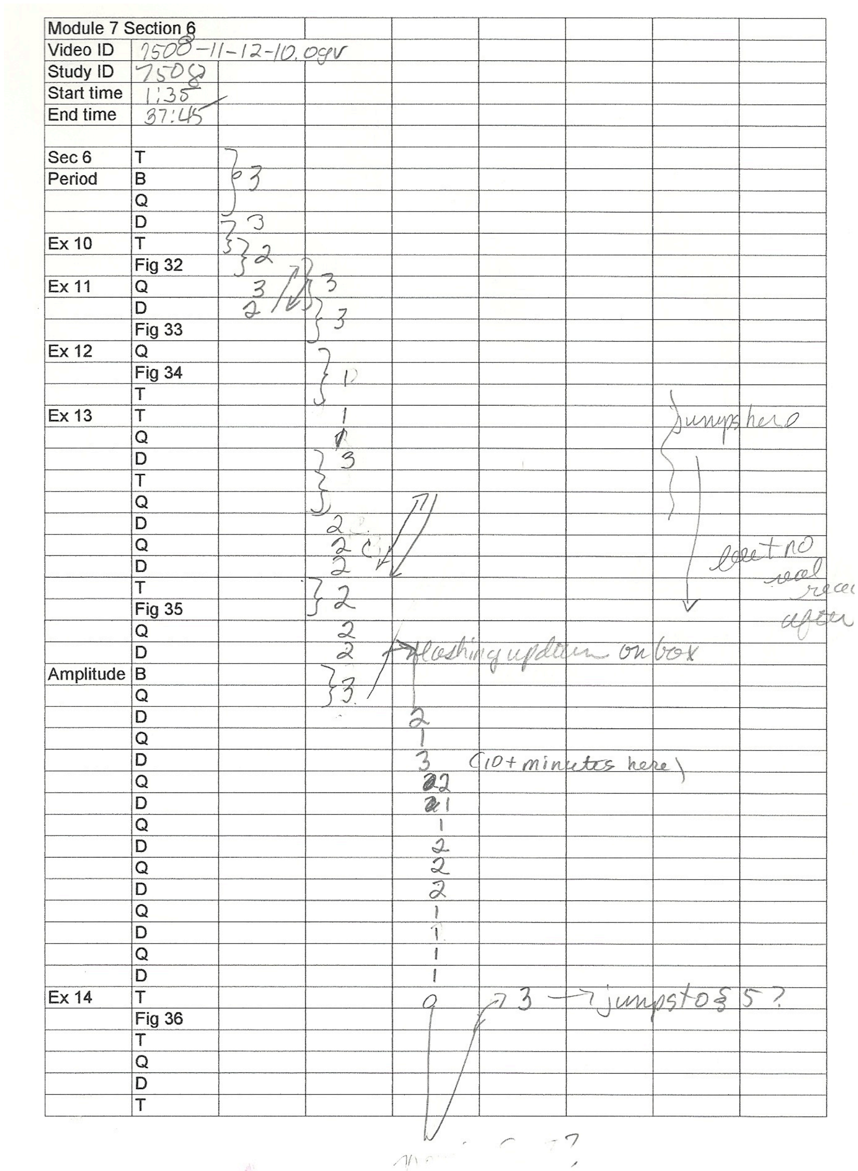


Figure 4: Example of coding for video screen capture. Arrows indicate student returning to previously encountered text or scrolling ahead.

Coding on videos where flashlettes appeared contain additional coding as to whether the opened answer was right (R) or wrong (W). (See Figure 5)

Question	2		
Answer	W	W	R
Feedback	1	2	

Figure 5: Example of additional coding for flashlette activities. Here the student

appeared to read the question, chose an incorrect first response (W), skimmed the feedback (1), chose a second incorrect response (W), read the feedback (2), and then chose the correct response (R).

Interviews: Interviews have been summarized but not yet been transcribed. Interesting comments were noted.

Results and Observations

After each reading session the students were asked to complete an After Reading Survey (ARS). One of the questions on this survey was for the student to rate his/her understanding of the section he/she just read. (See Figure 5).

i. 1 = I don't understand anything I just read.
ii. 2 = I understand a little, but not very much.
iii. 3 = I understand about half of what I read.
iv. 4 = I understand most of what I read.
v. 5 = I understand what I read perfectly.

Figure 5: Rating scale students used to rate understanding of material just read.

Not unexpectedly, students' beliefs about their level of understanding were often overestimates. For instance, in Week 4, the pre and post-tests had 6 questions for the section on Long-run Behavior: Horizontal Asymptotes & Vertical Asymptotes. A student had scores of 1,1,0,3,0,0 on the pre-test and 1,1,1,2,1,1 on the post-test yet rated his/her understanding as a 4 indicating the student felt he/she understood most of what he/she read. This is similar to findings by Chi, et al. (1989) that poor students were less aware of not understanding than good students.

If we use the student perception of understanding as a measure of how difficult the students felt the sections were to read/understand, then not all sections read appear to have the same level of difficulty—students "understood" some sections better than others as indicated in Table 4 below.

Looking at the self-rating column there seem to be two distinct categories of understanding. There are 5 sections with ratings of 3.8 or higher (which we might categorize as "easier" for the students feel like they understand), and 6 sections with ratings of 3.1 or lower which we might categorize as "harder" for students to fell like they understood), and no ratings in between. This should be useful feedback to the authors as to which sections students find more difficult to understand.

Module/Section (Week)	**Average Self rate understanding**	**N**
M4 S6 (1)	4.3	31
M5 S1&2 (2)	3.8	33
M5 S6&7 (3)	3	8
M5 S8&9 (3)	2.7	1

		9
M6 S3&4 (4)	3.1	9
M6 S5&6 (4)	4.2	19
M7 S2 (5)	4.1	7
M7 S3 (5)	3.8	17
M7 S5 (6)	2.8	4
M7 S7 (6)	2.7	13
M7 S6 (7)	2.9	19

Table 4: Student self-rating of understanding by section read

Analyzing the pre- and post-test scores by looking at the change from pre to post score by question we have the data summarized in Table 5 below. The Treatment Group is the group of students that viewed the text with the flashlette activities, the Control Group is the group of students who viewed the text without the flashlette activities during the last four weeks of the study. During the first three weeks of the study, all students viewed the text without flashlettes.

Post – Pre	First 3 weeks (All)		Last 4 Weeks (Control)		Last 4 Weeks (Treatment)	
	Count	Percent	Count	Percent	Count	Percent
+4	24	6.2%	7	3.4%	38	12.0%
+3	38	10.0%	33	15.9%	20	6.3%
+2	41	10.6%	24	11.6%	34	10.8%
+1	89	23.1%	26	12.6%	56	17.7%
0	162	42.1%	92	44.4%	147	46.5%
-1	29	7.5%	17	8.2%	14	4.4%
-2	1	0.3%	3	1.4%	4	1.2%
-3	1	0.3%	5	2.4%	2	0.6%
-4	0	0%	0	0%	1	0.3%

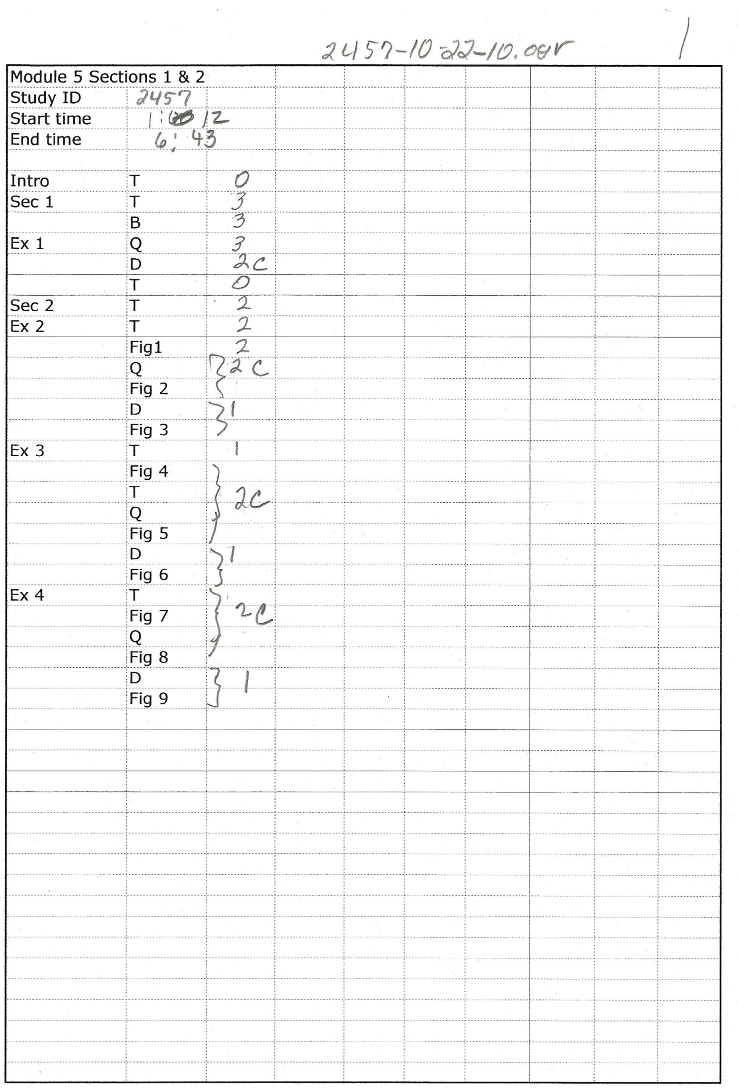
Table 5: Count of numbers of matched pre-/post-test question point improvement

An initial observation is that the reading did not harm the students. Just under 50% of the students in each group showed an increase between pre and post-test scores and over 40% showed no change. No change does not necessarily mean the student made no progress, because a pre and post-test score of 4 (indicating understanding) would have resulted in a zero in the table above. In fact, 63 of the 162 no change scores in the first 3 weeks came from pre and post-test scores of 4. Similarly in the last 4 weeks, 95 of the 239 no change scores came from pre and post-test scores of 4. The percent of questions where the scores dropped is small, and mostly appears in drop of one point. This was mainly due to minor errors where the pre-test and post-test had similar understandings, but the post-test showed a minor calculation error. For example, when the sample scoring example that received 4 points was given above, the post-test for the same person had the following work and received a score of 3 because the final exponent on c should have been 3 instead of 2.

3. Write as the logarithm of a single expression: $\log_2(b-4) - \log_2(b+4) - 3\log_2(c)$

$$\log_2 \frac{(B-4)}{(B+4)c^{2}}$$

There will be more analysis done when these scores are integrated with the screen capture video where we hope to capture some measure of apparent student engagement.



2457-10-22-10.ogv

Module 5 Sections 1 & 2		
Study ID	2457	
Start time	1:12	
End time	6:43	
Intro	T	0
Sec 1	T	3
	B	3
Ex 1	Q	3
	D	2c
	T	0
Sec 2	T	2
Ex 2	T	2
	Fig1	2
	Q	2c
	Fig 2	
	D	1
	Fig 3	
Ex 3	T	1
	Fig 4	2c
	T	
	Q	
	Fig 5	
	D	1
	Fig 6	
Ex 4	T	2c
	Fig 7	
	Q	
	Fig 8	
	D	1
	Fig 9	

Figure 6: Example of a coding where a student just reads straight through.

From the screen capture videos, we can see different reading techniques. Some students seem to just read straight through the material. Some students go back frequently to re-read material. Many students scroll to the end of reading apparently to see how much further there was to read. Figure 6 shows what a coding some one more or less reading straight through the material might look like. Figure 7 shows a student going back in the material, rereading or reviewing material previously encountered.

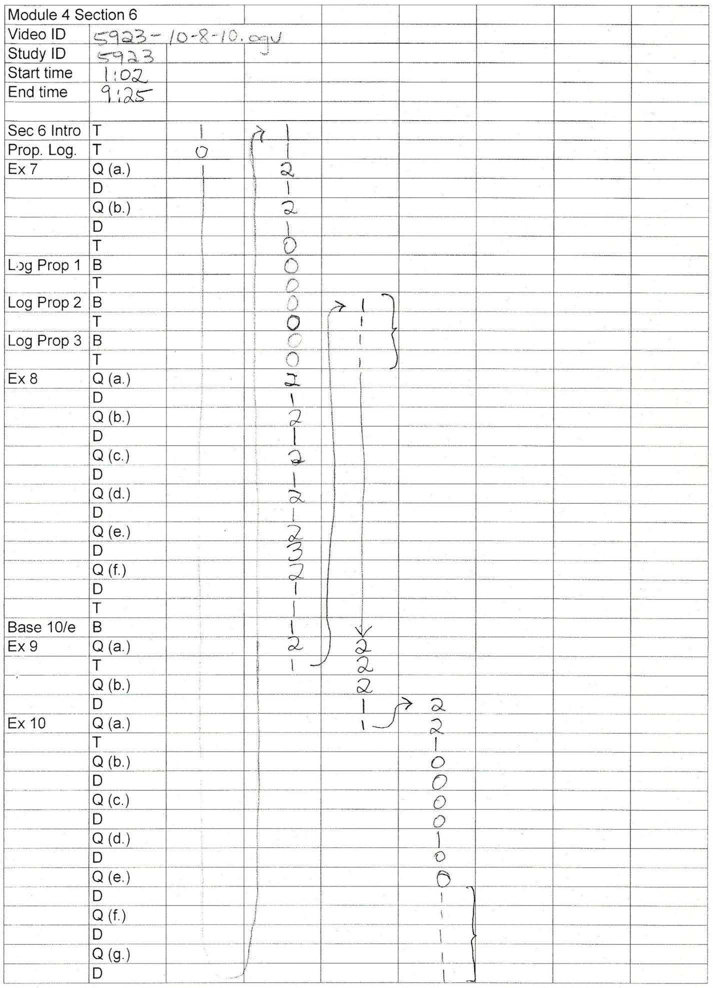


Figure 7: Example of a student who both scrolled to the bottom of the section and went back to re-read a few times.

We have observed that students may read less carefully when they believe they understand the material. Further analysis here is needed. We also believe we will be able to give a rough estimate of how long students are willing to read carefully at a single sitting. In Figure 7 when the student reaches Example 10, the student clearly begins skimming and does not pay close attention to the remaining text or participate in the questions in the text.

There were different approaches to answering flashlettes. So far, using open-ended axial coding (Strauss & Corbin, 1990), five different approaches have been identified, although the actual number of each is still being determined. These are described in Figure 8.

Reading approach	Typical coding

Skipping	Question: 0 Answer: Feedback:
Gaming (guessing at answer until correct)	Question: 1-2 Answer: W W R Feedback: 0 0
Got it! (or Lucky Guess)	Question: 2 Answer: R Feedback: 0
Using Feedback	Question: 2 Answer: W W R Feedback: 1 2
Re-reading	Question: 2B Answer: R Feedback: 1

Figure 8: Typical codings for the flashlettes

Finally, interviews were conducted with 13 of the students. Five were in the control group, eight were in the treatment. Although the interviews have yet to be transcribed, 12 of the 13 students stated they did not usually read the section before or after it was covered in class, even though each stated that when they did, what went on in class made more sense. Although we wish this were different, it is not different from what happens with standard textbooks.

Maybe the most interesting observation to come from the interviews is that students are quite willing to work, on their own, a calculation type problem (e.g. What is the maximum distance right of the vertical diameter?—see Figure 1), in either the control or treatment group, but questions which required an explanation (e.g. What does it mean to evaluate cos(1.4) in the context of the problem? –see Figure 2) students were unwilling to pause and think about, much less try to formulate an answer. They claimed they usually just opened the answer (in the drop down form), then read back through the response. We hope to quantify this behavior as we complete our analysis of the current data.

We have some preliminary answers to our major research questions. To the

question,
How are students interacting with the Pathways online precalculus text? They are interacting with the text when they read it, but their interaction is still not particularly effective in understanding the concepts covered.

To the question, How effective are their apparent reading strategies to the learning of students from this online textbook? Their strategies are not as helpful as one would hope, but there is hope that in developing activities that will help students work toward more interaction with "explain" type problems, new reading strategies can be developed to help students monitor their understanding better.

Implications for Further Research & Teaching

This research project is a preliminary step for identifying and constructing activities that promote effective reading strategies and that can be embedded in online mathematics textbooks. At this stage, we are restricting our activities to multiple-choice questions. There is a need for research that identifies statistically valid response choices that capture common student errors and ways of thinking so that appropriate sequences of hints can be designed. For instance, certain incorrect answers might be best addressed by posing hints that promote cognitive conflict with that particular way of reasoning.

We would also like to explore how students who rely on embedded scaffolded tasks to read their textbooks effectively can be graduated to the adoption of their own reading strategies that are consistent with reading for understanding. To address this issue, both the timing and manner in which the activities are faded need to be investigated.

Finally, this research has implications for how teachers can connect with the reading aspect of their students' instruction. At present, in order to check whether students have completed a reading assignment, many teachers resort to giving quizzes during (valuable) class time on the relevant material. Online texts can be designed to capture and log student actions, and so provide indicators of whether (and how) students are completing their reading assignments.

Summary

At the heart of our project is the goal of helping students become more effective readers of introductory level mathematics texts. In order to achieve this goal, we are harnessing the affordances of technology, and exploring the ways that activities can be embedded within online textbooks. Although the goal of these activities is to foster reading with understanding, we do not anticipate that they will produce "cautious readers." Instead, our much more modest hope is that we can help students turn over a new page in the way they interact with their textbooks.

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EXPLORING THE VAN HIELE LEVELS OF PROSPECTIVE MATHEMATICS TEACHERS

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This research project aimed to explore the van Hiele levels of undergraduate mathematics majors, most of whom were prospective mathematics teachers. The participants were taking a proof-intensive Euclidean geometry course that relied on technology, The Geometer's Sketchpad (KCP Technologies, 2006), to help them make and prove conjectures. Data was collected from classes in consecutive years, the first with 21 participants, the second with 24 participants. Data collection included a pre- and a posttest of participants' van Hiele levels. Data analysis suggests similar results for both sets of participants in that the course had greater influence on the van Hiele levels of female participants. Results also suggest that the van Hiele test instrument used for this study operated well with university students.

Keywords: Geometry, van Hiele levels, teacher preparation, secondary

In 1957, Pierre Marie van Hiele and Dina van Hiele-Geldof, mathematics educators in the Netherlands, developed a learning model for geometry as their doctoral thesis. The van Hieles defined what are now known as *the van Hiele levels of development in geometry*, which, according to van Hiele-Geldof's thesis, as cited in Fuys, Geddes, and Tischler (1984), are hierarchical. In other words, one cannot reach a higher level before mastering the lower levels first. Furthermore, van Hiele-Geldof's dissertation indicates that the language acquired at one level differs from that at a higher level (Fuys, Geddes, & Tischler, 1984).

Altogether, there are five van Hiele levels (VHLs), originally numbered 0 through 4 by the van Hieles and renumbered 1 through 5 in most American publications. In this study, as discussed by Senk (1989), the 1 through 5 numbering scheme will be used to allow an extra level, level 0, when the first VHL, called *basic* by Pierre Marie van Hiele, is not attained. The five van Hiele levels are summarized in Table 1 from several authors' interpretations of the definitions (Burger & Shaughnessy, 1986; Mayberry, 1983; Mistretta, 2000). Exact definitions can be found in van Hiele-Geldof's doctoral thesis (Fuys et al., 1984) and a more detailed list of behaviors at each level can be found in Usiskin (1982).

Table 1. Van Hiele Levels of Development in Geometry

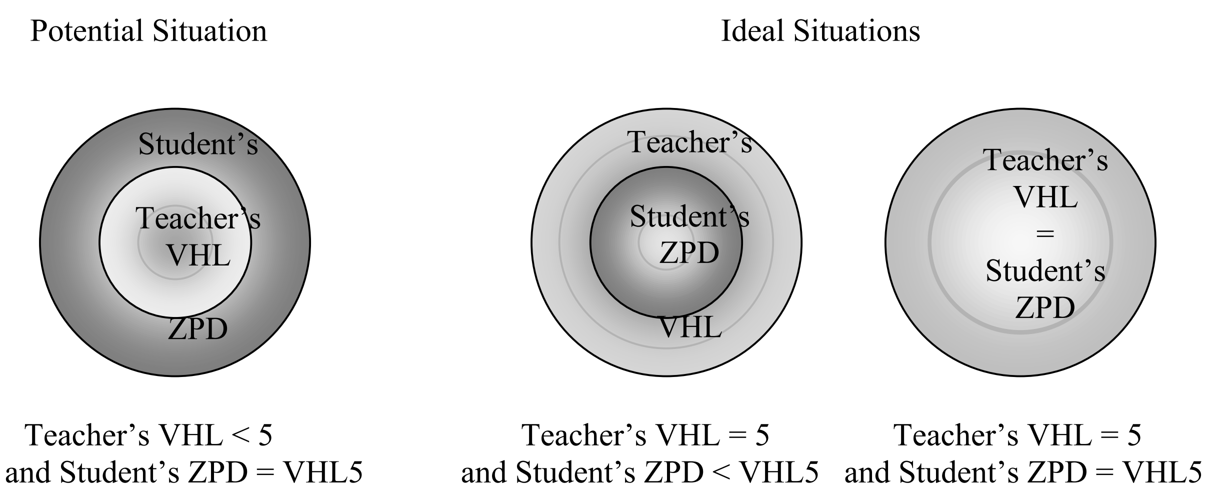
Level[1]	*Name*	*Characteristics*[2]
0	Pre-VHL	Student does not meet van Hiele basic level, that is, VHL 1.
1	Visualization[3]	Students visualize geometrical figures as a whole and recognize them by their particular shape.
2	Analysis	Students recognize the geometric properties of the

		different figures and are able to analyze the figures separately, but do not yet make connections between figures.
3	Abstraction	Students recognize relationships between figures and between properties of different figures.
4	Formal deduction	Students can write proofs and should provide justifications for each step in the proof.
5	Rigor	"… student understands the formal aspect of deduction… [and] should understand the role and necessity of indirect proof and proof by contrapositive" (Mayberry 1983, p. 59). Students can understand non-Euclidean geometries.

Notes. 1: This choice of levels was made to account for any student who did not meet the first van Hiele level.
2: The characteristics are drawn from Burger and Shaughnessy (1986), Mayberry (1983), and Mistretta (2000).
3: Level 1, visualization, corresponds to van Hiele's level 0 which he called the basic level.

As seen in past research, the van Hiele level or the level of competence in geometry of some teachers (male and female) is not at the highest level (Mayberry, 1983; Sharp, 2001; Swafford, Jones, & Thornton, 1997; van der Sandt & Nieuwoudt, 2003) thus possibly hindering the learning of geometry of some students. The conflict may arise when there is a discrepancy between the van Hiele level of the teacher and the zone of proximal development (ZPD; Vygotsky, 1987, p. 209) of the student as depicted in Figure 1.

Figure 1. ZPD VS Teacher's VHL - Potential and Ideal Situations



The authors believe that during the "'sensitive period of learning'", as cited by Vygotsky (1987, p. 212), the development of a student in geometry may be limited by the teacher's own ability to complete related tasks independently. We expect that the teacher should be knowledgeable and competent in geometry and we consider that the teacher's level of knowledge and competency may be equivalent to the teacher's van Hiele level of development. So, suppose that all five van Hiele levels of development are within the reach of the students, that is, within their zone of proximal development and that geometry instruction occurs during the sensitive period. Then, if the teacher's VHL is not at level 5, we argue that the van Hiele level of development of the teacher may limit the students' learning. Refer to the potential situation depicted in Figure 1. However, if the teacher's VHL is at the highest level, students' development is not compromised. Refer to the ideal situations depicted in Figure 1. It thus becomes ideal for all prospective mathematics teachers to achieve the highest van Hiele level of development, that is, level 5 or rigor.

While it is ideal for all prospective teachers to be at van Hiele level 5, gender differences favoring males are almost twice as large in geometry as in other areas of mathematics (Leahey & Guo, 2001). However, the findings reported in the literature suggest variations in gender differences. It also suggests that the differences are mostly in spatial visualization tasks. For example, in his study of 128 high school geometry students, Battista (1990) found that "males scored significantly higher than females on spatial visualization, geometry achievement, and geometric problem solving" (p. 54). In fact, on average, males had better spatial visualization skills than females and they always scored higher than females. However, Senk and Usiskin (1983) studied high school geometric proof writing abilities, which they consider as a high-level cognitive task requiring some spatial ability. While overall geometry performance has not been analyzed by gender, they found no gender differences in achievement in geometric proof writing at the end of a one-year geometry course even though females started the year with generally less geometry knowledge.

This review of literature only found a few peer-reviewed published studies involving the van Hiele levels in geometry of preservice or inservice teachers. Among them, one study has been conducted on VHLs of preservice elementary teachers (Mayberry, 1983), one on VHLs of inservice middle-grade teachers (Swafford et al., 1997), one on the geometry content knowledge of Grade 7 prospective and inservice teachers (van der Sandt & Nieuwoudt, 2003), two on three-dimensional geometric thinking of preservice teachers (Gutiérrez, Jaime, & Fortuny, 1991; Saads & Davis, 1997), and one on developing geometric thinking of inservice K-7 teachers (Sharp, 2001), but none specifically on the VHLs of prospective secondary mathematics teachers in a proof-intensive college Euclidean geometry course..

Since the two higher VHLs involve proof-writing abilities and all prospective secondary mathematics teachers, who are mathematics majors, have completed a high school geometry course, it makes sense that instruction that focuses on proof-writing activities may lead to an increase in VHLs. Therefore, the purpose of this study was to explore the van Hiele levels of prospective mathematics teachers before and after an upper-division undergraduate Euclidean geometry course that focused on geometric proof writing and to verify the choice of Usiskin's (1982) test to explore the VHLs of undergraduate students.

The Study

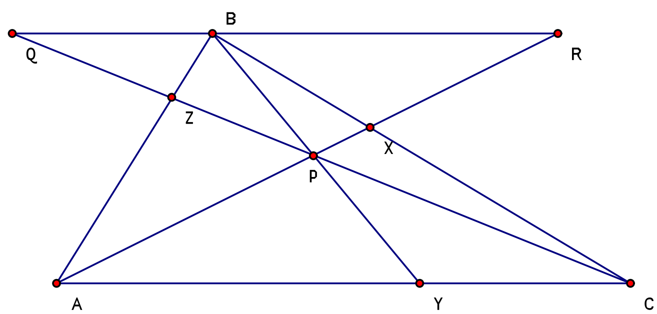
This research was conducted as a quantitative study using a pre- and posttest design with a convenience sampling as defined by Creswell (2005). Participants were enrolled in Euclidean geometry courses in consecutive years in a four-year Master's degree granting university located in the central coast of California. The first class included 21 participants, while the second class included 24 participants. Of the 45 participants, 40 were math majors and almost all were preparing to become a secondary mathematics teacher.

During both quarters the course was taught over a ten-week period, and met four times a week for fifty-minute sessions. The prerequisite is a course in methods of proof in mathematics, which focuses on instruction of logic and proof techniques. In addition, familiarity with the dynamic geometry software *The Geometer's Sketchpad* (GSP; KCP Technologies, 2006) is recommended and the geometry course is mandatory for math majors preparing to become secondary teachers.

Pedagogically, the classes could fully be described as proof-intensive (by their content) and as inquiry-oriented and technology-based (by their approach). A typical class meeting included working on instructor developed, inquiry-oriented activities using *The Geometer's Sketchpad*, and presenting and discussing solutions to the activities. A typical inquiry-oriented activity is shown in Figure 2.

Figure 2. A Typical Inquiry-Oriented Class Activity

1. (Based on Ch. 2 Activity 6 [Reynolds & Fenton, 2006]) Given with $\overleftrightarrow{QR} \parallel \overleftrightarrow{AC}$ and that $\overline{AX}, \overline{BY}$ and $\overline{CZ}$ are concurrent at *P*. Find all pairs of similar triangles in the picture below.



2. Draw an arbitrary . Let X be a point on $\overline{BC}$ and Z be a point on $\overline{AB}$ with $\overleftrightarrow{AX}$ and $\overleftrightarrow{CZ}$ intersecting at P. Construct $\overleftrightarrow{BP}$ and label its intersection with $\overline{AC}$ as Y. Now calculate $\frac{m\overline{AZ}}{m\overline{ZB}}, \frac{m\overline{BX}}{m\overline{XC}}$ and $\frac{m\overline{CY}}{m\overline{YA}}$ and find the product of the ratios. Write a conjecture about this situation and vary some points to be sure the conjecture holds. Use your work from #1 to prove the conjecture.

3. Your conjecture in #2 is called Ceva's Theorem. Write the converse of Ceva's Theorem and use GSP to decide if you believe the converse is true. If it is not true find a counterexample, if it is true prove it.

The first question in the activity is based on an exploration from the book *College Geometry Using The Geometer's Sketchpad* by Reynolds and Fenton (2006) and the goal is to lead to the proof of Ceva's theorem that is discovered in the second problem. The third problem required students to explore the converse of Ceva's theorem and to either provide a counterexample or justification of the converse. Participants used GSP throughout this exploration as a tool to make conjectures for #2 and #3 and as a tool to inform their formal proofs for #2 and #3. Participants' conjectures and formal proofs for #2 and #3 and the solution to #1 were discussed as a class through presentation of their work.

Activities, similar to the one in Figure 2, were completed in groups during the quarter using one to three class periods each. These activities were generally inquiry-oriented, that is participants were engaged in sense making activities that required them to use GSP to make and formally prove conjectures. Every activity required the proof of at least one conjecture and formal proof writing was the main form of assessment for the course. Participants presented their work regularly and presentations often resulted in rich mathematical conversations and multiple avenues to prove conjectures being explored.

Other activities expected of participants in the course included reading assignments and working on problems assigned from the book *College Geometry Using The Geometer's Sketchpad* (Reynolds & Fenton, 2006) in the first class and *Roads to Geometry* (Wallace & West, 2004) in the second class. Among all the problems assigned, problems were selected to create homework assignments completed individually and projects, made up of more challenging problems than the homework assignments, could be completed in groups of up to three participants. In addition to homework and projects, two one-hour exams were administered each quarter and the participants completed a three-hour final examination.

The Van Hiele Test

After examining several documents written by the van Hieles and describing behaviors at each van Hiele level, Usiskin (1982) developed a test instrument to assess the van Hiele level of an individual. Although this instrument was primarily devised with high school students in mind, it has been used, with permission from the authors, for this study (S. Senk, personal communication, November 19, 2007).

Whether the subjects involved would constitute an appropriate reference base for the study using Usiskin's test was considered since the subjects involved have all completed a one-year high school geometry course. However, even though the van Hiele levels have been defined while studying high school students, Pierre-Marie van Hiele, as quoted by Usiskin, believed that the highest level is "'hardly attainable in secondary teaching'" (1982, p. 12). Furhermore, Mayberry (1983), who devised her own test instrument, found that "70% of the response patterns of the students who had taken high school geometry were below Level III" (equivalent to level 4 in this study), and "only 30% were at Level III" (pp. 67-68). Therefore, it seemed adequate to use Usiskin's test with the participants in this study.

Time constraints in preparing a van Hiele level test instrument and in-class time usage were also key factors in the selection of a test instrument. Prior to choosing Usiskin's test, other test instruments were considered. For instance, Burger and Shaughnessy (1986), as cited by Jaime and Gutiérrez (1994, p. 41), developed a testing instrument to assess van Hiele levels, but its administration, through an interview, would have required more time to conduct. Mayberry's (1983) 128-item test was discarded for the same reason. Usiskin's test was readily available and it is a timed-test limited to 35 minutes.

Therefore Usiskin's (1982) 25-item questionnaire was used to measure the van Hiele level of development of the participants before and after the college geometry course.

Results

Each participant was assigned a raw score (out of 25) and a VHL similar to what Usiskin (1982) calls a "classical van Hiele level" (p. 25). Each group of five questions in Usiskin's test corresponds to a different VHL. For example, questions 1 to 5 correspond to VHL 1, questions 6 to 10 to VHL 2, and so on. Each test was examined by groups of five items. For a participant to be assigned a level, say n, at least four items must have been answered correctly at level n and at each preceding level. If a participant answered less than four questions correctly at level 1, then level 0 was assigned. The 4-item criterion was chosen over the 3-item criterion because random guessing on the test was not expected from the participants in this study and a higher mastery level was expected considering all the participants had completed a high school geometry course.

Analyzing the Test

Before analyzing the data with respect to our research purpose, we became interested in confirming our decision to use Usiskin's van Hiele test (1982) with mathematics majors. First, we noted that two participants did not meet the basic VHL and that none of the participants answered all the questions correctly on the pretest, thus allowing a measurement of growth on the posttest. Second, we implemented a Guttman scalogram analysis similar to that of Mayberry (1983) to determine whether the van Hiele levels as tested by Usiskin's test form a hierarchy. In order to do that, we created sequences in the same manner as Mayberry (1983) based on participants meeting the 4-item criterion for the set of five questions corresponding to each van Hiele level. For example, a sequence of 11101 means that a particular participant met the 4-item criterion at every van Hiele level except at VHL 4. In addition, this sequence is considered to have one error since the participant did not meet the level-4 criterion, but did meet the level-5 criterion. In this way, errors were counted as the number of 0s that appear before the last 1 in the sequence. Using the sequences generated, reproducibility factors (Reps), as described by Mayberry (1983), were calculated to give the percentage of VHLs that occur hierarchically in the sequences. The reproducibility factors were judged with the scale of 0.90 or better as suggested by Torgerson, as cited by Mayberry (1983). As can be seen in Table 2, these reproducibility factors confirm that the van Hiele levels, as tested by Usiskin's test, form a hierarchy in all cases except perhaps for male participants on the posttest in the first class. In our opinion, this Rep factor (0.867) in conjunction with the other 17 reproducibility factors over 0.90, and the results briefly mentioned previously, imply that Usiskin's van Hiele test operated adequately for our research population in terms of the hierarchical nature of the VHLs.

Table 2. Reproducibility Factors

Test	*Males*		*Females*		*Combined*	
	1st Class (12)	*2nd Class (8)*	*1st Class (9)*	*2nd Class (16)*	*1st Class (21)*	*2nd Class (24)*
Pretest	*0.96*	*0.975*	*0.91*	*0.90*	*0.9429*	*0.925*
Posttest	*0.86*	*0.925*	*0.95*	*0.9125*	*0.9048*	*0.916*
Combined Tests	*0.916*	*0.95*	*0.93*	*0.90625*	*0.9238*	*0.92083*

Once the functionality of Usiskin's test was confirmed, we turned to analyzing the data with our initial research purpose in mind. Essentially interested in exploring the van Hiele levels of prospective mathematics teachers before and after a proof-intensive geometry course, we began by analyzing the raw scores and the VHLs on the pretest and posttest. We then continued with analyses of changes in raw scores and VHLs, and changes by VHLs.

Raw Scores and van Hiele Levels

As suggested in past research, males outperformed females on the pretest in both classes (Battista, 1990; Hyde, Fennema, & Lammon, 1990; Leahey & Guo, 2001;

Halpern et al., 2007). However, females outperformed males on the posttest in the first class. The male and female average scores on the pretest and posttest can be seen in Table 3. With respect to the entire first class, the average raw score increased from 20.38 on the pre-test to 21.05 after the course and the second class saw an increase from 20.04 on the pretest to 21.71 on the posttest.

Table 3 also includes participants' average van Hiele levels as determined by the pretest and posttest and Table 4 includes counts of VHLs for each class.

Table 3. Participants' Averages of Raw Scores and van Hiele Levels

	1st Class				*2nd Class*			
	Males (12)		*Females (9)*		*Males (8)*		*Females (16)*	
Score Type	*Pre*	*Post*	*Pre*	*Post*	*Pre*	*Post*	*Pre*	*Post*
Raw Scores	21.08	20.92	19.44	21.22	20.75	22.375	19.69	21.375
Van Hiele Levels	3.67	3.25	2.56	3.44	3.75	3.875	2.875	3.67

Table 4. VHL Counts

	1st Class				*2nd Class*			
VHL	*Males (12)*		*Females (9)*		*Males (8)*		*Females (16)*	
	Pre	*Post*	*Pre*	*Post*	*Pre*	*Post*	*Pre*	*Post*
0	0	0	1	0	0	0	1	0
1	0	1	0	1	0	1	2	3
2	0	2	1	1	1	1	1	2
3	7	5	7	2	3	1	8	2
4	2	1	0	3	1	0	2	1
5	3	3	0	2	3	5	2	8

The VHL averages in Table 3 from the pretest imply that males in both classes began the course with higher VHLs than the females. Considering Table 3 and Table 4, the male VHL average from the pretest for the first class was 3.67 and all 12 male participants were at VHL 3 or higher, while the female VHL average was 2.56 and all female participants were at VHL 3 or lower. The data in the second class is slightly different: on the pretest 4 out of 8 males scored at VHL 4 or higher while 4 out of 16 females scored at VHL 4 or higher. Considering the first class, 67% of the participants started the course at VHL 3 and only 23% were at level 4 or 5. However, after the course 43% of the participants were at level 4 or 5. The second class began with 46% of the participants at level 3 and 33% at levels 4 and 5. However, after the course 58% were at level 4 or 5. As can be seen in Table 3, after the course the average female VHL was higher than the males' for participants in the first course and only slightly lower for participants in the second course.

Analyses of Changes

Counts of changes in raw scores and VHLs are summarized in Table 5. Some difference between the classes can be recognized from this data. In the first class only 3 males increased their raw score and 5 out of 12 decreased their raw score from pretest to posttest. However, in the second class 7 out of 8 males increased their raw score. These changes in raw scores translated to 4 males from the first class having a lower VHL after the course, but only 1 male from the second class has a lower VHL on the posttest. We see consistent changes between the females in both classes with the majority maintaining or improving both their raw score and VHL. This analysis will continue by class.

Table 5. Counts of Changes from Pretest to Posttest

Change Type	*1st Class*				*2nd Class*			
	Raw Scores		*van Hiele levels*		*Raw Scores*		*van Hiele levels*	
	Male	*Female*	*Male*	*Female*	*Male*	*Female*	*Male*	*Female*
Did worse	5	1	4	2	0	2	1	3
Maintained	4	0	7	1	1	3	5	5
Did better	3	8	1	6	7	11	2	8

First Class

While four male raw scores remained the same on the posttest as on the pretest and three increased, it is important to note that every male participant missed at least one item that they had answered correctly on the pretest. However, while all the females' raw scores changed from the pretest, three females did not commit any new errors on the posttest.

Examining the data from the perspective of total correct answers for males and females also helps describe the changes from pretest to posttest. As a group, the 12 male participants missed 47 items on the pretest while the 9 female participants missed 50 items on the pretest. Thus the male participants had the potential to increase their total raw score by 47 points on the posttest and in actuality their total raw score decreased by 2 points. By contrast, the 9 female participants increased their total raw score by 16 of the 50 potential points. These changes in raw score correspond to the male VHLs changing by –0.42 or almost the equivalent of half a VHL on average, while the female VHLs changed by 0.88, the equivalent of almost one full VHL.

Interestingly, two participants, one male and one female, increased their raw score on the van Hiele test from pretest to posttest, but saw their VHL decrease. Both participants were at a van Hiele level of 3 after the pretest. The male's raw score increased from 18 to 21 yet his VHL decreased to 1, while the female's raw score increased from 19 to 21 and her VHL decreased to 2. For the male participant, the VHL decrease was a result of answering two questions incorrectly at VHL 2 that he had answered correctly on the pretest. The increase in raw score for this participant can be attributed to answering four questions correctly at VHL 5 and one additional question at VHL 3 that he did not answer correctly on the pretest. The decrease in VHL for the female participant is due to answering one additional question incorrectly at VHL 3 that she answered correctly on the pretest. Her increase in raw score is attributed to two questions at VHL 4 and one question at VHL 2 that were answered incorrectly on the pretest.

Second Class

While all the males improved or maintained their raw scores from pretest to posttest only three male participants did not commit any new errors on the posttest. Fourteen of the sixteen females improved or maintained their raw score from the pretest to posttest and similarly only three females did not commit any new errors on the posttest. Examining the data from the perspective of total correct answers for males and females also helps describe the changes from pretest to posttest. As a group, the 8 male participants missed 34 items on the pretest while the 16 female participants missed 85 items on the pretest. Thus the male participants had the potential to increase their total raw score by 34 points on the posttest and in actuality their total raw score increased by 13 points. Similarly, the 16 female participants increased their total raw score by 27 of the 85 potential points. These changes in raw score caused the male VHL average to increase from 3.75 to 3.875 and the female VHL average to increase from 2.875 to 3.67, almost the equivalent of one VHL.

Interestingly, there were 3 participants, 1 male and 2 females, in the second class who actually increased their raw score from pretest to posttest and decreased their VHL on the posttest. The male was at level 3 on the pretest and at level 1 on the posttest, but his raw score increased from 19 on the pretest to 21 on the posttest. The decrease in VHL was due to the male incorrectly answering a question at VHL 2 that he had previously answered correctly on the pretest. The increase in raw score was due to answering two questions correctly on the posttest that he had answered incorrectly on the pretest, at VHLs 3 and 5.

The first female increased her raw score from 12 on the pretest to 15 on the posttest, while her VHL decreased from 2 to 1. As for the male, this decrease was due to the female answering one question at VHL 2 incorrectly on the posttest that she had answered correctly on the pretest. She also answered six questions correctly on the posttest that she had missed on the pretest (one at VHL 1, one at VHL 3, three at VHL 4, and one at VHL 5) and answered two additional questions incorrectly that she had formerly answered correctly, at VHLs 3 and 5. The other female scored 18 and 22, respectively, on the pretest and on the posttest, and saw her VHL decrease from level 3 to level 2. This was due to this female participant incorrectly answering one question in the VHL 3 section of the posttest that she had previously answered correctly. The change in her score was due to correctly answering six questions that she had not answered on the pretest (one at VHL 2, three at VHL 4, and two at VHL 5), and answering two questions incorrectly that she had previously answered correctly at VHLs 3 and 5.

Performance at Each van Hiele Level

The fact that some participants in both classes had an increase in raw score accompanied by a decrease in van Hiele level suggested that it may be interesting to look at participants' performance by groups of questions relating to each van Hiele level. Tables 6 through 9 show the performance on each set of questions for the male and female participants respectively by class.

Table 6. Analysis at Each van Hiele Level – Males in 1st Class

	Van Hiele Level									
	1		*2*		*3*		*4*		*5*	
Statistic	*Pre*	*Post*	*Pre*	*Post*	*Pre*	*Post*	*Pre*	*Post*	*Pre*	*Post*
Average	5	4.83	5	4.58	4.67	4.33	3.25	3.25	3.25	3.92
4-item criterion met	12	12	12	11	12	10	5	4	5	9
Paired *t*-	0.1661		0.0538		0.2199		1.0		0.0433	

test	Not stat.sign.	Not quite stat.sign.	Not stat.sign.	Not stat.sign.	Stat.sign.

Table 7. Analysis at Each van Hiele Level – Males in 2nd Class

	Van Hiele Level									
	1		*2*		*3*		*4*		*5*	
Statistic	*Pre*	*Post*	*Pre*	*Post*	*Pre*	*Post*	*Pre*	*Post*	*Pre*	*Post*
Average	5	5	4.62	4.75	4.0	4.63	3.63	3.89	3.5	4.25
4-item criterion met	8	8	8	8	7	7	4	6	4	6
Paired *t*-test	1.0 Not stat.sign.		0.6168 Not stat.sign.		0.2600 Not stat.sign.		0.6875 Not stat.sign.		0.1759 Not stat.sign.	

Tables 6 and 7 show data for the males from both classes. The data suggests that both before and after the course the male participants performed well on the questions related to van Hiele levels 1 to 3 with slight decreases at each level on the posttest in the first class. The data also suggest little change at VHL 4, none in the first class and a slight increase for the second class. The analysis suggests a statistically significant change in the performance at level 5 for the first class and a similar change for the second class, although not statistically significant. VHL 5 is "rigor" and relates to the proof-based nature of the course. In the first class, the average number of questions correct at VHL 5 increased from 3.25 on the pretest to 3.92 on the posttest with only 5 participants meeting the 4-item criterion on the pretest and 9 participants meeting the 4-item criterion on the posttest. A similar change at level 5 is evident in the second class as well with the average score increasing from 3.5 to 4.25 and the number of participants meeting the 4-item criterion increasing from 4 to 6.

The data in Tables 8 and 9 suggest a similar situation for the female participants; however, both classes saw either no change or slight increases in the averages at van Hiele levels 1 through 3. Unlike the male participants, the female participants showed no change at level 5, but statistically significant change at VHL 4. VHL 4 relates to formal deduction and mathematical proof, again a part of the focus of the proof-intensive geometry course. For the first class the number of correct answers increased on average by more than one (out of five) question at level 4. Also, no females met the 4-item criterion at level 4 on the pretest, but six out of nine met the criterion on the posttest. For the second class there was an increase of one and one-half questions on average from pretest to posttest with only 3 participants meeting the 4-item criterion on the pretest and 11 meeting the 4-item criterion on the posttest.

Table 8. Analysis at Each van Hiele Level – Females in 1st Class

	Van Hiele Level									
	1		*2*		*3*		*4*		*5*	
Statistic	*Pre*	*Post*	*Pre*	*Post*	*Pre*	*Post*	*Pre*	*Post*	*Pre*	*Post*
Average	4.67	5	4.78	4.77	4.22	4.56	2.44	3.56	3.33	3.33
4-item criterion met	8	9	9	8	8	8	0	6	3	3
Paired *t*-	0.1950		1.0		0.2815		0.0304		1.0	

test	Not stat.sign.	Not stat.sign.	Not stat.sign.	stat.sign.	Not stat.sign.

Table 9. Analysis at Each van Hiele Level – Females in 2nd Class

	Van Hiele Level									
	1		*2*		*3*		*4*		*5*	
Statistic	*Pre*	*Post*	*Pre*	*Post*	*Pre*	*Post*	*Pre*	*Post*	*Pre*	*Post*
Average	4.69	4.75	4.5	4.56	4.44	4.44	2.5	4.0	3.5	3.69
4-item criterion met	15	16	14	14	13	13	3	11	8	10
Paired *t*-test	0.7855 Not stat.sign.		1.0 Not stat.sign.		1.0 Not stat.sign.		0.0031 stat.sign.		0.6384 Not stat.sign.	

Discussion

Although change patterns in raw scores and VHLs between the pretest and the posttest are slightly different for the two classes, there are some interesting parallels when examining participants' performance by VHL. As shown in Tables 3 and 4, both raw score and VHL averages decreased for males (0.16 and 0.42 respectively) and increased for females (1.78 and 0.88 respectively) in the first class. To the contrary, both raw score and VHL averages increased for both males (2.125 and 0.125 respectively) and females (1.685 and 0.795 respectively) in the second class.

Looking at the combined male and female results after a ten-week proof-intensive geometry course, the influence on the van Hiele levels of development of prospective mathematics teachers cannot be stated as statistically significant. The difference in the average van Hiele level before and after the course is only 0.14 for the first class and 0.5 for the second class.

However, some influence has been observed on the VHLs by gender, more notably for females. Comparing pre- and post-VHLs shows an average increase of 0.88 for females with 6 out of 9 females improving their VHL in the first class and an average increase of 0.795 with 8 out of 16 improving their VHL in the second class. For males in the first class the VHL average decreased by 0.42 and only 1 out of 12 males improved his VHL. In the second class the VHL average increased by 0.125 and only 2 out of 8 males increased their VHL.

It should be noted that results related to the female participants were statistically significant for both classes. A paired *t*-test of the female raw scores showed that the change between pre- and posttest results is statistically very significant in the first class (p =0.0035, α =*0.05*) and similarly in the second class (p =0.0032, α =*0.05*). The overall change in female VHLs is statistically significant in the first class (p =0.0379, α =*0.05*) only when the results of one candidate (female participant number 6) are removed from the female population. Female participant 6 is the participant whose raw score increased and van Hiele level decreased. In addition, paired *t*-tests ran on female VHLs for each VHL separately showed a statistically significant change (p =0.0304, α =*0.05*) for the first class at level 4 and similarly (p =0.031, α =*0.05*) for the second class.

Pre- and posttest results for males and combined genders in raw scores and VHLs have not produced any statistically significant changes. However, paired *t*-tests ran on male VHLs for each VHL separately showed a statistical significance (p =0.0433, α =*0.05*) at level 5 for the first class only.

The unexpected result of the male participants, an overall decrease of almost one half

VHL in the first class and a minimal increase of one-eighth of a VHL in the second class, is hard to explain. Perhaps the test instrument was not adequately testing the levels at which some male participants were working? Usiskin (1982) mentioned that his test instrument could not be very well defined to test the highest level since the van Hiele theory itself is not defined at the highest level in enough detail. Van Hiele, as cited by Usiskin (p. 14), expressed doubts on the theory at the highest level. However, these arguments seem to be contradicted by the performance of the male participants on the level 5 questions on the posttest. Male participants in both classes averaged almost one more question correct on the posttest at level 5 and the number of males meeting the 4-item criterion at level 5 changed from five on the pretest to nine on the posttest in the first class and from four on the pretest to six on the posttest in the second class. It is possible that the male participants in the first class hurried through the "easy" questions on the posttest, but, like the male participants in the second class, demonstrated by their performance at VHL 5 that they are able to use "rigor" in their geometric reasoning. This ability can likely be attributed to the focus on mathematical proofs and the rigor required during the course.

While the participants had to brush up on general geometric concepts more independently (levels 1 through 3), the course in this study focused on proof-writing abilities (levels 4 and 5). The results show that the females had significantly more room for improvement at VHLs 1-3 from the pretest to the posttest. Also, based on the change in female performance at level 4 and male performance at level 5, it may be plausible to assume that Usiskin's test was nevertheless operating reasonably well at the two higher VHLs. The gain obtained by the male participants at level 5 was likely offset by mistakes that were not made on the pretest at VHLs 1-3. However, the female participants who had more room for improvement at lower VHLs and demonstrated significant improvement at level 4 might have made their gain more easily and accurately measurable by Usiskin's test. For Usiskin, this was not an issue, as he knew his participants would mostly be working at one of the first three levels.

Finally, some findings in this research are consistent with the findings of prior research. For instance, the results on the pretest are consistent with Leahey's and Guo's (2001) findings where male students did better than female students in geometry at the end of high school. As in Senk and Usiskin (1983), females and males performed (almost) equally well in geometric proof writing at the end of a geometry course. Additionally, as in this study where, in general, the females' performance has improved substantially, Ferrini-Mundy and Tarte, as cited by Leahey and Guo, found that girls' performance improved after learning spatial-related strategies. This may correspond to the use of *The Geometer's Sketchpad* in this course as a tool to visualize geometric objects and form conjectures about specific situations.

Furthermore, in a study of middle school students, Kenney-Benson, Pomerantz, Ryan, and Patrick (2006, p. 16) found that girls are more mastery oriented than boys are. Similarly, in a study of college students, Zusman, Knox, and Lieberman (2005) found that "women [are] significantly more likely than men to attend class, take notes, buy/read the book" (p. 625). Perhaps these findings were also present here and the female students were more dedicated to their learning. This may partially explain the significant increase in female scores and VHLs.

Conclusion

In this specific situation the instructor entered the course with the hope that both males and females would perform equally well and all participants would be operating at the highest VHL by the end of the course. Some of the results of this research were surprising and somewhat unexpected. The males' advantage on the pretest was consistent with past research, but unexpectedly, females finished with more significant increases in raw scores and VHLs. Additionally, one might have expected that after such an intensive course in geometric proof-writing, the van Hiele levels would have been at their highest for every participant. Surprisingly, it was not the case: males' VHL averages were 3.25 and 3.875 and females' was 3.44 and 3.67 at the end of the study. These averages are not at the highest VHL. Combining the two classes, only 3 out of 20 males improved their VHL while 14 out of 25 females improved theirs. Furthermore, only 18 out of the 45 participants are at the highest VHL, which is only 40% of the participants in the two classes. This allows one to wonder why. Could it be that, as forewarned by Dr. Senk (S. Senk, personal communication, November 19, 2007), the test instrument was not perfectly suited for this investigation? Alternatively, could it be that the higher the VHL, the higher the difficulty to reach it (Usiskin, 1982)? In turn, it makes one wonder what would have to be taught, and how, for prospective secondary mathematics teachers to achieve the highest van Hiele level.

After reviewing the results of this study, one legitimate question arises: are these prospective mathematics teachers well prepared to teach geometry at the high school level even if none of them scored perfectly on Usiskin's van Hiele test? Swafford, Jones, and Thornton (1997) showed that after increasing the geometry content knowledge of middle-grade (4-8) teachers as well as increasing their awareness of how students learn geometry, their teaching practices, when giving geometry lessons, were substantially and positively modified (pp. 480-482). Consequently, their students were exposed to more geometry activities and concepts, and their teaching performance moved closer to the ideal situations depicted in Figure 1.

It is easy to argue that the geometry content knowledge of the females in this study was increased and it is possible that this will influence how they teach geometry in the future. However, based on the overall results from the van Hiele test, a similar argument is hard to make about the male participants. It is argued, though, that based on the performance of the males on the posttest at level 5, the proof-intensive geometry course changed their content knowledge as well.

Finally, for teacher educators, the outcome of this research may raise questions about the geometry preparation that will best serve prospective teachers in their future practice. The course described here, while inquiry-oriented and proof-intensive, focused on content beyond the level of a typical high school Euclidean geometry course. It is possible that this choice in content coverage influenced the performance at VHL levels 1 through 3. Maybe a more axiomatic development of Euclidean geometry from scratch would better serve prospective teachers and help move them towards VHL 5. Alternatively, maybe the most likely scenario is that an undergraduate course develops prospective teachers thinking up to VHL 4 with the hope that, as they continue to learn beyond their undergraduate education, they will reach VHL 5.

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DYNAMIC VISUALIZATION OF COMPLEX VARIABLES: THE CASE OF RICARDO

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Quantitative reasoning combined with gestures, visual representations, or mental images has been at the center of much research in the field of mathematics education. In this research we extend these studies to include the arithmetic of complex numbers and the analysis of complex valued functions. Our data consists of videotaped interviews with experts, including mathematicians, physicists, and graduate students. Microethnography and phenomenological methods were used to analyze and interpret the data. In this case study we synthesize how one mathematician, Ricardo, employs geometric representations, gestures, metaphor, diagrams, and models to describe his understanding of complex variables topics. Linear transformations were fundamental in his connections between analytical and geometrical representations. He routinely characterized complex linear functions as linear transformations on $\mathbf{R}^2$ *with added symmetries. These findings may serve as a foundation for creating teaching experiments to help students develop geometrical representations of the mathematics behind complex variables.*

Key words: Complex variables, Embodied cognition, Experts, Perceptuo-motor activity

Introduction

The study of learning about numbers and their arithmetic operations is one of the best-developed fields in mathematics education research. The literature goes beyond the four basic operations to include composing and decomposing of whole numbers (Kilpatrick, Swafford, & Findell, 2001), verbal number competencies (Baroody, Benson, & Lai, 2003), ordering and comparing (Brannon, 2002), modeling and visual representations of the operations of whole numbers (Sowder, 1992). The research is not limited to whole numbers; rational numbers are part of the extensive literature related to number sense (Steffe & Olive, 2010). The studies on rational numbers entail investigating students' ability to create word problems that require division or multiplication of two fractions as well as students' visual representations of multiplication and division of two fractions. Studies of quantitative reasoning have elaborated the role of forming a mental image of the measurable attributes in a situation and conceiving of the relevant operations and relationships among these quantities (Thompson, 1994; Moore, Carlson, & Oehrtman, 2009). A natural extension to these studies is to investigate similar characteristics for complex numbers. This research is an effort to extend these studies and to contribute to body of knowledge on quantitative reasoning.

A long-term goal of our research is to create a framework, based on empirical evidence that describes how one perceives big ideas from complex variables. In an effort to contribute to this goal, we conducted an exploratory study to investigate the nature of experts' geometric reasoning about complex variables. In this report we describe one mathematician's flexibility with multiple representations of complex numbers and their operations including complex-valued functions. For our participant, Ricardo, these representations went beyond symbolism to include diagrams, gestures, perceptuo-motor activity, and metaphors. We explore how geometric representations, gestures, verbiage, and symbolism supports Ricardo's reasoning and how he uses these modalities to convey his understanding. As part of our results we found that complex linear transformations were central to Ricardo's reasoning about the arithmetic of complex numbers and analysis of complex valued functions.

Lierature Review

In this section we describe the few pieces of literature that hypothesize about students' understanding of complex numbers and a handful of studies that begin to provide empirical evidence about students' perspective of complex number arithmetic. We also include a description on how historical perspectives may influence the teaching and learning of complex variables. We begin by summarizing Sfard's (1999) theoretical article where she argued for the need for students to become more flexible in moving between operational and structural conceptions of complex numbers. She encouraged viewing the operational and structural components of complex numbers as complementary pieces rather than as dichotomous components. Researchers and instructors could support this perspective by integrating the two representations, for example through geometric illustrations of the operations since "visualization, … makes abstract ideas more tangible, and encourages treating them almost as if they were material entities" (Sfard p. 6). In order to transition from an operational to a structural perspective of complex numbers, Sfard posed three stages that students must navigate in order to develop their understanding of complex numbers.

The first stage is *interiorization*, which occurs when a process is performed on a familiar object. For the case of complex numbers Sfard claimed students who are just becoming proficient in using square roots, would be at the interiorization stage. *Condensation* is the second stage and it occurs when the learner is able to view a process as a whole without the tedious details. For example, students may continue to view 5+2i as a shorthand for certain procedures, but they would still be able to use this symbol in multi-step algorithms. The third stage, *reification*, is achieved when the learner has the ability to view a novel entity as an object-like whole. Learners who are at this stage would recognize 5+2i as a legitimate object that is an element of a well-defined set. According to Sfard (1999) this stage occurs as an "instantaneous leap" much like an "aha moment."

Danenhower (2006) incorporated both Sfard's (1999) framework and APOS theory to examine undergraduates' willingness and ability to convert variations of the fraction $\frac{a+ib}{c+id}$ to either Cartesian or polar form. Variations of the fraction included taking the absolute value of the numerator, raising the factors in the numerator and denominator to a power, and expressing the denominator in terms of sine and cosine, as well as other combinations of these variations. Danenhower found his participants were not flexible in moving between different representations and they did not tend to consider geometric representations, which would have made some items much easier to convert. Although the undergraduates were able to work with the polar form, they did not favor this representation due to the required use of trigonometry. Students were easily able to work with and frequently chose to work with the Cartesian form even though it was not as efficient as polar form. In summary, Danenhower found his participants worked with the Cartesian form of complex numbers at the object level, but could only work at the process level when complex numbers were represented in polar form. Both Danenhower's work and Sfard's analysis is restricted to introductory-level concepts about complex numbers. We intend to elaborate the interplay between more advanced concepts of complex numbers and complex-valued functions.

Lakoff and Núñez (2000) also offered a framework for the conceptual development of complex numbers. Their framework entails a conceptual blend of the real number line, the Cartesian plane, and rotations combined with the use of metaphor for number and number operations. Similar to historical descriptions, they portrayed multiplication of a real number x by -1 as a rotation of 180° to obtain $-x$. Thus, multiplying a number by i is equivalent to rotating by 90° counterclockwise. The beauty of this description is that mathematically it works, but empirical evidence suggests students do not view multiplying a number by -1 as a rotation of 180°, rather they perceive it as a reflection (Conner, Rasmussen, Zandieh, & Smith, 2007). This might be explained by the fact that students are focused on the real number line rather than the Cartesian plane.

In a more recent study, Nemirovsky, Rasmussen, Sweeney, and Wawro (in press) described the results of a teaching experiment with prospective secondary teachers enrolled in a capstone course. The goal of the teaching experiment was to create an instructional sequence that allowed students to create and discover the conceptual meaning behind adding and multiplying complex numbers. In this phenomenological study, the researchers incorporated microethnography to portray students' body activities over short time periods (between two to three minutes). These depictions included language use, gaze, gestures, posture, facial expressions, tone of voice, etc. As a result of their study, the researchers found perceptuo-motor activity was central in:

1. conceptualizing addition and multiplication of complex numbers,
2. communicating geometric representations for adding and multiplying complex numbers, and
3. creating a learning environment that influenced the development of structural components behind adding and multiplying complex numbers.

This study provides a hypothesis about how students make sense of arithmetic operations of complex numbers based on empirical data. As such it may provide insight into how best to introduce complex numbers to students besides as a tool for solving $x^2+1=0$. This study is particularly relevant to our research because we intend to create similar teaching experiments for advanced topics based on what we learn from experts. Incorporating reconstructed historical pieces of the development of complex numbers may also engender structural understanding of this area of mathematics (Glas, 1998).

Historically, the introduction and initial development of complex numbers was purely algebraic to resolve the issue of finding the real solution to certain cubic equations. Even after the square root of negative numbers was introduced, mathematicians such as Cardan found such numbers to be *sophistic* because they could not attach a physical meaning to these numbers (Nahin, 1998). These mathematicians tended to ignore the conceptual difficulties of these numbers and proceeded to apply the procedures "mechanically" (Glas, 1998, p. 368). It was Wallis who first dedicated much of his career attempting to represent the square root of a negative number through geometric constructions. Although, Wallis made progress his work was not convincing to other mathematicians or himself. It was more than a hundred years later that Wessel introduced the interpretation of placing $i=\sqrt{-1}$ at a unit distance from the origin on an axis perpendicular to the real number line to form the complex plane in which multiplying by *i* geometrically corresponds to a rotation of 90º counterclockwise. This representation allowed mathematicians to begin to think about complex numbers as vectors, which in turn led to geometric representations of the arithmetic operations of complex numbers. These models prompted mathematicians to prove that extended theories of complex numbers (i.e., quarternions, Cauchy-Riemann equations) are consistent and preserve the structure of the complex number system. For example, Hamilton's discovery of the quarternions was a result of wondering what would rotate vectors in three-dimensional space. Such historical developments may provide insights into how "concepts and theories can be best brought to light" for students (Glas, 1998, p. 377). Our theoretical perspective, discussed below, is in line with the small body of literature on historical and individual conceptual development of complex variables as described above.

Theoretical Perspective

As mentioned above Sfard (1999) did not view the operational and structural components of complex numbers as dichotomous, rather she viewed them as complementary pieces. In the same vein, some researchers (Lakoff & Núñez, 2000; Noe, 2006) argued that the way in which we perceive a mathematical concept and the way in which we physically interact with this concept are intertwined rather than disjoint, a perspective referred to as embodied cognition. Lakoff and Núñez contend that

> many cognitive mechanisms that are not specifically mathematical are used to characterize mathematical ideas. These include such ordinary cognitive mechanisms as those used for the following ordinary ideas: basic spatial relations, groupings, small quantities, motion, distributions of things in space, changes, bodily orientations, basic manipulations of objects (e.g., rotating and stretching), iterated actions, and so on. (p. 28)

Lakoff and Núñez offered four conceptual mechanisms typically employed in advanced mathematical thinking. The mechanisms are image schemas, aspectual schemas, conceptual metaphor and conceptual blend. Image schemas serve as a link between language and reasoning as well as language and visual perception. The notion of vision can be extracted from physical and real objects or something that is imagined i.e. a mental image. Aspect schemas occur when concepts that we associate as an action or event are conceptualized through motor-control schemas. As a result one communicates, prepares, gestures, etc. with a similar motor activity. In other words "the same neural structure used in the control of complex motor schemas can also be used to reason about events and actions" (Lakoff & Núñez, p. 35). The use of conceptual metaphors have been researched for over three decades in the domain of cognitive science and it has been established that the use of metaphors is systematic and unconscious. It has also been established that our personal and everyday experiences shape the metaphors that we use to convey abstract concepts through more concrete domains. The fourth mechanism, conceptual blend, entails combining "two distinct cognitive structures with fixed correspondence between them" (Lakoff & Núñez, p. 48). Metaphorical blends refer to a cognitive blend where the correspondence incorporates a metaphor, and are frequently used in the teaching, learning and understanding of mathematics.

By taking the perspective of embodied cognition we were better able to investigate how our participants incorporated general cognitive mechanisms used in their everyday nonmathematical world to craft their mathematical understanding and to communicate mathematical ideas related to complex variables. In the following section we discuss our research methods and follow this with our results.

Methods

We used purposeful sampling methods (Patton, 2002) to select our participants, which entailed selecting participants who we knew, either from personal experiences or through recommendations, possessed and articulated a rich geometrical understanding of complex variables. Although we interviewed several mathematicians, a physicist, and graduate students, this report details Ricardo's responses. Ricardo has over 20 years experience teaching undergraduate and graduate courses in complex analysis and his research interests include applications of partial differential equations, complex analysis, and Fourier analysis. He is known for having an extraordinary talent for teaching through metaphor and relating analytic and geometric representations in his teaching. The following quote describes Ricardo's commitment to and enthusiasm for developing students' visualization in a third semester calculus course:

> Because 3D visualization is a skill that needs constant nurturing, I have always made special efforts to engage their [students'] geometrical imagination, using a combination of low-tech and high-tech tools: by constructing paper, string, and wooden models of the surfaces we study, by drawing detailed multi-colored chalk diagrams that expose the geometrical structure of the objects under investigation, and by designing computer-based interactive labs and demonstrations that help students view geometrical objects from multiple perspectives. Each semester I hear gasps of sheer delight when students see a mathematical formula blossom on their computer monitor into a beautifully curved geometrical surface that they can explore at any level of magnification by rotating and zooming.

Data were collected through 90-minute video-taped interviews conducted by Hortensia and Michael. The interview questions (see Appendix 1) were structured to elicit how the participants geometrically perceived arithmetic and analytic concepts related to complex variables. The arithmetic questions centered on geometric representations of adding, multiplying, dividing, and exponentiating complex numbers. The analytic questions focused on geometric interpretations of continuity, the Cauchy-Reimann equations, differentiation, and contour integration of complex valued functions. At times, we probed in order to stimulate geometrical and visual explanations that our participants might provide to a novice. The process for analyzing and interpreting the data included transcribing the interviews while depicting where the participant incorporated visuals such as gestures, diagrams, models, and metaphors. After this, the research team used microethnography and phenomonological methods to analyze the video. This entailed watching and analyzing segments of the video repeatedly in order to capture and describe Ricardo's gestures and to synthesize his geometric explanations. We selected the segments based on their relevance to our research questions. We attempted to capture Ricardo's use of geometric representations throughout the entire interview unlike Nemirovsky et al. (in press) who only used small segments of the data. Upon completion of analyzing and interpreting the video we met with Ricardo in order to member check and to clarify any remaining questions. For example, at one point in time Ricardo made a movement with his hand for which we hypothesized several interpretations, so we asked him to explain what he was visualizing while he made the gesture. In the following section we summarize Ricardo's responses.

Results

In order to convey Ricardo's remarks we begin with a summary of his multiple representations for complex numbers and operations of complex numbers and complex valued functions. We follow this with a discussion of how these concepts and representations were embedded in his discussions related to differentiation of complex valued functions as well as his global perspective of complex analysis. It is important to note that we are not suggesting a hierarchical or dichotomous perspective of concepts. Rather, the data suggest that for Ricardo these representations and concepts were interwoven and complex linear transformations were the thread that allowed him to braid his perspectives.

Multiple Representations of Complex Numbers: Ricardo's flexibility to move from one representation of complex numbers to another was apparent in that he spontaneously recognized that some representations were more useful for given situations. During his interview, Ricardo represented complex numbers using vector notation, Cartesian and polar form, but he also made use of ordered pairs and points on the Argand plane. First and foremost he viewed complex numbers as vectors with the added property of multiplication and he commented that this was how he introduced complex numbers to students. Ricardo indicated that he only used the Cartesian form of complex numbers to add and subtract complex numbers and connected this to the geometric interpretation of vector addition and subtraction using a parallelogram. He also used his index finger or whiteboard marker to represent a vector and rotated his finger or the marker to indicate the rotation of a vector when multiplying by *i* (see *Figure* 1). For multiplication and division of complex numbers, Ricardo remarked that he preferred polar form because this allowed him to "visualize each of the pieces [the radius and the angle of the complex number] in my head." In essence it allowed him to recognize the "rotation and magnification factor." This was especially evident when we asked him to interpret the graduate students' processes for division of complex numbers. He had to pause and think about the geometry behind the graduate students' work before responding to Hortensia's question.

> Hortensia: Some graduate students attempted to locate 1z by rewriting it as zz2, what do you visualize when you see this notation?
>
> Ricardo: It's probably pure psychology, but I don't really process that one as quickly as the other one – the polar description, but the geometric description would be to first reflect across the real axis and then shrink it down by a scaling factor.

Complex variables textbooks rarely write complex numbers as ordered pairs, but Ricardo found this notation helpful in cases where he also represented complex linear transformations with matrices, especially in his discussion of the Cauchy-Riemann equations. Ricardo also thought of complex numbers as points representing a location on the Argand plane. He used this representation when he considered mappings and their input and output values in the plane. These representations are discussed further when we share his multiple illustrations of operations involving complex numbers and complex valued functions and when we portray his global view of complex analysis.

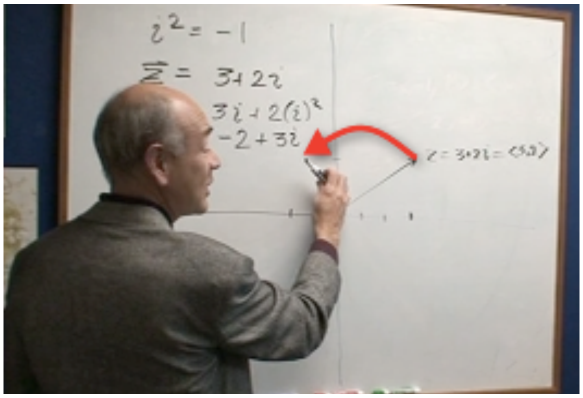

Figure 1. Using a marker to represent the rotation of a vector.

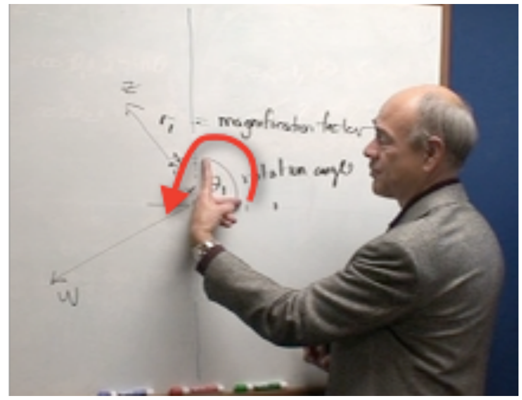

Figure 2. Rotating the first quadrant.

Multiple Representations of Operations: Two concepts seemed to be at the heart of Ricardo's ability to visualize and to bring to life complex variables concepts. The first was his recognition that multiplication by a complex number signified a rotation and dilation and the second was his conceptual understanding of functions. Ricardo persistently gave geometric interpretations and made corresponding gestures exemplifying multiplication by a complex number throughout the interview. Ricardo explained that he taught multiplication of complex numbers by first explaining how *i* acted on a specific complex number such as 3+2□ as depicted in *Figure* 1. After this he explained how multiplication acted on a frame, and consequently the entire Argand plane. In *Figures* 2 and 3, Ricardo configured his middle and index fingers as perpendicular vectors to represent a frame for the Argand plane and rotated his fingers to illustrate how multiplication by *i* would simultaneously transform both. He re-emphasized this representation when he discussed the geometric representation of multiplication of two general complex numbers *z* and *w*. After he described how multiplication acts on a frame, Ricardo progressed to illustrating rotating the entire plane. He used gestures similar to those shown in *Figures* 2 and 3 and swept his fingers around the entire plane. Another gesture that he seemed to favor to indicate a rotation of the plane was a fanned hand, which he rotated around the Argand plane as shown in *Figure* 4. He used this motion when he discussed the Cauchy-Riemann equations and when he discussed the notion of differentiation of complex valued functions.

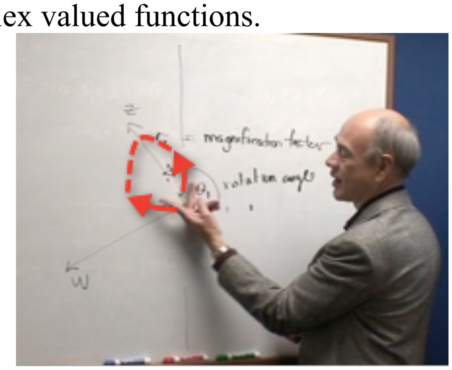

Figure 3. Rotating a frame to the second quadrant.

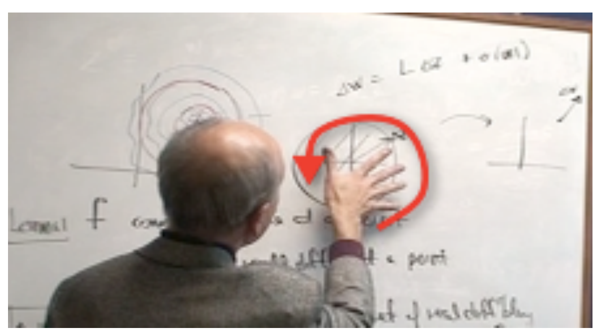

Figure 4. Rotating the Argand plane.

Ricardo also used hand gestures to represent expansions. There were two gestures that were prominent in his interview. The first began with a closed fist, which opened up to extended fingers as illustrated in *Figure* 5. The second gesture started with his hands together (see *Figure* 6) and then he pulled his hands away from each other. He repeatedly made these rotation and expansion motions when discussing multiplication. It was clear from his remarks and gestures that he perceived z as an operator and w as an operand in the product of zw, where z was the function that rotated and expanded the element w.

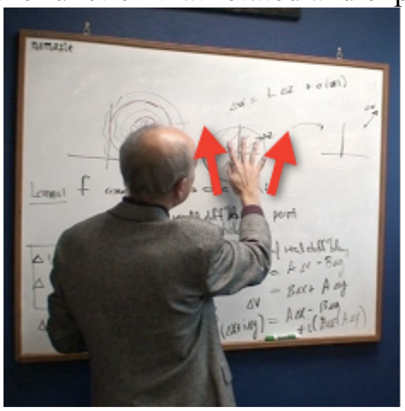

Figure 5. Expansion starting with fist.

Figure 6. Expansion starting with hands together.

The function concept was the second component that appeared to form Ricardo's geometric interpretations of the operation on complex numbers and concepts of complex variables. For example when we asked him about the location of □□, he incorporated the compositions of functions by first considering mapping of the points z, followed by rotations and dilations, and finally illustrating how exponentiation results in a spiral (see *Figure* 7). The following is his commentary that corresponds to *Figure* 7.

> If □=□□□□, then □□□=□□□+□(□+2□□) so if I plot all these points, all the values of log of z, they are separated by 2π in the imaginary direction. So it's a whole arithmetic progression in the complex plane that we're looking at. So when we multiply by w, in general if w is a complex number then it is going to be a rotation and expansion of this pattern here (points to 1st diagram). So when I look at *wlnz*, then possible interpretations of that are other arithmetic progressions on the plane and the spacing can be different from before because we performed an expansion by the magnitude of w. Now we want to exponentiate that (points to 2nd diagram) and that's actually going to be a logarithmic spiral (creates 3rd diagram).

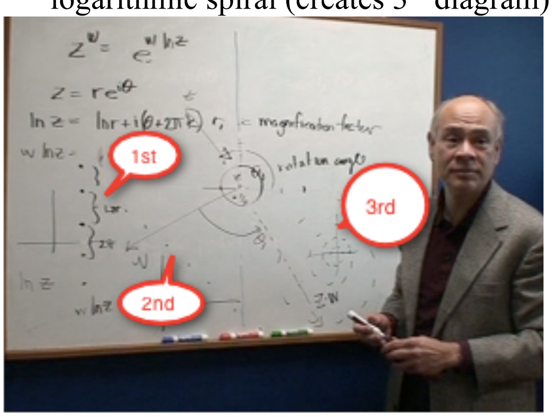

Figure 7. Illustrating □□.

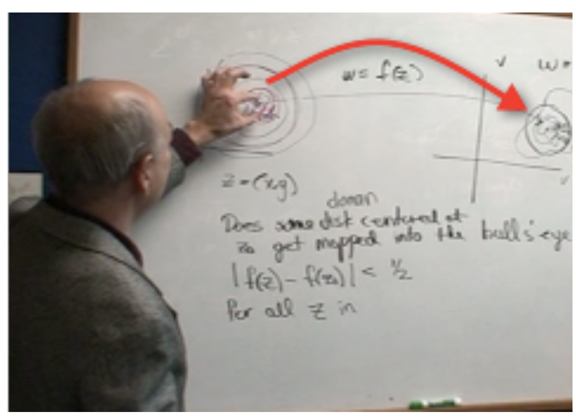

Figure 8. Demonstrating continuity by moving a chunk of space

Ricardo also demonstrated flexibility with multiple representations of operations when he responded to the continuity, Cauchy-Riemann equations and differentiation questions. In his geometric interpretation of continuity of complex valued functions he described how points that are close together in the domain get mapped to points that are close together in the range. Ricardo elaborated using the metaphor of a bulls' eye target centered around □□*0*=□*0*, where one would need to determine the radius of a disk centered at □*0* so that all the points in the disk get mapped inside the bulls' eye zone. As part of this metaphor he used his right index finger to illustrate mapping a point from the disk to the bulls' eye zone. This finger flicking movement illustrated the function acting on the domain point-by-point, but he also illustrated how it could map "*chunks of space*" at a time, by forming a c-shape with his left index and thumb moving it over to the range (see *Figure* 8). Ricardo continued to incorporate gestures when he discussed the continuity of the function □□=□□□□ at $z = 0$.

After reducing the function to □□=□□□□, he explained how one could graph this on a 3-D coordinate system, but before graphing it, he illustrated the output through gestures. The gesture appeared to be an infinity sign as shown in Figure 9, as if tracing the real-valued graph of $z = \cos 2\theta$ above the unit circle. After gesturing with this motion he moved to a physical model, which was the graph of two periods of the cosine wave going around a cylinder (in this case a thermos). He then proceeded to draw the vectors with different heights on the whiteboard. During the follow-up interview, he provided another model using paper, and explicitly confirmed the correspondence between his infinity-sign gestures and the thermos model. These models are shown in *Figures* 10 and 11. The arrow in *Figure* 10 outlines part of the curve he drew. During the follow-up interview Ricardo provided another metaphor that he found useful for describing continuity; it involved a painter's palette and count-by-numbers painting.

Ricardo's explanation of the geometry behind the Cauchy-Riemann equations was another example where he provided rich and multiple representations of his understanding. His representations included, matrices, systems of equations, ordered pairs, mappings, metaphors, gestures, and symbolism involving Landau notation. He commented that the underlying geometry of the Cauchy-Riemann equations occurred in the system of equations:

Δ□=□□+□□
Δ□=□□+□□

where Δ□=Δ□+□Δ□ is of first-order. He stated how

> this system of equations can be described by multiplication of a complex number if and only if this 2x2 matrix [matrix constructed from system of equtions] has certain symmetries. If we are lucky then □□ is the derivative of some constant times □. These symmetries are what allow one to deduce the Cauchy-Riemann equations.

As Ricardo discussed the symmetries he placed his hands at the center of the system of equations and pulled his hands away from each other in a diagonal direction to indicate the symmetries as illustrated in *Figure* 12. After this Michael asked Ricardo to re-interpret this with a picture on the complex plane, which generated a discussion about linear approximations.

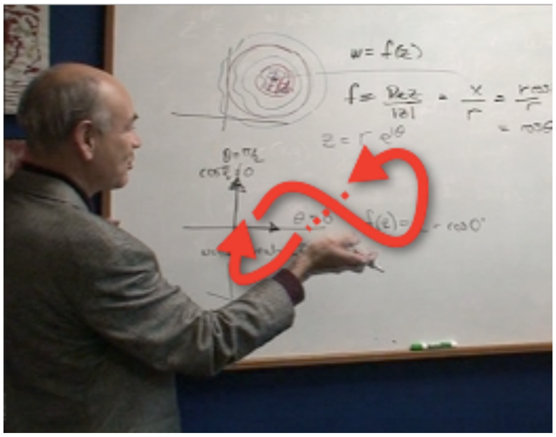

Figure 9. Discussing the continuity of □□=□□□□.

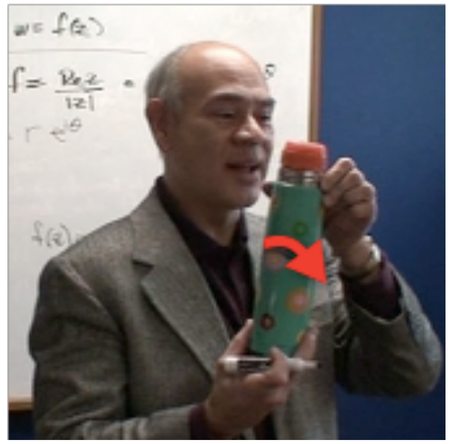

Figure 10. Physical model of □□=□□□□.

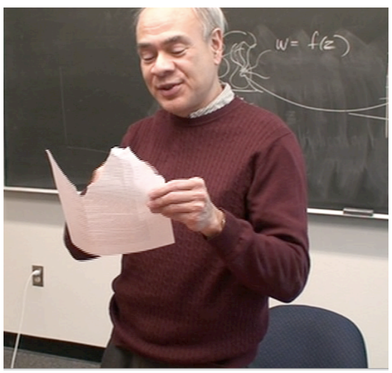

Figure 11. Physical model of □□=□□□□.

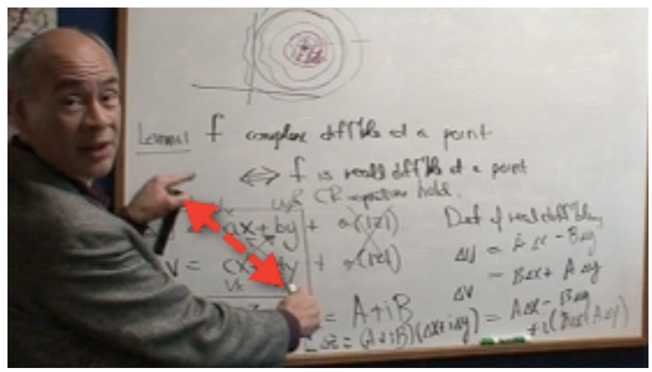

Figure 12. Depicting the symmetries of the C-R equations.

Ricardo discussed the first-order approximation Δ□=□Δ□ as a complex linear map , defined by a complex multiplier □=□+□□. Upon writing this he rotated his hand over Δ□=□Δ□, in the same manner as shown in *Figure* 4. He proceeded to explain how a disc with radius of □□ gets "rotated and expanded out" to produce the image in the range. After this description he used a turntable metaphor in which he said,

> it's like having a turntable that you can spin and expand and that is what the complex linear mapping is doing. So it's taking a patch of the plane and almost treating it like a rigid body rotating it, but then expanding it.

Figure 13, illustrates how he turned the turntable in a similar manner in which one might turn the steering wheel of a car. He followed this with his standard rotation and expansion gestures with spread fingers as described above. In this discussion, Ricardo commented that Δ□ is determined by the derivative as the product of some constant *L* and Δ□, which segued into the next interview question about differentiation.

As part of his geometric interpretation of complex differentiation, Ricardo reiterated that in general

> differentiation for any mapping means that a small patch can be approximated by what geometrically corresponds to a rotation and expansion. So small displacement vectors here (in the domain) are going to get rotated as a rigid body and then expanded uniformly.

As before, he explained this while using his rotation and expansion gestures. For the specific example of □□=□*2* and □′□= *2*□, Ricardo explored mappings of different disks centered at given point □*0* on the unit circle. He indicated that □*0* would get mapped to a point □*0* that has twice as big an angle due to the 2 in the exponent and that the disk centered at □*0* would double in radius and "get spun around" due to the 2*z* in the derivative with $|z|=1$. He illustrated this by drawing a small disk around □*0*, and noted that small displacements Δ□ in the domain will result in corresponding displacements Δ□ in the range. Specifically, the vector Δ□ will be rotated by 2arg(*z*) from the vector Δ□. Ricardo went on to remark that as one rotates around the unit circle in the domain of *f*, the amount of rotation varies depending on the angle of the point □*0*. As he stated this, he traveled around the unit circle in the domain with his left hand. He then moved to the range and explained that the "dial [image vector of z0] would get spun around to get different circles." This is an example where he used his left index finger to represent the vector and he moved it in a counter-clockwise direction around the unit circle in the range (see *Figure* 14).

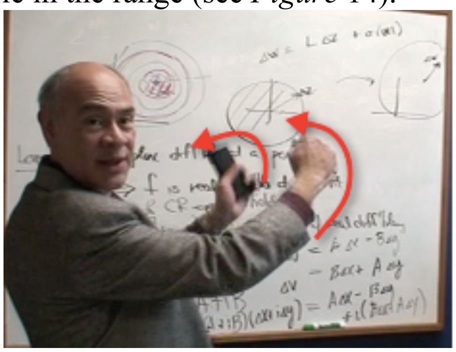

Figure 13. Turntable metaphor for complex linearization.

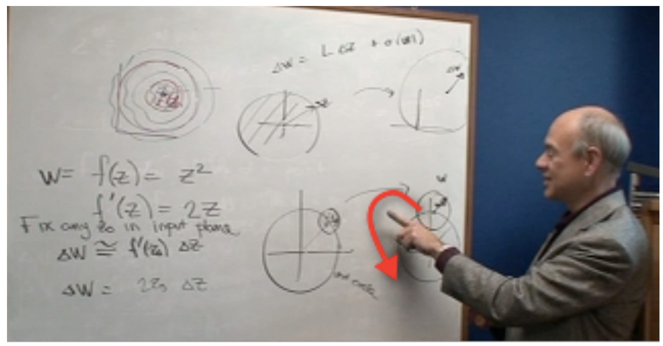

Figure 14. Spinner metaphor to describe differentiation at a point.

Real Analysis vs. Complex Analysis: We noticed that during the initial interview Ricardo always discussed and expressed rotation before dilation, with one exception. In the follow-up interview when we asked why this might be the case, the following dialogue occurred:

> Hortensia: Are expansion and rotation equally important or is one more important than the other for you?
>
> Ricardo: Theoretically they are equally important, but in my own psychology I am more interested by the rotation because it is more dramatic than the dilation (as he gestured as shown in *Figures* 4 and 5).

Although Ricardo initially called this a "personal quirk" he then emphasized how he perceived complex analysis as an extension of real analysis. This was prominent in his geometric interpretation of the product of two complex numbers, because in the real number system, multiplication only results in a dilation, whereas, multiplication of two complex numbers results in both a rotation and dilation. In the follow-up interview he emphasized this point,

> when you're analyzing complex differentiable maps if the derivative happens to be real then it's just a pure dilation (while he gestured as shown in Figure 5) and that is sort of the degenerate case and that's why I tend to look at the more general case that includes a rotation.

For Ricardo, differentiation in both systems represents an approximation for local linearity. He went on to say, "I think of it [complex linear maps] as a real linear map that has certain symmetries."

Figure 15, shown below, is an attempt to encapsulate Ricardo's perceptions of the arithmetic of complex numbers and the analysis of complex variables. Central to the model is linear transformations, since this appeared to be the foundation for Ricardo's geometric interpretation and visualization. Furthermore, it was the link between arithmetic and analysis and the link between real analysis and complex analysis. The model also exemplifies the significance that Ricardo placed on rotations and dilations – one can see the ample connections to and from this concept. Ricardo exemplified the four schemas described by Lakoff and Núñez (2000), and the dotted lines in the model depict how these schemas were not used in isolation. For example, Ricardo used the metaphor example of the Bulls' eye in conjunction with the thermos and paper cut-out concrete descriptions as part of his explanation about the continuity of complex-valued functions. We would like to re-iterate that Ricardo did not appear to possess a hierarchical view of these ideas; instead his representations appeared to be threaded together as one concept. This of course makes it even more difficult to portray Ricardo's perceptions through a model.

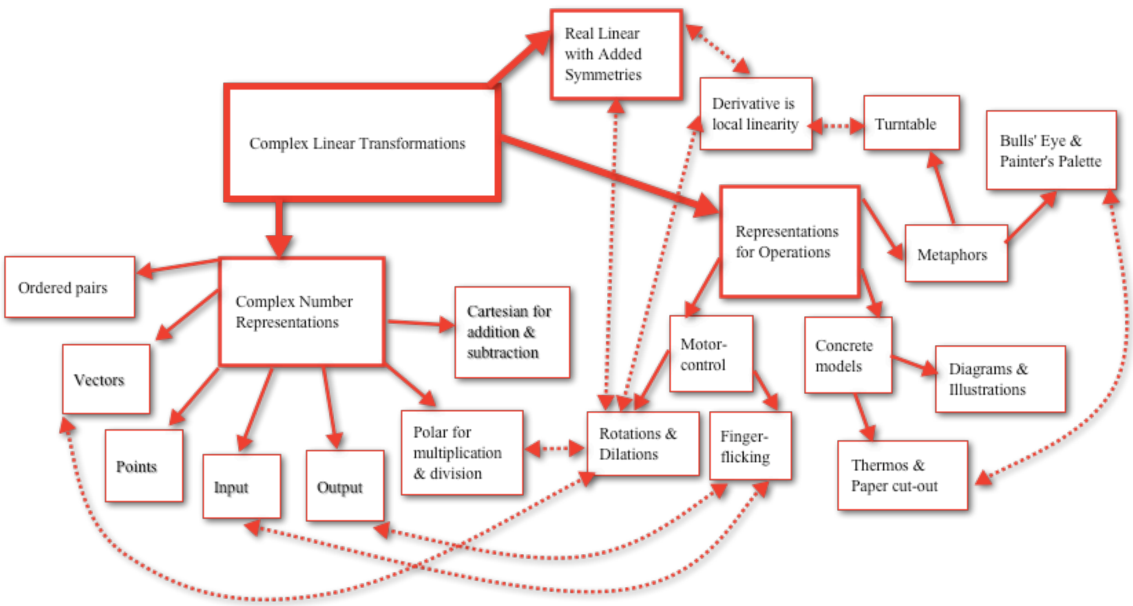

Figure 15. Illustration of Ricardo's geometric interpretation of complex variable.

Discussion and Future Directions

While Danenhower (2006) found undergraduates did not recognize the usefulness and appropriateness of different representations for complex numbers (i.e. Cartesian or polar), Ricardo easily navigated between different representations. He was conscious of the representations that he chose to implement in different situations; his purposeful and consistent choice appeared to be based on how the representation would aid Ricardo in visualizing the mathematics. The visualization and representation of the arithmetic and analysis of complex numbers and complex-valued functions went hand-in-hand. Sfard's (1999) work supports the non-dichotomous nature of Ricardo's understanding since both the visualization and representation of complex numbers or functions complemented one another. For example, given Ricardo visualized division of complex numbers as a rotation in the counterclockwise direction followed by scaling compelled him to represent *1*□ as *1*□□−□□. Similarly, even though he considered graphing □□ as "tricky," he used the fact that □□could be reinterpreted as a composition of multiple simpler transformations in order to reconstruct a visual diagram of the points represented by this expression. The ability to visualize each transformation allowed Ricardo to reify this novel mathematical situation.

Complex linear transformation played an unmistakable role in how Ricardo perceived complex variable topics and he consistently employed multiple conceptual mechanisms as described by Lakoff and Núñez (2000) to communicate his understanding. Ricardo incorporated image schemas that were concrete and abstract in nature. For example, his paper model and drawings and illustrations on the whiteboard and on the thermos were very concrete and easily visible. On the other hand his rotation and expansion gestures, his tracing along the diagonal to indicate the symmetries of the Cauchy-Riemann equations, his finger-flicking to illustrate mappings of points, and gestures for mapping regions on the plane were much more perceptuo-motor in nature and could be classified as aspect schemas since they were communicated through physical actions. Ricardo also conveyed his ideas through metaphors and conceptual blends that incorporated metaphors. These metaphorical blends appeared in his discussion of continuity through a bulls' eye target and a painters' palette as well as in his explanation of differentiation via a turntable and a spin-dial. Although we did not discuss Ricardo's explanation of complex integration, he also used a very elaborate metaphor. This metaphor referred to the path of a ship that traveled on a sea devoid of landmarks, a compass that deviated from true north, and a cable that expanded or contracted depending on water temperature. This description also relied heavily on his imagery of complex linear transformations discussed above. Although Ricardo's interview provided us with rich information and dynamic illustrations, it also triggered further questions.

One question that immediately comes to mind is are students attuned to Ricardo's conceptual mechanisms? As researchers, we only noticed some gestures after viewing the video numerous times. Thus, it is difficult to know if students actually connect the gestures to the mathematical content, especially if the gestures are different. As such, a follow-up research question is to investigate how, if at all, students make sense of others' embodied concepts that are expressed through conceptual mechanisms. A similar line of inquiry might be to investigate if and how experts, such as Ricardo, make a concerted effort to develop students' geometric interpretation of complex variables topics, especially those related to analysis. Our immediate plans are to complete the analysis of the interviews with the other mathematicians, physicists and graduate students and synthesize the results. We expect to obtain wide variations in the personal models such as Ricardo's illustrated in *Figure* 15. The next phase of our research will be to conduct guided reinvention teaching experiments with undergraduates to explore their geometric interpretations of the arithmetic and analysis of complex variables. Guided reinvention teaching experiments (*Gravemeijer* et al., 2000) are "a process by which students formalize their informal understanding and intuitions" (p. 237). These teaching experiments may allow us to capture a glimpse of students' embodied concepts of complex variable topics.

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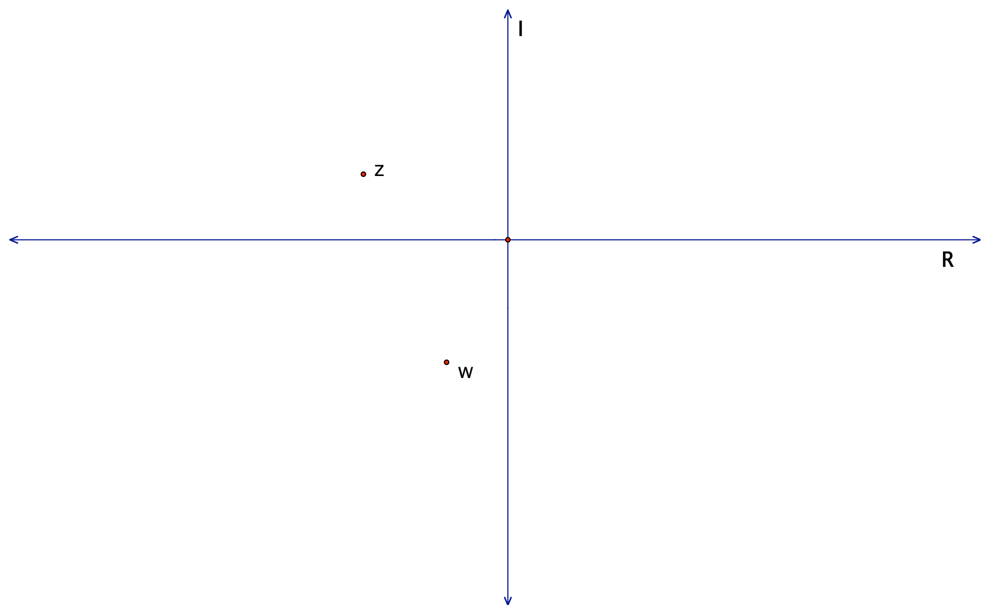
Appendix 1: Interview Questions

1. Below are two complex numbers z and w. Determine and explain how you know where each of the following are located. How do you think of these operations algebraically, geometrically, and in terms of $\mathrm{Re}^{i\theta}$?



   a. $z+w$
   b. zw
   c. $1/z$
   d. z^w

2. In calculus, we sometimes use the idea of "tracing the graph of function and not lifting our pencil" to convey the concept of continuity. What geometric representation or explanation might be useful to understand continuity of complex valued-functions?
   a. Consider the function $f(z) = \frac{\mathrm{Re}\, z}{|z|}$. Is there a way to define the function at $z = 0$

      in order to make the function continuous? Why or why not?

   b. Consider the function $f(z) = \frac{z\,\mathrm{Re}\, z}{|z|}$. Is there a way to define the function at $z =$

      0 in order to

      make the function continuous? Why or why not?

3. Give a geometric reasoning as to why it is enough for a real-differentiable

function $f(z) = u(x,y) + iv(x,y)$ to satisfy the Cauchy-Riemann equations in order for it to be complex differentiable throughout some ε neighborhood of a point $z_0 = x_0 + iy_0$? Recall that the Cauchy-Riemann equations are: $u_x = v_y$ & $u_y = -v_x$

4. If $f(z) = z^2$, then
   a. What does it mean that $f'(z) = 2z$?
   b. What does it mean that $f'(2) = 4$?
   c. What does it mean that $f'(i) = 2i$?
   d. What does it mean that $f''(z) = 2$?
   e. What does the derivative of a complex function represent? Is it the slope of a line?
5. Sometimes in calculus we can interpret the definite integral of a real-valued function to represent the area under the curve. What geometric representation or explanation might be useful to understand the complex number obtained as an answer to a definite integral of a complex valued-functions?
6. How is complex variables useful in the real world? What are some applications of complex variables? How does it fit into the scheme of mathematics?

EFFECTIVE STRATEGIES THAT UNDERGRADUATES USE TO READ AND COMREHEND PROOFS

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In this paper, we examine the strategies that two successful undergraduate students used to comprehend six mathematical proofs. These students spent nearly as much time studying the theorem as they did studying the proof, both by reformulating the theorem and trying to understand why it was true prior to reading the proof. When reading the proof, these participants attended to proof frameworks, partitioned the proof, and considered specific examples. How these ideas might be used to improve students' proof comprehension is discussed.

Key words: Proof; Proof comprehension; Undergraduate mathematics education

1. Introduction

In mathematical practice, proof plays both a role in the generation and communication of mathematical knowledge. For the individual mathematician, proof can be used as a tool, often an exploratory one, for coming to understand a piece of mathematics in several ways; proof can be used to gain conviction that a theorem is true, develop insight into why mathematical relationships hold, and discover new methods for attacking mathematical problems (see, for instance, de Villiers, 1990; Rav, 1999; Schoenfeld, 1994). For a community of mathematicians, proof is the predominant means of *communicating* mathematics. Most scholarly presentations and articles in professional mathematics are largely comprised of theorems and their proofs.

Proof is expected to play similar roles in mathematics classrooms, particularly in students' advanced mathematics courses (i.e., upper-level proof-oriented university mathematics courses). For elementary and secondary mathematics classrooms, the NCTM (2000) argues that justifying and proving should play an important role in all students' mathematics education throughout the curriculum; the NCTM advocates that all students appreciate the need for proof and be able to construct deductive arguments by the time they complete 12th grade (p. 56). Proof assumes an even larger role in advanced mathematics courses, where students' ability to prove theorems about the concepts covered in the course is often the primary means of assessing students' performance.

Not only are students expected to construct proofs in their advanced mathematics courses, but proof is also the primary means by which mathematics is conveyed to students. Lectures in advanced mathematics courses usually are delivered in a definition-theorem-proof format (e.g., Dreyfus, 1991; Weber, 2004, 2010), where students spend considerable time observing the professor present proofs to them. Textbook presentations typically place a similar emphasis on proof (e.g., Raman, 2004). Mathematics professors list a variety of reasons for engaging in this practice, including providing students with an understanding of why important theorems are true, illustrating new proof techniques to students, exposing them to proofs that have cultural significance in the mathematics community, and increasing students' appreciation of proof in general (Weber, 2010).

While there has been a great deal of educational research on proof, Mejia-Ramos and Inglis (2009) note that most of this research has concerned students' construction of proofs, with several researchers noting there is comparatively little research on students' reading of proofs and arguing more research in this area is needed (Hazzan & Zazkis, 2003; Mamona-Downs, 2005; Mejia-Ramos & Inglis, 2009; Selden & Selden, 2003; Weber, 2008). Mejia-Ramos and Inglis (2009) further note that research studies on students' reading of proofs have generally focused on the ways that students' *evaluation* of proofs rather than their *comprehension* of proofs. That is, researchers have typically investigated what types of arguments students found convincing or judged to be valid proofs. However, there are few studies on students' understanding of the proofs that they read. As an important goal of presenting proofs to students is to increase their mathematical understanding, the lack of research on students' proof comprehension represents an important void in the literature. The goal of this paper is to address this void by delineating strategies that undergraduate students can use to improve their comprehension of the proofs that they read.

2. Research context

2. 1. Overarching goals of our research program

The study reported in this paper was conducted in support of a larger research program. The overarching goal of this research program is to design and assess instruction that will improve undergraduate mathematics majors' ability to comprehend mathematical proofs. To achieve this goal, our research team seeks to (a) document the difficulties that undergraduates have in understanding mathematical proofs (see Weber, 2009, in press, for the results of this research), (b) identifying strategies that successful undergraduates use to overcome these difficulties, and (c) designing instruction that can enable less successful students to effectively employ these strategies to increase their comprehension of proofs. The report in this paper deals with the second goal of this research program—identifying strategies that successful students use to overcome these difficulties. We address this issue by reporting a case study of how two successful students read six mathematical proofs.

2. 2. Theoretical perspective

To identify strategies that help students understand a proof, it is necessary to first clarify what it means to understand a proof. In this paper, we adopt the theoretical model of understanding by Mejia-Ramos et al (2010). In this model, proof can be understood at an elemental level that focuses on the meanings of, or connections between, individual statements in the proof. Here, understanding can be assessed in terms of (a) meaning of terms and statements, (b) logical statuses of statements within a proof framework, or (c) how new statements were justified from previous statements. Proofs can also be understood in global terms, which include (a) understanding a summary of the proof, (b) being able to transfer methods of the proof to prove other theorems, (c) breaking the proof into components, or (d) applying the ideas of the proof to a specific example. A rationale for why these types of questions are important, as well as a scheme for generating these types of questions, is provided in Mejia-Ramos et al (2010). For the sake of brevity, they will not be discussed in this paper.

2. 3. The students

This study took place at a large state university in the northeast United States. Two students, Kevin and Tim (both names are pseudonyms), participated in this study. The students were each paid $20 an hour in exchange for their participation. Kevin and Tim were both mathematics majors in their senior year at the time of the study; both were also concurrently enrolled in the mathematics teacher education program at that university and preparing to become high school mathematics teachers.

Kevin and Tim were specifically invited to participate in this study for several reasons. Both students displayed an interest in mathematics, as well as an eagerness to study it, in mathematics education courses taught by the first author of this paper. In these classes, both students were articulate, successful, and willing to share their reasoning. Further, Kevin and Tim each participated in (separate and independent) research studies in the past at the university where this study took place and provided interesting and useful data on how some students were able to learn mathematics. In summary, Kevin and Tim were not typical students, but specifically recruited because we felt it was likely they would articulate interesting and effective proof-reading strategies.

During their interviews, Kevin and Tim revealed that they had taken advanced mathematics courses together in the past and were sometimes members of the same study group. We were not aware of this at the beginning of the study, but do not think this threatens the validity of our findings.

2. 4. The materials

All materials can be found in the Appendix of this paper. The materials consisted of six proofs of six theorems. Throughout the remainder of the paper, we refer to the n^{th} theorem that the participants saw as Theorem n and its associated proof as Proof n. We designed these proofs to have the following features: (a) the mathematical content of the proof relied on introductory calculus and basic number theory (so participants' difficulty with the proofs would presumably not be due to a lack of content knowledge), (b) the proofs were of interesting propositions whose veracity would not be immediately obvious to an undergraduate student, and (c) the proof employed an interesting and non-standard technique. For each proof, we generated a set of proof comprehension questions using the proof comprehension model put forth by Mejia-Ramos et al (2009).

2. 5. Procedure

Kevin and Tim met with the first author of the paper for two task-based interviews, each of which lasted approximately two hours. In the interviews, Kevin and Tim were presented with a proof and asked to "think aloud" as they read and studied the proof. They were asked to read the proof until they felt that they understood it and, after they read the proof, they would be asked a series of questions about it to test their comprehension. When Kevin and Tim felt they understood the proof, the interviewer took the proof from the students, told the students he would ask them questions about the proof, and informed them that after the questioning was over, they would have the opportunity to read the proof again and change any of their answers to these questions. In this study, Kevin and Tim answered nearly every question correctly without the proofs in front of them, so there was no reason for them to change their answers upon viewing the proof again. This finding confirms that Kevin and Tim were effective at reading the proofs for comprehension. The participants were first asked the open-ended questions that follow each proof. They were then asked to answer the multiple choice questions that followed each proof. This process was then repeated. Each two-hour interview consisted of Kevin and Tim reading and evaluating three proofs.

2. 6. Analysis

As noted in the introduction, there are few research articles on proof comprehension (Mejia-Ramos & Inglis, 2009) and we are not aware of any research on the strategies that students should use to read proofs for comprehension. Consequently we did not have any pre-existing categories in mind when analyzing this data and opted to use an open coding scheme in the style of Strauss and Corbin (1990).

In a first pass through the data, we independently noted each attempt that Kevin and Tim made to make sense of the theorem statement or the proof and provided a summary of the students' behavior. (Here, "attempt" was construed broadly to mean anything beyond a literal reading of the text). After these summaries were produced, the authors met to discuss their findings.

From here, it was noted that Kevin and Tim's proof-reading could be divided into four phases: (a) studying the theorem, (b) reading the proof, (c) re-reading and summarizing the proof, and (d) critically evaluating the proof. These phases are discussed in more detail in section 3.1. Within each phase, similar proof-reading attempts were grouped together to form categories of the proof-reading strategies that Kevin and Tim employed. These strategies are discussed in detail in section 3.2. After categories were named and defined, we again independently viewed the videotape, coding for each instance of the proof-reading strategies. We then compared notes and discussed disagreements until they resolved. Most disagreements were the result of oversight on one of our parts. After our coding, Kevin and Tim were again interviewed about whether the strategies we observed were commonly used and why they engaged in those proof-reading strategies.

3. Results

3. 1. Phases of proof reading

Kevin and Tim's proof-reading could be partitioned into four distinct phases:

(a) *Studying the theorem*. This phase consisted of Kevin and Tim's behavior prior to reading the proof and was comprised of actions such as attempting to understand the theorem or see why the theorem was true.

(b) *Reading the proof*. This phase occurred as Kevin and Tim made their first pass through the proof.

(c) *Re-reading the proof*. On some occasions, after Kevin and Tim read and understood each step of the proof, as well as the proof as a whole, they would return to the beginning of the proof to re-read it. In some instance, they would simply check each step again. In others, they would form a high-level summary of the proof.

(d) *Critically evaluating the proof*. For Proof 1 and Proof 5, Tim would note apparent inconsistencies that he observed in the proof—either between assertions within the proof or between the arguments employed in the proof and his prior knowledge. Kevin and Tim would then discuss how those inconsistencies could be resolved.

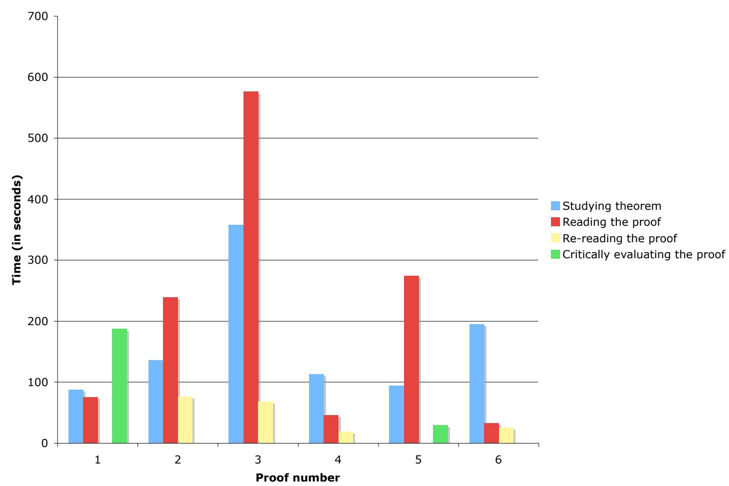


Figure 1. Time spent on each activity for each proof

The total amount of time spent on each phase of the activity for each proof is presented in Figure 1. We note two things about Figure 1. The first is the amount of time spent reading each proof. For each proof, Kevin and Tim spent between three minutes (proof 4) and over 16 minutes (proof 3) studying the proof, with an average of 7 minutes and 20 seconds per proof. This behavior seems to be atypical of how undergraduates studied proofs, as other studies report undergraduates generally spending under five minutes, and often less than two minutes, studying proofs (e.g., Selden & Selden, 2003; Weber, 2010). It is not clear if Kevin and Tim behaved differently from the undergraduates in Selden and Selden's (2003) and Weber's (2010) studies because they were unusually thoughtful and deliberate or because of differences in the task design (e.g., reading the proofs in pairs rather than alone or being told there would be a comprehension test after the proofs were read).

The second thing we note is the amount of time Kevin and Tim spent studying the theorem prior to reading the proof. Kevin and Tim averaged nearly three minutes (164 seconds) studying the theorem prior to reading the proof, spending nearly six minutes studying the statement of proof 3. While the participants spent a plurality of their time reading the proof itself (47%), it appears that the activity of studying the theorem was important as well. This suggests that proof-reading strategies should not only focus on how students read the proof of a theorem but also on how they comprehend the theorem.

3. 2. Strategies for proof-reading

In this section, we present the strategies that Kevin and Tim used in proof-reading. A summary of the students' strategy usage by proof is presented in Table 1.

Table 1. Kevin and Tim's proof-reading strategies

Proof-reading Strategy	Proof 1	Proof 2	Proof 3	Proof 4	Proof 5	Proof 6
Studying the theorem						
Rephrasing the theorem		X	X	X	X	
Attempting to prove theorem	X	X	X	X	X	X
Reading the proof						
Identifying proof		X			X	

framework								
Partitioning the proof				X			X	X
Checking inferences with examples			X			X		
Re-reading proof								
Provide high-level summary		X			X			
Critically examines proof	X					X		

3. 2. 1. Studying the theorem

3. 2. 1. 1. Understanding the theorem. Kevin and Tim would not proceed to reading a proof until they felt they understood what the theorem was asserting. This was illustrated sharply when the participants read the theorem statement for Proof 3, which introduced the concept of *k*-tuply perfect numbers. When Kevin suggests they read the proof, Tim stops him and says they first have to figure out what the theorem was saying. The theorem statement was:

We say that n is k-tuply perfect if and only if $\sigma(n)=\sum_{d|n} d=\mathrm{k}n$.

Theorem: If n is 3-tuply perfect and 3 does not divide n, then $3n$ is 4-tuply perfect.

After spending some time trying to figure out exactly what was being asserted, Kevin became frustrated and suggested reading the proof.

Kevin: Well we just have to agree with the proof

Tim: Yeah, but we have to figure out what it [Theorem 3] is saying, so hold on.

3. 2. 1. 2. Reformulating the theorem. A common strategy that Kevin and Tim used to understand a theorem was to reformulate it. In some cases, this involved rephrasing the formal theorem less formally in their own words. For instance, when reading Theorem 3 (stated above), Kevin expressed the theorem as follows.

Kevin: So if n is 3-tuply perfect, so that's if all the divisors including itself add up to 3n, and if that's the case, and 3, just the number 3, does not divide whatever n is, then 3n, then 3 times it is 4-tuply perfect.

In other instances, Kevin and Tim would express what it would mean for the theorem to be true symbolically. For instance, Theorem 4 asserts "There is a real number whose fourth power is exactly one larger than itself". After reading the theorem, Kevin represented the theorem algebraically.

Kevin: So we're comparing x to x^4, right?

Tim: OK.

Kevin: And the x^4 power is exactly one larger than itself, right? So x + 1 is equal to x^4?

Tim: Is that right? [pause] Yeah.

Kevin: So this [$x^4 - x - 1$] would equal zero. So if we subtract x and 1 from both sides. So we know that $x^4 = x + 1$ based on the theorem…

Tim: We want to show that.

Kevin: We want to show that, right. So it should be based on something like this, and we would have to solve this.

In this instance, the reformulation that Kevin proposed suggested an approach for constructing the proof—namely, Theorem 4 was equivalent to showing that $x^4 = x+1$ had a solution. In fact, this was the approach used in Proof 4.

3. 2. 1. 3. Seeing why the theorem is true. Prior to reading the proofs of the theorems, Kevin and Tim would always attempt to prove the theorems themselves. For instance, consider how Kevin and Tim reasoned about Theorem 2, which asserted that if (a, b, c) is a Pythagorean triple, then c must be odd.

Tim: Before we do the proof, we're going to use some, what do you call it? Mod 2?
Kevin: Modular arithmetic?
Tim: I think so. Maybe.
Kevin: Why do you say we should be, we will be using…
Tim: Well you can also use 2k+1 and 2k and wind up being plus one at the end.
Kevin: Yeah.
Tim: OK. If a and b are both even and if you square them, those are both even and you add them, that's also even so c squared would become even. And they're not coprime.
Kevin: Exactly.
Tim: And the square root of an even number is…
Kevin: Right. If a and b are both even, c has to be odd or they're not coprime. So we can consider the other cases where either a and b are both odd or one is odd and the other is even. I mean, I assume that's what the proof is going to be. Those three cases, right? They're both even so c has to be odd. That one's easy..
Tim: Well odd squared plus odd squared is odd [sic] and two odds make an even.
Kevin: 2k + 1 squared plus 2l + 1 squared
Tim: And they can't be both odd… [pause] *That's interesting.* Anyway, anyway…
Kevin: We should probably just read. [italics are our emphasis]

In this excerpt, Tim suggests using parity as an approach to seeing why Theorem 2 is true. Parity-based arguments can be easily used to show why the theorem is true in all cases except when a and b are both odd. When seeing how this line of argumentation could not be used to address this case, Tim remarks, "That's interesting". Realizing why the obvious approach to proving Theorem 2 was insufficient appeared to motivate Tim and Kevin's reading of the proof.

Tim and Kevin's work may have had an additional benefit as well. When reading case 3 of the proof, Kevin remarked, "If a is odd, and then they go through the whole rigamorole, oh and they just said what I said. I agree with this". Having produced a similar argument himself, Kevin was able to quickly skim and comprehend the argument in the proof.

A second illustration of Tim and Kevin attempting to prove a theorem occurred when they read Theorem 1, which asserts that "$4x^3 – x^4 + 2\sin x = 30$ has no solutions".

Tim: As x gets really big, it gets dominated by the negative x to the fourth term. And it's a parabola going down basically and it's going to get modulated a little bit.
Kevin: Right. And sine of x is…
Tim: Periodic.
Kevin: "It is periodic so that wouldn't really affect it too much… out of the two functions, f(x) is the trumping one.
Tim: So in the long run, it's going…
Kevin: It's really f(x) that matters.
Tim: And the question is, does it reach 30.

Like their attempt to prove Theorem 2 presented above, a complete proof was not reached. However, in this case, Kevin and Tim recognized the high-level ideas of the proof—since $2\sin x$ is periodic (actually, more importantly, but not stated, $2\sin x$ is bounded), the key to the proof is showing that $4x^3 – x^4$ has a sufficiently low bound as well. The details in the proof itself, such as using differentiation to find the critical points of $f(x)=4x^3 – x^4$, can be seen as supporting these high-level goals.

Another benefit of attempting to prove the Theorems prior to reading the proof is that these efforts sometimes helped Kevin and Tim determine what proof frameworks (in the sense of Selden and Selden, 1995) would be appropriate to use in a proof. Consider Tim's proof attempt for Theorem 3:

Tim: So 3n is 4-tuply perfect because the sum of all the numbers that divide 3n, including 3n, sums to 12n, right?

Kevin: Say what?

Tim: Use the pen, the power of the pen… [Tim begins writing] such that d divides 3n is 4 times 3n… for n is 3-tuply. What do you think about that?

Tim: The numbers that divide n is 3n, right? Now if you take 3n, that's 4-tuply because sigma of 4 of 3n is the sum of the divisors of 3n.

Kevin: But we're proving that it's 4-tuply.

Tim: Right, right. This is true is what I'm trying to say. I'm just trying to say that if this is true, then this is true… I'm not sure if I'm dong the implications right.

Although Tim was unable to make much subsequent progress on constructing the proof, it did allow him and Kevin to recognize that they needed to show the factors of $3n$ summed to $12n$ using the fact that the sum of the factors of n summed to $3n$, which was the method used in Proof 3.

Tim discussed several benefits of attempting to prove a theorem before reading its proof. He notes, "it makes you more of an active proofreader". Also, Tim noted that a practical reason for reading proofs is so he could expand his arsenal of proving techniques, noting one goal when reading a proof was "understanding the tricks and techniques so I can use them again". Mathematicians also list this as an important reason for reading proofs (Weber & Mejia-Ramos, 2011). Tim claimed that if he considered how an argument would proceed before reading it, it would help him "learn to do it [himself]"..

3. 2. 1. 4. Discussion.

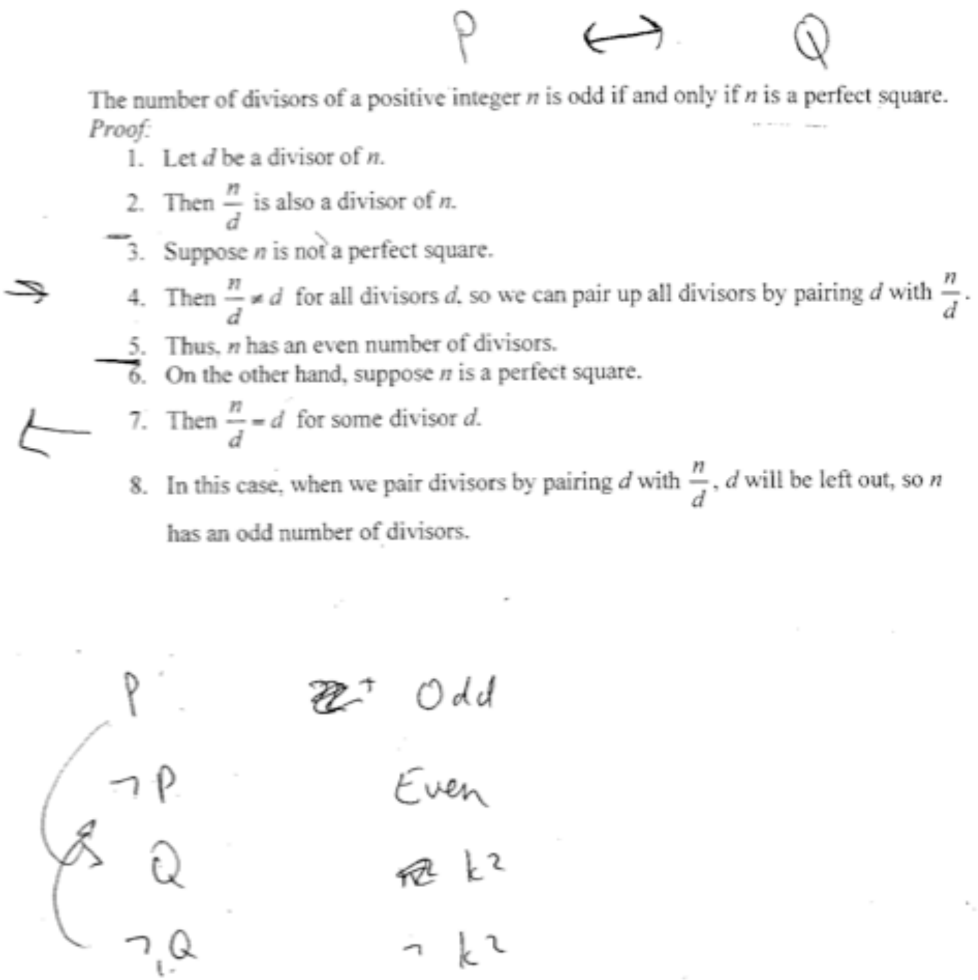
Before reading a proof, Kevin and Tim would sometimes reformulate the theorem being proven and would always attempt to prove the theorem themselves. Reformulation not only allowed Kevin and Tim to gain a meaningful sense of what the theorem was asserting, but also in some cases, suggested a method of proof. Dahlberg and Housman (1998) suggested that reformulations of definitions of concepts may help students construct proofs about that concept. It may also be the case that reformulating theorems can aid in the comprehension of proofs. Kevin and Tim consistently tried to prove the theorems themselves before reading the proof. This appeared to have a number of benefits, including motivating their proof-reading, allowing them to anticipate the higher-level structure of the proof, recognizing the proof frameworks that were employed in the proof, and appreciating the methods used in the proofs that they read.

3. 2. 2. Reading the proof

3. 2. 2. 1. Attending to proof frameworks

When reading the proofs, Kevin and Tim sometimes explicitly attended to what claim (or sub-claim) was being proven, what proof technique was being employed, and what the hypotheses and conclusions would have to be given that claim and proof technique. This was most clearly illustrated in their reading of Proof 5, where Tim was confused by the overarching structure of the proof. Kevin explicitly attends to each of the issues described above, as is illustrated both in their conversation and their written work on Proof 5 presented in Figure 2 below.

Figure 2. Kevin and Tom's written work on Theorem 5 and Proof 5



$P \longleftrightarrow Q$

The number of divisors of a positive integer n is odd if and only if n is a perfect square.

Proof:

1. Let d be a divisor of n.
2. Then $\frac{n}{d}$ is also a divisor of n.
3. Suppose n is not a perfect square.
4. Then $\frac{n}{d} \neq d$ for all divisors d, so we can pair up all divisors by pairing d with $\frac{n}{d}$.
5. Thus, n has an even number of divisors.
6. On the other hand, suppose n is a perfect square.
7. Then $\frac{n}{d} = d$ for some divisor d.
8. In this case, when we pair divisors by pairing d with $\frac{n}{d}$, d will be left out, so n has an odd number of divisors.

P	$\cancel{\mathbb{Z}}^{+}$ Odd
$\neg P$	Even
Q	k^2
$\neg Q$	$\neg k^2$

Kevin: From [lines] two to four, it's doing the proof by contradiction. Suppose n is not a perfect square. […]

Tim: Suppose n is not a perfect square. So you're saying, suppose not this?

Kevin: For which one?

Tim: So you're saying, suppose n is not perfect, right, and that's the opposite of the right side.

Kevin: Cause it's dichotomous. If it's not a perfect square, then it's even. So therefore if P implies Q, then not Q implies not P, right? Contrapositive.

Tim: So we're saying this is P [writes P abover "the number of divisors of a positive integer n is odd], this is Q [writes Q over "n is a perfect square"].

Kevin: OK sure. So P is a positive integer, right? Positive integer?... Right?... Positive integer?

Tim: Is odd?

Kevin: Right. Sorry. P is odd, right? So then not P would be even. So Q would be a perfect square, k squared. And not Q would not be k^2 or whatever. So here, [referring to lines 3-5] it shows not Q implies not P.

Tim: And this part of the proof ends at 5 [Tim draws horizontal dashed between lines 2 and 3 and between lines 5 and 6 to partition the two parts of the proof]

Kevin: So it shows not Q implies not P and therefore P implies Q.

Tim: OK.

Kevin: So it's proving it forwards. [Tim rights a right arrow next to lines 3, 4, and 5]. It's the forwards way. [Reading line 6] On the other hand, suppose n is a perfect square. So now it's going to prove the backwards.

Selden and Selden (2003) and Weber (2009) found that students would often accept a proof of a conditional statement as valid in cases where the proof began by assuming the conclusion of the conditional statement and deduced the antecedent—i.e., the argument proved the converse of the statement. Both research teams took this as evidence that participants did not attend to the proof framework of the proofs that they were reading. The excerpt above illustrates that Kevin and Tim explicitly attend to the assumptions and conclusions of the proof to understand how a valid proof technique is being employed.

3. 2. 2. 2. Partitioning the proof

For proofs that could be broken into different modules, Kevin and Tim would sometimes partition the proof to see when one part of the proof began and the other part ended. For instance, the first eight lines of Proof 3 consisted of proving the lemma "if *m* and *n* are relatively prime, then $\sigma(m)\sigma(n) = \sigma(mn)$" and the last three lines of the proof were applying that lemma to Theorem 3. When reading the proof, Tim suggests partitioning the proof into parts and initially skipping the proof of the lemma.

> Kevin: Is this of the lemma?
> Tim: This is the proof of the lemma.
> Kevin: Where does the proof of the lemma end?
> Tim: Good question.
> Kevin: Is there more? [checks other side of the sheet]
> Interviewer: The proof of the lemma ends at step 8.
> Kevin: So why isn't this just the entire proof?
> Tim: Because when you use the lemma, you prove this.
> Kevin: So the actual proof is just like two steps. Well three steps. Well, nine to eleven, so it's four steps. OK. [Kevin begins reading the proof, starting at the lemma]
> Tim: So, let's not read the lemma first.
> Kevin: You have to read the lemma.
> Tim: Let's read the lemma later. Trust me.

In this instance, partitioning the proof into sections allowed Kevin and Tim were able to discern the high-level structure of the proof before reading its technical details. Their later comments revealed that Kevin and Tim both agreed that seeing how the lemma was used in the proof motivated their need to read the proof of the lemma. Kevin claimed a benefit of partitioning the proof into parts was that it helped break complicated proofs into chunks that were manageable for him.

3. 2. 2. 3. Using examples to make sense of statements within the proof

For Proofs 3 and 5, when there was confusion about an inference within a proof, Kevin and Tim would attempt to resolve this confusion by seeing how the step applied to a specific example. For instance, the first two lines of Proof 5 were: "Let *d* be a divisor of *n*. Then *n*/*d* is also a divisor of *n*". Tim hesitated upon reading this statement, prompting Kevin to add:

> Kevin: So, say 7 is a divisor of 21. So, to get the other factor, it's 3.
> Tim: Right, right.

Kevin and Tim both emphasized the importance of using examples to make sense of problematic aspects of the proof with Kevin noting that while numbers cannot be used to prove a general statement, they are "really good to conceptualize it". In general, Kevin and Tim both emphasized that considering examples was essential when trying to make sense of abstract mathematics (particularly with proofs involving summations) because without using examples to make these ideas concrete, they would be unable to follow what was asserted within a proof. It is interesting to note that some mathematicians also claim to often work through a proof with a specific example (Weber & Mejia-Ramos, 2011).

3. 2. 2. 4. Summary of proof-reading strategies

Kevin and Tim displayed three proof-reading strategies in their initial reading of the proofs—identifying the assumption and conclusions of the proof framework being used, partitioning the proof into parts, and using examples to understand problematic aspects of the proof. The first two strategies concern understanding the high-level structure of the proof—namely what are the main components of the proof and what is the logical structure of each of those components. The third concerns local details of the proof by trying to understand individual statements.

3. 2. 3. Summarizing the proof

After reading each step of the proof, Kevin and Tim would sometimes return to the beginning of the proof and re-read it. In some cases, this process seemed merely to remind themselves of what the proof was saying and to check that each step was valid (possibly in preparation for the comprehension test they would be given shortly afterwards). In other cases, Kevin and Tim would give a summary that expressed the high-level ideas of the proof that went beyond describing what was in each individual step in the proof. After reading Proof 2, Kevin and Tim summarized it as follows:

> Tim: There are three possibilities
>
> Kevin: And two can't exist
>
> Tim: A and b are both even. A and b are both odd. A and b are even odd. The first two aren't possible, so it's got to be a even and b odd or b even and a odd.
>
> Kevin: And those are probably interchangeable.
>
> Tim: And the result of that is c is odd. And that's what we wanted to show.
>
> Kevin: Right. It has to be odd because it has the plus one.

Note that in this summary, the participants broke the proof into three separate cases and quickly dismissed two cases as impossible. Kevin ended the transcript by describing the method used to show the third case implied the desired result. When summarizing Proof 4 (which was only four lines long), Kevin strips away the specificity of the point to describe the overarching method used in the proof:

> Kevin: Right, so given that we know it's a continuous function, and some value makes it negative and some value makes it positive, so since it's continuous, it's got to go through zero at some point.
>
> Tim: Right.
>
> Kevin: And it doesn't actually say what it is, but that's irrelevant.
>
> Tim: [laughing] Right. All the [inaudible] wants to know is, "Does it exist?"
>
> Kevin: Right. Sounds good.

Note that Kevin observes an important detail of the proof. The root of the function cannot determined from the proof (i.e., the proof is not constructive) but both Kevin and Tim recognize that this is nonetheless sufficient to prove the theorem. In the excerpts highlighted above, we observe how the act of summarizing provided Kevin and Tim with insight about the higher-level structure and method involved in the proofs that they read.

3. 2. 4. Critically evaluating the proof

After reading the proof, Tim would sometimes notice what he thought could be an inconsistency in the proof. In Proof 1, it is deduced that the function $f(x) = 4x^3 - x^4$ diverged to negative infinity as x approached either positive infinity or negative infinity. It was also deduced that $f(x)$ contained exactly two critical points. Tim noted that if critical points were synonymous with turning points, this would be a contradiction.

> Tim: There are two critical points, so it's sort of like at the top and it sort of like

flattens out and keeps going down. Like that [draws a graph with three turning points].

Kevin: You mean like if it doesn't line up? You mean if it were like f(x) and g(x)?

Tim: No, no, a critical point is like where the derivative is equal to zero, which means its like horizontal slope. So I was just thinking it's like... I don't want to be too…

Kevin: [referring to Tim's graph] There would be three here. Here, here, and here.

Tim: [referring to the proof] But there's only two [critical points].

Kevin: Oh you mean a sort of saddle point?

Tim: Yeah, I forgot the word.

Kevin: Like x^3.

Tim: Right, there was only two critical points, so I was sort of like… [draws graph with one turning point, one saddle point, and diverging to infinity in both directions]

The resolution to the apparent contradiction was that a critical point is not necessarily a turning point. It could also be a saddle point. After reading Proof 5, Tim asks where the assumption in Theorem 5 that *n* is a positive integer was used in the proof. This might seem to be an inconsistency with Tim since proofs usually make use of all of their hypotheses.

Tim: Actually, real quick, can we talk about one more thing? Why does it matter that it's a positive integer?

Kevin: Well if it's a negative integer n, then you're dealing with negative factors.

In both instances, Tim pointed out what seemed to be an inconsistency that Kevin was able to help him resolve.

4. Discussion

4. 1. Caveats

This paper sought to identify effective proof reading strategies that could be taught to undergraduates in future teaching experiments. Of course, as with any case study, there are caveats due to the small sample size of this study. In particular, it is possible that using different proofs, or performing this study with different students, would yield different proof-reading strategies. Hence the proof-reading strategies described in this paper are not meant to constitute an exhaustive list, but can be used as a starting point for future teaching experiments.

4. 2. Viability of these strategies

There are (at least) three hypotheses for why the proof-reading strategies described in this paper might not be appropriate to teach to undergraduates:

(i) The strategies used in this paper are unique to Kevin and Tim and are not used by many other students who are effective at reading proofs.

(ii) Most undergraduates already use these proof-reading strategies. Hence, teaching undergraduates to use these strategies would not improve their proof-reading substantially. The keys to Kevin and Tim's success lie elsewhere.

(iii) Kevin and Tim's use of these strategies was helpful, but the implementation of these strategies relied on Kevin and Tim's conceptual understanding of the mathematics involved and/or with their experience in constructing proofs. Students without such a background might not benefit from using these strategies.

We discuss each hypothesis below. Regarding (i), in separate papers, we have examined the ways that professional mathematicians read proofs (Weber, 2008; Weber & Mejia-Ramos, 2011) and note consistencies between Kevin and Tim and the mathematicians' behavior. In particular, like Kevin and Tim, the mathematicians would also identify proof frameworks (Weber, 2008), partition proofs (Weber & Mejia-Ramos, 2011), and use examples to understand problematic aspects of the proofs they were reading (Weber, 2008; Weber & Mejia-Ramos, 2011). Further, there is some evidence that mathematicians would try to prove the theorem themselves to motivate the proofs that they read and summarize the proof by seeing how different sections of the proof related to one another (Weber & Mejia-Ramos, 2011). Hence some of the strategies that Kevin and Tim used where consistent with mathematicians who are presumably experts at comprehending proofs.

We also note that some of the strategies that Kevin and Tim used were consistent with suggestions from mathematics educators. For instance, Leron (1983) argued that proof comprehension would be facilitated by first presenting the high-level ideas of the proof and then supporting sub-arguments to carry out these high-level ideas. Kevin and Tim's partitioning of the proof could, in a sense, be viewed as an attempt to structure a proof themselves. Selden and Selden (2003) emphasized the importance of identifying proof frameworks when reading a proof and Kevin and Tom explicitly attended to this.

Finally, we note that Kevin and Tim's learning strategies seemed consistent with the proof comprehension model of Mejia-Ramos et al (2009). The strategies they used seemed to have direct benefits for understanding the meaning of a theorem, identifying proof structures, providing high-level summaries of the proof, describing the general method used in a proof, and applying the ideas of the proof to a specific example. It is possible that the assessment items that were based on Mejia-Ramos et al's model influenced the participants' proof-reading behavior. However, we note that some of the proof-reading strategies were used when reading Proof 1 (prior to receiving any assessment items) and the participants described the strategies that they used in this study as ones they used frequently.

Regarding hypothesis (ii), the studies of Selden and Selden (2003) and Weber (2010) suggest that undergraduates do not use the proof-reading strategies described in this paper. For instance, both Selden and Selden (2003) and Weber (2010) noted that many undergraduates ignore proof structure when reading a proof, preventing them from recognizing some proofs as invalid. Selden and Selden (2003) also observed that students tended to focus on local details of the proof (i.e., specific inferences and calculations) while ignoring higher-level details. In Weber's (2010) study, participants rarely considered examples or graphs when reading a proof, something that Kevin and Tim did in this study.

We do not have data to assess the viability of hypothesis (iii). Whether undergraduates possess the background knowledge to successfully implement Kevin and Tim's proof-reading strategies, or whether they can be taught this knowledge in a reasonable period of time, are questions that we will investigate in future teaching experiments.

4. 3. Teaching suggestions

From our perspective, one of the most surprising findings of our study is the importance that Kevin and Tim paid to understanding a theorem and attempting to generate a proof themselves before reading the proof in the text. In lectures, professors might consider spending some time reviewing the meaning of the theorem they are going to prove. Selden and Selden (1995) document that undergraduate students often are unable to extract the logical meaning of informal mathematics statements. It may be worthwhile for professors to guide students through this process. Professors may also want to give students the opportunity to understand why a theorem is true and generate their own proofs of the theorem before presenting a proof in the class. Kevin and Tim cited many benefits to doing this, including motivating the need for a proof and appreciating the higher-level ideas and methods used in the proof. Similarly, Leron and Dubinsky (1995) suggest that presenting a formal proof of an important theorem to a class after they have extended experience exploring that theorem as this might lead students to appreciate and understand the proof better.

Having students read and discuss proofs collaboratively might also benefit students. Throughout our study, we saw Kevin and Tim working together to overcome confusions that arose as they were reading the proof. Finally, as Conradie and Frith (1990) suggest, asking students deep and meaningful questions about the proofs that they read might not only encourage students to study these proofs more, but also inform them about what it means to understand a proof.

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APPENDIX: Theorems and proofs used in this study

Theorem 1.

Claim: $4x^3 - x^4 + 2\sin x = 30$ has no solutions.

Proof 1.

Consider the functions $f(x) = 4x^3 - x^4$ and $g(x) = 2\sin x$..

Since *f(x)* is a polynomial of degree 4 whose leading coefficient is negative, $f(x) \to \infty$ as $x \to \infty$ and $x \to -\infty$. Hence, *f(x)* will have an absolute maximum. Taking the first derivative of f(x) yields $f'(x)=12x^2 - 4x^3$. Setting $f'(x)$ equal to zero and solving for x yields $x = 0$ and $x = 3$. These are the critical points of *f(x)*. The absolute maximum must occur at a critical point. $f(0) = 0$. $f(3) = 27$. Hence, $f(x) \le 27$ for all x.

The range of sin x is [-1, 1]. Hence the range of 2sin x is [-2, 2]. Therefore $g(x) = 2 \sin x \le 2$ for all x.

$4x^3 - x^4 + 2\sin x = f(x) + g(x) \le \max(f(x)) + \max(g(x)) = 27+2= 29<30$. Therefore $4x^3 - x^4 + 2\sin x = 30$ has no solutions.

Theorem 2.

Let a, b, and c be natural numbers. (a, b, c) is a Primitive Pythagorean triple if two conditions hold:

(1) $a^2 + b^2 = c^2$

(2) a, b, and c are co-prime (i.e., there is not a common factor of a, b, and c other than 1)

For example, (3, 4, 5) and (5, 12, 13) are Pythagorean triples. (9, 12, 15) is not a Pythagorean triple because 3 is a factor of 9, 12, and 15. (4, 5, 7) is not a Pythagorean triple because $4^2 + 5^2 = 41 \neq 49 = 7^2$.

Theorem: If (a, b, c) is a Primitive Pythagorean triple, then c is odd.

Proof 2.

Lemma 1: $a^2 \equiv 1 \pmod 4$ if a is odd and $a^2 \equiv 0 \pmod 4$ if a is even.

Proof. Suppose a is odd. Then a = 2k+1 for some integer k. $a^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1$. So $a^2 \equiv 1 \pmod 4$. Suppose a is even. Then a = 2k for some integer k. $a^2 = 4k^2$ so a is divisible by 4 and hence $a^2 \equiv 0 \pmod 4$.

Lemma 2: If a^2 is odd, then a is odd.

Proof by contraposition. It suffices to show that if a is not odd, then a^2 is not odd. This is equivalent to showing if a is even, then a^2 is even. Suppose a is even. Then a^2 is even since the product of two even numbers is even. This proves the claim.

Proof of theorem (by cases).

There are three cases. Either a and b are both even, a and b are both odd, or a is even and b is odd (or vice versa).

Case 1 (a and b are both even). Since a and b are even, a^2 and b^2 are even. Hence $a^2 + b^2$ is even. If (a, b, c) is a Pythagorean triple, $a^2 + b^2 = c^2$ so c^2 is even. By lemma 2, since c^2 is even, c must be even. Hence, a, b, and c are all even and hence all have a factor of 2. Thus (a, b, c) is not a primitive Pythagorean triple. There are no primitive Pythagorean triples if a and b are both even.

Case 2 (a and b are both odd). By lemma 1, since a and b are odd, $a^2 \equiv 1 \pmod 4$ and $b^2 \equiv 1 \pmod 4$. Thus, $a^2 + b^2 \equiv 2 \pmod 4$. If (a, b, c) is a Pythagorean triple, $a^2 + b^2 = c^2 \equiv 2 \pmod 4$. By Lemma 1, we know that c^2 cannot be congruent to 2 (mod 4). Thus there are no primitive Pythagorean triples if a and b are both odd.

Case 3. (a is odd and b is even, or vice versa). Without loss of generality, assume a is odd and b is even. Then a^2 is odd and b^2 is even. Hence $a^2 + b^2$ is odd. If (a, b, c) is a Primitive Pythagorean triple, $a^2 + b^2 = c^2$ is odd. Since c^2 is odd, by Lemma 2, c is odd.

Hence, the only way to a Pythagorean triple can be formed is if a is odd and b is even or a is even and b is odd. In either case, c is necessarily odd.

Theorem 3.

We say that n is k-tuply perfect if and only if $\sigma(n)= \sum_{d|n} d =kn$.

If n is 3-tuply perfect and 3 does not divide n, then $3n$ is 4-tuply perfect.

Proof 3.

1. **Lemma:** If m and n are relatively prime, then $\sigma(mn)=\sigma(m)\sigma(n)$.
2. *Proof:* let the prime factorization of n be $n=p_1^{a1}p_2^{a2}...p_k^{ak}$. .
3. Then the divisors of n are all numbers of the form $d=p_1^{b1}p_2^{b2}...p_k^{bk}$.where $0 \le b_i \le a_i$.
4. But these numbers are precisely the terms in the expansion of the product: $(1+p_1+p_1^2+...p_1^{a1})(1+p_2+p_2^2+...p_2^{a2})\ldots(1+p_k+p_k^2+...p_k^{ak})$.
5. Thus, it must be that $\sigma(n)=(1+p_1+p_1^2+...p_1^{a1})(1+p_2+p_2^2+...p_2^{a2})\ldots(1+p_k+p_k^2+...p_k^{ak})$
6. Since each set of parentheses contains a geometric sum, we can rewrite this as $\sigma(n) = \frac{p_1^{a_1}-1}{p_1-1} \bullet \frac{p_2^{a_2}-1}{p_2-1} \cdots \frac{p_k^{a_k}-1}{p_k-1}$.
7. For n and m relatively prime, we will have $mn=p_1^{a1}p_2^{a2}\ldots p_k^{ak}q_1^{b1}q_2^{b2}\ldots q_j^{bj}$ where $n=p_1^{a1}p_2^{a2}\ldots p_k^{ak}$ and $m= q_1^{b1}q_2^{b2}\ldots q_j^{bj}$.
8. Then
$$\sigma(mn) = \frac{p_1^{a_1}-1}{p_1-1} \bullet \frac{p_2^{a_2}-1}{p_2-1} \cdots \frac{p_k^{a_k}-1}{p_k-1} \bullet \frac{q_1^{a_1}-1}{q_1-1} \bullet \frac{q_2^{a_2}-1}{q_2-1} \cdots \frac{q_j^{a_j}-1}{q_j-1} = \sigma(m)\sigma(n)$$.
9. With this lemma in hand, suppose that $\sigma(n) = 3n$ and n is not a multiple of 3.
10. Then $\sigma(3n)=\sigma(3)\sigma(n)$.
11. However, $\sigma(3)=1+3=4$. So $\sigma(3n)=4\sigma(n)=4(3n)$,which makes $3n$ 4-tuply perfect.

Theorem 4.

There is a real number whose fourth power is exactly one larger than itself.

Proof 4.

1. Let $f(x)=x^4-x=1$.

2. $f(1)=-1$ and $f(2)=13$.
3. By the intermediate value theorem, there must be some number c such that $1<c<2$ and $f(c)=0$.
4. c has the desired property.

Theorem 5.
If every even integer greater than 2 can be expressed as the sum of two primes, then every integer greater than 5 can be expressed as the sum of three primes.

Proof 5.
1. Assume every even integer greater than 2 can be expressed as the sum of two primes.
2. Let $n>5$.
3. We proceed by cases.
4. Case 1: n is even.
5. Then n-2 is even and greater than 2.
6. So $n-2=p_1+p_2$, with p_1 and p_2 prime.
7. Then $n = p_1+p_2+2$, a sum of three primes.
8. Case 2: n is odd.
9. Then $n - 3$ is even and greater than 2.
10. So $n-3=p_1+p_2$, with p_1 and p_2 prime.
11. Then $n = p_1+p_2+3$, a sum of three primess.

Theorem 6.
The number of divisors of a positive integer n is odd if and only if n is a perfect square.

Proof 6.
1. Let d be a divisor of n.
2. Then n/d is also a divisor of n.
3. Suppose n is not a perfect square.
4. Then $n/d \neq d$ for all divisors d, so we can pair up all divisors by pairing d with n/d.
5. Thus, n has an even number of divisors.
6. On the other hand, suppose n is a perfect square.
7. Then $n/d = d$ for some divisor d.
8. In this case, when we pair divisors by pairing d with n/d, d will be left out, so n has an odd number of divisors.

STUDENT UNDERSTANDING OF INTEGRATION IN THE CONTEXT AND NOTATION OF THERMODYNAMICS: CONCEPTS, REPRESENTATIONS, AND TRANSFER

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Students are expected to apply the mathematics learned in their mathematics courses to concepts and problems in physics. Little empirical research has investigated how readily students are able to transfer their mathematical knowledge and skills from their mathematics classes to other courses. In physics education research, few studies have distinguished between difficulties students have with physics concepts and those with either the mathematics concepts, application of those concepts, or the representations used to connect the math and the physics. We report on empirical studies of student conceptual difficulties with (single-variable) integration on mathematics questions that are analogous to canonical questions in thermodynamics.

Key words: [Physics, integrals, conceptual understanding, representations, transfer]

Introduction

Mathematics is a vital part of how physics concepts are represented (e.g. equations, graphs and diagrams) and how problems are solved, all across the curriculum. It allows students to simplify the analysis of complex problems by representing complicated conceptual physics problems as a relatively simple relationship between variables. Appropriate interpretation of these representations requires recognition of the connections between the physics and the mathematics built into the representation as well as subsequent application of the related mathematical concepts (Redish, 2005).

Students are expected to apply the mathematics learned in their mathematics courses to concepts and problems in physics. Despite the fact that students are expected to carry out such interdisciplinary study as a matter of course, little empirical research has investigated how readily students are able to transfer their mathematical knowledge and skills from their mathematics classes to other courses.

With many physics topics, specific mathematical concepts are required for a complete understanding and appreciation of the physics. Meltzer (2002) has shown a link between mathematical acumen and success in an algebra-based physics class. Tuminaro and Redish (2007) have combined the frameworks of resources (Hammer, 1996), epistemic games, and framing to analyze student use of mathematics in physics. To date, however, there have only been a few physics education research (PER) studies exploring physics students' difficulties with calculus concepts (Black and Wittmann, 2009; Cui, Rebello, and

Bennett, 2007; Pollock, Thompson, and Mountcastle, 2007; Rebello, Cui, Bennett, Zollman, and Ozimek, 2007).

Our work aims to identify the extent to which mathematical understanding affects physics conceptual knowledge, specifically in the context of upper-level thermal physics. We have two main research questions: What specific difficulties do advanced-level undergraduate students have when learning thermal physics concepts? To what extent do students' mathematical understandings influence their responses to physics questions?

Background

One mathematical concept that has emerged as an area for investigation in our research is that of integration. In thermodynamics, two- or three-dimensional graphical representations of physical processes are especially useful in helping to understand these processes. These diagrams can contain information about the thermodynamic "path" followed in a process, regions of different phase, and critical behavior. Diagrams that plot pressure *versus* volume, known as *P-V* diagrams, are canonical representations of physical processes as well as of the corresponding mathematical models in thermodynamics. Information can quickly and easily be obtained from these representations. A point on this diagram represents the pressure and volume values for a particular equilibrium state of a system. Thermodynamic processes (reversible processes, at any rate) can be represented as curves on the diagram tracing the "path" the system took to go from its initial state to its final state during a specific process. One important physical quantity that can be obtained from this diagram is the thermodynamic work done (energy transfer via mechanical forces) on a system undergoing a thermodynamic process. The work done on the system is defined as the integral of pressure with respect to the volume:

$$\mathrm{W} \equiv \int PdV$$

Thus physicists can quickly determine the work done by a calculation, if the function *P(V)* is known for a given process; qualitative determinations can be made by evaluating the area under the curve for a particular process.

Questions that have focused on student understanding of physics and that have featured integration have proved troublesome for students at the introductory level (Meltzer, 2004). Meltzer gave students a *P-V* diagram in which an ideal gas undergoes two different thermodynamic processes, but each process starts in the same state and ends in the same state. For the situation described, he figure is shown in Fig. 1(a). The most common incorrect response and reasoning was that the works were equal, based on the initial and final states being the same for each process. There is a reasonable physics-based interpretation for this reasoning, namely that students are effectively treating work as a function of state (which would only depend on the "endpoints") rather than a process-dependent quantity.

Previous findings on student understanding of integral calculus concepts in research in undergraduate mathematics education indicate that students do not possess the necessary knowledge to allow them to successfully complete problems involving concepts of integration, especially with regards to considering the integral as the area under the curve. The literature in mathematics education repeatedly documents the lack of student understanding of the relationship between a definite integral and the area under the curve (Grundmeier, Hansen, and Sousa, 2006; Orton, 1983; Thompson, 1994; Vinner, 1989). These include student difficulties with recognition of integrals as limits of

(Riemann) sums (Orton, 1983; Sealey, 2006); student confusion about the concept of "negative area" for integrals of curves that fall below the *x*-axis either conceptually (Bezuidenhout and Olivier,

2000), computationally (Orton, 1983; Rasslan & Tall, 2002) or both (Hall, 2010). Thompson and Silverman (2007) showed that the reliance on *area under curve* reasoning may limit applicability of the conception of integrals.

Researchers at the University of Maine asked Meltzer's questions in upper-division thermal physics courses and found similar difficulties and reasoning patterns that Meltzer found (Pollock, Thompson, and Mountcastle, 2007). We wondered if some of the conceptual difficulties might originate in the confusion over the mathematics involved. In addition to replicating Meltzer's experiment in our upper-level thermodynamics course, we designed qualitative questions regarding comparisons and determinations of the magnitudes and signs of integrals, without any physics context, that are analogous to Meltzer's questions. We call these "physicsless physics questions" (Christensen and Thompson, 2010). The physicsless physics version uses notation that is more consistent with representations used in physics than math. By asking the physicsless phyiscs question we hope to further isolate mathematical difficulties that may underlie observed physics difficulties.

In the physicsless physics question related to Meltzer's work question, students are asked to determine the sign of the integral of each of two functions from the same starting point (*a*) to the same ending point (*b*), and to compare the magnitudes of the integrals:

$$I_1 \equiv \int_a^b f(y)dy \, ,$$

$$I_2 \equiv \int_a^b g(y)dy \, .$$

One version of the figure provided to students for this question is shown in Fig. 1(b). We administered the analogous math version of the question as well as the physics-less questions to the students in our upper-division thermodynamics courses (Christensen and Thompson, 2010). Many of the students used "endpoint" reasoning to state that the integrals were equal, in complete analogy to the reasoning seen in physics questions.

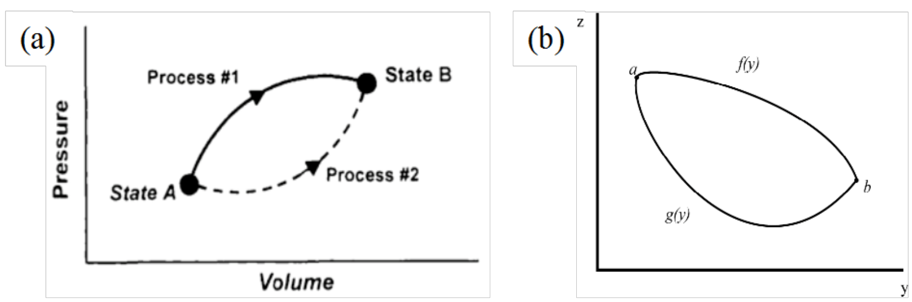

Figure 1. Diagrams used on questions described in the text. (a) Pressure-Volume diagram used to compare work done by identical systems over two different thermodynamic processes, from Meltzer (2004); (b) one version of figure used in "physicsless physics question" to compare values of the integral of the two functions shown over the same interval.

The survey results from the paired physics and physicsless questions among physics students show that some of the difficulties that arise when comparing thermodynamic work based on a pressure-volume (*P-V*) diagram may be attributed to difficulties with the *mathematical* aspect of the diagram, in particular with the correct application of an understanding of integrals, rather than (or in addition to) physics conceptual difficulties.

The original target population for our earlier research was student in upper-division physics courses. However, as we explored the mathematical aspects in more detail, we have included students in mathematics courses as part of our study as well. We also asked the physicsless physics question to students at the end of the third semester of calculus: a greater proportion of calculus students used "endpoint" reasoning than did physics students.

The current study extends earlier work in two ways. First, we follow up on the results from the physicsless physics question in calculus classes. We explore the extent to which explore students' understanding of the mathematics using a more conventional graph to start with. By removing the unfamiliar representation, we can start to determine the extent of students' ability to interpret graphs in the context of integration.

Experiment Design

We use the empirical framework of *specific difficulties* (Heron, 2003) to guide analysis of data and descriptions of student reasoning in our research. We start with targeted, context-dependent results and then generalize across contexts, seeking larger patterns of student responses in our data. Our emphasis is on gathering and interpreting empirical data that can act as a foundation for future studies on reasoning in physics and for curriculum development to address specific difficulties.

The original target population for our earlier research was student in upper-division physics courses. However, as we explored the mathematical aspects in more detail, we have included students in mathematics courses as part of our study as well. The study described here includes students in single-variable integral calculus (Calculus 2) and in multivariate calculus (Calculus 3). We were also curious about the extent to which introductory math students may have a different concept image of integrals, or certain facets of integrals, than physics students who recently learned the requisite math concepts; thus we also included students in our study who were taking the second semester of the calculus-based introductory physics course (physics 2), who had taken previously or were taking concurrently calculus 2.

It was suggested by our colleagues in the RUME community in particular that features in the physicsless physics question (Fig. 1(b)), because they lack the conventions from mathematics for a representation of an integral of a single-variable function, may be cueing students to try different mathematical approaches, in particular analysis of the task using line integrals. Following up on these suggestions and others from previous work, we created an analogous math version of the question (Fig. 2) that would provide a more mathematically acceptable representation for the question. In particular, the analogous math version of the question uses *y* and *x* as the axis labels, has each function extend beyond the intersection points, and shows the limits of the integral on the *x*-axis, as values of *x* rather than points on the graph.

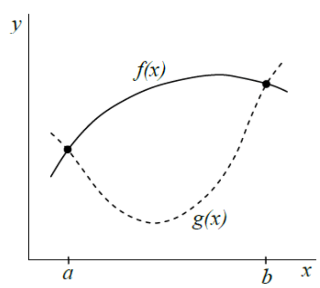

Figure 2. Diagram used for the Analogous Math version of the question described in the text.

We report here on two separate but related studies as extensions of the work reported earlier (Christensen and Thompson, 2010). In the first study, we gave the physicsless physics question to students in the last week of instruction in both Calculus 2 and Calculus 3, to see if the representation cued different knowledge in the different courses. To follow up on previous results and to see if different disciplinary populations would produce different results, the analogous math question was given near the end of the semester to a separate group of Calculus 3 students as well as a group of students at the end of Physics 2. Responses to the physicsless physics question and the analogous math question were analyzed for patterns and categorized.

To probe more deeply into the stability and the nature of students' concept image of these integral questions, seven interviews were carried out, using a think aloud protocol (Ericsson and Simon, 1998). These interviews were designed to build on the replies from the survey results. Interview subjects were taking physics 2; interviews occurred near the end of the semester. Each of the 7 subjects was asked to solve two of the written survey questions at the start of the interview. The interviewer then questioned students in order to follow up on their responses, and asked them to explain their reasoning in more detail. Based on the category of reasoning students provided in the initial conversation, the interviewer asked specific follow-up questions taken from a large set of possible questions. The follow-up questions were chosen to probe the stability and the nature of the reasoning that students were using to answer the first question.

Written Survey Data

The physicsless physics question shown in Fig. 1(b) was given to 41 students in calculus 2 and 56 students in calculus 3. When asked about the sign of the integral, just over three quarters of the calc 2 students and just over two thirds of the calc 3 students were correctly able to identify both integrals as being positive (first two rows of Table 1). When asked to compare the magnitude of the integrals for *f(y)* and *g(y)*, about 90% of calc 2 students were able to answer the question correctly, while about 60% of calc 3 students were able to correctly answer the question (first two columns of Table 2).

Table 1. Results for students who answered that both integrals are positive for the different versions of the integral question.

	Both Integrals Positive
Physicsless Calc 2 (N=41)	78%
Physicsless Calc 3 (N=56)	68%
Analogous Math Calc 3 (N=95)	77%
Analogous Math Physics 2 (N=95)	78%

Table 2. Results for magnitude comparison of the integrals in the written surveys.

	Physicsless Physics Calc 2 N=41	Physicsless Physics Calc 3 N= 56	Analogous Math Calc 3 N=95	Analogous Math Physics 2 N=95
Greater	**87%**	**58%**	**73%**	**73%**
Less than	0	2	7	3

Equal	3	7	9	1
Not enough	0	9	9	3
Other/No Ans	1	8	1	7

One interesting finding is that calculus 2 students markedly outperformed calculus 3 students in the magnitude comparison task. One explanation of the difference comes from the reasoning students gave. About 20% of calc 3 students said that I_1 was less than I_2, while none of the calc 2 students gave this response. A significant fraction of the reasoning given for this response in calc 3 was that *g(y)* appears to be longer in length than *f(y).* This is consistent with the idea that some students are interpreting the task as the evaluation of a line integral, an idea further supported by the absence of such reasoning in the calc 2 population, who have not yet learned about line integrals. Additional reasoning for *less than* includes a comparison of the "curvatures" (i.e., second derivatives) of the two curves: since *g(y)* has a larger magnitude of curvature, it has a greater integral. We note that a far smaller fraction of students said that the integrals would be equal. We attribute this to the asymmetry of this version of the question, since previous versions with symmetric curves (as in the physics version) yielded higher fractions of *equal* responses. Overall these results suggest that the details of the representation affect the responses. This has strong implications for the teaching of the representations in physics classes.

In a separate survey, the analogous math question was given to 95 students in calculus 3 and a separate 95 students in physics 2. We find that in both classes, over three quarters of the students are able to correctly determine the signs of both integrals, and about three quarters of the students are able to correctly compare the magnitudes of the integrals. The reasoning given with answers was not always clear, but they could roughly be put into a number of categories: area under the curve, position of the function, and shape of the curve (which includes both the slope and the curvature). Just over 50% explicitly used the word "area" in their reasoning. As mentioned above, while there is evidence for these categories of reasoning, the written data do not provide definitive proof that students have a concept image that is consistent with their explanations. The interviews provided more insight into student reasoning categories.

Interview Results

We found it difficult to understand the concept image (Vinner, 1989) that students from the survey answers alone. While 50% of students use the term *area* to describe the area under the curve, in many other answers it is hard to tell whether area under the curve is a part of the concept image. To probe more deeply into the stability and the nature of students' concept image of integrals in these questions, seven interviews were carried out, each with a think-aloud protocol (Ericsson and Simon, 1998). These interviews were designed to build on the responses from the survey results.

We are currently analyzing the seven interviews. Early into observing the interviews two observations have been notable: (1) students are unable to use mathematical reasoning to explain why the integral in the negative-*y* direction is negative when looking at the diagram, (2) while unable to explain the negative area under the curve mathematically, students are more comfortable talking about negative area – the area when the function is below the *x*-axis – when discussing the question in a physics context.

Interpretation of sign of integral by reversing the direction of integration

Freddie, one of the participants in the interviews, correctly identifies that the integrals are both positive. The interviewer then asks Freddie what the sign of the integral is if the integration direction is reversed (i.e., so that the integral is taken from *b* to *a* instead of *a* to *b*). Given a positive function, this method will yield a negative integral.

I: "So if I ask you to compare the integration from this point [points to f(b)] to this point [points to f(a)]." [The interviewer then writes $I \equiv \int_b^a f(x)\,dx$ *on the board.]*

F: "Ok so... Given that the integral, the way you have it set up from what I can tell, is used to find the area under the curve it should still be positive…"

Using the graphical representation, Freddie is fairly certain that the reversed-direction integral is still positive. But then Freddie incorrectly invokes the Fundamental Theorem of Calculus, comparing the endpoint values of the function (*f(x)*, the integrand) rather than the integrated function (*F(x)*). For the function Freddie was given, *f(a)* was less than *f(b)*, so that the difference would be negative. Freddie uses this to state his response:

F: "Based on that it would look like it would probably be negative."

Another student, Simon, also correctly identifies the initial integrals as positive. When asked to consider the reversed-direction integral, Simon explicitly uses the Riemann sum to think about the sign of the integral. He draws "columns" and states that the width of the columns are "your little *dx*s," and recognizes the arbitrariness of *dx*: "you can change your dx depending on, you know, whether or not you want to change by so much or so little…"

Further discussion of the process of integration elicits some (also incorrect) ideas about the Fundamental Theorem of Calculus (FTC) and the Riemann sum concept, in which he states that if "your last [column] is then […] smaller than your first [column], then I would, I would think you are going to get a negative value [for an integral]." Like Freddie, Simon confused *f(x)* with *F(x)* in invoking the FTC and came up with the correct sign for an incorrect reason. But again, Simon is conflicted between the answers obtained using graphical and symbolic representations.

The interviewer prods Simon more about the situation to get a sign for the reversed-direction integral. Simon responds:

S: "I feel that it should be positive because technically it shouldn't matter how you count this together right? … If you counted this way [moving his hand from right to left] or you count this way [moving his hand from left to right across the diagram] and you keep the dx *the same, you should find the same area."*

Simon seems to be adding the "columns" used for the Riemann sum in his description. For Simon, these columns seem to represent positive quantities, independent of the direction.

Both Freddie and Simon use the FTC to reason that the reversed-direction integral is negative, and also use the graphical represetation to reason that the area under the curve is positive regardless of integration direction.

Interpretation of sign of integral below the x-axis using math and physics contexts

Freddie was asked about the sign of the integral for a function whose values were negative over the range of integration (but the direction of integration was in increasing *x*). He invoked *area under curve* reasoning to interpret the integral on the graph, but initially he had difficulty in identifying the integral as negative in this case. When presented with a sine curve, Freddie stated the following while contemplating the negative part of the integral:

F: *"In order to get negative area it is not... conceptually, looking at like a plot of land, it would be an impossibility. However, we are looking at something like a voltage; voltages can very easily go negative because we only have them in reference to what we called to be ground."*

In this case Freddy felt more comfortable using voltage to interpret the negative area under the curve than giving a mathematical explanation. Freddy also used the equation to infer that the area under the curve must be negative, while initially looking at the diagram: he felt that the area must be negative. The reasons given by students in interviews show that graphically, students find it difficult to account for a negative integral using only mathematical knowledge, but when physical meaning is attributed to the variables in the graph, students are better able to interpret the integral of a negative function.

Conclusions

The survey results build on the work of Christensen and Thompson (2010) to show that students entering thermodynamics have non-trivial difficulties with the math they need to understand systems that are represented like the one in Figure 1(a). One of the reasons for the revised surveys was to test how the students' knowledge of calculus might affect their responses to the physicsless physics and analogous math questions. Calculus 3 students were outperformed by Calculus 2 students, most likely due to the additional knowledge of line integrals brought to bear on the task by some of the calc 3 students. Our preliminary results suggest that calculus 3 students performed better on the analogous math question than on the physicsless physics question. This is consistent with the idea that the features of the analogous math question leave less ambiguity about the nature of the representation itself, and thus students are less likely to misinterpret the task as a line integral task. However, the sample sizes are not large enough to make more than tentative conclusions. Further surveys will be asked this year to follow up on the ones already asked.

We have made a preliminary review of the interviews, with specific emphasis on aspects of student interpretations of negative integrals, either as integrals of positive functions taken from a higher-valued limit to a lower-valued one (i.e., a "right-to-left" or reversed-direction integral) or as integrals of functions that are negative within the range of integration (i.e., integrals of negative functions).

Prevalent throughout the interviews was attention to the idea of reversing the direction of the integral, which is a relevant procedure in some physics contexts. Prior research has not addressed student interpretation of this method, however. Interview results suggest that most students struggled with this task. One possible reason for this may be students failing to think of the differential *dx* as a negative quantity when the direction of the integration is reversed. This certainly appeared to be the case more explicitly for Simon's example. There are implications here for student understanding of the Riemann sum and the integral as the limit of that sum. Additionally, we find students incorrectly invoking the Fundamental Theorem of Calculus to reason about the sign of reversed-direction integrals. (This particular confusion with applying the FTC to values of the integrand has been seen by Pollock, and is reported in his thesis (Pollock 2008).)

The reasons given by students in interviews show that graphically, students find it difficult to account for an integral of a function that is negative within the range of integration using only mathematical knowledge; this result is consistent with literature from the RUME community (Bezuidenhout and Olivier, 2000; Hall, 2010; Orton, 1983;

Rasslan & Tall,
2002). However, students who attribute physical meaning to the variables in the graph are better able to interpret this type of integral and make sense of the negative "area."

We will pursue these observations in more depth with newly revised survey questions and with further analysis of these interviews. In this analysis, we will also follow up on how students use physics concepts compared to math knowledge when discussing the idea of negative area, as seen in Freddie's interview.

Finally, we feel that this work has relevance to the cognitive framework of *transfer in pieces* (Wagner, 2006), ideally suited for the study of knowledge transfer and the context-sensitivity of mathematical knowledge. The interviews to date have largely focused on students' mathematical ideas, and the analysis has focused on student knowledge of content and representations. We plan to extend the analysis and to carry out additional interviews, to probe students' transfer between mathematics and physics, through the transfer-in-pieces lens.

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EXTENDING A LOCAL INSTRUCTIONAL THEORY FOR THE DEVELOPMENT OF NUMBER SENSE TO RATIONAL NUMBER

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We report on the process by which we extended a local instruction theory for number sense development from the whole-number domain to the rational-number domain. Students involved in an earlier teaching experiment developed improved number sense, particularly in the form of flexible mental computation. The previous research was informed by a conjectured local instruction theory and informed the refinement and elaboration of that local instruction theory. The present study concerns a recent classroom teaching experiment in which envisioned learning routes that were developed in the context of whole-number mental computation and estimation were applied to reasoning about fraction size. In this way, the application of the local instruction theory was extended from whole-number sense to rational-number sense.

Keywords: Local instruction theory, number sense, prospective teachers, rational number

Gravemeijer (1999, 2004) introduced the construct of *local instruction theory* (LIT) and has encouraged the development of LITs to support reform efforts in mathematics education. A *local instruction theory* consists of “the description of, and rationale for, the envisioned learning route as it relates to a set of instructional activities for a specific topic” (Gravemeijer, 2004, p. 107). We have written about our LIT for number sense development (Nickerson & Whitacre, 2010), which informed instruction in a previous classroom teaching experiment (Cobb, 2000). Our LIT was refined on the basis of the results of that teaching experiment. In our previous work, the focus was on developing number sense in prospective elementary teachers within the whole-number domain, and the particular focus was on mental computation and computational estimation. Our recent research involved extending the LIT for number sense development to the rational-number domain, with a focus on students’ reasoning about fraction magnitude.

Background

This study represents the latest phase in an ongoing design research effort concerning the number sense development of prospective elementary teachers. In this section, we the stage for the present study with a brief review of relevant literature and a description of our approach to instructional design.

In our work, we conduct design research in the form of classroom teaching experiments, which are reflexively related to theory building (Cobb & Bowers, 1999). Design research occurs in cycles of instructional design, classroom teaching experiments, and data analysis (Gravemeijer, 1999). We see these cycles occurring at two distinct grain sizes: (1) The cycles are classroom teaching experiments, which are informed by and inform the LIT; (2) Within a given classroom teaching experiment, the LIT informs the design of hypothetical learning trajectories (Simon, 1995). A hypothetical learning trajectory (HLT) is a vehicle for planning the learning of particular mathematical topics. An HLT consists of the goal for students’ learning, the mathematical tasks, and hypotheses about the process of students’ learning.

Note that a local instruction theory is distinct from a hypothetical learning trajectory in two ways: (1) an LIT tends to describe an instructional sequence of longer duration; (2) an HLT is situated in a particular classroom, whereas an LIT is not (Gravemeijer, 1999). An LIT for the development of number sense would need to be generalizable to underlie many mathematical topics. Local instruction theories are an important part of our work as mathematics teacher educators teaching prospective elementary school teachers.

In order to teach mathematics effectively, elementary teachers need to understand elementary mathematics deeply (Ball, 1990). However, prospective and practicing elementary teachers often know the procedures of elementary mathematics, but do not understand the material conceptually (Ball, 1990; Ma, 1999). In fact, studies of preservice elementary teachers have found that this population tends to exhibit poor number sense, *even after* having completed their required college mathematics courses (Tsao, 2005; Yang, 2007; Yang, Reys, & Reys, 2009). In light of these findings, our research has focused on activities associated with improving the number sense of prospective elementary teachers.

Number Sense

One major area of focus in the number sense literature has been an investigation of the computational strategies that students use. Good number sense is associated with flexibility, which is exhibited in the use of a variety of computational strategies (Heirdsfield & Cooper, 2004). This is in contrast to individuals who exhibit little flexibility. In mental computation in older children and adults, inflexibility often manifests in the use of the mental analogues of the standard paper-and-pencil algorithms. While the standard algorithms can be useful, skilled mental calculators tend to select a strategy for an operation based on the particular numbers at hand. Furthermore, the strategies that these individuals use often stray far from standard, as in reformulating computations or rounding and compensating (Carraher, Carraher, & Schlieman, 1987; Greeno, 1991; Hierdsfield & Cooper, 2002; 2004; Hope & Sherrill, 1987; Markovits & Sowder, 1994; Reys, Rybolt, Bestgen, & Wyatt, 1982; Reys, Reys, Nohda, & Emori, 1995; Sowder, 1992; Yang, Reys, & Reys, 2009). The literature also suggests that these strategies should develop in an environment in which students are encouraged to develop their own strategies, share these strategies, and reflectively and collectively discuss the merits of different approaches (McIntosh, 1998; Sowder, 1992).

In 2005, we conducted a classroom teaching experiment in a mathematics content course for prospective elementary teachers. This course was the first in a sequence of four content courses and had a focus on number and operations. In that study, we focused on mental computation as a microcosm of number sense. Instruction was guided by our local instruction theory for the development of number sense. Thirteen students participated in pre/post interviews in which they were given story problems to be solved mentally. In addition to coding for the particular strategies that participants used, we coded these as belonging to more general categories of strategies indicating the extent to which the person's approach is tied to (or departs from) the standard algorithm for a given operation. We used a scheme of Markovits and Sowder (1994) to categorize mental computation strategies as *Standard*, *Transition*, *Nonstandard*, and *Nonstandard with Reformulation*. Nonstandard strategies are those that diverge substantially from the standard algorithms. The use of such strategies suggests an understanding of the operation that is not bound to any particular algorithm. As such, nonstandard strategies are associated with number sense (Markovits & Sowder, 1994; Yang, Reys, & Reys, 2009).

When the results were reviewed with respect to the Standard-to-Nonstandard framework, it was evident there was a rather dramatic shift in the strategies used by the 13 interview participants. In the first interview, Standard strategies were most common. Participants used few strategies for the operations, and often relied on the mental analogues of the standard written algorithms. In the second interview, by contrast, the most common category of strategy was Nonstandard with Reformulation – at the other extreme of the continuum. Thus, participants shifted from using the most standard to the least standard strategies, which suggests that their understanding of the operations moved from being bound to the standard algorithms to being unconstrained by these (Whitacre, 2007). These results were encouraging and led us to pursue further research concerning the number sense development of prospective elementary teachers.

LIT for the Development of Number Sense

Our local instruction theory for the development of number sense is organized around three major goals: (1) Students capitalize on opportunities to use number-sensible strategies; (2) Students develop a repertoire of number-sensible strategies; (3) Students develop the ability to reason with models. The third of our goals was influenced by one of the design heuristics of Realistic Mathematics Education (RME): students and teachers develop models *of* their informal activity, which become models *for* more formal mathematical reasoning (Gravemeijer, 1999; 2004; Stephan, Bowers, Cobb, & Gravemeijer, 2003). In this short paper, we focus on Goals 1 and 2. Table 1 describes the instructional activities and corresponding envisioned learning route for Goal 1. We think about the route to Goal 1 as a progression in which students move from being reliant on the standard algorithms toward understanding numbers and operations independently of those algorithms. In a class in which number sense development is a major goal, students initially are not accustomed to exercising their number sense when opportunities arise. The instructional activities in Table 1 are intended to encourage students to recognize and take advantage of those opportunities.

Table 1. Route to Goal 1.

Instructional Activities	Envisioned Learning Route
Instructor identifies and engineers opportunities for computational reasoning	
Students are invited to engage in computational and quantitative reasoning	Many students initially rely on standard algorithms
Students are encouraged to carry sense making to solutions with nonstandard strategies	Students use their own nonstandard strategies
Students engage in computational reasoning in a variety of contexts	Students capitalize on opportunities to use number-sensible strategies

Table 2 describes the instructional activities and corresponding envisioned learning route for Goal 2. This progression focuses on students' explicit awareness of strategies through discourse concerning these. We envision students moving from thinking/talking about how they solved a particular task to looking/discussing strategies that they and their peers have used across a set of tasks. As students reflect on these strategies and make them objects of discourse, they develop an explicit repertoire of strategies that they can draw from, depending on the details of the task at hand.

Table 2. Route to Goal 2.

Instructional Activities	Envisioned Learning Route
Instructor anticipates the nonstandard strategies that students might use	

Class negotiates records of strategies and initiates the practice of naming	Students may initially name strategies in ways tied to specific examples
Instructor and students negotiate differences and relative efficacy of strategies	Students name strategies according to essential characteristics
Instructor and students make strategies objects of discourse	Students develop a repertoire of number-sensible strategies

Note that these tables differ somewhat from those presented in Nickerson and Whitacre (2010) in terms of both format and content. The progressions remain very much the same, but we have modified our language in order to broaden the scope of the LIT. We use the term *computational reasoning* to encompass various activities that involve reasoning about numbers and operations and that have been associated with number sense. Among these are: mental computation, computational estimation, and reasoning about fraction magnitude.

Applying the LIT to Reasoning about Fraction Magnitude

Originally, our LIT was developed with a focus on whole-number mental computation. In the 2010 study, we sought to extend it to the rational number domain, and we chose to focus on students' reasoning about fraction magnitude. This area is related to mental computation in that it too consists of tasks for which there are a variety of strategies that can be used. There are traditional procedures, which are often taught in school, as well as various nonstandard strategies. Behr, Wachsmuth, Post, and Lesh (1984) touted the importance of reasoning about fraction magnitude as a prerequisite to reasoning meaningfully about operations involving fractions. Furthermore, fraction estimation and comparison tasks have been used in assessments of students' number sense (Hsu, Yang, & Li, 2001; Reys & Yang, 1998; Yang, 2007).

A framework of Smith (1995) informs our thinking concerning reasoning about fraction magnitude. Smith groups fraction comparison strategies into four categories, which he calls *perspectives*. These perspectives serve not only to categorize strategies but also to highlight commonalities in reasoning across groups of strategies. Strategies such as converting to a common denominator or converting to a decimal belong to the *Transform* perspective. They involve transforming one or both fractions in some way in order to facilitate the comparison. One can also compare fractions without performing any sort of transformation. One way to do this is to apply the *Parts* perspective, wherein the fractions are interpreted in terms of parts of a whole. This perspective alone is sufficient for relatively simple cases, such as comparing fractions that have the same numerator or same denominator.

The *Reference Point* perspective involves reasoning about fraction size on the basis of proximity to reference numbers, or *benchmarks* (Parker & Leinhardt, 1995). For example, using the *residual strategy*[15] for comparing fractions, one compares the difference of each fraction from a common benchmark number, typically 1: To compare 7/8 an 6/7, we can notice that 7/8 is 1/8 away from 1, whereas 6/7 is 1/7 away from 1. Since 1/8 is less than 1/7, 7/8 is closer to 1, and therefore larger (Yang, 2007). The *Components* perspective involves making comparisons within or between two fractions, as in coordinating multiplicative comparisons of numerators and denominators. For example, in order to compare 13/60 and 3/16, we can notice that $13 \times 5 = 65 > 60$, whereas $3 \times 5 = 15 < 16$. It follows that 13/60 is greater since its numerator-denominator ratio is less extreme.

[15] Smith refers to this as the *reference point* strategy, as do Behr, et al., 1984.

In designing instruction, these perspectives informed our decisions relative to tasks, number choices, and anticipated student reasoning. We mapped out the envisioned learning routes described in our LIT in terms of the evolution of these perspectives and of particular strategies within each category.

Although Smith (1995) does not describe the perspectives or particular strategies belonging to his framework in a hierarchical way, we view the Reference Point and Components perspectives as generally more sophisticated categories of reasoning about fraction size.[1] There is support for this in the literature. For example, Yang (2007) considers the residual strategy to be Number sense-based, as opposed to Rule-based. We posit that there is a general correspondence between Smith's perspectives and the Standard-to-Nonstandard framework, described earlier. In particular, the Transform and Parts perspectives correspond more or less to the Standard and Transition categories of strategies, while the Reference Point and Components perspectives correspond to Nonstandard strategies (with or without reformulation). We do not intend by this a one-to-one mapping of categories, but a more general grouping into Standard (including Transition) and Nonstandard. Smith distinguishes between *instructed* and *constructed* strategies for comparing fractions, based on the degree to which he found support for these in popular textbooks. He found little support for Reference Point or Components strategies in texts. Smith's constructs of instructed and constructed marry well with the categories of Standard and Nonstandard.

Based on our previous teaching experience, together with a review of the literature, we expected that the prospective elementary teachers would come to our course with limited number sense and would tend to apply standard algorithms for comparing fractions (Newton, 2008; Yang, 2007). That is, their reasoning about fraction magnitude would, for the most part, fall under the Transform and Parts perspectives. Pilot interviews that we conducted with preservice elementary teachers who had completed their mathematics content courses confirmed this expectation. In our instructional sequence, we aimed for the more sophisticated strategies to eventually be used by students and established for the class by mathematical argumentation. In particular, we sought to engage students in reasoning about fraction size from Smith's Reference Point and Components perspectives. Tasks were designed and sequenced so as to begin with students' current ways of reasoning and to provide opportunities for reasoning about fraction magnitude in new ways. The design and sequencing of tasks involved consideration of these perspectives relative to number choices, contexts, and anticipated student reasoning, and guided by the envisioned learning routes described in our LIT. In the instructional sequence, we aim for these more sophisticated strategies to eventually be used by students and established for the class by mathematical argumentation.

Instruction

The course was intended to foster the development of number sense. In particular, the broad instructional intent was for students to come to act in a *mathematical environment* in which the properties of numbers and operations afforded a variety of computational strategies, as opposed to one in which mathematical operations map directly to particular algorithms (Greeno, 1991). Topics in the curriculum include quantitative reasoning around various story problems, place value, meanings for operations, children's thinking, standard and alternative algorithms, representations and values of rational numbers, and operations involving fractions. Our planning of instruction involved identifying in the curriculum opportunities to engage students in computational reasoning, as well as to facilitate rich discussions concerning students' strategies. For both mental computation and fraction comparison, shared sets of strategies were established via mathematical argumentation. These strategies were given agreed-upon names, and the class maintained a list of strategies with examples of each.

Preliminary Results

Our analysis of collective activity during the rational number unit is ongoing. At this point, we sketch the progression of fraction comparison strategies that were used by students and articulated in whole-class discussion over the first three days of the unit.

Amongst other tasks, students were asked to compare pairs of fractions, to order sets of more than two rational numbers (most of them represented as fractions), to reason about placement of fractions on a number line, to engage with others' reasoning about fraction magnitude, and to explicitly discuss and name their strategies for comparing fractions. On the second and third days of the unit, these activities often involved placing paper tags representing fractions on a string that represented a number line. The string was set up in the corner of the classroom and stretched from one wall to an adjacent wall. Tags were placed on the string to indicate the locations of those numbers on the number line.

Progression of Strategies

The "part-whole" meaning of a fraction was discussed in class on the first day of the rational number unit. Students were asked to discuss the meaning of a fraction, and the instructor recorded ideas on the board. A formal definition from the book was also presented. Students appealed to this part-whole meaning in many of their arguments concerning fraction magnitude. On the first day of the unit, two basic strategies for comparing fractions were used by students, discussed, justified, and named. These addressed the cases in which the fractions to be compared had either a common denominator or common numerator. Students named the strategy for the common denominator cases "Equal slices, more pieces." This name reflects a Parts perspective, wherein the common denominator is taken to indicate that same-sized "slices" of the whole are being considered, so that the greater numerator indicates the number of slices of that fixed size. Similarly, the strategy for reasoning in the common numerator cases was named "Equal number of pieces, different sizes."

Early instances of these basic strategies were described in terms of parts of a whole, often using contexts such as pies, cookies, or pizza. For example, Nancy described her reasoning to compare 3/4 and 3/7:

> Okay, well, if you take a pizza, and it's like the same size no matter how many pieces you cut it into. If you cut into four pieces, then they'd be bigger because you have less pieces. If you cut into seven, there's gonna be more, but each one is gonna be smaller. And, regardless of how many pieces it's cut into, you're only gonna eat three. Would you rather have three of the bigger ones or three of the smaller ones?

Arguments similar to Nancy's were articulated repeatedly in whole-class discussion. These arguments involved Parts reasoning and, in particular, were predicated on the idea that the number of pieces that a whole is partitioned into is inversely proportional to the size of those pieces. This idea was articulated repeatedly (in students' language) and functioned to justify "Equal number of pieces, different sizes" arguments.

By the second day of the unit, the class began pointing to the strategy names as a succinct way of describing students' reasoning. That is, the phrase, "We used 'Equal number of pieces, different sizes'" came to stand in for an argument such as Nancy's. There were also instances in which this strategy was used without being stated explicitly, especially when it was used in support of other, more elaborate arguments.

Building on their Parts comparisons, students repeatedly used the strategies of converting to a common denominator or common numerator. Whereas the procedure of converting to a common denominator was familiar to students from their previous mathematical education, converting to a common numerator was new for many students. Both of these strategies manifested in the discourse not as algorithms but as arguments that built on "Equal slices, more pieces" and "Equal number of pieces, different sizes," together with conversions to equivalent fractions, which were discussed and established early in the fraction unit. On the second day, these conversion strategies were named as "Make same number of pieces" and "Make same sized pieces." These names reflect a Parts interpretation of fractions, which tended to undergird students' arguments concerning fraction magnitude.

On the third day of the unit, students began to articulate Reference Point strategies. The first of these was the residual strategy: students compared proper fractions by comparing the residuals (distances from 1) of each. This strategy initially arose in the task of ordering 7/8, 9/10, and 8/9. Here, the residuals were unit fractions, and students compared these by established Parts reasoning. Students then extended the residual strategy to cases involving less trivial residuals, which sometimes required more sophisticated means of comparison. Other variations included comparing to benchmarks other than 1, such as 1/2 and 1/4; comparing distances above, rather than below, a benchmark; and *straddling* a benchmark, in which two fractions are compared by identifying one of them as less than the benchmark and the other as greater than it (Smith, 1995). The class eventually named this group of strategies "Difference Compared to Benchmark."

We offer an example of relatively advanced Reference Point reasoning from Day 3 of the fraction unit. Students were given the task of ordering 15/29, 4/9, and 8/15, and placing these on the number line. The class first established through argumentation that 4/9 was less than 1/2, while 15/29 and 8/15 were both greater than 1/2. They then had to find a way to compare 15/29 with 8/15. In whole-class discussion, Rebecca (R) and Julie (J) presented their argument that 8/15 was greater, while the instructor (I) facilitated the discussion:

> R: So, we think that 8/15 is larger than 15/29, uh, because we converted 8/15 – we multiplied the 8 and the 15 to get 16/30.
>
> J: Times two.
>
> R: Times two. Sorry.
>
> I: Does everybody see that step? Equivalence?
>
> R: And then compared that to 15/29. And we imagined a pie [circular spread gesturing] and shaded in the areas. And in both, the 16/30, there's 14 pieces shaded.
>
> J: Not shaded.
>
> R: Not shaded. And the same 14 pieces over there [referring to 15/29 as the instructor is notating this information on the whiteboard]. So, and then, more area is missing from – since the pieces are bigger in 14/29 than in 14/30.
>
> I: So, your conclusion then is that there's more missing here [pointing to 14/29 and the words "more missing" written on the whiteboard].
>
> R: Yeah.
>
> J: Yes.
>
> I: And how does that help you decide which of these is greater?
>
> J: Which means that 15/29 is not as great as – sorry, less than.
>
> I: Is less than 8/15?
>
> J: Yeah.

Rebecca and Julie's argument employs a circular area model for fractions ("a pie") under the Parts perspective. The students made use of the residual strategy. This instance of the strategy is particularly elaborate since the residuals are large, and a transformation is required in order for the residuals to be compared in the manner of "Equal number of pieces, different sizes."

Figure 1. Rebecca argues that 8/15 is larger than 15/29

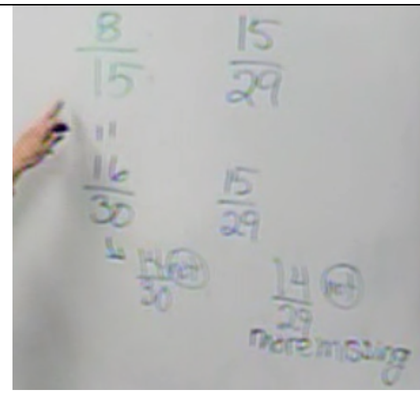

Figure 2. The instructor notates Rebecca and Julie's argument

Discussion

We present the example of Rebecca and Julie's argument for a few reasons:

1. The strategy that the students articulated is sophisticated and nonstandard.
2. The strategy is presented in the form of a complete, valid mathematical argument.
3. The students' argument builds on ideas that have been previously established in whole-class discussion: The residual strategy had been used previously in simpler cases. Producing equivalent fractions had been discussed and justified previously. Also, common numerator comparisons had been used repeatedly and crystallized in the form of a named strategy. This kind of comparison was used as an ancillary strategy in this case, and this seems to suggest that it functioned "as if shared" (Rasmussen & Stephan, 2008) by this point, although further analysis is needed to establish this.
4. At the same time, this example conveys the tentative nature of students' reasoning. Rebecca has some difficulty putting a complete argument into words (e.g., she says "shaded" when she means *not* shaded). Her facial expressions and voice inflection also suggest that this explanation requires care. Julie uses the language "not as great" rather than *less than*. These characteristics of the discourse are to be expected of prospective elementary teachers who are operating at the frontier of their knowledge of fraction magnitude. That is, they appear to be in the process of learning something.

Conclusion

We began this study with the idea of extending a local instruction theory for the development of number sense to the domain of rational numbers. In that vein, we shifted our focus from whole-number mental computation to reasoning about fraction magnitude. Smith's framework informed our thinking about strategies and perspectives involved in comparing fractions. The research literature, together with our pilot interviews, enabled us to anticipate prospective elementary teachers' initial reasoning about fraction magnitude. Our pre-instruction interviews enabled us to refine our expectations further. We designed the instructional sequence on the basis of students' starting points, conceived in terms of Smith's framework as essentially the Transform and Parts perspectives, and with the goal of building toward the Reference Point and Components perspectives. Specific instructional activities were then crafted with this broad progression in mind.

We found that we could apply the LIT to the rational number domain. By this, we mean that we found the LIT useful for our instructional design purposes during the course. The envisioned learning routes, incorporating Smith's framework, seemed to provide a viable way of conceptualizing students' learning. Our preliminary findings are encouraging, and our ongoing analysis involves documenting the collective activity in the class, using the methodology of Rasmussen and Stephan (2008). This analysis will enable us to investigate the relationship between the LIT and the actual classroom activity that it informs. This, in turn, will allow us to further elaborate the LIT in our continuing design research efforts concerning number sense development.

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