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FOREWORD

The research reports and proceedings papers in these volumes were presented at the 14th Annual Conference on Research in Undergraduate Mathematics Education, which took place in Portland, Oregon from February 24 to February 27, 2011.

Volumes 1 and 2, the RUME Conference Proceedings, include conference papers that underwent a rigorous review by two or more reviewers. These papers represent current important work in the field of undergraduate mathematics education and are elaborations of the RUME conference reports.

Volume 1 begins with the winner of the best paper award, an honor bestowed upon papers that make a substantial contribution to the field in terms of raising new questions or gaining insights into existing research programs.

Volume 3, the RUME Conference Reports, includes the Contributed Research Reports that were presented at the conference and that underwent a rigorous review by at least three reviewers prior to the conference. Contributed Research Reports discuss completed research studies on undergraduate mathematics education and address findings from these studies, contemporary theoretical perspectives, and research paradigms.

Volume 4, the RUME Conference Reports, includes the Preliminary Research Reports that were presented at the conference and that underwent a rigorous review by at least three reviewers prior to the conference. Preliminary Research Reports discuss ongoing and exploratory research studies of undergraduate mathematics education. To foster growth in our community, during the conference significant discussion time followed each presentation to allow for feedback and suggestions for future directions for the research.

We wish to acknowledge the conference program committee and reviewers, for their substantial contributions and our institutions, for their support.

Sincerely,

Stacy Brown,
RUME Organizational Director & Conference Chairperson

Sean Larsen,
RUME Program Chair

Karen Marrongelle
RUME Co-coordinator & Conference Local Organizer

Michael Oehrtman
RUME Coordinator Elect

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Making the familiar strange: An Analysis of Language in Postsecondary Calculus Textbooks then and now

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Three calculus textbooks covering a span of about 40 years were examined to determine whether and how the language used has changed given the reform movement and the impetus to make mathematics accessible to all. Placed in a discourse analytic framework using Halliday"s (1978) theory of functional components –ideational, interpersonal and textual, and using the exposition of the concept of a function as a unit of comparison, the study showed that language is an integral indicator of the author"s view of mathematics and an important factor for textbook adoption in the pursuit of student success.

Keywords: discourse analysis, calculus textbooks, language of mathematical discourse

INTRODUCTION

In the late 1980s, the Calculus Consortium at Harvard (CCH) was funded by the National Science Foundation to redesign the Calculus curriculum with a view to making Calculus more applied, relevant, and accessible. The intent was to re/think and re/present the content so as to focus on real-world applications, to emphasize concepts and graphical representations, and to take advantage of the increasingly sophisticated technology. Calculus is now presented in a manner radically different from the traditional approach of abstraction, formal notation and symbolism, and algebraic conventions.

The goal of this research is to see whether and how calculus textbooks designed for the postsecondary level in „regular" Calculus courses have changed over the years with respect to the language used in the exposition and by inference, the view of mathematics manifested. One concept, that of a function and in particular its definition, is chosen and used to trace the dimensions of the language over the years and the consequent shifts in the view and presentation of mathematics in calculus textbooks. The research questions are: Has the language of calculus textbooks changed over time and if so, in what ways? Has the language changed from one that is exclusive (mathematics as an elite subject with an elite community) to one that is inclusive and accessible to all? From the language, how are the authors" views of mathematics characterized and how have they changed over time?

The three textbooks I have chosen are *Calculus* by Spivak (1967), *The Calculus of a Single Variable with Analytic Geometry*, 5th edition by Leithold (1986), and *Single Variable Calculus: Early Transcendentals*, 5th edition by Stewart (2003). Textbooks may be studied subjectively to describe the interaction between the student and the written material or to describe teachers" use of textbooks and the subsequent effect on the teacher (Remillard et al, 2009). However, following Herbel-Eisenmann (2007), I seek to examine the „voice" of calculus textbooks over the years as *objectively given structure* (emphasis in the original, p.396). This examination will be placed in a discourse analytic framework which attends to the aspects of text relating to language, voice, agency and identity.

ANALYTIC FRAMEWORK

Language has been increasingly seen as an important issue relating to mathematics teaching and learning. Rowland (2000) emphasizes two principles in studying language: the linguistic principle ("language as means of accessing thought") and the deictic principle (language as a means of communication and a „code to express and point to concepts, meanings and attitudes‟) (p. 2). In his *Language as a Social Semiotic*, Halliday (1978) identifies three functional components or functions of language– the ideational, the interpersonal, and the textual –from which meaning is apprehended. The ideational functional component of the text answers the questions: What is the view of mathematics as presented in the text? How is the subject of mathematics envisioned in the mind of the author of the text and in what style is it rendered? The interpersonal functional component describes the social and personal roles and relationships among the authors and readers. Evidence of this function is discerned by considering the use of personal pronouns (first, I/we/us/our, and second person, you), imperatives, and modality. The textual functional component describes the content matter or the mathematics presented in the text, the theme and modes of reasoning, the arguments and their forms, and the narratives of mathematical activity.

Each of the textbooks will be examined as to the "voice" that emerges, the extent of agency, and the construction of the identity of the reader by the text.

METHOD

The data consists of the 10 – 14 pages from the each of the three Calculus textbooks that cover the exposition of the concept of a function. Exposition includes the preliminary introductory commentary and the definition (or definitions) of a function. I mined the relevant pages carefully with respect to the linguistic markers for the three functions as articulated by Halliday.

FINDINGS AND DISCUSSION

Table 1 gives the results of the comparison of the textbooks across markers for the functional components of language with respect to the concept of a function.

Table 1. Comparison across markers for the functional components.

	Spivak (1967)	Leithold (1986)	Stewart (2003)
Pronouns - 1st person	we/us/our 32 instances	we/us 5 instances	we/us 24 instances
Pronouns – 2nd person	you 9 instances	None	you 3 instances
Imperatives Inclusive	let‟s 1 instance	call, compare, let, note, observe, recall 6 instances	consider, determine, let, notice, remember 7 instances
Imperatives Exclusive	None	find, read 4 instances	draw, find, sketch, use 6 instances
Modal verbs	May 2 instances	None	None
Questions	2	None	1
Conditionals	If 6 instances if … then 10 instances	Given 3 instances given that 2 instances	If 3 instances if … then 4 instances

Beginning with the interpersonal component, the most striking occurrence is that of 32 instances of first-person pronouns in Spivak as compared with five in Leithold and 23 in Stewart. In Spivak, there were 29 uses of *we*, two of *us* and one of *our*. From the opening paragraph in his liberal use of *we* and *us*, Spivak sets the tone of including the reader in his deliberations. Spivak clearly views the reader as someone who is part of the community of people doing or studying mathematics. Another possible reading is that the use of *we*, *us*, and *our* suggests a more general form indicative of the register of mathematicians. In comparison, the five occurrences of *we* in Leithold read clinically as in „we see that" or „we observe that". The use of personal pronouns indicates the presence or absence of humans in the activity and the implied distance and degree of formal relationship between the author and the reader (Morgan, 1996). Leithold deploys his words in a detached „scientific" manner, the very opposite of the kind of writing that Burton and Morgan (2000) exhort mathematicians to adopt.

The frequency of imperatives in a text indicates the degree to which the author wishes to draw the reader"s attention to a point in the text (note that, observe that), to encourage the reader to reflect (consider, compare, recall, remember), or to give a simple command (find, sketch, use). Both Leithold and Stewart use a similar number of imperatives that indicate the usual textbook framing (consider, notice, observe, recall) and that signal the ability of the author (determine, evaluate, find, sketch, use) to tell the reader what to do. It is note-worthy that Spivak does not use any of these imperatives but still manages by his use of personal pronouns to convey a sense of introducing the reader to and including the reader in the activity that mathematicians undertake.

Modality, as a feature of language, enables authors and speakers to express their feelings, values, attitudes, and judgments about the propositions in their texts. Demonstrations of modality include modal auxiliary verbs such as „may" and „can", adverbs relating to the uncertain state of knowledge such as „possibly" and „maybe", the use of moods and tenses, and the use of hedges (Rowland, 2000, p. 65). For these three textbooks there was little or no evidence of modality. There were two instances of „may" in Spivak („You may feel that we have also reached…" and „Two consolations may be offered", p. 45). These have nothing to do with the mathematics involved but indicate concern for and offer solace to the reader. Leithold and Stewart offer no suggestion that that there is any uncertainty related to mathematical activity and by their lack of use of modality, indicate a view of mathematics that strongly holds to an absolute, ideal perspective.

For the textual component, all three authors use the mode of discourse characterized by exposition (evident of the raison d'être of the textbook) in laying out a clear and concrete treatment of the subject matter. Questions as evidence of a conversational or dialogic style of exposition were barely used; there were two questions in Spivak, none in Leithold and one in Stewart.

The ideational functional component in each of the three textbooks is very nearly identical in that the authors" content and meaning are similar. Each author is interested in communicating the content of the concept of a function and introducing the objects and relations that are under consideration when discussing the concept of a function. Each encodes in the text his individual vision of mathematics. The view of mathematics evinced in all three is fixed, absolute, and formal.

As seen from these linguistic markers, the tenor of the language in evoking the relationship between the author and the reader in the three textbooks is markedly different. Spivak and Leithold are diametrically opposite in the use of the first and second person pronouns and imperatives in engaging and addressing the reader with Stewart striking a moderate note in this regard. In summary, the three textbooks are similar in their theme and message but differ considerably in the interpersonal component with Stewart capturing a moderate position between what may be considered the extremes of linguistic markers by Spivak and Leithold.

IMPLICATIONS

The language of mathematics is often seen as foreign with its own lexicon, grammar, and modes of argument. More than being able to negotiate the language, students of mathematics must become fluent in it. Bakhtin declares that „[e]ach text presupposes a generally understood (that is, conventional within a given collective) system of signs, a language (if only the language of art)" (1953/1986, p. 105). Hence the mathematics textbook has a conventional system of signs which is part of a language that is to be understood if one wishes to be a member of the community involved in mathematical activity.

The differences in language in a textbook account for much of the reader"s regard for the textbook. In this paper I have teased out the subconscious linguistic markings in the text and have shown that there is more to the text than meets the eye; that what we have taken as familiar is indeed strange: a nebulous complex of beliefs and ideas about mathematics which we adopt and perpetuate without realizing the implications and consequences. This analysis suggests that it behooves us as teachers to re/examine our practices in making textbook choices for the betterment of ourselves and our students and to be aware of the functions and forms of language that subtly maintain hegemonic practices in the teaching and learning of mathematics.

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The Effectiveness of Blended Instruction in Postsecondary General Education Mathematics Courses

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Abstract

Despite best efforts, hundreds of thousands of students are not succeeding in postsecondary general education mathematics courses each year. Low student success rates in these courses are pervasive, and it is well documented that the nation needs to improve student success and retention in general mathematics.

Using data from 11,970 enrollments in College Algebra, Foundations of Mathematics, and Elementary Calculus from fall 2007 to spring 2010 at the University of Memphis, we compare the impact of the Memphis Mathematics Method (MMM), a blended learning instructional model, to the traditional lecture teaching method on student performance and retention.

Our results show the MMM was positive and significant for raising success rates particularly in Elementary Calculus. In addition, the results show the MMM as a potential vehicle for closing the achievement gap between Black and White students in such courses.

Key Words

Calculus, general education mathematics, classroom research, teaching experiment

Introduction

In the U.S., students who pursue a postsecondary baccalaureate degree are required to complete at least one general education mathematical science course. Low student success rates in these courses are pervasive, and it is well documented that the nation needs to improve student success and retention in general mathematics. National recognition of the poor success rates has resulted in vigorous debate and a series of proposed reform models over the past two decades, usually as curricular reform or delivery reform. Particular attention has been paid to reforming College Algebra and Calculus curriculum and pedagogies. Technology focused reforms have included attempts to change instructional delivery methods by training students to use technology to solve problems (Lavicza, 2009; Heid & Edwards, 2001; Smith, 2007), using technology as an instructional tool (Peschke, 2009; Judson & Sawada, 2002; Caldwell, 2007; Fies & Marshall, 2006), or using a technology based assessment system (Zerr, 2007; Nguyen, Hsieh, & Allen, 2006; Vanlehn, et al., 2005).

In this paper, we report results comparing the impact of the Memphis Mathematics Method (MMM), a blended learning instructional model, to the traditional lecture teaching method on student performance and retention in general education mathematics courses at the University of Memphis (UM). The comparison includes a total of 11,970 enrollments in College Algebra, Foundations of Mathematics, and Elementary Calculus from fall 2007 to spring 2010. Results indicate that the MMM is effective in increasing student achievement and retention.

The Memphis Mathematics Method

The MMM substitutes traditional lecture-style instruction with a brief introduction of a topic followed by a laboratory session requiring students to complete classroom-based assignments using MyMathLab software. During the short lecture, instructors introduce a concept and provide examples that emphasize the use of mathematical techniques to solve problems motivated by other sciences. The remaining class time is dedicated to solving problems using the MyMathLab software. Over the course of a 15-week semester, students log 30 hours of class time practicing problems on MyMathLab. In addition to its use as an instructional tool, instructors use MyMathLab for course management and grading.

Data and Methods

The MMM intervention was piloted at UM in 2007 in a specialized Developmental Studies Program in Mathematics (DSPM) College Algebra course, which combined a remedial Intermediate Algebra course with a regular College Algebra course. Students were eligible for the DSPM course only if their ACT scores would have required them to take remedial Intermediate Algebra. Based on positive student outcomes during the initial pilot, UM expanded MMM in 2008 to regular sections of College Algebra; regular and DSPM sections of Foundations of Mathematics; and regular sections of Elementary Calculus. Instructors in both DSPM and regular MMM-taught sections reported anecdotal evidence of greater student engagement. There were 11,970 enrollments in the sections across the three courses. Of these, 10,424 enrollments were in regular sections while 1,546 enrollments were in DSPM sections.

We analyze data from College Algebra, Foundations of Mathematics, and Elementary Calculus from fall and spring semesters beginning in 2007 and ending in 2010. These data contain information about student characteristics, student performance, and teaching methodology.

Dependent variables.

To gauge student success in the three courses, we define an indicator variable "success" coded as 1 if a student obtains a passing grade and 0 otherwise. The variable success thus combines the effects of changes in pass rate and changes in dropout rate.

In addition, we are interested in separately determining the effects of the MMM pedagogy on dropout rates. We define an indicator variable "dropout" coded as 1 if a student dropped out of a course and 0 if a student completed the course. Success and dropout serve as our dependent variables in this study.

Independent variables.

We include the student's gender, the student's racial/ethnic background (White, Black, Hispanic, and Other), and the student's prior mathematics knowledge as measured by their ACT math score, as three independent variables in the analysis. In addition, we control for whether a student is repeating the course and define an indicator variable "redo" coded as 1 if a student has attempted the course before and 0 if this is their first attempt. Also, an indicator variable for whether a student was exposed to the conventional or to the MMM pedagogy is included in the analysis.

Estimation approach.

To estimate the effects of MMM on student success and dropout rates in these courses, we fit a total of 10 regressions – four interactive models for remedial courses and six interactive models for non-remedial courses.

Results

Descriptive results. Of the 11,970 enrollments in College Algebra, Foundations of Mathematics, and Elementary Calculus at UM from fall 2007 to spring 2010, 5,530 ended in a passing grade reflecting a 54% success rate over the three courses. Of these 11,970 enrollments 1,596 ended when the student withdrew from the course.

For every course, we found that the percentage of students who withdrew from the MMM classes is lower than in the traditional classes. With respect to performance, more students were passing in MMM classes than in traditional classes. In DSPM courses for Foundations of Mathematics, for example, 56.7% of students received passing grades, while only 60.7% passed the equivalent MMM classes. Furthermore, a striking difference of grades across instructional methods is seen in Elementary Calculus. Approximately 49% of students in traditional courses passed while about 72% passed when exposed to the MMM teaching methodology.

Additionally, we compared the percentage breakdown of student performance and retention by racial/ethnic background for each course, and see that racial disparities between Black and White students in performance seem to be greatly reduced in the MMM classes. For example, across all three regular courses, Black students pass at a rate of 39.9% when taught using traditional pedagogy compared to 56.2% when using MMM. This difference is staggering. Also, in DSPM courses, Black students dropout at a rate of 10% for the MMM method compared to a rate of 14% for traditional teaching. In traditional DSPM College Algebra, 49.7% of Black students received passing grades compared to 64.4% of White students; that is, there is a 14.7% differential between Black and White students. In the equivalent MMM courses, however, this differential is only 7.7%. In traditional Elementary Calculus, the racial disparity between Blacks and Whites is completely erased with 75.7% of Black students and 68.9% of White students receiving passing grades.

With respect to withdrawal rates, in traditional Calculus, 22.4% of Black students dropped compared to 15.4% of White students, while in the MMM calculus courses, only 6.8% of Blacks withdrew compared to 9% of Whites. These results indicate that the MMM is a potential vehicle for decreasing the achievement gap. These relationships are further examined in the following section using regression.

Regression results. The regression output is illustrated in Table 1.

Succeed. Female students in each course have a higher chance at succeeding than their male counterparts, and the higher a student's ACT score the higher the likelihood of succeeding in a course. We find that students who were retaking a course have significantly lower odds of succeeding compared to those taking a course for the first time. With respect to the racial/ethnic disparities, we see that under conventional instruction Black students have 38%, 29%, and 49% lower odds of succeeding than White students in Foundations, College Algebra, and Elementary Calculus, respectively. Other student have 79% higher odds than White students to succeed in Calculus.

The MMM teaching pedagogy is significantly effective in increasing the odds of succeeding in Calculus — students exposed to the MMM have 78% higher odds of succeeding than those in traditional Calculus. Furthermore, the large magnitude and significance of the interaction of teaching method and race illustrates a particular benefit of this teaching method for Black students. In Elementary Calculus, Black students instructed via MMM have 779% (computed as 1.78*4.94 -1) higher odds of succeeding than Black students receiving conventional instruction.

Dropout. Columns 6-8 show that regular female students have a lower probability of dropping Calculus compared to male students. We find a strong ACT score effect illustrating that students with higher ACT scores have lower odds of dropping out. Students who are retaking a course are more likely to persist in Calculus and have 29% lower odds of dropping out.

Black/White differentials persist when comparing the probabilities of dropping out. Black students in College Algebra have 31% lower odds of dropping out compared to White students. The MMM is positive and significant for students taking Calculus. Calculus students in the MMM are about 48% lower odds of dropping out with respect to conventionally taught students. This positive finding provides evidence that the MMM is effective in increasing retention.

Discussion & Conclusion

Despite best efforts, hundreds of thousands of students are *not* succeeding in postsecondary general education mathematics courses each year. Our results suggest that MMM was positive and significant for raising success rates particularly in Elementary Calculus. In addition, the results show the MMM as a vehicle for closing the achievement gap between Black and White students in such courses. Overall, our data suggest that MMM increases success and decreases dropout rates for these general education mathematics courses. The positive results may be attributed to the structure and interactive nature of the MMM which forces a daily involvement on the part of the student. This type of active engagement along with the use of technology is in-line with reform pedagogy. The MMM implementation has resulted in overall improved student success in Elementary Calculus, lower dropout rates in College Algebra, and lower costs.

Table

Table. Succeed & Dropout Regressions

	Succeed					Dropout				
	Regular			DSPM		Regular			DSPM	
Variables	Foundations	Algebra	Calculus	Foundations	Algebra	Foundations	Algebra	Calculus	Foundations	Algebra
Female	1.18*	1.38***	1.47***	1.89**	1.33**	0.89	0.80*	0.72***	0.68	0.84
Act Math Score	1.13***	1.17***	1.11***	1.19**	1.15***	0.93***	0.88***	0.93***	0.79*	0.92
Redo	0.66***	0.36***	0.92	1.06	0.68*	1.27	1.21	0.71**	1.75	1.02
Black	0.62***	0.71***	0.51***	0.84	0.30***	0.90	0.69***	1.33**	0.79	1.98
Hispanic	0.64	0.66	0.92	0.54	0.91	1.20	0.93	1.88	NA	4.20
Other	0.98	0.81	1.79**	NA	0.24	0.84	1.02	0.65	NA	NA
MMM	1.03	1.24	1.78***	1.09	0.51*	0.49*	0.21	0.52**	1.55	1.19
Black*MMM	1.13	1.01	4.94***	1.06	2.84**	1.78	5.09	0.40*	0.37	0.60
Hispanic*MMM	1.10	NA	0.82	NA	2.50	1.65	NA	1.31	NA	NA
Other*MMM	NA	NA	1.00	NA	19.04*	NA	NA	4.33	NA	NA
Constant	0.14***	0.06***	0.11***	0.05*	0.29	0.58	1.84	0.91	7.31	0.31
N	2,984	3,102	2,595	309	983	2,984	3,102	2,595	305	959

*** p<0.01, ** p<0.05, * p<0.1

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Obstacles to Teacher Education for Future Teachers of Post-Secondary Mathematics

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The purpose of this study was to uncover issues and difficulties that come into play as mathematics graduate students develop their views of their roles as university teachers of mathematics. Over a six-month period conversations were held with mathematics graduate students exploring their experiences and perspectives of mathematics teaching. Using hermeneutic inquiry and thematic analysis, the conversations were analyzed and interpreted with attention to themes and experiences that had the potential to influence the graduate students' ideas about and approaches to teaching. Using Lave and Wenger's notion of legitimate peripheral participation, themes that are explored in this paper are the replication of mathematics teaching practice and identity, and resulting feelings of resignation. It is hoped that this research will contribute to the understanding of teaching and learning in post-secondary mathematics as well as provide guidance in structuring post-secondary teacher education in mathematics.

Keywords: post-secondary, mathematics graduate students, community of practice, teacher identity

Introduction and Purpose

Mathematics departments are often one of the largest departments within institutions of higher education, providing prerequisite courses for students in diverse disciplines such as engineering, psychology, chemistry, business, medicine, and education. Almost seventy-five percent of mathematics PhDs will become professors at post-secondary institutions dedicated to undergraduate education rather than research (Kirkman et al., 2006). Consequently, the teaching of mathematics at the university level is quite important in undergraduate education, and professors, instructors, and graduate teaching assistants in mathematics have a wide-reaching influence on the education of future researchers, teachers, and mathematicians (Golde & Walker, 2006). However, the format of post-secondary mathematics teaching has remained problematic for undergraduate success in mathematics and the sciences (Alsina, 2005; Kyle, 1997; NSF, 1996).

The preparation of the future mathematics professoriate has recently become a subject of investigation. In particular, the development of mathematics graduate students' teaching practices has become a focus for mathematicians and mathematics educators. Recent research into mathematics graduate students' teaching has examined their classroom practices and possible connections between their practices and beliefs about teaching and learning. Researchers concluded that newly acquired positive attitudes and beliefs about teaching mathematics did not bring about hoped for changes to graduate students' teaching practices (Belnap, 2005; Speer, 2001). Although the mathematics graduate students in at least one study developed a new vocabulary for discussing teaching, these students also reported that they maintained a lecture-style form of instruction (Belnap, 2005). Other research has shown that enrollment in a course in pedagogy also did not produce expected changes to mathematics graduate students' teaching practices (DeFranco and McGivney-Burelle, 2001).

In light of these conclusions, the purpose of this research study was to learn about the obstacles and issues that might exist for mathematics graduate students that could prevent teacher preparation programs from taking root and being successful. To uncover these potential barriers, this study was undertaken with the following questions in mind: How do graduate students come to understand their roles as mathematics teaching assistants and possible future professors of mathematics? How might experiences and interpretations of experience serve as obstacles to teacher education programs for these future teachers of post-secondary mathematics?

Theoretical Framework

Lave and Wenger (1991) have offered the term *legitimate peripheral participation* in relation to a community of practice to name one central process by which novices gain knowledge and understanding about the practices of a community. Lave and Wenger claimed "even in cases where a fixed doctrine is transmitted, the ability of the community of practice to reproduce itself through the training process derives not from the doctrine, but from the maintenance of certain modes of coparticipation in which it is embedded" (p. 16). Moreover, within the framework of legitimate peripheral participation exist issues of identity where Lave and Wenger describe how "the development of identity is central to the careers of newcomers in communities of practice" where "learning and a sense of identity are inseparable" (p. 115). As such, the concept of legitimate peripheral participation offers an interesting perspective for understanding what might be happening for the mathematics graduate students as they progress through their programs. Legitimate peripheral participation prompts an interesting question for this study: How might the attention to legitimate peripheral participation in a mathematics department prevent graduate students from adopting alternate modes of teaching?

Mode of Inquiry

As hermeneutics "holds out the promise of providing a deeper understanding of the educational process" (Gallagher, 1992, p. 24), hermeneutic inquiry was chosen as the mode for exploring the experiences that mathematics graduate students face in their programs. Hermeneutics helps to understand how we create and find meaning through experience and social engagement (Brown, 2001). Davis (2004) offered a description of hermeneutics as a mode of inquiry that asks "What is it that we believe? How did we come to think that way?" (p. 206). Hermeneutic inquiry into mathematics graduate students' understanding of their possible future roles as professors compelled a look at what is present in departments of mathematics that might cause them to adopt the teaching methods that persist as part of their role in maintaining "certain modes of coparticipation."

Carson (1986) and van Manen (1997) propose conversation as a mode of doing research within hermeneutic inquiry to uncover interpretations and understanding of experience. For this study, a series of five audio-recorded semi-structured, recursive conversations were conducted with the research participants, all of whom were mathematics graduate students in a doctorate granting university. Each conversation was analyzed by the researcher, who listened for the topics of conversation attended to by the research participants. The participants had the opportunity to review the analyses in a collaborative effort to refine the reporting of their experiences. Because of its recognition of the interpretive work of data analysis, Braun and Clark's (2006) six-stage process for thematic analysis was coupled with hermeneutic inquiry. The stages of thematic analysis are in accord with Laverty's (2003) description of a hermeneutic project where "the multiple stages of interpretation allow patterns to emerge" (p. 23). Combining these two notions, the themes and the participants' comments within each theme were analyzed using a hermeneutic, interpretive lens.

Results

The participants in this study lacked a forum to discuss their views, explore different ideas for teaching, and were not provided mentorship for their teaching duties. They were left to creating meaning amongst themselves, relying solely on the reproduction of the teaching and a unitary identity they observed. They resigned themselves to a notion that there was only one way to teach mathematics and one way to be as a professor of mathematics. These conclusions are explored in the themes below.

Replication of identity and practice

The replication of mathematics professors' identity and teaching practices resonated in the conversations with the research participants. Similar to Lave and Wenger's (1991) idea that communities "reproduce themselves" (p. 121), the post-secondary teaching of mathematics, as viewed by the participants, appeared to be a practice of replication, a reproduction of others' teaching. Specifically, one participant spoke of the structure of all mathematics courses as "definition, theory, example," while another participant described teaching as "You just do examples," pointing to a replication of the fixed structures of mathematics texts and courses as the legitimate form of teaching practice. Other participants acknowledged the replication of legitimate practice seen in calculus courses, with one stating "It's easy to keep teaching calculus like this. We've been doing it forever" and another asking "How many ways can you skin a calculus class?" Beyond replication of teaching practice, though, was also a notion of replication of identity. Jardine (2006) has written that in mathematics there exists a "mood of detached inevitability: anyone could be here in my place and things would proceed identically" (p. 187), signaling the replication of identity amongst mathematics teachers. This view echoed in the language of professor A and professor B used by one of the participants: "You could teach a little bit better, but I don't know how much variety you can actually put in. How much different is professor A from professor B?" which spoke to an interchangeability between professors, as though their identities might be so alike or the differences so insignificant that it would not matter who was in the classroom.

Resignation

The act of replication of mathematics teaching and the thought of taking on a particular identity in mathematics evolved into feelings of resignation among the participants. With regard to his current role as a graduate student, one participant said, "You can't have an opinion; you can't have anything except the fact that 'yeah, this is true.'" Here it seemed that this participant was resigned to a passive position within his role as mathematics graduate student, and that he must accept the ways he could participate in the department. Further, when speaking about the possibilities for his future teaching practice and, in particular, about the use of discussion in a mathematics classroom, he said, "that's never going to happen in math," a statement that expressed a resigned view that there are no alternative possibilities for what can occur in mathematics classrooms. Concerning his own observations of the ways in which the undergraduates were being taught by professors in the department, another participant remarked "I might have the same complaints, but there's nothing I can do about it," signaling a resignation to being unable to change the way mathematics courses are taught or structured. With regard to his own teaching, another participant spoke of how he could not work "outside of a certain box" in the department. As a result, he no longer appeared to have a concern for his teaching, saying, "I would not be able to change things even if I wanted to." When this participant spoke of his hopes for his future career as an academic, teaching was no longer of consequence to his success as a mathematician and future professor. In the final year of his doctoral program, this participant was an illustration of what Lave and Wenger (1991) refer to as the "transformation of newcomers into old-timers" (p. 121) and how "an extended period of legitimate peripherality provides learners with opportunities to make the culture of practice theirs" (p. 95).

Implications of the study

The goal of this project was to understand what the obstacles might be for post-secondary mathematics teacher education. The participants in this study did not report a public statement or acknowledgement that they had to abandon other ideas about teaching and that they should no longer consider teaching important, but they interpreted their lives in mathematics to be restricted to a particular way of being and of teaching mathematics. Thus if the current structures and suggestions of what is important to graduate study in mathematics remain in place, it is unlikely that new teacher education programs that are established for mathematics graduate students will produce hoped for changes to teaching in post-secondary mathematics.

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Designing and Implementing a Limit Diagnostic Tool

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Abstract:

The purpose of this study is to create and utilize a tool that evaluates students' comprehension of the logical structure and implications of the formal definition of limit. This study continues the trajectory of recent limit research involving classroom-based interventions that reveal student metaphors and conceptions (Boester, 2010; Oehrtman, 2009; Roh, 2008, 2010). The diagnostic tool, based on seven concepts embedded in the formal definition, uses a set of delta/epsilon diagrams that students must explain, either accepting them as correct, or augmenting them to make them correct. The assessment was used after giving students in a conceptually-based calculus class a problem meant to introduce the logical structure of the formal definition. While students did not spontaneously show many of the concepts based on the problem alone, an interview protocol following the assessment prompted the students to rethink the implications of the problem, thus promoting the missing concepts.

Keywords: calculus, limit, assessment, conceptual decomposition

Cornu (1991) first summarized research on students' spontaneous conceptions, mental models, and epistemological obstacles concerning the formal definition of limit. Since then, research has grown from cataloging misconceptions (Bezuidenhout, 2001; Davis & Vinner, 1986; Tall & Vinner, 1981) to describing possible frameworks of student conceptions (Cottrill et al., 1996; Lakoff & Núñez, 2001; Williams, 1991, 2001). Some of the most recent research (Boester, 2010; Oehrtman, 2009; Roh, 2008, 2010) has described classroom-based interventions that both further our understanding of students' conceptions of limit, while documenting how and why limits were taught to students using particular problems, activities, or manipulatives.

In order to assess the effectiveness of emerging pedagogical strategies for limit instruction, it would be nice to have a generic tool to gauge students' comprehension of the logic contained within the formal definition. The purpose of this study is to create such a tool, then use it to assess students' comprehension of the logical structure and implications of the formal definition of limit following a classroom-based intervention.

Limit Diagnostic Tool

Using the following statement of the formal definition of limit at a point

$$\lim_{x\to a} f(x) = L \text{ means that}$$
$$\forall\varepsilon > 0,\ \exists\delta > 0, \text{ such that } 0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

the concepts contained within this definition, which should be assessed by the diagnostic tool, need to be established. Through discussion with other limit researchers, the following list was created:

1) We control delta, not epsilon.
2) Delta interval must fit inside (cannot be outside) epsilon interval.
3) Delta interval can be strictly inside epsilon interval.
4) Delta and epsilon do not have to be equal.
5) In order for the limit as x approaches a to be L in the continuous case, $f(a) = L$.
6) The length of the interval on each side of a / L must be the same, because you can be the same distance (delta / epsilon) away in both directions.
7) For a non-linear graph, one side of the delta interval may be the same as the epsilon interval, but for the other side, the delta interval may be strictly inside the epsilon interval.

A written assessment, consisting of six standard delta/epsilon diagrams (an example is shown in Figure 1), was then created to test for these seven concepts. For each diagram, the student must either confirm that this pairing of delta and epsilon in the diagram is appropriate for the given graph of a function, or explain why it is not appropriate and correct the diagram to create an appropriate pairing. For instance, if a student states that the diagram in Figure 1 is not an appropriate pairing, and proposes redrawing the delta interval to "correct" the diagram in Figure 2, this would reveal that the student has failed to grasp concept #3, that the delta interval can be strictly inside the epsilon interval. A student could alternatively redraw the epsilon interval to line up with the delta interval, showing that they have also failed to grasp concept #1, that we can only change delta.

In addition to the written assessment, an interview protocol was also created. After a few preliminary questions eliciting feedback about limits in general, students are asked to explain their responses to all six diagrams. The interviewer questions their responses only after the student has completed explaining all the diagrams, in order to provide a baseline for their responses. For correct responses, the questions probe for recognition of the embedded concepts. Did the student actually answer the question following the intent of the concept? In other words, were the students' beliefs robust enough to withstand deeper questioning, or were they easily malleable by the researcher? For example, if a student correctly states that the diagram in Figure 1 follows the definition, the protocol indicates to show the student Figure 2 and ask "Another student might say that the delta is too small, and should be bigger to match the epsilon. Do you need to do this?" For partially correct or incorrect responses, the questions probed for the related concepts in their apparent absence. Would students recognize or maintain their misconceptions of the definition? If a student proposed Figure 2 as a correction to Figure 1, the student would be asked "Is it ok for the delta interval to be inside of the epsilon interval? Do delta and epsilon have to be equal?" (This addresses concept #4, as well as concept #3.)

Implementation of the Tool Following a Classroom Activity

This diagnostic tool was first used in Math 348, Concepts of Calculus for Middle School (Pre-Service) Teachers, a course taught by the researcher during the Spring 2010 quarter at a mid-size, Midwestern university. Math 348 focuses on the ideas, rather than the procedures, of calculus. Even though the goal of the course is to enable students to recognize how the fundamental idea of change relates to the functions commonly presented in middle school curricula, limits *are* introduced in the course, mainly to promote derivatives and integrals later on.

Limits were first covered informally, based on a dynamic, approaching conception. Students were asked to solve routine limit problems for continuous, discontinuous, and piecewise-defined functions. Then the formal definition was introduced through a classroom activity centered around a story problem originally created for a teaching experiment (Boester, 2010). The bolt manufacturing problem allows students to explore the logical structure of the formal definition by thinking about the functional relationship between the input and output of a factory that makes bolts. After allowing students to discuss the bolt problem in groups, a whole class discussion (led by the researcher) was held to come up with the following statement:

> For every bolt length tolerance, there exists a raw materials tolerance, so that if an amount of raw material that falls within the raw material tolerance is put into the machine, the length of the bolt produced falls within the bolt length tolerance.

Written on the board next was a delta/epsilon diagram, whose graph had a slope roughly equal to one so that the delta and epsilon intervals would match and be equal. (The researcher tried to match the diagrams drawn by the students during their small group discussions of the problem.) This diagram was initially labeled with the terms from the bolt problem. The formal definition of limit was then written on the board. Finally, the researcher walked the students though a mapping of each representation onto the other, showing in particular how the symbols of the definition matched the pieces of the statement and the diagram.

The concepts on the above list were intentionally not highlighted during instruction. While all of these concepts are embedded in the bolt problem, could students actually unpack them without further explicit instruction? The students were reminded, however, to think of the bolt problem while constructing their answers and explanations for each diagram.

Students were then given the assessment tool as a written, optional homework assignment for extra credit. (Students did not have to agree to participate in the study to complete the assignment and receive the extra credit.) Those students who completed the assessment were also given the opportunity to be interviewed. Out of 24 students enrolled in the course, 13 returned the written survey, and 8 of those sat for a videotaped interview. For the students who were interviewed, their remarks were transcribed and compared to their written solutions in two passes, once for their initial explanations, and a second time when their responses were probed by the researcher. For those students that were not interviewed but turned in the assessment, their written remarks were coded as if they had sat for the first pass of the interview.

Results

A preliminary analysis of the results has revealed that the students struggled with the assessment. Most students showed that they grasped concept #2, that the delta interval cannot be outside the epsilon interval. Interviewed students attributed their explanations of concept #2 to the bolt problem, that an amount of raw materials too far away from the target amount would produce a bolt with a length outside the acceptable length range. Students also showed that they grasped concept #4, that the delta and epsilon intervals do not have to be equal. One student explained that delta and epsilon are describing different qualities, so why should they have to be equal?

However, few students expressed knowledge of concepts #1, #3 and #5. Evidence of this was shown through changing the epsilon interval to correct a diagram, expanding a delta interval to match an epsilon interval (as in Figure 2), and not recognizing that a misalignment of a and L in a diagram was a fundamental error, even if the delta interval was still within the epsilon interval. The lack of these concepts seemed to hinder their grasp of concepts later on the list (and tested in later diagrams in the assessment). While the last two errors could be attributed to metaphorical “baggage” of the bolt problem (in the context of the problem, these are acceptable or even desired actions), students should have recognized that the bolt manufacturer cannot control the customers’ demands on the accuracy of the bolt length (epsilon), only their own measurement error of the raw material (delta).

Even with this disappointing performance, the second pass of the interview showed entirely different results. When probed, although a few students backed away from correct concepts, every student gained several concepts. The students’ responses to the probes were consistently framed in the context of the bolt problem, which clearly helped the students come to grasp the concepts they heretofore lacked. Even with the researcher being careful to closely adhere to the interview protocol, six of the eight students left the interviews having expressed every concept on the list. Thus, while the bolt problem itself may not be sufficient to instill the logical structure and implications of the formal definition of limit, the problem clearly primed students for a discussion of the diagrams used in the diagnostic tool. This suggests that an instructional sequence involving the bolt problem should not be considered complete without an assignment like that of the limit diagnostic tool, and a discussion of the assignment that questions the reasoning of correct and incorrect responses.

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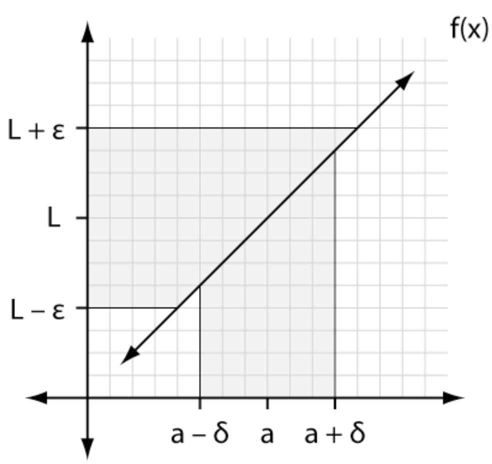

Figure 1. The delta/epsilon diagram from Question #2.

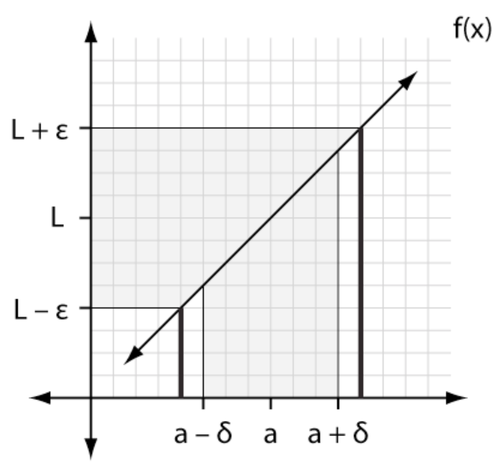

Figure 2. A proposed student correction to the delta/epsilon diagram in Question #2.

Title: Assessing Active Learning Strategies in Teaching Equivalence Relations

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Contributed Research Report

Abstract: In this study, students in transition-to-proof courses were introduced to equivalence relations either using a traditional classroom lecture or using small group learning activities. Students' understanding of equivalence relations were then assessed using task-based interviews aimed at assessing concept image, concept definition, as well as concept usage in terms of writing proofs. The students involved in small group activities made stronger connections to partitions and were more successful in writing proofs. In addition, the concrete learning activities gave many participants a strong prototypical example that aided in encapsulating the essential features of an equivalence relation.

Key words: transition to proof, classroom teaching experiment, concept definition

The notion of equivalence plays a role in understanding relationships between a wide variety of mathematical objects, such as fractions, equations, and vectors. This fundamental idea is formalized in the notion of an equivalence relation. Students are typically introduced to the formal definition of an equivalence relation in a transition-to-proof course. In a study involving a transitions course, Chin and Tall (2001) point out that the idea of relations is one of the more difficult concepts for students to understand. In particular, while many students were able to recall that equivalence relations must be reflexive, symmetric, and transitive, the standard definition of a relation as a subset of the cross product had little concrete meaning. Further, for most students, the conceptual link between equivalence relations and partitions was fairly limited. In terms of proofs, Chin and Tall (2000) also observed that many students attempted demonstrate that a relation was an equivalence relation using informal rather than formal, definition-based arguments. The goal of this study is to assess the impact of introducing equivalence relations and partitions using small group learning activities. In particular, when compared to a standard lecture, do small group activities improve student understanding of equivalence relations and partitions? Further, does an understanding developed through small group interaction lead to greater success in writing formal proofs?

Background

In his study of a transitions course, Moore (1994) identified several difficulties that students had in writing proofs. These included not knowing the concept definition, having inadequate concept images, and not knowing how to use the definitions to structure their proofs. Building on the distinction between concept definition and concept image, Moore introduced the notion of concept usage. Concept usage refers to the way one operates with the concept in generating examples or in writing proofs. Taken together, Moore referred to the concept definition, image, and usage as the concept-understanding scheme. He noticed that students had difficulty remembering formal, abstract definitions without an informal understanding of the concept: "The students often needed to develop their concept images through examples, diagrams, graphs and other means before they could understand the formal verbal or symbolic definitions" (Moore, 1994, p. 262). In terms of constructing an understanding of a new concept, Dahlberg and Housman (1997) outlined four basic strategies that students use when presented with a new concept definition: example generation, reformulation, decomposition and synthesis, and memorization. Their research indicated that the strongest evoked concept image arose from example generation. With this in mind, the group activities to introduce equivalence relations were designed to help students develop an informal understanding of the concept definition through example generation and exploration.

Methodology

Participants in the study were undergraduate students enrolled in either a lower division discrete mathematics course or an upper division transition-to-proof course. During the first year of the study, students in both courses were introduced to equivalence relations and partitions using a traditional lecture format, with one fifty minute class period devoted to equivalence relations and another devoted to partitions. Using the standard definition, equivalence relations were introduced as a subset of the cross product satisfying the reflexive, symmetric, and transitive properties. The instructor used several examples to illustrate the meaning of each requirement and how it could fail. There was good classroom interaction, with a variety of questions and discussion involving both the instructor and the students. During the second lecture, after defining and illustrating the notion of a partition, the instructor proved the standard theorem connecting equivalence relations and partitions. During the second year of the study, students worked in small groups during the two class periods. Activities on the first day related to equivalence relations while the second day focused on partitions. Students were given the formal definitions, and the quantifiers in each requirement were verbally emphasized. After this introduction, the rest of the class period was devoted to small group interactions. Beginning with a variety of colored shapes, the students were asked to determine whether "same color" and "differ in exactly one attribute" were equivalence relations. Using these same shapes, students were asked to formulate other examples and non-examples of equivalence relations. Students then considered several relations defined symbolically, such as aRb if and only if $a+b=2n$ for $n \in \mathbb{Z}$, and were asked to convince each other that the given relations were or were not equivalence relations. Students participated in similar activities regarding partitions and, using some concrete examples, convinced themselves that equivalence classes naturally give rise to a partition. In addition, they discovered that a collection of sets that do not form a partition do not give rise to an equivalence relation. The theorem relating equivalence relations and partitions was not proven, but was stated as a result that generalized the examples they had investigated. During the group activities, the instructor spoke with individual groups to help clarify any questions or to encourage rigor in their arguments.

Assessment

Approximately three to four weeks after the classroom lecture or group activities, students' understanding of equivalence relations was assessed using task-based interviews. 17 students from the lecture sections and 21 students from the active learning sections participated in the interviews. These occurred outside of class, lasted approximately a half an hour, and were videotaped for further review. Participants were first asked "what is an equivalence relation" and, in attempting to get a better picture of their concept image, they were asked to describe any other related concepts, ideas, or illustrations that they thought of when hearing the term "equivalence relation." In terms of concept usage, students were given two relations and asked to determine whether or not they formed equivalence relations. For each relation, participants were first asked to give an example of two elements that were related and two other elements that were not related. This was to ensure that there was no confusion regarding the notation or definition of the relations. Students were asked to talk out loud as they worked on the problems, and they were prompted for additional clarification when their reasoning was unclear.

Results

In terms of being able to define an equivalence relation, the vast majority of students in both courses remembered the three words reflexive, symmetric, and transitive. When asked to describe each requirement, most students gave symbolic definitions that were largely correct. The most common mistakes involved the use of quantifiers. This is similar to the results in Chin and Tall (2000, 2001), and there was little difference between the written definitions given by students in the traditional lecture sections versus those with small group activities. When asked about any other related concepts or ideas that popped into their head when they heard the term "equivalence relation," there was a difference observed between the lecture sections and the active learning sections. Partitions were mentioned by only 4 of the 17 participants in the traditional lecture sections, while 10 of the 21 students in the active learning sections made a connection with partitions. Finally, several students made comments about the examples used in their initial encounter with equivalence relations. Two students in the lecture sections indicated that equivalence relations involved ordered pairs, while almost a quarter of the students in the active learning sections described sorting colored shapes into related groups of objects. For example, in describing other ideas related to equivalence relations, one student remarked that "Well, definitely the shape and colors concept is the strongest ... it does help to remember kind of the flavor of relation that we're describing."

In terms of concept usage, students were asked to prove or disprove that two relations formed equivalence relations. The first relation, aRb if and only if $a \cdot b \geq 0$ for $a, b \in \mathbb{R}$, is not transitive (take $a = -1$, $b = 0$ and $c = 1$ so that $ab \geq 0$ and $bc \geq 0$ but $ac \ngeq 0$). Students in the active learning sections were more successful at noticing that transitivity fails, with several students immediately considering the transitive property as they suspected it might be problematic. In contrast, all students in the traditional lecture proceeded by checking reflexive first, symmetry second, and transitive third. Only one student, a student in the active learning section, made a connection with partitions before demonstrating that transitivity fails: "Obviously, the negative numbers are related to each other and also the positive numbers are related to each other. The problem is zero itself ... it doesn't create a partition because zero is related to any positive or any negative." The second relation participants considered, mRn if and only if $m^2 = n^2$ where $m, n \in \mathbb{Z}$, does form an equivalence relation. The students in the active learning sections were more successful at writing a correct proof of this fact. In particular, only 4 of 17 participants in the lecture sections presented a correct proof compared with 14 out of 21 in the active learning sections. Finally, in terms of proof schemes (as described in Harel and Sowder, 2007), almost half of the students in the traditional lecture sections gave only an empirical argument while almost all of the students in the active learning sections attempted a deductive argument.

Discussion

When compared with a traditional lecture, small group learning activities involving equivalence relations and partitions led students to a more interconnected concept image as well as greater success in writing proofs. In addition, sorting concrete shapes into groups of equivalent objects seemed to provide many students with a prototypical example that embodied the formal definition while making a clear link to partitions. Typically, one of the goals of a transitions course is to help students in working with abstract mathematical ideas, where meaning is often derived from the formal definition. In this sense, classroom activities where students are given an opportunity to explore formal definitions in small groups supports the goals of a transitions course. Further, this study demonstrates that these same types of activities can help students in taking a more formal, deductive approach to mathematical argument and proof. Finally, in terms of limitations of this study, it is important to note that the small group learning activities occurred during two days of a semester where almost all classes were taught in a modified lecture format. It is unclear how students would perform in a course where most classes involved active group learning. However, this study does suggest that small group activities where students generate and explore mathematical definitions can be an effective tool for teaching certain concepts within a lecture style transitions course.

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Surveying Mathematics Departments to Identify Characteristics of Successful Programs in College Calculus

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Sally Jacobs, Jessica Ellis, and Eric Weber

This report highlights the completion of the first step of a large national investigation of mainstream Calculus I that aims to identify the factors that contribute to student success in Calculus I. Calculus I is *the* critical course on the road to virtually all STEM majors. Even students who do well in it often find the experience so discouraging that it leads to a change of career plans. We have very little data on the preparation and aspirations of the students who enroll in this course or of the factors that contribute to success in calculus. This five-year NSF funded project begins to fill this gap in knowledge.

The report provides a brief overview of departmental and instructional factors that influence student persistence and success in college and university calculus. We also describe the processes of developing a suite of six survey instruments to assess the characteristics of calculus instruction at colleges and universities across the nation. Since the six surveys (department chair, calculus coordinator, instructor pre, instructor post, student pre, and student post) followed the same development process we will limit our discussion in this report to the student pre- and post-surveys. The survey development process began with the development of a taxonomy of critical attributes of successful calculus programs that have been reported in the literature. This was followed by cycles of item construction, clinical interviews with survey respondents and item refinement until survey items assessed the intended taxonomy variables and the survey question and answer formats were interpreted consistently with the designers' intent. The research that guided the item development, including the format choice and validation cycles will be described with findings that reveal items that are effective for gaining information about calculus instruction and its impact on students. Pre-course surveys have been administered to over 10,000 students across the nation and post-survey data will be available at the end of the fall, 2010 semester.

Brief Summary of the Literature

Over the last 25 years, various studies about student persistence in college in general and in STEM studies in particular have converged on a nearly common set of clearly identifiable factors that contribute to student persistence. Broadly, these factors pertain to (a) a strong sense of community and self-perception of identity with that community, (b) departmental or institutional supports for learning, and (c) instructional behaviors that meet students' intellectual needs, promote greater learning and develop student self-confidence. In the area of college calculus as well as secondary mathematics leading to calculus, research findings also highlight the effect of pedagogical issues that affect students' understanding of the key ideas of the course (e.g., limit, derivative). Numerous studies have also highlighted the importance of students' development of problem solving behaviors and habits of mathematical thinking that are consistent with ones held by acting mathematicians and scientists. Yet another vast area that has been shown to effect students' learning and mathematical self efficacy include the quality of interactive engagement within the classroom and intellectual demand put on students in homework and within the classroom. (e.g., homework, explanations). Other areas that have been particularly influential in affecting student persistent include: i) student self-efficacy relative to mathematics, ii) student and teacher beliefs about the nature and methods of mathematics, and iii) student self-identity with the culture of mathematics.

Research findings reveal many more variables than are feasible to include on the CSPCC Survey. This resulted in our considering the extent to which research indicates that a particular variable is a powerful factor in learning calculus or in student persistence in calculus. Other criteria we took into account include: (1) How amenable is this factor to actual change or manipulation? (i.e. can the instructor or department do anything about it?); (2) How hard is it to answer the question? (respondent burden); and (3) How confident are we that students will give us a truthful answer? (expected reliability).

Our literature review (see bibliography) guided our choice of variables to include in our surveys. These variables have been articulated in the form of a taxonomy. The student post-survey variables characterize both the dependent and independent variables that we hypothesize (based on our review of the literature) are critical for student success and continued mathematics study.

Taxonomy Keyed to Student Post-Survey

What follows is a curtailed taxonomy that shows only the major dimensions. As noted below, most dimensions has several subcategories. The full taxonomy will be presented with the full report and related to the literature.

<u>Potential Dependent Variables</u>

A. Course grade and intention to take Calc II (with 4 subcategories)
B. Impact of Calc I course on student (with 4 subcategories)
C. Student self-perception of knowledge/skills in calculus

<u>Potential Independent Variables</u>

A. Student Beliefs and Affect (with 5 subcategories)
B. Perceived Behaviors and Values of the Calculus Instructor (with 4 subcategories)
C. The Role of Homework and Exams (with 4 subcategories)
D. The Role and Behavior of the Student in Learning (with 6 subcategories)
E. Supports for Students (with 2 subcategories)
F. Readiness for Calculus (Pre-survey) (with 3 subcategories)
G. Readiness for Calculus (Post-survey)

Format and Design of Survey

While the variables embedded in the CSPCC Survey questions relate to factors identified from the literature, the format and design of the questions are consistent with recommended practice in survey design (Colton & Covert, 2007; Fowler, 1995; Saris & Gallhofer, 2007). Depending on the information sought by each question, a specific question format was selected as deemed most appropriate (Likert, contrasting alternative, categorical, matrix configuration). Professional advice was also sought from Dr. Jillian Kinzie, Associate Director, Indiana University Center for Postsecondary Research and NSSE Institute, whose area of expertise is survey design. In addition, we have been in periodic communication with Co-PIs Phil Sadler and Gerhard Sonnet regarding survey formatting and data processing plans. They have also reviewed drafts of the survey taxonomies and instruments, and have offered suggestions for survey question refinement.

Survey Development

The project team developed the following five surveys: i) course coordinator, ii) instructor pre- suvey, iii) instructor post-survey, iv) student pre-survey and v) instructor post-survey. The development process for each instrument involved cycles of: i) constructing items for each taxonomy item for each survey; ii) conducting clinical interviews with a talk-aloud protocol with subjects for each of the respective surveys (i.e., course coordinators, instructors and students); refining the item questions and answer choices based on analysis of clinical interview data.

Each survey included multiple question types, including likert, contrasting alternative, and categorical. The question format for each taxonomy variable was considered with the format choice relying on which format would provide the most valid and reliable data relative to that variable. As one example, the contrasting alternative format is more reliable in instances where likert scales are ambiguous because different survey respondents construct different images of what it means to select a value in the scale. They are also more valid in instances where the goal is to gain information about the relative degree to which students agree with two common alternatives to a particular statement. In the case of the contrasting alternative format a brief description is provided for each end of the survey scale so that the respondent is clear on what it means to select that answer choice. As one example, we devised a contrasting alternative item that provides two alternatives about what a score on a mathematics exam is measuring because these two choices were revealed during interviews to be the most common uses of exams.

Example *constrasting altetnative* item type that appears on both the student pre- and post-survey.

30. My score on my mathematics exam is a measure of how well

(a) I understand the covered material.	1 O	2 O	3 O	4 O	(b) I can do things the way the teacher wants.

The conference presentation and report will provide additional exemplars of survey items for each of the four surveys and some time will be allotted for participants to react to the taxonomy and sample items.

Significance

The potential significance of this five-year study is very strong and members of the RUME community will be interested in the progress and outcomes of this national project. Through the policies and publications of the Mathematical Association of America, the results of this project will effect calculus instruction and curricular development across the nation by providing knowledge of approaches to teaching calculus that are more successful, with particular attention paid to the differential effects of racial/ethic and gender variables. This report represents the first in a series of reports at the RUME conference detailing the progress and results of this important work.

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Translating Definitions Between Registers as a Classroom Mathematical Practice

Paul Dawkins

Abstract: Many have noted that mathematical definitions constitute a duality between a category of objects and the definition that delineates that category (Alcock & Simpson, 2002; Edwards & Ward, 2008; Mariotti & Fischbein, 1997; Tall & Vinner, 1981). Prior research has readily identified conflict between these two elements of students' conceptions, but reliable mechanisms for explaining and resolving such conflicts are still forthcoming. The present study observed a real analysis classroom in which the duality was embodied and addressed directly in class dialogue and activities. Particularly, three linguistic registers (metaphorical, common, and symbolic) arose to express different aspects of the definitions themselves (conceptual and formal). Translation across these registers provided a mechanism by which some students were able to segue their concept image and concept definitions successfully. Some students corrected errors in their concept image as a result of this practice.

Keywords: mathematical defining, real analysis, translating definitions, harmonisation, classroom communication

Mathematical definitions constitute an integral part of proof-based mathematics, but often receive less pedagogical attention than do the theorems and proofs built thereupon. This may result from the fact that, in the logical realm, definitions require no proof and do not add to a body of deductive theory (nothing is true with a definition that was not true without it) (Mariotti & Fischbein, 1997). In the cognitive realm however, nothing can be proven about a category or property that has not been defined, thus definitions are of premium pedagogical importance.

This emphasizes definition's role, which is to delineate a particular category of objects or aspect of a class of objects. Definitions thus constitute a duality between the class or concept being identified and the formal definition used to isolate that class (Alcock & Simpson, 2002; Edwards & Ward, 2008; Tall & Vinner, 1981). Many previous studies reveal the pertinence of this duality within definitions in light of the conflict and divergence of the two aspects.

Mariotti and Fischbein (1997) propose that, at least in the realm of plane geometry, concepts have two cognitive aspects: the *figural* aspect that relates to the concrete and visual nature of the concept (members of a category) and the *conceptual* aspect that expresses the abstract and theoretical nature of the concept (the property establishing category membership). Though they note the common discord between these two aspects, the authors propose that the process of defining involves "harmonisation" between the two aspects by ferrying back and forth mentally between the two viewpoints until they can be brought into sufficient agreement. In one reported interchange between students, Mariotti and Fischbein (1997) note how one students was focusing on the figural aspect while the other focused on the conceptual, such that their dialogue embodied direct interaction between these two aspects within the class discussion.

The present study identifies a classroom mathematical practice that developed in an undergraduate real analysis classroom and provided a mechanism for harmonisation between student's concept image and concept definition. The classroom discourse and activities openly addressed the duality that existed within definitions and the classroom mathematical practice of translating definitions through three linguistic registers helped them mediate between the aspects.

Methods

This study represents part of a larger study of classroom communication in undergraduate real analysis. This study's data comes from two 15-week semester courses of real analysis taught by the same instructor at a medium sized university in the southwest. The course included a proof-based development of real numbers, sequences, limits of functions, continuity, and uniform continuity. All class meetings were observed and written records were kept of the overall classroom dialogue and activities. A set of student volunteers (about 5 per semester) were interviewed weekly throughout the semester regarding their understanding of course content and classroom dialogues and activities. The interviews particularly focused on the appearance of classroom diagrams, lines of reasoning, and language as students articulated their own understanding and reasoning or as they worked on presented tasks.

Results

The three registers of definition articulation appeared early in and throughout each semester of study. For example, the class formulated the definition of one-to-one as:

1. If the function is viewed as arrows being shot from the domain to the target, then no one in the target gets hit with two arrows (metaphorical register).
2. No two inputs have the same output (common register).
3. For every x_1, x_2 in the domain such that $f(x_1)=f(x_2)$, $x_1=x_2$ (symbolic register).

The metaphorical register played different roles in the discussion of different definitions. Dawkins (2009) presents a detailed account of these classes' metaphor use and comprehension.

During the second semester of study, the professor discussed with the students how the limit of a sequence should be defined without presenting the definition itself. One student suggested that if the limit was like a party, then you could find the party by seeing where all of the people (the elements) are. The professor adopted this language and began to verbally explain the sequence definition in terms of people going to a party. She wrote on the board a common register definition that stated, "A sequence converges to the real number L if we can make the terms of the sequence stay as close to L as we wish by going far enough out in the sequence." She translated this statement verbally into the metaphorical domain saying it is only a party if for any size party you pick, after some point everyone shows up at the party. In another formulation, she said "only finitely many guys can be outside the room for you to have a party." She then proceeded to translate the common language definition on the board into symbolic language replacing "far enough out in the sequence" with index terminology and "as close to L as we wish" with epsilon neighborhood terminology. This pattern of translation between the metaphorical (party, time), common language (close enough, far in the sequence), and symbolic (epsilon intervals, indices) appeared repeatedly in the classroom discussion across the various topics of the course and throughout the discussion of sequence limits.

When asked three days after the introduction of the party metaphor what sequence convergence means, Tidus explained the definition primarily in the common language register. He did not directly reference the party language, but rather said "at some point" adopting a time-based metaphor for sequences and said "numbers will be in 4's neighborhood" treating the neighborhood as a place rather than a set.

When I asked Tidus later in that same interview to explain to me the roles epsilon, K, and n played in the definition, he said, "K represents how far n has to go on the number line to get into the epsilon neighborhood." Tidus expressed a correct correspondence between the mixed metaphorical and common register explanation he had previously provided of sequence convergence and its symbolic register translation in terms of indices.

After the class took the test over sequence convergence (about two weeks later), I asked Tidus to explain the definition of sequence convergence and he said, "you pick an epsilon. For any epsilon that you pick, an infinite amount of terms will be in that epsilon neighborhood and a finite amount of terms will be outside." However, when I asked him about the role of epsilon, K, and n, he explained by giving me the formal definition. When asked how he understood the definition, he accurately elaborated his understanding in terms of the party metaphor.

During the first semester, the professor sought to help students identify which functions are uniformly continuous by describing that they contained a steepest point. She indicated that this distinction showed why $\sqrt{x}$ is uniformly continuous though $\ln(x)$ is not. During interviews, three students reasoned from this criterion that if the point (0,0) is deleted from $\sqrt{x}$, then the function no longer uniformly continuous because it has no steepest point. Only one of these three, Aerith, then looked at the formal definition and concluded that:

> Well, cause by definition it says x_1-x_2 will be less than delta and $f(x_1)$-$f(x_2)$ will be less than epsilon and there is like you can find two points from here and that holds the definition. I just think when, it's like, this thing basically is saying when x_1 and x_2 getting closer, the image of, I mean the value of these two points will getting close, too.

Aerith translated the formal definition back into the common register and concluded that $\sqrt{x}$ would not cease to be uniformly continuous by the deletion of a point.

Discussion

Both Tidus and Aerith mirrored the classroom mathematical practice of translation across registers as they discussed their understanding of analysis definitions. Tidus began expressing himself in a mixture of the metaphorical and common registers, and initially was unable to articulate the symbolic register definition. He could explain the correspondences between his less formal definition and the symbolic definition's elements. Over time, he constructed his concept definition, but did so with strong ties to his concept image expressed in the metaphorical and common registers. It appears that the translation process provided a means of harmonisation between his concept image and concept definition of sequence convergence. Also, he gave more prominence over time to the formal, symbolic definition rather than the metaphorical.

Aerith, along with several classmates, developed a misconception in their concept image of uniform continuity based upon the professor's non-standard criterion of a "steepest point." Though not a metaphor per se, this alternate notion does not directly represent the concept or definition of uniform continuity and requires some translation. Once Aerith translated the formal definition into the common register, she was able to develop an alternative to the "steepest point" within her concept image ("when x_1 and x_2 getting closer, the image of, I mean the value of these two points will getting close, too") that helped her correct her misconception.

The professor engaged the class in the process of developing definitions to describe the behavior they observed in sets of examples. The dialogue separately referenced the "idea" and the "definition." In this way, she introduced the duality of definitions into the consensual domain. The "idea" was usually expressed using the metaphorical or common register, while the "definition" arrived by the end of the discussion in the symbolic register. The mathematical metaphors (Dawkins, 2009) employed in the classroom such as the party metaphor or the steepest point criterion had the effect of helping students develop their concept image of the particular concept. Similarly, the common register articulated the "idea" or the concept image. Thus, as Figure 1 presents, the dual aspects of definitions were acknowledged in the classroom discourse as the "idea" and the "definition." The linguistic registers embodied the dual aspects in the classroom discussion. The need to coordinate the two aspects (harmonisation) motivated the classroom mathematical practice of translation.

Thus, the classroom mathematical practice of translation between linguistic registers appears a viable tool for guiding students toward harmonising their concept image and concept definitions. This appears reasonable in light of the fact that this process encourages students to develop their concept definition out of their concept image, or at least with many connections between, rather than the two being introduced into the mind separately as when the formal definition is introduced in final form at the beginning of the discussion (Pinto & Tall, 2002).

This point also may explain the difficulty students experienced in working with uniform continuity. Most of the course definitions prior to this one had been introduced to students at the calculus level such that students already had concept images of function and sequence limits. Most students had no exposure to uniform continuity, and so they simultaneously constructed their concept image and concept definition from the professor's explanations and explorations.

Mariotti and Fischbein (1997) observed that classroom dialogue in which students embodied the different aspects of a definition seemed to promote harmonisation between the aspects. The ability of distinct linguistic registers to embody the different aspects of a definition appeared to promote similar negotiation both in the classroom setting and for individual students as they reasoned about classroom definitions in isolation. Further research is needed to identify the reliability of and conditions upon this classroom mathematical practice as a tool for helping students construct definition understanding and harmonise the dual aspects thereof.

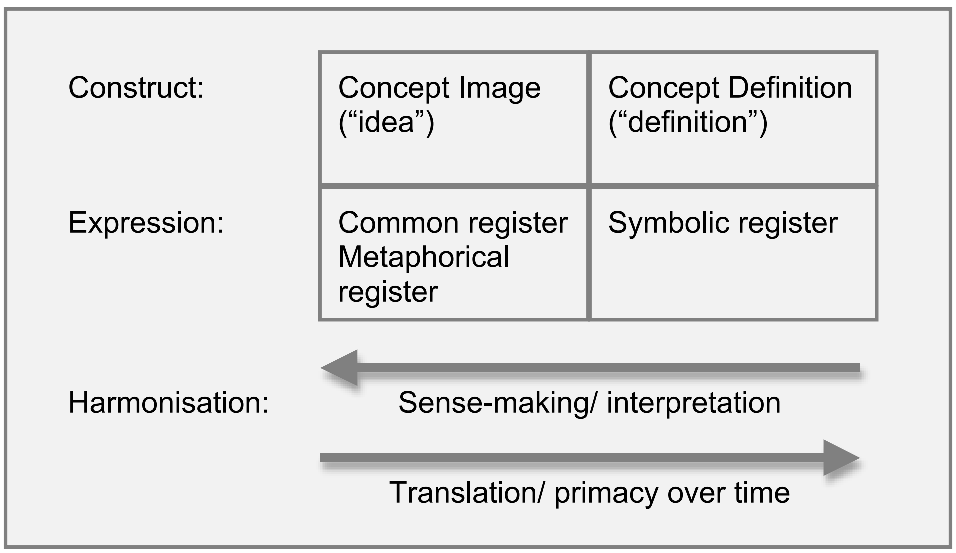

Figure 1: Dual Aspects of Defining

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The Role of Conjecturing in Developing Skepticism: Reinventing the Dirichlet Function.

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The study presented in this research report was born out of the desire to develop pathways for students from informal to formal modes of thinking. The data from this report stems from a series of small group interviews using a process of guided reinvention incorporating frequent student conjectures in order reinvent the definitions of limit and continuity. During these interviews, students used the practice of skepticism in order to suspend judgment on various mathematical statements. In the process of exploring a developed conjecture, the students' suspension of judgment allowed them to alter their initial beliefs about the nature of continuity and their interactions with functions.

Keywords: Conjecturing, Skepticism, Calculus, Continuity, Dirichlet Function

The study reported on in this presentation was born out of the need to develop pathways for students from intuitive to advanced mathematical thinking. As noted by Gravemeijer and Doorman (1999) "guided reinvention offers a way out of the generally perceived dilemma of how to bridge the gap between informal knowledge and formal mathematics" (p. 112). In spring 2010 four small groups of multivariable calculus students participated in a series of eight interviews aimed at reinventing the core concepts of limit and continuity. The groups each experienced different levels of uncertainty and skepticism throughout the project and frequently used the practice of conjecturing to reframe and clarify their skepticism.

According to Dewey (1933) uncertainty plays an integral role in cognitive development as it is the primary component in reflective thinking, which he describes as "a state of doubt, hesitation, perplexity, [and] mental difficulty" (p. 12) accompanied with actions seeking to resolve these feelings of doubt and uncertainty. Similarly Cornu (1991) set forth a model of learning marked by overcoming cognitive obstacles which may be characterized in part by the uncertainty that they create in those encountering them.The theoretical framework employed in this report was set forth by Zaslavsky (2005) and expanded by Brown (2010).

Zaslavsky (2005), in her work, describes the origins of uncertainty in different mathematical tasks. She details three types of mathematical tasks that lead to different types of uncertainty: competing claims, unknown paths or questionable conclusions, and non-readily verifiable outcomes. Brown (2010) expanded upon Zaslavsky's work by recognizing that in Zaslavsky's work uncertainty was coupled with a *lack of belief* about the truth of the premise at hand. Brown went on to define skepticism as doubt coupled *with a belief* regarding the truth of the premise at hand. She further describes skepticism "as a state of being; that is, a collective or individual can, at a particular point in time, both obtain evidence for a conjecture and view the conjecture as of unknown truth value" (p. 2). She goes on to point out that viewed from this lens, skepticism can be perceived as the classroom practice of "suspending judgment against a backdrop of empirical or experiential evidence" (p. 2).

This research report presents the results from three of the above mentioned interviews during which the students involved spontaneously reinvented the Dirichlet Function, confronting several of their preconceived, informal beliefs regarding the concept of function. Throughout the semester, and in particular, throughout the interviews, students were encouraged to develop and write conjectures which captured their beliefs about the task. The students were then encouraged to attempt to either resolve the conjectures or refine their conjectures into new statements to be resolved.

The interviews used in this study were developed from the perspective of guided reinvention with the use of both predetermined and spontaneous context problems (Gravemeijer and Doorman, 1999). The context problems provided by the researcher were designed to complement the student's conjecturing activities by providing environments of uncertainty for the students. The students were then asked to participate in the process of creating and resolving and refining conjectures in order to address their uncertainty about the problem.

Much like the students in Brown's (2010) study, the students observed in this study frequently encountered uncertain situations with a well-developed belief about the truth of the premise being discussed. However, through the disciplines of skepticism and conjecturing, they were able to suspend their judgment on the premise in order to fully resolve the conjecture at hand. In the three interviews considered for this report, this process of suspending judgment allowed the students to further explore and refine their conjectures and eventually demonstrate that their initial beliefs about the nature of continuity were false.

This practice of skepticism and conjecturing exposed several of the students' naïve beliefs about the nature of continuity. Nunez (1999) in his argument that all mathematics is the result of embodied experience proposes two metaphors for continuity based on embodied experiences: *natural continuity* which is based on the metaphor that a continuous graph is the result of motion along that line and *Cauchy-Weierstrauss continuity* which is based on the metaphors that a line is a collection of gapless points with continuity defined as the preservation of closeness between those points. Based on these descriptions, the group in question unanimously mirrored the metaphor of natural continuity described by Nunez, thus leading them to believe that continuity must only exist on open intervals. However, though the use skepticism this group was able to reinvent the Dirichlet function (a function which takes the value of 1 for all rational numbers and 0 for all irrational numbers) and as a result alter their metaphor of continuity.

Additionally, the change in metaphor corresponded to a change in the way the students interacted with functions. The interviews exposed a view of function as existing outside the students' control. However, the change in metaphor resulted in a shift in control towards the students, allowing them to have power over the individual values of the function, thus allowing them to embrace the formal definition of function.

This research report is a case study analysis that took place in the first iteration in a process of development research aimed at better understanding how students transition from informal to formal understanding of calculus. The author welcomes all suggestions for improvement in future iterations of the study.

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Toulmin Analysis: A Tool for Analyzing Teaching and Predicting Student Performance in Proof-Based Classes

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Abstract: This paper provides a method for analyzing undergraduate teaching of proof-based courses based on Toulmin's model of argumentation. The paper then describes how that analysis can be used as a predictor of subsequent student proof-writing performance and shows that the predictions are reasonable approximations of students' subsequent proof-writing. The method of analysis was developed via research in a lecture-based abstract algebra class, it has application, to any lecture-based, proof-intensive course. This method provides one possible way to directly link classroom teaching activities to subsequent student performance that would force instructors to assume more responsibility for their students' demonstrated end-of-course performance.

Keywords: proof, Toulmin analysis, abstract algebra, classroom research

Introduction and research questions

It is suggested in numerous studies (Dreyfus, 1999; Dubinsky, et. al, 1994; Leron, Hazzan, & Zazkis, 1995; Weber, 2001) that students are not learning at the level that faculty desire. What needs to change? Advisory reports from national research associations have called upon faculty to move away from the lecture format and adopt other teaching methods (National Science Foundation (*NSF*), 1992; Mathematical Sciences Education Board (*MSEB*), 1991). However, instructors often believe that students bear the responsibility of learning (Wu, 1999). New insight is needed to help these instructors understand how their actions in the classroom affect their students' ability to master the material. This study will develop new tools needed to analyze lecture-based teaching and will directly connect instruction to student learning in proof-based courses. In particular the study:

1) Uses Toulmin's (1969) model of argumentation to analyze the teaching of proof-based undergraduate courses taught via lecture.
2) Proposes that students in the class will adopt the argumentation methods as modeled by their instructor.
3) Analyzes student work to determine whether the Toulmin-analysis of teaching does predict students' proof-writing behavior.

1 Literature

Despite the lack of use of non-traditional curricula and pedagogies, the *NSF* and professional organizations such as the Mathematical Associan of America (*MAA*) continue to fund the development of new curricula and professional development in the hope of changing collegiate mathematics teaching and improving student learning. Each of these curriculum projects has promised to disseminate their work, typically via professional development activities at mathematics conferences. This curriculum development and implementation model almost exactly mirrors the process used at the K-12 level in which "many efforts over the past decade or so have been aimed at providing well-designed curricula for school mathematics… Each has spawned an industry of workshops and conferences focused on helping teachers prepare to use the materials in their classrooms" (Sowder, 2007, p. 177). It has been shown that these efforts "have not been particularly successful in educational projects" (Richardson & Placier, 2001, p. 906); that is, they have not affected meaningful change in teaching practices.

New studies of pedagogical methods suggest that it is essential that teachers reflect on their beliefs and practices in order to affect meaningful change (Richardson & Placier, 2001). Yet, at the undergraduate level faculty generally hold the belief that "the professor gives an outline of what and how much students should learn, and students do the work on their own outside" of the class meetings (Wu, 1999, p. 267). This contract implies that as long as the instructor delivers a clear lecture and communicates appropriately, the students are responsible for their own failure to fully comprehend the instructor's intent or apprehend the deeper structure of the material as well as adopt the mathematical behaviors, such as proof-writing strategies, that the instructor models.

I posit that students are appropriating some of the modeled behavior, but not always the aspects that faculty believe are most important. This argument is supported by new research on transfer-in-pieces (Wagner, 2006). This research claims that experts and novices perceive the same aspects of an instructor's presentation as having differing levels of importance, and as a result attend to different aspects. Wagner (2006) then demonstrated that students would transfer aspects of a structure from one problem to another, even when then they seem mathematically inappropriate from an expert's perspective.

In a proof-based course, such as abstract algebra, the instructors are modeling proof-writing strategies and the types of arguments that they expect from their students. Thus, we should be able to observe which aspects of proof-writing and types of arguments are appropriated by the students. After analyzing the classroom teaching via Toulmin analysis (1969), this paper analyzes student work and then provides a preliminary means of linking the analyses of classroom instruction with student learning.

2 Data and Methodology for analysis

I took detailed field notes while observing 18 class meetings. I also collected demographic information on all students in the class and work from 6 participants of 12 students on a content assessment that required them to write a ring-theory proof.

I transcribed all text on the board in addition to classroom dialogue. I reviewed all classroom video recordings and made a log of all episodes that included proof-writing or presentation. Criteria for proof-production or presentation was rather straightforward. An incident was logged as such when any member of the class community was writing or showing a formal mathematical proof that drew on symbolic notation and logical reasoning. To pursue the question of whether students had appropriated the proof-writing behavior that Dr. Tripp had modeled required analysis of the proofs she wrote during class and the student's work outside of class. I analyzed all such arguments using Toulmin's (1969) model into one of the following: data, warrant, backing, qualifier or conclusion. I also noted when an aspect of argumentation was written or spoken aloud. In this paper I present two sets of arguments, the first set was chosen because, collectively, they were the only observed instance of Dr. Tripp demonstrating all of the properties for a sub-ring proof. The second set of examples was chosen because they mirrored the sub-ring property proofs in a different structure. I employed a Toulmin analysis on the student work similar to that used to analyze instruction.

3 Results

3.1 Dr. Tripp's presentation

In the proof to be discussed in this paper we see the following patterns in Dr. Tripp's written presentation of mathematical property-verification proofs (excluding spoken comments):

	Data	Warrant	Backing	Qualifier	Conclusion
Written	13	5	0	0	13

A similar pattern was recapitulated across all of Dr. Tripp's proof-writing. When Dr. Tripp models proof writing, she always writes the data and conclusion but she was never observed to write a backing or a qualifier. Moreover, she only infrequently wrote warrants but, when her spoken comments were included in the analysis she included warrants in more than half of all arguments. Because the students were only submitting written work, the table above does not take into account the fact that Dr. Tripp always spoke warrants aloud, nor does it reflect any of her statements of backing.

When we consider this mixed pattern of writing out versus only speaking the warrant for a particular piece of data as modeling the written arguments that she expected of her students, we should expect some mixed results. In terms of writing the data and conclusions, Dr. Tripp has always modeled writing those and, as a result, we should see that students always or almost always write the data and conclusion. But, her mixed writing of warrants may not provide her students a consistent model for their own work. Finally, we should expect that the students never, or almost never, write backing or a qualifier in their proofs.

3.2 Students' work

When all the student data on property-verification arguments is aggregated, without reference to the validity of the student's claim, we see the summary of written argument elements table below.

	Data	Warrant	Backing	Qualifier	Conclusion
Stated	32	14	2	1	30
Implied	1	18	7	4	3

The first observation is that the students almost universally wrote both the data and conclusion of their arguments. They were much more varied in writing out warrants or, at least, implying warrants in their workof the 33 written arguments, only 14 included a written warrant. Finally, the students almost never wrote out a backing or a quantifier, writing only two backings and one quantifier out of 33 arguments. In short, this is exactly what we would have expected from the students given Dr. Tripp's given Dr. Tripp's presentations of written proof.

4 Significance and directions for further research

This paper makes two significant contributions. First, I showed one way to draw upon Toulmin's (1969) model of argumentation to analyze proof-based courses including traditional taught abstract algebra courses. Moreover, Toulmin's model helped explain the relationship between the instructor's written proof and the classroom dialogue that she led. When Dr. Tripp's presentation is taken as modeling the type of mathematical behavior that she wants her students to demonstrate we would infer that she wants her students to always be able to articulate the data, warrant and conclusion of an argument.

Second, I analyzed student work by again drawing upon Toulmin's (1969) model for argumentation. When taken as a whole, the students' collective proof-writing that they also wrote a level of detail similar to that modeled by their instructor and used argumentation elements at a similar rate as Dr. Tripp. That is, the students almost perfectly demonstrated that they had appropriated Dr. Tripp's modeled proof-writing in terms of the level of detail that they included in their written work.

This research immediately suggests two future directions for research; both directed towards better understanding the development of students' mathematical proficiency. First, this use of Toulmin's framework to analyze teaching was helpful in making sense of some aspects of Dr. Tripp's writing and her classroom dialogue, but cannot explain all aspects of her modeling of proof-writing. We need significant research that studies teaching of proof (Harel & Sowder, 2007; Harel & Fuller, 2009) and, in particular, lecture-based teaching of proof. Moreover, we need new theoretical lenses to make sense of what lecture-based teachers are doing in classes and that also provide a means to explain student mathematical proficiency.

Furthermore, it is also worth pursuing the idea that teachers consciously model appropriate mathematical behavior for their students as a means of making sense of lecture-based undergraduate courses. For example, we should be exploring how instructors model for their students other fundamental mathematical skills such as exploring definitions and examples, organizing and linking knowledge, and abstraction and generalization from examples to name but a few. In general, significantly more research is needed to explain traditional lecture-based instruction of proof-based content courses and abstract algebra courses in particular.

Lastly, in this study I reported seemingly inconsistent behavior on the part of the instructor which was linked to inconsistent behavior on the part of the students. We might suggest that classroom professors make explicit statements, like, "listen to the types of questions I ask you and ask myself, these are the kinds of things *you* should be asking while you write proofs." Then, "to decide when you should write down the answers to these questions you should

…" This type of meta-level dialogue could lead the students to a better understanding of when warrants and backing need to be explicitly stated in a proof as well as decreasing their writing of invalid or incorrect proofs.

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A Multi-Strand Model for Student Comprehension of the Limit Concept

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Abstract: In analyzing interview transcripts to assess student understanding of limits for first year calculus students, the application of the 7 Step Genetic Decomposition created by Cottrill, et. al. (1996) indicated that the interviewed students possessed no higher than a 3rd step understanding. Despite an inability to clearly articulate their understanding in terms of the expected lexicon, several students were able to create valid examples and counterexamples while justifying their answers. This suggests that these students possessed a better understanding of the limit concept than they were able to articulate. Thus, this study concludes that there exists additional criterion that should be taken into account in order to accurately diagnose student understanding of the limit concept. In particular a model for student understanding of limits should contain strands reflecting the student's method for solving a problem involving limits, the student's justification for the solution, and the applicability of the student's method and justification within the context of the problem.

Keywords: limits, student understanding, calculus, interview methodology

Introduction and research questions

The failure rate in undergraduate calculus courses is traditionally quite high (Ferrini-Mundy & Graham, 1991) and many reasons have been offered as explanations for this phenomenon. Insufficient or unsatisfactory background courses and student difficulty with the important concepts that form the underpinnings of calculus are just a few of the explanations that have been put forth (Burton, 1989; Ferrini-Mundy & Graham, 1991). In particular, Davis and Vinner (1986) point out that students' troubles with the concept of limits may play a strong role in the rate at which they fail out of calculus. Research suggests that a strong conceptual understanding of this abstract concept is needed in order for students to understand the topics which follow, i.e. derivatives and integrals (Bezuidenhout, 2001; Hardy, 2009; Orton, 1983). Since typical undergraduate courses in calculus often rely on the limit concept to explain continuity of functions, define derivatives, and define integrals, a student holding misconceptions of the limit concept runs the risk of developing flawed conceptions of these later topics that will negatively impact the rest of his or her mathematical understanding. Must this understanding include symbolic proficiency with the formal definition of limit or can a student develop a strong conceptual understanding of the limit concept that does not rely on the symbolic interpretation?

Up to now, the most comprehensive model of the stages of understanding students must pass through before achieving mastery of the limit concept has been the genetic decomposition created by Cottrill, et. al. (1996). However, this decomposition seems to account for only one strand of the many that make up the web of student understanding of the limit concept. Inspired by the model for comprehension of mathematical proof designed by Mejia-Ramos, et. al. (2009), this study seeks to identify additional strands of knowledge needed by students to fully master the limit concept.

1 Literature

Several researchers suggest that most of the research conducted on student understanding of the limit concept seems to fall into one of two or three categories. These classifications can be summarized as research on the informal notions of limit held by students, research on how students reason about limit in the context of the formal definition, and research into the obstacles students face as they try to make sense of the limit concept (Swinyard, 2009; Williams, 2001).

Much of the research that has been conducted has focused on developing classification schemas or identification rubrics to identify commonly held conceptions and describe the levels of understanding students might hold regarding the limit concept. Throughout the available research on student understanding of limit the most influential categorization seems to be the 7 Step Genetic Decomposition devised by Cottrill, et. al. (1996). Through their work to identify student conceptions and misconceptions about the limit concept they developed a 7 step classification for the phases students undergo as they make sense of the formal limit definition for themselves (pp. 177 - 178). As evidenced by their decomposition, their emphasis is on students starting with an *x*-value understanding which then culminates in a formal definition.

Swinyard (2009) attempted to enhance their 7 step genetic decomposition by filling in more detail in the last few stages of the genetic decomposition, steps 5 – 7, which focus on "the transition from informal to formal reasoning" (p. 20). According to the work of Swinyard these stages can be enhanced by the consideration of how students "define limit" and "define closeness in a concrete and increasingly restrictive manner" (pp. 22, 24). This study addresses the following questions: Is the genetic decomposition sufficient for determining a student's understanding of the limit concept? If not, what other strands should be taken into account when assessing what a student understands about the limit concept?

2 Data collection and methodology for analysis

A multiple choice written assessment was administered to 9 recitation sections of a first semester undergraduate calculus course. Individuals that represented low, medium, and high levels of understanding based on their assessment responses were then contacted to participate in follow up interviews. During the interviews the students revisited on their answers to the assessment items before being presented with two novel tasks.

During the first analysis of the transcribed interviews, a combination of both Cottrill, et. al.'s (1996) genetic decomposition and Swinyard's (2009) enhancements was used to interpret student answers and to categorize what understandings of the limit concept were held by the interviewees. Though it was possible to identify which stage of understanding the interviewed students reached, the decomposition did not seem to take into account several features of the students' explanations. Thus a second analysis was conducted which focused on identifying and defining other strands of knowledge held by the students that could reflect their understanding of the limit concept as depicted in their explanations.

3 Results

The first analysis of the transcribed interviews suggested that the interviewees had only reached the 3rd step of the 7 Step Genetic Decomposition. This finding was primarily based on the student responses to the second task, an item adapted from Bezuidenhout (2001). In this task the students were confronted with the limit of a difference quotient involving a function *f* and asked to determine the value of the limit, if it existed, based on a table of values for the function and its derivative. All interviewed students chose to find the limit of the whole expression rather than acknowledging that the problem could be interpreted as the limit of a combination of functions. Thus their answers failed to meet the criteria for step 4 of the genetic decomposition, "perform actions on the limit concept by talking about, for example, limits of combinations of functions" (Cottrill, et. al., 1996, p. 178).

However, several aspects of understanding arose during the first analysis of the interviews that did not seem to be adequately addressed by the combined genetic decomposition. The genetic decomposition did not account for the types of examples the students used during their explanations, the high level justifications offered for the existence of a limit, or their choice of methods for approaching the task of finding the limit. The second analysis then focused on these issues to identify and define the following strands of knowledge: the student's method for solving the presented task, the student's justification for the final answer, and the applicability of the student's method and justification in the context of the task.

There are many valid methods students are taught in order to determine the limit of an expression. These methods include the utilization of graphs and tables, but also encompass applications of stronger results such as L'Hopital's Rule. Thus one strand of student understanding addresses sophistication of the method the student selects for evaluating the limit. For instance, in one interview a student, Jim, utilized his knowledge that the existence of the derivative of the function in the table implied that the function was continuous, and subsequently that the limit of the function was equal to the value of the function at the given *x* value. This method for solution surpassed the demonstrated logic of his interviewed peers who focused on values provided in the table when they faced this task. Hence, there is an argument to be made for students' understanding being reflected by the method of solution they choose.

Another strand accounts for student justification, that is, the student's ability to correctly use the chosen method based on the information he or she perceived as applicable. The graphical examples and counterexamples offered by the interviewees as they responded to first task would be assessed along this dimension. The interviewees' decision to utilize L'Hopital's rule in the second task, after a direct evaluation of the limit of the difference quotient failed, was a selection of an appropriate method on the applicable data. However, as their computations were incorrect, they failed to reach the desired result and this would count as a failure of justification. Another example of justification that occurred in the second task was when Jim explained his use of the knowledge that the existence of the function's derivative at $x = 2$ implied both the continuity of the function and the existence of its limit.

The final strand is applicability which addresses the student's ability to see the relevance of given data and his own knowledge in the context of the problem. It is this factor that separated several of the interviewees' solutions and allowed the second analysis to distinguish between their respective understandings of limits. In one instance two students had provided very similar graphical counterexamples to a statement they were trying to prove true. Only one of them recognized the applicability of the counterexample to the task and used it to show the statement in question was false. The other interviewee failed to see the applicability of her example to the task at hand and persisted in trying to show the statement was true. This issue was also evident in both Jim's ability to see the relevance of his knowledge to the solution of the second task and Jim's selection of an inappropriate example for reasoning about the veracity of statements in the first task.

4 Significance and directions for further research

This study has opened several avenues for future research. Of particular importance is the consideration that the current genetic decomposition (Cottrill, et. al., 1996) alone is not sufficient for determining a student's understanding of the limit concept. Student understanding is comprised of many different strands and students may have developed methods for solving limit problems that do not rely on their appropriation of the formal limit definition. Are there limitations to the types of problems these methods can solve? Of particular interest would be problems that seem to require use of the formal definition but are solvable by students with a weak understanding of the formal definition.

This study suggested at least three strands that are evident when students interact with problems involving limits. Further research is necessary to verify the existence of these strands in a larger population and determine whether there are additional strands that need to be added to this tentative model. Such research should also refine the currently identified strands, especially the applicability strand. Clearly the function concept is strong for some of the interviewees as they constructed clear counterexamples that showed how a function's limit might not equal the value of the function at a particular x-value. However, the students' apparent inabilities to perceive that these were counterexamples and then use that knowledge to reach the correct conclusions in the first interview task is troubling. Additional study of this strand may also help educators understand why students select inappropriate examples when trying to make sense of a problem they are trying to solve.

Ultimately, this multi-strand perspective of student understanding could be a valuable aid to teachers. Knowledge of these strands could be incorporated into assessments which would enable instructors to determine their students' levels of understanding. Not only would this allow instructors to tailor their lectures to address any areas of weak understanding demonstrated by the students, but it would also serve as a guide toward the desired level of student understanding of the limit concept.

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Authority in the Negotiation of Sociomathematical Norms

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In 1996, Yackel and Cobb introduced the study of sociomathematical norms in an attempt to understand how students' mathematical autonomy might be fostered by their mathematical beliefs and values and to make sense of the complexity of mathematical activity in the classroom. They defined sociomathematical norms to be "normative aspects of mathematics discussion specific to students' mathematical activity (p. 461). In other words sociomathematical norms can be seen as the reoccurring mathematical aspects of discourse that focus on mathematical thinking rather than thinking about mathematics. For example, the social norm that you should justify your answer does not, by itself, insure that your justifications will be accurate, rigorous, or convincing. However, a sociomathematical norm that defines what constitutes a convincing argument can be introduced to set an expectation in the classroom that encourages strong mathematical activity in the form of justification (Yackel & Cobb, 1996; Kazemi & Stipek, 2001).

We propose that four components must be identified for an expectation to qualify as a sociomathematical norm. *The four components of a sociomathematical norm are: 1. a mathematical expectation is set forth, 2. a mathematical interpretation of the expectation occurs, 3. the expectation is agreed upon, and 4. the expectation is validated as legitimate.*

Researchers have documented sociomathematical norms introduced by a research team (McClain & Cobb, 2001), teachers (Yackel, Rasmussen, King 2000), or students (Hershkowitz & Schwarz, 1999). Once introduced, sociomathematical norms are negotiated and re-negotiated by various participants in the class. The negotiation process often allows the expectation, embedded within the sociomathematical norm, to become clear and understandable as it is interpreted by both teacher and students so that it is genuinely agreed upon. However, recently Levenson, Tirosh, and Tsamir, (2009) found that teacher endorsed norms, enacted norms, and student perceived norms may all be different within the same classroom. In this case, the expectation in teacher endorsed norms was not genuinely agreed upon.

In previous work, we suggested that authority in the classroom hinges on three major concepts: authority relation, legitimacy, and change (Gerson & Bateman, submitted June, 2010). The authority relation is a relationship between two or more people, with at least one person acting as the bearer of authority and at least one person acting as the receiver of authority. The bearer of authority makes a claim; the receiver of authority recognizes the claim as legitimate and is influenced to change his or her behavior. In traditional classroom settings, authority is usually hierarchal with the teacher acting as bearer of authority and the student acting as receiver of authority (Herbel-Eisenmann, Wagner, & Cortes, 2008). In inquiry-based classroom settings the bearer and receiver of authority are more fluid roles taken on by both teacher and student at different times (Hamm & Perry, 2002; Wilson & Lloyd, 2000). We define legitimacy of authority as "the knowledge, skills, position, or experiences that influence a person or group within an authority relationship (Gerson & Bateman, submitted June 2010)."

In addition we defined four general types of authority in the mathematics classroom by the knowledge, skills, position, or experiences that legitimize the authority: Hierarchal, Expertise, Mathematical, and Performative (Gerson & Bateman, submitted 2010). Briefly, Hierarchal authority is the authority a person holds because of their position in the class (e.g. as an instructor, or presenter). Expertise Authority is legitimized by the perceived expertise of the bearer either by proving their mathematics expertise or by having ownership in the creation of a solution. Mathematical authority is legitimized by mathematical argument and justification. Performative authority is legitimized by the ability to engage the class. These authorities can all be held by both instructors and students.

Method

Our research is set in a teaching experiment in a university honors calculus class by Hope Gerson and Janet Walter. Students worked on tasks designed or selected to elicit conceptually important calculus content without prior instruction. The corpus of data from this study is taken from two, 2-hour class periods in Calculus II taught in the fall of 2007. The data were chosen for two reasons. First they occurred early in the semester as sociomathematical norms were still being actively negotiated. Second, a compelling episode occurred at the beginning of the second day, where Michael, a student in the class, introduced a new way of thinking about what constitutes a mathematical difference and how mathematical difference should be explored. We recognized this episode as pertaining to the negotiation of sociomathematical norms and wanted to further understand the dynamics in play, in particular, under what authority are sociomathematical norms introduced and negotiated?

We analyzed four hours of videotape gathered on January 29 and 31, 2007. Members of the research team transcribed and independently verified videodata. Together, the authors used open coding on one-half hour of videodata to identify key ideas, such as authority, agency, social norms, and sociomathematical norms. Later, the authors independently coded surrounding episodes. We, then, came back together to build consensus about which codes were important, how to define them and how to recognize them in the data. This helped us refine our definition of sociomathematical norms and to more accurately recognize them in the data. We then used axial coding to look for patterns in the data.

Analysis and Discussion

In the following excerpt, after two groups presented their solution to the same task, Michael introduced a new expectation (lines 1 and 4) for what constitutes a mathematical difference. Heber and Tyler interpreted the expectation and began to negotiate with Michael the meaning of the new expectation.

1 Michael: Now, we it looks like we've got two different equations? *[expectation]*
2 Heber: Yeah. *[interpretation]*
3 Tyler: But they're the same. *[interpretation]*
4 Michael: Um, they're not the same equation, they both model something different, and I'm 90 percent sure that I know what that is, the difference. *[negotiation of meaning]*

In order for this new expectation that two equations are different if they "model something different" to become normative, Michael's expectation needed to be interpreted, agreed upon, and legitimized.

For the next ten minutes, Michael and the class continued to negotiate the meaning of Michael's expectation that they explore what the two different equations model. For example in the next excerpt, Michael re-states the expectation embedded in a mathematical argument in line 7, and Robert interprets that to mean solving part a, and expresses that he understands Michael's expectation.

5 Michael: if you just get the volume function and just start evaluating the volume function *[mathematical argument]*

6 Derrick: [inaudible] you could get a general equation *[interpretation]*

7 Michael: Right, but I'm saying take an indefinite in, er a definite integral of their equation. What would that model? What, if you plug in the value of one, into their indefinite integral, what does that represent? *[mathematical argument and restatement of the expectation]*

8 Robert: 'Cause, 'cause do you want me to solve part a? 'cause it [inaudible] [moves thumb and index finger together] I see what you're asking. *[interpretation]*

Michael's initial statement in line 4, that "they both model something different" did not supply enough mathematical information for Robert and the rest of the class to build a mathematical interpretation of the expectation, nor to judge whether it was legitimate. Michael's argument both made explicit how he wanted to compare the two equations, and offered legitimization of the expectation through mathematical authority. Although Michael held granted authority to present his ideas to the class, and expertise authority for his own solution, it was Michael's mathematical authority legitimized through his mathematical argument that allowed others to interpret and agree on the expectation.

We found similar results with the other sociomathematical norms that were introduced by students. The fact that mathematical authority played so large a role in building a mathematical interpretation, agreeing on the interpretation , and validating the expectation lead us to believe that mathematical authority is important in the negotiation of sociomathematical norms. While any authority could validate an expectation, we found that mathematical authority, when activated, played a role in every component of the sociomathematical norm.

When instructors introduce sociomathematical norms, as in the study by Levenson, Tirosh, and Tsamir (2009), we suspect that their hierarchal authority and expertise authority legitimize the expectation before its meaning is fully interpreted. Therefore students are likely to accept the norm before they agree on its mathematical meaning. But when students introduce the expectation, it opens the path for mathematical authority to legitimize the claim. We suggest creating a classroom environment where students rather than the teacher are encouraged to initiate and negotiate sociomathematical norms will lead to better agreement on the expectations among the members of the class. We also believe that if teachers introduce a sociomathematical norm, they should be aware of the potentially obstructive role their hierarchal authority and expertise authority may play in the negotiation of that norm.

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Student Understanding of Eigenvectors in a DGE: Analysing Shifts of Attention and Instrumental Genesis

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This study examines the potentialities of the theory of instrumental genesis and shifts of attention in analysing students' evolving understanding as they interacted with a dynamic geometry representation of eigenvectors and eigenvalues. Although the former theory provides a framework to analyse students' interactions with tools and transformation of tools into instruments, it makes an assumption about the role of instrument in cognitive development. According to Verillon and Rabardel (1995), the founders of the theory, the role of instrument in cognitive development is a sensitive point. I thus explore the complementary use of the theory of instrumental genesis with the theory of shifts of attention to enable an analysis of students' cognitive development in a digital technology environment.

Keywords: Technology, linear algebra, instrument and attention

The integration of digital technology in mathematics education has given rise to continuing research concerning mostly students' use of digital tools to develop an understanding of mathematical ideas and objects. Researchers have proposed several theoretical perspectives for analysing the interactions between tools and the student (Guin and Trouche, 1999; Kieran and Drijvers, 2006; Arzarello et al., 2002; Falcade et al., 2007). In this study my focus is on the potentialities of the use of the theory of instrumental genesis to analyse students' mathematical knowledge acquisition.

The theory of instrumental genesis (Verillon & Rabardel, 1995) draws on actions and procedures taken by a student to use a tool. The tool can be transformed into an internally oriented tool (instrument of semiotic mediation) by the process of internalization (Vygotsky, 1978) that occurs through semiotic processes. For example, given a specific task in a dynamic geometry environment, the dragging tool can be transformed into a sign referring to the idea of function as covariation between dependent and independent variables (Falcade et al., 2007). The development of instrumental genesis is a complex process that depends upon several factors such as potentialities and constraints of the tool, actions and procedures taken by the student, the student's knowledge of mathematical concept in the task, and also the student's awareness of the affordances of the tool. The two interconnected components of instrumental genesis, instrumentalization and instrumentation, are used to describe the processes involved in the interactions between the student and the tool. The instrumentalization process, directed toward the tool, involves the development of skills to use the tool, the personalization and the transformation of the tool. It is about what the student thinks the tool was designed for and how the student uses the tool. It therefore calls upon attending to tool use. The instrumentation process, directed by the tool, involves the constraints and potentialities of the tool that shapes the student's knowledge acquisition (Trouche, 2005). This involves a shift of attention from tool use to what the tool can do, so that the tool becomes not the object of attention, but something that focuses and directs attention in particular ways, a mediating tool. The two components are concerned mostly with processes involved in transforming a tool into an instrument, not the role of instrument in knowledge acquisition. As researchers point out, the role of instrument in cognitive development is a sensitive point (Verillon and Rabardel, 1995), and the theory of instrumental genesis has shortfalls in putting forward the potentialities of instrument in the development of mathematical thinking.

On the other hand, among theories on cognition, John Mason's theory of shifts of attention appears to be more descriptive in terms of revealing the developmental process of mathematical being. Mason (2008) believes in the power of awareness and its education. Awareness refers to what enables us to act, calling upon our conscious and unconscious powers, and sensitivities to detect changes and to choose proper actions in certain situations (Gattegno, 1987; Mason, 2008). To educate awareness is to draw attention to actions which are being carried out with lesser or greater awareness. Attention can be drawn not only to mathematical objects, relationships and properties, but also to manifestations of mathematical themes, and to heuristic forms of mathematical thinking (Mason, 2008). According to Mason, the structure of attention comprises macro and micro levels; what is being attended to is as important as how it is being attended to. At the macro level, Mason describes the nature of attention as follows: "attention can vary in multiplicity, locus, focus and sharpness" (p.5). At the micro level, he distinguishes five different states of attending: holding wholes, discerning details, recognizing relationships, perceiving properties and reasoning on the basis of agreed properties. Holding wholes is when a student gazes at a definition, collection of symbols and/or diagram. The student may not focus on anything in particular, while 'waiting for things to come to mind'. Looking at the wholes, the student may discern and identify useful sub-wholes or details. Discerning details is a process that participates in and contributes to subsequent attending. As the student discerns details, she may recognize relationships between symbolic and geometric representations of mathematical concepts. When she becomes aware of possible relationships in the particular situation, she may perceive these as instantiations of a property. As she continues attending, she can use the perceived properties as a basis for mathematical reasoning. It is noteworthy that the described states of attention are not levelled or ordered. They often last for a few micro-seconds and alternate among other states. Those that become stable and robust against alteration for varying periods of time may block further development of awareness (Molina and Mason, 2009).

As summarized above, Mason's theory provides a framework for analysing students' attention in a mathematical activity. However, given the important role of the digital tool in the DGE-based activity, I want to take into consideration the interaction between the student and the tool. In the context of using DGS then, how might the relationships between instrument and attention be conceived, and, in particular, what might the effect of instrument of semiotic mediation be on shifts of attention?

To identify the states of attending, one may analyse semiotic resources such as gesture and discourse used by a student in a paper-pencil environment. However, my empirical study shows that in a digital technology environment, the instrumental genesis also causes shifts of attention. This suggests combining the theory of instrumental genesis with the theory of shifts of attention to enable analysing cognitive development in a digital technology environment. In particular, my empirical data supports the conjecture that analysing processes of instrumentalization reveals evidence of shifts of attention.

I interviewed a total of eight undergraduate linear algebra students who had all successfully completed a Linear Algebra course. They were given a worksheet containing a formal definition of eigenvectors and eigenvalues. They were then given a sketch designed to enable exploration of eigenvectors and eigenvalues for given 2×2 matrices using *The Geometer's Sketchpad* software (Jackiw, 1991). As shown in Figure 1, the sketch includes a draggable vector x, as well as a non-draggable vector Ax. As, vector x is dragged about the screen the vector Ax moves accordingly. The sketch also includes numeric values of the matrix-vector multiplication (Ax). The user can change the values of matrix A.

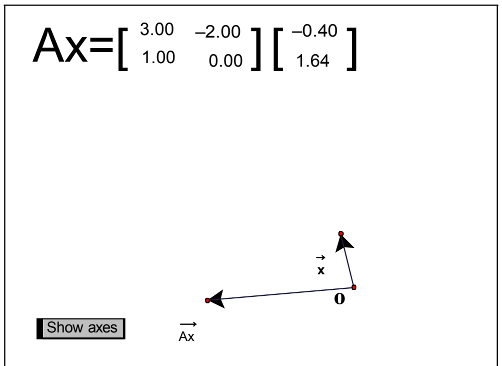

Figure 1. A snapshot of eigen sketch

In my analysis, I looked at students' actions with the sketch, different dragging strategies, and their ways of communicating orally about their interactions. Mason's theory of shifts in attention enables an analysis of students' interactions with the sketch as well as a description of their mathematical awareness. These shifts were made evident in part by the students' changing focus of attention from the definition to the sketch, but also in their different dragging strategies, which they used to identify eigenvectors and to explore the relationship between eigenvectors and eigenvalues.

In my presentation (and my extended paper), I will illustrate details of my analysis of students' actions with the dragging tool and argue that the dragging tool that was transformed into an instrument mediated students' conceptualization of the concept of eigenvectors and eigenvalues.

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University Students' Understanding of Function is Still a Problem!

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Abstract

A research study was designed using the conceptual model consisting two cells of concept images and concept definition developed by Vinner (1983) and has been used by many researchers since then, to investigate students' understanding of different concepts of calculus. A related literature review made us believe that students' understanding of function as one of the pillar of calculus is still problematic. 53 first year university students participated in this study that its purpose was to shed more light into the students' understanding of function in terms of their concept images and concept definitions. The study showed that the most common concept images of function among the students were having a rule, and using a machine as a metaphor for a function. The study also indicated that a concept image of having a rule for each function acted as an obstacle for students to understand the concept definition of function.

Key words: Conceptual Model, Concept Image, Concept Definition, Function, Calculus..

Introduction

Long time ago (1988), the first author did a research about students' understanding of calculus focusing on two fundamental concepts of calculus that are function and derivative. The study carried out at the University of British Columbia in Canada in which, the drop out rate of calculus by then was about %50 (1988). For that study, she used constructivism (Kilpatrick, 1987) as a general theoretical framework and used task based interviews (Jones, 1985) and adopted the idea of teacher as researcher and model builder (Cobb & Steffe, 1983) during the interviews. Finally, Gooya (1988) used a conceptual model consisting of concept image and concept definition developed by Vinner (1983). The finding of that study revealed that the nature of university students' conceptual understanding of function was as follows:

- *A number of students held proper concept images of function which should lead to the development of an appropriate concept definition.*
- *Few of the students, understood function as a relation between two variables (without having the restriction that for each x there is only one y.)*
- *For some students, a function was only considered to be an algebraic function.*

(p.104).

In 2008, both authors felt that university students' understanding of function is still a problem! The second author was high school teacher interested to find out that why despite teaching calculus to students majoring mathematics and physics or natural sciences at secondary school, they still have difficulty understanding it and the first author's experience was that this difficulty still exists at the university level as well. And this was the beginning of our explicating journey to find out more about this extremely important issue in the teaching and learning of calculus. So we did start our journey in the following way.

We conducted a study that its main purpose was to investigate the first year university students' understanding of function. 53 first year university students completing Calculus 1 and Foundation of Mathematics- from three universities in Tehran- participated in this study that its aim was to shed more light into university students' understanding of function in terms of their concept images and concept definitions. The research was exploratory in nature and the data were collected through 8 carefully designed questions. Those who participated in this study were volunteered students majoring mathematics at their universities and all studied calculus at high school. For this purpose, we took Harel's (2004) advice and developed aforementioned conceptual model (Vinner, 1983) to explore students' common concept images and concept definitions regarding the concept of function. Further, Bingolbali and Monaghan (2007) have also mentioned that this conceptual model is still has its own place among mathematics education researchers and the first paper that introduced this model is among the classics of mathematics education research. The data for this study were collected through a set of eight carefully designed questions based on the research findings in this field. For instance, the first three questions asked the students to explain that "whether the showing graphs represent a function or not and why", and the fourth question asked "is there exist a function that all its values are equal?" The next three questions gave us a chance to explore the students' concept images of the function. The purpose of the last question was to investigate the students' personal concept definitions of function.

After the analysis of the data, the students' concept images and their personal concept definitions were categorized according to the above conceptual model. The result was similar in various ways with what was found by Gooya in 1988 and that became our concern about teaching and learning of calculus. In particular, the findings showed that one of the major concept images among the students was that having a rule for function, is a relation between two things, and it is recognized by the test of perpendicular line. However, the students did not consider "arbitrariness" as one of the important characteristics of function – meaning that the value of a function at any given point is independent of the value at other points, and the domain and range can be arbitrary sets. Indeed, the idea of having a rule for function was in contradiction with their images about the arbitrariness of the correspondence of function. This concept image acted as an obstacle for the formation of the formal concept definition of function. In addition, "univalence" (meaning that for each element in the domain, there is a unique element in the range) as another key element in the formal concept definition of function, had not a solid formation in the students' concept images. Therefore, the students' concept images did not provide a proper foundation for them to understand the two essential features of the concept of function namely; the arbitrariness and the univalence.

However, there are many potential opportunities in high school and university calculus textbooks to develop and enrich students' concept images but they were mainly used as examples for different features and characteristics of the concept of function. Thus to conclude, the researchers speculate that teaching plays a major role to cause students' difficulties with understanding *function* and they suggest further research to attest this speculation.

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THE LIMIT NOTATION: WHAT IS IT A REPRESENTATION OF?

Contributed research report

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Abstract

Student difficulties with the notion of limit are well-documented by research. These studies suggest that students mainly realize limits through dynamic motion, which can hinder further realizations of the concept. Some studies mention the overemphasis on the dynamic aspects of limits in classrooms but research on the teaching of limits is quite scarce. This work investigates the development of discourse on limits in a beginning-level undergraduate calculus classroom with a focus on the limit notation and uses a communicational approach to learning, a framework developed by Sfard (2008). The study explores how the limit notation is utilized by an instructor and his students and compares the realizations of limit in their discourse. The findings indicate that the shifts in the instructor's word use when talking about the notation supported students' realizations of limit as a process despite the frequency with which the instructor talked about limit as a number in his discourse.

Keywords: teaching of calculus, limits, the limit notation, discourse analysis

THE LIMIT NOTATION: WHAT IS IT A REPRESENTATION OF?

Introduction

Being the building block of many fundamental calculus concepts, the notion of limit has drawn significant attention from researchers and student difficulties about the notion are well-documented by research. These studies suggest that dynamic motion dominates students' realizations of limit, which can interfere with other aspects of limits such as the formal realization of the concept (Bezuidenhout, 2001; Tall, 1980; Tall & Schwarzenberger, 1978; Tall & Vinner, 1981; Williams, 1991). In particular, the representational tools (e.g., verbal, visual, and symbolic) used by students while thinking about limits may lead to additional difficulties (Bagni, 2005; Cottrill et al., 1996; Williams, 1991). Further, some of the problems students encounter as they work on limits result from difficulties related to the underlying concepts such as functions and the notions of infinitely large and small (Parameswaran, 2007; Sierpińska, 1987). Therefore, the concept of limit presents students with two challenges: the need to make the transition from its intuitive to formal realization, and the need to cope with the compatibility of the conceptual and representational tools within the intuitively realized aspects of limits.

Some researchers highlight that the intuitive aspects of limits are perpetuated in teaching and curriculum. Parameswaran (2007) considered the reliance of calculus textbooks on graphing as problematic since it can lead to the incorrect idea that limit is a process of approximation. Cornu (1991) mentioned that "in teaching mathematics, certain aspects of the limit concept are given greater emphases, which are revealed by a review of the curriculum, the textbooks and examinations" (p. 153). Bezuidenhout (2001) argued the learning and teaching approaches stressing the instrumental rather than conceptual aspects of limits can result in students' realization of the notion as isolated procedures.

Although existing studies imply possible links between instruction and students' realizations of the limit concept, there is not extensive research on the teaching of limits to justify these claims. This work is part of a case study that investigates the development of the discourse on limits in a beginning-level undergraduate calculus classroom. The study uses a communicational approach to learning, a framework developed by Sfard (2008), to focus on the elements of one instructor's and his students' discourse on limits. In this paper, the main focus is on the limit notation as a symbolic representational tool in the discourse of limits since, besides graphing, it is the main visual mediator with which ideas about limit are communicated. The study addresses the following questions: (a) How is the limit notation utilized by the instructor in a beginning-level undergraduate calculus course and what kinds of realizations of limit does the notation support?, and (b) How do the elements of the instructor's discourse on the limit notation compare and contrast with the students' discourse?

Theoretical framework

One of the highlights of the *commognitive* framework (Sfard, 2008) is the interrelationship between communication and thinking. By defining thinking as the individualized form of communication, Sfard (2008) argues that the "cognitive processes and interpersonal communication processes are thus but different manifestations of basically the same phenomenon" (p. 83). Given this, the term *commognitive* entails the combination of the terms *cognitive* and *communicational*. This framework considers discourse as its central unit of analysis in which the main focus is on activities, patterns of interaction and communicational failures. Sfard (2008) defines the term discourse as " the different types of communication set apart by their objects, the kinds of mediators used, and the rules followed by participants and thus defining different communities of communicating actors" (p. 93).

The commognitive framework views mathematics as a particular type of discourse, which is distinguishable by its *word use*, *visual mediators*, *routines*, and *narratives*. Although number or quantity related words can be found frequently in daily life, "mathematical discourses as practiced in schools or in academia dictate their own, more disciplined uses of these words" (Sfard, 2008, p. 133). Given the abstract nature of mathematical objects, *word use* is a critical element of a mathematical discourse because possible differences in participants' use of those words can hinder mathematical communication. An important feature of mathematical *word use* is objectification. Objectification results in replacing the talk about processes and actions with states and objects (Sfard, 2008). For a mathematical discourse, objectification is a means for formalization and enhances the effectiveness of our communication. However, the objectified mathematical discourse is abstract and hides the discursive layers and metaphors it is composed of. Therefore, being explicit about the underling discourses and metaphors of an objectified mathematical concept can be quite important for students at the beginning stages of their learning.

Visual mediators refer to the visible objects created and operated upon for the sake of communication. Daily life discourses are generally mediated by the images of concrete objects whereas mathematical and scientific discourses are often mediated by symbolic artifacts. *Routines* refer to the set of metarules that characterize the patterns in the activity of participants of a discourse. *Narrative* is “any sequence of utterances framed as a description of objects, of relations between objects, or of processes with or by objects, that is subject to *endorsement* or rejection with the help of discourse-specific substantiation procedures” (Sfard, 2008, p. 134, italics in original). Narratives of a given discourse that are endorsed by the majority of the discourse community, in particular by “experts”, are considered as “true”.

The focus of this paper is on one instructor's and his students' use of the limit notation as a *visual mediator* as well as their *word use* and *endorsed narratives* associated with the notation to explore the similarities and differences between the instructor's and students' discourse.

Research methodology

The participants of this study were one calculus instructor and his section of undergraduate students taking a beginning-level calculus course in a large Midwestern university. For the instructor's discourse, the data consisted of video-taped classroom observations and field notes. The observation data consisted of eight 50-minute sessions in which the instructor discussed limits and continuity. For the students' discourse, part of the data included 23 students' responses to a diagnostic survey, which was taken from Williams (2001). The survey informed the selection of four students for an individual task-based interview session. The data for the analysis of the interviews came from students' written work and field notes taken during the interviews. The interviews were audio-taped and lasted between 53-76 minutes. Participation in the survey and the task-based interviews was voluntary.

The transcripts for the video and audio-taped sessions included what the participants said and what they did. Therefore, the transcripts also coded participants' actions as they were referring to the limit notation. For this study, the units of analyses were the instructor's and students' word use, and the limit notation as a visual mediator. Both the instructor's and the students' word use was analyzed with respect to the degree of objectification in their discourse on limits. The word use on limits was considered objectified if the participants talked about limit as an end-state or a number; it was considered operational if participants talked about limit as a process. Particular attention was also given to the use of metaphors and endorsed narratives underlying participants' word use. The analysis then focused on the similarities and differences between the instructor's and the students' discourse on the limit notation.

Results

The analysis of the instructor's overall discourse on limits revealed that he mostly talked about limit as an end-state of the limiting process: a specific number. In the context of the limit notation, however, he shifted his word use and referred to limit as a process based on dynamic motion. The analysis also showed that the instructor's word use on the limit notation depended on the following three mathematical contexts: computing limit at a point; limit at infinity; and infinite limits. In each of these contexts, the ways he talked about limits and infinity as end-states or processes differed. However, the shifts in his word use remained implicit for the students.

The analysis of the diagnostic survey and the individual interview sessions showed that, unlike the instructor, students rarely referred to the limit L as a number when talking about the limit notation $\lim_{x \to a} f(x) = L$. Instead, they adopted the elements of the instructor's discourse that referred to limit and infinity as processes. Therefore, although the instructor could flexibly talk about limit and infinity as processes or as end-states depending on the context, the notions remained as processes in students' discourse.

In summary, although the instructor's discourse on limits was mainly objectified, the shifts in his word use when talking about the limit notation supported students' operational word use. As a result, the students heavily relied on the metaphor of continuous motion whereas the instructor alternated between the metaphors of motion and discreteness. Moreover, the students only endorsed the narrative *limit is a process*, whereas the instructor mainly endorsed *limit is a number*.

Conclusions and implications for mathematics education

The students in the study developed the realization of limit as a process despite the instructor's general word use on limits, which was objectified. Talking about the limit notation was one mathematical context in which the instructor's word use alternated between the objectified and operational aspects of limits. Note also that the operational and objectified word use on limits utilize distinct metaphors: the former is based on the metaphor of continuous motion whereas the latter is based on the metaphor of discreteness. The tacit nature of these metaphors and students' adoption of the instructor's operational word use as the dominant means with which to talk about limits signal the importance of explicitness during instruction. As the insiders and the experts of the mathematical discourse, instructors can “lose the ability to see as different what children cannot see as the same” (Sfard, 2008, p. 59). This study provides evidence of the ways in which connected aspects of a mathematical concept can remain distinct and implicit for learners.

In addition, the study problematizes the utilization of the limit notation. The instructor primarily used the limit notation $\lim_{x \to a} f(x) = L$ to represent the end result of a limiting process (the limit is *equal to L*). Yet, students used it to represent the limiting process “the function *f(x) approaches the limit L as x approaches a”* (Hughes-Hallett et al., 2008; Thomas et al., 2008). So, although a symbolic and abstract visual representation, the limit notation might inevitably support dynamic motion and assumptions about continuity that underlie students' intuitive realization of limits.

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Student Outcomes from Inquiry-Based College Mathematics Courses: Benefits of IBL for Students from Under-Served Groups

Marja-Liisa Hassi, Marina Kogan, Sandra Laursen

Abstract

Our large, mixed-methods study examines cognitive and affective outcomes of inquiry-based learning (IBL) in a variety of undergraduate mathematics courses at four universities. Student outcomes are measured by pre/post-survey items, self-reported gains and historical transcript data. Students in IBL courses report higher cognitive and affective gains than do non-IBL students. IBL students also report increase in motivation and interest, whereas non-IBL students' motivation drops after mathematics courses. The historical transcript data also shows IBL students' higher interest compared to their non-IBL peers. These benefits of IBL instruction are especially important for women and low achieving students, who are often under-served by the traditional college mathematics courses. Our findings suggest that IBL instructional methods support positive learning outcomes in various groups of students, including those under-served and under-supported by the traditional college mathematics courses.

Keywords: inquiry-based learning, mixed methods, learning outcomes, undergraduate students

Introduction

Inquiry-based learning (IBL) refers to teaching and learning approaches that engage undergraduates in learning new mathematics by exploring mathematical problems, proposing and testing conjectures, developing proofs or solutions, and explaining their ideas. Thus students both "learn new mathematics through engagement in genuine argumentation" and come to "see themselves as capable of reinventing mathematics and to see mathematics itself as a human activity" (Rasmussen & Kwon, 2007, p. 190). Such approaches are supported by current socio-constructivist views of learning that emphasize individual constructions and ways of thinking and learning developed in social interactions in classrooms (Bransford et al.,1998; Cobb et al., 2000; Davis, et al., 1990). For college students in science and engineering, inquiry appears to be more effective than traditional instruction at improving academic achievement and developing problem-solving (Prince & Felder, 2007). However, fairly little empirical evidence exists to demonstrate the impact of IBL methods on student learning in college mathematics. Exceptions include studies by Smith (2006), Jensen (2006), Kwon, Rasmussen and Allen (2005), Ju and Kwon (2007), and Rasmussen et al. (2006). These studies suggest that undergraduate students' ideas of mathematics, proofs, and their own role in doing mathematics can be affected by the social norms and classroom practices that emphasize student activity, problem-based learning, and classroom discussions. This raises interesting new questions: how and to what extent do IBL experiences influence undergraduate students' motivation, achievements, and choices in learning mathematics?

Our group has conducted a large, mixed-methods study of IBL mathematics courses taught at four campuses where "IBL Centers" have been established. The courses range from introductory to advanced college mathematics and target varied audiences including math majors, science and engineering majors, and pre-service teachers. Observation, survey, interview and test data were gathered from over 100 course sections across two years, most from

IBL sections but also from non-IBL sections of the same courses, where these were available. In addition, student academic records for over 5000 students were obtained so that we could examine patterns in student achievement and course-taking following an IBL (or non-IBL comparative) course. In this report, we consider multiple measures of two main types of outcomes, broadly described as cognitive and affective outcomes, for students from these IBL courses and comparison sections. We examine key differences among student groups that suggest that IBL methods particularly benefit some groups of students who are often under-served by traditionally taught college mathematics courses: women and low-achieving students.

Methods

The study used several different measures for cognitive outcomes, including self-reported learning gains from surveys, academic achievement measures from transcripts, and test data from a subset of courses. Multiple measures for affective outcomes included self-reported affective changes from pre/post survey items and pursuit of additional mathematics courses, which we took as a proxy for increased interest in mathematics or commitment to it as a discipline, in parallel to survey items that explored these interests. We also explored cognitive and affective gains, and how these came about, in interviews with 68 IBL students.

Pre- and post-surveys were obtained from 800 IBL and 400 non-IBL students on cognitive, affective and social aspects of student learning and experiences during their math course. Longitudinal measures are based on pre/post items grounded in theory and constructed to probe students' mathematical beliefs, affect, goals and strategies of learning and problem solving on a seven-point Likert scale. Gains in understanding, thinking, attitudes, confidence and capabilities are measured at the end of courses on a five-point scale from "no gain" to "great gains" that is based on the SALG instrument (Student Assessment of their Learning Gains, 2008), developed to gather formative and summative data on classroom practices. The composite variables were constructed on the basis of the designed scales, exploratory factor analyses, and item analyses. The surveys also gathered information on students' personal and mathematical backgrounds and were matched using a unique identifier.

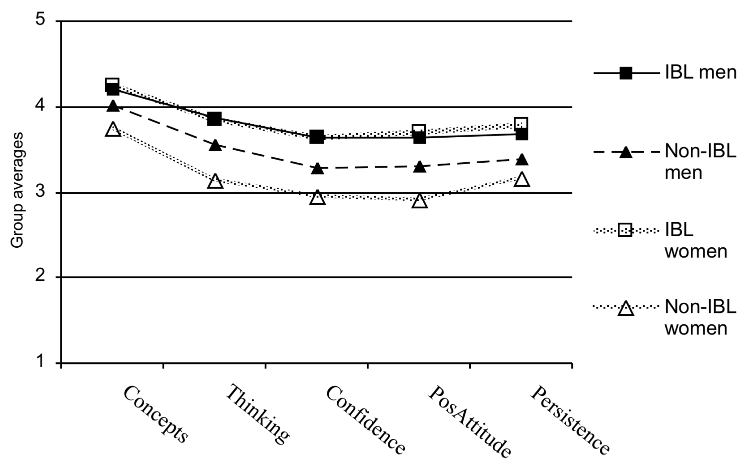
Historical transcript data for 5563 students at 3 campuses included mathematics courses taken, grades obtained, majors and minors, and some backgrounds (by academic term) for samples of students who took an IBL or non-IBL version of the same course in specific semesters, and allowing time for most students to complete subsequent mathematics courses and college degrees. Composite variables were constructed to measure students' incoming mathematical background, overall academic preparation, course outcomes, and post-course outcomes, such as number of additional math courses taken, average grades in all, required, and elective courses. For both survey and transcript data, results are based on statistical analysis including descriptive statistics and parametric or non-parametric tests.

Findings

Survey measures provide the strongest measures of both cognitive and affective outcomes for IBL students, but academic records and test data provide several points of corroborating evidence. Overall, IBL students reported higher gains than their non-IBL peers on both cognitive and affective survey measures. For example, IBL students reported higher gains in understanding concepts, mathematical thinking, confidence in doing and communicating about mathematics, persistence, and positive attitude about mathematics learning. Moreover, IBL students preserved their high motivation and increased their interest in college mathematics, whereas non-IBL students' motivation to graduate in mathematics clearly dropped during a conventional course. Pre-service teachers benefited less from the IBL instructional approaches than the non-teaching track IBL students.

Some IBL instructors interviewed for this study hypothesized that women would especially benefit from the collaborative style and confidence-building typical of IBL courses. They suggested that, while high-achieving students were not harmed by IBL courses, and often enjoyed them very much, students with more modest records of achievement would benefit most from this teaching style. We thus examined survey data for these sub-groups. Both men and women in IBL courses reported higher learning gains than their non-IBL peers, but the gains for women were striking. IBL women scored high on all cognitive and affective gains, whereas non-IBL women reported the lowest gains. This strongly indicates that women are underserved by non-IBL courses, whereas they clearly benefit from the IBL experience.

Differences between IBL and non-IBL students are statistically highly significant (p<.001) both among men and women.



Breaking out the students by prior achievement levels (using their self-reported college GPAs) is also illuminating. We divided the groups in rough thirds, according to their self-reported college GPA: top (≥3.8), high (3.0-3.79), and moderate or low (<3.0). It appeared that lower achieving students' cognitive gains were higher in IBL courses. The results indicate that traditional methods benefit stronger students the most. While IBL methods are beneficial to all types of students, the learning gains are greater for IBL students who started with the lower scores. These differences in gains were particularly apparent among pre-service teachers. However, as first-year students did not report a previous college GPA, the sample size is smaller for this result and do not reflect situation for all students.

Analysis of academic records data indicates that some of these gains may outlast the course itself. For example, for one campus with large IBL enrollments, we divided students in rough thirds based on their math GPA prior to the IBL class (or comparable non-IBL section): high (>3.4), medium (2.5-3.4), and low (<2.5). Our analysis shows that the low-scoring IBL students get higher average grade on the later required math courses than their non-IBL peers. Thus, IBL experience boosts achievement for the initially low-achieving students, while traditional courses show no such benefit. On the other hand, there is no evidence that the IBL methods disadvantage medium and high-achieving students. On the contrary, our analysis indicates that high-achieving IBL students take significantly more IBL-method math classes than their non-IBL peers. As greater number of math classes taken (including IBL-style) represents greater interest in mathematics, the analysis shows that taking an IBL course fosters greater interest in mathematics among high-achieving students. In sum, the previously low-scoring students benefit in achievement, while the high-scoring student get a boost in interest and motivation. Thus, the faculty prediction on the benefits of IBL is supported by our transcript data.

The evidence to date from math test results is less detailed. However, the above findings are corroborated by results of a pre/post test of mathematical knowledge for teaching (Hill, Schilling & Ball, 2004) given to students in IBL courses for pre-service teachers. The pre-to-post improvement in test score was greatest for students who answered fewer than 50% of items correct on the pre-test. That is, low-achieving students in IBL math courses for teacher preparation made greater gains than did their higher-scoring peers.

In sum, multiple measures of students' cognitive and affective outcomes from college mathematics courses taught with IBL methods indicate that students benefit from these approaches to teaching and learning. Indeed, in no case do the student outcomes favor the non-IBL group. And two groups of students who are often under-served by traditional courses benefit in particular from their experiences in IBL classrooms: women, who in many departments are underrepresented in mathematics, and students who are not already high-achievers in mathematics. Such positive outcomes of IBL instruction in college mathematics should justifiably get attention of undergraduate mathematics educators. IBL provides powerful tools for enhancing learning outcomes of undergraduate mathematics students, especially those under-served by the traditional college mathematics courses.

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On exemplification of probability zero events.

Simin Chavoshi Jolfaee

Abstract: In this study the example space of pre-service secondary teachers on probability zero events is examined. Different aspects of such events as perceived by the respondents are discussed and their perception of impossible events versus improbable is studied. The examples are categorised in terms of the type of sample space and once again categorised in terms of how do they fit the classic definition of probability. The role of measure theory to approach probability is briefly looked at via the examples. Meanwhile the participants' understanding of "more complicated" is explored and different ways they add complexity to their examples are analysed.
Keywords: example space, classic probability, impossible events.

Background

Extended attention to probability and statistics in school curriculum resulted in renewed interest in these topics in mathematics education research. Despite the variety of studies that explore understanding of probability concepts among students and teachers, (Lester 2006), little or no attention has been paid to zero probability events. My study aims at addressing this deficiency.

Theoretical framework

The importance of experiencing with examples has always been dealt with in theories and frameworks for describing the learning of mathematics. Watson & Mason (2005) define a concept as being aware of dimensions of possible variation and with each dimension, a range of permissible change within which an object remains an example of the concept. They also develop the idea of example space as collections of certain types of examples and suggest this idea as central in teaching and learning. Another study highlights that when invited to construct their own examples, learners both extend and enrich their personal example space, but also reveal something of the sophistication of their awareness of the concept or technique (Bills, 2006).

Goldenberg and Mason shed more light on the construct of example space and on how it can inform research and practice in the teaching and learning of mathematical concepts (Goldenberg, 2008).

Methodology: Participants and Task

The participants of this study were pre-service secondary school teachers (n=30), holding Majors or minors in mathematics or majors in science. There were asked to respond in writing to the following task. The time for completing the task was not limited.

Give an example of an event with probability zero.

Give an example of a more complicated event with probability zero.

The task was followed by classroom discussion around the general notion of probability zero events and the given examples.

The research questions were:

How do pre-service teachers interpret and exemplify probability zero events in variety of situations?

What is their personal example space with regard to events with probability zero?

Data analysis: First examples

The data were first analysed in terms of the respondents' perception of probability, which appeared to be in accord with the classical interpretation. The reasoning behind the given examples made it explicit that their common perception of probability is of a fraction (the ration of favourable events to all possible events). According to the different ways a fraction could be equal to zero (exactly or approximately), the examples were categorised into three groups:

- Zero divided by a non-zero: this type of examples was called "logically impossible" events. This category dominated the participants' example spaces. (Example: Rolling a 7 with a standard die for instance). Number of examples in this category: 50.
- A non-zero number divided by a constant large number: this type was referred to as "estimated to be zero" probability events. (Example: Flipping 10 coins all sitting in heads, an event with a probability 0.00097, which is estimated to be zero). Number of examples in this category: 3.
- A nonzero number divided by a sequence of numbers tending to infinity: this group of examples was called "events with probability converging to zero at limit". (Example: tossing a fair coin infinitely many times, all of them sitting in head). Number of examples in this category: 5

From a theoretical account a fourth type of examples was introduced as "measure-theoretically explainable probability zero". (Example: picking a certain number from a given interval of real numbers.) Two examples could fit this category, however, no evidence to a reference to measures in the sense that distinguishes a set of countable points versus a set of uncountable points was given. However, the classroom discussion suggested that this type of probability zero events could be understood from the point of view of each of the three aforementioned categories.

Moreover, the examples were examined in terms of the probability generators used to make a random experiment. The impact of classical textbook objects for teaching probability on the example space of the teachers is conspicuous.

From 60 examples, 32 use dice, 14 use coins, 8 use marbles in a bag (or equivalent variations of it), one uses a spinner, one uses a deck of cards, 2 use picking random numbers and 2 use real life objects such as vending machine and street crossway.

Data analysis: Second examples

Watson and Mason (2005) discuss the "give another example" strategy as a powerful instructional tool. From the examination of second examples in this study it turned out that in 24 out of 30 cases the first and second examples fell in the same category. I further examined how the participants have made their second example "more complicated".

It turned out that combining is quite a popular technique to get more complicated events. A total of 20 examples out of 30 were combining two events in order to give an example of a more complicated event.

Three different types of combination have been recognizable from the data:

The impossible-possible combination:

In this type of examples the impossible event described in first example is frequently used as the impossible component; first example is rolling a 7 with a fair die while second example is asking for rolling a 5 and then rolling a 7 with a fair die.

The impossible-impossible combination:

Some participants have conceived “more complicated” as an event even less likely to happen than their first impossible event. The second example is a combination of two probability zero events.

First example: Getting infinitely many 1’s when rolling a fair die infinitely many times1.

Second example: Getting all faces when flipping a coin infinitely many times while getting infinitely many 1’s when rolling a fair die at the same time.

The possible-possible combination with empty intersection:

Another way to get a “complicated” event was to combine the possible events in the sample space such that their intersection is empty, which at the same time makes the event logically impossible. The frequent example of this type was getting both 3 and 4 at the same time when rolling a fair die once.

As a second technique to add more complexity, some participants have used generalization; the second example is a generalized form of the first, so it is perceived to be both a zero probability event and a more complicated one. First example: rolling two dice and getting (6,7), second example: Rolling two dice and getting (i,j) such that i+j=13, for example. As Watson & Mason suggest, leading the learners toward generalization is one of the merits of asking for another or for a more complicated example.

Data Analysis: Number treatment

Any task designed for different research questions that deals in a way with numbers could reveal some by-product facts about people’s perceptions on numbers and part of their real number sense. The task described in this proposal is no exception. One of such interesting by-products is the different treatment of numbers found in two of the examples: in both examples the random experiment wsa to pick a random number from a real interval and the probability zero event was to pick a certain pre-determined number, 4.3275 and 1.0000097 respectively. It could not be helped but notice that the examples are of the same nature: they provide “safe” examples of numbers that are not likely to be picked. However, both respondents were aware of the fact that picking any number has the same probability zero, but they might feel that the numbers like 0,1,2 or 1/3 are not safe enough to mention. My conjecture is that the reason for such preference may be in that frequently students are asked to locate integers and simple fractions on the number line, but they are never asked to locate 1.0000097. The first numbers are then analogous to big bold dots or thick dashes on the number line; they are ‘exposed’ numbers as opposed to ‘anonymous’ numbers living safely in the oblivion of atom-size inseparable habitants of real line.

An interesting issue related to the definition of probability zero event surfaced in a discussion with participants and will be included in my presentation.

Summary:

The example space of 30 pre-service secondary teachers on probability zero events was studied through the examples they were asked to generate. Their example space was found to be rather limited and dominated by the standard probability teaching examples. Also the ‘expert’ example space appeared to be missing. ‘Complicatedness’ of examples was mostly represented in combination and generalization.

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Differences in Beliefs and Teaching Practices between International and U.S. Domestic Mathematics Teaching Assistants

Minsu.Kim

International Mathematics Teaching Assistants (MTAs) and U.S. domestic MTAs are an indispensable part of mathematics departments regarding teaching a substantial portion of undergraduate students. Because MTAs' beliefs are significant to their pedagogical methods, this study examines the contrast between international and U.S. domestic MTAs' beliefs and teaching practices. This research aims to answer the following questions: 1) What are the differences in beliefs and teaching practices between international and U.S. domestic MTAs? and 2) How are MTAs' different teaching practices shaped by their beliefs? The goals of this study are to help understand international and U.S. domestic MTAs' different approaches to education. The results indicate significant differences between the two groups centered on how they taught students to understand definitions and problems and how they motivated students to learn mathematics. The findings also describe MTAs' beliefs in relationship with their teaching practices.

Keywords: U.S. domestic mathematics teaching assistants (MTAs), international mathematics teaching assistants (MTAs), beliefs and teaching practices

After the graduate assistantship program was founded in the late 1800s, several researchers increased their interests in mathematics teaching assistants (MTAs) regarding diverse roles in universities and their potential influence on undergraduate education (Belnap, & Allred, 2006; McGivney-Burelle, DeFranco, Vinsonhaler, & Santucci, 2001; Latulippe 2007; Speer, Gutmann, & Murphy, 2005). Because MTAs teach a substantial portion of undergraduate students, MTAs' teaching practices are major potential factors that directly influence the students' perspective on mathematics and achievement in mathematics education (Commander, Hart & Singer, 2000; Speer, Gutmann, & Murphy, 2005). International MTAs also have become an indispensable part of mathematics departments. In the last two decades, international MTAs have been counted as a high percentage of the teaching assistants' population in mathematics departments in the U.S. Being interested in MTAs, I studied the literature related to MTAs' instructional practices, which contends that a variety of factors influence teachers' practices. In particular, teaching assistants' beliefs strongly influence their teaching practices (Speer, 1999, 2005, 2008; Thompson 1984, 1992). Because McGivney-Burelle, DeFranco, Vinsonhaler, & Santucci (2001) and Twale, Shannon, and Moore (1997) suggest that different educational experiences and philosophies influence MTAs' beliefs and pedagogical methods, I believe that there are significant differences in beliefs and teaching practices between international and U.S. domestic MTAs. The aim of this research is to answer the following two research questions: 1) What are the differences in beliefs and teaching practices between international and U.S. domestic MTAs? and 2) How are MTAs' different teaching practices shaped by their beliefs? To adequately answer these research questions, definitions and classifications of beliefs from the literature were used. In mathematics education, researchers defined beliefs as personal philosophical conceptions, ideologies, worldviews and values that shape practice and orient knowledge (Aguirre and Speer, 1999; Ernest, 1989; Speer, 2005). According to their definitions, beliefs are classified based on beliefs about mathematics, teaching, student learning and students (Cooney 2003; Cooney et al. 1998; Cross, 2009; Ernest 1989; Speer 2005, 2008; Thompson 1992).

To obtain my theoretical framework, based on Crotty's description, I have the objectivism view in epistemology. Since phenomena have meaningful entities of consciousness and experience, respectively, researchers find the objective truth and meaning of certain phenomena (Crotty, 1998, p.6). When certain phenomena are verified, the statement becomes meaningful and truthful. Even though research is able to attain the cause of the origin by being verified, I believe it is impossible to be only verified by experience based on Crotty's explanation about postpositivism. Researchers can only uncover approximate truth of phenomena instead of not finding the accurate truth with certainty of phenomena in the human experiences (Crotty, 1998, p29). Therefore, as a postpositivist, I believe that knowledge is created by the approximate cause or truth of phenomena through uncovering. Although phenomena cannot be verified by accurate truths or meanings, the research of the phenomena is important for the postpositivism perspective because researchers will discover approximate meanings and truths. Thus, the research explains well the phenomena and provides opportunities for readers to understand and accept these as knowledge. It is hard to determine the truths of the differences even though I discover regular patterns of the differences between MTAs' beliefs and teaching practices. For example, we do not have tools to determine accurately MTAs' beliefs. In addition, their beliefs often are inconsistent with their behaviors. Even though my research will not be verifying truths of the differences, I am able to discover regular differences. Through postpositivism and the uncovering of the differences in MTAs' beliefs and teaching practices, the answers to my research questions become knowledge and may help us understand what the differences in beliefs and teaching practices between international and U.S. domestic MTAs are.

As a case study in a qualitative research project, this study uses purposeful sampling (Creswell, 2007, p. 125). According to criterion sampling (Creswell, 2007, p.127), based on three criteria, I selected my participants: twelve MTAs that consist of six international and six U.S. domestic MTAs at the University of Oklahoma. The first criterion was that MTAs were in the Mathematics department at the University of Oklahoma. The second was MTAs' nationalities, such as international and U.S. domestic MTAs. One of the two groups was U.S. domestic MTAs who were born in the U.S., completed high school in the U.S., and spoke English as their native language. The other group was international MTAs who were born outside of the U.S., completed high school out of the U.S., and were non-native speakers of English. The third was that MTAs taught their own class during the spring semester of 2010.

Through triangulating (Creswell, 2007, p.209), I employed three different data sources: observation, questionnaires, and interviews with a digital voice recorder. From these three research instruments, data were gathered with the following procedures: 1) Observations and making condensed field notes and expanded field notes, 2) Questionnaires, and 3) Interviews with the participants with a digital recorder and transcripts of the digital voice recorder. After teaching observations, data were collected by using the aforementioned preceding, followed by an interview to not influence the participant's teaching. First, I observed my participants' classes for one class period during the spring semester in 2010 at the University of Oklahoma. I did not participate in their classes and made condensed field notes. I gathered the data of the questionnaire and then interviewed them in my office or their offices. The total time of the questionnaire (less than 15 minutes) and interview (less than 45 minutes) was less than one hour. I provided the questionnaire first because my participants were able to readily think about their teaching practices and beliefs before the interview. The interview was semi-structured with 12 open-ended questions with a digital voice recorder. The interview questions were six questions about their teaching practices and six questions about beliefs. I took notes in shorthand during the interviews. In addition, I did appropriate reaction and follow-up to probe questions to elaborate meanings of their responses.

I conducted my research with the intent of finding patterns and finally identify salient themes by inductive analysis. I frequently looked over the expanded field notes from observations, transcripts from interviews, and questionnaires. Using NVivo 8, software for analysis, through the transcripts, I made twelve sections based on the number of interview questions. In addition, I put codes on the expanded field notes to find their pattern about teaching practices. From the questionnaires, I could support the data of beliefs on the transcripts. I identified tentative codes from the database and reduced and combined the codes as I continued to review and re-review my database.

From analysis of the data, I have found the significant differences in teaching practices and beliefs between the two groups centered on how they taught students to understand definitions and problems and how they motivated students to learn mathematics. The international MTAs believed that understanding concepts were fundamental to learn mathematics. If students knew and understood concepts, they could solve all kinds of problems. According to the international MTAs' beliefs about teaching, they believed that teachers' abilities (background knowledge) and preparations of brief explanations of concepts were important for effective teaching of mathematics. In the literature reviews, beliefs strongly influence teaching practices. My results also support the statement. In addition, MTAs' beliefs about mathematics, teaching, students' learning, and students have close relationships with teaching practices. Thus, there is consistence between beliefs and teaching practices of international and U.S. domestic MTAs.

The international MTAs used problems as supplements to help students understand concepts because their intent was more for students to understand concepts, not problem solving. To help students to understand concepts, the international MTAs emphasized clear explanations of concepts and adjusted to the students' level. On the other hand, the U.S. domestic MTAs taught students to understand material by solving problems for students instead of spending much time explaining concepts. In addition, through solving problems, they showed that mathematics is useful and valuable. The U.S. domestic MTAs provided problems as much as they could that stressed main points because they wanted their students to understand concepts from the problems. In addition, the U.S. domestic MTAs believed that students were able to improve pattern recognition by solving many problems.

Regarding methods of how to motivate their students to pay attention in class and learn mathematics, the international MTAs used simple examples for motivation and asked students to solve them because the international MTAs focused on students understanding concepts. On the other hand, the U.S. domestic MTAs focused on explaining why concepts were useful and valuable to motivate students to learn mathematics. They stimulated students' motivation for learning mathematics and paying attention in class through explaining why these concepts are needed and why these problems are important. The U.S. domestic MTAs emphasized reasons to learn mathematics. Therefore, the different beliefs about mathematics, teaching, learning, and students significantly influence different teaching practices between international and U.S. domestic MTAs.

I anticipate that from the findings of the first research question, people in the academic community will gain an increased awareness of not only U.S domestic MTAs' but also international MTAs' teaching practices. In particular, the findings contribute to the academic community's knowledge of MTAs' practices and beliefs. The findings of the second research question provide opportunities to understand the relationships between MTAs' practices and beliefs, and support other researchers' assertions that beliefs have a noteworthy influence on MTAs' practices. In addition, I believe that this study will contribute essential resources for the body of knowledge about MTAs and the creation or adaptation of professional development programs for MTAs. By acknowledging international and U.S. domestic MTAs' different instructional practices and beliefs, mathematics departments will gain insight into the proper support needed by MTAs to improve their teaching methods. This information provides a good opportunity for readers to understand the differences between international and U.S. domestic MTAs' beliefs and teaching practices. This research will contribute to MTAs' teaching and knowledge, and will encourage faculty to be interested in the professional development of MTAs.

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Improving the Quality of Proofs for Pedagogical Purposes: A Quantitative Study

Yvonne Lai, Juan-Pablo Mejia Ramos and Keith Weber

In university mathematics courses a primary means of conveying mathematical information is by mathematical proof. A common suggestion to realize learning goals related to proof is to increase the quality of the proofs that we present to students. The goal of this paper is to investigate evidence related to the question: What changes to a proof do mathematicians believe will improve the quality of a proof for pedagogical purposes?

We present quantitative findings that corroborate hypotheses generated by a qualitative study of features of proofs that mathematicians find pedagogically valuable. Our work examines hypotheses related to typesetting, brevity, and the framework of a proof. One of our findings suggests that there is not a consensus among mathematicians what level of justification is desirable or necessary for the purposes of teaching undergraduates, though there may be common themes in the warrants they give for the level chosen.

Key words: proof evaluation, mathematicians, proof revision, quantitative study.

1. Introduction

In university mathematics courses, a primary means of conveying mathematical information is by mathematical proof. We hope that these proofs can convince students that a theorem is true, illustrate why a theorem is true, or illustrate new methods of reasoning (e.g., de Villiers, 1990; Hanna, 1990; Hanna & Barbeau, 2008). Unfortunately, both empirical and anecdotal evidence indicate that these learning goals often are not realized. Selden and Selden (2003) argue that as undergraduates do not have the ability to differentiate valid and invalid proofs (Selden & Selden, 2003; Weber, 2010), they cannot be gaining legitimate mathematical conviction from the proofs that they read. Many researchers report that students find the proofs they read to be confusing or pointless (e.g., Harel, 1998; Hersh, 1993; Porteous, 1986; Rowland, 2001).

A common suggestion to improve this situation is to increase the quality of the proofs that we present to students. For instance, some researchers argue that proofs should be more closely tied to informal arguments (Hersh, 1993), make explicit the proof's overarching structure while suppressing logical details (Leron, 1983), or illustrated with a carefully chosen generic example (Rowland, 2001). We attempt to build on this work by addressing the question: What types of modifications do mathematicians believe will improve the quality of a proof for pedagogical purposes?

Last year, at the 2010 RUME Conference, we presented the results of a qualitative study in which we asked eight mathematicians to revise two proofs intended for a calculus course for second- or third-year mathematics majors. We suggested a number of features in proofs that mathematicians find pedagogically valuable (see Lai & Weber, 2010). However, due to small sample sizes and the qualitative nature of our study, we qualified our findings as "grounded hypotheses". The goal of this study is to test these hypotheses with a larger number of mathematicians. Specifically, we aim to test the following hypotheses:

(H1) A proof for undergraduates can be improved if a hypothesis and conclusion statements are added to the proof that make explicit the proof framework (in the sense of

Selden & Selden, 1995) being employed. (In many undergraduate proofs, these are not explicitly stated and the proof framework is implicit).

(H2) Emphasizing important equations in a proof via typesetting will improve the quality of the proof because it will make clear the proof's main ideas.

(H3) Adding extra justification to support an assertion can improve the clarity of a proof if that justification might be difficult for a student to infer on their own.

(H4) Including unnecessary irrelevant computations or assumptions in a proof will make the proof worse since this will unnecessarily lengthen the proof and confuse students.

2. Theoretical assumptions

This work is based on three theoretical assumptions. First, understanding teachers' pedagogical beliefs is essential for modifying teachers' behavior (e.g., Aguirre & Speer, 1996). It follows that understanding what mathematicians believe constitutes a good proof for pedagogical purposes is necessary if we want to change the way that proofs are presented in university classrooms. Second, as experienced practitioners, mathematicians are a useful source of pedagogical content knowledge (Alcock, 2010). Consequently, mathematicians' views on how proofs might be improved are useful considerations for mathematics educators to consider. Third, small-scale qualitative studies are essential for developing grounded hypotheses in mathematics education research; however, these hypotheses need to be rigorously tested.

3. Methods

To test the hypotheses, we conducted an internet-based study. To seek participants, we sent e-mail requests to 28 mathematics departments inviting their faculty members, post-docs, and Ph.D students to participate in our study, providing a link to the website that they could click on to participate in the study. 110 mathematicians chose to participate. Methodological details about measures we took to insure the validity of this study, as well as empirical evidence that this approach is valid, are similar to the methods and arguments given in Inglis and Mejia-Ramos (2009).

In this study, participants were then shown the "master proof" below and told they would be asked whether changes to the proof would make it "less or more understandable to second or third year undergraduate students".

Proposition. *If $f : \mathbb{R} \to \mathbb{R}$ is differentiable and $f'(x) > 0$ for all $x \in \mathbb{R}$, then f is injective.*

Proof. Let $x_1, x_2 \in \mathbb{R}$, where $x_2 > x_1$. The Mean Value Theorem implies there exists $x_3 \in [x_1, x_2]$ such that $f'(x_3) = \frac{f(x_2)-f(x_1)}{x_2-x_1}$. Since, by hypothesis, $f'(x_3) > 0$ and $x_2 - x_1 > 0$, then $f(x_2) - f(x_1) = f'(x_3)(x_2 - x_1) > 0$. Therefore $f(x_2) \neq f(x_1)$. $\square$

Figure: "master proof" shown to participants.

The participants were then shown a screen with the master proof on top and a modified version of the master proof with the modifications in blue at the bottom and were asked to judge whether the changes made the proof "significantly better", "somewhat better", "the proofs were the same", "somewhat worse", or "significantly worse" (which we coded as 2, 1, 0, -1, or -2 respectively). This process was repeated five times with five different modified proofs. The order in which each proof was received was randomized.

(M1) We presented a proof where we added the sentence, "To show f is injective, we must show that $f(x_1) \neq f(x_2)$ after the first sentence of the master proof and the sentence "It follows that f is injective" as the last sentence of the proof. If H1 is correct, the participants should judge M1 to be an improvement over the master proof.

(M2) The formulas, $f'(x_3) = f(x_2) - f(x_1)/(x_2 - x_1)$ and $f(x_2) - f(x_1) = f'(x_3)(x_2-x_1) > 0$, were re-formatted to appear centered as their own lines. If H2 is correct, the participants should judge M2 to be an improvement over the master proof.
(M3) The last sentence of the proof was re-written as "Since $f(x_2) - f(x_1) \neq 0$, $f(x_2) \neq f(x_1)$". This added an extra justification that had previously been implicit to the proof. If H3 is correct, participants should judge M3 to be an improvement over the master proof.
(M4) We added the phrase "so $x_2 - x_1 = f(x_2) - f(x_1)/f'(x_3)$" immediately before the sentence beginning with "Since". While this inference is correct, it is not useful in the proof. If H4 is correct, participants should judge M4 as worse than the master proof.
(M5) We added the phrase "f is a real valued function" after the phrase "Since, by hypothesis". This assumption was not relevant to the subsequent argumentation. If H4 is correct, participants should judge M5 as worse than the master proof.
Finally, we gave the participants the option of commenting on why they made the judgment that they did.

3. Results

A repeated measures ANOVA revealed a main effect based on the modifications the participants' received ($F(327, 4) = 231.7$, $p<0.001$), meaning participants did not judge all modifications to be of equal quality. A summary of the results is given in the table below.

Condition	Mean score	# participants who thought proof was better	# participants who thought proof was worse
M1	1.29*	97	4
M2	1.05*	88	2
M3	0.02	41	40
M4	-1.66*	6	98
M5	-1.12*	7	94

* Indicates a mean score statistically different than zero with $p<0.001$.

These results confirm the predictions based on H1, H2, and H4. However, participants' responses for M3 fail to confirm H3 (the hypothesis that an extra justification will help students). Among the participants, 41 mathematicians thought adding the extra justification in M3 to the proof would make it better, with some giving reasons such as "I can imagine students being confused by the last step and this change would make it clearer". The 41 who thought the change lowered the quality of the proof gave responses that they thought students should make this inference easily, that students should be pushed to make these inferences on their own, or that the inference added was worded poorly. This suggests there is not a consensus among mathematicians for what level of justification is desirable or necessary for the purposes of teaching undergraduates.

4. Significance and future research

These results confirm several of the hypotheses proposed by the authors based on a qualitative study at last year's RUME conference (Lai & Weber, 2010). In particular, mathematicians believe that, for purposes of pedagogy, brevity is a desirable attribute of a proof but adding proof frameworks improves their quality. Formatting a proof by centering important equations also improves their quality. An interesting future research question is if the changes the mathematicians endorse would improve students' comprehension of proofs.

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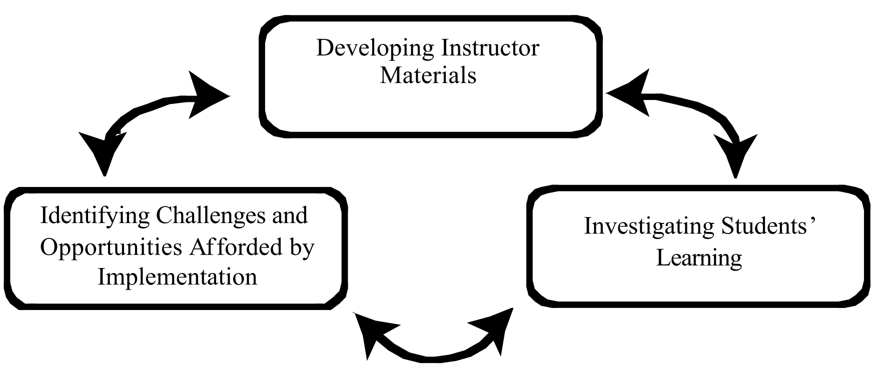
Putting Research to Work: Web-Based Instructor Support Materials for an Inquiry Oriented Abstract Algebra Curriculum

Sean Larsen, Estrella Johnson and Travis Scholl

For several years we have been researching students' and instructors' experiences with an inquiry-oriented group theory curriculum. This research has resulted in a number of insights; include findings that may be significant only for instructors and students engaged with this specific curriculum as well as findings that appear to have broader significance. We are putting this research to work as we develop web-based instructor support materials to accompany the curriculum. These materials include 1) information about the rationale for each task/sequence, 2) insights about how student thinking related to the task/sequence, and 3) discussion of task/sequence implementation considerations. In this presentation we will share some of our findings (both general and specific) and illustrate how we have incorporated these findings into the web-based instructor support materials in the form of text, video-clips, and images culled from our research efforts.

Key Words: abstract algebra, curriculum, teaching, student thinking

Questions or issues explored by the research. This report comes from a three-year collaborative project involving mathematics educators, mathematicians, and community college faculty members. The larger project has three primary goals, all related to a set of inquiry-based group theory curriculum materials. These goals are to 1) identify the challenges and opportunities that are likely to arise as different instructors implement the materials, 2) develop instructor support materials to meet the challenges and take advantage of the opportunities and, 3) investigate how students' learning is enhanced by the curriculum materials. These three project activities are tightly interrelated with each informing the other.



In our presentation we will touch on each of these activities. We will describe some insights from our investigations into students' learning as they interact with the materials. We will discuss some challenges/affordances we have identified in our observations of instructors' efforts to implement the curriculum. And we will demonstrate the online materials that we have developed to support instructors by providing selected video-clips, images, and exposition culled from our research efforts.

Relation of this work to the research literature / Theoretical perspective. Our work draws from and builds on two kinds of previous research. First, our work is related to research on undergraduates' thinking and learning in the area of abstract algebra and related concepts including functions and operations. In particular both our early design work and our ongoing investigations of students' learning have been informed by previous work on students' learning of the specific concepts such as isomorphism (e.g., Leron, Hazzan, & Zazkis, 1995) and quotient groups (e.g., Asiala, Dubinsky, Mathews, Morics, & Oktac, 1997). Second, the work is part of a growing effort to develop undergraduate mathematics instruction that is consistent with the design principles of the instructional design theory of Realistic Mathematics Education (e.g. Rasmussen & King, 2000; Rasmussen & Marrongelle, 2006; Zandieh & Rasmussen, 2010). Like these researchers, we have been concerned with both developing instructional approaches that support students' reinvention of mathematics and with contributing to the ongoing development RME theory.

Research Methodology. Our research program has featured multiple methodological approaches. The original design of the curriculum was supported by small-scale teaching experiments conducted with pairs of students as well as whole-class teaching experiments. Data resulting from these teaching experiments include video-recordings, students' written work, and researcher notes. More recently we have video-taped mathematicians teaching with the materials and conducted video-taped debriefing/planning sessions with them. Typically our analysis of the video data been inspired by the iterative approaches described by Cobb & Whitenack (1996) and Lesh & Lehrer (2000). More specifically, during early design stages, initial passes through the data would feature a search for informal student strategies that anticipated the formal mathematics targeted for reinvention while in subsequent passes we would strive to understand how these ideas could be evoked and how they could be leveraged to support the reinvention of the formal mathematics. Our more recent analyses have been focused on the mathematician's instructional moves. In this case, early passes were conducted to identify instances where implementation differed significantly from what was anticipated while later passes were conducted to search for explanations for the deviations. The web-based instructor support materials feature video-clips, exposition, and images that have been drawn from all of these component research activities.

Results. We will share various results from our research efforts. Some of these will be of a general nature and have implications beyond the context of our specific abstract algebra curriculum. For example, a finding that is emerging from our investigation of mathematicians' efforts to implement the curriculum is that what Ball, Thames, & Phelps (2008) refer to as *knowledge of content and students* is particular important for supporting the kind of *generative listening* (Yackel, Stephan, Rasmussen, & Underwood, 2003) required for successfully building on students' mathematical contributions (Johnson, Larsen, Rutherford, 2010). We will describe a couple of instances in which (in our analysis) mathematician's ability to listen generatively to their students was either supported or constrained by their knowledge (or lack of knowledge) of content and students. Then we will describe our general approach to providing information about students' thinking through the web-based instructor support materials and we will look specifically at our approach to addressing the issues involved in the shared episodes.

In addition to such general findings, we have accumulated a large number of smaller findings regarding how students will likely approach the various instructional tasks and what kinds of difficulties may emerge as they do so. We will share a selection of findings of this type and illustrate how they have informed the design of the curriculum itself and how they have been integrated into various aspects of the instructor materials.

Applications to/implications for teaching practice and future research. Our presentation is best conceptualized as a research-to-practice report. We will be describing our research program and some of our findings, but our main purpose is to share our efforts to develop the web-based instructor support materials. Our hope is that the presentation will contribute to ongoing efforts in our community to find ways to broaden the impact of instructional innovations emerging from our research.

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Students' Modeling of Linear Systems: The Car Rental Problem

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Abstract: In this talk, we characterize the nature of students' thinking about real-world problem situations that mathematicians might choose to reason about using ideas from linear algebra such as eigen theory, matrix equations, and/or systems of linear equations. We documented students working in groups on the "Car Rental Problem," a task that our research team specifically designed to elicit students' thinking about problem contexts that might be modeled in the aforementioned ways. We will describe the models students create to reason in this problem context, illustrating the variety in the final solutions of four different groups of linear algebra students and discuss the trends that appeared across the four groups as they worked toward their solution. Our analysis follows Lesh & Kelly's (2000) multi-tiered approach, and will focus on the mathematical topic areas drawn upon, the inscriptions created, and the quantitative reasoning that the students engaged in as they worked toward a solution.

Key Words: Linear Algebra, Modeling, Student Thinking

Students' Modeling of Linear Systems: The Car Rental Problem

Introduction:

Linear algebra has the potential to provide students with powerful tools for analyzing and understanding systemic problems in many areas of mathematics, engineering, and sciences. These tools include the use of systems of linear equations, matrices, and eigen theory for modeling real world phenomena. Research shows that students struggle to bridge their informal and intuitive ways of thinking with the formalization of concepts in linear algebra (Carlson, 1993; Dorier, Robert, Robinet & Rogalski, 2000).

Research Objective:

The central objective of this work is to characterize the nature of students' thinking about real-world problem situations that mathematicians might choose to reason about using ideas from linear algebra such as eigen theory, matrix equations, and/or systems of linear equations. The reason for this is two-fold: first, it offers insight into what it means to understand these ideas at a very fundamental level. Second, it offers insight into the informal and intuitive ways students have for thinking about these ideas -- ways that might then be leveraged instructionally. To this end, we documented students working in groups on a task that our research team specifically designed to elicit students' thinking about problem contexts that might be modeled in the aforementioned ways. We will refer to this task as the Car Rental Problem. In our talk, we will describe the models students create to reason in this problem context. We will illustrate the variety in the final solutions of four different groups of students and discuss the trends that appeared across the four groups as they worked toward their solution. The three questions that will guide our analysis are: (1) What mathematical topic areas do students draw upon to reason about such situations? (2) What are the nature and role of the inscriptions students develop to structure their thinking about such situations? (3) What role does quantitative reasoning play in the development of students' solutions?

In the car rental problem, students are presented with a scenario where there is a car rental company that has three locations in a city. Patrons of the company are allowed to return cars at any of the three locations and the problem describes what percent of cars from each location are returned where (see Figure 1).

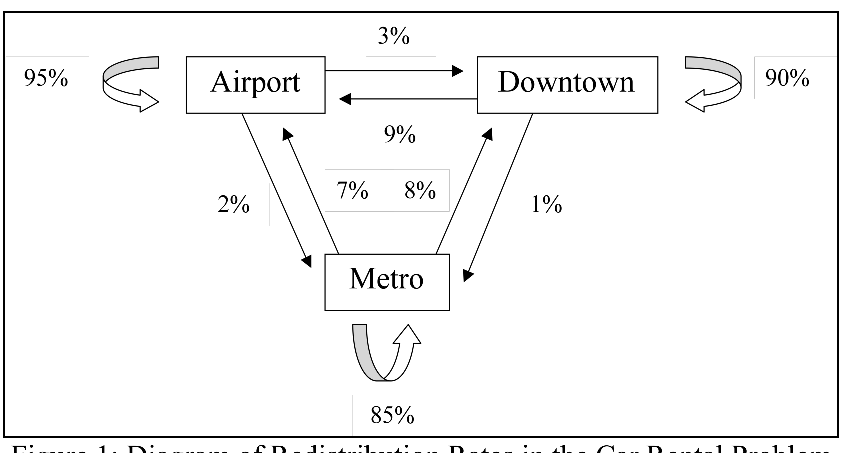

Figure 1: Diagram of Redistribution Rates in the Car Rental Problem

In particular, each week, about 95% of the vehicles rented from the Airport location are returned at the Airport location, about 3% rented at the Airport are returned Downtown, and about 2% of the cars rented from the Airport location are returned at the Metro location. Students are given an initial distribution of cars, and asked to describe the long-term distribution of the cars if the cars are returned at the described rates. They are also asked whether changing the initial distribution of cars would change the long-term distribution, and told to develop a business plan for the company so that they do not have to keep reshuffling the cars.

Theoretical Lens: Modeling & Quantitative Reasoning

The Models and Modeling (M&M) perspective adopts the view that many mathematically significant ideas that need to be learned by students originate from real world contexts, and that meaningful learning occurs when students are given the chance to reason about such ideas in their rich contexts (Lesh & Doerr, 2003). This work draws heavily on the literature and research tools developed in association with the M&M perspective in two important ways. First, we appeal to Lesh and Doerr's characterization of a model: "*Models* are conceptual systems… that are expressed using external notation systems, and that are used to construct, describe, or explain the behaviors of other system(s) – perhaps so that the other system can be manipulated or predicted intelligently" (2003, p. 10). Second, we have relied on the design principles associated with this perspective to inform the design of the task posed to students in this study.

In order to simplify our analysis of students' problem-solving efforts, we decided to focus largely on the quantitative reasoning that was related to each group's final solution. According to Kaput (1998), "Quantitative reasoning… can be regarded as modeling – building, usually in several cycles of improvement and interpretation, mathematical systems that act to describe and help reasoning about phenomena arising in situations" (p. 16). In our analysis we chose to use Smith and Thompson's (2008) description of quantitative reasoning; namely, quantitative reasoning is reasoning with and about quantities, where quantities "are measureable attributes of objects or phenomena; it is our capacity to measure them – whether we have carried out those measurements or not – that makes them quantities" (p. 10).

Methods

The students in this study were enrolled in an introductory undergraduate linear algebra course at a public university in the southwestern United States during the Spring of 2007. Successful completion of two semesters of calculus was prerequisite to the course, so students had a strong mathematical background. There were 32 students in the class, 8 of whom participated in the problem-solving interviews used as data for this study. Information is not available on the breakdown of the majors of the students in the class as a whole. However, of the 8 participants, 4 were electrical or computer engineering majors (one of whom had a double major in mathematics), 3 were computer science majors, and 1 was a graduate student in the business school. In the problem solving interviews, students worked in groups of 2-3 (although one student ended up working individually because the others in his group did not show up for their interview) for approximately 90 minutes. Data were collected on four groups of students working on the car rental problem. The first two groups completed the task about halfway through the semester, and the other two groups completed the task at the end of the semester. The interviews were videotaped and student work was collected.

Data were analyzed using Lesh and Kelly's (2000) multi-tiered approach to create accounts of the models created by each group of students. While our accounts of each group individually focus primarily on their final solution to the problem, we looked for themes across the problem solving sessions in order to identify points of commonality among the groups with regards topic areas, inscriptions, and quantitative reasoning.

Results

In the interest of space, we will focus our discussion here on just the first of our three research questions. Our first question is "What mathematical topic areas do students draw upon to reason about such situations?" Viewing each group's final solution individually illustrates the variety of topic areas that students drew upon. The students in Group 1 drew heavily on ideas from calculus to reason about the patterns they saw, considering them as sequences whose rates of change were decreasing. The students in Group 2 drew on their knowledge of computer programming and created general algebraic expressions for the computations to be performed. The students in Group 3 focused on a (perceived as constant) weekly rate of cars gained or lost, and drew on ideas from psychology and business (considering customer's needs and desires and working to meet them while maintaining a profit). The student in Group 4 drew primarily upon ideas from linear algebra, using a matrix to model the system of linear equations and reason about the system's behavior.

However, looking across the problem-solving activity of all groups showed that Group 1's idea from calculus (namely the general idea that 'if the change in the number of cars from one week to the next is decreasing, then the number of cars must converge') was echoed by nearly every group, even though none of the other groups appealed to the argument in their final solution. For example, early in his interview Matthew (Group 4) made a passing comment "Metro looks like it's decreasing the amount it's going down. Downtown is decreasing the amount it's going up, and so is airport. It seems like they're going to converge somewhere maybe." Group 3 offered a similar explanation very late in their interview (after they had written their final solution and been pushed by the interviewer to try to extend and generalize their argument). The only group that did not offer an argument of this nature was Group 2, who had focused their efforts on their computer simulation and never considered the difference in cars from one week to the next at a given location as a quantity of interest.

In our talk, we will delve into the second and third research questions as well, using examples of student work to illustrate themes that emerged across groups. We will illustrate the ways in which students' inscriptions (especially their symbolic expressions) served to support their quantitative reasoning. We will also argue that in students' solutions, quantitative reasoning served as a basis for the aforementioned symbolic expressions students developed to aid computation and further symbolization. Across groups, we show how the inter-relatedness of the quantities created a need for computational efficiency as well as symbolism that (1) supports computational needs, (2) represents interrelatedness of quantities, and (3) aids in conceptualization of the system as a whole.

Final Remarks

In a way, this work serves to illustrate the ways in which students draw on their experiences and coordinate multiple topic areas as they engage in new mathematical problems. Here, this highlights the need for students to expand their mathematical horizons with the additional computational and notational tools linear algebra has to offer. Having conducted this analysis, we are now exploring ways in which this task can be leveraged to help students develop ideas about matrix multiplication as a tool to aid in computation and modeling of systemic level change.

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Navigating the Straits: Critical Instructional Decisions in Inquiry-Based College Mathematics Classes

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Abstract

Inquiry-based learning (IBL) approaches engage college mathematics students in analyzing and solving problems and inventing and testing mathematical ideas for themselves. But to effectively apply IBL teaching methods, instructors must make good decisions both in planning their syllabus, assignments, and assessment before the term begins, and in the moment, as they monitor classroom progress, manage interpersonal dynamics, and decide what to do when things do not go as planned. Using interview data from 40 IBL instructors at four campuses, including graduate teaching assistants and faculty at a range of experience levels, we identify critical instructional decisions that can affect the success of IBL classes. We describe why these decisions are more salient in IBL classrooms than in those using lecture-based methods, and we examine patterns in instructors' ability to identify these issues for themselves and suggest appropriately nuanced solutions to common IBL classroom dilemmas.

Keywords: inquiry-based learning, teaching assistants, faculty, qualitative methods, instructional methods

Introduction

The term inquiry-based learning (IBL) is used to describe approaches to college mathematics that place student discovery of mathematical ideas at the center of the classroom. Rather than emphasizing rote memorization and computation skills, IBL methods seek to help students develop critical thought processes by exploring ill-defined problems, applying logic, making and analyzing arguments. Moreover, by building students' confidence in their abilities to generate and critique ideas and to solve problems independently, IBL methods help to foster students' creativity, persistence and intellectual growth (e.g., Buch & Wolff, 2000). Like "discovery learning" (Bruner, 1961), "problem-based learning" (Savin-Baden & Major, 2004), and other "inductive teaching" approaches (Prince & Felder, 2007), IBL invites students to work out ill-structured but meaningful challenges. In mathematics, IBL approaches are often derived from the work of Texas mathematician R. L. Moore, but while Moore emphasized individual learning, modern implementations of IBL in mathematics draw importantly on social learning perspectives (e.g., Lave & Wenger, 1991; Vygotsky, 1978).

Our research group has studied IBL classrooms at four university "IBL Centers" funded by a private foundation. Since 2004, a cadre of faculty on each campus has been engaged in developing inquiry-based undergraduate mathematics courses for upper- and lower-division mathematics majors, science and engineering students taking math as a cognate, and pre-service teachers. These courses engage students in creating, exploring and communicating mathematical ideas, guided by faculty and critiqued by peers. Our large, mixed-methods study of these courses includes two years' of survey, interview, and test data from over 100 discrete class sections, and academic records from over 6000 students. Much of our study focuses on student outcomes of IBL courses; here we highlight instructors' experiences in teaching IBL mathematics courses.

Methods

Interviews were conducted with 40 IBL instructors at four campuses,18 graduate student teaching assistants (TAs) and 22 faculty. The general term 'instructors' recognizes the range of classroom roles represented; some graduate students were lead instructors, and others, though nominally TAs, had more IBL teaching experience than did their faculty member. Faculty included pre-tenure, tenured, non-tenure-track, and visiting (postdoctoral) faculty. The semi-structured interview protocol established instructors' career status and IBL involvement; explored instructors' teaching style, beliefs, and specific classroom practices; and asked for their observations of student learning gains (or lack of gain) and personal or professional benefits and costs to themselves. Most of the interviews were conducted in person; a few were done by phone. Interviews of 45-70 minutes were digitally recorded and transcribed *verbatim*. These data are complemented by a set of individual and focus group interviews with 68 IBL students.

The text data were coded using a mix of inductive and deductive codes. A total of 164 codes were generated under six main domains (Spradley, 1980). Instructors' observations of student learning gains and learning processes were coded using a scheme previously developed for coding student data, so that these two data sets could be compared. Instructors' reports about their own teaching practices, context, and educational beliefs, and about the outcomes of IBL teaching for themselves and their departments, were coded and subjected to taxonomic analysis to develop sub-categories and identify analytical themes. Here we focus on the broad category of codes labeled as "teaching processes"; this category included 14 subcategories and constituted the bulk of the coded data, over 900 coded passages or individual instructor observations.

Findings

In interviews, instructors often told us in copious detail about specific practices, such as how they graded homework assignments, how they assigned class participation points, or how they selected students to present at the board. After puzzling for a while over the evident importance of these details to instructors, we came to recognize that, collectively, these reports delimited a shared set of teaching concerns. Each instructor described an idiosyncratic practice developed for a particular course, student audience, and personal style, and seldom couched these in philosophical terms; but as analysts, we could abstract from these narratives the teaching dilemmas that every instructor had to resolve. The general issues do not differ from those encountered by more traditional instructors—choices about curriculum, classroom atmosphere and management, and student assessment. But for IBL instructors, these become "critical instructional decisions" that are especially salient for several reasons:

- Most decisions become *more explicit*: Traditional teaching may be based on received wisdom held by both students and instructors, often unquestioned, about how things are or should be. Instructors may not have faced these choices explicitly in prior teaching.
- Decision-making becomes *more dynamic*: With class activities and pacing often in students' hands, IBL instructors must respond in the moment rather than follow prepared notes. This requires alertness to the classroom atmosphere and attention to student responses.
- Some aspects of class are *more sensitive* to teaching decisions: Because IBL classrooms rely more on collaborative learning, decisions that affect student participation and the classroom

atmosphere may have greater consequences for the success of a course. Choices about these factors intersect extensively with issues of individual accountability and work load. With greater responsibility for everyday work, students' need for available and appropriate help can become more acute, affecting the use and tenor of office hours.

- Some issues require that *new solutions* be found: As learning goals shift away from content coverage and toward skills such as constructing and communicating mathematical arguments, past solutions (e.g., assessing learning by timed, individual tests) may no longer measure what instructors value or may not mesh well with students' experience of the class as a whole.

We will present a research-based framework, drawn from the interview data, that organizes the critical decisions that IBL instructors must make in designing and running their courses, including setting the tone and expectations for students; managing interpersonal dynamics; balancing student accountability and participation; setting curriculum; and evaluating student learning. We will use examples from student and instructor interviews to show how and why instructors' decisions on these points become critical for the effectiveness of an IBL class. In some cases, the decisions are highly interdependent. For example, practices intended to increase student accountability for the homework that is the basis of the next day's class work can make it difficult to establish a positive and collaborative classroom atmosphere, but too little attention to accountability can lead to student over-dependence on others to do the work and thereby reduce the level of class participation. The instructor is thus easily caught between a rock and a hard place—the straits noted in the title.

The interview data also indicate some group differences in instructors' decision-making. For example, local "styles" of IBL teaching on each campus affected the choices that instructors made. Sometimes these shared styles meant that colleagues were good sources of teaching advice and wisdom, but sometimes they also constrained the solution space, restricting the range of teaching choices that were seen as possible or desirable. Instructors also varied in their ability to identify critical classroom issues for themselves. TAs in particular were often able to identify subtleties in how classroom decisions affected student behavior. For example, in explaining "what worked" or did not work in their classroom, TAs were more likely to offer explanations that hinged on the nature of instructional decisions or the quality of their implementation, while faculty more often gave explanations in terms of student characteristics, such as work ethic, ability and preparation. TAs' distinctive perspective appeared to arise from multiple sources: their dual classroom roles as teachers and observers; their nearness to the student experience—both as recent undergraduates themselves and from direct work with students in office hours or help sessions; and their lower identity investment in the perceived success of the course.

Previous studies have taken a close look at particular instructors' practices in one or two classrooms (e.g. Weber, 2004; Rasmussen & Marrongelle, 2006). Here, by examining the practices of a large sample of teachers and across a wide range of IBL styles, we establish that certain teaching issues are quite commonly confronted by IBL instructors. Knowing *what* these issues are, and how they appear to mathematics instructors, we have the chance to better understand *how* particular instructional choices may affect student outcomes in positive or negative ways. The findings thus contribute to the small existing body of empirical research on the "unexamined practice" of college mathematics teaching (Speer, Smith & Horvath, 2010). Lastly, our findings have significant practical implications for designing professional development to increase the uptake and improve the implementation of IBL methods across the

U.S. We seek to develop a framework that can help IBL instructors consider and analyze their instructional decisions, and predict (or at least attend to) the classroom consequences of their choices; and that can provide them with a common language to recognize shared problems and swap solutions despite differences in student audience, course content, and institutional setting.

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Student Perceptions of an Explicitly Criterion Referenced Assessment Activity in a Differential Equations Class

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This report presents the findings of a study into the perceptions held by students regarding the use of criterion referenced assessment in an undergraduate differential equations class. Students in the class were largely unaware of the concept of criterion referencing and of the various interpretations that this concept has among mathematics educators. Our primary goal was to investigate whether explicitly presenting assessment criteria to students was useful to them and guided them in responding to assessment tasks. The data and feedback from students indicates that while students found the criteria easy to understand and useful in informing them as to how they would be graded, it did not alter the way the actually approached the assessment activity.

Keywords: differential equations, assessment experiment, criterion referenced assessment

Introduction

Generally speaking, Australian Universities impose or very strongly encourage the use of criterion referenced or standards based assessment in the courses that they offer. At the authors' home institution this is no different with the University's Manual of Policies and Procedures stating that the University "has adopted a criterion-referenced approach to assessment where assessment is based on pre-determined and clearly articulated criteria and associated standards of knowledge, skills, competencies and/or capabilities." In a sense, this directive has been largely ignored in the context of many quantitative courses such as those in mathematics and science by offering justifications that in quantitative studies assessment responses are either right or they are wrong and that is sufficient for a criterion. In this study we report on the successful implementation of elements of criterion referenced assessment into a Differential Equations course that goes beyond simple "right-wrong" criteria while maintaining the mathematical integrity of the assessment program. Furthermore, we present findings based on quantitative and qualitative feedback from students regarding their perceptions of criterion referencing and how it is used in guiding their learning throughout the course.

It is important to place this study in context by comparing the assessment experiment with the methods previously used to assess students in the course. Over approximately the past 10 years, the course has been taught by a number of people, however the assessment strategy has essentially been to employ 1-2 assignments (problem solving tasks with a 2-4 week completion timeframe) and a mid-semester and final examination. These tasks generally contribute 30-40% (assignment) and 60-70% (examination) of the final grade for the course, respectively. Grading of all tasks has been quite traditional in the sense that the academic responsible for assessment writes their own set of "correct" solutions and assigns points or marks throughout the solutions corresponding with reaching certain points in the solution process.

In the assessment experiment reported on in this paper, we have attempted to maintain the previously employed assessment program as much as possible. In particular, we maintained progressive, non-examination assessment of 40% and used mid-semester and final examination contributing 60% of the students' final grades. However, we implemented an explicit criterion referenced method of grading students in the assignment tasks completed during semester. This involved presenting students with a set of criteria and standard definitions in addition to the actual problems to be solved. Students were provided with details of exactly how responses to the mathematical problems would be graded and how translation between the mathematics and the standards and criteria would be carried out.

Our goals in conducting this experiment fall into two main areas: to gauge students' perceptions regarding criterion referenced assessment and its usefulness, and to a lesser extent, effecting culture change among mathematics academics. With regard to students' perceptions, we investigated how students viewed the understandability and the usefulness of criterion referencing and how they employed the additional information provided to them via the criteria and standards definitions in directing their learning and assessment responses. Implicitly, we believe that such an investigation and its results can then be used to effect culture change among mathematics teachers at universities by changing the way they view criterion referenced assessment, taking CRA from a directive imposed by administrators to a useful tool for mathematics learning.

Introductory literature review and placement of this research

Niss (1998, in Pegg 2003, p.228) notes that mathematics assessment identifies and appraises the knowledge, insight, understanding, skill and performance of a student. Pegg however points out that this is not in fact the reality of assessment in mathematics and that rather, it is most often concerned with reproduction of facts and computational skills or algorithms (Pegg 2003). It is our contention that this is how previous years' assessment programs for the course under investigation have been presented to students. In the assessment experiment discussed in this report, we attempt to explicitly link the subtasks of the assessment activities with the learning outcomes of the course, which include such concepts as knowledge, insight and understanding in addition to skills. In this way we believe that our assessment becomes more of an educational tool for students than it has been in previous versions of the course, and that it allows for a more "constructive alignment" (in the sense of Biggs, 1996) of the content, pedagogy and assessment.

Criterion referenced assessment involves determining the extent to which a learner achieves certain predetermined goals or criteria, importantly, without reference to the performance of others (Brown, 1988; Harvey, 2004; TEDI, 2006). The implementation of CRA involves the design or statement of a set of learning outcomes for a course, design of a program of assessment to obtain information about a student's performance in relation to the learning outcomes, and the presentation of a criteria set and definition of standards which serves to both inform students how their performance will be judged and to provide directions for assessors.

Pegg (2003) notes that while the movement towards assessment based on outcomes and standards (rather than individual comparison) did initially have some basis in research regarding student learning, the links remain tenuous. As such, there is debate among teachers and academics alike as to whether the claims regarding the benefits of criterion referenced assessment are supported by strong research. Through research such as that presented in this study, we attempt to provide a research base that advocates the benefits and warns of the pitfalls of criterion referenced assessment in the undergraduate mathematics classroom.

Theoretical perspective/conceptual framework

In this study we carry out descriptive research related to questions around student perceptions and criterion referenced assessment. This descriptive research involves statistical and textual analysis/synthesis of data collected from a student population undertaking a course in differential equations in an attempt to understand student perceptions and provide guidance for academic staff in undertaking more useful assessment in mathematics courses.

Methods

We have used two primary data sources, one quantitative and one qualitative, in an attempt to address our research goals regarding student perceptions of criterion referenced assessment. The first source was a survey allowing free-text responses on two questions of interest, while the second was a 10-item survey using a 5 level Likert scale. Both surveys were conducted at the end of the course of study, following the provision of feedback to students on the criterion referenced items and also following the post mid-semester exam feedback sessions. All 56 students in the cohort were offered the chance to respond, with a 30% response rate achieved. Another source of data that will be commented upon, although to a lesser extent, are the assessment responses themselves. Numerical and statistical analyses of the Likert-survey were conducted, while textual analysis and synthesis was carried out on the free-text responses.

Results

Quantitative data collected via the second of the student surveys indicates that while students found assessment criteria easy to understand and useful in informing them as to how they would be graded, it did not alter the way the actually approached the assessment activity. Qualitative feedback from almost 100% of respondents indicated that in general the criteria provided were not used to determine how a student would approach individual questions or the assessment tasks as a whole. Interestingly, a similar percentage of students stated that they found CRA beneficial as it made the process of allocating scores by graders much clearer. A small percentage of students indicated that they did refer to the criteria sheets after the tasks were graded in order to get a different, higher level representation of where they had made errors in their responses.

Implications/Applications

This research study has opened up new questions for future research. For example, we are now considering the impact on graders/academics and the usefulness they perceive in employing criterion referenced assessment.

With regard to application in the classroom in the future, both the qualitative and quantitative data indicate that students and graders alike, need to be explicitly informed exactly why they are provided with criteria and how they can be used to assist learning. In particular, guiding them in their response attempts (showing them what the grader will deem to be "important") and also aiding them in understanding the feedback they receive following the grading of their work. Furthermore, the actual construction of the criteria and standards is by no means straight forward – but it is important, because these are exactly the types of judgements we are normally making in an implicit, contentcentred manner. Academic staff need to be closely guided in the development of these elements of any criterion referenced assessment strategy.

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Reaching out to the Horizon: Teachers' use of Advanced Mathematical Knowledge

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This paper explores teachers' use of advanced mathematical knowledge (AMK) – that is, the knowledge acquired during undergraduate university or college mathematics courses. In particular, our interest is in the use of advanced mathematical knowledge as an instantiation of knowledge at the mathematical horizon (KMH). With this tie to undergraduate mathematics education, we re-conceptualize the notion of knowledge at the mathematical horizon and illustrate its value with excerpts from instructional situations.

Key words: advanced mathematical knowledge; horizon knowledge; group theory; calculus

Knowledge at the mathematical horizon is a category of teacher's knowledge included by Hill, Ball and Schilling (2008) in their refinement of Shulman's classic categorization of teacher's Subject Matter Knowledge (SMK) and Pedagogical Content Knowledge (PCK). While Hill et al. (2008) develop several subcategories of SMK and PCK, they omit elaboration on KMH – Knowledge at the Mathematical Horizon. Our interest, in this paper, is to explore the idea of teachers' knowledge at the mathematical horizon, what is there, how it may be used, and to what benefit. We develop a notion of KMH, which is influenced by educational and philosophical perspectives, and explore examples that illustrate how KMH in conjunction with knowledge acquired in undergraduate studies at university or college – that is, Advanced Mathematical Knowledge, AMK (Zazkis and Leikin, in press) – may be used to advantage in teaching and in teacher education.

At the horizon of teachers' knowledge

Ball, Thames, and Phelps (2008) describe horizon knowledge as "an awareness of how mathematical topics are related over the span of mathematics included in the curriculum (p. 403), while Ball and Bass (2009) explain in more detail the notion of mathematical horizon:

> We define horizon knowledge as an awareness – more as an experienced and appreciative tourist than as a tour guide – of the large mathematical landscape in which the present experience and instruction is situated. It engages those aspects of the mathematics that, while perhaps not contained in the curriculum, are nonetheless useful to pupils' present learning, that illuminate and confer a comprehensible sense of the larger significance of what may be only partially revealed in the mathematics of the moment (p. 6).

They further describe their idea of horizon knowledge as consisting of four elements (ibid):

1) A sense of the mathematical environment surrounding the current "location" in instruction.
2) Major disciplinary ideas and structures
3) Key mathematical practices
4) Core mathematical values and sensibilities

Their attention, however, seems to be focused on teachers' knowledge of *students'* mathematical horizon. For instance, they remark, "that teaching can be more skillful when teachers have mathematical perspective on what lies in all directions, behind as well as ahead, for their pupils, that can serve to orient their navigation of the territory" (p.11).

Horizon knowledge also has philosophical roots in Husserl's notions of 'inner' and 'outer' horizon. Briefly, Husserl's notion of inner horizon corresponds to aspects of an object that are not at the focus of attention but that are also intended, while the outer horizon of an object includes features which are not in themselves aspects of the object, but which are connected to the world in which the object exists (Follesdal, 1998, 2003). Connecting this notion to mathematics, and to Ball and Bass's (2009) description, we interpret the inner horizon of a mathematical object as the features of the object which are not at the focus of attention, but which surround its current "location", and this includes major disciplinary ideas and structures. The outer horizon, that which is not part of the object but is connected to the 'world', includes key mathematical practices, values, and sensibilities.

In this paper we extend the idea of knowledge at the mathematical horizon by focusing on teachers' 'inner' horizon knowledge and exemplifying the value of knowledge acquired during teachers' undergraduate studies in mathematics (their AMK) as it informs their understanding of the mathematical environment and major disciplinary ideas and structures.

Horizon and AMK

Teachers' horizon knowledge is, for us, deeply connected to their knowledge of advanced (university/college level) mathematics – that is, to their AMK (Zazkis and Leikin, in press). We consider application of AMK in a teaching situation as an instantiation of KMH. Our view is influenced by the metaphor of horizon as a place "where the land meets the sky" and we interpret this as the place where advanced mathematical knowledge of a teacher (the sky) appears to meet mathematical knowledge reflected in school mathematical content (the land). In what follows, we offer two examples of teachers' KMH.

Example 1

Miss Scarlett's Grade 12 students had just finished a unit on inverse functions. The unit test had been poorly done; Miss Scarlett observed several instances of confusion in notation and this was leading to miscalculations among other errors. The majority of her students were writing $1/f(x)$ where they meant $f^{-1}(x)$, and she suspected that students were unclear as to when the reciprocal of a function was, or was not, also its inverse.

Miss Scarlett decided to take time clarifying this confusion when taking up the test. She illustrated with a handful of examples instances when the reciprocal and the inverse are the same function and when they are not. She also recalled students' work with reciprocal and inverse of numbers, noting that the reciprocal of a number depends on the operation of multiplication, but that the inverse of a number can refer to its additive inverse or its multiplicative inverse (the latter being equivalent to the reciprocal).

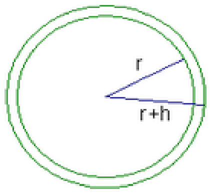
The concept of inverse is one that is prevalent in many mathematics courses, however it was during a university course in group theory that Miss Scarlett acquired an understanding of the inverse of a group element with respect to the particular operation of that group. For example, the set of integers with a corresponding operation of addition has a group structure – it includes an identity element, is closed with respect to addition, and necessarily contains inverses but *not* reciprocals. Miss Scarlett drew on this understanding to help her address her students' confusion. A similar instance of confusion and resolution was reported in (Zazkis and Zazkis, 2011) where a teacher used her understanding of group theory to help her student interpret the meaning of an exponent of negative one.

Example 2

During a lesson on applications of derivatives, Mrs. Peacock's pre-calculus students were given a set of 'real-world' problem in which they were to take derivatives of various formulae. The lesson was designed to reinforce calculation techniques through application to standard word problems. The students were unfamiliar with limits, as it was not part of the course curriculum.

As the class worked on their exercises, one student noticed when working with the sphere and circle, that the derivative of the volume formula yielded the formula for surface area, and the derivative of the area formula yielded the formula for circumference, respectively. That is, $\frac{dV}{dr} = \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$ and $\frac{dA}{dr} = \frac{d}{dr}(\pi r^2) = 2\pi r$. The student asked why this relationship held for the sphere and the circle, and not in other cases such as with the cube and square.

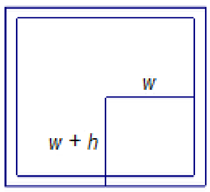
The connection between surface area and volume is one that Mrs. Peacock made during a university calculus course. She recalled a geometric representation for the derivative of a circle's area, and was aware of an analogous argument for the derivative of a sphere's volume. Mrs. Peacock understood the significance of the diagram:



and knew that the derivative of the volume (for e.g.) could then be defined as:

$$\lim_{h\to 0}\frac{\frac{4}{3}\pi(r+h)^3-\frac{4}{3}\pi r^3}{h}=\lim_{h\to 0}4\pi r^2+4\pi rh+\frac{4}{3}\pi h^2=4\pi r^2.$$

She further knew that it was possible to similarly represent and define the derivative of the volume of a cube. She recalled the figure:



where w is equal to half the length of one side. From this diagram, she knew that the derivative of the volume of the cube could be written as:

$$\lim_{h\to 0}\frac{8(x+h)^3-\ 8x^3}{h}=\lim_{h\to 0}24x^2+24xh+\ 8h^2=24x^2.$$

While it was beyond the scope of the lesson to introduce the definition and calculation of limits in this class, Mrs. Peacock gave an intuitive and geometric explanation for why this relationship holds. It was her knowledge of mathematics acquired in her university studies that heightened her awareness of the important observations her student had made and of the potential connections that might result.

Discussion

The two examples presented above illustrate how teachers' knowledge of major disciplinary ideas and structures beyond what was addressed in the secondary school curriculum (e.g. group structures and inverses; derivatives and geometric interpretations of limits) were useful in a teaching situation. The knowledge they acquired in university – their AMK – was not at the focus of their attention, however they were able to recognize its applicability and connection to the mathematics in question, and to access it easily and flexibly in order to address students' questions. In particular, Miss Scarlet and Mrs. Peacock were able to apply their AMK in a way that resonated with their students, and that showed their "sense of the mathematical environment surrounding the current 'location' in instruction" (Ball and Bass, 2009, p.6).

Although related to the work of Ball and Bass (2009), our notion of knowledge at the mathematical horizon differs from what they describe as "a kind of elementary perspective on advanced knowledge" (2009, p. 10). Rather, we see it as an advanced perspective on elementary knowledge. That is, as advanced mathematical knowledge (AMK) applied to ideas in the elementary or secondary (or undergraduate) curriculum. The two examples focus on what we interpret as the "inner" horizon of function or derivative – aspects of these mathematical objects that are beyond the scope of the student, but that are fundamental to the object and within the grasp of the teacher. We seek to further explore teachers' and prospective teachers' KMH with a particular focus on what could be included in undergraduate mathematics education and teacher preparation in order to encourage a flexible use of KMH.

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Students' Reinvention of Formal Definitions of Series and Pointwise Convergence

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The purpose of this research was to gain insights into how calculus students might come to understand the formal definitions of sequence, series, and pointwise convergence. In this paper we discuss how one pair of students constructed a formal ε-N definition of series convergence following their prior reinvention of the formal definition of convergence for sequences. Their prior reinvention experience with sequences supported them to construct a series convergence definition and unpack its meaning. We then detail how their reinvention of a formal definition of series convergence aided them in the reinvention of pointwise convergence in the context of Taylor series. Focusing on particular x-values and describing the details of series convergence on vertical number lines helped students to transition to a definition of pointwise convergence. We claim that the instructional guidance provided to the students during the teaching experiment successfully supported them in meaningful reinvention of these definitions.

Keywords: Reinvention of Definitions, Series Convergence, Pointwise Convergence, Taylor Series

Introduction and Research Questions

How students come to reason coherently about the formal definition of series and pointwise convergence is a topic that has not be investigated in great detail. Research into how students develop an understanding of formal limit definitions has been largely restricted to either the limit of a function (Cottrill et al., 1996; Swinyard, in press) or the limit of a sequence (Roh, 2010). The general consensus among the few studies in this area is that calculus students have great difficulty reasoning coherently about formal definitions of limit (Bezuidenhout, 2001; Cornu, 1991; Tall, 1992; Williams, 1991). The majority of existing research literature on students' understanding of sequences and series concentrates on informal notions of convergence (Przenioslo, 2004) or the influence of visual reasoning or beliefs (Alcock & Simpson, 2004, 2005). Literature on pointwise convergence is typically in the context of Taylor series addressing student understanding of various convergence tests (Kung & Speer, 2010), the categorization of various conceptual images of convergence (Martin, 2009), the influence of visual images on student learning (Kidron & Zehavi, 2002), and the effects of metaphorical reasoning (Martin & Oehrtman, 2010). We recruited a pair of students from a second-semester calculus course incorporating approximation and error analysis as a coherent approach to developing the concepts in calculus defined in terms of limits (Oehrtman, 2008). The goal that they reinvent the formal definitions of sequence, series, and pointwise convergence. For this paper we posed:

1. What are the challenges that students encountered during guided reinvention of the definitions for series and pointwise convergence?

2. What aspects of the students' definition of sequence convergence supported their reinvention of series convergence? What aspects of the students' definition of series convergence supported their reinvention of the definition of pointwise convergence?

Theoretical Perspective and Methods

To investigate our research questions, we adopted a *developmental research* design, described by Gravemeijer (1998) "to design instructional activities that (a) link up with the informal situated knowledge of the students, and (b) enable them to develop more sophisticated, abstract, formal knowledge, while (c) complying with the basic principle of intellectual autonomy" (p.279). Task design was supported by the *guided reinvention* heuristic, rooted in the theory of Realistic Mathematics Education (Freudenthal, 1973). Guided reinvention is described by Gravemeijer, K., Cobb, P., Bowers, J., and Whitenack, J. (2000) as "a process by which students formalize their informal understandings and intuitions" (p.237).

The authors conducted a six-day teaching experiment with two students at a large, southwest, urban university. The full teaching experiment was comprised of six, 90-120 minute sessions with a pair of students who were currently taking a Calculus course whose topics included sequences, series, and Taylor series. The central objective of the teaching experiment was for the students to generate rigorous definitions of *sequence convergence*, *series convergence*, and *pointwise convergence*. The research reported here focuses on the evolution of the two students' definitions of series and pointwise convergence over the course of the last three sessions of the teaching experiment following the students' reinvention of a formal definition of sequence convergence. The design of the instructional activities was inspired by the *proofs and refutations* design heuristic adapted by Larsen and Zandieh (2007) based on Lakatos' (1976) framework for historical mathematical discovery.

The teaching experiment activities on series began with students producing and subsequently unpacking details of convergent series graphically. We then asked the students to generate a definition by completing the statement, "A series converges when…" To address pointwise convergence, we asked the students to produce a graph of e^x with several approximating Taylor polynomials and discuss several details of convergence on the graph. The students where then prompted to talk about what Taylor series were, and finally instructed to produce a definition for Taylor series convergence. The majority of each session consisted of students' iterative refinement of a definition and the unpacking of their intended meanings for individual elements within each definition.

Results

The reinvention of series convergence began at the end of day 3 and continued on into the 4th day. The students initially drew a graph of an alternating series, and after considering the harmonic series, they remembered that the series was divergent. When considering other formulas they were unable to produce another graph besides alternating series graphs. However, when prompted to not focus on finding a formula, the students compared these graphs to sequences and expressed that series graphs "are harder to throw out there." After they stopped focusing a formula they were able to produce a series graph increasing toward 7 and partial sums eventually became constant. Afterwards, these graphs were available for the students to refer to when defining series convergence. The students' initial definition of series convergence to 7 was simply that a series converges when "the a_n's are going to 0 and s_n's are going to 7." On day 4, after briefly looking at their graphs of series from the previous day, the students almost immediately started to perceive "graphically" series as "very similar to sequences because you could still set the error bound within certain- whatever range you want- any error bound, and then determine the point N where all the partial sums are within the error bound." Likewise, the students stated the meaning of series convergence in terms of terminologies and notations from the approximation frame. They also reinterpreted each element (N, error bounds, quantifiers) from their prior definition of sequence convergence as elements in a definition for series convergence. Furthermore, they recognized the need to replace a_n with the partial sum s_n. However, the students did not *just* change a_n to s_n, they considered each dot in the series graph as representative of partial sums, "You're adding a_1 to a_2 to a_3 to get each one of these dots on the graph of a series." After a few revisions, they constructed a definition for series convergence as follows: "A series converges to U when $\forall \varepsilon > 0$, there exists some N s.t. $\forall n \geq N$ $|U - S_n| \leq \varepsilon$."

In initially discussing Taylor series, the students employed informal reasoning as they described various graphical attributes of Taylor polynomials approaching e^x. While giving these informal descriptions the students were not attending to the convergence of Taylor series for particular values of the independent variable. When they were prompted to discus error, however, one of the students suggested considering a specific point, and they subsequently highlighted errors as vertical distances between the values of e^x and a Taylor polynomial at a particular x-value. Even though their focus had moved to a particular x-value, they continued to employ informal reasoning that entailed Taylor polynomials and generating functions as being exactly the same once the graphs were "on" each other. They reasoned that this "on-ness" would occur for a Taylor polynomial of relatively low degree for x-values close to the center while a larger degree was needed for x-values away from the center. Only once they had used a Taylor series equation for e^x to find an explicit series for e did they realize that a finite number of terms merely approximated e^x because the remaining terms not used in the approximation "had value."

Once the students recognized that focusing on a single x-value produced a series, they attempted to leverage their definition of series convergence to define Taylor series convergence. During this process they demonstrated considerable confusion between the independent variable, x, and the index, n. One of the students questioned the existence of an N for which all subsequent Taylor polynomial approximations would be within a given error bound of e^x, but in her explanation N appeared to correspond to some lower bound of x-values rather than n-values. After being instructed to explain their definition of series convergence using only a vertical number line, the students recognized the role of N on a vertical number line. This shift to viewing series in a vertical orientation eventually freed them to see the graphs of Taylor series as comprised of convergent series at each x-value where N is dependent upon x as well as ε. Subsequently they expressed a need to "expand" their series definition to capture all x-values. In their first attempt they simply added $\forall x$ at the end of their definition, but they later expressed discomfort with finding one N such that all subsequent Taylor polynomials would be within ε of e^x for all x. The students then quickly latched onto a suggestion to move $\forall x$ to the beginning of their definition, acknowledging how this movement expressed the dependence of N upon x. Their final definition of pointwise convergence in the context of a Taylor series with an infinite interval of convergence was as follows: "A Taylor series converges to $f(x)$ when $\forall x$, $\forall \varepsilon > 0$ there exists some N such that $\forall n \geq N$ $|f(x) - S_n(x)| \leq \varepsilon$."

Even though this definition for Taylor series convergence captures much the formal meaning of pointwise convergence, one student commented that the consecutive universal quantifiers felt "goofy." Even so, they continued to view it as best capturing Taylor series convergence.

Conclusion and Discussion

It is remarkable that the students reinvented and unpacked the formal definition of series and pointwise convergence within such a short time. The students faced challenges that ranged from seeing graphical attributes of series and Taylor series to the ordering of quantifiers. We claim that the instructional guidance provided to the students during the teaching experiment successfully supported them engaging these challenges and their subsequent reinvention of these definitions. First of all, the instructors' asking students to produce graphs of series and Taylor series convergence gave students a reference point for which they could refer to during the construction of their definitions. Second, the prior activity of defining sequence convergence became a means for supporting the students' definition of series convergence as they recognized similarities between sequence and series convergence in the context of their graphs and their definitions. Similarly, their prior activity of defining sequence and series convergence supported students' definition of pointwise convergence. Finally, their emerging approximation scheme helped the students to meaningfully recognize similarities between definitions and interpret each component within a definition. The approximation terminology that they had learned from class allowed them to meaningfully interpret the role of approximations, error, and error bounds in and across definitions.

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An Analysis of Examples in College Algebra Textbooks for Community Colleges: Opportunities for Student Learning

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We report on an investigation of the quality of instructional materials available to students in community colleges as part of a larger research study that seeks to characterize mathematics instruction in community college with courses that prepare students to take a calculus sequence. One of such courses is College Algebra.

The rising costs of higher education have made the community college a natural, and in many cases, the only, option for completing postsecondary studies (Dowd et al., 2006). This makes analysis of the resources used in mathematics instruction timely. We focus on College Algebra, because the number of students taking this class is large, the cost of teaching the course is relatively low, and many programs have College Algebra as a prerequisite to other mathematics courses and to courses outside of mathematics (Gordon, 2008; Katz, 2007, Lutzer, Rodi, Kirkman, & Maxwell, 2007). Also, the influence of introductory courses such as College Algebra is significant to a student's life-long attitude to mathematics (Barker, Bressoud, Ganter, Haver, & Pollatsek, 2004).

Textbooks are an important resource for students and instructors in community colleges. We focus on textbooks because they portray information that is presumed to be relevant for learning about the subject matter (Herbel-Eisenmann, 2007; Howson, 1995; Love & Pimm, 1996) and because they are a source of examples and exercises for instruction; as such, however, they afford probabilistic rather than deterministic opportunities to learn mathematics (Mesa, 2004; Valverde, Bianchi, Wolfe, Schmidt, & Houang, 2002). Thus what we infer are possibilities rather than definitive influences on teaching and learning.

We focus on the examples within the textbook for three reasons. First, they are usually intended to be representative of the work that students need to do–they correspond to those portions of the textbook "that demonstrate the use of specific techniques" (Watson & Mason, 2005, p. 3)—and as such, they are most likely to contain explicit information that will help students in solving similar problems. Second, instructors use examples in the textbook as part of their lectures, some times changing them slightly, so that students have later access to more than one illustration of how to solve a given problem. Third, instructors indicate that students, rather than reading the exposition, rely primarily on examples in order to work out homework problems (Mesa & Griffiths, 2010). Thus, if examples are to be used by students as models of thinking through problems, we ask, what the cognitive demands of the examples are, to what extent can they assist students in learning strategies for controlling the solutions to mathematical problems, what are the types of answer students are expected to generate, and what are the connections made among different representations. This study explores what textbooks offer and what they omit. We believe that by understanding the characteristics of textbooks, instructors can gain a clearer perspective on how this resource supports their practice.

Methods

We focus on seven College Algebra textbooks used by at least twelve large community colleges used in a Midwester state. We concentrated on three topics--transformation of graphs, exponential functions, and logarithmic functions--because they are both foundational for further study of calculus and emerge as key in solving many mathematical and "real world" applications. All text identified in the corresponding sections of each textbook as examples was analyzed using four frameworks. Cognitive Demand captures the level of complexity of tasks using four categories, Memorization, Procedures Without Connections, Procedures With Connections, and Doing Mathematics (Stein, Smith, Henningsen, & Silver, 2000). The second framework, Controlling the Work, examines explicitness of solutions, and focused on only four aspects: Further Elaboration, Correctness, Suggestion to Check, and Plausibility or Interpretation (Mesa, 2010). The third framework is Types of Representation and we sought to determine how frequently different representations (symbols, tables, graphs, numbers, and verbal) are given in statements of the examples and in their solutions. The last framework we applied is Types of Response. We sought to characterize the type of answer expected: Only Answer, Answer and Mathematical Sentence, Answer and Graph, Explanation or Justification, and Making a Choice (Charalambous, Delaney, Hsu, & Mesa, 2010). These analyses were chosen, because as a group they speak about the complexity of mathematical activity that is offered to students in the examples. The inter-rater reliability between two coders ranged from 74% to 96% across analyses.

Results

1. Cognitive Demand (N=348)	
Memorization	0 (0%)
Procedures Without Connections	312 (90%)
Procedures With Connections	34 (10%)
Doing Mathematics	2 (1%)

2. Controlling the Work (N=348)	
Further Elaboration	33 (9%)
Correctness	32 (9%)
Suggestion to Check	8 (2%)
Plausibility or Interpretation	0 (0%)

3. Types of Response (N=348)	
Only Answer	176 (51%)
Answer and Mathematical Sentence	53 (15%)
Answer and Graph	102 (29%)
Explanation or Justification	21 (6%)
Make a Choice	6 (2%)

4. Types of Representation (N=348)		
	In the Statement	In the Solution
Symbols	254 (73%)	145 (42%)
Tables	17 (5%)	42 (12%)
Graphs	36 (10%)	130 (37%)
Numbers	107 (31%)	248 (71%)
Verbal	68 (20%)	31 (9%)

Figure 1. Results from the four analyses with the seven textbooks.

We found no memorization examples in these seven college algebra textbooks. However, nearly 90% of the examples were coded as procedures without connections and very few examples that would be categorized as more demanding. This trend was observed in all seven textbooks, with the percentage of examples requiring procedures without connections ranging from 75% to 100%. Only 2 examples were open enough to be coded as doing mathematics, and these appeared in one textbook.

Across textbooks, less than 10% of the examples modeled the four strategies that allow the solver to control their work. There were differences at the textbook level, with one textbook including seven Further Elaboration examples, but no Correctness examples and two textbooks accounting for seven of the eight Suggestion to Check examples in the corpus. Thus, in spite of these examples using real world applications, very little of that content was used to control the correctness of the solution.

We also found that 46% of the examples expected an answer only response. Examples asking for explanation or justification and asking for making a choice were relatively rare (from 0% to 15% and 0% to 6%, respectively) across all seven textbooks. All the textbooks had more Answer and Graph responses than Answer and Mathematical Sentence response.

Finally, symbols were most frequently used in the example statements, and numbers are most frequently requested in the solution. The second most dominant form of representation was numbers in the statement and the symbols in the solution. Tables, graphs, and verbal representations appeared less frequently in both the statement and the solution. With a couple of exceptions, the use of symbols and numbers in the statement and solution was common.

Discussion

In many cases, examples applied the concepts or formulas explained in the textbook. Although we found several examples that had features of reform-oriented tasks (e.g., ‘real-world’ contexts or use of technology), not all of those problems had high-cognitive demand tasks. While applying procedures without connections is an important activity, concentrating only on these less demanding examples can restrict students’ perception on what mathematics is (Stein et al., 2000). One possible reason for these results might be that the reform movement for two-year colleges (Blair, 2006) has not yet influenced the textbooks, in spite of this document being available for authors. Another possibility is that instructors and colleges might prefer to adopt textbooks that are more traditional, or that they, in making decisions, focus on other aspects such as the number of problems for the students.

The analysis of strategies for controlling the work shows that in general the examples do not provide explicit information about the meanings of an answer or about ways to make sure answers are correct, which reduces students’ opportunities to learn to use these strategies when solving problems. It might be possible that authors expect instructors teach these strategies during instruction; however, if it is true that students rely on examples to learn to do homework, an important opportunity is missed by not adding this information to the examples. Considering that students tend to go back to examples when they meet difficulties doing homework, increasing the frequency of further elaboration, checking or suggesting correctness, interpretation, and examining plausibility could be a way to enhance students’ learning of these strategies.

Textbooks in this study have a large number of examples looking for only answer and few asking for explanations or justifications. Considering that students rely on examples to learn and practice the mathematics they are not yet competent with, having them paying more attention to explanation and justification will help students and instructors uncover misconceptions. Sometimes, students arrive at the right answers with erroneous understanding (Erlwanger, 1973). In addition it will be difficult to gain proficiency in explaining and justifying when the focus is mostly on finding an answer to a problem. Using problems that has explanation or justification as the type of response is also beneficial to teachers in terms of the acquisition of *pedagogical content knowledge* (Cohen, Raudenbush, & Ball, 2003). Attention to explanation and justification gives teachers a better access to their students’ struggles, by making explicit the path students took to find the answer.

While the analysis of the expected response type contains some information about representations, our final analysis provides a better picture. The low percentage of tables, graphs, and verbal representation suggests that few connections are made across these types of representations. What is desirable for students is not only to become proficient at interpreting a situation with multiple representations, but also to generate models of real-life situation using multiple representations (NCTM, 2000). The ability of using numerical, graphical, symbolic, and verbal representations is expected to the students taking mathematics-intense courses in community colleges (Blair, 2006).

Our findings suggest that examples in these textbooks are not very demanding and that they do not assist students in developing strategies for controlling their work. Moreover, students are not asked to reflect their reasoning nor to actively make connections among different representations. While these findings are particular to College Algebra textbooks used in community colleges it would be important to know if that is the case for other college textbooks. Research is needed to find out whether this is the case for other textbooks for different topics at the community college level (horizontal) and different levels of algebra courses (vertical, at different institutions). How to introduce these changes needs to be researched, in order to provide instructors guideline for improving opportunities for students to learn college algebra in community colleges.

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Promoting Success in College Algebra by Using Worked Examples in Weekly Active Group Work Sessions

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At a research university near the east coast, researchers have restructured a College Algebra course by formatting the course into two large lectures a week, an active recitation size laboratory class once a week, and an extra day devoted to active group work called Supplemental Practice (SP). SP was added as an extra day of class where the SP leader has students to work in groups on a worksheet of examples and problems, based off of worked example research, that were covered in the previous week's class material. Two sections of the course was randomly chosen to be the experimental group and the other section was the control group. The experimental group was given the SP worksheets and the control group a question and answer session. The experimental group significantly outperformed the control on a variety of components in the course, especially when SP attendance was factored into the analysis.

Keywords: College Algebra, Cognitive Science, Worked Examples, Large Lecture Supplemental Sessions

INTRODUCTION

A Commitment to America's Future: Responding to the Crisis in Mathematics and Science Education states that ``nationally 22% of all college freshman fail to meet the performance levels required for entry level mathematics courses and must begin their college experience in remedial courses'' (p. 6). The enrollment in college algebra has grown recently to the point that nationally there are estimated 650,000 to 750,000 students per year (Haver, 2007) and has surpassed the enrollment in Calculus recently. Although there are almost three fourths of 1 million students enrolling in college algebra, it is estimated conservatively that 45% of these students fail to receive a grade of A, B, or C and can reach percentages in the sixties at some colleges. To address this non-success of students at a large research university in the eastern part of the United States, faculty members teaching Applied College Algebra have implemented a new structure in the course that emphasizes active learning through a day called Supplemental Practice.

BACKGROUND AND BRIEF LITERATURE REVIEW

Supplemental Practice Structure

The idea of Supplemental Practice, denoted SP, was implemented during the fall 2004 and was modeled after Supplemental Instruction (Arendale, 1994; SI Staff, 1997). The normal structure of the Applied Algebra Class that consisted of three lectures a week morphed into a structure of two lectures a week in a large lecture room, and an active laboratory class once a week in computer classrooms where students meet in smaller groups. The lab class was held on Tuesdays while the lecture class was held on Mondays and Fridays. The SP days on Wednesdays were originally added to the weeks' schedule to help lower-achieving students. This was done by requiring students that scored lower than an 80 on a placement exam or scored lower than a 70 on any regular exam, to attend the SP sessions. Starting in the fall 2006 semester, the SP sessions have since morphed into active problem session days modeled after the cognitive science "worked-out example" research. The worked-out example research ask students to study a worked out example for a particular topic, ask questions about anything in the example that they do not understand, and finally work a similar example without reference to the worked out example nor other outside sources (Cooper and Sweller, 1985; Ward and Sweller, 1990; Zhu and Simon, 1987; Carroll, 1994, Tarmizi and Sweller, 1988). Most all of "worked-out example" research has been in a laboratory setting rather than in classroom settings. In this research, the researcher randomly designated one of the course sections as the control group and the other two sections as the experimental group. In the experimental group, the students were given a worksheet at the beginning of the SP day and ask to work in groups to complete the worksheet. Three to four class assistants circulated around the room to answer any student questions about the worksheet. In the control group, a graduate student organized a question and answer session during the extra day instead of giving a worksheet to the students. Students were able to get any question answered, but the graduate student only answered student questions and did not generate questions themselves. For the most part, the graduate student spent all of the class time answering student generated questions. The research questions that will be addressed in the research are the following:

1. Do students in the experimental group earn a significantly different course grade/exam scores/quiz scores/etc… than students in the control group?
2. What are students overall perceptions and experiences of the SP sessions?

Past research on worked-out examples in mathematics has been conducted in a laboratory setting. This research is conducted in a large lecture classroom setting and concentrates on determining if worked-out examples helps promotes success in the course. In addition, past worked-out example research in mathematics has not dealt with college mathematics courses, classes in a large lecture setting, or implementing an extra day of class to focus on working with students to master material. The research could be valuable to other researchers that are working to promote student success in large lecture classes.

METHODOLOGY

The setting for the research was a college algebra course with an annual enrollment of around 1000 students. This course is one of three different types of college algebra courses at the university. One type of college algebra is called a 3-day algebra course that comprises of two lectures a week in a large lecture setting and one day a week in the lab where students actively work in smaller group math labs. The second type is the college algebra 4-day course in which this study took place in. The 4-day college algebra course is the same as the 3-day except the 4th day is spent in SP. The final type is a 5-day college algebra course that is comprised of 5 lectures a week in a class size of approximately 40 students. Each type of college requires specific placement exams scores. The 3-day algebra course requires the highest placement score and the 5-day algebra course requiring the lowest placement score.

One of the three sections of College Algebra was randomly selected as the control group and the other two sections served as the experimental group. Quantitative data (course scores, supplemental days attended, class attendance, total points,…) was collected for each student in both the control and experimental groups and analyzed at the end of the semester. There were similar demographics in both the control and experimental groups.

RESULTS

The researcher compared the data for the control and experimental groups to determine whether there was any significance in total course points, exam scores, quiz scores, and lab scores. Students in the experimental group significantly outperformed students in the control group on total points on exams and quizzes, final exam, exam 3, and quizzes ($p < 0.05$) using a t-test. Using previous data on SP sessions, the researcher has established in previous semesters that students who voluntarily attend eight or more SP sessions are more successful on passing the class than students who attend seven or less days of SP sessions. When the researcher includes only the students that have attended eight or more SP sessions in the experimental group and compares these students to the control group, students in the experimental group significantly outperform ($p < 0.01$) students in the control group on everything (total course points, each exam, final, laboratories, and quizzes) except on the first exam using a t-test. The researcher believes that the reason the first exam is not being significant is because students are just being introduced to the active SP sessions and worked-example worksheets and there are only two active SP sessions before the first exam.

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Relationships between Quantitative Reasoning and Students' Problem Solving Behaviors
Contributed Research Report

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Abstract: This presentation reports on the results of a study into precalculus students' reasoning when solving novel problems. The study intended to identify students' mental actions that support or hinder their ability to provide meaningful and correct solutions, while also characterizing the role of quantitative reasoning in the students' solutions. Analysis of clinical interviews with each student revealed that a student's propensity to reason about quantities and a problem's context significantly influenced his or her problem solving approach. Students who spent a significant amount of time orienting to a problem by identifying quantities and relationships between quantities leveraged the resulting mental images throughout their problem solving activity. Contrary to this, students who focused on recalling procedures and performing calculations spent little time reasoning about a problem's context and encountered difficulty providing meaningful and correct solutions. These findings offer insights into the relationship between students' reasoning and their problem solving behaviors.

Key Words: Precalculus, Problem Solving, Student Reasoning, Quantitative Reasoning

Introduction

Problem solving has been a focus of mathematicians and mathematics educators for well over the past half-century. The focus on problem solving has ranged from suggesting that curricula be designed to promote learning through problem solving (NCTM, 2000) to characterizing the problem solving processes and performance of students and mathematicians (M. Carlson, 1999; M. P. Carlson & Bloom, 2005; Lester Jr., 1994; Pólya, 1957; Schoenfeld, 2007). Such investigations have labeled problem solving as a complex process of interrelated factors and phases, including planning, monitoring, affect, and orienting. Recent studies (M. Carlson, 1999; M. P. Carlson & Bloom, 2005; Schoenfeld, 2007) have begun revealing the intricate role these factors play in problem solving, while emphasizing the importance of exploring students' problem solving behaviors and the role of problem solving in learning mathematics.

Contributing to the body of research on problem solving, recent reports (Moore, Carlson, & Oehrtman, 2009; Smith III & Thompson, 2008) have illustrated the importance of quantitative reasoning in students solving novel problems. These reports describe that a student's mental image of a problem's context (e.g., a mental scene consisting of quantities and relationships between quantities) significantly influences his or her solution to the problem. This finding highlights the delicate and complex nature of problem solving, and advocates the need to further investigate the role of quantitative reasoning in problem solving and learning mathematics.

This study sought to build on the current body of problem solving research by investigating precalculus students' reasoning as they engaged in problem solving activity. The goal of the study was to identify relationships between quantitative reasoning and students' behaviors during the various problem solving phases identified by Carlson and Bloom (2005). In doing so, this study's findings add to the limited knowledge on students' problem solving behaviors at the secondary and undergraduate mathematics level. The results presented in this paper focus on various behaviors that occur during the problem solving phases, and how a student's propensity to reason quantitatively influences the mental actions driving these behaviors. These results offer insights into the types of reasoning that either hinder or support students' problem solving abilities, and how these reasoning patterns influence each problem solving phase.

Background

In an attempt to provide a finer characterization of problem solvers' cognitive processes, Carlson and Bloom (2005) investigated the problem solving activity of 12 mathematicians. Drawing from analysis of interviews with the mathematicians, as well as previous research on problem solving (Lester Jr., 1994; Pólya, 1957; Schoenfeld, 2007), the authors created the *Multidimensional Problem-Solving Framework*. This framework identifies multiple problem solving cycles within four problem solving phases: orientation, planning, executing, and checking. Additionally, Carlson and Bloom's study revealed various problem solving attributes (e.g., monitoring and affect) that influence a problem solver's behaviors.

Carlson and Bloom (2005) noted that much is still to be learned relative to the problem solving processes of students, as their study focused on mathematicians. In response to this call, Moore, Carlson, and Oehrtman (2009) examined precalculus students' problem solving behaviors. Findings from this study identified the critical role of quantitative reasoning (Smith III & Thompson, 2008) when a student orients to a novel problem. The students involved in the study often constructed incorrect mental images of a problem's context when orienting to a problem. The students subsequently constructed incorrect solutions, where these solutions were consistent with their images of the problem's context. After the students reflected on their solutions and refined their images of a problem's context, they corrected their solutions to reflect their modified quantitative structures. These actions enabled the students to provide meaningful explanations of their corrected solutions. These findings illustrate the importance of the orientation phase, as well as a need to further explore the role of quantitative reasoning in problem solving.

Methods and Subjects

The subjects of this study were nine undergraduate precalculus and college algebra students at a large public university in the southwest United States. The students were chosen on a voluntary basis and they received monetary compensation for their participation. Clinical interviews (Clement, 2000; Goldin, 2000) were conducted with each student, during which they were asked to solve a set of novel problems. During the interviews, the interviewer prompted the students to explain their thinking in order to gain insights into the reasoning processes driving their problem solving behaviors. Due to the cognitive nature of problem solving, the clinical interview setting was critical in identifying reasoning that would not have been revealed in a classroom setting or collected student work. Also, this study rested on the stance that each student engages in unique reasoning, and hence the clinical interview methodology offered data that enabled characterizing each student's reasoning processes.

The data was analyzed following an open coding approach (Strauss & Corbin, 1998). The students' behaviors were analyzed in an attempt to determine the mental actions that contributed to their solutions. The mental actions inferred from the students' behaviors were then characterized in terms of the problem solving phases identified by Carlson and Bloom (2005). This phase of the data analysis involved identifying how the students' mental actions influenced their behaviors during the four problem solving phases. This approach to analyzing the data enabled classifying how various reasoning patterns related to the students' problem solving behaviors. Lastly, the students' mental actions and problem solving behaviors were compared and contrasted. This stage of analysis led to the finding that the students held varying problem solving dispositions that paralleled their propensity to engage in quantitative reasoning.

Results

Analysis of the students' solutions revealed that their propensity to engage in quantitative reasoning significantly influenced the nature of their problem solving behaviors and their ability to provide meaningful solutions. Students who extensively focused on a problem's context developed a mental image of the context that they leveraged during the problem solving phases. Contrary to this, when students focused on performing procedures and calculations, they did not build an image of a problem's context that supported their solution process.

When orienting to a problem, students with a propensity to focus on a problem's context frequently drew and labeled a diagram of the situation. This act included identifying known and unknown measurements and discussing various relationships between quantities. As these students continued to focus on a problem's context, they were observed revisiting the problem statement to identify the goal (and sub-goals) of a problem in terms of the quantities of the situation. During the planning phase of problem solving, they continued to spend a significant amount of time reasoning about a problem's context. They planned their solutions by identifying relationships between quantities and reasoning about these relationships in ways that enabled them to *anticipate* performing calculations. By reasoning about relationships between quantities without performing numerical operations, the students were able to engage in the *conjecture-imagine-evaluate* cycle identified by Carlson and Bloom (2005) to mentally play out their solutions. Similarly, these students recalled formulas during the planning phase and described these formulas in terms of the quantities of the situation. This enabled the students to use formulas to represent relationships between values without having to evaluate the formulas.

When executing their planned solutions, the students described calculations in terms of the quantities of the situation and consistently illustrated the quantity referenced by a newly obtained value. Also, due to their calculations being grounded in quantitative relationships, the students constructed a quantitative meaning for a value before obtaining a numerical result. That is, the students did not need to determine the meaning of a result of calculating, as they had developed a meaning previous to the calculation. The students' images of the problems' contexts also supported their monitoring the appropriateness of the calculated values. When they obtained values that were not consistent with their image of a problem's context, the students returned to the context to further orient to the problem, check their solution, and modify their solution if needed. These actions enabled the students to identify incorrect solutions and use the context of the problem to justify alterations to their solution.

Students with a propensity to reason about calculations and procedures engaged in problem solving behaviors significantly different to the behaviors previously outlined. When orienting to a problem, students with a tendency to focus on calculations and procedures often drew a diagram, but they infrequently labeled known and unknown values on the diagram and spent limited time verbally discussing a problem's context. Instead, they regularly referred to previously completed problems deemed similar to the current problem. Subsequently, these students attempted to recall the steps or calculations made when solving a similar problem. In the cases that they recalled previous solutions, the students progressed to the executing phase without further explaining or analyzing the recalled solution. In the cases that they could not recall a previous solution, they experienced difficulty progressing and suggested calculations to the interviewer (who did not provide feedback). When asked to explain a meaning of their suggested calculations, these students expressed a need to first calculate a numerical value, as opposed to attempting to explain the calculation previous to performing the calculation.

After executing a suggested calculation, the students experienced difficulty determining how the obtained value related to a problem's context or the goal of the problem. The students frequently gave multiple meanings to the determined numbers (e.g., using a number to refer to multiple lengths), and the students relied on the aesthetic quality of their answers (e.g., values not "too big" or "too small") to check their solutions. In the cases that the students believed their solution was incorrect, they looked to the interviewer for assistance or attempted to recall another procedure.

Conclusions and Implications

The varying problem solving approaches exhibited by the students of this study reveal how a student's problem solving disposition can influence his or her ability to solve novel problems. These insights should inform curriculum designers and teachers about the reasoning abilities and problem solving behaviors they should strive to engender in students. Also, students were sometimes observed alternating problem solving dispositions from problem to problem, as well as within a single problem. Further research should explore reasons for such transitions, and the instruction necessary to promote students developing a disposition that supports their constructing meaningful and correct solutions to novel problems. Students with a quantitative disposition also appeared to be more reflective during their problem solving activity. This may have been a result of their reasoning creating a foundation for reflective actions. Future research should investigate this phenomenon, and its implications for using problem solving to promote students learning mathematics.

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The Physicality of Symbol-Use: Projecting Horizons and Traversing Improvisational Paths Across Inscriptions and Notations

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The way people use symbols and drawings has an intrinsic physicality. Viewed as an extension of gesture-making, symbol-use can give us insight into how symbol-users experience the mathematics at hand. Using a theoretical framework of embodied cognition, we explore this matter by conducting a phenomenological analysis of a 2-minute selection from an interview with a topologist about one of his published papers. We propose an interpretation of the mathematician's symbol-use in terms of two related constructs: *realms of possibility* in what the mathematician perceives as available to him and *paths* within and between these realms. Both of these are projected onto the writing surface and embodied through gestures, speech, eye gaze, and many other means. We explore the origins and relevance of these in our presentation.

Keywords: Embodied cognition, phenomenology, gesture, mathematicians

Traditionally, concepts have been understood in contrast to percepts (e.g. Kant, 1998). A growing body of research questions this division suggesting that the way we think about something, the way we perceive it, and the way we can and do physically interact with it are all inextricably intertwined (Noe, 2006). For instance, when we first learn to drive, the car feels like a foreign object and other vehicles suddenly seem much larger and more dangerous than when we were passengers. Yet as we become more familiar with driving, the car comes to feel like an extension of us and we learn to perceive and think about vehicles with a different sense of ourselves than when we were passengers or students of driving.

The split between concepts and percepts has been reflected in cognitive science as an opposition between modal (i.e. perceptual) and amodal (i.e. conceptual) systems (Barsalou, 1999). One way to argue for the inseparability between concepts and percets is to state there are no amodal systems in which we do logical reasoning, inference and so on, but instead that all activity is perceptuo-motor activity. Our reasoning about even the most highly abstract topics manifests through a partially covert sense of what we can do and perceive with the representations we use for said topics. It is in this sense that we say that cognition is embodied.

This leaves us with a question: How do those who are skilled in highly abstract forms of reasoning embody their thinking about those abstract topics? We know that this must occur through their interactions with the representations they use for the ideas in question, but how do those interactions contribute to the way in which the abstractions are understood and used?

We explore these questions in a case study of a mathematician explaining an aspect of his published work. We asked him to choose a paper he considered interesting or significant, did our best to understand the paper over the course of a few weeks, and then conducted a video-recorded unstructured interview (Bernard, 1988) in which we asked him to explain the paper as he thought of it. We watched the subsequent video several times to select segments for microanalysis (Erickson, 2004), choosing the segments based on which ones seemed most likely to give fruitful insight into the embodiment of abstract mathematics. With the 2-minute segment in question, we alternated between examining the microanalysis individually and discussing our examinations as a team. In our individual examinations, we would generate possible descriptions of the mathematician's actions based on what we knew about his background, the demands of ongoing circumstances (e.g. his reacting to the interviewer's questions), and the multiple unintended contingencies arising moment by moment. In our collective discussions we would share each other's examinations and explore the implications of one another's observations in light of the data on hand, with the goal of generating compelling and viable accounts of this mathematician's experiences allowing us insight into the nature of how abstract thought can be embodied. While this is an case study of a single subject, a microethnographic analysis has the potential to broaden our perspectives and to suggest new interpretations which may enrich our understanding of how anyone grapples with mathematical problem solving.

Our analysis has highlighted two related constructs that we'd like to share in this presentation. The first we term *realms of possibility*. A crucial observation arising from numerous phenomenological investigations is that what we perceive is not given merely by light, sound, and so on but is also saturated with our anticipations of how we might be able to interact with and change that which we perceive (Gallagher & Zahavi, 2008; Husserl, 1913/1983; Merleau-Ponty, 1962). The collection of such anticipations often presents itself to the individual as being a kind of space, just as we have a sense of the space in which we could move a chair and sit ourselves upon it. But just as there are limitations to how you anticipate being able to move a given chair, these realms of possibility have a kind of boundary, which Husserl (1913/1983, p.52) referred to as a "horizon". We find that the mathematician in our study consistently defined these horizons between realms as he experienced them by creating gaps in the blackboard or drawing dividing lines on it and reinforcing them with his gestures, gaze, and placement and orientation of his body.

The second construct is that of *paths*, both within and between realms of possibility. In order to actualize his explanations, the mathematician has to "travel" within the realms he describes. This "travel" occurs via gestures, speech, gaze trajectory, inscription on the blackboard, and so on. Some of these paths follow the symbols and drawings in the order in which they were inscribed, whereas others get overlaid on an existing inscriptional surface along temporal sequences that differ significantly from the order in which they were generated. Both the travel along and redefinition of paths occurs through the mathematician's physical interactions with the symbols, such as when he seemingly runs into a difficulty with his exposition, physically steps away from the blackboard to gesture an explanation that gets around the difficulty, and then physically returns to the blackboard and manually puts his explanation into the symbols already written. The accompanying speech makes a corresponding shift as well; in this particular example, the mathematician switched to the subjunctive ("If you wanted to…") until he physically reconnected his talk and gesture back to the symbols on the board with which he was making his original point. This is just one of several different kinds and uses of paths that we've noticed as defining methods of travel within and between realms of possibility in this episode.

In exploring these matters, we hope to contribute to basic research that can help frame mathematical activity in ways that are both practical for researchers and consistent with the mounting evidence supporting the close connection between concepts, perception, and physical action. These theoretical constructs – realms of possibility and paths within and between them – provide us with a way of perceiving some of the bodily interactions that individuals can have with mathematical entities. Further exploration of these and related constructs has the potential to provide a rich account of how collegiate mathematics is practiced while remaining true to the inseparability of mind and body.

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From Intuition to Rigor: Calculus Students' Reinvention of the Definition of Sequence Convergence

Contributed Research Report

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Abstract

Little research exists on the ways in which students may develop an understanding of formal limit definitions. We conducted a study to i) generate insights into how students might leverage their intuitive understandings of sequence convergence to construct a formal definition and ii) assess the extent to which a previously established approximation scheme may support students in constructing their definition. Our research is rooted in the theory of Realistic Mathematics Education and employed the methodology of guided reinvention in a teaching experiment. In three 90-minute sessions, two students, neither of whom had previously seen a formal definition of sequence convergence, constructed a rigorous definition using formal mathematical notation and quantification nearly identical to the conventional definition. The students' use of an approximation scheme and concrete examples were both central to their progress, and each portion of their definition emerged in response to overcoming specific cognitive challenges.

Keywords: Limits, Definition, Guided Reinvention, Approximation, Examples

Introduction and Research Questions

A robust understanding of formal limit definitions is foundational for undergraduate mathematics students proceeding to upper-division analysis-based courses. Definitions of limits often serve as a starting point for developing facility with formal proof techniques, making sense of rigorous, formally-quantified mathematical statements, and transitioning to abstract thinking. The majority of the literature on students' understanding of limits (Bezuidenhout, 2001; Cornu, 1991; Davis & Vinner, 1986; Monaghan, 1991; Tall, 1992; Williams, 1991) describes informal student reasoning about limits, with particular attention given to the myriad of student misconceptions. However, there is a paucity of research on student reasoning about formal definitions of limits. The general consensus among the few studies in this area seems clear – calculus students have great difficulty reasoning coherently about the formal definition (Artigue, 2000; Bezuidenhout, 2001; Cornu, 1991; Tall, 1992; Williams, 1991). What is less clear, however, is how students come to understand the formal definition. Indeed, this is an open question with few empirical insights from research to inform it (Cottrill et al., 1996; Roh, 2008; Swinyard, in press). Oehrtman (2008) proposed a coherent approach to developing the concepts in calculus through a conceptually accessible framework for limits in terms of approximation and error analysis. Students were recruited to participate in our study from a course that relied heavily on Oehrtman's approach. This study addressed the following research questions:

1. What are the cognitive challenges that students encounter during a process of guided reinvention of the formal definition for sequence convergence?
2. What aspects of their concept images do students evoke during this reinvention?
3. How do students' evoked concept images and their solutions to cognitive challenges encountered support more advanced mathematical thinking about limits of sequences?

Theoretical Perspective and Methods

We adopted a *developmental research* design, described by Gravemeijer (1998) "to design instructional activities that (a) link up with the informal situated knowledge of the students, and (b) enable them to develop more sophisticated, abstract, formal knowledge, while (c) complying with the basic principle of intellectual autonomy" (p.279). Task design was supported by the *guided reinvention* heuristic, rooted in the theory of Realistic Mathematics Education (Freudenthal, 1973). Guided reinvention is described by Gravemeijer, K., Cobb, P., Bowers, J., and Whitenack, J. (2000) as "a process by which students formalize their informal understandings and intuitions" (p.237).

The authors conducted a six-day teaching experiment with two students at a large, southwest, urban university. The full teaching experiment was comprised of six 90-120 minute sessions with a pair of students currently taking a Calculus course whose topics included sequences, series, and Taylor series. The central objective of the teaching experiment was for the students to generate rigorous definitions of *sequence*, *series*, and *pointwise convergence*. The research reported here focuses on the evolution of the two students' definition of sequence convergence over the course of the first three sessions of the teaching experiment. The design of the instructional activities was inspired by the *proofs and refutations* design heuristic adapted by Larsen and Zandieh (2007) based on Lakatos' (1976) framework for historical mathematical discovery. Activities commenced with students generating prototypical examples of sequences that converge to 5 and sequences that do not converge to 5. The majority of each session then consisted of the students' iterative refinement of a definition to fully characterize sequence convergence. The students were to evaluate their own progress by determining whether their definition included all of the examples of convergent sequences and excluded all of the non-examples.

Results

Three broad areas of findings emerged from our data analysis: the role of students' use of examples, the effect of a scheme for limits based on approximation language, and the students' adoption and appreciation of quantifiers and efficient mathematical expressions.

The Role of Examples. The students' reinvention efforts were aided considerably by the presence of the examples they constructed at the start of the experiment. These examples served as sources of cognitive conflict when their definition failed to fully capture the necessary and sufficient conditions under which sequences converge. For example, the students' initial definitions were predictably couched in language that was vague, intuitive, and dynamic. Their first written definition was "A sequence converges to 5 as $n \to \infty$ provided that the number approaches or is 5 and no other number." The students immediately identified weaknesses in this definition as they applied it to their examples that increase monotonically to 4, alternate around 5 or behave erratically before eventually looking like a standard example of a convergent sequence. Having identified these weaknesses, they also looked to their examples to provide direction for their revisions. This pattern of evaluating and refining their definitions against the examples repeated over 18 cycles during the first three days of the teaching experiment.

The Effect of an Approximation Scheme for Limits. The students' familiarity with a previously established approximation scheme mirroring the structure of the formal definition but framed in more accessible terms (Oehrtman, 2008) provided students significant leverage for i) focusing on relevant quantities in the formal definition, ii) fluently working with the relationships between these quantities, and iii) making the necessary but difficult cognitive shift to focus on N as a function of ε (Roh, 2008; Swinyard, in press). For example, during the first 12 minutes of the teaching experiment, the students did not invoke language about approximations to describe aspects of a sequence $\{a_n\}$. During this time they did not discuss or represent the quantity $|a_n - 5|$ in any form and all descriptions of convergence involved informal dynamic language. But once they invoked an approximation scheme, they described the limit as the value being approximated, the terms a_n as the approximations and the distance between them as the error which they immediately represented as $|a_n - 5|$. These ideas became an integral part of their arguments and the students shifted to discussing how close the terms needed to get to 5 to consider the sequence convergent. After another 14 minutes, the students invoked the idea of an error bound (corresponding to ε in the formal definition) to address this question and focused on how to make the error smaller than this bound. Nine minutes later, they introduced the idea of there being "some point n" (corresponding to N in the formal definition) at which this must happen. Afterwards, they consistently reasoned that this "point n depends [on] what the acceptable error is." For the remainder of Day 1 and throughout Days 2 and 3 of the teaching experiment, the students continued to rely on this approximation scheme to describe the relevant quantities and to keep track of the relationships among them.

Adoption of Quantifiers and Mathematical Expressions. Powerful use of logical quantifiers and mathematical expressions emerged only after the students had i) fully developed the underlying conceptual structure of convergence in informal terms, ii) wrestled with the problem of how to rigorously express those ideas, and iii) seen the quantifiers and expressions as viable solutions to these problems. Early in the first day of the teaching experiment, one student recalled the use of universal and existential quantifiers. While she used them correctly neither student applied them to resolve any problem they were wrestling with and they soon dropped the quantifiers. On Day 3 of the teaching experiment, the students were consistently verbalizing all elements and appropriate logic of the ε-N definition, but lacked the terminology or notation to construct what they considered an acceptable written definition. As they struggled with these issues, brief reminders of the quantifiers they had used earlier but discarded were seized upon as perfect solutions to their difficulties. Ultimately the students settled on the definition

"A sequence converges to U when $\forall\ \varepsilon$, there exists some N, $\forall\ n \geq N$, $|U - a_n| < \varepsilon$."

The students expressed strong appreciation for the power of the quantifiers and mathematical notation in their definition, citing multiple problems that each part efficiently resolved.

Limitations, Implications and Conclusions

The two students in this teaching experiment had only experienced instruction aimed at developing a systematic approximation scheme for reasoning about limits for a portion of one calculus course. Consequently, it is not surprising that they did not immediately invoke this scheme as they began to wrestle with generating a definition of sequence convergence and that the scheme emerged in pieces. Nevertheless, it did not take them long to turn to approximation ideas, and each portion of their evoked scheme emerged in response to particular problems for which it was well-suited to address. We note that these students progressed much more quickly towards a formal definition and through resolving several cognitive challenges than students not introduced to the approximation framework (Swinyard, in press). Once evoked, the students' ideas about approximation remained consistent, and their images and application of their scheme was sufficiently strong to provide them considerable guidance and conceptual support for reasoning about the formal definition.

This study drew from data collected in a teaching experiment with only two students and we acknowledge that each individual will follow unique paths. Further, orchestrating this type of discussion for an entire class will certainly involve significant differences from what was possible with focused attention on two students. Nevertheless, these students' reinvention of the definition serves not only as an existence proof that students can construct a coherent definition of sequence convergence, but also as an illustration of *how* students might reason as they do so. Our findings shed light on several relevant cognitive challenges engaged by the students, how they resolved these difficulties, and the resulting conceptual power derived from their solutions. These results are guiding our future work to develop, evaluate and refine classroom activities for introductory analysis courses.

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How Intuition and Language Use Relate to Students' Understanding of Span and Linear Independence

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This report describes a case study in an undergraduate elementary linear algebra class about the relationship between students' understanding of span and linear independence and their intuition and language use. The study participants were seven students with a range of understanding levels. The purpose of the research was to explore the relationship between students' "natural" thinking and their conceptual development of formal mathematics and the role of language in this conceptual development. Findings indicate that students with low indicators of intuition and stronger language skills developed better understanding of span and linear independence. The report includes possible instructional implications.

Keywords: Intuition, Language use, Linear algebra, Linear independence, Span

In an essay about his experiences teaching linear algebra, David Carlson (1997) posed a question that has become emblematic of students' learning in linear algebra: Must the fog always roll in? This question, he writes,

> refers to something that seems to happen whenever I teach linear algebra. My students first learn how to solve systems of linear equations, and how to calculate products of matrices. These are easy for them. But when we get to subspaces, spanning, and linear independence, my students become confused and disoriented. It is as if a heavy fog rolled over them, and they cannot see where they are or where they are going. (p. 39)

Research into the teaching and learning of linear algebra has spanned several decades, but the issue of how to clear the fog for students is still outstanding. In this report, I describe a research study designed to contribute to the understanding of how students learn concepts in linear algebra.

The purpose of this study was to address two outstanding issues in the learning of advanced mathematics. The first issue is a theoretical difference between the ways in which students learn "naturally" and the formal structure of mathematics, and how this difference may or may not influence students' mathematical understanding. The second issue is the relationship between students' language use and their mathematical understanding and how this might relate to students' natural ways of learning. My research question was:

> How do students' intuition and language use relate to the nature of their understanding of span and linear independence in an elementary linear algebra class?

Existing research supports the existence of the issues this study was designed to address. In his epilogue of *Advanced Mathematical Thinking*, Tall (1991) noted that many of the book's contributors believed students' difficulties in learning advanced mathematics could be explained by the discrepancies between the way students viewed mathematics and classroom instruction, which is often based on the formal structure of mathematics. More recently, in their discussion of advanced mathematical thinking, Mamona-Downs and Downs (2002) suggested traditional teaching of mathematics does not "connect with the students' need to develop their own intuitions and ways of thinking" (p. 170). An impediment to developing instructional theory based on students' intuitions is an incomplete understanding of how people develop abstract mathematical knowledge. Pegg and Tall (2005) compared several theories of concept development and derived a fundamental cycle of concept construction underlying each of the theories. However, there is no consensus on the mechanism of how this concept development occurs. Some evidence exists to suggest language may play a role in this development (Dehaene, Spelke, Pinel, Stanescu, & Tsivkin, 1999; Devlin, 2000). Pugalee (2007) contends "language and competence in mathematics are not separable" (p. 1). MacGregor and Price (1999) and Boero, Douek, and Ferrari (2002) believe that metalinguistic awareness is necessary for students to coordinate the various notation systems in mathematics. Yet, little research exists that explores the relationship between students' language abilities and mathematics learning (Barwell, 2005; Huang & Normandia, 2007; MacGregor & Price). Interestingly, though, just as mathematics education researchers have found a contrast between intuitive thinking and formal mathematics, language researchers have found this same contrast between everyday language use and the demands of formal school language (Schleppegrell, 2001, 2007). It is possible, then, that language plays an important role in how students move from intuitive, everyday thinking to understanding formal mathematical concepts and theory.

The literature about learning linear algebra in general and learning about span and linear independence specifically reflects the issues reported in the literature about intuition in learning mathematics. This includes students' difficulty with understanding and using formal definitions (Medina, 2000) and students relying upon surface features, prototypical examples, and intuitive models rather than conceptual understanding (Harel, 1999; Hristovitch, 2001; Medina). Lacking in this literature, though, is a clear picture of the interaction between instruction, students' intuition, and the nature of students' understanding. In particular, it does not reveal the components of understanding of span and linear independence that are sufficient for an elementary linear algebra class nor the individual differences in intuition and language use that may account for variation in student learning.

The theoretical perspective for this research was the *emergent perspective* described by Cobb and Yackel (Cobb, 1995; Cobb & Yackel, 1996; Yackel & Cobb, 1996). The emergent perspective is a type of social constructivism that coordinates the social and psychological (individual) views (Cobb & Yackel). The *interactionist* view of classroom processes (Bauersfeld, Krummheuer, & Voigt, 1988) represents the social perspective, while a constructivist view of individuals' (both students and teacher) activity (von Glasersfeld, 1984, 1987) represents the psychological perspective. I used the case study methodology for this research and delimited the setting of the study to one elementary linear algebra class. Broadly, the unit of analysis for this study was individual students. However, in alignment with my research question, I focused my analysis on students' understanding of span and linear independence and on their intuition and language use related to these understandings. I analyzed the overall level of students' understanding for the first four weeks of the course and then selected a set of seven students to represent as much as possible maximum variation in understanding levels.

This research depended on being able to operationalize the constructs of understanding, intuition, and language use. Based on the literature and the nature of my data, I found each of these constructs to be multi-dimensional. I defined understanding as the composition of definitional understanding and problem solving skills. Each of these elements had multiple components. Intuitions fell into two categories: self-evident intuitions and surface intuitions, with each category consisting of three different sub-types of intuitions. The salient characteristics of language use were understandability, completeness, and vocabulary use.

The overall findings of this research indicated an association between the quality of students' language use and the quality of their understanding. That is, the students with stronger language skills generally exhibited better understanding of span and linear independence. There was also an association between the degree to which a student's cognition had intuitive indicators and the quality of his/her understanding. The more a student's thinking had intuitive characteristics, the less likely he/she was to develop good understanding of span and linear independence.

A more detailed picture of the findings is as follows. Students' understanding was either functional or problematic. Students with fair or weak problem solving skills were classified as having problematic understanding, while those with good or strong problem solving skills were classified as having functional understanding. The quality of students' definitional understanding determined the level of understanding within each category. Within the functional category, students had strong, good, or fair definitional understanding. Within the problematic category, students had weak or poor definitional understanding. Students with functional understanding had low self-evident intuition indicators, while students with problematic understanding had medium or high self-evident intuition indicators. Students with fair, weak, or poor definitional understanding had more surface intuition indicators than students with strong or good definitional understanding. The quality of students' written explanations was associated with the students' level of understanding. However, language use quality more closely aligned with students' definitional understanding than with their problem solving skills.

There were several finding about the nature of students' learning of span and linear independence. While many students could learn the procedures related span and linear independence, some students struggled to develop conceptual understanding. In addition, many students eschewed knowing and understanding formal definitions in favor of using their own intuitive pseudo-definitions. Students who failed to develop conceptual understanding of foundational concepts, such as linear combination and solution, failed to develop conceptual understanding of span and linear independence. Students who were unclear about the objects associated with span and linear independence (e.g., did not associate linear independence with a set of vectors) did not reify these concepts, but instead viewed these concepts primarily as procedures.

The findings suggest possible classroom implications. While none of the instructional methods are new, this research may underscore their validity in supporting students' learning of mathematics by reducing the role of interfering intuitions. Instructional recommendations include helping students develop metacognitive awareness (Fischbein, 1987) and implementing compare and contrast activities (Marzano, Pickering, & Pollock, 2001). Several researchers have outlined more elaborate instructional tools. These include the instructional practices developed by researchers studying the role of beliefs in mathematics (Muis, 2004), conceptual change in science and mathematics (Vosniadou & Vamvakoussi, 2006), and in reducing misconceptions in mathematics (Stavy & Tirosh, 2000). In order to help students develop their language skills, which in turn may support their mathematical learning, it may be helpful to provide opportunities for students to engage oral and written language practice.

The study has several limitations. Because it was conducted in a single class, the findings may have limited transferability. Also, the nature of the data sources (student work and student interviews) may have limited the validity of the findings. Future research may refine or extend this study's findings in other linear algebra classes. It may also be fruitful to explore this research question in other advanced mathematics classes, such as abstract algebra and analysis.

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The Impact of Technology on a Graduate Mathematics Education Course

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Given the rise in distance delivered graduate programs, educators continue to seek ways to improve teaching and learning in an online environment. In particular, the need for high quality K-12 teachers requires superior teacher-education programs that model good instructional practice, especially in mathematics. In this article, the instructor of a mathematics education course describes the opportunities and difficulties he encountered in designing and implementing an online course for inservice mathematics teachers. In addition to anecdotal evidence from class observations, researchers collected survey data from participants. Results of these data are presented and used with the instructor's reflections to make specific recommendations for improving the course and to offer insight to others using distance-learning technology to teach graduate mathematics education courses.

Key Words: online professional development, mathematics teacher education, teaching geometry

The advances in online technology continue to transform how university faculty can provide teacher professional development (Hramiak, 2010). Advocates of online teacher education maintain that it "holds the possibility of developing not only vibrant explorations of knowledge and practice in the content area, but also communities of learners and practice, and lifelong learning perspectives and skills" (King, 2002, p. 224). Concurrently, problems in the design and implementation of online courses may hinder learners in these environments. Given the demand for high-quality teachers, online courses appear to be an increasingly popular way to provide teacher professional development (Signer, 2008). However, there is a clear need for continuing research in online teacher professional development to ensure that it is meeting the professional needs of teachers (Dede, Ketelhut, Whitehouse, Breit, & McCloskey, 2009).

The purpose of this paper is to present results of an investigation into the design and implementation of an online mathematics teacher education course for secondary inservice teachers as part of the Mathematics Teacher Leadership Center (Math TLC). The Math TLC is a collaboration among the University of Northern Colorado, the University of Wyoming, and partner school districts in Colorado and Wyoming in the United States and is funded by a National Science Foundation[1] Mathematics and Science Partnership. One goal is to help develop culturally competent master teachers to work locally, regionally, and nationally to improve teacher practice and student achievement. Designers of the course relied on recommendations from the literature including purposeful attention to instructor roles and community. Researchers administered a survey to course participants to obtain feedback from teacher-participants about their attitudes about the impact of technology on their learning. With these empirical results as well as observations and notes taken during the semester the instructor offers recommendations to improve the mathematics education course he taught.

Literature

In examining the available literature concerning online teacher education programs, two emerging themes are particularly helpful to frame the design of the teaching geometry course: instructor roles and community. The roles of distance education instructors, while similar to face-to-face instructor roles, have the added dimension of the necessary use of technology. Maor (2003) and Johnson and Green (2007) categorized the roles of distance education instructors as pedagogical, managerial, social, and technical. The *pedagogical role* entails all of the abilities involved in delivery of content, included the ability to make instructional decisions, develop appropriate learning tasks, facilitate learning, and assess for understanding. The *managerial role* comprises the abilities to administer the course, including the skills plan the scope and sequence of the online course, monitor the teaching and learning processes, and manage the constraints of the course, including the timeline. The *social role* includes the ability to provide one-on-one, emotional support and advising to participants. The *technical role* includes the proficiencies involved in the decision-making process of selecting technology, the aptitude to use technology, and the ability to trouble-shoot problems with the technology quickly so that participants may remain focused on learning the material. Each of these roles – pedagogical, managerial, social, and technical – is thought to encompass the duties and tasks of the instructor and, when performed professionally and proficiently, is assumed to ensure a positive learning environment.

Equally important to the success of online courses is the presence of learners' sense of community, which is typically defined as feelings of trust, belonging, commitment, and shared

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goals among online learners (Shea, Li, Swan, & Pickett, 2002). Rovai (2002b) claims the sense of connection among learners helps overcome feelings of isolation caused by physical distance. Other researchers find that graduate students in online programs that have a higher sense of community also have lower attrition rates, increased student learning outcomes, and higher levels of satisfaction and engagement (Lui, Magjuka, Bonk, & Seung-hee, 2007; Shea et al., 2006).

Research Methods

There were 22 participants in the course consisting of inservice secondary (grades 7-12) mathematics teachers working towards a master's degree of the Math TLC. Because of the relatively sparsely populated nature of northern Colorado and Wyoming, the participants were spread out geographically over the two states, though all learners were within 250 miles of one other. The participants had met in person during a six-week session the previous summer, about seven months prior to the start of the course.

The teaching geometry course took place during a 15-week semester in the spring of 2010 and focused on current research and practices of teaching, learning, and assessing geometry in secondary schools. The course was conducted completely online with both asynchronous and synchronous components. Participants used the course management system Blackboard to access course materials and submit assignments as well as to post occasionally on assigned discussion board topics. The participants of the course used the online collaboration software Elluminate to meet virtually every Monday night in a webinar, where live audio and video conferencing was used to facilitate real-time class discussions, small group work, and lecture. The instructor frequently used Elluminate to poll participants for informal feedback as well as put participants into small groups for discussion, followed by whole-group discussion paired with the lesson's PowerPoint slides visible to everyone. Participants received the PowerPoint slides and other required readings electronically, prior to the start of the webinar as recommended by Hofmann (2004). Virtual office hours were held on Elluminate and email was used regularly for the instructor and participants to communicate.

Survey data were collected at the end of the semester through an electronic survey with quantitative questions. Thirteen of the 22 participants completed the survey, with questions focusing on the implemented technology of the course. The instructor and graduate assistant of the course took notes during the semester about the structure and effectiveness of the course webinars and recorded the webinars for later viewing. Both qualitative and quantitative data were used in the investigation.

Results

Overall findings from the survey and observational data indicated participants had successful learning experiences with the class. Most participants indicated they were satisfied with all four roles of the instructor; specifically participants rated technological and pedagogical roles highest, with social and managerial roles receiving positive but more widely distributed responses. Positive feelings of community were indicated by most participants, though one individual reported feelings of isolation from the rest of the class. In addition to the survey results from the participants, the instructor provided reflective comments on the four roles based on the survey results.

With respect to the pedagogical role, course participants generally held favorable views of the course design, including learning tasks and weekly webinar interactions. The instructor was a veteran instructor of the teaching geometry course. However, the instructor felt this role was time consuming mainly because the course expectations were set too high. The time commitment involved for the instructor, as well as the participants, exceeded that of a two-credit master's-level course, including the design of the weekly tasks and the assessment of participants' work. Additionally, a few participants indicated they were only sometimes satisfied with the amount of contact they had with the instructor. On reflection, the instructor indicated this is an aspect that needs improvement, as research indicates online instructors must work harder than face-to-face instructors to establish rapport and open lines of communication with learners (Rovai 2001, 2002a, 2002b; Shea et al., 2006).

The technical role was also a significant aspect of the duties of the instructor. Course participants generally thought that technology was used to promote learning. Although the instructor was a novice user of Elluminate, the instructor felt that decisions involving the use of polling as a formative assessment, the use of breakout rooms for small-group discussion, and the capability of multiple video and audio interactions contributed to these positive learning experiences. Additionally, a majority of participants indicated that technology concerns sometimes interfered with their understanding. The instructor felt that more experience with the technology may increase his ability to act as technical advisor and be a better initial source for solving technical problems.

The managerial and social roles had the added restriction of the separation between the instructor and the participants compared to face-to-face instruction. For example, managing the weekly webinars required explicit attention to environmental norms, such as the use of the chat box and the use of video for course discussion; whereas in face-to-face instruction most classroom norms in college classrooms are implicit based on common educational experiences. The newness of these experiences may have contributed to participants' varied views on interactions during the webinars. Participants, however, generally viewed the social role of the instructor as favorable.

Implications and Future Research

From the data gathered over the duration of the semester, the instructor compiled recommendations for future instructors of online mathematics education courses in this program. The use of break-out sessions and polling in the course was deemed important by both instructor and participants not only as tools for learning but also for community building. Teacher-participants considered a sense of community an important factor in their learning, a finding supported by literature (Rovai 2001, 2002a). The results indicated most learners were satisfied with the amount of community they felt, though a few were only slightly satisfied. Now aware of both the importance and the challenge of building community in online courses, the instructor suggests that this aspect of the course be a focus in the future.

Overall, the instructor and the participants thought that the course was educationally successful. In the future, the instructor plans to continue incorporating technology that fosters knowledge and community building. Evaluating his own teaching using the four instructor roles was helpful in identifying strengths and areas for improvement in the online course, and he recommends this approach for other educators.

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Student teacher and cooperating teacher tensions in a high school mathematics teacher internship: The case of Luis and Sheri

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1. Introduction

Teacher educators and teachers alike acknowledge students' clinical experiences, especially their teaching internship experience, as one of the most important and influential experiences in their teacher education programs (Byrd & McIntyre, 1996; Wilson, Floden, & Ferrini-Mundy, 2002). Despite the importance of teaching internships, there has been little research on prospective teachers teaching internship experience (Peterson, Williams, & Durrant, 2005; Wilson, et al., 2002). Although this lack of research exists across disciplines, it is especially acute in secondary mathematics education (Mtetwa & Thomspon, 2000; Rhoads, Radu, & Weber, in press). We seek to address this void with the present study.

The teaching internship is a time in which many teachers develop their philosophy of teaching. The cooperating teacher with whom a student teacher is placed may contribute to the development of this philosophy. In a survey of 63 secondary mathematics student teachers, interns cited their cooperating teacher as having the greatest influence on their teaching philosophy (Frykholm, 1999). However, sometimes this influence may not be a positive one. Ensor (2001) described how one secondary mathematics student teacher rarely taught in a manner that aligned with her own teaching philosophy. Ensor hypothesized that this may have been due to the cooperating teacher's different philosophy. In a more recent study with nine secondary mathematics student teachers, Rhoads, Radu, and Weber (in press) found that student teachers felt that having teaching philosophies that differed from their cooperating teacher was not problematic, as long as they were given freedom to try out their own teaching methods.

The feedback that cooperating teachers provide interns also affects student teachers' pedagogical and mathematical development. Peterson and Williams (2008) presented a case study of two secondary mathematics student teachers. One teacher was paired with a mentor who challenged her to think deeply about the mathematics she was teaching, but the other student teacher was paired with a cooperating teacher whose feedback focused on classroom management issues. This second student teacher missed key opportunities to develop his mathematical knowledge for teaching. Other researchers have suggested that mathematics-specific feedback is rare in the student teaching experience (Fernandez & Erlbigin, 2009).

Freedom of teaching methods and the feedback that student teachers receive from their cooperating teachers are just two of many factors that can affect a student teacher's experience. In a previous study, we found a wide variance in the quality of prospective teachers' internship experiences (Rhoads, et al., in press). Some students reported having positive experiences where they learned a great deal. Others reported having negative experiences where they had tense relationships with their cooperating teachers and felt constrained in the teaching methods they were allowed to apply. The purpose of this paper is to understand such a negative relationship in more detail. We do this by presenting a case study of a student teacher and a cooperating teacher who had a difficult relationship, focusing on what issues may have contributed to these difficulties.

2. Research methods

Context. This data came from a larger study that took place at a large northeastern state university. In the fall of 2009, there were seven prospective high school mathematics teachers enrolled in a five-year mathematics education program at this university. To understand their teaching internship experiences, we interviewed all seven of these students, along with six of their cooperating teachers and three of their supervisors, about their teaching internship experience. In this paper, we focus on one student, Luis, and his cooperating teacher, Sheri. (Luis and Sheri are pseudonyms.)

Luis and Sheri. Luis worked with two cooperating teachers, Sheri and Anya. Anya declined to be interviewed but gave Luis very favorable evaluations. By most accounts, Luis was an exceptional student. His GPA as a mathematics major was nearly 3.9; he earned A's in all of his teacher education classes, and the teachers of his mathematics education classes raved about his performance; and his student-teaching supervisor gave him very high evaluations, saying he could develop into a "master teacher". Sheri was viewed by her principal and Luis' supervisor as an effective teacher; by her account, she had worked successfully with two student-teachers in the past.

Data and analysis. At the end of the semester, Luis and Sheri individually met with the first author for a semi-structured interview about their experiences during Luis' teaching-internship. Questions were based on the preliminary findings reported in Rhoads, et al. (in press) and focused on their overall experience, their relationships with one another, the freedom Luis was permitted in the classroom, and the feedback Luis received. Analysis of these interviews was conducted by the authors in the style of Strauss and Corbin (1990); the findings of this analysis were then compared with the interview with Luis' supervisor as well as written artifacts that we collected (i.e., Sheri and others' formal feedback of Luis and 20 pages of hand-written feedback that Sheri provided to Luis). Once our tentative conclusions were reached, the first author again interviewed Luis to see if he felt these findings were accurate and to ask about issues we found ambiguous. This data was used to amend our findings.

3. Results

Although Luis and Sheri both professed to respect one another and not dislike each other personally, each reported having a difficult internship experience. We identified seven causes of tension between them: (a) different perspectives on how much freedom Luis was allowed, (b) disagreements about what mathematics should be emphasized in Luis' teaching, (c) Luis' failure to understand students' thinking, (d) Luis' difficulties with time management, (e) Sheri's propensity to interrupt Luis during his lessons, (f) Luis receiving little feedback from Sheri late in the semester, and (g) a tense personal relationship between them.

In the presentation, we will illustrate each of these points in detail. For the sake of brevity, we discuss only three in this proposal.

Freedom of teaching methods. Sheri taught primarily with the use of PowerPoint slides. She also required that Luis have his notes prepared in a format that could be readily displayed to students. However, Sheri felt Luis had freedom because he could prepare his notes and solutions to in-class problems using PowerPoint, overhead slides, or in some other format. In this way, Sheri believed she allowed Luis freedom in "pretty much everything" in teaching. In contrast, Luis felt constrained that he could not be responsive to the students, in part because he could not work through in-class problems in real time. This cause of tension suggests that what might constitute freedom to a cooperating teacher might be quite restrictive for a student teacher.

What mathematical topics should receive emphasis. Luis taught precalculus with Sheri and believed it was important to prepare the students mathematically for higher courses, such as calculus. For example, when Luis taught addition of functions, he encouraged students to think critically about the domain of a sum of two functions with different domains. During this lesson, Sheri interrupted Luis to say "you're spending way too much time on this". In her interview, Sheri expressed that many of her students were not going to take calculus and so the ideas that Luis emphasized were unnecessary, confusing to students, and too time consuming. Many mathematics educators would likely prefer the ideas that Luis emphasized in his teaching, and this points to the possible conflicts between the goals of mathematics educators and those of cooperating teachers in the internship experience.

Common difficulties of beginning teachers. Both Sheri and Luis acknowledged Luis had difficulty with time management and understanding student thinking. However, Luis was not alone in this regard. All the student teachers that we interviewed had similar difficulties, and other cooperating teachers and supervisors found this to be normal. One difference in Sheri and Luis' interactions was she thought that this was a serious shortcoming that prevented Luis from teaching her class competently.

4. Significance

Typically, in the United States, cooperating teachers receive little or no formal preparation informing them of how to be effective cooperating teachers or even telling them what to expect (Giebelhaus & Bowman, 2002). Recently, some researchers have urged for the development of such preparation programs (Feiman-Nemser, 2001; Giebelhaus & Bowman, 2002). Our results suggest what might be included in such programs. First, cooperating teachers should be aware of what difficulties student interns are likely to have so they do not find these difficulties to be problematic. Second, cooperating teachers should be encouraged to allow student teachers sufficient freedom to try out the ideas they learned in their teacher education programs. Third, mathematics educators and cooperating teachers should be encouraged to discuss their philosophies and goals regarding the student-teaching experience. Such conversations may not lead to consensus, but could lead to a mutual understanding and help to avoid some of the tension that we saw with Luis and Sheri.

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Promoting Students' Reflective Thinking of Multiple Quantifications via the Mayan Activity

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The aim of this presentation is to introduce the Mayan activity as an instructional intervention and to examine how the Mayan activity promotes students' reflective thinking of multiple quantifications in the context of the limit of a sequence. The students initially experienced a difficulty due to the lack of understanding of the meaning of the order of variables in the definition of convergence. However, such a difficulty experienced was resolved as they engaged in the Mayan activity. The students also came to understand that the independence of the variable ε from the variable *N* is determined by the order of these variables in the definition. The results indicate the Mayan activity promoted students' reflective thinking of the independence of ε from the variable *N* and helped them understand why the order of variables matters in proving limits of sequences.

Keywords: Quantification, Reflective Thinking, Proof Evaluation, Convergent Sequence, Cauchy Sequence

Introduction

The purpose of this paper is to introduce the Mayan activity as an instructional intervention and to give an account of its effect on students' understanding of multiple quantifications in the context of the limit of a sequence. The ε-N definition of the limit of a sequence is of fundamental importance and is very useful in studying advanced mathematics; however, many students encounter difficulties when learning the ε-N definition (e.g., Mamona-Downs, 2001; Roh, 2009, 2010). In particular, students' difficulty is caused by their lack of understanding of multiple quantifications in general (Dubnisky & Yiparaki, 2000) as well as the logical structure of the ε-N definition (Durand-Guerrier & Arsac, 2005). Many students cannot perceive the importance of the order between ε and N in the ε-N definition, and they cannot recognize the independence of ε from N (Roh, 2010, Roh & Lee, in press). Accordingly, in order to improve students' understanding of the ε-N definition of limit, it is important to enable the students to understand the role of multiple quantifiers in the definition. The Mayan activity is specially designed with the intention of helping students understand the *independence of* ε *from* N in the ε-N definition of the limit of a sequence. By comparing students' responses before and after the Mayan activity, this study addresses the following research question: How do students develop their understanding of the role of the order of variables in the ε-N definition via the Mayan activity?

Theoretical Perspective

The theoretical perspective is based on Dewey's theory of reflective thinking. According to Dewey (1933), when an individual is opposed to his or her knowledge or belief, he or she experiences perplexity, difficulty, or frustration; then in the process of resolving it, the reflective thinking is necessarily accompanied. Dewey divides reflective thinking into three situations as follows: The *pre-reflective* situation, a situation experiencing perplexity, confusion, or doubts; the *post-reflective* situation, a situation in which such perplexity, confusion, or doubts are dispelled; and the *reflective* situation, a transitive situation from the pre-reflective situation to the post-reflective situation. In addition, Dewey characterized the reflective situation in terms of suggestions, intellectualization, hypotheses, reasoning, and tests of hypotheses by actions, which are not always in the order but some phases can be omitted or include sub-phases. In line with this perspective, the Mayan activity introduced in this paper was designed to provide arguments described in a way that ε is selected depending on N and against student knowledge or belief about limit, and to present a tractable context later in which the students can properly activate their reasoning and perceive the independence of ε from N, hence resolve their perplexity, difficulty, or frustration about their problem.

Research Methodology

The research was conducted as part of a design experiment (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003) at a public university in the USA. The tasks designed were iterated 4 times from fall 2006 to spring 2010. Such an iterative nature of the design experiment allowed for frequent cycles of prediction of student learning, analysis of student actual learning, and revision of the tasks. This paper reports two studies from the design experiment: Study 1 in the fall semester of 2006 and Study 2 in the spring semester of 2010. The participants were mathematics students or preserive mathematics teachers, and had already completed calculus and a transition-to-proof course. The author of this paper served as the instructor in both studies. The classes in both studies mainly followed an inquiry approach, in which students often made conjectures, verified their argument, or evaluated whether given arguments were legitimate as mathematical proofs. In this manner, the students studied the limit of a sequence and its related properties, in particular, the ε-N definition, and its negation, and limit proofs using the ε-N definition. Also, the similar discussions related to Cauchy sequences followed prior to the days of this study.

In Study 1, the instructor asked the students to evaluate Statement 1: *If a sequence* $\{a_n\}_{n=1}^{\infty}$ *in* $\mathbb{R}$ *is a Cauchy sequence, then for any* $\varepsilon > 0$*, there exists* $N \in \mathbb{N}$ *such that for all* $n > N$*,* $|a_n - a_{n+1}| < \varepsilon$. After the group discussion about Statement 1, the instructor asked the students to evaluate Ben's argument: *Consider* $a_n = 1/n$ *for any* $n \in \mathbb{N}$. *Since the sequence* $\{a_n\}_{n=1}^{\infty}$ *is convergent to 0, it is a Cauchy sequence in* $\mathbb{R}$. *Let* $\varepsilon = 1/\{(N+1)(N+2)\}$ *for all* $N \in \mathbb{N}$. *Let* $n = N+1$. *Then* $n > N$. *But* $|a_n - a_{n+1}| = |a_{N+1} - a_{N+2}| = 1/\{(N+1)(N+2)\} = \varepsilon \geq \varepsilon$. *Therefore, Statement 1 is false.*

On the other hand, the Mayan activity (Roh & Lee, in press) implemented in Study 2 consists of three steps: The first is to evaluate Sam's argument and Bill's argument. Sam's argument is a proper argument showing the sequence $\{1/n\}_{n=1}^{\infty}$ converges to 0 whereas Bill's argument draws an erroneous conclusion that $\{1/n\}_{n=1}^{\infty}$ does not converge to 0 by selecting ε dependently on N; the second is to evaluate the Mayan stonecutter story (see Figure 1) in which the priest's argument is compatible to Bill's argument, but is relatively easier than Bill's or Ben's argument to track on the logical error made by reversing the order of two variables from the craftsman's argument in the story; and the third is to evaluate Statement 1and Ben's argument to Statement, which were used in Study 1. Comparing results from Study 1 with those from Study 2, this paper addresses the role of the Mayan activity as an instructional intervention in promoting students' reflective thinking of the independence of ε from N.

The Mayan stonecutter story

One of the famous Mayan architectural techniques is to build a structure with stones. These stones were ground so smoothly that there was almost no gap between two stones. It was even hard to put a razor blade between them. One day a priest came to a craftsman to request smooth stones.

Craftsman: *No matter how small of a gap you request, I can make stones as flat as you request if you give me some time.*

Priest: *I do not believe you can do it. If I ask you to flatten stones within 0.01 mm, you won't be able to do it.*

Craftsman: *Give me 10 days, and you will receive stones as flat as within 0.01 mm.*

Ten days later, the craftsman made two stones so flat that the gap between them was within 0.01 *mm*. On the 11th day, the priest came to see the stones and argued that,

Priest: *These stones are not flat within 0.001 mm. What I actually need are stones as flat as within 0.001 mm.*

Craftsman: *Okay, if you give me 5 more days, I can make the stones as flat as within 0.001 mm.*

Five days later, the craftsman made the two stones so flat that the gap between them was within 0.001 *mm*. On the 16th day, the priest came to see the stones and argued that,

Priest: *But these stones are not flat within 0.0001 mm and I meant 0.0001 mm. You don't have that kind of skill, do you?*

If the priest keeps arguing this way, is the priest really fair showing that the craftsman does not have the ability to flatten stones within any margin of error?

Figure 1. The Mayan stonecutter story.

Results and Discussions

It is expected that when two conflict arguments to each other are suggested, students can recognize that at least one of the arguments is false. However, it is not assured that they will select the true statement between the two conflict arguments. In Study 1, many students initially accepted Statement 1 as a true statement, but they reversed their determination of Statement 1 to accept Ben's argument. Although the students had considerable experiences with rigorous proofs about the convergence of sequences and their reasoning was proper in deriving the truth of Statement 1, they had deficiency of perception of the independence of ε from N, and could not give their refutation against invalid conclusions derived from allowing ε to be selected dependently on N. This result from Study 1 indicates that in order to properly promote students' reflective thinking of the independence of ε from N, it is needed to exclude the possibility that students can accept an argument, such as Ben's argument, that is described by choosing ε dependent on N, hence to be false.

In Step 1 of the Mayan activity implemented in Study 2, two conflict arguments were also given to students: One is Sam's argument that students can be convinced of the truth of its conclusion, and the other is Bill's argument that is contradictory to Sam's argument by choosing ε dependent on N. Unlike Study 1, students in Study 2 could perceive that Bill's argument induces an erroneous conclusion. Pointing out that a negation was attempted in Bill's argument, the students also intellectualized the problem of Bill's argument, and took note of that a negation was tried in Bill's argument. It indicates that they were beyond just suggesting the invalidity of Bill's argument, but further explored intellectually the problem of Bill's argument. Nonetheless, similar to the students in Study 1, the students in Study 2 were unable to find the logical fallacy in Bill's argument. When a student Matt asked "how did he [Bill] not correctly [conclude] it? I guess that's part of the question here," other students encountered a difficulty in explaining the reason why such an erroneous conclusion could be derived. These students did not develop any proper hypothesis and did not make any proper reasoning, to the problem of Bill's argument. Consequently, they failed to resolve their perplexity caused from Bill's argument.

It is worth noting that in Step 2 of the Mayan activity while evaluating the priest's argument in the Mayan stonecutter story, the students in Study 2 instantly suggested the priest unfair. In addition, they perceived that the priest attempted to disprove the craftsman's claim, and intellectualized that in order to disprove the craftsman's claim, the priest should prove the negation of the craftsman's claim. After comparing the negation of the craftsman's claim and the priest's argument in terms of quantified statements, the students recognized that the order between the margin of error and time in the priest's argument was reversed from that in the negation of the craftsman's claim. The students then hypothesized that the reversal of the quantifiers in the priest's argument entailed the illogical conclusion that the priest made. They also reasoned out that while attempting to disprove the craftsman's claim, the priest generated an irrelevant argument to the negation of the craftsman's claim. Eventually the students found why the priest's argument is invalid. As a consequence, they came to understand why the order of variables in these arguments is improperly determined. Furthermore, in Step 3 of the Mayan activity, the students were convinced of their reasoning by confirming that the reversal of the order of the variables in Ben's arguments is the same logical problem as that in priest's argument.

The results from this study indicate that the Mayan activity played a crucial role as an instructional intervention in promoting students' reflective thinking and helping them understand the role of the order of variables in the ε-N definition. The Mayan activity enables students to experience first-hand the meaning of the independence of ε from N. In fact, the activity introduces the Mayan stonecutter story from which students concretely realize the problem of describing ε dependently on N. In addition, the priest's argument is logically *compatible* with Bill's argument but is *tractable* so that students easily understand the logical structure and perceive the logical fallacy in the argument. Furthermore, the stonecutter story is a *transferrable* context in the sense that students can properly link the variables (gaps between stones and days) in the priest's argument to the variables (ε and N) in Bill's argument.

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How Mathematicians Use Diagrams to Construct Proofs

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1. Introduction

One of the primary goals of undergraduates' upper-level mathematics courses is to improve their abilities to construct formal proofs. Unfortunately, numerous studies reveal that mathematics majors have serious difficulties with this task (e.g., Moore, 1994; Weber, 2001). While there has been extensive research documenting undergraduates' difficulties with proof construction, research on how undergraduates can or do successfully construct proofs has been limited.

One approach that several researchers recommend is for students to base their formal proofs on diagrams and other informal arguments (e.g., Gibson, 1998; Raman, 2003). These recommendations are supported by the theoretical advantages afforded by visual reasoning (Alcock & Simpson, 2004; Gibson, 1998), successful illustrations of students using visual arguments as a basis for formal arguments (e.g., Alcock & Weber, 2010; Gibson, 1998), and the fact that mathematicians claim to use diagrams extensively in their own work.

However, for this to be useful pedagogical advice, more research is needed on *how* students can effectively use diagrams in their proof construction. Researchers such as Pedemonte (2007) and Alcock and Weber (2010) have noted that students find it difficult to translate an informal visual argument a formal proof. Also, several studies have failed to find a correlation between students' propensity to use diagrams and their success in proof-writing (e.g., Alcock & Simpson, 2004; Alcock & Weber, 2010; Pinto & Tall, 1999). If undergraduates are to successfully use diagrams as a basis for their proofs, they need to have a better understanding of how diagrams can be useful in proof construction and the skills needed to express and justify inferences drawn from a diagram in the language of formal mathematical proof. The goal of this presentation is to investigate these issues by analyzing ten mathematicians' behavior as they complete a non-trivial proof construction task that invites the construction and use of a graph.

2. Theoretical assumptions

This paper is based on the assumption that a goal of instruction in advanced mathematics courses is to lead students to reason like mathematicians with respect to proof (a position endorsed by Harel & Sowder, 2007), realizing these goals requires having a more accurate understanding of mathematical practice than we currently have (a position argued by the RAND Mathematics Study Panel, 2003), and we can improve our understanding of mathematical practice by carefully observing mathematicians engaged in mathematical tasks (see Schoenfeld, 1992).

3. Research Methods

Data collection. Ten mathematicians participated in a study in which they were asked to "think aloud" as they proved that the sine function was not injective on any interval of length greater than π. They were told to produce a proof suitable for an undergraduate textbook for second and third year mathematics majors. This task was chosen because we anticipated the participants would likely draw a graph of the sine function, quickly become convinced that the theorem was true as a result of inspecting this graph (or prior to constructing it), but nonetheless have some difficulty producing a formal argument that

this was true. We note in the results section that our assumptions proved to be accurate. All interviews were videotaped.
Analysis. Analysis was conducted in the style of Weber and Mejia-Ramos (2009). We first noted every inference the participant made while constructing the proof, where an inference could be a mathematical assertion (e.g., $\sin(x + \pi) = -\sin x$), a proving approach (e.g., use a proof by cases, use a calculus-based derivative argument), or an evaluation of either (e.g., a conjecture is not true, $\sin(x + \pi) = -\sin x$ is true but not useful to prove the claim). For each inference, we coded whether the inference was made from inspecting the appearance of the graph, a logical deduction from some other inference, recall, or from some other source (e.g., a metaphor, some other diagram they constructed). Also, for each inference, we noted what previous inferences that the new inference was based upon. Once this was coded, we looked at the final proof and determined the chain of inferences used to produce this written argument. Consequently, for each inference we coded, we determined whether it was part of a chain of argumentation that led to the final proof or constituted a "dead-end" (i.e., was not directly used to produce the final argument). Finally, for each inference that was based on a graph, we used an open-coding scheme to categorize how the graph was used to support this inference.

4. Results

This was a surprisingly challenging task for mathematicians. One participant was unable to complete it successfully and several other mathematicians produced invalid proofs. Nine of the ten participants spent between 9 and 40 minutes in completing this task. During their proof construction processes, most drew inferences or suggested proof approaches that did not play a role in the construction of the proofs they wound up producing, suggesting that translating the conviction they obtained from the graph to a formal proof was not direct or straightforward.

The participants used the graph for six purposes:
(a) noticing properties and generating conjectures of the sine function that might be useful for the proof (e.g. sine is periodic with period 2π),
(b) representing or instantiating an assertion or an idea on the graph,
(c) disconfirming conjectures that are not true (e.g., one participant initially conjectured $\sin(\pi + x) = \sin x$ and used the graph to reject this conjecture)
(d) verifying properties that they deduced through logic,
(e) suggesting proving techniques (such as using the periodicity of the sine function or forming a case-based argument) to prove the theorem,
(f) using the graphs as a justification for claims they wished to make (e.g., noting that a student could see that a claim was true by inspecting the graph).
The extent of participants' graph usage. The extent of graph usage varied greatly by participant, with some frequently interacting with the diagram and others making little use of the diagram after it was drawn. Some proofs were not based on any inferences that were derived from the graph, suggesting that not all mathematicians write their proofs based on the visual arguments they used to obtain conviction.
Skills needed to use visual diagrams in proof-writing. We identified a number of skills that the participant used to utilize the graphical inferences they made into their formal proofs. These skills included access to a number of domain specific proving strategies (e.g., for a continuous function, proving injectivity and monotonicity are equivalent),

fluency in algebraic manipulations, and translating logical statements into equivalent statements that are easier to work with.

The limited use of the graph in the final product. Only one participant included the diagram in the proof that he would present in the textbook. This illustrates how mathematicians may, perhaps unintentionally, mask the informal processes they use to create formal arguments when presenting proofs to their students. When this was pointed out to them, some viewed the lack of a graph as a shortcoming of their presentations while others did not.

5. Significance

The participants' difficulties with this task shows how challenging it is to base a formal proof on visual evidence. Hence, it should be no surprise that students also find this process difficult. This study describes the specific ways in which the visual diagrams were used by the participants to construct their proofs. It can be beneficial for instructors to make students aware of these purposes. The variance in the extent of graphical usage is consistent with the arguments of others that there is no single way that mathematicians engage in doing mathematics; some mathematicians use diagrams regularly in their mathematical work while others do not (e.g., Pinto & Tall, 1999; Alcock & Inglis, 2008). Finally, the skills that we outlined are important for students to master if they are to successfully use diagrams in their own proof-writing.

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Exploring the van Hiele Levels of Prospective Mathematics Teachers

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This research project aimed to assess the influence of an inquiry-oriented, technology-based, proof-intensive geometry course on the van Hiele levels of prospective mathematics teachers. Data was collected in an upper division geometry course taught from an inquiry-oriented perspective. The course relied on technology (The Geometer's Sketchpad) to help students make and prove conjectures. Data was collected from classes in consecutive years, the first with twenty-one participants and the second with twenty-four participants. Most participants were prospective secondary mathematics teachers. Data collection included a pre- and a posttest of participants' van Hiele levels. Data analysis suggests similar results for both sets of participants in that the course had greater influence on the van Hiele levels of female participants. Results also suggest that the van Hiele test instrument used for this study operated well with university students.

Geometry, van Hiele levels, teacher preparation, secondary

This research was conducted as a quantitative study using a pre- and a posttest design with a convenience sampling as defined by Creswell (2005). The participants were from two geometry classes, in consecutive calendar years, in a four-year Master's granting university located in the central coast of California. Only one section of the course is offered per calendar year and every student enrolled was offered to participate in this study. Twenty-one students participated in the data collection from the first class and 24 students participated from the second class. Of the 45 participants from the two classes the majority had declared an interest in teaching secondary mathematics and some were considering teaching at the community college level. The course was taught each time over a ten-week period, and met four times a week for 50-minute sessions. The prerequisites for this geometry course included a course in methods of proof in mathematics, which focuses on instruction of logic and proof techniques. In addition, this geometry course is mandatory for mathematics majors in the teaching concentration while open to other students who have met the prerequisites. The purpose of this study was to assess whether a proof-intensive geometry course, taught from an inquiry-oriented, technology-based perspective, has any influence on the van Hiele levels of prospective mathematics teachers and whether the influence, if any, varies by gender.

In 1957, Pierre Marie van Hiele and Dina van Hiele-Geldof, mathematics educators in the Netherlands, developed a learning model for geometry as their doctoral thesis. They defined what are known as "the van Hiele levels of development in geometry", which, according to van Hiele-Geldof's thesis, are hierarchical (cited in Fuys, Geddes, & Tischler, 1984). Altogether, there are five van Hiele levels (VHLs): 1) visualization - students visualize geometrical figures as a whole and recognize them by their particular shape; 2) analysis - students recognize the geometric properties of the different figures and are able to analyze the figures separately, but do not yet make connections between figures; 3) abstraction - students recognize relationships between figures and between properties of different figures; 4) formal deduction - students can write proofs and should provide justifications for each step in the proof; 5) rigor - "… student understands the formal aspect of deduction… [and] should understand the role and necessity of indirect proof and proof by contrapositive" (Mayberry, 1983, p. 59), and students can understand non-Euclidean geometries. These definitions were gathered from several authors' interpretations of the five van Hiele levels (Burger & Shaughnessy, 1986; Mayberry, 1983; Mistretta, 2000). Exact definitions can be found in van Hiele-Geldof's doctoral thesis (Fuys et al., 1984), and a more detailed list of behaviors at each level can be found in Usiskin (1982, pp. 9-12).

As seen in past research, the van Hiele level or the level of competence in geometry of some teachers is not at the highest level (Mayberry, 1983, pp. 67-68; Sharp, 2001, p. 201; Swafford, Jones, & Thornton, 1997, pp.469-470), thus possibly hindering the learning of geometry of some students. A conflict may arise when there is a discrepancy between the van Hiele level of the teacher and the zone of proximal development (ZPD) (Vygotsky, 1987) of the student. We expect this conflict to be mitigated if a teacher is at VHL 5.

While it is ideal for all prospective teachers to be at VHL 5, gender differences favoring males are almost twice as large in geometry as in other areas of mathematics (Leahey & Guo, 2001). Furthermore, even though the findings reported in the literature suggest variations in gender differences, the differences are mostly in spatial visualization tasks (Battista, 1990). Senk and Usiskin (1983) studied high school geometric proof writing abilities, which they consider as a high-level cognitive task requiring some spatial ability. However, while overall geometry performance has not been analyzed by gender, they found no gender differences in achievement in geometric proof writing at the end of a one-year geometry course even though females started the year with generally less geometry knowledge (p. 193).

This review of literature only found a few peer-reviewed published studies involving the level of content knowledge in geometry of prospective or practicing teachers. Among them, one study has been conducted on VHLs of prospective elementary teachers (Mayberry, 1983), one on VHLs of practicing middle-grade teachers (Swafford et al., 1997), and one on developing the geometric thinking of practicing K-7 teachers (Sharp, 2001), but none on the influence of an inquiry-oriented, technology-based, proof-intensive geometry course on VHLs of prospective secondary mathematics teachers.

After examining several documents written by the van Hieles and describing behaviors at each van Hiele level, Usiskin (1982) developed a 25-item test instrument to assess the van Hiele level of an individual. Although this instrument was primarily devised with high school students in mind, it has been used, with permission from the authors, for this study (S. Senk, personal communication, November 19, 2007). Whether the subjects involved would constitute an appropriate reference base for the study using Usiskin's test was considered since the subjects involved have all completed a one-year high school geometry course. However, even though the van Hiele levels have been defined while studying high school students, Pierre-Marie van Hiele, as quoted by Usiskin, believed that the highest level is "hardly attainable in secondary teaching" (1982, p. 12). Furthermore, Mayberry (1983), who devised her own test instrument, found that "70% of the response patterns of the students who had taken high school geometry were below Level III" (equivalent to level 4 in this study), and "only 30% were at Level III" (pp.67-68). Time constraints in preparing a VHL test and in-class time usage were also key factors in the selection of a test instrument. Burger and Shaughnessy (1986), as cited by Jaime and Gutiérrez (1994, p. 41), developed a test to assess VHLs, but its administration, through an interview, requires more time to conduct. Mayberry's (1983) 128-item test was discarded for the same reason. Usiskin's test was readily available and it is a timed-test limited to 35 minutes.

Finally, in 1990, Usiskin and Senk confirmed the validity of Usiskin's test even though they were aware of a better instrument, the RUMEUS (Research Unit for Mathematics Education at the University of Stellenbosch) test. Smith, as cited by Usiskin and Senk (1990), admitted that Usiskin's test was quicker and more convenient to apply in addition to being shorter than the RUMEUS test which he had used in a comparative study with Usiskin's test (p.245). It was thus decided to move forward with Usiskin's test to assess the van Hiele levels of development in geometry in a post-secondary setting.

Usiskin's test was administered during the first and last class meetings as a pre- and posttest. During class, students typically worked on inquiry-oriented activities using the dynamic geometry program *The Geometer's Sketchpad* (GSP) (KCP Technologies, 2006). The activities were generally completed in groups and provided the foundation for the inquiry-oriented, technology-based nature of the class as participants were expected to make and prove conjectures from their exploration with the dynamic geometry software. After students engaged with the activities, they were regularly asked to present their conjectures and proofs to the class, which often resulted in multiple avenues to prove the conjectures being explored. These activities, presentations, and class assignments make up the proof-intensive nature of the course.

Before analyzing the data with respect to our research purpose, we became interested in verifying the hierarchical nature of Usiskin's van Hiele test (1982) with our participants. We implemented a Guttman scalogram analysis similar to that of Mayberry (1983) to determine whether the VHLs as tested by Usiskin's test form a hierarchy. The scalogram analysis implied that Usiskin's van Hiele test operated adequately for both of our sets of participants in terms of the hierarchical nature of the VHLs.

To interpret the results of the pre- and posttests, each participant was assigned a raw score (out of 25) and a VHL similar to what Usiskin (1982) calls a "classical van Hiele level" (p. 25). The 4-item criterion (p. 24) was used since random guessing was not expected from the participants in this study and a higher mastery level was expected considering all the participants had completed a high school geometry course. Each group of five questions in Usiskin's test corresponds to a different VHL (questions 1 to 5 correspond to VHL 1, questions 6 to 10 to VHL 2, and so on). For a participant to be assigned a level, say *n*, at least four items must have been answered correctly at level *n* and at each preceding level. If a participant answered less than four questions correctly at level 1, then level 0 was assigned. The table below summarizes the raw scores and VHLs from both sets of data.

	Data Set 1				**Data Set 2**			
	Male		***Female***		***Male***		***Female***	
	Pre	*Post*	*Pre*	*Post*	*Pre*	*Post*	*Pre*	*Post*
Raw Score	21.08	20.92	19.44	21.22	20.75	22.375	19.69	21.375
VHL	3.67	3.25	2.56	3.44	3.0	3.875	2.875	3.56

Beyond the analysis of raw scores and VHLs, we decided to look at the data by VHL to document changes, especially related to the proof-based nature of the course, levels 4 and 5. For both sets of participants, the females made statistically significant gains at VHL 4 and little change at all other VHLs. For the first group of participants, the males made statistically significant gains at VHL 5 with very little change at any other VHL. Similarly, although not statistically significant, the male participants in the second group made substantial gains at

VHL5 with their average increasing from 3.5 out of 5 questions correct to 4.25 out of 5 questions correct.

Some findings in this research are consistent with the findings of prior research. For instance, the results on the pretest are consistent with Leahey's and Guo's (2001) findings where male students did better than female students in geometry at the end of high school (p.721). As in Senk and Usiskin (1983), females and males performed (almost) equally well in geometric proof writing at the end of a geometry course. Additionally, as in this study where, in general, the females' performance has improved substantially, Ferrini-Mundy and Tarte, as cited by Leahey and Guo (p. 721), found that girls' performance improved after learning spatial-related strategies. This may correspond to the use of *The Geometer's Sketchpad* in this course and other teaching strategies used by the professor including the inquiry-oriented nature of the course. While the results of this research suggest a positive change in participants' VHLs, the small number of participants at VHL 5 continues to raise the question about the best manner to assess prospective teachers' preparedness to teach geometry.

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Introduction

Several researchers (Dorier, Robert, Robinet, & Rogalski, 2000; Harel, 1989, 1990; Harel & Kaput, 2002; Hillel, 2000; Sierpinska, 2000) have indicated the need to integrate students understanding of algebra, geometry and symbolic formalism in order to help students use linear algebra to solve problems and do proofs. The results from these studies provide powerful evidence of students' difficulties and the challenges inherent in learning linear algebra. Recently, researchers (Larson, Nelipovich, Rasmussen, Smith, & Zandieh, 2008; Possani, Trigueros, Preciado, & Lozano, 2009) have used modeling and instructional design based upon realistic situations in order to deal with integrating the algebraic, geometric and formal aspects of linear algebra . These approaches to teaching linear algebra allow for students to interact with one another, examine the situation from a variety of mathematical positions, and create meaning that is integrated and deep. In this talk, I will answer two questions: What are the activities that students engage in as they learn to symbolize vector spaces in $\mathbf{R}^n$ using realistic situations intended to promote the integration of formal linear algebra, algebraic symbolism and geometric intuition? And, what is the process by which the classroom community developed these activities and how does this process reflect the moment-to-moment and context dependent needs of that community? Answering these questions can provide teachers the ability to be responsive to student needs and thinking as they lead their classrooms in symbolizing vectors and vector equations. As well, it can provide instructional designers with valuable insight into how classroom communities integrate informal and formal aspects of linear algebra.

Theoretical Perspective

A potential consequence of researching student work on complex activities in complex mathematics is that classroom mathematical activity from this perspective requires examining how meaning for mathematical objects gets generated over time as a process of collective action and negotiation. In undergraduate mathematics, several studies have examined how classroom communities generate meaning for mathematics (Rasmussen & Blumenfeld, 2007; Rasmussen, Zandieh, King, & Teppo, 2005; Rasmussen, Zandieh, & Wawro, 2009; Stephan & Rasmussen, 2002) and the role that gesture plays in argumentation(Marrongelle, 2007; Rasmussen, Stephan, & Allen, 2004). Because of the multiple voices present in the classroom, meaning from a collective perspective is never really fixed. At a given moment, for a given task, the researcher might be able to say that students are utilizing a certain meaning or engaging in a certain activity, but that meaning is undergoing a constant process of construction and deconstruction. According to Wenger (1998), the process by which members of a community come to understand a particular artifact or concept is via the process of the negotiation of meaning. Negotiation of meaning implies that meaning is created over time as a process of give and take between members of the community. The classroom community I examined spent several class periods discussing and arguing about the creation, use and interpretation of symbols in the classroom.

In this analysis, I use the term activity to signify the collections of meanings and practices that students created, used and yielded as interpretations when working with vector spaces in $\mathbf{R}^n$. The use of the term activity is purposeful here as it indicates a frame for action that is both goal directed and the product of cultural mediation(Lave & Rogoff, 1984). As well, any set of activities has associated with a set of goal directed actions that make up that particular activity. Hence, when characterizing an activity, it is essential to indicate not only what is being done, but also to what end is that action being done.

Methods

The following analysis is based upon data gathered during a classroom teaching experiment (Cobb, 2000) conducted at a southwestern research university. This study was part of a larger study that followed an introductory linear algebra course over the course of an entire semester. The study examined eight days from that semester-long class, focusing on classroom sessions that dealt with material germane to the study, including vectors, vector equations, linear dependence/ independence, span and basis. Each classroom session was videotaped and student work and daily reflections were collected and used for triangulation purposes. . The classroom sessions in this study focused on two sets of tasks. The first set of tasks, which took place over the first 3 weeks, involved an imagined scenario involving two or three modes of transportation, symbolized by vectors, and the ability of a rider to get around in two and three dimensions using these modes. This scenario was used to teach the symbolic system of vectors and vector equations, solution methods using Gaussian elimination, linear independence and dependence, and span. It also served as a springboard for formalized linear algebra. The second set of tasks, which took place in the 13th and 14th weeks, focused on basis and change of basis and integrated the language and imagery from the first set of tasks into class discussion. Furthermore, in the third and fourth week of the semester, 3 focus group interviews were conducted, and in the final week of semester, three more were conducted. The focus groups had students address the norms of the classroom and their understanding and use of symbolic expressions. Focus group participants were chosen reflect various ability levels and because of their membership in various small groups in the class. Video-recordings were made for these focus groups and written work was collected

Analysis of this data was three-phased. In phase-one, the six focus group interviews were analyzed with regards to students' creation, use and interpretation of vectors and vector equations. Then, whole class and small group episodes were coded using a modified Toulmin scheme (Rasmussen & Stephan, 2008). The use of this scheme was intended to locate meanings that were functioning-as-if-shared in the classroom community by identifying data, claims, warrants or backings that either shift roles within a chain of arguments or cease to require further justification by members of the classroom as they are used in later arguments. The analysis of argumentation focused on activity with students' symbolizing, but also included meanings generated by students with regard to these symbols. This analysis was compared against the focus group analysis creating a narrative for the meanings that these students developed for the symbolic system. This narrative illustrated what the meanings were, how they came to be and the ways that students used them to solve problems in linear algebra. Finally, the whole class analysis was compared against the focus group analysis in order to insure that the two analyses were consistent.

Results and conclusions

Analysis of the focus group interviews and whole class sessions yielded three distinct, but integrated activities for the symbols for vector spaces in $\mathbf{R}^n$.

- Drawing and Interpreting Lines In Space
- Coordinating Slopes
- Generating Linear Combinations

The first activity, called "drawing and interpreting lines in space" was utilized as students were constructing geometric intuitions and was most prevalent when working directly with the geometry of $\mathbf{R}^2$ and $\mathbf{R}^3$. When engaged in this activity, students coordinated the lines in space in order to reach specific destinations or to generally specify where on the plane a set of vectors could reach. The directionality of the vector specifies where on the plane or in 3-space a vector allows the student to reach a destination. The use of this meaning was prevalent when discussing the parallelogram rule for vector addition and early in the class when solving for scalar multiples. Scalars were used to represent numbers of iterations of these vectors, while addition of the vectors is used in order to coordinate discrete distances in potentially differing directions.

The second activity: "coordinating slopes": reflects the use of vectors component-wise and grouping them together by common ratios. Frequently, the goal for this activity was to create relationships between two or more vectors and draw conclusions based upon those relationships. Although the term slopes often indicates geometric interpretations, for this class the term was more algebraic in its connotation. A slope was the specific relationship between the components of a vector. However, students did not find these slopes for individual vectors, but rather established a vectors slope as an equivalence class. If one vector could be expressed as a scalar times another vector, then those two vectors were members of the same equivalence class, called like "slopes." Vectors with like ratios between their components supplied redundant information, as they did not allow for movement in differing directions. This redundancy of information became a precursor for students' meanings for linear dependence, as students noticed that vectors that had like slopes allowed for movement away from the origin in one direction and movement back to the origin by multiplying by a negative scalar.

The classroom community developed the third activity, generating linear combinations, when they needed to create more generalized meanings and communicate those meanings with others. From a student perspective, algebraic relationships or geometric interpretations are either too imprecise or lack the ability to communicate an entire range of possibilities that a set of vectors might provide. Thus, the language of linear combinations provided a precise and fully generalized way of expressing mathematical solutions and relationships. It is important to note, however, that these relationships did not begin formal in nature, but instead became formal as students developed meaning for formal definitions and notation. When engaged in this activity, students used scalar multiplication and addition in conjunction with one another to identify specific properties of sets of vectors, including whether or not the set of vectors was linearly dependent or independent and what space the set might span.

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Abstract: The Mathematical Sophistication Instrument (MSI) measures the extent to which students' mathematical values and ways of knowing are aligned with those of the mathematical community based on eight interwoven categories: patterns, conjectures, definitions, examples and models, relationships, arguments, language, and notation. In this paper, we present the results of a study designed to explore whether students' scores on the MSI improved during their introductory college mathematics courses. A large sample of five sections of a first course for elementary education majors, five sections of College Algebra, and seven sections of mathematics for liberal arts majors completed the instrument both at the start and end of the spring 2009 term. Results showed that students in courses where instructors used inquiry-based pedagogies scored markedly higher on the instrument at the end of the semester than at the start. In courses where instructors used traditional pedagogies, only slight changes in scores were observed.

Keywords: inquiry-based pedagogy, teacher knowledge, mathematical enculturation, autonomy, mathematical sophistication

Background and Framework: In previous research (Seaman & Szydlik, 2007), we studied the ways in which preservice teachers learned mathematics by observing their attempts to understand ideas in arithmetic and number theory using a teacher resource website. Results suggested that our participants were profoundly mathematically unsophisticated; they displayed a set of values and tools for learning mathematics that was so different from that of the mathematical community, and so impoverished, that they were essentially helpless to create fundamental mathematical understandings.

Based on our comparison of the 2007 participants' mathematical behaviors and beliefs with those of mathematicians, we created a framework to define a construct that we termed *mathematical sophistication.* The construct is defined in terms of beliefs about the nature of mathematical behavior, values concerning what it means to know mathematics, avenues of experiencing mathematical objects, and distinctions about language and notation. Specifically, we proposed the following list of values and behaviors that indicate mathematical sophistication.

1) Seeking to understand patterns based on underlying structure.
2) Making and testing conjectures about mathematical objects and structures.
3) Creating mental (and physical) models and examples and non-examples of mathematical objects.
4) Using and valuing precise mathematical definitions of objects.
5) Valuing an understanding of why relationships make sense.
6) Using and valuing logical arguments and counterexamples.
7) Using and valuing precise language and having distinctions about language.

8) Using and valuing symbolic representations of, and notation for, objects and ideas.

Increasing students' mathematical sophistication became an articulated goal of the mathematics for elementary education sequence at our university. All instructors of those courses committed to using inquiry-based pedagogies; their students solved novel problems in small groups and then discussed their solutions, strategies, and reasoning as a class. Furthermore, instructors made the values of the mathematical community more overt. For example, making sense of definitions, discussing the value of pattern-seeking and generalization, and studying distinctions between inductive and deductive reasoning became explicit topics of that sequence. These goals are aligned with demands that teachers understand the rich connections among mathematical ideas; bridge gaps between students' use and standard mathematical use of notation and language; and model and request the mathematical behaviors of sense making, conjecturing, and reasoning (CBMS, 2001). (For a comprehensive overview of the literature on teacher knowledge see Hill et. al., 2007.)

In order to measure changes in mathematical sophistication in our students, we developed a twenty-five item, multiple-choice, paper-and-pencil Mathematical Sophistication Instrument (MSI) based on the above framework. Items were developed by, or in consultation with, eight mathematicians. Our attempt was to make the items substantially free of specific mathematics content. For example, consider the below MSI item designed to assess students' abilities to make sense of a new definition and the meaning of "or" in a mathematical context:

A number is called **normal** if it is less than 10 or even. According to this definition, of the numbers 5, 8, and 24,

a) Only 5 and 8 are normal.
b) Only 8 is normal.
c) Only 5 and 24 are normal.
d) All of these numbers are normal.

In Fall 2007 a large sample of students in their mathematics for elementary teachers courses completed the instrument during the first month of the semester. Twelve students (four who scored in the top quartile, four who scored in the middle half and four who scored in the lower quartile on the instrument) were interviewed to determine whether the level of sophistication shown by the students as they explained their thinking was reflected by their performance on the items. The MSI was revised based on that data.

In fall 2009 we assessed both the validity and reliability of the updated instrument with a large sample of undergraduates at a Midwestern comprehensive state university (Szydlik, Kuennen, & Seaman, 2010). In order to assess the validity of the instrument, course instructors rated the mathematical sophistication of their students based on our framework, and instructor ratings were compared with student scores on the items. Results suggest that the MSI is a valid measure of sophistication as defined by the eight categories. In pilot testing, the MSI has obtained Kronbach Alphas between .053 and 0.73.

Methods for the Current Study: In spring 2010 we sought to investigate whether students' scores on the MSI improved during their first course for preservice teachers: Number Systems. That semester Number Systems was taught by four different instructors, and all sections (116 students) participated. We formed two comparison groups for the research: a sample of 116 students taking a first liberal arts mathematics course (with four different instructors), and a sample 97 of students taking college algebra (with two different instructors). All three courses had the same prerequisite. The MSI was administered in classes at both the start and at the end of the semester in all participating sections. Almost all students present chose to participate.

Results: Each MSI item was scored 1 point for the most sophisticated answer (as determined by the mathematicians) and 0 points for all other response options. Cumulative pre-test scores ranged 1 to 19 (out of 25 points) and post-test scores ranged from 2 to 19 points. Students in all groups showed significant gains on the MSI during the spring 2010 term. This is not surprising; since the same instrument was used for both the pre- and post-tests, we expected gains. However, as shown in the table below, students in both Number Systems and the liberal arts mathematics course obtained important and highly significant gains on the MSI ($p < .0001$), whereas the students in College Algebra showed only modest changes. These results remained even upon deleting four MSI items with mathematics content explicitly addressed in one or more sections. For example, one instructor of liberal arts mathematics included a graph theory unit, and since two graph theory items appeared on the MSI to measure students' abilities to make sense of a new definition, those items were deleted (for everyone) in a reanalysis of the data.

Course	**MSI Score at the start of the term**	**MSI Score at the end of the term**	***p*-value**
Number Systems ($n = 116$)	Mean = 7.74 Stand. Dev. = 2.83	Mean = 10.01 Stand. Dev. = 3.65	$p < 0.00001$
Liberal Arts Math ($n = 116$)	Mean = 8.11 Stand. Dev. = 3.42	Mean = 9.12 Stand. Dev. = 3.60	$p < 0.0001$
College Algebra ($n = 97$)	Mean = 7.37 Stand. Dev. = 3.09	Mean = 7.90 Stand. Dev. = 3.17	$p < 0.04$

MSI Scores by Course

Conclusions: Because the instrument is substantially free of relevant mathematics content topics, we assert that gains on the MSI are due primarily to differences in the ways students in the courses *experienced* mathematics. According to an instructor questionnaire, and informal observations and discussions of teaching, inquiry-based pedagogies were used almost exclusively by instructors in all sections of Number Systems and were used on most days by the instructors of liberal arts mathematics. College Algebra was taught using traditional lectures.

This work suggests two conclusions. First, measurable changes in student sophistication can be affected during the course of a semester; and second, those changes appear to be the result of the students having engaged in mathematically authentic behaviors in the

classroom. In our presentation we will share our instrument, a
connections between students' mathematical experiences in tl
MSI.

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Individual and Collective Analysis of the Genesis of Student Reasoning Regarding the Invertible Matrix Theorem in Linear Algebra

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Abstract: I present research regarding the development of mathematical meaning in an introductory linear algebra class. In particular, I present analysis regarding how students–both individually and collectively–reasoned about the Invertible Matrix Theorem over the course of a semester. To do so, I coordinate the analytical tools of adjacency matrices and Toulmin's (1969) model of argumentation at given instances as well as over time. Synthesis and elaboration of these analyses was facilitated by microgenetic and ontogenetic analyses (Saxe, 2002). The cross-comparison of results from the two analytical tools, adjacency matrices and Toulmin's model, reveals rich descriptions of the content and structure of arguments offered by both individuals and the collective. Finally, a coordination of both the microgenetic and ontogenetic progressions illuminates the strengths and limitations of utilizing both analytical tools in parallel on the given data set. These and other results, as well as the methodological approach, will be discussed in the presentation.

Key words: linear algebra, individual and collective, genetic analysis, argumentation, Toulmin scheme, adjacency matrices.

The Linear Algebra Curriculum Study Group (Carlson, Johnson, Lay, & Porter, 1993) named the following as topics necessary to be included in any syllabus for a first course in undergraduate linear algebra: matrix addition and multiplication, systems of linear equations, determinants, properties of $\mathbf{R}^n$, and eigenvectors and eigenvalues. Some of the specific concepts involved in the aforementioned topics are: (a) span, (b) linear independence, (c) pivots, (d) row equivalence, (e) determinants, (f) existence and uniqueness of solutions to systems of equations, (g) transformational properties of one-to-one and onto, and (h) invertibility. These concepts, in addition to others, are the very ones addressed and linked together in what is referred to as the Invertible Matrix Theorem (see Figure 1). The Invertible Matrix Theorem (IMT), which consists of seventeen equivalent statements, is a core theorem for a first course in linear algebra in that it connects the fundamental concepts of the course.

I take the perspective that the emergence and development of mathematical ideas occurs not only for each individual student but also for the classroom as a collective whole. Many researchers acknowledge in the role of the collective on the mathematical development of a learner and vice versa (Hershkowitz, Hadas, Dreyfus, & Schwarz, 2007; Rasmussen & Stephan, 2008; Saxe, 2002). Through this viewpoint, the interrelatedness of the individual and the collective come to the fore, highlighting how the activity of one necessarily affects that of the other. These two forms of knowledge genesis—on an individual and on a collective level—are inextricably bound together in their respective developments. Therefore, in order to gain the most fully developed understanding of the emergence, development, and spread of ideas in a particular classroom, analysis along both individual and collective levels, over the course of the semester, is warranted and necessary.

This presentation will highlight portions of my dissertation research, which has two main aspects: (a) research into the learning and teaching of linear algebra, and (b) research into analyzing the development of mathematical meaning for both students and the classroom over time. The two research questions that guide my dissertation work are the following:

1. How do students –both individually and collectively—reason about the Invertible Matrix Theorem over time?
2. How do students—both individually and collectively—reason with the Invertible Matrix Theorem when trying to solve novel problems?

The first research question investigates the connections that are made, on both the individual and the collective level, between the various statements in the IMT. The second research question investigates the ways in which students, on both the individual and the collective level, use the IMT as a tool for reasoning about new problems. During my presentation, I will discuss results from both individual and collective-level analyses from question one.

Background and Methodology

The theoretical perspective on learning that undergirds my work is the emergent perspective (Cobb & Yackel, 1996), which coordinates psychological constructivism (von Glasersfeld, 1995) and interactionism (Forman, 2003; Vygotsky, 1987). In honoring the importance of both psychological and social processes, the emergent perspective posits that:

> The basic relationship posited between students' constructive activities and the social processes in which they participate in the classroom is one of reflexivity in which neither is given preeminence over the other...A basic assumption of the emergent perspective is, therefore, that neither individual students' activities nor classroom mathematical practices can be accounted for adequately except in relation to the other." (Cobb, 2000, p, 310)

From the perspective that learning is both an individual and a social process, investigating the mathematical development of students necessarily involves considering the individual development of students as well as the collective activity and progression of the community of learners in which the individuals learners participate. Thus, in studying the development of reasoning regarding the Invertible Matrix Theorem, both levels of development will be analyzed.

The overarching structure of my analysis is influenced by a framework of genetic analysis that delineates multiple levels of investigation. Saxe (2002) and his colleagues (Saxe & Esmonde, 2005; Saxe, Gearhart, Shaughnessy, Earnest, Cremer, Sitabkhan, et al., 2009) investigated knowledge development through the notion of cultural change. Particular to development in the classroom, the authors investigated how researchers could collect data (how much, from what sources, etc.) and conduct analyses that would allow them to make descriptions of how individuals' ideas develop in the classroom over time, given that the classroom is also changing over time. As a response, they suggested analyzing human development over time from three different strands, providing researchers a way to account for some of the complex factors of development. *Microgenesis* is defined as the short-term process by which individuals construct meaningful representations in activity, *ontogenesis* as the shifts in patterns of thinking over the development of individuals, and *sociogenesis* as the reproduction and alteration of representational forms that enable communication among participants in a community (Saxe et al., 2009, p. 208). I focus on and adapt the first two strands in my own analysis.

The data for this study comes from a semester-long classroom teaching experiment (Cobb, 2000) conducted in a linear algebra course at a large university in the southwestern United States. Students enrolled in the course had generally completed three semesters of calculus and were in their second, third, or fourth year of university. Furthermore, the majority of students enrolled in the course had chosen engineering (computer, mechanical, or electrical), mathematics, or computer science as their major course of study at the university.

In order to address the *individual* components in the proposed research questions, I focused on five of the students enrolled in the linear algebra course. All five sat at the same table during class, which is one of three tables that are videorecorded during every class period for the duration of the semester. In order to collect data relevant to these five individuals and their establishment of meaning regarding the IMT, I collected four sources of data: video and transcript of whole class discussion, video and transcript of their small group work, video and transcript from their individual interviews, and various written work. Individual interview data comes from two semi-structured (Bernard, 1988) interviews, one conducted midway through the semester and one conducted at the end of the semester.

In order to collect data relevant to the *collective* establishment of meaning regarding the IMT, I collected video and transcript of whole class discussion and small group work, photos of whiteboard work, and written work from in-class activities. As stated, portions of 12 class days are analyzed, which were the days that the IMT was explicitly addressed during whole class discussion.

In order to investigate how students reasoned about the IMT over time, I utilize five analytical phases, and each has both an individual and a collective level. The five phases are: 1) Microgenetic analysis via the construction of adjacency matrices; 2) Microgenetic analysis via the construction of Toulmin schemes of argumentation; 3) Ontogenetic analysis of constructed adjacency matrices; 4) Ontogenetic analysis of constructed Toulmin schemes,; and 5) Coordination of analysis across the two analytical tools. As highlighted in the five phases, I employ two main analytical tools: adjacency matrices and Toulmin's (1969) model of argumentation. Adjacency matrices are representational tools from graph theory used to depict how the vertices of a particular graph are connected (e.g., Frost, 1992). These matrices can be used to represent data from a variety of graph forms. In my dissertation, I create adjacency matrices that correspond to directed graphs in which the vertices are the statements in the Invertible Matrix Theorem (or students' explanations of those statements) and the edges are directed in such a way as to match the implication offered by the student. The developed adjacency matrices are n x n, where n is the number of recorded relevant yet distinct statements made by students in any given explanation. The rows are the 'p' and the columns are the 'q' in statements of the form "p implies q" or "another way to say p is q." Adjacency matrices are used as a tool to analyze explanations that explicitly address how students connect the ideas of the Invertible Matrix Theorem, as well as to analyze arguments made at the collective level during whole class discussion. These arguments are comprised of statements from one or many students in the class as meaning is negotiated collectively through participation in the classroom.

The second main analytical tool I use is Toulmin's (1969) model of argumentation, which describes six main components of an argument: claim, data, warrant, backing, qualifier, and rebuttal. The first three of these—claim, data, and warrant—are seen as the core of an argument. According to this scheme, the claim is the conclusion that is being justified, whereas the data is the evidence that demonstrates that claim's truth. The warrant is seen as the explanation of how the given data supports the claim, and the backing, if provided, demonstrates why the warrant has authority to support the data-claim pair. This work has been adapted by many in the fields of mathematics and science education research as a tool to assess the quality or structure of a specific mathematical or scientific argument and to analyze students' evolving conceptions by documenting their collective argumentation (Erduran, Simon, & Osborne, 2004; Krummheuer, 1995; Rasmussen & Stephan, 2008; Yackel, 2001). While the Toulmin model has proven a useful tool for documenting mathematical development at a collective level (e.g., Stephan & Rasmussen, 2002), I utilize Toulmin's model to analyze structure of individual and collective exchanges both in isolation and as they shift over time.

While Phases 1 and 2 are comprised of many discrete analyses, Phases 3 and 4 are compiled from the results of Phases 1 and 2. In Phase 3, shifts in form and function of how students reason about reason with the various concepts in the IMT over time are analyzed by considering qualitative changes in constructed adjacency matrices from Phase 2. This type of analysis is what Saxe (2002) refers to as *ontogenetic analysis*. Phase 4, on the other hand, considers the individually constructed Toulmin schemes from Phase 2 as a whole. This sort of analysis, at the collective level, is consistent with the work of Rasmussen and Stephan (2008) in identifying classroom mathematics practices. Finally, Phase 5 combines the work done in parallel with adjacency matrices and Toulmin schemes on both the microgenetic level (comparing the results of Phases 1 and 2) and the ontogenetic level (Phases 3 and 4). In other words, Phase 5 consists of cross-comparative analyses, for any given argument or collection of arguments, of the results from both analytical tools (adjacency matrices and Toulmin schemes).

Results

The cross-comparison of results from the two analytical tools, adjacency matrices and Toulmin's model, provides a rich way to investigate the content and structure of arguments offered by both individuals and the collective. A coordination of both the microgenetic and ontogenetic progressions illuminates the strengths and limitations of utilizing both analytical tools in parallel on the given data set. Analysis reveals rich student reasoning about the IMT that may not be apparent through use of only one analytical tool. For instance, adjacency matrices proved an effective analytical tool on arguments consisting of multiple connections that were for explanation, whereas Toulmin models proved illuminating for arguments with complex structure for the purposes of conviction. These and other results, as well as my methodological approach, will be discussed during my presentation.

The Invertible Matrix Theorem

Let A be an $n \times n$ matrix. The following are equivalent:

a. The columns of A span $\mathbf{R}^n$.
b. The matrix A has n pivots.
c. For every $\mathbf{b}$ in $\mathbf{R}^n$, there is a solution $\mathbf{x}$ to $A\mathbf{x}=\mathbf{b}$.
d. For every $\mathbf{b}$ in $\mathbf{R}^n$, there is a way to write $\mathbf{b}$ as a linear combination of the columns of A.
e. A is row equivalent to the $n \times n$ identity matrix.
f. The columns of A form a linearly independent set.
g. The only solution to $A\mathbf{x}=\mathbf{0}$ is trivial solution.
h. A is invertible.
i. There exists an n x n matrix C such that $CA = I$.
j. There exists an n x n matrix D such that $AD = I$.
k. The transformation $\mathbf{x} \rightarrow A\mathbf{x}$ is one-to-one.
l. The transformation $\mathbf{x} \rightarrow A\mathbf{x}$ maps $\mathbf{R}^n$ onto $\mathbf{R}^n$.
m. Col $A = \mathbf{R}^n$.
n. Nul $A = \{\mathbf{0}\}$.
o. The column vectors of A form a basis for $\mathbf{R}^n$.
p. Det $A \neq 0$.
q. The number 0 is not an eigenvalue of A.

Figure One: The Invertible Matrix Theorem

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Using The Emergent Model Heuristic to Describe the Evolution of Student Reasoning regarding Span and Linear Independence

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Key Words: Linear algebra, Student Reasoning, Realistic Mathematics Education, Inquiry-Oriented Instruction

A prominent problem in the teaching and learning of K-16 mathematics is how to build on students' current ways of reasoning to develop more generalizable and abstract ways of reasoning. This problem is particularly pressing in undergraduate courses that often serve as a transitional point for students as they attempt to progress from more computationally based courses to more abstract courses that feature proof construction and reasoning with formal definitions. One such course is that of introductory linear algebra. A promising aspect of linear algebra, however, is that it presents an array of applications to science, engineering, and economics, providing instructional designers with opportunities to use these applications to motivate and develop mathematical ideas. The purpose of this talk is to report on student reasoning as they reinvented the concepts of span and linear independence. The reinvention of these concepts was guided by an innovative instructional sequence known as the Magic Carpet Ride problem, whose creation was framed by the emergent models heuristic (Gravemeijer, 1999) of the instructional design theory of Realistic Mathematics Education (Freudenthal, 1991). The sequence makes use of an experientially real problem setting (in the sense that students can readily engage in the task) and aids students in developing more formal ways of reasoning about vectors and vector equations. Thus, during our talk we will:

1. Explain how this instructional sequence differs from a popular "systems of equations first" approach and why this conscious change was made;
2. Present the instructional sequence via the framing of the emergent models heuristic; and
3. Provide samples of students' sophisticated thinking and reasoning.

Literature Review

In addition to research that categorizes student difficulties in linear algebra (e.g., Dorier, 1995; Harel, 1989; Hillel, 2000), more recent work has examined the productive and creative ways that students are able to interact with the ideas of linear algebra. For instance, Possani, Trigueros, Preciado, and Lozano (2010) analyzed the use of a teaching sequence that began with a real life problem and reported on student progress as they advanced through different solution strategies. In a similar spirit, Larson, Zandieh, and Rasmussen (2008) reported a key idea that emerged as a central and powerful way in which students came to reason and eventually develop the formal ideas and procedures for eigenvalues and eigenvectors. Complementary to these two veins of research, we report on students' activity as they both reinvent and reason with the notions of span and linear independence.

The instructional sequence that was developed to foster student reinvention of these ideas does so within the first five days of the course, prior to any explicit treatment of Gaussian elimination. This is in contrast to a widespread tendency to begin the semester with systems of linear equations and Gaussian elimination (e.g., Anton, 2010; Lay, 2003). One possible reason for beginning the course in this manner is to build from students' prior experiences with solving systems of linear equations. We strongly agree with beginning a course with content that has an intuitive basis for students. Our instructional sequence, however, relies on a different intuitive background from which to build and structure an introductory linear algebra course. Our approach begins by focusing on vectors, their algebraic and geometric representations in $\mathbf{R}^2$ and $\mathbf{R}^3$, and their properties as sets. We contend that this switch not only fosters the development of formal ways of reasoning about the 'objects' of linear algebra, namely vectors and vector equations, but also instigates an intellectual need (Harel, 2000) for sophisticated solution strategies, such as Gaussian elimination. These aspects will be elaborated upon during the presentation.

Theoretical Background

Drawing on the work of Freudenthal (1991) and the instructional design theory of Realistic Mathematics Education (RME), we take the perspective that mathematics is first and foremost a human activity of organizing mathematical experiences in increasingly sophisticated ways. A central RME heuristic that captures this perspective is referred to as “emergent models.” This heuristic offers researchers and teachers a way to design and trace ways that students can build on their current ways to reasoning to develop rather formal mathematics. In RME the term model has a specific meaning. In particular, Zandieh and Rasmussen (2010) define models as student-generated ways of organizing their activity with observable and mental tools. Observable tools refer to things in the environment, such as graphs, diagrams, explicitly stated definitions, physical objects, etc. Mental tools refer to ways in which students think and reason as they solve problems—their mental organizing activity. Following Zandieh and Rasmussen, we make no sharp distinction between the diversity of student reasoning and the things in their environment that afford and constrain their reasoning.

The emergent model heuristic involves the following four layers of increasingly sophisticated mathematical activity: Situational, Referential, General, and Formal. Situational activity involves students working toward mathematical goals in an experientially real setting. Referential activity involves models-of that refer (implicitly or explicitly) to physical and mental activity in the original task setting. General activity involves models-for that facilitate a focus on interpretations and solutions independent of the original task setting. Formal activity involves students reasoning in ways that reflect the emergence of a new mathematical reality and consequently no longer require support of prior models-for activity. The model-of/model for transition is therefore concurrent with the creation of a new mathematical reality.

Methods

The classroom sessions analyzed for this presentation come from a classroom teaching experiment (Cobb, 2000) conducted in the spring of 2010 at a southwestern research university. This classroom was the third iteration of a semester-long classroom teaching experiment in linear algebra. Video-recordings were made of each classroom episode. Transcriptions were then made from the videos. Daily reflections and homework were also collected.

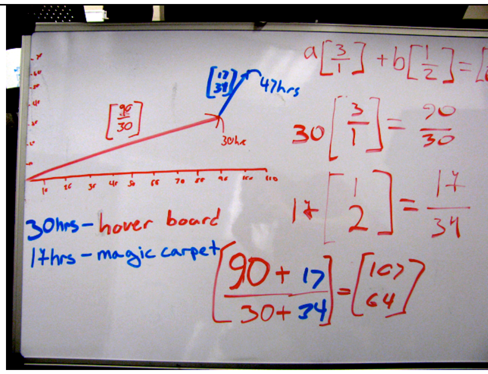
Results

This section discusses how student reasoning progressed through each of the four levels of activity throughout the semester, but especially in relationship to the tasks that students worked on during the first five days of class. Given space limitations, we provide more detail on student reasoning at the beginning of the task sequence. Note that we spent approximately one day per task during the semester.

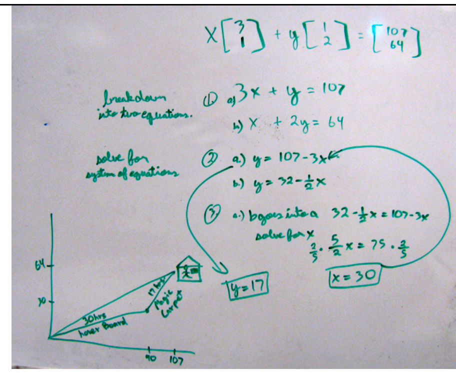
Situational and Referential Activity. The student thinking on the first two tasks was primarily Situational activity in that students focused on engaging in solving problems in the Magic Carpet Ride task setting. However, even at this level students were developing symbolic and graphical inscriptions that were models of their thinking and that the teacher was able to label with the terminology of the mathematical community such as linear combination and span. During the third and fourth tasks, student reasoning was more explicitly Referential as students used their experience in the Magic Carpet Ride setting to create a definition for the linear dependence of two vectors and as they worked to interpret the definition of linear independence in terms of the Magic Carpet Ride scenario.

> *TASK 1*. You are given a hover board and a magic carpet. The hover board can move according to <3, 1> and the magic carpet according to <1, 2>. If Old Man Gauss lives in a cabin 107 miles East and 64 miles North, can you get there with the board and carpet?

This activity helped students explore the notion of a linear combination of one or two vectors in $\mathbf{R}^2$, including its symbolic and graphical representations. The figure below provides two examples of student thinking on this problem. On the left students use a non-standard symbolic vector notation and a guess and check methodology. On the right the students converted their vector equation into a system of equations and solved for the appropriate weights.



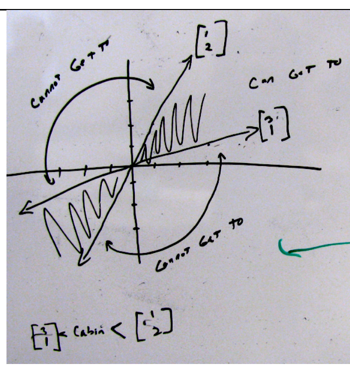
Guess and check via vector weighting



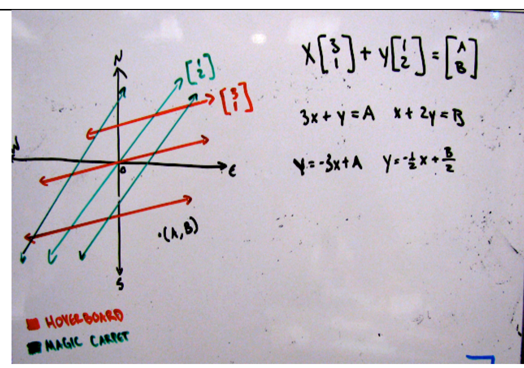
Vector equation then system

> *TASK 2*. Are there some locations where Gauss can hide and you cannot reach him from your home with these two modes of transportation?

This extension pushes students to explore how a linear combination of two vectors can encompass all points in $\mathbf{R}^2$ and introduces the term span. The figure below provides two examples of student thinking on this problem. Notice that the board on the left indicates that this group of students thought that they could only get to points within the double funnel using the two modes of transportation, whereas the group on the right used a grid to illustrate that they could reach any point on the plane.



The Double Funnel



The Grid

> *TASK 3*. You still have two modes of transportation, but now you cannot get everywhere. What are the possible vectors for the movement of the hover board and magic carpet now?

In discussing which sets of vectors span all of $\mathbf{R}^2$ and which do not, students defined linear dependence for pairs of vectors. In particular, students determined that if two vectors are multiples of each other, then they are linearly dependent.

TASK 4. You may travel each mode of transportation only once. Can you start and end back at home?

This activity allows for the introduction of the formal definition of linear independence. Students were asked to interpret this formal definition in terms of the Magic Carpet Ride task.

General Activity. In task 5, students are given a series of questions that asks them to create a linearly independent (or dependent) set of 2 (or 3 or 4) vectors in $\mathbf{R}^3$. Some students were able to develop conjectures about what must be true a set of vectors to span a space. One such conjecture was that to span $\mathbf{R}^n$, one must have n vectors and they must be linearly independent. This is General activity since the students are now working with vectors without referring back explicitly to the Magic Carpet activity as they explore properties of these sets of vectors.

Formal Activity. Formal activity occurs much later in the term as students are able to use definitions of span or linear independence in the service of making other arguments without having to explicitly recreate or reinterpret those definitions.

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