PROCEEDINGS OF THE 14TH ANNUAL CONFERENCE ON RESEARCH IN UNDERGRADUATE MATHEMATICS EDUCATION

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FOREWORD

The research reports and proceedings papers in these volumes were presented at the 14th Annual Conference on Research in Undergraduate Mathematics Education, which took place in Portland, Oregon from February 24 to February 27, 2011.

Volumes 1 and 2, the RUME Conference Proceedings, include conference papers that underwent a rigorous review by two or more reviewers. These papers represent current important work in the field of undergraduate mathematics education and are elaborations of the RUME conference reports.

Volume 1 begins with the winner of the best paper award, an honor bestowed upon papers that make a substantial contribution to the field in terms of raising new questions or gaining insights into existing research programs.

Volume 3, the RUME Conference Reports, includes the Contributed Research Reports that were presented at the conference and that underwent a rigorous review by at least three reviewers prior to the conference. Contributed Research Reports discuss completed research studies on undergraduate mathematics education and address findings from these studies, contemporary theoretical perspectives, and research paradigms.

Volume 4, the RUME Conference Reports, includes the Preliminary Research Reports that were presented at the conference and that underwent a rigorous review by at least three reviewers prior to the conference. Preliminary Research Reports discuss ongoing and exploratory research studies of undergraduate mathematics education. To foster growth in our community, during the conference significant discussion time followed each presentation to allow for feedback and suggestions for future directions for the research.

We wish to acknowledge the conference program committee and reviewers, for their substantial contributions and our institutions, for their support.

Sincerely,

Stacy Brown,
RUME Organizational Director & Conference Chairperson

Sean Larsen,
RUME Program Chair

Karen Marrongelle
RUME Co-coordinator & Conference Local Organizer

Michael Oehrtman
RUME Coordinator Elect

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Students' Logical Reasoning in Undergraduate Mathematics Courses

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This narrative describes a pilot study (the author is conducting) to prepare for a proposed study to be implemented at Salisbury University in the academic year (2010-2011). As such, it is a Preliminary Research Report. The proposed study will be done in an attempt to describe undergraduate Salisbury University mathematics/computer science students' understanding of logical inference.

It is well-known that there is an extensive list of studies that have been conducted on proof and logical inference. Many of those studies will be cited and included in the literature review for the proposed study. For this brief report , only a few most relevant sources have been cited.

Theorems in mathematics are often stated in an "if-then" form, and this type of statement is often called a "conditional statement." If someone wants to become a student of mathematics, it is imperative to distinguish between this technical form of statement and that same form as used in everyday normal speech. In everyday speech one can glean from the context the meaning that is intended even if the form of the statement is not really correct. In mathematics, however, it becomes important to know precisely what the form of the statement implies (or does not imply).

In 1984, the author (Austin, 1984) conducted a study in an attempt to assess the level of understanding that undergraduate students at a state university seemed to possess in this regard. Responses were collected from 493 students enrolled in mathematics courses and from a sample of 219 students selected randomly from the student population.

In this study, students were given the following four items (reasoning patterns):

(1) Detachment (Modus Ponens):

If the couch is soft, then it is Linda's. The couch is soft. Is the couch Linda's?

(a) yes (b) no (c) not enough clues

(2) Conversion:

If the water is warm, it is dirty. The water is dirty. Is the water warm?

(a) yes (b) no (c) not enough clues

(3)Contraposition (Modus Tollens) :

If the shoes are big, they are John's. The shoes are not John's. Are the shoes big?

(a) yes (b) no (c) not enough clues

(4) Inversion:

If the man is old, he is sad. The man is not old. Is he sad?

(a) yes (b) no (c) not enough clues

Frequencies (and percentages) of correct responses were reported, and some observations were made. As one might expect, students fared best on detachment, and not so well on the other three reasoning patterns. Although responses from those students enrolled in mathematics courses were somewhat better than those in the random sample, the difference was not strikingly impressive.

The author also participated in another study that sought to describe pre-service elementary teachers' understanding of logical inference, and the qualitative portion of this study is available (Hauk, et.al. ,2008). The quantitative portion of the study is yet to appear. The qualitative study documents the fact that students appear to be at different levels in their understandings of logical inference, and based on the conclusions from this study, several implications for teaching mathematics to these learners are suggested.

The 1984 study was a descriptive one, and no attempt was made to discover why students thought (in the ways they must have) relative to their responses. Also, no real theoretical framework was used to interpret or shed light on any of the findings. In 1984 it seems that few, if any, researchers were using mixed methods in their research, and though much research was being done in mathematics education, there were not many research studies being conducted in undergraduate mathematics education. Since that time the research milieu has changed in that today many research studies employ mixed methods and undergraduate mathematics education research has become an active area of inquiry.

APOS (Action-Process-Object-Schema) theory describes a possible way that learners progress in their attainment of mathematical concepts (Dubinsky & McDonald, 2001). One of the salient aspects of APOS theory is that it posits that learners can be at different levels in their understandings of mathematical ideas/concepts. Like other theories of learning, this constructivist theory also embodies a progression from lower levels to higher levels of attainment. Although it is understood that it is probably not the case that everyone's learning journey fits into this model in a lockstep way, the APOS theory provides a means or lens by which or through which empirical results might be viewed or interpreted.

Balacheff's levels of proof understanding is another model with four stages (Balacheff, 1988). The four levels described in this model are (a) naïve empiricism or "proof by example" strategies; (b) crucial experiment (includes generation of counterexamples); (c) generic examples; and (d) thought experiment (learner arrives at structured deductive logical forms. Again, there is this progression from low levels to higher levels, and gives a background for assessing levels of understanding of proof.

This preliminary report describes a pilot study in which data has been collected on the four reasoning patterns (as the 1984 study) from students enrolled in one introductory statistics class (freshman class), two statistics laboratory classes (freshman/sophomore class), one class in a statistics class for mathematics and computer science majors (sophomore class), and a class in abstract algebra (junior/senior class). In addition to having the students give their responses to the multiple-choice items for the four reasoning patterns, students were asked to record their thoughts (thinking) as to why they chose the response they did. Students were asked if they found some items easier than others. If they did find items easier, they were asked why they thought they were easier? All of the data in the pilot study was collected by written information given by the students in a classroom setting.

The research questions for this pilot study were the following:

1. Are there observable differences between the responses for the four different classes? If so, how can these be characterized?
2. Can different levels of learning be detected (as posited by APOS theory)? How do these levels (if detected) compare to the level of the course? Do they match well or is there a mismatch between the two?

3. If the four reasoning patterns are ordered based on performance in each of the classes, is the order the same for all classes or are there different orders for different classes? How do the orders compare/contrast with the ideas of easiness? Does the performance on the four reasoning patterns match the ideas of easiness well or not?

Method

For the pilot study data were collected using a survey instrument and given to students enrolled in the classes. No formal instruction of logical reasoning had been given in any of the classes prior to the collection of the data. At this point, the data have not been analyzed, but findings from the pilot study will be presented at the RUME Conference (2011).

For the proposed study, data will be collected in the academic year (2010-2011) at Salisbury University using students enrolled in several undergraduate mathematics courses. Data will be collected using the refined survey instrument and by interviews. The author seeks suggestions and ideas from anyone at the RUME conference, especially for the interview protocols. Since the study is somewhat "fluid" at present, all suggestions for questions for the interview protocols will be welcomed and any ideas for conducting the study are solicited.

Questions:

1. What are your ideas for questions for the protocols?
2. What are suggestions you have for methodology?

References:

Austin, H.A. (1984) An assessment of mathematical implication in college students. *International Journal of Mathematical Education in Science and Technology*, 15,327-338.

Balacheff, N. (1988). Aspects of proof in pupils" practice of school mathematics. In D. Pimm (Ed. aMatnd trans.), *Mathematics, teachers and children*. Kent, UK: Open University.

Dubinsky, E. & McDonald, M. (2001) APOS: A constructivist theory of learning in undergraduate mathematics education research. In D. Holton, M. Artigue, U. Kirchgraber, J. Hillel, M. Niss & A. Schoenfeld et al. (Eds), *The teaching and learning of mathematics at university level: An ICMI study (pp. 273-280).* Kluwer, Netherlands.

Hauk, S., Brown A., Tsay J., Barzilai, H. Austin, H. (2008). Preservice Elementary Teachers" Understanding of Logical Inference. *Investigations in Mathematics Learning, 1, 1-34.*

Examining Personal Teacher Efficacy Beliefs and Specialized Content Knowledge of Pre-service Teachers in Mathematical Contexts

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Abstract: *This study addressed the following research question: To what extent are K-8 pre-service teachers' personal mathematics teacher efficacy beliefs aligned with their content knowledge for teaching mathematics? 18 K-8 pre-service teachers enrolled in a teacher preparation mathematics content course completed semi-structured interviews and follow-up written assessments in which efficacy beliefs and content knowledge regarding specific mathematical teaching scenarios were assessed. Preliminary analyses indicate that the efficacy beliefs of pre-service teachers with low content knowledge vary according to the nature of the teaching scenario. Consequently, the extent to which teacher efficacy beliefs and knowledge are aligned for these pre-service teachers depends on the mathematics involved.*

Keywords: pre-service teachers, teacher efficacy beliefs, mathematical content knowledge

Teacher efficacy beliefs are considered an important topic of study, in part because of the apparent positive correlations between teacher efficacy beliefs and a variety of desirable outcomes including student achievement (Ghaith & Yaghi, 1997; Riggs & Enochs, 1990; Ross, 1992). While there is an increasing body of literature on teacher efficacy beliefs, few researchers have examined the potential links between pre-service teachers' mathematics teacher efficacy beliefs and mathematical content knowledge. In fact, investigating the extent to which pre-service teachers' teacher efficacy beliefs are aligned with their mathematical content knowledge for teaching is an important area of research with implications for understanding how teachers reflect on and learn from their teaching practices.

Relation of this work to the research literature

A process of intentional, careful reflection followed by changes in practice is an important part of becoming a more effective teacher (Hiebert et al., 2007). Teachers who already believe that they are highly effective would seem to be less likely to engage in this type of careful reflection. Personal teacher efficacy is defined as “a teacher’s belief in his or her skills and abilities to be an effective teacher” (Swars, 2005, p. 139). Then, it is likely that many teachers who exhibit high levels of personal teacher efficacy actually view themselves as highly effective teachers. As such, teachers with high levels of personal teacher efficacy might see little need to engage in careful reflection regarding the extent to which their teaching practices are successful. Indeed, previous studies both indicate the benefits of lower teacher efficacy as a mechanism for fostering teacher reflection and demonstrate the potential downsides of high teacher efficacy (Brodkey, 1993; Wheatley, 2002; Wheatley, 2005).

Specialized content knowledge (SCK) for teaching mathematics (Ball, 2008) is one aspect of teaching about which teachers might need to reflect and improve. Recent research provides empirical evidence that teachers’ levels of *mathematical knowledge for teaching*, of which specialized content knowledge is an important subcomponent (Ball, 2008), might be particularly important (Hill, Rowan, & Ball, 2005).

A connection between teacher efficacy beliefs and specialized content knowledge worthy of empirical examination becomes apparent when one considers results from previous empirical work on the relationship between individuals’ self-assessments of knowledge and their performance on various tasks. Wheatley (2000) discovered that teachers’ efficacy beliefs can be "poorly grounded" (p. 19), partially because teachers might not be aware of "their own lack of knowledge" (p. 19). Then, it is possible that teachers with low specialized content knowledge for teaching mathematics might be overconfident in their teaching abilities, and thus exhibit high levels of personal teacher efficacy. In other words, teachers' teacher efficacy beliefs might be unaligned with their specialized content knowledge. Indeed, Stevenson et al. (1990) discovered a similar disconnect regarding American children's mathematical knowledge and their mathematical performance, with American students tending to overestimate their mathematical abilities more than students from China or Japan.

One might then ask why it matters whether or not teachers' teacher efficacy beliefs and content knowledge are aligned. To begin, teachers with unaligned teacher efficacy beliefs and specialized content knowledge might have difficulty in recognizing when they are overconfident in their teaching abilities. Kruger and Dunning (1999) found that not only do individuals overestimate their abilities in given situations, but that such overestimation might be due in part to lack of knowledge hampering individuals' abilities to evaluate their own skills accurately. Then, it is possible that teachers with low content knowledge could have difficulty in assessing their teaching abilities realistically.

If this is the case, it is likely that teachers who feel they are highly efficacious yet have low specialized content knowledge will exhibit two characteristics. First, they might be less effective teachers than they actually believe themselves to be. Second, and more importantly, such teachers might not only have insufficient motivation for engaging in reflection on their practices, but also have difficulty in assessing when changes in practice are needed even when reflection takes place (see, e.g., Borko et al., 1992).

Implications for teacher educators

Pre-service teachers are a particularly important population to study, as the teacher efficacy beliefs of these teachers are still developing (Swars, 2005). If it is in fact the case that pre-service teachers with high teacher efficacy beliefs and low specialized content knowledge have insufficient motivation for engaging in reflection on their practices, and also have difficulty in assessing when changes in practice are needed even when they do reflect on their practices, teacher educators would surely want to know that this is the case. That is, an efficacy beliefs-knowledge relationship that hinders rather than promotes teacher reflection is not a relationship that will help teachers improve over time. Studies that explore the extent to which pre-service teachers' efficacy beliefs and content knowledge are aligned have the potential to uncover potentially unhelpful relationships between teacher efficacy beliefs and content knowledge, and consequently can inform teacher educators' understandings of how to help pre-service teachers develop more helpful efficacy beliefs-knowledge relationships.

Thus, the research question of interest in this study is the following: To what extent are K-8 pre-service teachers' personal mathematics teacher efficacy beliefs aligned with their content knowledge for teaching mathematics?

Research methodology

The participants were 18 sophomore undergraduates enrolled in a K-8 teacher preparation program in a medium-sized state university in a Mid-Atlantic state. All of the participants in this study were enrolled in the third course of a three-course sequence of mathematics content courses at the time of their participation. These 18 pre-service teachers were selected randomly from the total 47 pre-service teachers enrolled in the third content course.

Pre-service teachers first participated in a 90-minute semi-structured interview in which they were asked to respond to four written mathematical teaching scenarios, the *teaching scenario tasks*. Each scenario consisted of a written student question about a given fractions task. For each of the teaching scenarios, pre-service teachers were first asked to write a written response regarding what they would do as the teacher in the given situation. Then, they were asked to rank the effectiveness of their response on a scale of one to five, and to list any factors that contributed to their rating. Finally, the author asked participants to explain their responses verbally. All interviews were audio-recorded, with the recordings used to supplement written pre-service teacher responses. Following the semi-structured interview, pre-service teachers completed a 60-minute written assessment containing *SCK written tasks*. These tasks contained the same mathematical content as those used in interviews but contained no teaching context.

Pre-service teachers' personal teacher efficacy beliefs were operationalized in two ways. For each interview task, pre-service teachers' reported rankings of their effectiveness were taken as a measure of their personal mathematics teacher efficacy. Additionally, pre-service teachers' rankings were coded according to the factors pre-service teachers mentioned as contributing to their teacher efficacy. Four categories emerged from the interview data: *content knowledge*, *pedagogy*, *students*, and *other*. Pre-service teachers both mentioned factors that made them feel more efficacious and factors that made them feel less efficacious in their responses to teaching scenario tasks.

Specialized content knowledge (SCK) was operationalized with an approach similar to that used in Morris et al.'s work (2009). That is, the author constructed a list of mathematical subcomponents relevant to each teaching scenario task. Participants then received a score of 0, 1, or 2 for each subcomponent based on the quality of their responses. Five mathematical subcomponents were identified for each of tasks 1 and 2, so SCK scores could range from 0 to 10 on these tasks. Four subcomponents were identified for tasks 3 and 4, so SCK scores could range from 0 to 8 on these tasks. This scoring was computed for both the teaching scenario and SCK written tasks.

Based upon their scores on the teaching scenario tasks, pre-service teachers were separated into *higher* and *lower* SCK groups for each task using a median split in order to examine alignment of beliefs with level of SCK. The same was done for scores on the SCK written tasks. Then, pre-service teachers were grouped into four categories for each task: *higher/higher*, *higher/lower*, *lower/higher*, and *lower/lower*. The groups of particular interest were the higher/higher and lower/lower groups, as these pre-service teachers displayed a more reliable level of SCK across the two assessments. Additionally, for each task, Spearman's rho was computed to examine potential correlations between pre-service teachers' efficacy rankings and their SCK scores.

Inter-rater reliability scores for 20% of scores were obtained. The percent agreements reported here were computed across the interview and parallel written assessment tasks. The agreements for tasks 1, 2, 3, and 4 were 80%, 80%, 81%, and 81% respectively.

Results of the research

Only preliminary results are given here as this study is a work in progress. More detailed analyses are currently underway. For task 2, efficacy rankings and SCK scores were significantly negatively correlated (rho = -0.486, p = 0.041). That is, apparently, pre-service teachers tended to be somewhat overconfident regarding their responses on this task, as pre-service teachers with higher confidence ratings tended to exhibit lower levels of SCK in their responses to the teaching scenario.

Additionally, the nature of the task apparently influenced the extent to which lower SCK pre-service teachers mentioned content knowledge as making them feel less efficacious in their teaching scenario responses. The percents of pre-service teachers in the lower/lower SCK group that mentioned content knowledge factors making them feel less efficacious on tasks 1-4 were 50%, 17%, 100%, and 83% respectively. Pre-service teachers with relatively low SCK presumably should not mention their own math content knowledge as positively contributing to their sense of efficacy, so one could say low SCK pre-service teachers who cited content knowledge as making them feel more efficacious exhibit efficacy beliefs that are not aligned with their SCK. This might indicate that the extent to which the efficacy beliefs of pre-service teachers with lower SCK are aligned with their content knowledge depends upon the contextual features of the teaching scenario.

Questions for the audience

- What might be the implications of this work for researchers of teacher efficacy beliefs? for teacher educators? for designers of professional development?
- To what extent are the data presented convincing?
- What additional data would be useful to collect in future work in order to address the research question more thoroughly?

References

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389-407.

Borko, H., Eisenhart, M., Brown, C. A., Underhill, R. G., Jones, D., & Agard, P. C. (1992). Learning to teach hard mathematics: Do novice teachers give up too easily? *Journal for Research in Mathematics Education, 23*(3), 194-222.

Brodkey, J. J. (1993). Learning while teaching: Possibilities and problems. *Teacher Education Quarterly, 20*(1), 63-70.

Ghaith, G., & Yaghi, H. (1997). Relationships among experience, teacher efficacy, and attitudes toward instructional innovation. *Teaching and Teacher Education, 13*(4), 451-458.

Hiebert, J., Morris, A. K., Berk, D., & Jansen, A. (2007). Preparing teachers to learn from teaching. *Journal of Teacher Education, 58*(1), 47-61.

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal, 42*(2), 371-406.

Kruger, J., & Dunning, D. (1999). Unskilled and unaware of It: How difficulties in recognizing one's own incompetence lead to inflated self-assessments. *Journal of Personality and Social Psychology, 77*(6), 1121-1134.

Ma, L. (1999). *Knowing and teaching elementary mathematics : Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Erlbaum

Morris, A. K., Hiebert, J., & Spitzer, S. M. (2009). Mathematical knowledge for teaching in planning and evaluating instruction: What can preservice teachers learn? *Journal for Research in Mathematics Education, 40*(5), 491-529.

Riggs, I. M., & Enochs, L. G. (1990). Toward the development of an elementary teacher's science teaching efficacy belief instrument. *Science Education, 74*(625-637).

Ross, J. A. (1992). Teacher efficacy and the effects of coaching on student achievement. *Canadian Journal of Education, 17*, 51-65.

Stevenson, H. W., Lee, S.-Y., Chen, C., Stigler, J. W., Hsu, C.-C., Kitamura, S., et al. (1990). Contexts of achievement: A study of American, Chinese, and Japanese children. *Monographs of the Society for Research in Child Development, 55*(1/2), 1-119.

Swars, S. L. (2005). Examining perceptions of mathematics teaching effectiveness among elementary preservice teachers with differing levels of mathematics teacher efficacy. *Journal of Instructional Psychology, 32*(2), 139-147.

Wheatley, K. F. (2000). Positive teacher efficacy as an obstacle to educational reform. *Journal of Research and Development in Education, 34*(1), 14-27.

Wheatley, K. F. (2002). The potential benefits of teacher efficacy doubts for educational reform. *Teaching and Teacher Education, 18*, 5-22.

Wheatley, K. F. (2005). The case for reconceptualizing teacher efficacy research. *Teaching and Teacher Education, 21*(7), 747-766.

The Effects of Online Homework in a University Finite Mathematics Course
Preliminary Report

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Over the past 15 years, mathematics departments at colleges and universities have begun to incorporate the use of online homework systems in a variety of lower-level mathematics courses. Several online or web-based homework systems exist, including WebAssign, MyMath, DRILL, and WeBWorK, and most have been developed by textbook companies for use in introductory college-level mathematics courses. Web-based homework systems typically allow the instructor to create homework sets from a list of problems that have been pulled from the textbook. While all students typically receive the same set of problems, the numbers present in the problem will typically be randomized for each student (Denny & Yackel, 2005). The instructor may limit the number of attempts the student will be allowed on each problem and can specify the amount, and type, of feedback the student receives after an incorrect attempt. Typically, the numbers within a given problem are randomized so that different students will be working slightly different exercises. Overall, the purpose of the online homework system is to provide students with multifaceted, technology-based opportunities, rich with immediate feedback, to engage with course material outside of class.

Several studies have found that online or web-based homework systems effectively promote engagement with course material and may result in gains in both knowledge and skill, as they are measured inside the classroom. Denny & Yackel (2005) found that students attempt online homework problems at very high rates and their increased practice may translate into greater content-related knowledge and skill. Zerr (2007) found that by providing students with detailed feedback for incorrect responses when using the online system, and allowing several attempts for successful completion of each online assignment, that student learning in an introductory calculus course improved. When students are allowed several attempts at answering a question, without penalty, they may develop the tenacity required to solve more complex problems (Denny & Yackel, 2005). Moreover, the participants in Zerr's (2007) study overwhelmingly indicated that they felt their time spent using the online homework system was productive and worthwhile.

Web-based homework systems may not work equally well for all students, in all mathematics classes, however. Hirsch and Weibel (2003) found that the effectiveness of online homework greatly depended on the number of problems that students attempted; the more problems attempted the more correct answers students were able to provide. Zerr (2007) also noted that web-based homework systems may be most beneficial for students without prior college experience. Students with prior college experience using a web-based homework system actually tended to perform more poorly on exams and quizzes than students completing homework in a more traditional format (Zerr, 2007). The fact that students have occasionally been found to have lower performance when engaging with web-based homework systems may be accounted for, in part, by the fact that with web-based homework systems there is little opportunity for instructors to provide students with the tools they need to further their conceptual understanding of topics (Hauk & Segalla, 2005). On web-based homework, students will not receive credit based on work shown or not shown. When the work for a problem is not examined or graded by an instructor, it might be difficult for an instructor to pinpoint the conceptual issues a student is having. And, many instructors in mathematics would agree that an emphasis of product over process is misplaced. When a student performs poorly on an online homework assignment, it would likely be the student's responsibility to seek out the instructor's expertise since the instructor need not be a part of the homework feedback loop if a web-based homework system is in use. In this way, online homework systems may actually inhibit collaboration between students and instructors. Hauk and Segalla (2005) found no significant difference between the performance of web-based and traditional homework sections of students taking college algebra. Moreover, web-based homework systems may do little to challenge the commonly-held notion that mastery of college-level mathematics involves little more than rote computation (Hauk & Segalla, 2005).

There are aspects of web-based homework systems that make them very attractive for instructors and administrators. From the instructor's perspective, they spare the teacher the time and tedium inherent in collecting and grading paper-based homework (Denny & Yackel, 2005; Hauk & Segalla, 2005). In addition, these online tools can be programmed to prompt students who have answered a question incorrectly to 'watch' a similar problem be worked through online or to look at a certain example in the text that will provide some guidance on the problem. In effect, online homework students may save the instructor some of the time and effort they typically exert in addressing students' more basic questions about course content (Hauk & Segalla, 2005). Meanwhile, administrators may appreciate the online system since it means they will not have to hire graders, and such systems may be perceived by students as an appropriate and forward-thinking use of technology. In addition, the cost of the product is paid directly by the student.

Even though colleges and universities have been using web-based homework systems in introductory mathematics classes for more than a decade, consensus has yet to be reached on how, or even whether, web-based homework systems compare to that of more traditional approaches in facilitating student learning outcomes. Although some researchers have found benefits for both students and instructors using web-based approaches, others have found none; hence, questions remain concerning the effectiveness of such a tool. It is possible students using an online homework system are not encouraged to be systematic in their approach to problems, as written evidence of their process is not required when using an online homework system. Moreover, if students are not thoroughly documenting the process used to solve a problem, they will not be able to use their work as a study prep. Nor would they readily and easily be able to communicate with their instructor about the difficulties they are having either in computation, process, or conception. A final cause for concern not already addressed in the literature is the student's awareness of their level of understanding of the material. Allowing a student multiple attempts on online homework problems is valuable in that it allows students the opportunity to immediately learn from their mistakes, correct themselves, and be rewarded for this effort. This often means the students' online homework scores are quite high (90% and higher is common). Is it possible that, as a result of high on-line homework scores, students feel they understand the material at a higher level than they actually do? These students may be developing a false sense of confidence after earning high homework scores, only to be defeated on exam day. These are the kinds of observations that are routinely made by instructors teaching courses that contain a web-based homework component.

Our study seeks to add to the body of research examining the effectiveness of web-based homework systems by examining the performance of students taking Finite Mathematics at a medium-sized private university in the Midwest. One group of 24 students was assigned homework using a web-based homework system (WebAssign); the other group of 24 students completed and turned in tradition pencil and paper homework which was graded by the instructor. Both groups were assigned the same set of problems, simply in different formats. Both sections were taught by the same instructor and in the same format; common exams were given to both sections. This study will investigate the effectiveness of the online homework system by comparing the individual final exam items and overall exam performance of the students in the web-based and traditional homework sections. This analysis will be completed in order to better understand whether web-based homework is equivalent to (or better than) traditional homework for facilitating learning overall. Additionally, the researchers will examine and compare the types of questions that traditional and web-based homework students tend to get correct (or incorrect) in order to gain insight into the depth of learning that may be promoted using either homework system. Student perceptions of how the class structure, materials, and assigned work impact their learning and classroom experience will also be analyzed and compared in an attempt to gain greater insight into the learning preferences of the contemporary undergraduate student in entry-level mathematics courses. This study is preliminary and data is currently being analyzed; statistical results and relevant recommendations for practice and further study will be forthcoming.

<u>Questions for audience consideration:</u>

1. From the professor's perspective, what are the biggest benefits and drawbacks to web-based homework systems and traditional homework formats?
2. From the student's perspective, what are the biggest benefits and drawbacks to web-based homework systems and traditional homework formats?
3. What are the benefits and drawbacks of each system in facilitating breadth of learning?
4. What are the benefits and drawbacks of each system in facilitating depth of learning?
5. Do web-based systems promote the kind of engagement that leads to long-term learning or just short-term competency, and how could we capture this information?
6. Can we identify categories of questions that are better served by using web-based systems, traditional systems?
7. How would we go about this categorization?

Bibliography

1. Student perceptions of the web-based homework program WeBWorK in moderate enrollment college algebra courses, by S. Hauk and A. Segalla. *J. of Computers in Mathematics and Science Teaching*, (2005), 24(**3**), pgs 229-253.
2. The effectiveness of online homework in an introductory science class, by R. Allain and T. Williams. *J. of College Science Teaching*, May/June 2006, pgs. 28-30.
3. Implementing and teaching with WebWorks at Mercer University, by J. Denny and C. Yackel. *Proceedings of the 2005 ASCUE Conference*, pg. 85-93.
4. A quantitative and Qualitative Analysis of the effectiveness of online homework in first-semester Calculus, by. R. Zerr. *J. of Computers in Mathematics and Science Teaching* (2007) 26(**1**), pgs 55-73.
5. Statistical evidence that web-based homework helps, by L. Hirsch and C. Weibel. *Focus*, Feb. 2003, pg. 14.

Building Knowledge within Classroom Mathematics Discussions

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Over the past decade, a growing trend has been to center instruction around student learning, rather than teacher performance. Such instruction often elicits student participation through various classroom activities: answering questions; solving problems; having students do board work; working on collaborative tasks in groups or pairs; making and testing conjectures; presenting ideas, proofs, and solutions; and debating. Through these and other classroom activities, students are expected to engage in the learning process, participate in mathematical thinking, and contribute to classroom discourse.

The presence of these instructional forms in the classroom does not always indicate quality instruction or guarantee quality student involvement. Cursory classroom observations might not reveal students' mathematical thinking or engagement. For example, working in groups, carrying on discussions, or answering questions does not necessarily mean that they are engaged in mathematical thought. We would need to know: What is the nature of the group's task? What are they discussing? How thought provoking are the questions? and What is the nature of students' responses?

In general, determining the quality of student involvement and their role in building classroom knowledge requires us to answer deeper questions about the discourse: How are learners contributing to the discussion? What is the nature of those contributions? What role are they playing in the discussion? and What significance and impact do their contributions have on the developing content? Addressing these questions for discussions in the mathematics classroom setting is the aim of this paper.

Within a different context and social group, I was recently able to address these exact questions, while developing an analytical framework. While studying discussions among practicing teachers who participated in a professional development program, Belnap and Withers developed an analytical framework for identifying the origin of a discussion's content and how each individual contributed to that knowledge (Belnap & Withers, 2010).

This framework is based on a view of discourse that integrates aspects of various learning perspectives: constructivism (Cobb, 1994; Cobb & Yackel, 1996; Ernest, 1996; Sfard, 1998; Zevenbergen, 1996), the social perspective (Cobb, 1994; Cobb & Yackel, 1996; Lerman, 1998, 2000; Sfard, 1998), socioconstructivism (Lerman, 1998, 2000; Cobb & Yackel, 1996; Cobb, Jaworski, & Presmeg, 1996), and agency (Walter & Gerson, 2007). This view is that discourse involves the mutual construction of both individual and social knowledge; it is a social activity shaped by participants' involvement. At the same time, participants willfully act and construct their own knowledge from their involvement in the discourse.

From this perspective, the discussion's text represents a form of social knowledge constructed by the willful actions of its contributors. Stemming from social linguistics and the work of Nassaji and Wells (2000), Wells (1996), the framework Belnap and Withers developed both grew from and illuminates this idea. It describes how each individual's contribution links to the contributions of other participants, building the conversation's content (Belnap & Withers, 2010; Belnap, 2010).

As detailed in Belnap (2010), when individuals take turns in a discussion, they make willful contributions to the growing text, making *moves*. In building the discussion's content, each move has a function, determined by its *action* (i.e. how it affects the growing text) and its *target* (i.e. any content receiving the action).

Based on function, there are 13 different move (or function) categories, clustered into five main groups or types: *anchoring, valuing, altering, requesting,* and *contentless* moves. *Anchoring moves* present new ideas, opening potential lines of discussion. *Valuing moves* address the value, validity, or correctness of existing contributions, focusing on assessing, supporting, refuting, or otherwise affecting the credibility of prior contributions. *Altering moves* develop the content of existing contributions by adding to, modifying, or clarifying it. *Requesting moves* (including, but not limited to questions) solicit content. Finally, *contentless moves* either do not directly develop a discussion's content or are counted as such.

This framework provides a means of ascertaining the discussion's content structure. Each move's action and target describes a linkage between moves. Using these linkages to chain moves together breaks the discussion into *fibers*, each representing the development of a single idea.

Using this framework allowed me to see both the structure and individual contributions' roles in content development. Identifying fibers allowed me to distinguish separate ideas or topics in complex conversations, facilitating the identification of productive conversations (i.e. those relevant to the purpose of the PD program). The distinctions among moves provided a means of identifying substantive contributions to the conversation; by counting this information for individual participants and contrasting the results, I ascertained the extent and nature of their involvement.

As a particular example, using this framework allowed me to determine information about participant involvement, individual conversational roles, and discussion characteristics in a recent analysis of one professional development session. Specific data and details are provided in Belnap (2010).

The framework provided an overview of participant involvement in the session. I found that all participants took an active role in developing the discussion's content. Each participant initiated some conversational fibers. All listened to and built off of the ideas of others. Finally, they each made efforts to explain and support their own and others' contributions.

On a more specific level, the framework revealed the nature and extent of individual involvement. I found that the facilitator's involvement mainly consisted of initiating and soliciting content; the extent of this involvement was limited, as he often took a back seat, avoiding direct control of the content, and allowing it to develop at the participants' discretion. Other participants took an active role in the conversation, with no one clear discussion leader. One (while not dominating the conversation) did play a leading role, initiating conversational fibers, conveying information, evaluating/refuting contributions, and integrating and building ideas, all more so than any other participant. By contrast, two participants seemed to hold back and contribute much less.

In addition to individual involvement, by revealing the conversation's composition, the framework provided the means of characterizing the discussion overall. The discussion could be characterized by conveyance, slight developing, and justifying a wide variety of ideas and opinions, with almost no discussion, change, and deliberation of ideas. A common pattern was that a conversational fiber consisted of an initial idea, justification with some addition of ideas or details, and then a topic shift. Little time was spent deeply investigating the many ideas initiated. Most content arose from spontaneous comments. There was a profound lack of inquiry and little disagreement and resolution of differences.

It is plausible that this framework can be used to answer similar questions and provide similar information regarding mathematical classroom discussions; this is the goal of this study. At the same time, differences in these contexts (the mathematics classroom verses a professional development program for teachers) are great, including: strongly rooted cultural norms, roles, responsibilities, and expectations; differences in the nature of the discussed content; and typical goals and objectives for the two contexts. With such strong differences, it is plausible that the framework may need to be modified to accurately reflect the content development of mathematical conversations.

Based on discussion and feedback from members of the research community, I am conducting a pilot study, to see how the framework can describe content development in a mathematics classroom. To do this, I have purposively selected a mathematics teacher who is well known for effectively engaging students in investigative tasks, orchestrating student centered classroom discussion, and establishing classroom discourse in which students listen to and respond to each other.

To test the analytical framework using typical qualitative methods. I will video tape an hour long class and apply the framework to the coding of the class' transcript, looking for contributions whose function may not be described well by the framework, modifying and reconceptualizing the move categories as necessary to account for these differences. Once I have completed analysis of the class, I will discuss the results with another researcher to gain an outside perspective and find concepts and ideas that may be missed, adjusting the framework as necessary. Next, to test the modified framework, I will apply the framework to the transcript of a second class to see if the framework accurately describes the discussion. Finally, I will analyze the resulting data to determine the extent to which the framework facilitates answering the questions posed earlier: How are learners contributing to the discussion? What is the nature of those contributions? What role are the students playing in the discussion? and What significance and impact do their contributions have on the developing content?

Data collection is currently beginning and preliminary results will be reached during December 2010 and January 2011. This paper and presentation will focus on discussing these preliminary results, examining potential information they give about the mathematics classroom, and beginning to contrast this with other frameworks that examine classroom discourse. As a preliminary presentation, participant discussion will also center on feedback, ideas, observations, and additional viewpoints on these same three issues. In particular, I will pose questions for discussion such as: What useful information can this framework provide for us as researchers? What interesting research questions could be answered utilizing this framework? How may this framework relate to other analytical or theoretical work? and What details may I have overlooked?

References

Belnap, J. K. (2010). Building classroom mathematical knowledge: Extending a framework for the co-construction of knowledge to the classroom context. In *Proceedings of the thirteenth conference for Research in Undergraduate Mathematics Education.* Raleigh, NC.

Belnap, J. K., & Withers, M. G. (2010). *Discourse analysis: Finding structure in the content of uncontrolled discussions.* Manuscript in preparation, University of Wisconsin at Oshkosh.

Cobb, P. (1994). Where is the mind? constructivist and sociocultural perspectives on mathematical development. *Educational Researcher*, *23*(7), 13–20.

Cobb, P., Jaworski, B., & Presmeg, N. (1996). Emergent and sociocultural views of mathematical activity. In L. P. Steffe, P. Nesher, P. Cobb, G. A. Goldin, & B. Greer (Eds.), *Theories of mathematical learning* (pp. 3–19). Mahwah, NJ: Lawrence Erlbaum Associates.

Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist*, *31*(3/4), 175–190.

Ernest, P. (1996). Varieties of constructivism: A framework for comparison. In L. P. Steffe, P. Nesher, P. Cobb, G. A. Goldin, & B. Greer (Eds.), *Theories of mathematical learning* (pp. 335–350). Mahwah, NJ: Lawrence Erlbaum Associates.

Lerman, S. (1998). Research on socio-cultural perspectives of mathematics teaching and learning. In A. Sierpinska & J. Kilpatrick (Eds.), *Mathematics education as a research domain: A search for identity* (p. 333-350). Great Britain: Kluwer Academic Publishers.

Lerman, S. (2000). The social turn in mathematics education research. In J. Boaler (Ed.), *Multiple perspectives on mathematics teaching and learning* (pp. 19–44). Westport, CT: Ablex Publishing.

Nassaji, H., & Wells, G. (2000). What's the use of 'triadic dialogue'?: An investigation of teacher-student interaction. *Applied Linguistics*, *21*(3), 376–406.

Sfard, A. (1998). On two metaphors for learning and the dangers of choosing just one. *Educational Researcher*, *27*(2), 4–13.

Walter, J. G., & Gerson, H. (2007). Teachers' personal agency: Making sense of slope through additive structures. *Educational Studies in Mathematics*, *65*(2), 203–233.

Wells, G. (1996). Using the tool-kit of discourse in the activity of learning and teaching. *Mind, Culture, and Activity*, *21*(2), 74–101.

Zevenbergen, R. (1996). Constructivism as a liberal bourgeois discourse. *Educational Studies in Mathematics*, *31*, 95–113.

Using Think Alouds to Remove Bottlenecks in Mathematics

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Abstract

Think alouds are a research tool originally developed by cognitive psychologists for the purpose of studying how people solve problems. The basic idea being that if a subject can be trained to think out aloud while completing a certain task then the introspections can be analyzed and may provide insights into misunderstandings as well as higher thinking. This talk is a preliminary report of a think aloud conducted with calculus students to understand their difficulties with work problems in integral calculus.

Keywords: Calculus, cognitive science, classroom research, think alouds

Students in integral calculus often face difficulties in problems involving applications to physics like work and pressure problems. It is unclear whether their difficulties are due to lack of understanding of the definition of the definite integral as a limit of Riemann sums or whether it is difficulty in actually applying the concept to a physical situation like a work problem. This study is a preliminary report of an investigation conducted using the think aloud (or protocol analysis) technique with second semester calculus students at a two year campus of the University of Wisconsin.

Think alouds are a research tool originally developed by cognitive psychologists for the purpose of studying how people solve problems. The basic idea being that if a subject can be trained to think out aloud while completing a certain task then the introspections can be analyzed and may provide insights into misunderstandings as well as higher thinking. Schoenfeld(1) has used verbal transcripts and protocol analysis to study mathematical problem solving .

The goal of this study was to answer the question "Why do calculus II students have difficulty solving Work problems?". This was a qualitative study. Six students of varying abilities were selected to participate in the study. Students were first trained to think aloud by being asked to solve simple linear equations. They were then given a series of work problems ranging from the simplest kind with a fixed force and a fixed distance to the more involved that had a variable force and/or a variable distance. The sessions were video recorded. During the sessions the students were not prompted in any way and nor were any interventions introduced. The only comments made by the instructor were to request the student to verbalize their thoughts if and when the student fell silent. The recordings were transcribed and screen shots of the diagrams were taken. The transcriptions were coded and analyzed.

A coding scheme (see Appendix 1) was developed to code the verbalizations using Polya's four step problem solving process. Next we needed to identify the codes that would enable us to answer our question. We highlighted three codes, strategy, mathematical argument and logical inference. These codes reflect the thought process of a mathematician while problem solving. Each problem was then assigned a rating from 1-5 for each of the selected codes using a rubric (see Appendix 2 for the rubric), with 5 representing the score of a mathematician and 1that of a novice problem solver. These ratings helped to identify possible bottlenecks in the problem solving process.

Preliminary analysis indicates that the students who got "1" under strategy had a common trait. They headed straight for the fluid slice in the pool type problems but were then confused as to what to do next. Therefore one possible bottleneck is students memorizing a fragment of the instructor's strategy without understanding the underlying connections. Four of the six students seemed to have a significant strategy but were unable to solve the problem correctly due to mistakes in calculating the volume of a generic slice or incorrectly calculating the weight of the slice. This suggests that students do not have a good handle on the basic mathematical tools that are considered essential at this level.

During regular assessment students often erase their wrong work, so we only see the end product which doesn't always help us to identify the bottleneck. With a think aloud we are able to see much more of the problem solving process, the students' struggles in formulating strategies and mathematical arguments and thus make the thinking process more visible.

Questions for discussion

1. Can we make our think aloud coding list portable for problem solving across the university mathematics curriculum?
2. As an intervention for lack of strategy, will a grading rubric which will ask students to actually write down the strategy (before starting the mathematics), encourage students to strategize more ?
3. Strategy, Mathematical Argument and Logical Inference are key thought processes of a Mathematician solving problems. Are there any others that should be considered in this analysis?

References

1) Lendol Calder and Sarah-Eva Carlson, "Using Think Alouds to Evaluate Deep Understanding", Handout at Faculty College 2009, Office of Professional and Instructional Development, University of Wisconsin System.
2) Alan Schoenfeld, "Mathematical Problem Solving", Academic Press, Inc, 1985.

Appendix 1: Coding Sheet based on Polya's four steps

A. Understanding the problem

I. Understanding the problem UP
II. Recall RE

B. Devising a plan

I. Initial Plan DP
II. Alternate Plan DAP

C. Carrying out the plan

I. Setting up the variable SV
II. Mathematical argument MA
III. Mathematical argument (In correct,) MA_I
IV. Questioning (Q)
V. Guess (G)
VI. Recognizing limits (H)
VII. Narration (N)
VIII. Uncategorized (X)
IX. Strategizing (ST)
X. Strategizing with Reflection (ST_R)
XI. Inference (geometry) I_G
XII. Inference (Reflexive) I_R
XIII. Inference (Previous) I_P
XIV. Rearranging Terms R_T
XV. Calculation (C)

D. Looking Back

XVI. Revision (R)
XVII. Reflection on concept (R_C)
XVIII. Reflection(Rf)

Appendix 2: Mathematical Argument Rubric

Representative Thought	Points
Focal mathematical Argument executed(correct) early, i.e. Mathematical procedures applied correctly at the appropriate steps to solve the problem correctly	5
Focal mathematical Argument executed(correct) , late Or Focal Mathematical Argument executed (correct)early except for non-conceptual mistakes	4
Mathematical argument with some focus correctness but has significant mistakes Could not completely carry out mathematical procedures	3
Mathematical argument , with little focus with minor parts which are correct	2
Unfocussed Mathematical argument containing a few correct components.	1

Strategy Rubric

Representative Thought	Points
Focal Strategy achieved early (professional)	5
Focal Strategy achieved	4
Focal Strategy achieved with uncertainties	3
Indication of a strategy in the problem solving process but is not the focal strategy nor does it contain parts of the focal strategy	2
Unfocussed /unsignificant strategy , i.e.No evidence of a strategy or procedure	1

An Investigation of Students' Proof Preferences: The Case of Indirect Proofs

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Abstract: This paper reports findings from an exploratory study regarding undergraduate natural sciences students' proof preferences, as they relate to indirect proof. While many agree that students dislike indirect proofs and fail to find them convincing, quantitative studies of students' proof preferences have not been conducted. The purpose of this study is to build on the existing qualitative research base and to determine if the identified preferences and conviction levels can be established as general tendencies among undergraduates. Specifically, the aim of the study is to explore two common claims: (1) students experience a lack of conviction when presented with indirect proofs; and (2) students prefer direct and causal arguments, as opposed to indirect arguments. The purpose of this preliminary report is to share findings from the proof preference pilot study.

> *"Why do I have to start with something that is not? … … However, the final gap is the worst, … … it is a logical gap, an act of faith that I must do, a sacrifice I make. The gaps, the sacrifices, if they are small I can do them, when they all add up they are too big."*
>
> (Fabio quoted in Antonini & Mariotti, 2008)

> *"The proof is by reductio ad absurdum, and reductio ad absurdum, which Euclid loved so much, is one of a mathematician's finest weapons"*
>
> (G. H. Hardy, *A Mathematican's Apology*, 2005/194, p. 19)

Research Goals & Questions

Research on undergraduate mathematics students' understandings of proof has continuously demonstrated that proof represents a significant barrier within the undergraduate mathematics curriculum (Selden & Selden, 2003; Harel & Sowder, 1998). Research suggests that students experience a variety of difficulties, including but not limited to difficulties moving between syntactic and semantic proofs (Weber & Alcock, 2004), transitioning from computational courses to proof-centered courses and topics (Moore, 1995), reading mathematical proofs (Selden & Selden, 2003), understanding the logical structure of proofs (Selden & Selden, 1995), developing appropriate proof schemes (Harel and Sowder, 1998), and interpreting proofs (Weber, 2001). Beyond difficulties producing and understanding proof, research suggests that not all forms of proof are equally difficult for students, with the most problematic being Mathematical Induction (Dubinsky, 1986, 1989; Movshovitz-Hadar, 1993a, 1993b; Fischbein & Engel, 1989; Harel, 2001; Brown, 2003; Harel and Brown, 2008) and Proof by Contradiction (Harel & Sowder, 1998; Antonini & Mariotti, 2008). Thus, little has changed since Robert and Schwarzenberger (1991) noted, "Research into students' ability to follow or produce proofs … confirms that students find proof difficult, with proofs by (mathematical) induction and proofs by contradiction presenting particular difficulties" (p. 130). Interestingly, the two forms of proof that Robert and Schwarzenberger (1991) highlight – *proof by mathematical induction* and *proof by contradiction* – are likely to be the two most ubiquitous forms of mathematical proof in the undergraduate mathematics curriculum. Thus, it seems likely that identifying factors contributing to students' difficulties with these particular forms might be critical to developing instructional innovations that foster students' transition to proof.

The Case of Indirect Proof

In relation to *indirect proof*, which we will take to include both proof of the contrapositive and proof by contradiction, qualitative studies have demonstrated that students experience a lack of conviction with respect to such proofs (Harel & Sowder, 1998; Antonini & Mariotti, 2008; and Leron, 1985). For example, Harel and Sowder (1998) found that many students in their teaching experiments preferred constructive proofs – proofs that directly construct mathematical objects rather than solely justify existence – and dislike proofs by contradiction. The following remark, made by a student, Dean, is provided as a typical example of students' views towards this form of proof: "I really don't like proof by contradiction. I have never understood proofs by contradiction, they never made sense" (Harel & Sowder, 1998, p. 272). In more recent work by Antonini and Mariotti (2008), involving clinical interviews with Italian secondary school and university students, it is argued that students' dislike of indirect proofs may be tied to a lack of intuitive acceptance regarding the equivalence of a particular mathematical statement and its contrapositive; that is, while students recognize the contrapositive and can evaluate the proof of the contrapositive, they find it difficult to accept such proofs as proofs of the original theorem, as indicated by Fabio's remarks at the beginning of this paper. Such an interpretation fits well with others' comments regarding indirect proofs. For instance, Leron (1985) argued that when engaging in such proofs "we must be satisfied that the contradiction has indeed established the truth of the theorem (having falsified its negation), but psychologically, many questions remain unanswered" (p. 323). Antonini and Mariotti (2008) suggest that in the case of statements for which there exists a direct proof, students may find the direct proof more intuitively acceptable.

While many agree that students dislike such proofs and fail to find them convincing, quantitative studies of students' proof preferences and conviction levels have not been conducted. The purpose of this study is to build on the existing qualitative research base and to determine if the identified preferences and conviction levels can be established as general tendencies among undergraduate natural sciences students. Specifically, the aim of the study is to conduct a quantitative validation study of two claims: (1) Students experience a lack of conviction when presented with indirect proofs; and (2) Students prefer direct and causal arguments rather than indirect arguments. The purpose of this preliminary report is to share findings from the pilot survey, which explored claim (2).

Considerations Regarding Indirect Proof

Indirect proofs, according to many (e.g., see Polya, 1957), occur in two forms: (a) *proof by contraposition*; and (b) *proof by contradiction*. The two forms of proof prove different yet logically equivalent statements. In the case of proof by contraposition, one proves the contrapositive of a statement rather than the original statement; i.e., one proves $\sim Q \Rightarrow \sim P$, rather than $P \Rightarrow Q$. Proof by contradiction, also referred to as *reductio ad absurdum*, entails proving $P \wedge \sim Q \Rightarrow Q \wedge \sim Q$ or that $P \wedge \sim Q \Rightarrow P \wedge \sim P$. Others studying indirect proof have opted to group the two forms of proof together (See Antonni and Mariotti, 2008). In the context of this study, the two forms of indirect proof are viewed as distinct in terms of their structure. This is not to say that the proofs do not overlap but rather that they do not lie in complete bijection. This follows from consideration of what one can assume at the outset of constructing such proofs. Here we see that one can assume ~Q, when constructing a proof by contraposition, while one can assume P∧~ Q, when constructing a proof by contradiction.

It is possible that being able to assume more initially (P∧~ Q) might better enable novices to construct such proofs. On the other hand, such proofs require one to assume "the absurd," which might make such proofs especially difficult for novices, for they require purely hypothetico-deductive thinking in the sense of Piaget. Another complicating factor is that indirect proofs take multiple and varied forms and are sometimes the only apparent or feasible approach. Take, for example, the task of proving the irrationality of the $\sqrt{2}$. One can either prove that for every pair of integers, p and q, $\sqrt{2} \neq p/q$ or one can assume there exists integers p and q such that $\sqrt{2} = p/q$ and arrive at a contradiction. To the experienced proof writer, the later may seem easier.

Finally, historically, the mathematics community has held discrepant views of proof by contradiction or "indirect proof." For instance, the famous mathematician, G. H. Hardy describes *reducio ad absurdum* as "one of a mathematician's finest weapons" (p. 19). In contrast, Polya (1957), in his discussion of "Objections" to indirect proof, states, "we should be familiar both with 'reductio ad absurdum' and with indirect proof. When, however, we have succeeded in deriving a result by either of these methods, we should not fail to look back at the solution and ask: *Can you derive the result differently*" (Polya, p. 169). Taking a more extreme stance, mathematicians, such as L. E. J. Brouwer, who in the early twentieth century were part of the intuitionist movement, rejected the law of the excluded middle and thus, proof by contradiction. Thus, as also noted by (Antonini & Mariotti, 2008) and (Mancuso, 1996), the mathematics community does not appear to be in harmony in terms of a general preference for indirect proof.

Methodology

To explore the claim that students prefer direct and causal arguments rather than indirect arguments an 8-item indirect proof survey was developed. This survey included three types of proof comparison tasks. Type I tasks as participants to compare a direct proof to an indirect proof and to indicate which argument they found more convincing. For example, a participant might be asked to compare a proof by induction (direct proof) to a proof that relied on the Well-ordering Principal (proof by contradiction). Type II tasks asked participants to compare a *Proof by Construction*, in which a mathematical object is constructed, to an *Existence Proof*; that is, to a non-constructive, indirect proof of existence. Type III tasks explored the idea that there might be psychological distinctions to be made between the two forms of indirect proof, and asked participants to compare a proof by contraposition to a proof by contradiction. In addition to the comparison tasks, Type IV tasks asked participants to select from among three statements which statement they would choose to prove. Specifically, students were asked to indicate: (a) if a given theorem could be proved by proving an alternative statement of the theorem; and, (b) which among the potential alternative statements they would choose to prove. Alternative statements were of the form $\sim Q \Rightarrow \sim P$ and "there exists no P such that, P∧~ Q." Participants of the study were undergraduate mathematics students, enrolled in post-calculus collegiate mathematics courses such as differential geometry, linear algebra, and knot theory. Responses were anonymous, with respondents simply indicating their major and year in school.

Findings

Preliminary findings from a cohort of students (n = 20) drawn from four advanced mathematics courses indicates that advanced, undergraduate mathematics students' proof preferences are not consistent across comparison type. In comparison tasks of Type I, participants preferred direct proofs, which relied on the Principal of Mathematics Induction, when such proofs were contrasted with a proof by contradiction, which relied on the Well-Ordering Principal. In comparison tasks of Type II, students overwhelmingly selected an existence argument, with an implicit proof by contradiction, when compared with a constructive proof. This finding conflicts with prior qualitative research. No trends were observed in Type III comparison tasks. Finally, regarding Type IV survey items, students overwhelming selected direct statements; that is, statements which did not include a negation. This finding aligns with prior qualitative research on indirect proofs.

Discussion

This paper presents a novel approach to studying students' proof preferences related to indirect proof. The findings reported in this paper are preliminary and should be viewed as such, in part, because the result are drawn from a small sample of advanced students and because the work is preliminary. Further work is needed both with this population and with other populations. Indeed, novice proof writers, students at the beginning of the undergraduate studies, may exhibit different proof preferences. It is interesting, however, that much of the qualitative work on indirect proof has stressed that students prefer direct and constructive arguments (Harel & Sowder, 1998; Antonini & Mariotti, 2008) yet, in the context of the survey, students responses indicated a preference for the existence proof. Finally, variations in students' proof preferences across task type suggest that students' proof preferences may be more nuanced than indicated by prior characterizations.

Audience Questions

1. *Students' proof preferences appear to be linked, in some cases, to students' self-reported "comfort level" with particular forms of proof (e.g., induction proofs), as indicated by students' survey comments. In such cases, is "preference" an appropriate characterization of students' responses?*

2. *Several students noted in the comment section that they prefer direct proofs to indirect proofs. Yet, these same students selected the existence argument, with an implicit indirect proof, over a direct, constructive proof. What can we infer from instances in which students' comments do not align with trends in their proof preferences?*

References

Antonini, S., & Mariotti, M. A. (2008). Indirect Proof: What is specific to this way of proving? *ZDM – The International Journal on Mathematics Education*, *40*, 401 – 412.

Brown, S. (2003). *The evolution of students understanding of mathematical induction: A teaching experiment*. Unpublished doctoral dissertation, University of California, San Diego and San Diego State University.

Harel, G. (2007). Students' proof schemes revisited. In P. Boero (Eds.), Theorems in school: from history, epistemology and cognition to classroom practice (pp. 65–78). Rotterdam: Sense Publishers.

Mancosu, P. (1996). Philosophy of mathematical practice in the 17th century. New York: Oxford University Press.

Dubinsky, E. (1986). Teaching Mathematical Induction I. *Journal of Mathematical Behavior*, 5, 305-317.

Dubinsky, E. (1989). Teaching Mathematical Induction II. *Journal of Mathematical Behavior*, 8, 285-304.

Fischbein, E., & Engel, I. (1989). Psychological Difficulties in Understanding the Principle of Mathematical Induction. In Vergenaud, G., et als. (Eds.): *Proceedings of the 13th International Conference for the Psychology of Mathematics Education* (pp.276-282). Paris, France.

Harel, G. (2001). The Development of Mathematical Induction as a Proof Scheme: A Model for DNR-Based Instruction. In S.R. Campbell and R. Zazkis (Eds.), *Learning and Teaching Number Theory: Research in Cognition and Instruction*. Monograph Series of the Journal of Mathematical Behavior, Vol. 2., 185-212.

Harel, G. & Brown, S. (2008). Mathematical induction: Cognitive and instructional considerations. In M. Carlson & C. Rasmussen (Eds.), *Making the Connection: Research and Practice in Undergraduate Mathematics*. Mathematical Association of America, 111-123.

Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. In A. Schoenfeld, J. Kaput, & E. Dubinsky (Eds.) *Research on Collegiate Mathematics Education III*. (pp. 234-283). Providence, RI: American Mathematical Society.

Leron, U. (1985). A direct approach to indirect proofs. *Educational Studies in Mathematics*, 16(3), 321 – 325.

Moore, R. (1995). Making the transition to formal proof. *Educational Studies in Mathematics*, *27*, 249-266.

Movshovitz-Hadar, N. (1993a). Mathematical Induction: A Focus on the Conceptual Framework. *School Science and Mathematics*, 93(8), 408-417.

Movshovitz-Hadar, N. (1993b). The False Coin Problem, Mathematical Induction and Knowledge Fragility. *Journal of Mathematical Behavior*, 12, 253-268.

Polya, G. (1957). *How to Solve It*. Princeton University Press.

Robert, A., & Schwarzenberger, R. (1991). Research in Teaching and Learning Mathematics at an Advanced Level. In D. Tall (Ed.), *Advanced Mathematical Thinking*. (pp. 127-139). Dordrecht, Netherlands: Kluwer.

Selden, A., & Selden, J. (1995). Unpacking the logical of mathematical statements. *Educational Studies in Mathematics, 29*, 123 – 151.

Selden, A., & Selden, J. (2003). Validations of proofs considered as texts: Can undergraduates tell whether an argument proves a theorem? *Journal for Research in Mathematics Education, 34,* 4-36.

Weber, K. (2001). Student difficulty in constructing proof: The need for strategic knowledge. *Educational Studies in Mathematics, 48*, 101-119.

Weber, K., & Alcock, L. (2004). Semantic and syntactic proof productions. *Educational Studies in Mathematics, 56*, 209 – 234.

COUNTING PROBLEM STRATEGIES OF PRESERVICE SECONDARY TEACHERS

ToddCadwalladerOlsker,ScottAnnin,andNicoleEngelke

"Counting problems" are a class of problems in which the solver is asked to determine the number of possible ways a set of requirements can be satis.ed. Students are often taught to use combinatorial formulas, such as permutation or combination formulas, to solve such problems. However, it is common for students to incorrectly apply such formulas. Heuristics, such as "looking for whether or not order matters," can be unhelpful or misleading. We will discuss an ongoing analysis of preservice and inservice secondary and community-college level teachers' responses to six counting problems in order to determine the strategy or formula used in attempting to solve the problem. We are particularly interested in whether or not an explicit statement about order "mattering" helps or hinders the participants' ability to choose an appropriate strategy.

Counting problems, which ask the solver to determine the number of possible ways to fulfill a set of requirements, are an important type of problem in combinatorics and discrete probability. Solving such problems involves combinatorial ideas such as the multiplication principle (also known as the basic counting principle), combination formula, permutation formula, or a mixture of any of these. These topics are included in the typical introductory material of probability and combinatorics courses at the undergraduate level, and of second-year algebra and statistics courses at the secondary school level.

Solving such counting problems can be quite troublesome for students. It can be very difficult to determine the combinatorial ideas that are appropriate for solving a given problem. (This problem is compounded by the fact that there are several different ways to solve most counting problems.) Suppose we consider the number of arrangements of r objects selected from a set of n distinct objects. Students are often taught a common heuristic for solving counting problems: for ordered arrangements, use the permutation formula $n!/(n-r)!$; for selections that are unordered, use the combination formula $n!/(n-r)!r!$. However, this heuristic can be unhelpful or even misleading in certain contexts. In order to successfully and consistently solve counting problems, students must have a much richer knowledge of these combinatorial ideas.

Therefore, it is important that preservice teachers, in particular, be able to confidently solve counting problems. Furthermore, it is critical that they be able to choose an appropriate strategy for solving a given problem. By *strategy*, in this context, we mean the combinatorial constructions (such as the aforementioned basic counting principle, combination formula, or permutation formula) applied in order to solve the problem. Currently, there is very little research that has been done in this area. Therefore, we are currently conducting a study to investigate the counting problem strategies used by preservice teachers. We are currently in the beginning stages of this project, but we will have collected a significant amount of data by the end of the current semester. We propose to speak on the design and preliminary results of this study at the upcoming conference on RUME.

There is a small body of recent literature on the strategies used in solving counting problems. Annin and Lai (2010) have identified some common errors made by students when solving particular types of counting problems. Godino et al. (2005) have looked at counting problems through an ontological-semiotic model. Two of the same authors examined the responses of secondary school students' responses in an earlier paper (Batanero et al., 1997).

At our institution (a large public university in the Southwest United States), there are several mathematics and education classes intended for preservice teachers at the secondary or community college levels. During the current semester, these courses include two undergraduate "capstone" courses for students in the teaching option of the mathematics major, two graduate courses in the master's degree program in teaching mathematics, two courses in the post-baccalaureate teaching credential program, and a workshop for current teaching assistants at the university. We will give a short "quiz" of six counting problems to students in each class. (There are some students who are enrolled in more than one of these courses; we will ask these students to take the quiz more than once, if they are willing. This will give us data on the consistency of the students' strategies.) We expect that all of the participants will have encountered the basic combinatorial ideas necessary to solve counting problems in at least one high-school level course and at least one college-level course.

The six problems chosen for the quiz are given below:

1) A scientist has six test tubes, labeled A-F. Each tube contains one liquid: water, sugar solution, hydrochloric acid, or chocolate milk. In how many ways can the scientist place liquids in the tubes so that exactly two tubes contain water?

2) A bag contains 26 marbles, labeled A through Z. In how many ways can six marbles be chosen, where each of the six chosen marbles is different and the order in which they are chosen matters?

3) Six college freshmen must each be assigned to one of ten available academic advisors. If each student is to receive exactly one advisor, and each is assigned to a different advisor, in how many ways can these assignments be made?

4) A painter has twelve colors of paint available. When painting a house, she needs to choose a main color, trim color, accent color, and siding color, and all of these colors must be different from one another. How many ways are there for the painter to pick colors for the house?

5) A youth hockey team has twelve members. How many ways are there to choose a starting lineup of center, left wing, right wing, left defense, right defense, and goalie, if the order in which these positions are filled does not matter?

6) A toddler is stacking colored blocks, which can be red, white, blue, or green. If the toddler makes a stack of eight blocks, how many ways are there to stack the blocks so that exactly three blocks are red? (The order in which the blocks are arranged matters.)

The problems chosen are intended to be unfamiliar to the participants, as we did not want the participants to rely on previously memorized strategies. Each of these six problems can be solved using several different strategies. Three of the problems make an explicit statement of whether or not "order matters" (that is, whether or not rearrangements of objects are to be counted separately), but are otherwise "pairwise isomorphic" to the three problems that do not include a specific statement about order: problems 1) and 6) are essentially the same, as are the pair 2) and 3) and the pair 4) and 5). All six problems are really asking for a number of possible permutations; Problems 1) and 6) allow for repetition. The statements regarding order in problems 5) and 6) are deliberately misleading. In this way, we intend to investigate how the context and wording of a counting problem, particularly the inclusion of an explicit statement about order, affect the strategies used by the participants. We expect that at least some of the participants will rely heavily on the heuristic of using a permutation formula when explicitly told that order matters, and a combination formula when explicitly told that order does not matter. We are interested to see how widespread the use of this heuristic is in our data, and the strategies used when an explicit statement of order is not given by the problem.

The challenges of teaching combinatorics, particularly the strategies for solving counting problems, are not surprising– few formulas and set procedures can be blindly applied without a careful understanding of the delicacies that exist. Subtle differences in wording or interpretation of the questions can lead to vastly different solution techniques and answers. This research attempts to gain a foothold on some of the challenges in this area with a vision of enhancing the quality of instruction in the area of counting problems and bringing to light some new ideas on how teachers can help students avoid the pitfalls described above.

Questions for the Audience:

- What other studies have examined the strategies used by students and teachers to solve counting problems?
- What other difficulties are often encountered in solving counting problems?
- What other heuristics are used to solve counting problems?

References

Annin, S. A. and Lai, K. S. (2010). Common errors in counting problems. *Mathematics Teacher*, 103(6):402–409.

Batanero, C., Navarro-Pelayo, V., and Godino, J. D. (1997). Effect of the implicit combinatorial model on combinatorial reasoning in secondary school pupils. *Educational Studies in Mathematics*, 32:181–199.

Godino, J. D., Batanero, C., and Roa, R. (2005). An onto-semiotic analysis of combinatoral problems and the solving process by university students. *Educational Studies in Mathematics*, 60:3–36.

How Do Mathematicians Make Sense of Definitions?

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ABSTRACT: It seems clear that students' activity while working with definitions differs from that of mathematicians. The constructs of concept definition and concept image have served to support analyses of both mathematicians' and students' work with definitions (c.f. Edwards & Ward, 2004; Tall & Vinner, 1981). As part of an ongoing study, we chose to look closely at how mathematicians make sense of definitions in hopes of informing the ways in which we interpret students' activity and support their understanding of definitions. We conducted interviews with mathematicians in an attempt to reveal their process when making sense of definitions. A striking observation relates to the role of examples. We will share a preliminary analysis of these interviews and engage the audience in reflecting on the ideas.

KEY WORDS: mathematical definitions, advanced mathematical thinking, mathematicians' practice, examples

How do we come to understand mathematical definitions? Is the process different for students than it is for mathematicians? What can be learned from the practice of mathematicians that could support students' learning? In their chapter on advanced mathematical thinking, Harel, Selden, & Selden (2006) identified mathematical definitions as one area of focus when comparing the activity of students with the practice of mathematicians. The constructs of concept definition and concept image have served to support analyses of both mathematicians' and students' work with definitions (c.f. Edwards & Ward, 2004; Tall & Vinner, 1981). Our current research attempts to bring such ideas together into explanatory models for mathematicians' and students' activity. In this presentation we will focus on mathematicians and how their ability to build adequate concept images might develop.

Mathematicians encounter definitions in their work in a variety of ways. There are definitions included in courses they teach, definitions proposed by other mathematicians, and perhaps even new definitions created in the course of their own research work. In instructional settings, mathematicians must decide how to present definitions to students. In the context of current mathematical work, mathematicians must judge the clarity and appropriateness of stated definitions. In preparing to share proposed definitions, mathematicians must also consider presentation, clarity, and usefulness. Each of these settings requires some level of making sense of a given definition within a mathematical setting. We set out to create an interview context in which aspects of this activity were brought out and thus became accessible for analysis. The interviews provided opportunities for the mathematicians to articulate their perspectives on making sense of definitions and to participate in definition-related tasks (Watson & Mason, 2005).

In the interviews, participants were first asked to describe how they make sense of new mathematical definitions and to provide a recent example of doing this, if possible. The second interview question asked participants to share how they support students' work with definitions. Participants were then asked to engage in an example-generation activity, and finally were given a formal definition from an unfamiliar context and asked to share how they would go about making sense of the definition. We will focus on participants' articulated and observed sense-making process and possible connections to their articulation of instructional practice.

Participants described their process of making sense of a new definition by sharing insights from a range of activities, including reading papers or textbooks and creating definitions within their research work. When presented with the unfamiliar definition, participants reflected on their thinking, identifying what they found challenging about the task, and what next steps they would take. Participants were told that additional information was available and would be provided if requested.

One of our main observations concerns the use of examples. When asked how they make sense of a new definition, participants immediately referred to the use of examples and repeated the importance of examples when describing their teaching. Given their emphasis on examples, it was reasonable to expect participants to use an example when presented with a new definition. Rather, participants focused on specific terms or notations within the statement, either working through these on their own or asking for supporting information. Given the immediacy of their reference to examples previously in the interviews, we found the tendency to *not* ask for an example initially surprising. Our analysis needed to account for this expressed relevance of examples and the seemingly contradictory behavior of not actually asking for an example. Two themes emerged from the analysis that served to coordinate our observations. We will present these themes first as an explanatory model and then compare the model to what the mathematicians said about their instructional practice.

Mathematicians make sense of definitions by situating the definition within a particular mathematical setting and considering the usefulness of the definition within that setting. Placing a definition within a setting involves a progression of previous definitions, notations, and examples. When presented with the unfamiliar definition, participants began by sorting through the specific terms and notations within the statement. This involved requesting and receiving various supporting definitions. Within this process, participants made references to their own previous knowledge or to contexts with which they were familiar. In particular, participants questioned things such as why some terms were presented in a specific way, or whether the definition needed to be as general as it appeared to be. In most cases, participants did not ask to see an example as part of this process. We see this as the mathematicians needing to situate the statement of the definition clearly within a mathematical setting to judge the value or usefulness of the definition.

Mathematicians use examples as a tool for understanding definitions. In discussing both their own work and their work with students, participants spoke about examples as key to building understanding. Examples should be chosen carefully so that they serve to draw attention to important aspects. In making sense of definitions, the participants said they use examples to confirm their understanding, often choosing "messy" examples to be sure they had not introduced inappropriate assumptions. Creating or considering non-examples was considered an essential component of understanding: "and the only way to get there is look at concrete examples and look at concrete non-examples." The use of examples and non-examples seemed critical for their own understanding and how they support students' understanding.

In this study, mathematicians placed definitions within a particular mathematical setting; in their general practice and in their instruction, this setting is already set. When teaching, they attend to presenting ideas in a logical progression so that students have the necessary pieces to understand definitions. In their own work, they are familiar with current terms and notations within their field, so the setting and progression are understood. Within the setting, examples and non-examples serve to spotlight key features of the definition; using a non-example can help to clarify why certain aspects of the definition are needed (why does the function need to be continuous here?). When we presented mathematicians with an unfamiliar definition, it was not situated within a particular setting. Therefore, they needed to understand the key components of the statement before they could use examples as a tool. That is, examples do not "carry" the definition entirely.

In the presentation we will engage the audience with the topic by asking them to reflect on their own practice, provide background on our study, share our preliminary analysis, and finally, ask the audience to provide feedback. The following prompts will be used.

1. Take a minute to think about what you do when you want to make sense of a new definition. How do you know when you understand a definition?
2. Take a minute to think about what you do to support students' understanding of mathematical definitions. How does this compare to what you do for yourself?
3. Do these themes resonate with your experience? How does this help us interpret students' mathematical activity and inform instructional practice?

References

Edwards, B. S. & Ward, M. B. (2004). Surprises from mathematics education research: Student (mis)use of mathematical definitions. *The American Mathematical Monthly, 111*(5), 411-424.

Harel, G., Selden, A., & Selden, J. (2006). Advanced mathematical thinking. In A. Gutierrez & P. Boero (Eds.), *Handbook of research on the psychology mathematics education: Past, present, future* (pp. 147-172). Rotterdam, The Netherlands: Sense Publishers.

Tall, D. & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*(2), 151-169.

Watson, A. & Mason, J. (2005). *Mathematics as a Constructive Activity: Learners Generating Examples*. Mahwah, N.J.: Lawrence Erlbaum Associates.

Material Agency: questioning both its role and meditational significance in mathematics learning.

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Abstract

Tools in the mathematics classroom are often not given the credence or the attention they warrant. Considering Vygotsky's view of mediation, tools may play a larger role in mathematics then originally thought. This preliminary report presents a framework for attempting to identify the implications of tools in student learning. Using Pickering's analytic framework (1995) distinguishing individual, disciplinary and material agencies, I am interested in how material agency takes form in the interaction of students with tools. While teaching an education class of pre-service mathematics teachers I will analyze their interactions with a Dynamic Geometric software, specifically Geometer's Sketchpad. In the process of solving a problem I will analyze students' engagement with the tool in terms of the different types of agencies, based on their spoken words and their actions in using the program.

Key words: agency, disciplinary agency, material agency, mediation, dynamic geometry software, Geometer's Sketchpad

Introduction

A tension has always existed between the advocates of mathematics as being more of a mental discipline and, both academics and pedagogues, who consider the physical role of objects, materials or machines playing a formative role in the learning of mathematics. While both sides recognize that tools play their role in the practice of mathematics, the mental mathematicians may consider that tools or machines play a small role to either simplify a calculation to arrive at a particular theorem or merely serve as a vessel that serves the sole purpose of "getting" to the mathematics. This attitude is not so much explicitly stated as it is practiced. Whether stemming from Plato's vision of mathematics as a separate, distinct and pure discipline, that is accessible solely through contemplation (Tarnas, p. 6), mathematical production acts often state no reference to materials or tools used in the process. While mental discipline advocates argue tools can cloud the very nature of mathematics, advocates for an object-oriented inquiry argue that tools or machines influence how we learn mathematics (Turkle, 2007) and are consequently worthy of study.

Analytic Framework

The implementation of tools or machines into mathematics classrooms and how they are used is a topic of interest: if mathematics learning is to be fully understood, the tools used in mathematical activity are not to be reduced to an avoidable step. Wertsch claims that one of Vygotsky's major themes in his theoretical approach was "…that an adequate account of human mental functioning must be grounded in an analysis of the tools and signs that mediate it" (in Daniels, 2008, p. 4). The framework that I would like to propose for analyzing tools in mathematics education is based on Pickering's distinction of agencies. Pickering (1995) has classified 3 types of agency: individual, disciplinary and material. While one would not usually think of materials or disciplines as having agency, Pickering describes the individual engagement with either of these agencies as a "…dialectic of resistance and accommodation" (p. 52). Pickering has referred to this interplay of resistance and accommodation as a "dance of agency". His view is that mathematics is a product of human activity and therefore individual agency plays a major role in any conceptual and/or material advancement. However, engagement with materials or conceptual systems is not a one-sided affair. In his argument for disciplinary agency Pickering describes how a conceptual system can "…carry human conceptual practices along…independently of individual wishes and intents" (p. 115). So although individuals exercise their agency in their intentions and actions, they are often met with resistance or an obstacle . This resistance is the agency of the material or conceptual system. The dance of agency is then enacted by having the individual accommodate their actions to appropriate the resistance. This dialectical interaction is the framework from which I would begin. When a student of mathematics is interacting with an object and an attempt is made by the individual to achieve a goal, any resistance to that goal is an example of material agency.

Boaler uses this framework to argue that disciplinary agency dominates the practices in a traditional classroom. Pickering sees this disciplinary agency as the negotiated rules and algorithms of mathematics. Thus if student are not given the chance to act, the math is given the status to direct and determine the practices of math classroom activity. Boaler argues that good classroom teaching would entertain a balance between the two agencies for both are important and essential. Both Pickering and Boaler however do not refer to material agency in mathematics. Pickering offers material agency as only being evident in scientific advancements. So while Pickering is focusing on the emergence of new ideas, theories, and practices I hypothesize that material agency does have significance in the practices of mathematics. Wagner also uses Pickering's framework by acknowledging disciplinary agency but appeals to material agency in mathematics and poses the question: "What is the nature of material agency in mathematics?" (Wagner, p. 43). I borrow from Wagner and ask the question: what is the nature and implication of material agency when students of mathematics are engaged in using a tool.

Proposed Study

While there are many tools and/or artifacts that have found themselves in different ways into the mathematical community I am choosing what could be termed a technological artifact. A dynamic geometry software (DGS) can be said to have been made to elicit determined geometrical principles. DGS's options and many features such as built in tools offer many choices for students to engage with. It is the choices they have that allows for them to exercise their own agency. This dialectic engagement is what I choose to focus on. While teaching an education class of pre-service mathematics teachers I will analyze their interactions in solving a problem by analyzing the data in terms of the different types of agencies, based mainly on their spoken words as well as their actions in using the program. Informal, ad hoc studies using Jing as a way of capturing both their dialogue as well as their activity within the program show evidence of material agency.

Questions

What does this framework offer that appropriation does not?

Is this a viable framework in mathematics education? How best to capture data for material agency? Does a DGS afford the opportunity to observe individual agency alongside material agency? How can one distinguish between disciplinary and material agency in the context of a DGS?

References:

Boaler, Jo. (2002). The development of disciplinary relationships: knoweldege, practice and identity in mathematics classrooms. *For the learning of mathematics* 22, 1, p. 42-47.

Daniels, H. (2008). *Vygotsky and research*. Routledge, London and New York.

Pickering, A. (1995). *The Mangle of Practice: Time, agency, and Science*. Chicago: The university of Chicago Press.

Tarnas, R. (1991). *The passion of the western mind: Understanding the ideas that have shaped our world view*. Ballantine Books, New York.

Turkle, S. (2007). *Evocative Objects: Things we think with*. (Ed.), MIT Press.

Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes*. Cambridge, MA: Harvard University Press.

Wagner, D. (2007). Students' critical awareness of voice and agency in mathematics classroom discourse. *Mathematical Thinking and Learning*, 9 (1), 31-50.

The Impact of Instruction Designed to Support Development of Stochastic Understanding of Probability Distribution

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Abstract:

Large numbers of college students study probability and statistics, but research indicates many are not learning with understanding. The concept of probability distribution undergirds development of conceptual connections between probability and statistics and a principled understanding of statistical inference. Using a control-treatment design, this study employed differing technology-based lab assignments and investigated the impact of instruction aimed at fostering development of stochastic reasoning on students' understanding of probability distribution. Participants were approximately 200 undergraduate students enrolled in a lecture/recitation, calculus-based, introductory probability and statistics course. This preliminary research report will discuss the framework used to develop the stochastic lab materials and preliminary results of an assessment of students' understandings.

Key words: Probability distribution, stochastic reasoning, technology-based instruction, instructional intervention.

Statement of research issue:

Large numbers of university students study probability and statistics (Moore & Cobb, 2000), but research indicates that many of these students exhibit difficulties in learning and applying probabilistic and statistical concepts (Garfield & Ben-Zvi, 2007; Shaughnessy, 1992, 2007). Inappropriate reasoning in probability and statistics is widespread and persistent across all age levels. After probability instruction, many post-calculus students demonstrate merely instrumental understanding (Skemp, 1976) and present notions about probability that are not aligned with formal probabilistic concepts (Barragues, Guisasola, & Morais, 2007). This study draws on constructivist and situated learning perspectives and assumes understandings are built through learning experiences, which are impacted by the learner, teachers, and the instructional material. The study assumes that: (1) teaching impacts learning and can facilitate learning with understanding; (2) effective teaching elicits students' pre-existing understandings and builds on that understanding; (3) effective teaching helps students develop deep knowledge connections in the context of a conceptual frame for the content domain (Bransford, Brown, & Cocking, 2000). This research was designed to evaluate the effectiveness of an instructional intervention that builds on students' initial understandings of probability and statistics and facilitates student understanding of content within a connected conceptual framework. The study seeks to measure and describe individual understandings of probability distribution.

The concept of probability distribution is a powerful springboard for the development of stochastic reasoning as it may facilitate making deep conceptual connections around probabilistic understandings related to variability, independence, sample space, and distribution (Liu & Thompson, 2007). Principled knowledge (Spillane, 2000) refers to an understanding of the ideas and concepts that support mathematical procedures. Principled knowledge of probability distribution not only refers to a conceptual understanding of the mathematical procedures used when solving probability problems, but also an understanding of connections between and within the constructs of probability, variability, and distribution.

This large-scale control-treatment study investigated the impact of an instructional intervention on post-calculus students' understandings of probability distribution. The treatment intervention consisted of lab materials designed to address stochastic reasoning and to support students' principled knowledge of probability distribution. The control lab materials reviewed prerequisite calculus content which students encounter in the course, thus controlling for quantity of instruction. This study addressed the following question: *What is the impact of an instructional intervention designed to support development of stochastic understanding of probability distribution of undergraduate students enrolled in an introductory calculus-based probability and statistics course?*

Summary of Related Research

Stochastic reasoning is grounded in conceptual connections between probability and statistics. To reason stochastically means conceiving of an observed outcome as but one expression of an underlying repeatable process that will produce a stable distribution of outcomes in the long run (Liu & Thompson, 2007). One reason why learners may experience difficulty with stochastic reasoning is because learning about random experiments through simulation or experimentation is not connected to learning about combinatorial schemes or tools such as tree diagrams in probability (Batanero, Godino, & Roa, 2004). Also, intuitive thinking based on experience with random generators appears to be disconnected to formal mathematical thinking about probability (Abrahamson, 2007). Making statistical inferences requires application of stochastic thinking for correct interpretation, and a stochastic conception of probability supports thinking about formal statistical inference (Liu & Thompson, 2007).

Research indicates that many post-calculus students, who are either currently enrolled in or have recently completed introductory probability and statistics courses, demonstrate probabilistic thinking and heuristical biases that are aligned with the thinking of novice learners in algebra-based classes and high school students (Abrahamson, 2007; Barragues, et al., 2007; Lunsford, Rowell, & Goodson-Espy, 2006). Even after instruction addressing probability, many post-calculus students still exhibit poor understandings of random phenomena and present mistaken conceptions of random sequences, insensitivity to sample size, and a deterministic bias. Research shows that post-calculus students who have completed a probability/statistics course still have difficulty with a modeling viewpoint and struggle to discriminate between empirical distributions and theoretical probability distributions (Barragues, et al., 2007). Research suggests that after completing an introductory, calculus-based probability and statistics course, most students are comfortable with formal mathematical manipulations of probability distributions and master algorithmic techniques, but they lack stochastic conceptions and deep conceptual understanding of probability distribution.

Research investigating development of post-calculus students' understanding of probabilistic concepts indicates that teaching is an important factor related to students' understandings of probability. Teaching which emphasizes procedures tends to result in instrumental understanding, whereas teaching which facilitates learner explorations of conceptual notions of probability as a distribution and its connection to mathematical theorems offers opportunities for students to build relational understanding (Skemp, 1976) in probability and statistics. A study of post-calculus engineering students' conceptions of probability found that conventional teaching can have a poor effect on students' probabilistic reasoning (Barragues, et al., 2007). Although not conducted in a classroom, the work of Abrahamson (2007) indicates that post-calculus learners can consolidate their intuitive notions of probability with their formal mathematical knowledge in the context of probability distribution. Still other research points to the promise of learners' engagement in tasks utilizing a computer-based dynamic statistical environment as a means towards facilitating development of notions of sampling distribution, variability, and inferential reasoning (Meletiou-Mavrotheris, 2003; Sanchez & Inzunsa, 2006).

Research Methodology:

This study compared the impact of differing instructional lab materials. The subjects were approximately 200 students enrolled in a calculus-based introductory probability and statistics course at a large, public university. The course setting consisted of two lectures with the same syllabus taught by mathematicians who covered the same content. Teaching assistants led accompanying recitations. One lecture had six recitations sections, and the other had four. Students were randomly assigned to a recitation section via their course registration. Each recitation section associated with a given lecturer was randomly assigned to either the treatment or control condition whereby a teaching assistant had both treatment and control recitations. This assignment balanced the treatment and control across lectures and recitation sites in order to mitigate confounding variables due to differences in teaching between the lecturers and between the teaching assistants. All students enrolled in a particular recitation received one type of lab material. The treatment group received lab materials designed to support stochastic reasoning, and the control group received lab materials which consisted of a review of calculus content used in the course. Students' understanding was measured via conceptual assessments in the form of an extra-credit quiz and course examinations. At the end of the study, selected students participated in interviews designed to provide insight into students' thinking and reasoning about conceptual assessment items.

Framework for Instructional Intervention:

The treatment instructional intervention implemented in this study consisted of six supplemental lab assignments aimed at the development of stochastic reasoning in the context of probability distribution. The design of these tasks was based on a hypothetical learning trajectory (Simon, 1995) of students' stochastic conceptions of probability (Liu & Thompson, 2007) which was adapted for use in the context of probability distribution. This study extended the research investigating the impact of bridging tools (Abrahamson & Wilensky, 2007) on college students' understanding of probability distribution into a classroom setting. Learners in the treatment sections engaged in technology-supported simulation tasks designed to elicit prior understandings of probability. These tasks required learners to consider juxtaposed constructs in the domain, such as theoretical versus empirical probability and independent versus dependent events. The approach was to have the learner decompose domain constructs into idea components and then use conceptual bridging tools to recompose the constructs using their intuitive and analytic resources. In order to control for instructional time, the control group received tasks which reviewed calculus content used in the course and covered topics such as integration using substitution and integration by parts. The instructional intervention material was designed to prepare students to learn from the lectures and therefore provide greater opportunity for students to make deeper conceptual connections (Schwartz & Bransford, 1998).

Implications of this research:

Given the evidence that many college students in probability and statistics classes are not learning with understanding, it is critical to investigate the effectiveness of approaches for teaching probability and statistics in ways that build on students' initial understandings and helps students understand the content not merely as facts to be memorized, but as connected concepts within a conceptual framework. Knowledge of whether instruction which is aimed at fostering stochastic reasoning impacts learners' understandings of probability distribution could inform future design of instruction and development of instructional materials in probability and statistics.

Discussion Questions:

- How might this framework (to be shared in the presentation) be extended for use when planning instructional strategies or in designing instructional material?
- How might this framework further inform the analysis of the conceptual assessment items? (preliminary findings and planned analysis will be presented)
- What are the further implications of these preliminary findings for instruction in probability and statistics?
- What are the advantages/disadvantages of utilizing technology-based instructional material to support lecture/recitation delivery of course material?
- What other issues related to student understanding of probability and statistics might be informed by this study?
- What are the implications of the degree of student understanding of prerequisite calculus procedures/concepts (as revealed in the control labs) for those teaching probability and statistics as well as for the teaching of calculus?

References

Abrahamson, D. (2007). *The real world as a trick question: Undergraduate statistics majors' construction-based modeling of probability.* Paper presented at the American Education Research Association, Chicago, IL.

Abrahamson, D., & Wilensky, U. (2007). Learning axes and bridging tools in a technology-based design for statistics. *International Journal of Computers for Mathematical Learning, 12*(1), 23-55.

Barragues, J., Guisasola, J., & Morais, A. (2007). Which event is more likely? Estimating probability by university engineering students. *Recherches en Didactique des Mathematiques, 27*(1), 45-76.

Batanero, C., Godino, J., & Roa, R. (2004). Training teachers to teach probability *Journal of Statistics Education, 12*(1).

Batanero, C., Tauber, L., & Sanchez, V. (2004). Students' reasoning about the normal distribution. In D. Ben-Zavi, & Garfield, J. (Ed.), *The challenge of developing statistical literacy, reasoning, and thinking*. Dordrecht, the Netherlands: Kluwer Academic Publishers.

Bransford, J., Brown, A. L., & Cocking, R. R. (Eds.). (2000). *How people learn: Brain, mind, experience, and school*. Washington, DC: National Academy Press.

Garfield, J., & Ben-Zvi, D. (2007). How students learn statistics revisited: a current review of research on teaching and learning statistics. *International Statistical Review, 75*(3), 372-396.

Liu, Y., & Thompson, P. (2007). Teachers' Understandings of Probability. *Cognition & Instruction, 25*(2/3), 113-160.

Lunsford, M., Rowell, G., & Goodson-Espy, T. (2006). Classroom research: Assessment of student understanding of sampling distributions of means and the central limit theorem in post-calculus probability and statistics classes. *Journal of Statistics Education, 14*(3).

Meletiou-Mavrotheris, M. (2003). Technological tools in the introductory statistics classroom: Effects on student understanding of inferential statistics. *International Journal of Computers for Mathematical Learning, 8*, 265-297.

Moore, D., & Cobb, G. (2000). Statistics and mathematics: Tension and cooperation. *The American Mathematical Monthly, 107*(7), 615-630.

Sanchez, E., & Inzunsa, S. (2006). *Meanings' construction about sampling distributions in a dynamic statistics environment*. Paper presented at the Seventh International Conference on Teaching Statistics, Salvador, Brazil.

Schwartz, D., & Bransford, J. (1998). A time for telling. *Cognition and Instruction, 16*(4), 475-522.

Shaughnessy, J. M. (1992). Research in probability and statistics: Reflections and directions. In D. A. Grouws (Ed.), *Handbook of research on mathematical thinking and learning* (pp. 465-494). New York: McMillan.

Shaughnessy, J. M. (2007). Research on statistics learning and reasoning. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning*. Charlotte, NC: Information Age Publishing.

Simon, M. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26*(2), 114-145.

Skemp, R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching, 77*, 20-26.

Spillane, J. (2000). Cognition and policy implementation: District policymakers and the reform of mathematics education. *Cognition & Instruction, 18*(2), 141-179.

Supplemental Instruction and Related Rates Problems

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In this study, we observed first semester calculus students solving related rates problems in a peer-led collaborative learning environment. The development of a robust mental model has been shown to be a critical part of the solution process for such problems. We are interested in determining whether the collaborative learning environment promotes the development of such a mental model. Through our observations, we were able to determine the amount of time students spent engaging with the diagrams they drew to model the problem situation. Our analysis strove to also determine the quality of the student interactions with their diagrams. This analysis provided insights about the mental models with which the students were working. Engaging students with complex, non-routine problems resulted in the students spending more time developing robust mental models.

Background

Supplemental Instruction (SI) workshops, based on the work of Uri Treisman in the early 1980s and developed by the University of Missouri, Kansas City (UMKC), have been highly successful at campuses across the country and also in neighboring campuses with student populations like those at our institution. The UMKC SI workshop is a structured learning environment where students gain additional experience in the subject matter taught in the course to which it is linked (Bonsangue, 1994). From these models, we have developed a version of the SI workshops that meets our students' needs. Students do not simply review course material or do homework in SI workshops, but undertake additional, challenging problems or assignments to build confidence in their abilities and to gain self-reliance. They engage in active and cooperative learning activities, utilizing peer facilitators as resource persons. The peer facilitator attends each class lecture so that the workshop problems are relevant to course assignments. In doing so, the peer facilitator also serves as a role model for SI students and creates an increased culture of accountability in the classroom.

In the Spring 2009 semester, our institution began expanding our workshop program, particularly in calculus. We ran 3 successful pilot SI workshops for first semester calculus. The success of those courses led us to run 4 and 5 first semester calculus SI workshops, respectively, in the Fall 2009 and Spring 2010 semesters. Our workshop model is such that two calculus courses (usually) taught by the same instructor feed into one SI workshop. The workshop is an optional one semester credit hour course; it is hosted by a junior or senior level math major and graded credit/no credit based solely on attendance and participation. In each workshop session, the peer-mentor facilitates student group work on topics that have been recently presented in lecture. During the Fall 2009 and Spring 2010 semesters, we observed how students solve related rates problems in this peer-led instructional setting with the purpose of examining how students think about and solve application problems. We were particularly interested in whether the collaborative environment of the workshop setting promoted the development of the students mental model of the problem situation.

Theoretical Perspective

White and Mitchelmore (1996) studied students understanding of related rates and max/min calculus problems. Their study used differently worded versions of four problems, which ranged from a word problem that required the student to model the situation and come up with the appropriate relation to an almost strictly symbolic version that merely needed to be manipulated. It was found that students performed better when there was less need for translation from words to symbols.

White and Mitchelmores study showed that students have a tendency for a manipulation focus, in which they base decisions about which procedure to apply on the given symbols and ignore the meaning behind the symbols. Interview comments showed that manipulation focus errors were not just bad luck, but that students were actively looking for symbols to which they could apply known manipulations.(p. 88) The researchers further described two other forms of the manipulation focus: 1) the x, y syndrome, in which students remember a procedure in terms of the symbols first used to introduce the concept without understanding the meaning of the symbols; and 2) the students fail to distinguish a general relationship from a specific value.

In her studies, Engelke (2004, 2007a, 2007b) found that students fixated on procedural steps which prevented them from building a mental model of the situation. Without having a mental model of the situation, the students were less likely to engage in transformational (Simon, 1996) and covariational reasoning (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002) about the problem situation. The ability to engage in transformational and covariational reasoning appeared to be necessary to identify relevant functional relationships. She suggested that a robust mental model facilitated being able to move flexibly between the geometric and algebraic representations which fostered being able to construct an appropriate formula in which one can think of each of the variables as a function of time.

Methods

In our study, the students are first semester calculus students that are coming from two or more standard lecture courses and have the option to participate in a Supplemental Instruction (SI) workshop. The SI sessions are hosted by junior/senior level math majors and are intended to further develop the students understanding of the material presented in the regular class sessions. During the SI sessions the students worked in small groups to solve problems provided by the SI leaders. During their time in the SI workshop, the students were filmed over several days, in which the students covered the chain rule, implicit differentiation, and related rates problems. Since the students were vocalizing and writing their ideas and thought processes, we were able to determine what types of reasoning structures the students were utilizing. These videos were transcribed for analysis, using pseudonyms for the students names. The coding process was done using atlas-ti software.

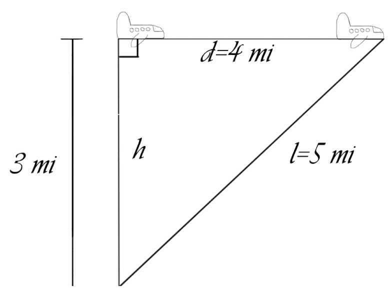
Results

The students were given the following problem: *A plane flying horizontally at an altitude of*

3 miles and a speed of 600 mi/hr passes directly over a radar station. When the plane is 5 miles away from the station, at what rate is the distance from the plane to the station increasing?

As we see in the transcript excerpt below, the students began by drawing a diagram to represent the problem situation and labeled the sides of the triangle that is being formed by the plane and the radar station with the variables l, h, and d. It was also determined that they should use the Pythagorean Theorem to relate the variables in the problem.

Student 3: The distance over time. Its miles like the island...okay so a plane is flying horizontally at an altitude of 3 miles [labels with d], at a speed 600 miles per hour. Passes directly over the radio station then is 5 miles away from the radio station, 5 miles. [calling the diagonal distance $l = 5$ miles] what is the rate, at what rate is the is the distance from the plane to the radio station increasing? [Also wrote $\frac{dd}{dt} = 600$ mi/hr and $\frac{dr}{dt} =?$ under the diagram] Ok. Were going to call this h [the vertical height of 3 mi]. Ok, someone. [Passing the chalk as per instructions from the SI leaders] Wait, wait, I forgot to do...never mind, the equation, the equation we are going to use to relate it all is the Pythagorean Theorem exactly.



Student 2: Does that look good? [wrote $3^2 + d^2 = 5^2$ to fill in $d = 4$ on the diagram]
Student 1: It looks great; I am just trying to
Student 3: Ok, can we just use 3 squared plus d squared is x squared, yes.
Student 2: Should I do like, should I differentiate it? ...
Student 1: Let me know if I am doing this right. We are looking for $\frac{dd}{dt}$, dont we need a derivative? [starts differentiating the Pythagorean theorem which he wrote as $a^2 + b^2 = c^2$]
Student 3: Why do a, b, c? I think it should really be the letters we have.
Student 4: Do we have this? We don't have this. [referring to $\frac{da}{dt}$]
Student 2: What we have is the second one.
Student 3: This still...if this still says a, I feel that this should say h, right? Not a right?

As the students start to incorporate the Pythagorean Theorem into the solution, there is some debate on the notation that is being used. Student 3 used d, h, and l as the variables, and Student 1 is now using a, b, and c. Even after Student 4 starts working, Student 3 is still worried about the mixed up variables.

There are three different ideas of what the Pythagorean Theorem should look like at this point in the problem solving process. Student 3 starts using $3^2 + d^2 = x^2$, but then changes her mind and thinks it should use the variables they have in their diagram. Student 1 is using $a^2 + b^2 = c^2$.

The variables used to write down the formula for the problem appear to be based on what has been used in previous problems for Student 1 and Student 3. However, Student 3 shifted her thinking to wanting the variables used in the formula to match the variables used in the diagram to represent the problem situation. In the end, it was decided that the variables in the formula should match the variables they had used in their diagram to get $3^2 + d^2 = l^2$. They differentiated their formula to obtain: $2d\frac{dd}{dt} = 2l\frac{dr}{dt}$. This brings up the question, where did the $\frac{dr}{dt}$ come from? Recall that when Student 3 drew the diagram at the very beginning of the problem solving process, she labeled $\frac{dr}{dt} =?$ under her picture. This could be an instance of the x, y syndrome described by White and Mitchelmore as the students wrote this for every problem. The students seemed to associate $\frac{dr}{dt} =?$ with the unknown rate regardless of which variables they had used in their diagram, so when they differentiated their formula, they had to accommodate this notation. The students successfully solved the problem; reporting that $\frac{dr}{dt} = 480$ mi/hr.

In their diagrams, these students were inconsistent in their use of variables, frequently not representing any of the changing quantities with variables but only labeling quantities with numerical values. While diagrams are being drawn, the students seem to be choosing a formula for the problem based on keywords about the "shape" that is mentioned in the problem rather than on the relationships that exist between variables in the problem. The absence of a robust mental model that incorporates variable names for changing quantities could be why the students had no issues writing down $\frac{dr}{dt} =?$ for each problem they solved. This was merely the notation for what they wanted to find rather than a representation of a rate connected to a particular changing quantity. These observations support Engelke's (2004, 2007a, 2007b) results that students are not adept at building mental models that support the problem solving process when solving related rates problems. Based on this evidence, we suggest that more time should be allotted for engaging students in building robust mental models that incorporate the relationships that exist between the diagram that is drawn, the variables that represent changing quantities, and an appropriate formula. We will present some ideas about how to improve students' mental models.

Questions for the audience:

1. What qualitative studies have been done on peer-led instructional settings, in calculus?
2. Is the problem solving process that occurs in peer-led instructional settings reflective of what occurs in the classroom?
3. How do you promote students' building of robust mental models?

References

Bonsangue, M. (1994). An efficacy study of the calculus workshop model. *CBMS Issues in Collegiate Mathematics Education*, *4*(117-134).

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, *33*(5), 352-378.

Engelke, N. (2004). Related rates problems: Identifying conceptual barriers. In D. McDougall (Ed.), *26th annual conference of the north american chapter of the international group for the psychology of mathematics education* (Vol. 2, p. 455-462). Toronto, Ontario, Canada: Ontario Institute for Studies in Education of the University of Toronto.

Engelke, N. (2007a). A framework to describe the solution process for related rates problems in calculus. In T. Lamberg & L. Wiest (Eds.), *29th annual meeting of the north american chapter of the international group for the psychology of mathematics education* (p. 441-448). Stateline (Lake Tahoe), NV: University of Nevada, Reno.

Engelke, N. (2007b). *Students understanding of related rates problems in calculus*. Doctoral dissertation, Arizona State University.

Simon, M. A. (1996). Beyond inductive and deductive reasoning: The search for a sense of knowing. *Educational Studies in Mathematics*, *30*, 197-210.

White, P., & Mitchelmore, M. (1996). Conceptual knowledge in introductory calculus. *Journal for Research in Mathematics Education*, *27*(1), 79-95.

Exploring student's spontaneous and scientific concepts in understanding solution to linear single differential equations

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In this study, we use the zone of proximal development to characterize students' spontaneous and scientific concepts of rate of change, rate proportional to amount, exponential function and long-term behavior of solutions for a system of one and two linear autonomous differential equations. Our focus on the dynamics of the differential equation systems is to investigate how these spontaneous and scientific concepts are incorporated from a system one linear differential equation into a larger system of two linear differential equations. We use and adapt previously used instructional activities from an inquiry-oriented differential equation course to help us gather our data by doing semi-structured interviews with five students. We present only preliminary findings on student's thinking of solutions mainly for single differential equations, with some insights of student thinking of solutions on a system of two differential equations.

KEYWORDS: Differential equations, solutions, rate of change, zone of proximal development

Research on the teaching and learning of differential equation concepts has recently appeared within the last decade (Habre 2000, Rasmussen, 2000, 2001, 2003, Rowland, 2006, and Rasmussen and Blumenfeld, 2007). These last studies show that students have difficulties acquiring meaningful understanding of the concept of derivative even though students can manipulate formulas algebraically or geometrically. For example, students have misconceptions in understanding the different components of a differential equation, such as having difficulties describing what the rate of change, variable and solution means within the context of the differential equation (Habre, 2000; Rasmussen and King, 2000; Rasmussen 2001). Rasmussen and Whitehead (2003) reported that similar cognitive difficulties were observed when students have to interpret solutions or varying rates for a system of two equations. In 2007, Rasmussen and Blumenfeld describe a teaching experiment about a spring-mass problem where students invented their own intuitive method for finding the eigenvectors for a system of two linear differential equations with constant coefficients. The authors found that students used proportional reasoning to help them understand the concept of eigenvectors. However, studies at this level, with a system of two differential equations, are very scarce.

This paper presents the preliminary results of an ongoing research investigation about the difficulties and ways of reasoning associated with understanding the dynamics of solutions and long term behavior of the autonomous linear system with one or two equations within different context representations (numerical, graphical, and algebraic). More specifically, these are the two main questions we want to explore:

1. What are students' spontaneous concepts about rate of change, "rate proportional to amount", exponential function, and long-term behavior of solutions to differential equations for systems of one and two equations?
2. Given a specific set of activities[1], what is the interplay between spontaneous and scientific concepts (within our focus)?

Our main motivation for this study is to explore student thinking and reasoning of rate proportional to amount in systems of linear differential equation with one and two equations. We used different mathematical representations so that students not only algebraically solve differential equations, but also given the opportunity to understand different patterns and relationships, and to model and predict general and qualitative behavior about solutions. We want to investigate the interplay of student's spontaneous and scientific concepts about rate of change, rate proportional to amount, and exponential function in connection to solutions to differential equations within both dimensions.

The theoretical foundations for this study follows: Vygotsky(1987)'s zone of proximal development and Steffe and Thompson (2000)'s teaching experiments. We use the framework of teaching experiments (Steffe and Thompson, 2000) not only to see first hand how students are constructing knowledge, but also to follow and not necessarily test a learning trajectory. According to Vygotsky, a learner develops meaning and understanding from mental processes and from a concept system (or from a structured concept system of ideas), in which both spontaneous and scientific concepts follow an interdependence learning development of a concept. A spontaneous concept originates when a person first encounters the new concept within empirical situations, while a scientific concept originates when a person first encounters the new concept in its generalized form. Both concepts follow interdependent paths of development in associating an object to its associated generalized concept. One of the attributes of a scientific concept is that it has to exist within a concept system so that connections of mental process can be made and generalized. Another attribute is that the learner has to operate with conscious awareness and volition in their thinking process. In contrast to all of these main attributes for the scientific concept, a spontaneous concept exist outside a concept system, it lacks some level of conscious awareness, with no generality and voluntary control involved.

Going back to our research questions, the second question is a very important question because not only we want to investigate characterization of spontaneous concepts to scientific concepts but also their influence on each other. Both spontaneous and scientific concepts required further thinking and reflection, so we anticipate these instructional activities will helps us get at those instances in students' learning development. We want to investigate what students can do with these concepts so that concepts are used consciously and with purpose (i.e. voluntary control). With "what can students do" we are referring to those mental images students developed in their learning process mediated by the spontaneous and scientific concepts. Hence, we are interested in the characterization of the zone of proximal development when learning differential equations given a specific set of instructional activities, and how certain concepts

[1] This specific set of activities were mainly developed by professor Chris Rasmussen

(especially rate and rate proportional to amount) are later incorporated into a more advanced system consisting of two differential equations.

For this paper, we will present preliminary findings with respect to our first question and some insights with respect to question two. Our preliminary findings indicate that students use differential equations (with one equation), $dp/dt = f(p)$ as a mathematical tool to predict numerical or graphical solutions, as reported in other prior studies. (Habre, 2000, Rasmussen, 2001, 2003, Rasmussen and Stephan, 2002). In general we expect students to be able to solve linear differential equations algebraically and qualitatively, but maybe not be consciously aware of why methods worked or what the solution actually means. We expect to see more cognitive difficulties when students make the transition from working with single differential equation to a system of two differential equations. Our long term goal for the study is to explore the effect of focusing on these different concepts given a set of particular set of instructional activities, and how it can help in improving the teaching and learning of differential equations.

Question for discussion:

1. Why is it useful to characterize spontaneous or a scientific concept of solutions in a single differential equation or from a system of two differential equations?

2. What is involved in creating a coherent learning trajectory for student when learning solutions from a single differential equation into a system of two differential equations? And what do we mean with a coherent learning trajectory?

References

Habre, S. (2000). Exploring Students' Strategies to Solve Ordinary Differential Equations in a Reformed Setting. *Journal of Mathematical Behavior,* **18**(4): 455-472.

Rasmussen, C. and Blumenfeld (2007). Reinventing Solution to Systems of Linear Differential Equation: A case of Emergent Models Involving Analytic Expressions (master thesis manuscript), San Diego State University.

Rasmussen, C. and M. Keynes (2003). Lines of Eigenvectors and Solutions to Systems of Linear Equations. PRIMUS **XIII**(4): 308-320.

Rasmussen, C. and K. King (2000). Locating Starting Points in Differential Equations: A Realistic Mathematics Approach. Journal of Mathematical Education in Science and Technology **31**(2): 161-172.

Rasmussen, C. and K. Whitehead (2003). Learning and Teaching Ordinary Differential Equations, MAA Online.

Rowland, D. and Z. Jovanoski (2004). Student Interpretations of the Terms in First-Order Ordinary Differential Equations in Modeling Contexts. *International Journal of Mathematical Education in Science and Technology* 35(4): 503-516.

Rowland, D. (2006). Student Difficulties with Units in Differential Equation in Modeling Contexts. *International Journal of Mathematical Education in Science and Technology* **37**(5): 553-558.

Steffe, L. P., & Thompson, P.W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A.E. Kelly(Eds.), *Research design in mathematics and science education.* Dordrecht, The Netherlands: Kluwer.

Stephan, M. and C. Rasmussen (2001). Classroom Mathematical practices in Differential Equations. *Journal of Mathematical Behavior* **21**: 459-490.

Thompson, P. W. (1990). A theoretical model of quantity-based reasoning in arithmetic and algebraic. Center for Research in Mathematics & Science Education: San Diego State University. (unpublished manuscript).

Thompson, P. W. (1994a). The development of the concept of speed and its relationship To concepts of rate. In G. Harel & J. Confrey (Eds), *The development of multiplicative reasoning in the learning of mathematics* (pp. 179-234). Albany, NY: SUNY Press.

Thompson, A. G. & Thompson, P. W (1995). A cognitive perspective on the mathematical Preparation of teachers: The case of algebra. In C. B. Lacampagne, W. Blair & J. Kaput (Eds.). *The algebra initiative colloquium*, 1, 95-116. Washington, DC: US Department of Education.

Thompson, P. W. (2008). Conceptual Analysis of Mathematical ideas: Some spadework at the foundation of mathematics education. Plenary paper delivered at the 32nd Annual Meeting of the International Group for the Psychology of Mathematics Education. In O. Figueras, J.L. Cortina, S. Alatorre, T. Rojano & A. Sepulveda (Eds)., *Proceedings of the Annual Meeting of the International Group for the Psychology of Mathematics Education,* (Vol 1, pp. 45-64). *Morelia, Mexico. PME.*

Vygotsky, L. (1987). The development of Scientific Concepts in Childhood. In R. Rieber and Carton (eds), in *The Collected Works of L.S. Vygotsky*. Vol 1, 167-241.

Concepts Fundamental to an Applicable Understanding of Calculus

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Abstract: Calculus is an important tool for building mathematical models of the world around us and is thus used in a variety of disciplines, such as physics and engineering. These disciplines rely on calculus courses to provide the mathematical foundation needed for success in their discipline courses. Unfortunately, many students leave calculus with an exceptionally primitive understanding and are ill-prepared for discipline courses. This study seeks to identify the fundamental calculus concepts necessary for successful academic pursuits outside the undergraduate mathematics classroom, describe appropriate understanding of these concepts, and collect tasks that elicit, document, and measure this understanding. Data were collected through a series of interviews with select undergraduate mathematics and other discipline faculty members. The data were used to build descriptions of and frameworks for understanding the calculus concepts and generate the pool of tasks. Implications of these findings for calculus curriculum are presented.

Keywords: Calculus, understanding, design research

Introduction

According to Ganter and Barker (2004):

> *Mathematics can and should play an important role in the education of undergraduate students. In fact, few educators would dispute that students who can think mathematically and reason through problems are better able to face the challenges of careers in other disciplines—including those in non-scientific areas. Add to these skills the appropriate use of technology, the ability to model complex situations, and an understanding and appreciation of the specific mathematics appropriate to their chosen fields, and students are then equipped with powerful tools for the future.*
>
> *Unfortunately, many mathematics courses are not successful in achieving these goals. Students do not see the connections between mathematics and their chosen disciplines; instead, they leave mathematics courses with a set of skills that they are unable to apply in non-routine settings and whose importance to their future careers is not appreciated. Indeed, the mathematics many students are taught often is not the most relevant to their chosen fields. For these reasons, faculty members outside mathematics often perceive the mathematics community as uninterested in the needs of non-mathematics majors, especially those in introductory courses.*
>
> *The mathematics community ignores this situation at its own peril since approximately 95% of the students in first-year mathematics courses go on to major in other disciplines. The challenge, therefore, is to provide mathematical experiences that are true to the spirit of mathematics yet also relevant to students' futures in other fields. The question then is not whether they need mathematics, but what mathematics is needed and in what context. (p. 1)*

These claims detail the rationale for The Mathematical Association of America's (MAA) Curriculum Foundations Project (http://www.maa.org/cupm/crafty/cf_project.html). This project studied the first two years of undergraduate mathematics curriculum. Portions of the mathematics community and its partner disciplines (e.g., biology, business, chemistry, computer science, several areas of engineering) worked together to generate a set of recommendations that have assisted mathematics departments plan their programs to better serve the needs of its partner or client disciplines (Ferrini-Mundy & Gücler, 2009).

The push to better serve the needs of client disciplines stemmed from the calculus reform efforts. Between the mid 1980's and the early 1990's, the undergraduate mathematics community engaged in a concentrated effort to overhaul the teaching and curriculum of beginning calculus (Ferrini-Mundy & Gücler, 2009). The heart of the reform was the concern over the depth and breadth of students' understanding of calculus (Douglas, 1986). This lack of understanding became especially apparent when students were asked to apply calculus in unfamiliar situations (Hughes Hallett, 2000).

As Ganter and Barker (2004) implied, client department faculty often complain that students are unable to apply calculus in the client coursework. Sometime this coursework asks students to use the calculus concepts in ways not familiar to them. For example, the minimization of average cost is done symbolically in calculus, whereas it is usually done graphically in economics (Lovell, 2004). At other times, even when the concept is used in a similar fashion, differences in notation or a lack of familiar cues (e.g., "maximum" or "minimum" in an optimization problem) derails students. Such difficulties in transferring knowledge between disciplines are stark indicators of a lack of understanding (Hughes Hallett, 2000). Thus, the reform called for fundamental changes in curriculum and pedagogy of beginning calculus. These changes emphasized conceptual understanding rather than procedural skills [1] (Ferrini-Mundy & Gücler, 2009).

Description of Study

The changes that have taken place during the reform years have placed greater emphasis on conceptual understanding (Hughes Hallett, 2000), but as Ganter and Barker (2004) point out, it has not been enough. So the question remains: what mathematics is needed and in what context? Following in the footsteps of the MAA's Curriculum Foundations Project, this study began exploring the potential disconnect between the calculus taught in the mathematics classrooms and the calculus needed outside the mathematics classroom at a particular undergraduate institution. Through exploring the disconnect, this study was able to identify some fundamental calculus concepts students need for successful academic pursuits outside the undergraduate calculus classroom, describe what it means to understand these concepts, and collect tasks that elicit, document, and measure student understanding of these concepts.

Describing the fundamental/core calculus concepts and creating the pool of tasks/activities constituted a design research study (Collins, 1992). In design research, the goal

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[1] Ferrini-Mundy an d Gücler did not define either *conceptual understanding* or *procedural skills*. To establish a common definition, for the purposes of this discussion, I refer the reader to the definitions offered by the MAA's Curriculum Foundations Project. Conceptual understanding is defined as the "broad understanding encompassing logical reasoning, generalization, and abstraction" (Kasube & McCallum, 2004, p. 109). Procedural skills are equated with computational ability (Kasube & McCallum, 2004, p. 109).

is to put people with different perspectives into situations that require them to express not only how they think about a concept, but to express it in a way that requires them to test and revise their way of thinking (Lesh, 2002). As such, each cycle included divergent ways of thinking, selection criteria for the most useful ways of thinking, and sufficient means of carrying forward the ways of thinking so they may be tested during the next cycle. Diversity, selection, and accumulation are necessary for iterative revisions to be passed forward.

Select faculty members at an engineering undergraduate institution participated in an iterative series of interviews during which they expressed, tested, and revised the descriptions of the fundamental calculus concepts, frameworks for understanding each concept, and associated tasks/activities. At this institution, all students are required to take two semesters of calculus and several calculus-based science and engineering courses. Mathematics and client department faculty were selected based on their proximity to the calculus courses and the client courses.

The interviews were designed around a series of concept descriptions, frameworks, and tasks developed by the researchers and/or adapted from the research of others. The intention was to provide scaffolding for the faculty to evaluate and recognize not only the necessary calculus concepts, but the ways in which the concepts need to be understood. Additionally, the tasks provided to and elicited from the faculty themselves served to provide a means to elicit, document, and measure the understanding students have of these concepts.

Results

The rounds of interviews addressed content and understanding. When faculty from mathematics and client departments were asked questions such as:[2]

- What conceptual calculus concepts must students master to be success in disciplines outside mathematics?
- What calculus (or mathematical) problem solving skills must students master to be success in disciplines outside mathematics?
- What broad mathematical topics must students master in the first two years? What priorities exist between these topics?
- What is the desired balance between theoretical understanding and computational skill? How is this balance achieved?

a dialogue centered on the fundamental calculus concepts emerged. This study will report on the blossoming of this dialogue into descriptions of essential calculus concepts and frameworks used to assess understanding of these concepts. For some of the concepts, tasks that elicit, document, and measure student understanding of the concepts were discussed and analyzed.

Implications

As stated before, calculus is an important tool for building mathematical models of the world around us and is thus used in a variety of disciplines, such as physics and engineering. These disciplines rely on calculus courses to provide the mathematical foundation needed for success in their discipline courses. Therefore, this study offers a collective vision to focus the content of beginning calculus courses on the meeting the needs of client disciplines. In the end, it is the mathematicians that have the responsibility to create courses and curricula that embrace the spirit of this vision while maintaining the intellectual integrity of mathematics. By explicitly

[2] Questions adapted from the MAA's Curriculum Foundations Project workshop questions (Ganter & Barker, 2004).

knowing what and how students should be prepared for client courses, teachers and curriculum developers of both calculus and client disciplines can work together to prepare students for academic success.

References

Collins, A. M. (1992). Toward a design science of education. In E. Scanlon & T. O'Shea (Eds.), *New directions in educational technology* (pp. 15-22). Berlin ; New York: Springer-Verlag.

Ferrini-Mundy, J., & Gücler, B. (2009). Discipline-Based Efforts to Enhance Undergraduate STEM Education. *New Directions for Teaching and Learning, 117*, 13. doi: 10.1002/t1

Ganter, S. L., & Barker, W. (2004). A collective vision: Voices of the partner disciplines. In MAA (Ed.), The Curriculum Foundations Project (pp. 1-13). Washington, D.C.: Mathematical Association of America's Committee on the Undergraduate Program in Mathematics.

Hughes Hallett, D. (2000). *Calculus at the Start of the New Millennium.* Paper presented at the Proceedings of the International Conference of Technology in Mathematics Education, Beirut, Lebanon.

Kasube, H., & McCallum, W. (2004). Mathematics. In MAA (Ed.), The Curriculum Foundations Project (pp. 109-113). Washington, D.C.: Mathematical Association of America's Committee on the Undergraduate Program in Mathematics.

Lesh, R. A. (2002). Research Design in Mathematics Education: Focusing on Design Experiments. In L. D. English (Ed.), *Handbook of international research in mathematics education* (pp. 27-49). Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Lovell, M. C. (2004). *Economics with calculus*. Hackensack, NJ: World Scientific Pub.

How Do iPads Facilitate Social Interaction in the Classroom?

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Abstract: *Traditionally, research on technology in mathematics education focuses on interactions between the user and the technology, but little is known is about how technology can facilitate interaction among students. In this preliminary report we will explore how students use iPads while negotiating mathematical meaning in a community of learners. We are currently studying the use of iPads in an introductory business calculus course. We will report on classroom observations and a series of small-group interviews in which students explore the concepts of local and global extrema. Our preliminary results are that the portability of iPads and the intuitive applications have allowed students to easily incorporate the iPad into their collaborations.*

Keywords: business calculus, social constructivism, classroom technology, iPad

Proposal: For the past half-century mathematics educators have been contemplating the role of technology in mathematics education. Recent decades have seen significant growth in student access to technology in the classroom. Among the key strands of research are:

- Handheld devices and calculators, e.g. (Burrill et al., 2002).
- Technology designed to accumulate real data for student exploration, e.g. (Konold & Pollatsek, 2002).
- Dynamic geometry software and other microworlds, e.g. (Jones, 2000).

Like the strands mentioned above, the bulk of research on technology in mathematics education focuses on interactions between the user and the technology. Little is known about how individuals use technology to interact with one another. However, the current generation of undergraduates is likely to incorporate technology throughout their social interactions with one another. In this preliminary report we will explore how students use handheld tablet devices while negotiating mathematical meaning in a community of learners.

Our theoretical perspective is built upon the significant body of research which views learning as an inherently social process, e.g. (Vygotsky, 1978; Cobb & Yackel, 1996; Stephan & Rasmussen, 2002). From this perspective, knowledge is socially constructed through interactions with other members of a learning community. Classroom technology may take on many different roles inside the community of learners, such as: a tool for computation, a medium for communication, a microworld for exploration, or an extension of the individual's voice in negotiating meaning. In particular, it is this final role that we intend to describe in greater detail through the results of a ongoing study into the use of iPads in undergraduate mathematics.

In the fall of 2010, two sections of Calculus for Business and Economics were chosen to be part of a university wide study of the effectiveness of the iPad as a classroom tool. The university distributed iPads to one section of 20 students along with two applications, a spreadsheet program called Numbers and a graphing calculator called Graphing Calculator HD. Students were able to use the iPads both inside and outside the classroom for the entire semester and returned them after the final exam. The second section of the course is using laptops throughout the course with Excel and a java graphing applet developed by one of our faculty members. The instructor, textbook, homework, quizzes, tests and in-class activities are identical for both sections. The course itself designed to allow students to reconstruct mathematical principles within a small group setting.

This report is focused on the degree to which the iPad enhances the classroom dialogue. We are using the following qualitative methods to conduct this study:

1. Classroom Observations: We are conducting classroom observations of both classes throughout the semester. We are most interested in recording student behavior during in-class activities in order to understand how the students work together within each section of the course.

2. Group interviews: We are meeting with a small group of 2-4 students and asking them to solve several questions related to the course. The focus is on how they use technology to help solve problems and whether the technology has an effect on their interactions.

3. Activity Logs: We are asking the students to keep a log of their use of any technology over the course of a few days. This helps us understand how students use technology in general, as well as specific technology for the course, on a daily basis.

Of particular interest to us is a series of small-group interviews focusing on the concepts of local and global extrema. Students often approach these concepts from a purely computational perspective, but would benefit from the use of technology to visualize the problem. This is an excellent opportunity to observe whether students will incorporate technology while negotiating the problem with their classmates. These sessions have been designed to illuminate student interaction involving technology.

Through classroom observations, we have already seen evidence of the positive role that the iPad can play in the classroom. In early lessons that did not necessarily require the use of technology, students chose to turn on the devices and explore the graphs of cost, revenue and profit functions without prior instruction on the application. We also observed a lesson on limits where a spreadsheet and graphing calculator is required. During the course of the lesson, we witnessed that the size and portability of the iPad allowed students to share their screens as part of their dialogue. The fact that the class is using a uniform device also facilitated students assisting each other in the learning process. Throughout the class activities the students were fully engaged and did not stray to online distractions.

Based on our experiences this semester, we would like to ask for feedback on future iterations of this study. We ask the audience to consider the following questions:

- Is there relevant literature that we have not considered?

- Are there other means of collecting data that we have not considered?

- Are there other topics in the business calculus curriculum that would help illuminate student interactions with technology?

- The university conducted a survey of general technology use for the students involved in the study. Should we use these surveys to classify students by technological comfort and track how that influences student interaction with the technology and each other?

- The criteria for the university-wide study included having one section taught with iPads and one section taught without. Is the comparison between the iPad section and the section where students use personal laptops of interest to the mathematical education community?

References

Burrill, G., Allison, J., Breaux, G., Kastberg, S., Leatham, K., & Sanchez, W. (2002). *Handheld graphing technology at the secondary level: Research findings and implications for classroom practice*. Texas Instruments.

Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist*, *31*(3), 175–190.

Jones, K. (2000). Providing a foundation for deductive reasoning: Students' interpretations when using dynamic geometry software and their evolving mathematical explanations. *Educational Studies in Mathematics*, *44*, 55–85.

Konold, C., & Pollatsek, A. (2002). Data analysis as the search for signals in noisy processes. *Journal for Research in Mathematics Education*, *33*(4), 259–289.

Stephan, M., & Rasmussen, C. (2002). Classroom mathematical practices in differential equations. *The Journal of Mathematical Behavior*, *21*(4), 459 – 490.

Vygotsky, L. S. (1978). *Mind in society: Development of higher psychological processes*. Cambridge, MA: Harvard University Press.

Evaluating Mathematical Quality of Instruction in Advanced Mathematics Courses By Examining the Enacted Example Space

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Abstract: In advanced undergraduate mathematics, students are expected to make sense of abstract definitions of mathematical concepts, to create conjectures about those concepts, and to write proofs and exhibit counter-examples of these abstract concepts. In all of these actions, students must be able to draw upon a rich store of examples in order to make meaningful progress.

We have created a methodology to evaluate what students might learn from a particular course by describing and analyzing the enacted example space (Mason & Watson, 2008) for a particular concept. This method will both give a means to create testable hypotheses about individual student learning as well as provide a way to compare disparate pedagogical treatments of the same content. Here, we describe and assess the enacted example space by studying the teaching of abstract algebra.

Keywords: example spaces, classroom research, teaching, evaluation, mathematical quality of instruction

0 Studying teaching and the quality of instruction

There is a desire to evaluate the quality of instruction and to compare teachers based on their classroom effectiveness. The interest in evaluating the quality of instruction has engendered papers exploring the concept of quality of teaching via conceptual and empirical processes (i.e., Fenstermacher, & Richardson, 2005), and handbooks for undergraduate faculty that present ways that they may evaluate their teaching (i.e., Angleo & Cross, 1993).

The desire to evaluate the quality of instruction is made more difficult when the pedagogical techniques that different teachers use are vastly different. Consider the case of two instructors teaching an undergraduate abstract algebra course; one employs a standard lecture method, whereas the other adopts an inquiry-based pedagogy. One series of assessment instruments has been created that measures how closely teachers follow the tenets of process-product instruction (e.g., Brophy & Good, 1986); these can inform the quality of lecture-based teaching. A second set of instruments measures how closely teachers follow reform-oriented practices (e.g., Horizon, 2000; Sawada & Pilburn, 2000). However, neither the measures of the process-product tradition nor those of the reform tradition would allow for meaningful judgments to be made about the quality of instruction offered to these two groups of students, who are learning under instructors with contrasting pedagogies. Yet, given the proliferation of inquiry-based curricula for undergraduate courses and the continuing predominance of the lecture method (Pemberton, et al., 2004), this is exactly the situation that we are faced with.

The Learning Mathematics for Teaching Project (*LMTP*) has argued that we should shift our attention away from these instruments, due to their failure to take into account “one critical aspect of mathematics instruction: its mathematical quality” (2010, p. 2). Following the LMTP definition, when we refer to the mathematical quality of instruction we mean “the nature of the mathematical content available to students during instruction” (LMTP, 2010, p, 6). This is meant to be independent of the instructional format, classroom environment, or level of discourse. We do not believe that these are unimportant to student learning. We believe that the quality and range of mathematical ideas that comprise a classroom experience have direct bearing on students’ ability to develop a rich understanding of mathematics regardless of the instructor’s pedagogical preferences.

1 Example spaces—Students’ range of thought, knowing what can vary, knowing what must stay the same

In advanced undergraduate courses, especially proof-based courses, increasing emphasis is placed on using examples as a pedagogical tool. For example, Alcock and Inglish (2008) examined doctoral students’ use of examples in evaluating the truth value of claims, Dahlberg and Housman (1997) found that students who generated their own examples were more likely to develop initial understandings of concepts, and Mason and Watson (2008) described ways to make use of the range of possible variation for pedagogical purposes.

We draw upon the enacted example space to measure and compare mathematical quality of instruction and resulting potential for student learning. We argue that this is an appropriate measure of quality due to the importance of examples for student understanding in proof-based courses, and assert that this measure is meaningful across pedagogical styles. We outline a methodology for using the enacted example space to describe potential and probable student learning, and finally we show the value of this methodology by using it to analyze one aspect of instruction in an introductory abstract algebra courses.

An example space is the "experience of having come to mind one or more classes of mathematics objects together with construction methods and associations" (Goldenberg & Mason, 2008, p. 189). This example space may include relatively frequently accessed members of the classes and less accessed members of the classes, and, via the construction methods, may include new members of the classes. The first important feature of an example space is that it purposefully includes construction methods and associations such as links to important theorems and relations to other constructs. These allow mathematicians to create new examples that meet specific criteria of theorems and to determine which classes of objects are most relevant in particular situations.

Mason and Watson (2008) point out two other important features of example spaces: what aspects of the examples the learner realizes can be varied, and what range of variation the learner believes is appropriate. For example, in the case of the definition of a group, when thinking about the possible aspects of a group it is possible to think about characteristics of the underlying set, the group itself, or of the behavior of specific elements.

2 Our methodology

Video data was digitized and Transana was used to code all incidents where an example or non-example was shown, constructed or analyzed in class. We created an example log, similar to Stephan and Rasmussen's (2008) argument log which characterized each example or non-example in four columns.

- Column 1: each example or non-example of the particular construct (in this case, an algebraic group).
- Column 2: counts the number of class meetings since the formal definition of a group (a written homework assignment was coded as occurring on the day that it was assigned).
- Column 3: description of the qualities of the example or non-example. In the case of examples, the third column described any additional qualities that the example possessed from a list that would be known to first semester algebra students by the midpoint of the semester (e.g., being a commutative group, a finite group, or a cyclic group). In the case of non-examples we described any properties of the construct that were missing as well as additional properties that the non-example possessed from the list above.
- Column 4: description of the manner in which the example or non-example was made part of the classroom discourse.

2.1 Our theory of measuring the enacted example space

We use three filters to assess the enacted example space and to describe the set of examples in that space: (1) *example neighborhood*, (2) *example construction*, and (3) *example function*. We define the *example neighborhood* as the entire collection of examples that the students are exposed to during the course of their studies. These may be concrete examples or relevant non-examples of a given concept. We analyze how the examples are organized on four levels: (1) who's on first? (2) temporal proximity (3) permissible variation and (4) variation constraint. We pay particular attention to the first few examples as instructors believe they are often the ones that students most closely link with a concept (Zodik & Zaslavsky, 2008). Dienes (1963) argued that students should see examples that vary only in a constrained manner so that they are able to determine what is structural and what is allowed to vary as well as to comprehend the range of permissible variation. Then, they should see other examples that vary along a different dimension. As a result, we argue that early examples that vary along too many dimensions may actually lower the mathematical quality of instruction. Similarly, a collection of examples that fails to support student construction of critical aspects of the construct will also lead to lower mathematical quality of instruction.

Secondly, we examine the *example construction* to support a particular concept. Example construction focuses on the range of possible variation to be included in the neighborhood of a particular example space. The analysis of example construction focuses on how particular examples are created and examines the tools for creating additional examples that students may derive from the creation of examples. The construct of example construction also makes possible mapping from concrete examples to a broad description of the example space that students may have the (perhaps untapped) ability to populate for themselves. In this way, the example space explicitly includes both the examples and the means of construction (Goldenberg & Mason, 2008).

Finally, *example function* situates the example in a particular area of the example space based on its frequency of use and exemplar status. In short, example function analyzes and describes how frequently a particular example or set of examples it called upon and in what contexts. In particular, we examine which examples are most frequently called upon. Frequently used examples may obtain "ready access" status for students (linked to Vinner's (1991) concept of *evoked concept image*). The frequency of use not only gives us a means to assess or predict the student's perception of the relative importance of each of the examples, but also a means to predict which examples can most readily function as an example for them. We assess separately using each filter, and then read them together to analyze the example space.

3 Using the method

The presentation will include a preliminary analysis of the teaching of one abstract algebra class. While data analysis has begun, it is not yet complete. Preliminary results include the fact that in one traditionally taught abstract algebra course, the example neighborhood for *group* was: $(\mathbf{Z},+)$, $(\mathbf{Q},+)$, $(\mathbf{Q}^*,\cdot)$, $(\mathbf{R}^*,\cdot)$, $(\mathbf{Z}_{12},+_{12})$ and $(\mathbf{Z}_n,+_n)$. For both of the multiplicative groups, the instructor initially proposed using the complete set of rational or real numbers and then noted that zero does not have a multiplicative inverse. He then demonstrated the construction of a new set, without zero, such that all elements have multiplicative inverses. Similarly, he introduced the set $A = \mathbf{R} - \{-1\}$ and as part of the class, constructed an operation, *, such that (A, *) is a group. We claim that the instructor demonstrated examples of groups as well as two different construction methods that are likely to have become part of the students' example spaces. But, we claim that the example space will not strongly support evaluation of conjectures because all of the examples are commutative groups.

4 Questions for discussion

1) While we believe this a helpful methodology for assessing the quality of instruction and, potentially, comparing different pedagogical treatments, we wonder if it is too narrow of a lens?
2) Similarly, is it too time-intensive to be useable?
3) Besides glaringly obvious teaching suggestions like, "include non-commutative examples early and often," what potential does this have for affecting instruction of either lecture or Inquiry based teaching? Further research?
4) What more should we be doing?
5) Can this methodology be adapted to other topics such as teaching proof?

References:

Alcock, L. & Inglis, M. (2008). Doctoral students' use of examples in evaluating and proving conjectures. *Educational Studies in Mathematics, 69*, 111-129.

Angelo, T.A. and Cross, K.P., *Classroom Assessment, Techniques: A Handbook for College Teachers.* Second Edition. Jossey-Bass. San Francisco CA (1993).

Brophy, J. E., & Good, T. L. (1986). Teacher behaviour and student achievement. In M. C. Wittrock (Ed.), *Handbook of research on teaching* (pp. 328–375). New York: MacMillan.

Dahlberg, R. P., & Housman, D. L. (1997). Facilitating learning events through example generation. *Educational Studies in Mathematics, 33*, 283–299. doi:10.1023/A:1002999415887.

Dienes, Z. (1963). *An experimental study of mathematics-learning*. London: Hutchinson Educational.

Fenstermacher, G. & Richardson, V. (2005). On making determinations of quality teaching. *Teachers College Record, 107*(1), 186-213.

Goldenberg, P. & Mason, J. (2008). Shedding light on and with example spaces. *Educational Studies in Mathematics, 69*, 183–194. DOI 10.1007/s10649-008-9143-3

Horizon Research. (2000). Inside the classroom observation and analytic protocol. Chapel Hill, NC: Horizon Research, Inc.

Learning Mathematics for Teaching Project (2010). Measuring the mathematical quality of instruction. *Journal of Mathematics Teacher Education*. DOI 10.1007/s10857-010-9140-1

Mason, J. & Watson, A. (2008). Mathematics as a Constructive Activity: exploiting dimensions of possible variation. In M. Carlson & C. Rasmussen (Eds.) Making the Connection: Research and Practice in Undergraduate Mathematics, Washington: MAA. p189-202.

Pemberton, J., Devaney, R., Hilborn, R., Moses, Y., Neuhauser, C., Taylor, T., Brady T., & Williams, R., (2004). *Undergraduate education in the mathematical and physical sciences: Interim report of the Joint Subcommittee of the NSF Education and Human Resources and Mathematical and Physical Sciences Advisory Committees*. Washington DC: National Science Foundation. http://www.nsf.gov/attachments/102806/public/JSACInterimReportEHR-MPSSpring2004.pdf. Accessed: 6-26-10

Rasmussen, C. & Stephan, M. (2008). A methodology for documenting collective activity. In A. E. Kelly & R. Lesh (Eds.). Handbook of design research methods in education: Innovations in science, technology, engineering, and mathematics learning and teaching (Pp 195-215). Mawah, NJ: Erlbaum.

Sawada, D., & Pilburn, M. (2000). Reformed teaching observation protocol (RTOP). Arizona State University: Arizona Collaborative for Excellence in the Preparation of Teachers.

Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 25-41). Dordecht: Kluwer.

Zodik, I. & Zaslavsky, O. (2008). Characteristics of teachers' choice of examples in and for the mathematics classroom. *Educational Studies in Mathematics, 69*, 165–182 DOI 10.1007/s10649-008-9140-6.

Technologizing Math Education: The case of multiple representations

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Abstract: *Technology is a cornerstone for NCTM and is agreed to be beneficial, but the level of effectiveness is still very vague. This research questions exactly how effective is technology in the mathematics classroom, and what are the definitive benefits. After studying over 300 articles, technology has proven to be beneficial in five ways: providing instantaneous visual feedback, creating student-centered learning environments, providing multiple representations of similar concepts, combining learning environments for generalizations, and retracing previous steps for self-assessment. The most frequently discussed topic was multiple representations, usually in the form of CAS and dynamic geometry systems. The research shows that providing multiple representations allows students with varying levels of intelligence to better understand tricky and abstract concepts.*

Keywords: Technology, Multiple representations, Multiple intelligences, technology effectiveness, mathematics education

One of the ultimate goals for general technology use, regardless of the subject, is the level of relevance to the student's natural surroundings. This research focuses on creating a framework evaluating the effectiveness of technology, utilized in teaching, learning, and curriculum development. There were two major questions that were central to this presentation: is technology use beneficial the classroom, and how exactly is it effective? During the course of the research, approximately 300 articles were reviewed, all of which being NCTM publications, PME proceedings, or ERIC database articles.

It is worth noting here that a major difficulty throughout these article reviews was due to the vague interpretations of the results. Others have expressed their difficulty in answering open-ended questions about technology's effectiveness, claiming that "research on this mode of teaching is sparse and open research questions are plentiful" (Engelbrecht & Harding, 2005). Many of these research-based studies do not completely answer the question of effectiveness, provide an adequate amount of quantitative results, nor show favor for technology use. The crux of the research focused on the effectiveness of technology, and even with the limited studies available on the topic of technology in the math classroom, fewer articles stress exactly *how* technology is beneficial in the classroom.

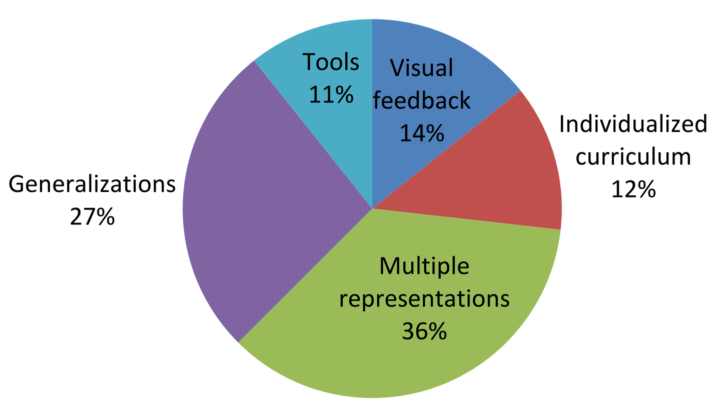
Our research project began with the review of over 300 articles that focus on technology and mathematics education. These studies were classified into five groups, revealing effectiveness of technological use to students learning of mathematics. First, technology has the capacity of providing *instantaneous visual feedback*, allowing the student to observe how a correct or incorrect input will alter the solution. Second, the use of technology assist in design of *student-centered learning* environments, allowing the student essentially personalize curriculum, focusing on student's individual needs. Third, technology provides *multiple representations* for the same content, allowing students to utilize a variety of tools, methods and algorithms to investigate mathematics, otherwise unavailable. Fourth, the *combination of learning environments* (technology and non-technology or two different technology programs) helps students create generalization of problems and allow them to solve similar yet more advanced problems. Finally, through the use of history, recordings and other technological remembering (memory) tools, students are able *to retrace the steps and reevaluate the solutions* to identify past mistakes and recognize patterns that will achieve success in the future.

The breakdown of the five subgroups is illustrated in figures 1 and 2:

Figure 1:

Topic	Number of Articles	Percentage(out of 300)
Instantaneous visual feedback	8	4.67
Individualize curriculum	7	2.33
Multiple representations/multiple intelligences	20	6.67
Combination of environments and generalizations	15	5.00
Tools (history, save, etc.)	6	2.00
Research without explicit assessment of technological benefits	244	80.33

Figure 2:Technology Impact in Learning



The aim of this presentation is to discuss one of the most popular and best represented categories of the five subgroups: *multiple representations*. This domain of the research focuses on a particular result of effective technology use; articles stress how technology deepens students' understand of mathematical concepts due to multiple ways of learning.

A number of positive results were discovered, revealing how technology supports the learning environments that accommodate students' diverse intelligence levels. Several examples are discussed, illustrating commonalities among outcomes. First is an experiment conducted by Pitta-Pantazi and Christou (2009). A pre- and post-test was administrated to forty nine 6^{th} graders before and after using dynamic geometry software Euclidraw Jr. After the pre-test, lessons focused on constructing lines, shapes, and angles were implemented. The students then took a posttest, which focused its results on how students did on the topic of area of triangles and parallelograms. From the results of the posttest, taken without a computer, students, who used the dynamic geometry in class, increased their scores compared to the pretest by a mean score of .10 to .25. What is unique about this study is that it focuses on multiple intelligence levels and methods students learn best, such as "analytic verbalisers" or "wholist imagers" (2008). All students, regardless of their primary method of learning, increased their mean scores compared to the pretest. In another experiment, Dugdale (1994, 2008) used the program Green Globs with 49 students, 25 in a geometry class and 24 in an Algebra II class. Students were to create functions, located in the designated place on the Cartesian plane, working for approximately three hours over a three-week span. After administering a pre- and posttest, students increased mean scores in both the Algebra and Geometry classes by 15% and 42%, respectively (2008). The Green Globs program allowed students to work on an individual basis and small group settings, and was able to contribute to the learning of both Algebra II and Geometry classes, according to the increase in the pre- and posttest. Finally, Borba and Confrey (1993) look at a case study and the uses of Function Probe through interviews which implement Function Probe's ability to make transformations of graphs and the corresponding tabular values. It this case study, participants were asked to predict tabular values from graph manipulation, and during each interview the participant's algebraic language increased. One particular individual was able to hypothesize about different properties of quadratics, using primarily the correlation between graph and table, and understanding their interdependence. These are a few examples of research on the topic, all of which highlight the effectiveness of using technology to enable the design of multiple representations to enhance learning.

Technologically enhanced learning environments impact the quality of student learning mathematics. By identifying ways that technology has proven to be effective, a foundation for bettering technological methodology can emerge. This presentation is a part of a larger research study that focuses on the design of the system of interpretive frameworks that enable the design of meaningful assessment of technological impact, and effective technologically-based learning situations. It will also enable us develop a better understanding of the terms of cyberlearning and cyberteaching.

References

Borba, M., & Confrey, J. (1993). The Role of the Teaching Experiment: Students' Construction of Transformations in a Multiple Representational Environment. Retrieved from ERIC database.

Dugdale, S. (1994). Using students' mathematical inventiveness as a foundation for software design: Toward a tempered constructivism. In G.W. Blume & M. K. Heid (Eds.), *Research on technology and the teaching and learning of mathematics: Vol. 2. Cases and perspectives* (pp. 106-109). Charlotte, NC: Information Age Publishing.

Engelbrecht, J., & Harding, A. (2005). Teaching Undergraduate Mathematics on the Internet. *Educational Studies in Mathematics*, *58*(2), 253-276. Retrieved from ERIC database.

Pitta-Pantazi, D., & Christou, C. (2009). Cognitive Styles, Dynamic Geometry and Measurement Performance. *Educational Studies in Mathematics*, *70*(1), 5-26. Retrieved from ERIC database.

Determining Mathematical Item Characteristics Corresponding With Item Response Theory Item Information Curves

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Abstract:
Tests in undergraduate mathematics courses are generally high stakes, and yet have low reliability. The current study aims to increase the reliability of such exams by studying the qualities of test items that determine the ability of the item to contribute to the information of the test. Using a three parameter item response theory model, 695 items contained in 25 different tests for 5 different first-year undergraduate mathematics courses have been analyzed to determine the ability of each item to contribute to the corresponding test's reliability. During the conference presentation, the speakers with solicit input from the participants regarding the types of qualities of these items that may contribute to their information index. These qualities may include cognitive, mathematical content, linguistic, or other descriptions.

Keywords: Assessment, test writing, item response theory

Tests comprise a major component of mathematics classes at the undergraduate level, particularly first and second year courses. The grades on tests range from 30% clear up to 100% of a student's final grade. However, very little is known about the reliability of such tests that can dictate whether students pass or fail a course, or can cause a student to need an additional year to complete college, adding thousands of dollars to the student's college expenses. Through an analysis of final exams in College Algebra and Business Calculus using a three-parameter item response theory (IRT) model for 1438 and 524 students, respectively, we have found that for a student receiving the border-line score to advance to the next course of 70%, the standard error is between 10% and 14%. In other words, the student's actual score is somewhere between an F and a B when taking into account measurement error. This type of reliability is unacceptable for such a high stakes exam. The goal of this current research program is to determine the characteristics of test items that contribute the most to improving this reliability.

There are several ways to test a student's knowledge of a particular subject, with multiple-choice and constructed response the two most popular. Constructed response items include any assessment where the test taker does not have a list of formulated responses from which to choose. These types of questions require more resources to administer and grade than multiple-choice with a constructed response test of equivalent reliability to a multiple-choice test taking from 4 to 40 times as long to administer and is typically thousands of times more expensive (Wainer & Thissen, 1993; Lukhele, Thissen, & Wainer, 1994). However, with the rise of homework response systems, this difference in administration and grading is becoming negligible. Our analysis includes both constructed response and multiple choice items used on tests in Remedial Mathematics, Intermediate Algebra, Finite Mathematics, College Algebra, and Business Calculus.

This study uses 695 items from twenty-five tests from five different first year mathematics courses to determine what characteristics contribute the most to the item providing information contributing to a test's reliability. Of these items, 18% were constructed response, 3% were true/false or yes/no items, and the remaining 79% were multiple-choice. For each test, a three-parameter IRT model (van der Linden & Hambleton, 1997, pp. 13-17) was used to determine the appropriate difficulty and discrimination parameters for the items, with the guessing parameter fixed at 0 for constructed response items, 0.25 for multiple choice items with four choices, and 0.5 for dichotomous response items. Using the parameters generated from the model, the item information function for each item was multiplied by the student ability distribution function for the corresponding test and then integrated over the range of student abilities to generate an item information index.

The item information indices ranged from essentially zero to 5.948, with a mean of 0.332, a standard deviation of 0.397, and a median of 0.251. Additionally, 60% of the items had an information index less than the mean, implying that less that 40%% of the items contributed nearly all of the reliability for each test. If instructors could know which attributes contributed to items having a high item information index, then more mathematics tests would have the reliability appropriate for such high-stakes testing.

During the presentation, we will study the items with high item information indices while answering the following questions.

1. What are the cognitive categories that might be contributing to an item's high information index?

   These cognitive categories could be based upon the structure of the observed learning outcome (SOLO) taxonomy (Biggs & Collis, 1982), Bloom's taxonomy (Engelhart, Furst, Hill, & Krathwohl, 1956), or the mathematical tasks framework (Stein & Smith, 1998). The challenge is that these taxonomies were designed for situations other than analyzing test items, with some of the taxonomies shown to actually be ineffective in accurately categorizing items to predict student cognitive processes as they work on such items (Chan, Tsui, Chan, & Hong, 2002; Gierl, 1997). However, this does not exclude them from possible effectiveness in the current context.

2. What are the content oriented categories that might be contributing to an item's high information index?

   While the goal of the current project is to discover ways of analyzing items that are independent of the mathematical topics assessed, there may be content oriented categorizations which contribute to an item's information index. One such possible example is rational expressions. Student's regularly have difficulty with fractions (Brown & Quinn, 2006), which may cause them to shut down when encountering rational expressions on a test and so may not perform as expected on such items even if the main goal of the item is to measure mathematical task distinct from rational expressions. On the other hand, such difficulty may contribute to the ability to differentiate students of various ability levels. Other similar topics may also exist and will be discussed among the participants.

3. What are the linguistic descriptors that might be contributing to an item's high information index?

   Translating between mathematical language, visual information, and descriptive language is challenging for many students (Arcavi, 2003; Capraro & Joffrion, 2006; Radford & Puig, 2007). This challenge may contribute to the ability to distinguish between top students and weaker students and so may contribute to an item's information index.

4. Are there other constructs or lenses through which the test items may be analyzed?

Other constructs exist that the researchers have not thought about and will be sought from the participants in the conference presentation.

While the line of research proposed for this conference presentation is very undeveloped, it is an area with great promise due to the increase in information provided by the use of computerized assessment systems used in large settings and has the potential to greatly influence the future of classroom assessment in the college mathematics classroom.

Works Cited

Arcavi, A. (2003). The role of visual representations in the learning of mathematics. *Educational Studies in Mathematics , 52* (3), 215-241.

Biggs, J. B., & Collis, K. F. (1982). *Evaluating the quality of learning: the SOLO taxonomy*. New York: Academic Press.

Brown, G., & Quinn, R. J. (2006). Algebra students' difficulty with fractions: An error analysis. *The Australian Mathematics Teacher , 62* (4), 28-40.

Capraro, M. M., & Joffrion, H. (2006). Algebraic equations: Can middle-school students meaningfully translate from words to mathematical symbols? *Reading Psychology , 27*, 147-164.

Chan, C. C., Tsui, M. S., Chan, M. Y., & Hong, J. H. (2002). Applying the structure of the observed learning outcomes (solo) taxonomy on students' learning outcomes: an empirical study. *Assessment & Evaluation in Higher Education , 27* (6), 511-527.

Engelhart, M. D., Furst, E. J., Hill, W. H., & Krathwohl, D. R. (1956). *Taxonomy of Educational Objectives: The Classification of Educational Goals (Handbook 1: Cognitive Domain)*. (B. S. Bloom, Ed.) New York: Longman.

Gierl, M. J. (1997). Comparing cognitive representations of test developers and students on a mathematics test with Bloom's taxonomy. *The Journal of Educational Research , 91* (1), 26-32.

Lukhele, R., Thissen, D., & Wainer, H. (1994). On the relative value of multiple-choice, constructed response, and examinee-selected items on two achievement tests. *Journal of Educational Measurement , 31* (3), 234-250.

Radford, L., & Puig, L. (2007). Syntax and meaning as sensuous, visual, historical forms of algebraic thinking. *Educational Studies in Mathematics , 66*, 145-164.

Stein, M. K., & Smith, M. S. (1998). Mathematical tasks as a framework for reflection: From research to practice. *Mathematics Teaching in the Middle School , 3*, 268-275.

van der Linden, W. J., & Hambleton, R. K. (1997). Item Response Theory: Brief History, Common Models, and Extensions. In W. J. van der Linden, & R. K. Hambleton (Eds.), *Handbook of Modern Item Response Theory* (pp. 1-31). New York: Springer-Verlag.

Wainer, H., & Thissen, D. (1993). Combining multiple-choice and constructed-response test scores: Toward a Marxist theory of test construction. *Applied Measurement in Education , 6*, 103-118.

Assessing the Effectiveness of an On-line Math Review and Practice Tool in Foundational Mathematics.

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Abstract:
Preliminary results of research into the effectiveness of an innovative on-line mathematics review and practice tool (www.mathessentials.ca) will be reported (data collection completion in Dec. 2010). The goal of the web-site is to provide students with the opportunity to review and practice developmental math skills (fractions, percents, etc.), thus filling in gaps in their knowledge. The development of the web-site begat the development of an innovative evaluation model, which can be used to evaluate online educational technologies. Key to the model is not simply evaluating improvement with pre/post test scores, or with anecdotal reports, but through tracking built into the site, which has the potential to provide a multidimensional view of improvement, usage and engagement (usability score). We believe that the web-site itself (support of student success) and the evaluation model ('gold standard' for evaluation of educational technologies) have implications for both teaching and research.

Keywords: online practice, developmental math, educational technology, introductory statistics

Introduction and Background

This research project arose out of a need for stronger basic math skills in developmental mathematics needed by students in Health Sciences, who were preparing for courses in introductory statistics. The initial focus was on two groups of students: 2-year diploma students studying Health Information Management (HIM) at George Brown College, and degree students in Nursing at George Brown College and York University all in Toronto, Canada.

HIM students have a one-semester course in foundations (developmental) math while Nursing students have no direct mathematics instruction as part of their coursework. In both cases the professors noticed that those students who struggled with basic math skills (e.g. fractions and percents) also struggled within the programs in general and in the introduction to statistics courses in particular. The hypothesis that strong basic math skills are a good predictor of success in introductory statistics was validated in research (Johnson&Kuennen 2006), thus our project was born. Introductory statistics is a key part of the program of study in HIM and Nursing even though their roles in the delivery of health care cannot be different. As front line patient care providers, nurses nevertheless need to be numerate and research savvy. HIM professionals as information managers, not only collect and prepare data for analysis, but participate in its dissemination and presentation as well.

Although the researchers recognize that some aspects of constructivist pedagogy are legitimate, especially in introductory statistics education (real data, problem based learning, emergent solutions to problems) this project has as one of its pillars the notion that many basic mathematical skills which are needed for introductory statistics education (e.g. fractions), need direct instruction and practice. The key seems to be in distinguishing which skills are biologically primary vs. secondary (Geary 1995).

> *"When one considers the pattern of ability development in children across cultures, it becomes clear that many cognitive abilities (e.g. language comprehension, habitat representation) are universal, whereas other abilities (e.g. word decoding in reading, geometry) only emerge in specific cultural contexts."* (Geary 1995 pg. 26)

Universal abilities are categorized as biologically primary and emerge without formal instruction and practice, whereas biologically secondary activities would be less likely to occur without instruction and practice. Counting the number of apples in a bag and dividing them among 3 friends is much closer to biologically primary than 3.572 ÷ .

We are firm believers in practice and formal instruction where warranted, and some of our students were actually demanding more practice. Instead of churning out paper practice sheets, we scoured the internet and for web-sties that could be accessible to our students. The only quality on-line practice vehicles that we encountered were tied to textbooks that were not appropriate to our students, and with a myriad web-instruction math sites providing only 5 or so practice questions, we embarked on the development of our own online resource. With funding from George Brown College, York University and the Inukshuk Foundation, a web-site On-line Math review Tool (OMT) was designed and developed in order to provide practice as well as instruction/review in basic mathematics skills.

We thought the OMT to be innovative, but were not sure about its utility. How to measure effectiveness of the site was a natural next step. Because no evaluation model existed for an educational technology that was unique in itself, we created our own. A review of the literature found many models for the evaluation and effectiveness of online courses, but not one geared specifically to on-line stand-alone educational technologies. We have reproduced our model below. Given the computing power available in contemporary web-site technologies, many aspects of our evaluation model were built into the OMT itself. It is important to note that we have recently discovered a paper by leading researchers in statistics education, Ooms and Garfield (2008) which describes a model that fits with ours very closely.

Basic principles of the On-line Math review Tool (OMT) (www.mathessentials.ca)

Format: Our basic principle in the development of the OMT was to make the site use voluntary, easily accessible (online and free), to provide the student with an experience that is individualized (choose which topics needed) and interactive (with multiple modes of interaction) and to provide the student with many randomly generated practice questions (at a variety of levels of difficulty) to work with. It was also necessary to develop an OMT that could be integrated into the curriculum of a course in foundational mathematics, or as an external add on to an introductory course.

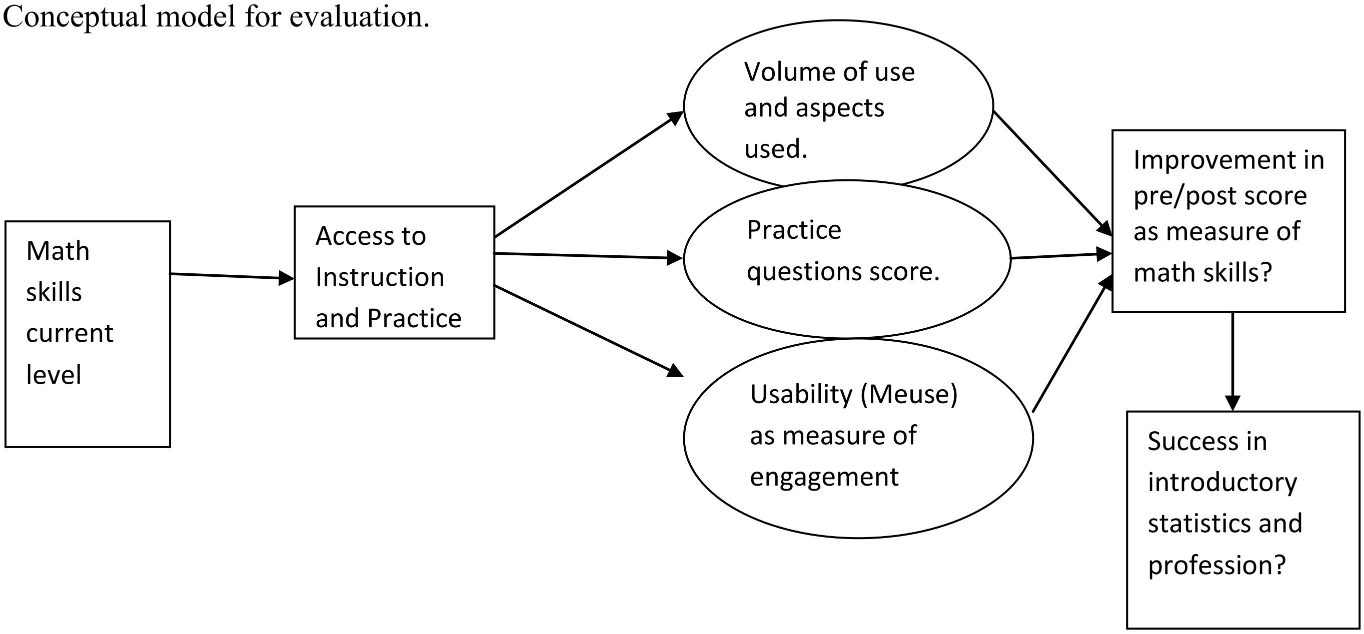
The reasons for the format parameters, stem from the experience of teaching developmental math at the college level. Even among those students who come into post secondary math (or math oriented) courses with solid foundations, many have gaps in one or another of the basic topics, and come to us with a myriad of learning styles and educational backgrounds. The OMT must be flexible and accommodating of the diversity of those who will use it.

Content: Since our focus was on improving the mathematics skills of students preparing for introductory statistics the content consisted of topics identified by research (and the investigators) as good predictors of success in introductory statistics. (Johnson&Kuennen 2006) A total of 17 modules with 6 submodules became the content of the mathessentials site.

Research Questions: a model for evaluating the On-Line Math review Tool (OMT)

1. Peer evaluation: used the *electronic Learning Object Peer Evaluation* questionnaire (eLOPE) designed by the researchers .
2. Volume of use.
3. Which aspects will students use?
4. Percent score in practice question usage broken up by difficulty level.
5. How do students see the site's usefulness? Utilized the *Math Essentials usability* (Meuse) tool developed by our team.
6. Improvement in test score (post – pre) based on (Johnson&Kuennen 2006).
7. Can we predict improvement in the pre/post test scores using any combination of 2,3,4 from above?
8. Will the tool be effective in improving the mathematical skills in the pre/post test?
9. Will student perceptions of OMT usefulness (Meuse) be related to any combination of ?

Conceptual model for evaluation.



Preliminary Results: The results are very preliminary as data collection is ongoing, with this round ending in December 2010. Currently a class of 58 students has been invited to use the site in a research capacity and we have 56 registered. We can report on the volume and aspects of use, while the more interesting questions will be answered after the end of semester and can be presented thereafter.

Volume and aspects of use as of Oct. 7, 2010.

Practice questions. 2284 practice questions attempted, 1160 of those are by 1 user, which is unexpected to say the least. The top 10 users make up 90.67% of the usage of the practice question portion of the site. 35 users have registered and completed the pretest, but have not tried any practice questions at all.

Videos: there were 43 video views in all, 2 students viewed all or part of >2 videos (16, 18) and 52 students watched 0 videos,

Games: 10 games were played in total as of Oct.7, 2010

Implications for teaching practice or further research

The implications presented herein come from the literature and from immersing ourselves in the process of developing and test running the OMT. We will be adding to the list as results become more firm.

For teaching:

1. We suggest that the results from a variation of the evaluation model by Ooms and Garfield(2008) be presented before we introduce new technologies in the classroom, or as peripherals to the classroom experience. All too often, we are expected to implement technologies based on anecdotal evidence.
2. We must be wary about the introduction of new technologies and strive as much as possible to study the way our students use these technologies. The surprising preliminary 'volume of use' results (e.g. 1 person responsible for over 50% of the usage) are any indication, simply providing access to learning technologies is not enough.
3. There is a potential for expanding the scope and capacity of the mathessentials.ca model, but we need to have evidence that it is needed.

For research:

1. Although Ooms and Garfield's (2008) iterative evaluation model may need to be adapted to make it useful to a wider range of educational technologies, we suggest that some form of hybrid of our model and theirs become the 'gold standard' by which research into online educational resources be evaluated.

References

Geary, D. C. (1995). Reflections of evolution and culture in children's cognition: Implications for mathematical development and instruction. American Psychologist, 50(1), 24-37.

Johnson, M. F. & Kuennen, E. W. (2006). Basic math skills and student performance in an introductory statistics course. *Journal of Statistics Education*, *14*(2). Retrieved from http://www.amstat.org/publications/jse/v14n2/johnson.html

Ooms, A, & Garfield, J. (2008). A model to evaluate online educational resources in statistics. *Technology Innovations in Statistics Education*, 2(1). Retrieved from: http://escholarship.org/uc/item/5rd175dg

The Nature and Effect of Idiosyncratic Examples in Student Reasoning about Limits of Sequences

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Abstract

We apply a Vygotskian perspective on the interplay between spontaneous and scientific concepts to identify and characterize calculus students' idiosyncratic use of examples in the process of trying to formulate a rigorous definition for convergence of a sequence. Our data is drawn from a larger teaching experiment, but analyzed for this study to address questions of the origins, nature, and implications of students' nonstandard ways of reasoning. We observed two students interpreting a damped oscillating sequence as divergent, drawing from considerations from an initial, intuitively-framed definition, but remaining persistent and consistent over the duration of multiple sessions. We also trace some of the implications of their idiosyncratic reasoning for their reasoning and ultimately for their definition of convergence. We conclude by posing several questions about the nature of such example use in terms of our Vygotskian perspective.

Keywords: Limits, Definition, Examples, Spontaneous and Scientific Concepts

Introduction and Research Questions

The research literature on students' understanding of limit concepts in introductory calculus courses is replete with idiosyncratic imagery related to informal notions and ontological commitments regarding infinity (Sierpinska, 1987; Tall, 1992; Tirosh, 1991), infinitesimals (Artigue, 1991; Tall, 1990, Oehrtman, 2009), the structure of the real numbers (Cornu, 1991; Tall & Schwarzenberger, 1978), incidental and misleading aspects of graphical representations (Monk, 1994; Orton, 1983), nonmathematical concepts such as speed limits, physical barriers, and motion (Davis & Vinner, 1986; Frid, 1994; Oehrtman, 2009; Tall, 1992; Tall & Vinner, 1981; Thompson, 1994; Williams, 1991), and epistemological beliefs about mathematics in general (Sierpinska, 1987; Szydlick, 2000; Williams, 1991, 2001). Little research, however, has provided an in-depth look at the origins, nature, and implications of a single idiosyncratic image over the course of significant reasoning and problem-solving activity. As part of a larger teaching experiment, we established protocols to attempt to identify idiosyncratic images, should they appear, as calculus students try to formulate a precise definition of convergence of a sequence. The aim of this study was to address the following research questions:

1. What idiosyncratic examples do students construct as the wrestle with reinventing a formal definition for sequence convergence?
2. What are the origins of these idiosyncratic examples?
3. What are the effects of students' idiosyncratic examples on their emerging understanding of a formal definition for sequence convergence?

Theoretical Perspective

We base our study design and analysis on Lev Vygotsky's (1978, 1987) characterization of conceptual development as a complex interplay between intuitive (spontaneous) and formally structured (scientific) thought. These two types of thought are distinguished by their relationship to the objects of reference and by the nature of thought available to them. Spontaneous concepts develop first through a direct encounter with the object and form the basis of experiential knowledge developed informally over long periods of time. They are intuitive in nature and can be applied spontaneously, without conscious reflection on their meaning, but are not available for application to problems in non-concrete situations. Scientific concepts emerge later through a mediated relationship to the object, such as a verbal definition. They are expressed and initially applied only in abstract ways affording quick mastery of operations and relationships, but they are disconnected from personal experience or meaning.

Especially within a field as structurally rich as mathematics, scientific concepts are distinguished by their systematicity. Within a spontaneous concept system, where the only relationships possible are relationships between objects (and not between concepts), verbal thinking is governed by the logic of graphic imagery and thus is highly dependent on perception. Corresponding concepts are presyncretic, that is, they are not tied to other concepts in meaningful ways. It is the appearance of higher order concepts that allows this to change; the unification of concepts within a single structure allows for the comparison and analysis of subordinate concepts. To recognize contradictions or evaluate one conceptualization against another, the individual must understand two different concepts as relating to the same thing within a single superordinate structure. Comprehending the structure of a scientific concept, therefore, requires the learner to develop higher levels of reasoning, to form new categories of relationships, and to generalize.

The strengths of the scientific concept are the weaknesses of the spontaneous concept, and vice-versa. By means of their complementarities, each one lays the foundation for the development of the other. The development of the scientific concept is mediated by the spontaneous concept as intuitive modes of analysis become available to it. The spontaneous concept is in turn transformed through this mediation much in the same way that one's native language is transformed when it mediates the learning of a foreign language. The structure provided by the scientific concept enables the spontaneous concept to grow and become more available to abstract functioning.

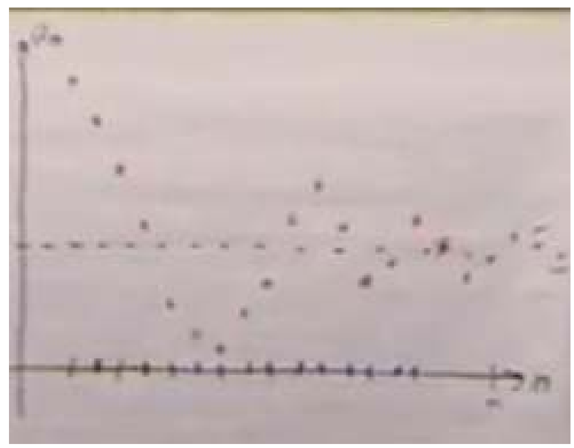
Outpacing of development, one purpose of instruction is to encourage in the student conscious awareness and volitional use of their spontaneous knowledge. This occurs as thinking is modeled within a system that is just beyond the current comprehension of the student but within their ability to imitate. Vygotsky argues that imitation is not an act of thoughtless mimicry but rather requires a beginning grasp of the structure of the system, noting that animals cannot imitate except through training. As opposed to performing a trained behavior, a student can only spontaneously imitate if the task lies within the zone of his or her own intellectual potential, the so-called "zone of proximal development."

Methods

The authors conducted a multi-day teaching experiment with two calculus students at a large, southwest, urban university. For this presentation, we report results from the first two 90-minute sessions of the study. The central objective of this portion of the teaching experiment was for the students to generate rigorous definitions of sequence convergence based on initial activity in which they generated examples of sequences that converge to 5 and sequences that do not converge to 5. These examples were then intended to i) motivate a need to generate a statement that was true of all sequences that converge to 5 but false of all sequences that do not converge to 5, ii) serve as images of their intuitive understanding of convergence against which they could test their definition, and iii) identify ways in which their definition needed refinement. The sessions were videotaped and transcribed, and we analyzed the data to first identify any examples of sequences that were improperly classified relative to their convergence or that were characterized in non-standard ways. We then identified all passages in the transcripts where students referred to these examples and looked for evidence of the origins of their non-standard interpretations, details of the ways in which they used these examples, and implications of their use of these examples.

Results

The students participating in the teaching experiment, pseudonyms Megan and Belinda, described several examples of sequence in non-standard ways during the teaching experiment, but most of these descriptions were not persistent. Nor did they use most of their idiosyncratic images to explicitly draw inferences about their definitions. One example, however, persisted over time and had wide-ranging implications for their reasoning and ultimately for their definition of a convergent sequence. Megan and Belinda both agreed that their example of a damped oscillating series (see Figure) would not converge because they noted



Megan: No matter how close it gets though there's going to be a point, you know where
Belinda: Where it's still moving away
Megan: It's still gonna come away every so often. And that, that coming away feels like it's not convergent.

Megan first suggested that this series would be divergent after trying to apply an early definition including the phrase "at some point N, it becomes closer and closer" to the example and concluding that since it was originally in their list of examples that converge to 5, it must eventually become monotonic. When one of the researchers asked "What if it continued to go away and come back and go away, but always going away less, would it be convergent?" she insisted it must then be placed in the category of examples not converging to 5, to which Belinda strongly agreed. Subsequently, both students continually returned to this example during the first two days of the teaching experiment and cited it's divergence as a reason to include statements like "and always gets closer to 5" or "$|5 - a_{n+1}| < |5 - a_n|$" in their definition of a sequence $\{a_n\}$ converging to 5. Even though they recognized that "for any chosen acceptable error range, there would be some point after which $|5 - a_n|$ does not exceed [that bound]," they still insisted that this sequence was divergent since the "errors don't get smaller."

Megan and Belinda were amazingly consistent in their nonstandard interpretation of this example and raised logical counterarguments to any suggestions made by the researchers why one might consider the damped oscillating series to be convergent. They even suggested adding the phrase "or is always 5" to their definition to include the constant sequence a_n=5 as convergent but still be able to keep their statement "$|5 - a_{n+1}| < |5 - a_n|$." Thus we see that although their reasoning is highly idiosyncratic, it is also systematically structured and applied volitionally with conscious awareness.

Questions

In our presentation, we will show video clips tracing the origins of Megan and Belinda's idiosyncratic interpretation of the convergence for a damped oscillating sequence, the nature of their arguments about the sequence, the effects they had on their definitions, and finally the method by which we convinced the students to alter their definition to include this sequence as convergent. We are interested on feedback from the audience on the following questions:

1. Is the reasoning of these students appropriately characterized as spontaneous, scientific, or neither? What are the implications of this?
2. Given the idiosyncratic but logical and consistent reasoning illustrated, what is the nature of the zone of proximal development for these students?
3. How might instruction mediate a productive interaction for these students with the standard interpretation held by the mathematics community?

References

Artigue, M. (1991). Analysis. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 167–198). Dordrecht, The Netherlands: Kluwer.

Cornu, B. (1991). Limits. In D. O. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 153-166). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Davis, R. B., & Vinner, S. (1986). The notion of limit: Some seemingly unavoidable misconception stages. *Journal of Mathematical Behavior, 5*, 281-303.

Frid, S. (1994). Three approaches to undergraduate calculus instruction: Their nature and potential impact on students' language use and sources of conviction. In E. Dubinsky, A. H. Schoenfeld, & J. Kaput (Eds.), *Issues in mathematics education: Vol. 4. Research in college mathematics education I* (pp. 69–100). Providence, RI: American Mathematical Society.

Monk, G. (1994). Students' understanding of functions in calculus courses. *Humanistic Mathematics Network Journal*, *9*, 21–27.

Oehrtman, M. C. (2008). Layers of abstraction: Theory and design for the instruction of limit concepts. In M. P. Carlson & C. Rasmussen (Eds.), *Making the Connection: Research and Teaching in Undergraduate Mathematics Education* (Vol. 73, pp. 65-80). Washington, DC: Mathematcial Association of America.

Orton, A. (1983). Students' understanding of differentiation. *Educational Studies in Mathematics, 14*, 235–250.

Sierpinska, A. (1987). Humanities students and epistemological obstacles related to limits. *Educational Studies in Mathematics, 18,* 371–397.

Szydlick, J. (2000). Mathematical beliefs and conceptual understanding of the limit of a function. *Journal for Research in Mathematics Education, 31,* 258–276.Tall, 1990, Tall, D. (1992). The transition to advanced mathematical thinking: Functions, limits, infinity, and proof. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 495–511). New York: Macmillan.

Tall, D., & Schwarzenberger, R. (1978). Conflicts in the learning of real numbers and limits. *Mathematics Teaching*, *82*, 44–49.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, *12*, 151–169.

Thompson, P. (1994). Images of rate and operational understanding of the Fundamental Theorem of Calculus. *Educational Studies in Mathematics*, *26*, 229–274.

Tirosh, D. (1991). The role of students' intuitions of infinity in teaching the Cantorian Theory. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 199–214). Dordrect, The Netherlands: Kluwer.

Vygotsky, L. (1987). The development of scientific concepts in childhood. In R. W. Rieber & A. S. Carton (Eds.), *Problems of general psychology: Vol. 1. The collected works of L. S. Vygotsky* (pp. 167–241). New York: Plenum. (Original work published 1934)

Vygotsky, L. (1978). *Mind in Society: The Development of Higher Psychological Processes.* Cambridge, Massachussetts: Harvard University Press.

Williams, S. R. (1991). Models of limit held by college calculus students. *Journal for Research in Mathematics Education, 22*(3), 219-236.

Williams, S. (2001). Predications of the limit concept: An application of repertory grids. *Journal for Research in Mathematics Education, 32*, 341–367.

Transitioning from Cultural Diversity to Intercultural Competence in Mathematics Instruction

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Abstract

We report on our work to build an applied theory for intercultural competence development for mathematics teaching and learning in secondary and tertiary settings. Based on social anthropology and communications research, we investigate the nature of intercultural competence development for mathematics instruction among in-service secondary mathematics teachers and college faculty participating in a university-based mathematics teacher professional development program. We present results from quantitative and qualitative inquiry into the intercultural orientations of individuals and subgroups (teachers, teacher-leaders, university faculty and graduate students) and offer details on the development of case stories for use in the professional development of mathematics university teacher educators, in-service teacher leaders, and secondary school teachers.

Keywords: secondary teacher preparation, cultural competence, intercultural development, cultural diversity

Transitioning from Cultural Diversity to Intercultural Competence in Mathematics Instruction
Preliminary Research Report[1]

I wanted to explain why some people seem to get a lot better at communicating across cultural boundaries while other people didn't improve at all, and I thought that if I were able to explain why this happened, educators could to a better job of preparing people for cross-cultural encounters. (Bennett, 2004, p. 62).

Relation of the Work to the Research Literature

While the significance of diversity as a factor in the education of children has been widely discussed for many years, the nature of "diversity" continues to evolve in U.S. classrooms (Aud, Fox, & KewalRamani, 2010). And, though a similar evolution in diversity is evident in school staffing among paraprofessionals, the teacher and principal populations continue to be more homogeneous than varied in terms of government-surveyed categories such as race, education, and socialization (Strizek, Pittsonberger, Riordan, Lyter, & Orlofsky, 2006). Since "culture" can include professional and classroom environments as well as personal or home experience, responding to it is a multi-faceted challenge (Greer, Nelson-Barber, Powell, & Mukhopadhyay, 2009). As Stigler and Hiebert (1999) noted after an international study of instruction, "teaching is a cultural activity…[and] recognizing the cultural nature of teaching gives us new insights into what we need to do if we wish to improve it" (p. 12). From anti-racism training to culturally responsive pedagogies, teacher education efforts have emerged largely from the same arena as teacher education itself: psychology. Yet, there is another area of the academy from which educators can draw great insight: anthropology (Ladson-Billings, 2001). That is, while psychology tackles the issue through a developmental approach to changing classroom *disposition* based on *behavior*, social anthropology provides a developmental continuum of *orientation* from a focus on *communication*. Several frameworks currently exist for professional contexts that involve understanding, interacting, and communicating with people from various cultures (e.g., from healthcare professions and international relations by governments; Bennett, 1993, 2004; Hammer, 2005, 2009; Kramsch, 1998; Leininger, 2002; Wolfel, 2008).

> *Short definition of culture*: A dynamic social system of values, beliefs, behaviors, and norms for a specific group, organization, or other collectivity; the shared values, beliefs, behaviors, and norms are learned, internalized, and changeable by members of the society (Hammer, 2009).

Conceptual Framework

Our work to build an applied theory for intercultural competence development for mathematics teaching and learning in secondary and tertiary settings is based on the *Developmental Model of Intercultural Sensitivity* (Bennett & Bennett, 2004). As a developmental model, it includes lower and upper anchor orientations, intermediate orientations, and descriptions of the transitions among the orientations. Additionally, we attend to discourse with the framing of communication dimensions for intercultural conflict resolution (Hammer, 2005). The continuum begins with a monocultural view based on the premise "Everybody is like me." This "denial" orientation (see Figure 1) may recognize observable cultural differences (e.g., distinctions in food or dress) but not notice complex difference (e.g., in values, beliefs, or communication norms) and will avoid or express disinterest in cultural difference. The transition to the next orientation comes with the

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Figure 1a.

Figure 1b.

Figure 1c.

Denial ⇨ Polarization ⇨ Minimization ⇨ Acceptance ⇨ Adaptation

recognition of difference, of light and dark in a situation (e.g., Figure 1a). The "polarization" orientation is driven by the assimilative assumption "Everybody should be like me and my group" and is an orientation that views cultural differences in terms of "us" and "them." Polarization can take the form of defense or reversal. Defense is an uncritical view toward one's own values and practices and an overly critical view towards others'. Reversal is a negatively judgmental take on the values and practices of the group with which one identifies and an uncritical view towards others'. Transitioning to the next level of development involves attention to nuance and awareness of norms. This middle orientation is "minimization," a lens for experience based on the notion that "Despite some differences, we really are all the same, deep down." Minimization attends to commonality and presumed universals (e.g., biological – we all eat and sleep; and values – we all know the difference between good and evil). The minimization orientation will, however, be blind to deeper recognition and appreciation of difference (e.g., Figure 1b, literally a "colorblind" view). Transition from minimization to an "acceptance" orientation involves mindful awareness of oneself as having a culture and interacting with other cultures (plural). While an acceptance orientation is aware of difference and the importance of relative context, how to respond and what to respond to, in the moment of interaction is still elusive. The transition to "adaptation" involves developing ethnorelative frameworks for perception that are responsive to a broad spectrum of intercultural interaction (e.g., the detailed and contextualized view in Figure 1c). Adaptation is an orientation wherein one may shift perspective, without losing or violating one's authentic self, and adjust communication and behavior in culturally appropriate ways. There are several ways that knowing one's orientation, or the normative orientation of a group, can inform teacher and researcher work.

Research Question & Methods

What is the nature of intercultural competence development for mathematics instruction among in-service secondary mathematics teachers and college faculty participating in a university-based mathematics teacher professional development program? Participants to date have been 26 in-service K-12 teachers and teacher leaders and18 university faculty and graduate students. All completed the *Intercultural Development Inventory* (IDI), a reliable and validated instrument for ascertaining a person's intercultural orientation and eliciting intercultural development goals (50 Likert-like items and 4 open ended items; Hammer, 2009). Each report from the IDI includes responses to open ended items along with quantitative information about *developmental orientation* (the orientation most likely at work in day-do-day interactions with others), *perceived orientation* (this is often a more advanced than the developmental orientation) as well as *trailing* orientation (a fallback that may come into play in situations high in conflict or stress) and *leading* orientation (often aligned with perceived orientation, this is at the leading edge of someone's intercultural competence and the target for development). We are using descriptive statistics, constant comparative coding, and cross coding to examine individual orientation profile results as well as group profiles for the teachers, teacher-leaders, and university staff. Our plan is to use case methods to link participant stories of intercultural challenges in teaching mathematics to activities for increasing intentionality for intercultural development. Ultimately, the results will include stories that illuminate aspects of orientations and transitions.

Results

In Figure 2 are the distributions among orientations for three groups. As a group, the teachers' orientation was normatively in polarization while the teacher leaders, as a group, were largely at the lower end of minimization and the university folk were largely in minimization.

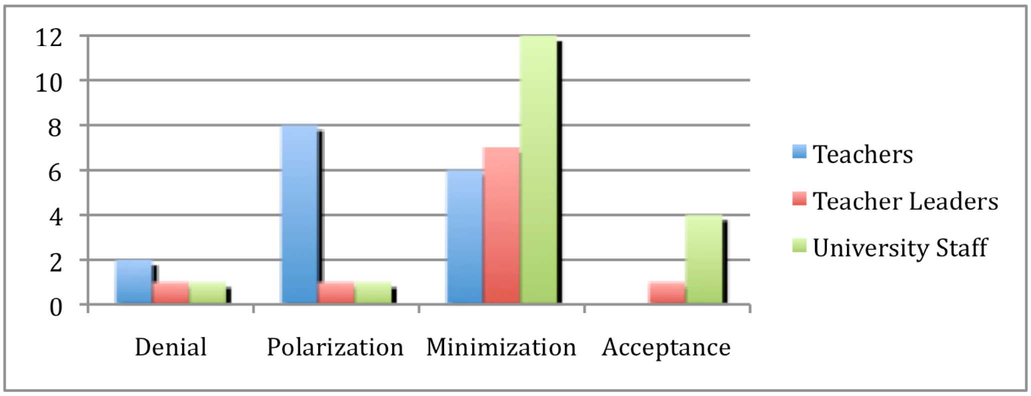

Figure 2. Distribution of Participant Developmental Orientations

Case stories are under development and will be completed and expanded for the final report and presentation. Below we offer one example. As part of the research process, we conducted group profile debriefing sessions with teachers, teacher leaders, and university staff. When debriefing, three common goals emerged from participants: (1) build *awareness of self as having a cultural lens for viewing the world*. (2) find guidance in the transitions from polarization through minimization and into acceptance, particularly how to *be mindful of one's cultural filter(s) for interacting with the world*; and (3) engage in *building a knowledge base about equity*.

Example Case Story. Helen is a public high school mathematics teacher in a socio-economically and culturally diverse community. She is teaching a consumer mathematics class with mostly seniors. Helen wants all her students to believe they have what it takes to succeed in college, so she has each student create a personal career finances portfolio. Students choose a job and a place to live after college. The portfolio is a report about living and working in this potential future career: starting pay for the job in that location, education required for that job, the cost of living in that location – including a budget for housing, food, transportation, and leisure. Helen's grading rubric has points for turning in a rough draft. Her intention is to provide opportunities for students: (1) to see themselves as college graduates, (2) to work with real-world values in creating a budget, and (3) to receive feedback on a draft so the final report will have a high score. Helen asks the class how the assignment is going and several student express frustration and confusion. She announces that she will be available after school to help in office hours and is disappointed that students do not take advantage of this opportunity. Helen gets frustrated when several students, who are not doing well, do not turn in a draft and do not come for help.

[Pause here and discuss what elements of the transition from polarization to minimization might support Helen to find a more satisfactory approach; what questions might need to be asked (and why)? What advice might Helen be ready to hear and act on?]

> Helen's colleague Lee offered her own experience from high school, explaining that "going to office hours" in her middle school was a form of punishment for misbehavior or low grades. In her first year of high school, the idea of going to a teacher's office hours voluntarily made no sense to her: "Why would someone purposely take what amounted to an oral exam? Just to let the teacher know what she did not know and then be criticized for not knowing it?" Helen's first reaction was to dismiss Lee's story. "That's not what my office hours are like, that's not what *I* do!" Lee nodded and said, "Yes, I know. But I'm not completely sure how I learned that what it meant in high school to seek help from a teacher could be different from what it meant in middle school. I've heard students talk about different reasons for not going to get help from teachers – like having a job or working with parents or friends instead or because there was difficulty communicating with the teacher. So, I'm not sure why your current batch of students is not coming to you for help, but there are probably lots of good reasons. Good to them, I mean." Helen shook her head, "That's too bad. Students should feel comfortable going to the teacher for help. Well, I can't help them if they don't come to see me. And, they won't come see me."

Indicators that Helen has an orientation of polarization–defense include her view that she is offering "opportunities" whether or not they are seen as opportunities by students. It could be that some of what Lee suggests is true, or that students in Helen's class were uncomfortable with her seeing their development process, or something else entirely. Discuss, again, what elements of the transition from polarization to minimization might help Helen, what questions might need to be asked (and why) along with advice Helen might be ready to hear and act on in the situation.

Implications/Applications for Research and Practice

A perennial challenge for any instructor is: how do I teach so that my students surpass me? What help in transitioning to global and ethnorelative mindsets can teacher educators offer if their own developmental orientations are more monocultural than intercultural? In terms of implications for research, what can researchers do to support their own growth as interculturally adaptive? For example, if researchers have a polarization orientation – where differentiating is essential – would instruments and observation protocols they designed do a good job of capturing the views and practices of teacher leaders in a minimization orientation (or vice versa)?

Questions for the Audience

1. The example given here is largely independent of mathematics content. What kind of story might foreground the intersection of content and culture in secondary mathematics, for example with the framing of teacher response to a student's questions about generating a polynomial from a graph where the x-intercepts are marked and labeled versus just marked.

2. In an editorial, Ball, Goffney, and Bass (2005) have argued that in addition to teachers being culturally aware, that it is important for *students* to build adaptive competence for mathematics:

> In a democratic society, how disagreements are reconciled is crucial. But mathematics offers one set of experiences and norms for doing so, and other academic studies and experiences provide others. In literature, differences of interpretation need not be reconciled, in mathematics common consensus matters. In this way, mathematics contributes to young people's capacity for participation in a diverse society in which conflicts are not only an inescapable part of life, but their resolution, in disciplined ways, is a major source of growing new knowledge and practice. … Important to our argument is that these skills and practices that are central to mathematical work are ones that can contribute to the cultivation of skills, habits, and dispositions for participation in a diverse democracy. (p. 4)

How might this perspective need to be revised or framed to be accessible to a teacher with a denial orientation? A polarization orientation? A minimization orientation?

3. How about new/pre-service teachers?

References

Aud, S., Fox, M., & KewalRamani, A. (2010). *Status and Trends in the Education of Racial and Ethnic Groups* (Tech. Report; NCES 2010-015). U.S. Department of Education, National Center for Education Statistics. Washington, DC: U.S. GPO.

Ball, D. L., Goffney, I. M., & Bass, H. (2005). The role of mathematics instruction in building a socially just and diverse democracy. *The Mathematics Educator, 15*(1), 2-6.

Bennett, M. J. (2004). Becoming interculturally competent. In J. Wurzel (Ed.), *Towards multiculturalism: A reader in multicultural education* (pp. 62–77). Newton, MA: IRC.

Bennett, M. J. (1993). Towards ethnorelativism: A developmental model of intercultural sensitivity. In R. M. Paige (Ed.), *Education for the intercultural experience* (pp. 21-71) Yarmouth ME: Intercultural Press.

Bennett, J. M. & Bennett, M. J. (2004). Developing intercultural sensitivity: An integrative approach to global and domestic diversity. In D. Landis, J. M. Bennett, & M. J. Bennett (Eds.), *Handbook of intercultural training* (pp. 147–165). Thousand Oaks, CA: Sage.

Greer, B., Nelson-Barber, S., Powell, A., & Mukhopadhyay, S. (2009)(Eds.), *Culturally responsive mathematics education*. Mahwah, NJ: Erlbaum.

Hammer, M. R. (2005), The Intercultural Conflict Style Inventory: A conceptual framework and measure of intercultural conflict approaches, *International Journal of Intercultural Research, 29*, 675–695.

Hammer, M. R. (2009). Solving problems and resolving conflict using the intercultural conflict style model and inventory. In M. A. Moodian (Ed.), *Contemporary leadership and intercultural competence* (pp. 219-232). Thousand Oaks, CA: Sage.

Kramsch, C. (1998). *Language and culture*. New York: Oxford University Press.

Ladson-Billings, G. (2001). *Crossing over to Canaan: The journey of new teachers in diverse classrooms.* San Francisco: Jossey-Bass.

Leininger, M. (2002). Culture care theory: A major contribution to advance transcultural nursing knowledge and practices. *Journal of Transcultural Nursing 13*(3), 189-192.

Strizek, G.A., Pittsonberger, J.L., Riordan, K.E., Lyter, D.M., & Orlofsky, G.F. (2006). *Characteristics of Schools, Districts, Teachers, Principals, and School Libraries in the United States* (Tech. Report; NCES 2006-313 Revised). U.S. Department of Education, National Center for Education Statistics. Washington, DC: U.S. GPO.

Wolfel, R. (2008). *Culture Cubed: Towards Three-Part Definition of Intercultural Competence* (Tech. Report). Center for Languages, Cultures, and Regional Studies Position Paper, U.S. Military Academy. West Point, NY.

The Treatment of Composition the Secondary and Early College Mathematics Curriculum

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While many studies have focused on student knowledge of function, few studies have focused on composition. This report describes a curriculum analysis of the treatment of composition in the secondary (algebra, geometry, algebra 2, precalculus) and early college (precalculus, calculus) mathematics curriculum. In this study composition is conceptualized as a sequence of functions and as a binary operation on functions. The curriculum analysis utilizes a framework of conceptual, procedural, and conventional knowledge elements as well as representations and types of functions. Preliminary data will be presented during the session and a discussion will center on conceptual, procedural, and conventional knowledge elements for composition.

Keywords: composition, curriculum analysis, conceptual and procedural knowledge, representations

The function concept is an essential topic to the undergraduate mathematics curriculum and to mathematics, in general. Freudenthal (1983) stated, "The strength of the function concept is rooted in the new operations - composing and inverting functions" (p. 523). The operation of composition is vastly different than the arithmetic operations (i.e., addition, subtraction, multiplication, division) that students encounter in elementary school. For example, composition is applied to mathematical objects such as functions (including constant functions), relations and transformation, but not to numbers. While educational researchers have extensively studied student knowledge of function (Carlson, 1998; Even, 1990, 1998; Ferrini-Mundy & Graham, 1991; Leinhardt, Zaslavsky, & Stein, 1990; Monk, 1994; Oehrtman, Carlson, & Thompson, 2008; Vinner & Dreyfus, 1989), research on the teaching and learning of the operation of composition has received little attention (Engelke, Oehrtman, and Carlson, 2005).

The existing research on composition has documented that the learning of composition is nontrivial for students. Research on the learning of topics built upon composition (i.e., chain rule) has reported that students' difficulties are related to a weak foundation of composition (Clark et al., 1997; Horvath, 2008). Studies of composition have focused on student knowledge or the output of learning. None have reported on the teaching of composition or on the written curriculum or the input of learning except for one study that focused on the genetic decomposition (Ayers et al., 1988). The study reported here is a beginning to fill this gap through a curriculum analysis on the treatment of composition in secondary (algebra, geometry, algebra 2, precalculus) and early collegiate (precalculus, calculus) mathematics textbooks. While the written curriculum does not determine what teachers teach or what students learn, the written curriculum influences both (Remillard, Herbel-Eisenmann, & Lloyd, 2009). The research questions guiding this work are: In what ways is the concept of composition developed across the algebra to calculus curriculum? Is there a difference between the way composition is treated in the secondary curriculum and college precalculus and calculus curriculum and if so, what are the characteristics of that gap?

In this study composition is conceptualized in two ways which relate to the notations of $g(f(x))$ and $(g \circ f)(x)$. First is the *sequence view of composition*. In this view, $g(f(x))$ denotes a sequence of functions where f corresponds x to $f(x)$ and g corresponds $f(x)$ to $g(f(x))$. Thus, the output of f, $f(x)$, is the input of g and it is the elements x and $f(x)$ that are being acted upon. In general, this view of composition describes composition as a sequence of recursive relations (including functions) where the input of the *n*th term is the output of the (*n-1*)th term. Carlson, Oehrtman, and Engelke (2010) described this as a process view of composition, while Harel and Kaput (1991) referred to this process of "acting on individual elements of [the] domain" and called it a *point-wise operation* (p. 84).

The other conceptualization of composition is the *operation view of function*. In this view, $(g \circ f)(x)$ is a binary operation on two functions, f and g, resulting in a new function $g \circ f$. In this case functions are the objects being acted upon and not simply their domain and range elements. Others who have written about replacing processes with objects include Asiala et al. (1996) using the term encapsulation, Sfard (2008) using the term reification, and Martin (1991) using the term of nominalization which is a specific case of Halliday's (1985, 1995) grammatical metaphor. The common feature among these perspectives is that processes (or verbs) are treated as entities (or nouns) which become the objects of other actions and procedures (or verbs). For example, the function $f(x) = x^2$ can be viewed as the process of corresponding any number to its square. Composing the function $g(x) = 2x + 4$ with $f(x)$ can be viewed as "plugging in" $f(x)$ into the x's in the $g(x)$ function resulting in $g(f(x)) = 2(f(x)) + 4$ or $g(f(x)) = 2(x^2) + 4$. In this situation, $f(x)$ is treated as an object and not as a correspondence between its domain and range.

Research on the sequence view of composition has reported students have interpreted the composition statement of $f(g(3)$ as the multiplication statement of $f(3) \cdot g(3)$ (Engelke et al., 2005, Meel 1999). These studies have reported students interpreting composition as multiplication while using formulas, graphs, and tables. Research has also shown that students have different success rates on composition problems in different representations. When asked to evaluate $g(f(2)$, Carlson et al. (2010) reported that 94% of students were successfully given two algebraic functions, 50% with graphical functions and 47% with tabular functions. Hassani (1998) reported students' success rates as 84%, 10%, and less than 50% for algebraic, graphical, and tabular, respectively. When the task was rephrased to evaluate $(g \circ f)(2)$ the success rates of students in Hassani's study changed to 35%, 25% and 33%, respectively. In an interview with a student in a developmental algebra course DeMarois & Tall (1996) reported that he was able to complete a composition task using the table with considerable guidance from the interview, was then unable to begin graphical composition task, but following that he was successful with minimal guidance on the algebraic composition task. This research implies that algebraic composition tasks are easier for students than other representations. One explanation has claimed that this is due to a curriculum that is heavily algebraic and that students have had more exposure and experience with dealing with the algebraic representation (Hitt, 1998). However, a curriculum analysis has not been conducted to empirically validate such claims.

Research on the operation view of composition has reported that students frequently implement this view by plugging in or substituting the one function for a variable in the other function (Ayers, et al., 1988; Carlson, 1998; Horvath, 2010; Uygur & Ozdas, 2007) or by interpreting composition as multiplication (Horvath, 2010; Meel, 1999). The difference in the multiplication between the sequence view and the operation view is that students not only multiply numbers, but are also multiplying objects such as functions. This interpretation appears symbolically as $(f \circ g)(x) = f(x) \cdot g(x)$.

This study uses the conceptual knowledge and procedural fluency framework to study curriculum materials. Many scholars have participated in the debate of conceptual and procedural knowledge. Piaget, Tulving, Anderson, Scheffler, and Skemp are a few who have done so. Hiebert and Lefevre (1986) described conceptual knowledge as knowledge that is rich in relationships and is like a network where both the vertices and the edges (words taken from Graph Theory) are essential and of equal importance. "In fact, a unit of conceptual knowledge cannot be an isolated piece of information" and the individual must consciously recognize links to other information (p. 4). They described procedural knowledge in two components. "One part is composed of the formal language, or symbol representation system, of mathematics. The other part consists of the algorithms, or rules, for completing mathematical tasks" (p. 6).

For this study of curriculum conceptual knowledge includes definitions and properties of composition such as the associativity and commutativity (in rare situations), the non-uniqueness of decomposition, etc. Procedural fluency elements performance of procedures and algorithms such as evaluate the composition, find the domain, decompose a function, etc. Vocabulary of important terms and notation is placed under the separate category called Conventional Knowledge Elements. This would include items such as the parenthetic $f(g(x))$ and circle, $f \circ g$, notations and what objects are described as being "composite." Other major categories in this study's framework are Representation (i.e., algebraic, graph, table) and Function Type (i.e., polynomial, trigonometric, exponential, logarithmic, piece-wise).

Method

In order to better understand the potential influence of the written curriculum on what opportunities to learn students have, this study analyzes the development of the concept of function composition in written curriculum over the span from Algebra to Calculus. High school curricula will be analyzed to study examples of the ways in which students are introduced to composition in high school (CCSS-M, 2010). The texts to be analyzed will include entire series of Algebra 1 and 2, Geometry, and Precalculus. The notion of composition is developed further in calculus which many students study in college. Thus, collegiate Precalculus and Calculus texts will also be analyzed. The duplication of the precalculus text at both the high school and college level will help identify any differences between the preparation for calculus at the different levels.

The two secondary mathematics curriculum series to be analyzed are *Glencoe/McGraw Hill Mathematics* (2010/2011) and the *CME Project* (2009). Glencoe Mathematics was chosen due to its large share of the secondary school market (see Dossey et al., 2008). The *CME Project* materials were chosen to be the second series because it has been developed more recently and have different features that provide a broader view of the treatment of composition across curricula. At the collegiate level, a widely used precalculus and calculus series was determined by surveying approximately 100 Department of Mathematics' websites and identifying the texts used for calculus and precalculus courses. The institutions chosen for the survey are those classified as very research intensive in the Carnegie Classification.This survey was conducted in June 2010. The survey results identified *Calculus: Early Transcendentals, 6th edition* (2008) by Stewart and *Precalculus: Mathematics for Calculus, 5th edition* (2006) by Stewart, Redlin, and Watson as the most widely used calculus and precalculus texts, respectively. The second precalculus and calculus text to be analyzed is *Functions Modeling Change: A Preparation for Calculus, 4th edition* (2011) by Connally, Hughes-Hallett, Gleason, et al. and *Calculus, 5th edition* (2009) by Hughes-Hallett, Gleason, McCallum, et al., respectively.

The content included in the analysis was determined by the following criteria. These criteria include both the explicit development and implicit use of composition. Any lesson that includes exposition regarding function composition in the student or teacher edition is considered to be explicitly developing the concept of composition. In those situations the entire lesson was included in the analysis. Lessons on function operations, inverse function, and composition of geometric transformations are examples of explicit development. For implicit uses of composition, only the sentence (if in the exposition) or the example will be included in the analysis. A few examples of implicit use of composition include translations of graphs, solving equations involving trigonometric functions with non-trivial angles. Exercises and review problems that explicitly or implicitly use composition were also included.

The results of this study will be important for secondary and university mathematics teachers as well as curriculum developers. It may reveal aspects of composition that are over- or under-emphasized. It will also inform college instructors on how students have been prepared by the secondary curriculum with respect to what is expected in the early calculus curriculum.

Questions to be posed to the audience

What do you consider to be a conceptual knowledge element or task for composition?
What topics (e.g., inverse function, chain rule) do you consider as explicit use of composition?

References

Asiala, M., Brown, A., DeVries, D. J., Dubinsky, E., Mathews, D., & Thomas, K. (1996). A framework for research and development in undergraduate mathematics education. *Research in Collegiate Mathematics Education,* 2, 1-32.

Ayers, T., Davis, G., Dubinsky, E., & Lewin, P. (1988). Computer experiences in learning composition of functions. *Journal for Research in Mathematics Education, 19*(3), 246-259.

Carlson, M. P. (1998). A cross sectional investigation of the development of the function concept. CBMS Issues in Mathematics Education Vol 7. 114-162.

Carlson, M., Oehrtman, M., and Engelke N. (2010). The precalculus concept assessment: A tool for assessing students' reasoning abilities and understandings. *Cognition and Instruction, 28*(2), 113-145.

Clark, J., F. Cordero, J. Cottrill, B. Czarnocha, D. DeVries, D. St. John, T. Tolias, & D. Vidakovic. (1997). Constructing a schema: The case of the chain rule. *Journal of Mathematical Behavior*. 16(4), 345-364.

Common Core State Standards for Mathematics (CCSS-M) (2010).

DeMarois, P. & Tall, D. (1996). Facets and layers of the function concept. In ---------- (Eds.) Proceedings of PME 20, vol 2. (pp. 297-304). Valencia.

Dossey, J., Halvorsen, K., & McCrone, S. (2008). *Mathematics education in the United States 2008: A capsule summary factbook*. Reston: National Council of Teachers of Mathematics.

Engleke, N., Oehrtman, M., & Carlson, M. (2005). Composition of function: Precalculus students' understandings. In Lloyd, G. M., Wilson, M., Wilkins, J. L. M., & Behm, S. L. (Eds.). *Proceedings of the 27th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Roanoke, VA.

Even, R. (1990). Subject matter knowledge for teaching and the case of functions. *Educational Studies in Mathematics, 21*(6), 521-544.

Even, R. (1998). Factors involved in linking representations of functions. *Journal of Mathematical Behavior*. 17(1), 105-121.

Ferrini-Mundy, J. & Graham, K. G. (1991). An overview of the calculus curriculum reform effort: Issues for learning, teaching, and curriculum development. *The American Mathematical Monthly*. 98(7), 627-635.

Freudenthal, H. (1983). *Didactical Phenomenology of Mathematical Structures*. Dordrecht: D. Reidel Publishing Company.

Halliday, M. A. K. (1985). *An introduction to functional grammar*. London: Edward Arnold.

Halliday, M. A. K. (1995). Language and the theory of codes. In A. R. Sadovnik (Ed.) *Knowledge and Pedagogy: The Sociology of Basil Bernstein* (pp. 127-144). Norwood, NJ: Ablex Publishing Corporation.

Harel, G. & Kaput, J. (1991). The role of conceptual entities in the construction of advanced mathematical concepts and their symbols. In D. O. Tall (ed.). *Advanced Mathematical Thinking*. (pp. 82-94).

Hassani, S. (1998). *Calculus students' knowledge of the composition of functions and the chain rule.* Unpublished doctoral dissertation, Illinois State University, Normal.

Hiebert, J., & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 1-27). Hillsdale, NJ: Lawrence Erlbaum Associates.

Hitt, F. (1998). Difficulties in the articulation of different representations linked to the concept of function. *Journal of Mathematical Behavior*. 17(1), 123-134.

Horvath, A. (2008). Looking at calculus students' understanding from the inside-out: The relationship between the chain rule and function composition. Proceedings of the 11th Annual Conference on Research in Undergraduate Mathematics Education, San Diego, CA.

Horvath A. K. (2010). Calculus students, function composition, and the chain rule. *Proceedings of the 32nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Columbus, OH.

Leinhardt, G., Zaslavsky, O., & Stein, M. (1990). Functions, graphs, and graphing Tasks, learning, and teaching. *Review of Educational Research, 60*(1), 1-64.

Martin, J. R. (1991). Nominalization in science and humanities: Distilling knowledge and scaffolding text. In E. Ventola (Ed.) *Functional and Systemic Linguistics* (pp. 307-337). Berlin: Mouton de Gruyter.

Meel, D. E. (1999). Prospective teachers' understandings: Function and composite function. *Issues in the Undergraduate Mathematics Preparation of School Teachers: The Journal*. 1, 1-12.

Monk, G. (1994). Students' understanding of functions in calculus courses. *Humanistic Mathematics Network Journal*. 9, 21-24.

Oehrtman, M., M. Carlson, and P. W. Thompson. (2008). Foundational reasoning abilities that promote coherence in students' function understanding. In *Making the Connection: Research and Practice in Undergraduate Mathematics, MAA Notes Volume 73*, eds. M. Carlson and C. Rasmussen, 27-41. Washington, DC: Mathematical Association of America.

Remillard, J. T., Herbel-Eisenmann, B. A., & Lloyd, G. M. (Eds.) (2009). *Teachers at work: Connecting curriculum materials and classroom instruction*. New York: Routledge Taylor, and Francis.

Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing.* Cambridge, UK: Cambridge University Press.

Uygur T., & Ozdas, A. (2007). The effect of arrow diagrams on achievement in applying the chain rule. *Primus, 17*(2), 131-147.

Vinner & Dreyfus, (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education, 20*(4), 356-366.

What Do We See? Real Time Assessment of Middle and Secondary Mathematics Teachers' Pedagogical Content Knowledge and Sociomathematical Norms[1]

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Abstract. The article reviews efforts to develop an observation protocol to assess the pedagogical content knowledge (PCK) and sociomathematical norms (SMN) that middle and high school teachers may develop over time as part of their participation in a master's program for secondary mathematics teachers. We observed each of 16 teachers in real time using the instrument, before involvement in the project and again after one year. Aspects of the protocol measure four critical components of PCK including curricular content, discourse, anticipatory, and implementation knowledge as well as some sociomathematical classroom norms. We present preliminary quantitative and qualitative analysis of the observations and discuss various challenges faced in the instrument development and its relation to similar protocols used by others previously.

Key Words: Pedagogical content knowledge, sociomathematical norms, inter-rater reliability, teaching moves

There have been several approaches to measuring the pedagogical content knowledge (PCK) of practicing teachers. Indeed, Hill, Ball, and Schilling (2008) and Hauk, Jackson, and Noblet (2010) have documented their development of written instruments designed to assess aspects of PCK. Both groups have developed theoretical frameworks for PCK that have similarities and some differences. One of the principle differences is that the Hill, Ball and Schilling linear model seeks to measure each of their proposed categories of PCK as distinct from each other, while Hauk, Jackson, and Noblet take a non-linear approach that presumes measurement overlap among categories.

Hauk, Jackson, and Noblet discuss PCK in terms of four components: curricular content, discourse, anticipatory, and implementation (action) knowledge. *Curricular content knowledge* is "substantive knowledge about topics, procedures, and concepts along with a comprehension of the relationships among them *as they are offered in school curricula*" (p. 2). *Discourse knowledge* "is about the culturally embedded nature of inquiry and forms of communication in mathematics (both in and out of educational settings" (p.2), and as such includes knowledge of mathematical syntax as a sub-category. *Anticipatory knowledge* "is an awareness of, and responsiveness to, the diverse ways in which learners may engage with content, processes, and concepts" (p. 3). *Implementation* or *action* knowledge "includes knowledge about how to adapt

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teaching according to content and socio-cultural context and *enact in the classroom* the decisions informed by content, discourse, and anticipatory understandings" (p. 3).

Both groups' written assessments use multiple choice items and are limited in measurement of action knowledge. Implementation knowledge is more challenging to assess as this type of knowledge requires actions executed in the classroom (i.e., teacher moves). That is, the written assessments could not test for this type of knowledge because it requires that the teacher act "in the moment." At best, any written item could only gauge what a teacher *might* do in certain situations (e.g., see Ball, Hill, & Schilling, 2008).

In order to validate their written instrument, Ball and others (Learning Mathematics for Teaching (LMT)*,* 2006) developed another instrument aimed at quantitatively measuring aspects of elementary and middle school teachers' classroom practice. Ten K-8 teachers who had taken a PCK test were videotaped for 3 times prior to, during, and after participation in professional development. Over the course of a year, a team of mathematicians, mathematics educators, mathematics teachers, and non-specialists analyzed the videos for various aspects of mathematics and mathematics teaching present in each lesson. A rubric was developed containing several items and video reviewers trained for and then coded each 5 minute segment of each lesson for X different categories of teacher move or classroom interaction. Each category had four possible codes: Present and Appropriate (PA), Present and Inappropriate (PI), Not Present and Appropriate (NPA) and Not Present and Inappropriate (NPI). LMT team leaders noticed early on a wide variability in how individuals coded lessons based upon the individuals' own professional backgrounds, and so to help ensure inter-rater reliability, the lessons were all recoded in pairs. A glossary describing each category (column) in the observation rubric was written, with each description giving some detail on when each code should be assigned during a segment.

Theoretical Perspective

Our research blends the Hauk et al., framework for PCK and the LMT instrument designed by the research team at the University of Michigan. We take the view that the teacher actions or moves (or the absence thereof) in the LMT protocol can be observed in the classroom, and that such actions or moves can be described (at least approximately) in a predetermined coding format *independently of the researcher involved*. Now, this is not to say that two different researchers may not observe and record different things (as frequently happened with the team at the University of Michigan and for our team) for a given segment, but, like the LMT tool, for an observation we would expect overall variation between observers to be minimal.

We use here the typologies of Hauk, Jackson, and Noblet. The reason is that any particular move that a teacher makes in the classroom can demonstrate multiple facets of PCK simultaneously, and hence we take their view that the strands of PCK are interwoven during instruction. Also, Hauk, Jackson, and Noblet make cultures in the classroom an explicit part of their definitions, which in turn may be part of teacher decisions to make certain moves in response to them.

The research questions for the work reported here are: How might we track the effects of professional development through changes in observed PCK and SMN? If traceable, how might professional development be designed to foster particular classroom moves through changes in PCK and SMN? Work on both of these questions continues, and we will primarily address the first here but some attention will be given to implications of current results for the second.

Methods

The research team at the University of Michigan point out in their technical report that there is a need to develop an instrument for doing observations in real time (Learning Mathematics for Teaching, 2006, p. 20). In order to address this need, we examined their observation protocol in some depth and determined which items were most appropriate given our focus on observing in secondary mathematics classrooms in real time (the LMT work was in grades K-8). Their protocol contained over 30 categories. To streamline for real-time observation we shortened to a protocol containing 20 items. Some of their categories were replaced or condensed in our version. For example, in the LMT version, the researchers created columns for the following: selection of correct manipulatives, and other visual and concrete models to represent mathematical ideas (their column II.e on sheet 2) and multiple models (column II.f on sheet 2). In our version, these two columns were condensed into the column that we titled multiple representations, which could include all of the things that the LMT team was looking for in II.e and f..

Great care was taken in finding an appropriate length of a segment to be viewed during the class. The team started with the 5 minute length that the LMT used for recorded sessions, but it soon became clear that a "5 minute on, 5 minute off" strategy in which the researcher would observe for 5 minutes and then record tallys on the protocol during the next 5 minute interval would result in 5 or fewer codings per class period for each category. Eventually, the team agreed upon observing for 3 minutes, and then recording for 3 minutes.

After the team started using the protocol, we began to reexamine the glossary that the LMT team had developed. We found that trying to use the instrument in real time created new challenges with respect to inter-rater reliability. In particular, the words "explicit" and "inappropriate" leave much room for interpretation even in the definitions provided by the LMT team. Though we used many of the same column categories and indentifying language as they did, we also saw it was important to craft definitions and create a new glossary. The idea was to create an instrument with sufficient examples and non-examples for each category that it could act as a coding book: a guide to the intended viewpoint of the protocol and how to observe through a particular lens. The eventual goal is to have an instrument that is terse but of sufficient detail that individuals can observe classrooms after a short calibration training paired with a practiced observer.

For example, while our glossary continues to be refined, we felt a need to be, well, more explicit about what "explicit talk about a topic or subject" meant. Currently, our glossary description of this category is: any utterance from student or teacher in which a topic or subject is stated verbally or in writing or by reference to a clear written or verbal precursor familiar to people in the room. In-vivo exemplars have been included in our glossary to demonstrate categories. For example, during one 3-minute segment, the teacher presented the Fundamental Theorem of Algebra. The exercise the teacher assigned called for students to find a polynomial of lowest degree with real coefficients that had certain prescribed roots. At one point, an exercise asked for a polynomial with roots $3i$, 4, and 5. The teacher produced a monic degree 4 polynomial with these 3 prescribed roots, and a student asked why it was necessary to have $-3i$ as a root when this number was not contained in the list. The teacher responded that since $3i$ was a root, its conjugate $-3i$ also had to be a root. The student again asked why this must be true when $-3i$ was not listed, and the teacher replied "because conjugates are always roots." The researcher coded this particular segment as NPI in the explicit talk about ways of reasoning column due to the teacher's not addressing directly the student's question (e.g., the idea that the requirement that the coefficients of the polynomial be real was connected to the need for the use of conjugate roots).

Each column of the protocol was assigned a quadruplet of the form (*c, d, a, i*) where the values of *c, d, a, i* were determined by the research team based on the descriptions of the categories Curricular Content, Discourse, Anticipatory, Implementation and the glossary description of the category represented by the column. The possible values for *c, d, a,* and *i* were 1 if a particular kind of knowledge was present in the observable category and a value of 0 otherwise. Research team members spent a significant amount of time on coming to a consensus on implementation knowledge and trying to understand how it actually gets demonstrated in the classroom. One challenge in defining this particular type of knowledge is that the other three are interwoven with it so much that at times it can be difficult to "tease apart" implementation knowledge from say anticipatory knowledge. After much discussion, we began to understand that implementation knowledge had to meet **both** criteria given in the definition by Hauk, Jackson, and Noblet (i.e., satisfying only one of the two pieces was not enough). This categorical inductive coding left some of the columns without non-zero alignment to any PCK codes. In reviewing what was left uncoded, it was apparent that all of these were related to the establishing of sociomathematical norms. One such example is the column titled "instructional time is spent on mathematics (>75%)" in which a segment being marked as PA indicates nothing in particular about a teacher's knowledge of teaching mathematics, but rather indicates something about what the teacher and students treat as acceptable time to spend on mathematics instruction, fitting Yackel and Cobb's (1996) classic definition of sociomathematical norm. One particularly interesting column titled "encourages diverse mathematical competencies" has a unique feature: we determined that this column loaded heavily on PCK by assigning it a quadruplet of (1,1,1,1) (and hence having all four components of PCK) as well as being a sociomathematical norm. The item loads in discourse because of the communication about the mathematics that occurs between a teacher and student or among students when the item is present, and it loads on curricular content knowledge as a teacher must know about the connections among different procedures and solutions that students may use in solving problems. This previous statement also shows that a teacher will demonstrate anticipatory knowledge in this item's presence as she must be aware of how the students may interact with the problem at hand in order to encourage the competencies (i.e. curricular content and anticipatory knowledge overlap for this category). The teacher then uses her curricular content and anticipatory knowledge to adapt her teaching in response to the diverse competencies that arise as well making choices for her instruction in encouraging these competencies, thereby demonstrating her implementation knowledge. The item is also a SMN since the presence of the item in a segment is illustrative of a shared meaning between teacher and student of what diverse competencies in the classroom are.

Results

As indicated above, the research is currently in the data analysis stage, which will be complete by January, and a summary of results will be offered at the conference.

Questions (a handout of the protocol will be provided to the audience)

1. If the goal of observation of teaching is basic research about the nature of teacher enactment of PCK and SMN for secondary mathematics instruction, what are the implications of the realities of classroom practice for the revision of the protocol?

2. If the goal of the observation is evaluation of the impacts of a professional development (PD) program in which the teacher has been participating (PD focused on PCK growth), what are the audience suggestions for the revision/streamlining of the protocol?
3. How might such a protocol be used to help pre-service and practicum teacher candidates to think about and prepare for teaching?

References

Hauk, S., Jackson, B., Noblet, K. (2010). No Teacher Left Behind: Assessment of secondary mathematics teachers' pedagogical content knowledge. *Proceedings for the 13th Conference on Research in Undergraduate Mathematics Education (electronic)*. PDF available at sigmaa.maa.org/rume/crume2010/Archive/HaukNTLB2010_LONG.pdf

Hill, H. C., Ball, D. L., & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education, 39*, 372-400.

Learning Mathematics for Teaching (2006). *A Coding rubric for Measuring the Mathematical Quality of Instruction* (Technical Report LMT1.06). Ann Arbor, MI: University of Michigan, School of Education.

Yackel, E. & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for research in Mathematics Education*, 27, 458-477.

Navigating the Implementation of an Inquiry-Oriented Task in a Community College

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Key Words: teaching, symmetry, community college, mathematical discourse

Abstract: Teachers implementing inquiry-oriented, discourse-promoting tasks can face a number of challenges (Speer & Wagner, 2009; Ball, 1993). In this study we will examine the challenges faced by two community college instructors as they implement such a task in a “transition to proof” course. In this task students initially use their informal ideas of symmetry to develop a criteria to quantify the symmetry of six figures (see Larsen & Bartlo, 2009), these criteria are then formalized into definitions for symmetry and equivalent symmetries. During this task a number of conflicts arise, and to resolve these conflicts the students engage in rich mathematical discourse. While this task and ensuing discourse offer opportunities for learning mathematics, they also offer significant challenges for effective implementation. We aim to identifying these challenges and the ways in which these challenges were navigated as the class worked towards formal definitions of symmetry and equivalent symmetries.

While working on a project aimed to develop a community college “transition to proof” course, bases on an inquiry-oriented abstract algebra curriculum, we began to wonder what sort of challenges the community college instructors would face as they navigated the curriculum. In order to begin looking at this question we decided to focus our attention on an inquiry-oriented task in which the students reinvent and define the concepts of symmetry and equivalent symmetries.

In this task students are initially given six shapes (see figure below) and are asked to arrange the figures from least to most symmetric. The students work on this task individually and then in small groups prior to a whole class discussion. The groups share how they ordered the figures and how they came to that decision. The students are then asked to determine a way to quantify the symmetry of each figure and, using their quantification criteria, the groups rank the figures and present both their criteria and their ranking to the whole class. Following these presentations the groups work to develop both a definition of what a symmetry is and what makes two symmetries equivalent (see Larsen & Bartlo, 2009).

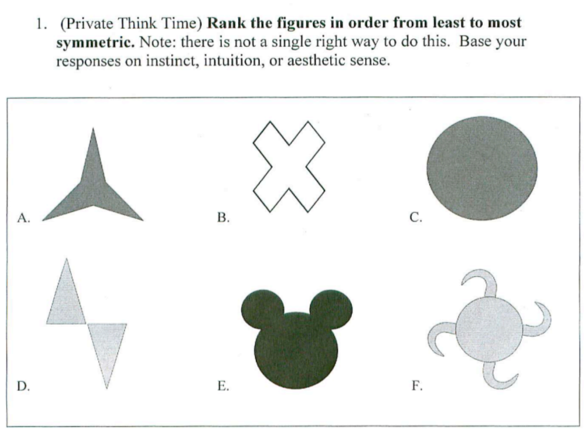

Fig. 1 Symmetry Task Launch

This task typically promotes rich mathematical discourse (Larsen & Bartlo, 2009) as the students work to resolve ranking discrepancies and develop definitions. Common points of contention are whether "doing nothing" should be counted as a symmetry, if a symmetry is a property of a shape or a function acting on the shape, and if symmetries are equivalent when they produce the same result or when the action done to the figure is the same (essentially making equality the criteria for equivalence).

By engaging in rich mathematical discourse, including questioning, challenging, and justifying, students can learn what it means to do mathematics (Stein, 2007). However, managing and facilitating student discourse comes with an array of challenges, such as respecting students as mathematical thinkers, even when their ideas are not in alignment with standard mathematics (Ball, 1993), and providing analytic scaffolding during whole class discussions to move the mathematical agenda forward (Speer & Wagner, 2009).

While such challenges have been documented and analyzed at the elementary and undergraduate level, in this study we are looking at how these challenges are addressed and negotiated by community college instructors, specifically related to this symmetry task. Through our analysis we aim to answer the following research questions:

1) What challenges do the two community college instructors we were working with face as they facilitate this discourse-prompting task?
2) How did these two community college instructors navigate and manage these challenges?

To answer these questions video data of the classrooms of the two community college instructors will be analyzed through iterative stages (Lesh & Lehrer, 2000). This analysis will focus on how the instructors implemented this task and facilitated the ensuing class discussion. Specifically, we will look at how the classes arrived at their formal definitions of symmetry and equivalent symmetries and how instructors use student thinking to formalize these definitions.

Questions for the Audience:

We had considered using data from a university professor's introduction to group theory classroom for comparison purposes.

- Would in make sense to compare how this task unfolded given the different student populations?
- Would it make sense to compare the two different teaching populations?
- If so, how might we compare between community college instructors and university professors in a way that does not cast the university instructor as the 'expert'?

References

Ball, D. L. (1993). With an Eye on the Mathematical Horizon: Dilemmas of Teaching Elementary School Mathematics. *The Elementary School Journal 93*(4), 373-397.

Larsen, S., & Bartlo, J. (2009). The Role of Task in Promoting Discourse Supporting Mathematical Learning. In L. Knott (Ed.), *The Role of Mathematics Discourse in Producing Leaders of Discourse* (pp. 77-98): Information Age Publishing

Lesh, R., & Lehrer, R. (2000). Iterative Refinement Cycles for Videotape Analysis of Conceptual Change. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of Research Design in Mathematics and Science Education* (pp. 992). Mahwah, NJ: Lawrence Erlbaum Associates.

Speer, N. M., & Wagner, J. F. (2009). Knowledge Needed by a Teacher to Provide Analytic Scaffolding During Undergraduate Mathematics Classroom Discussions. *Journal for Research in Mathematics Education, 40*(5), 530-562.

Stein, C. C. (2007). Let's Talk: Promoting Mathematics. In D. Weaver, T. Dick & N. Rigelman (Eds.), *Assessing the Quality and Quantity of Student Discourse in Mathematics Classrooms*. Portland, OR: RMC Research Corporation

Linking Instructor Moves to Classroom Discourse and Student Learning in Differential Equations Classrooms

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In undergraduate mathematics classrooms where instructors are beginning to focus on more student-centered instruction, teachers' moves foster mathematical discourse among students and teachers as a way to further the mathematics. While some are studying these teacher moves in K-12 classrooms, there seems to be little research focusing on this in the university classroom. We define a pedagogical content move to be a discursive or inscriptive act by an instructor that is purposely used to promote or further the mathematical agenda in the classroom (Lee, Keene, Lee, Holstein, Early& Ely, 2009). In an earlier paper, we presented several of these moves as identified in our data collection and analysis. In this proposal, we further this research by answering the question:

What specific links can be described between university professors' instructional moves and the discourse and learning in a classroom about one particular mathematical concept?

We have chosen parametric curves as the specific content to embed our work for two reasons. First of all, it is an overarching and important mathematics concept which appears in mathematics from precalculus through university level mathematics analysis. Specifically, in differential equations, as students learn how to find solutions, they are often represented as parametric equations and visualized as curves in two or three dimensions. Secondly, the authors have previously reported on research about how student come to visualize curves that are parameterized over time (Keene, 2007).

Literature Review

Parameter and parmetric curves. We define the concept of "parametric curves" to be representations in 2 or more dimensions of functions defined by two or more equations with the same independent variable. Often in differential equations this variable is time, but it is not a requirement to be a parametric curve. Research about student learning of parameter and parametric equations is limited. Student understanding of parameter was studied by Drijver (2001) who discusses how students understand parameter as place holder, changing quantity, and as generalizer. Keene (2007) also discusses the notion of parametric reasoning with time as the dynamic parameter. She provides the notion that parametric reasoning includes students' making time an explicit quantity, using and connecting qualitative and quantitative reasoning, and imagining the motion. Engelke (2007) introduces a framework for student understanding of related rates (of change), which closely links to the idea of parameter.

The idea that parametric curves are important to many areas of advanced mathematics has not led to significant research in their understanding. Some publishing appears about how to teach parametric equations using technology (Drivjers, 2001) but how students learn them in a classroom situation is missing.

Teacher pedagogy, discourse, and student understanding at the undergraduate level. Some prior research has begun to focus on pedagogical issues related to mathematics instruction at the undergraduate level. For example, in studying the implementation of the same differential equations curriculum materials, Wagner et al (2007) analyzed the specific problems a professor encountered in facilitating mathematical discussions. The professor in that study had taught DEs for many years from a traditional perspective and was new to inquiry-oriented instruction. In particular they found that the professor struggled to respond to unexpected student responses during whole class discussions. This is similar to the work of Bartlo et al (2008) that shows that the mathematics knowledge that a professor brings to an abstract algebra classroom is broad in certain ways but that there are pedagogical situations when building content connections and understanding student thinking is a challenge.

Additionally, other researchers have focused on discourse practices and moves in the mathematics classroom in K-12, as well as some at the university level. While general practices such as telling or revoicing have been carefully analyzed for their effects on the mathematical discussion (e.g., O'Conner & Michaels, 1993) most do not focus on the ways in which the instructor draws upon specific content knowledge when making a discursive move. Rasmussen, Marrongelle, & Kwon (2008) have developed an IODM (Inquiry Oriented Discursive Move) framework to analyze mathematical discourse. We are interested in using and modifying this framework to identify and analyze moves used by an instructor to introduce such tools and the mathematical content understandings that drive the move, specifically in terms of parametric equations and their representations as curves.

Methodology

Data collection was conducted in Spring 2008 in a college level Differential Equations class in the southeastern United States (enrollment of 25) using a classroom teaching experiment methodology (Cobb, 2000). Most students in the class were mathematics, science, or engineering majors, had finished Calculus III, and about one third of the students had taken at least one prior course with this particular mathematics professor. The professor had been using inquiry-oriented strategies in his other courses (e.g., Abstract Algebra, Mathematical Reasoning) for several years, but had only taught Differential Equations once about 7 years prior and was implementing an inquiry-oriented differential equations materials (Rasmussen, 2003) for the first time that semester. Prior to many teaching sessions, the professor met with one of the researchers to discuss the material to be taught and make a planned trajectory. They also met immediately after class for debriefing sessions to reflect on the lesson and discuss any issues or questions that arose that may affect the content and teaching strategies used for the next class.

The class was designed to be student centered and inquiry-oriented with each class session involving cycles of learning: whole class discussion, followed by small group discussion, followed by whole class discussion. The learning environment of the classroom established by the professor required students to discuss the mathematics they were learning, express their own ideas, and make sense of, and agree or disagree with others' ideas.

The data used for analysis for this paper was drawn from the videotaped class episodes, field notes from a non-participant observer, video/audio taped debriefing sessions held immediately after class and student work. To begin our analysis, we reviewed videotapes and field notes of class sessions throughout the semester. We identified episodes (short periods of classroom video) where it was noted that the class was discussing ideas about parametric equations, time as a parameter, graphing of parametric equations as curves or related ideas. Once these episodes where identified, we used a coding scheme that was both top-down (Miles and Huberman, 1994) and generative in nature (Strauss & Corbin, 1990). It was top-down in the sense that we used research in prior literature (Rasmussen, Marrongelle, & Kwon, 2008; Whitacre & Nickerson, 2009) to identify instances where the instructor was initiating a conversation, possibly using one of the typical discursive moves such as telling or questioning, interjecting something in a conversation using revoicing or using pedagogical content moves (Rasmussen & Marrongelle,2006;). The coding was generative in nature as we created and used codes of ways the instructor was drawing upon his own knowledge and what discourse the students and teacher participated in. After identifying these episodes, we used a comparison method to establish links in teacher moves and the discourse. One assumption we made in the analysis is that discourse is one lens on student thinking and that communication is thinking (Pea, 1993; Lampert & Cobb, 2003). After the links were established, we triangulated the analysis results with students' work we collected.

Results

The results of the analysis are not finished at this time; currently, we have linked at least two of the teacher moves to the discourse and student reasoning. By the conference, we plan to have more evidence to support these linkages and others.

First, the teacher focuses and uses student ideas and builds upon them in ways that allows the students in the classroom to understand. For example, if a student mentions in classroom conversation that they remember $x=f(t)$ and $y=g(t)$ when asked if they know about parametric equations, but cannot remember what they mean, then the teachers brings that idea to the front of the class (either himself or the student may speak, either might be appropriate). Because he knows that it is from a student, he then asks questions to either small groups or the whole class to elicit ideas. He then creates and asks a question (the teacher move) that engages students in thinking about this so they can reconstruct understandings and participate in the discourse around the concept.

Second, the teacher focuses on eliciting ideas from students that will allow them to build up their mathematical habits of mind. These habits of mind for this particular teacher involve developing an intuition to recognize when mathematical ideas are present in the current mathematical agenda that can connect on concepts from their earlier learning. We provide examples of this in detail and other results of the analysis.

Implications

By identifying one particular mathematical content strand that weaves through many areas of mathematics, this research is a good model for those interested in finding ways to strengthen student understandings across mathematics as a discipline. Additionally, offering ways that teachers can make explicit pedagogical moves in a university level classroom, whether it be student centered or more teacher centered, provides new ways to improve mathematics teaching at the undergraduate level. For example, if mathematics instructors think about specific ways they can order student answers that allows discourse and reasoning to move forward mathematically, this could be an important area for future research and professional development.

Additionally, another area for future research is pointed to by this report. Mathematicians are interested in how to assure that mathematics majors at the university have long lasting understandings that span the curriculum. If thinking deeply about one particular topic, and ways that teachers can support learning of that, is useful, then researchers may be able to use the technique in other mathematical conceptual areas.

References

Bartlo, J., Larsen, S., & Lockwood, E., (2008). Scaling up instructional activities: Lessons learned from a collaboration between a mathematician and a mathematics education researcher. In *Proceedings for the Conference on Research inUndergraduate Mathematics Education*. San Diego, CA. Available online at http://www.rume.org/crume2008/Bartlo_LONG.pdf.

Cobb, P. (2000). Conducting teaching experiments in collaboration with teachers. In E. Kelly & R. Lesh (Eds.), *Handbook of research design in mathematics and science education.* (pp. 307-334). Mahwah, NJ: Erlbaum

Cuoco, A., Goldenberg, E. P., & J. Mark. (1997). Habits of mind: an organizing principle for mathematics curriculum. *Journal of Mathematical Behavior*, 15(4), 375-402.

Drijvers, P. (2001). The concept of parameter in a computer algebra environment. In Chick, H., Stacey, K.,Vincent, J. & Vincent, J. (Eds.), *Proceedings of the 12th ICMI study conference the future of the teaching and learning of algebra, Vol 1* (pp. 221-227). Melbourne: The University of Melbourne.

Engelke, N. (2007). *Students' understanding of related rates problems in calculus. Published doctoral dissertation*. (doctoral dissertation, Arizona State University).

Keene, K.A. (2007) A Characterization of dynamic reasoning: Reasoning with time as parameter, *Journal of Mathematical Behavior*,26. 230-246.

Lampert, M., & Cobb, P. (2003). White paper on communication and language. In J. Kilpatrick, D. Shifter, & G. Martin (Eds.), *Principles and practices of school mathematics: Research companion volume* (pp. 237-249). Reston, VA: National Council of Teachers of Mathematics.

Lee, H. S., Keene, K. A., Lee, J. T., Holstein, K., Early, M.E., Eley, P. (2009). Pedagogical content moves in an inquiry-oriented differential equations class: Purposeful decisions to further mathematical discourse. In Proceedings of the *Annual Conference on Research in Undergraduate Mathematics Education conference*, Raleigh, NC.

Miles, M. B., & Huberman, A. M. (1994). *Qualitative data analysis: An expanded sourcebook.* Thousand Oaks, CA: SAGE Publications.

O'Connor, M. C., & Michaels, S. (1993). Aligning academic task and participation status through revoicing: Analysis of a classroom discourse strategy. *Anthropology Education Quarterly*, 24, 318–335.

Pea, R. D. (1993). Practices of distributed intelligence and designs for education. In G. Solomon (Ed.), *Distributed cognition* (pp. 47-87). New York: Cambridge University Press.

Rasmussen, C. (2003). *Differential equations instructional materials*. Non-published materials.

Rasmussen, C. & Marrongelle, K. (2006). Pedagogical content tools: Integrating student reasoning and mathematics in instruction. *Journal for Research in Mathematics Education, 37*. 388-420.

Rasmussen, C., Kwon, O. N., & Marrongelle, K. (2008). *A framework for interpreting inquiry-oriented teaching*. Proceedings of the Eleventh Special Interest Group of the Mathematical Association of America on Research in Undergraduate Mathematics Education.

Strauss, A. & Corbin, J. (1990). *Basic of qualitative research: Grounded theory procedures and techniques.* Thousand Oaks, CA: SAGE Publications.

Wagner, J., Speer, N. & Rossa, B. (2007). Beyond mathematical content knowledge: A mathematician's knowledge needed for teaching an inquiry-oriented differential equations course. *Journal of Mathematical Behavior, 26*. 247-266.

Whitacre, I. & Nickerson, S. (2009). *Measuring Inquiry-Oriented Teaching in the Context of TA Professional Development.* Proceedings of the Twelfth Special Interest Group of the Mathematical Association of America on Research in Undergraduate Mathematics Education.

Understanding and Overcoming Difficulties with Building Mathematical Models in Engineering: Using Visualization to Aide in Optimization Courses.

Rachael Kenney, Nelson Uhan, Ji Soo Yi, Sung-Hee Kim, Mohan Gopaladesikan, Aiman Shamsul and Amit Hundia

Introduction

Operations research—and in particular, optimization—is one of the key courses in many universities' engineering curricula. An optimization model (or mathematical program) is a mathematical representation of a decision-making problem, consisting of variables that reflect the decisions to be made, and an objective function to be minimized or maximized, subject to a set of mathematical constraints on the variables. Formulating a valid optimization model from a verbal description of a decision-making, problem is perhaps the most important skill taught in an optimization course aimed at undergraduate students, since excellent modeling skills are vital to putting optimization techniques into practice. However, though undergraduate engineering students have been engaging in modeling activities (i.e., mathematical "word" or "story" problems) since elementary school, many students find it difficult to learn how to build good optimization models. Many educators in operations research anecdotally report this phenomenon (e.g., Sokol 2005), but little work has been done on systematically understanding why optimization modeling is such a difficult skill to learn and how such insights can lead to effective modeling pedagogies. By effectively teaching optimization modeling skills, we can provide our students with a powerful set of tools that can help solve important, complex problems in engineering, mathematics, and management.

The objective of this study is to help undergraduate engineering students overcome their difficulties in optimization modeling by

- determining and understanding commonly made mistakes in optimization modeling;
- developing a visual, web-based environment that teaches students to formulate valid and tractable optimization models; and
- evaluating the effectiveness of the developed visual, web-based environment on learning modeling in optimization.

This preliminary proposal is intended to share work completed on the first two objectives and to generate discussions to help us better conceptualize the next stages of our project.

Literature Review

A modeling approach to teaching in engineering or mathematics puts the focus in problem solving on creating a system of relationships that is generalizable and reusable (Doerr & English, 2003). Contemporary approaches to solving mathematical story problems have emphasized the need for a proper conceptual understanding of the problem. However, the factors that inhibit such conceptual understanding are quite complex. Lucangelli, Tressoldi, and Cendron (1998) suggest that problem solving with modeling problems is more difficult than solving algorithms because it requires (a) comprehension of the text, (b) ability to visualize the data provided, (c) capacity to recognize the underlying structure, (d) ability to correctly sequence solution activities, and (e) ability to evaluate the procedures used. These skills are especially important when solving college-level word problems in engineering where the problem complexity is often increased, contributing to learners' difficulties with problem solving (Jonassen, 2000).

The ability to translate from one representation of a mathematical problem to another is critical to the problem solving process (Janvier, 1987). However, it has been shown that even after several years of schooling in algebra or calculus, students often cannot engage successfully in this translation (Clement, Lochhead, & Monk, 1981; Clement, 1982; Janvier, 1987; Arcavi, 1994). Learners need a way of "developing a cognitive representation of information in the story" (Jonassen, 2000, p. 79). That is, in order to be successful, problem solvers must have an accurate mental representation of the pattern of information indicated by the story problem (Hayes & Simon, 1976; Riley & Greeno, 1988; Jonassen, 2000).

Researchers have discussed the role of visual diagrams as a conceptual tool that promote students' construction of flexible and applicable concept images that allow for flexible problem solving and connection making (e.g. Dreyfus, 1994; Koedinger, 1994; Larkin & Simon, 1987). According to Dreyfus (1994), computer-designed diagrams can be thought of as cognitive tools that make it possible to represent mathematics with an amount of visual structure that we cannot readily achieve with any other medium. Koedinger (1994) has identified emergent properties of diagrams that that make them superior to a linear representation of information for many learning and reasoning activities. For example, they provide the potential for students to recognize relationships that may have otherwise gone unnoticed in a verbal or symbolic representation. This supports earlier findings from Larkin and Simon (1987) who identified a diagrams' superiority to verbal problem descriptions due to their usefulness for grouping together all useful information and for supporting a large number of perceptual inferences. Koedinger (1994) suggests that students are more practiced in relying on perceptual inferences than the corresponding symbolic inferences, making the former often seem easier for the learner.

Methodology and Preliminary Findings

One end-goal for our study is to develop a visualization tool that can aid in modeling. Before fully developing this tool, we first need to better understand students' experiences and practices when solving optimization modeling problems (specifically linear programming problems) and to identify common errors that a visualization tool could help correct. The following sections outline our procedures and findings for the first three phases of our work.

Taxonomy of Optimization Modeling Word Problems

As a first step, we looked at five optimization textbooks (Hillier and Lieberman 1995, 2001; Rardin, 1997; Srinivasan, 2007, Winston, 1994) to determine their categorizations of different linear programming models. After comparing these categorizations, we developed a preliminary, unified taxonomy of word problems, based on the types of constraints a problem requires (i.e., the *constraint patterns*). Then, we tested the validity of this taxonomy by solving approximately 35 word problems from the different textbooks and examining how each problem fit into our taxonomy. Throughout this process, we discovered that some constraint patterns needed to be more specific, and so we revised our taxonomy accordingly.

Our current version of the constraint pattern taxonomy consists of five categories: (a) *composition constraints* (indicated by terms such as "meets", "has only", and "more than"); (b) *balance constraints* (e.g. "Each *A* requires *x* number of…"); (c) *ratio constraints* (often includes a mixture of *A* and *B*); (d) *pattern-covering constraints* ("*x* people work this type of shift"); and (e) *time-based constraints* (e.g. investment problems). By identifying and categorizing the different types of constraints, we propose that we may be able to develop a more generalizable method for formulating mathematical models across all problem types.

Taxonomy of Common Student Errors

To identify students' common errors and difficulties with modeling problems, we analyzed three sets of quizzes (one question each) from two sections of an optimization course and three sets of similar word problems, given to students on their final exams in three different semesters. We first studied each response and recorded the specific errors each response contained, keeping track of similar errors between students. We then categorized these errors broadly, depending on where they appeared in the model: the decision variables, the objective function, or the constraints. During this process, we also kept track of summary statistics for each type of error.

The analysis of students' responses showed mistakes on 84% of the 374 total responses analyzed. We identified five categories for the taxonomy of mistakes: (a) *mathematical notation errors* indicate errors associated with, for example, missing or having too many summation signs, or reversing indices; (b) *comparison errors* indicate mistakes in the direction of the inequality sign; (c) *flow errors* usually occur when multiple statements relate to one constraint but students forget to take that into consideration; (d) *missing information errors* suggest a student ignored the type of constraint construction, such as a profit function equation where you need the revenue and cost equations; and (e) *decision variable errors* included missing decision variables from the objective function or constraints, replacing a decision variable with some other variable, or using incorrect parameters. We found that the majority of mistakes fit into either the mathematical notation error (25.40%) or decision variable error (20.32%) categories. Comparison errors were found least (4.55%). Our current data on student errors comes from students' work on only one type of constraint pattern – composition constraints. As our work continues, we will collect and analyze data related to other types of constraint patterns.

Development of a Visualization Tool

Based on these taxonomies, we designed a preliminary visualization scheme that could help students gain a better conceptual understanding of optimization modeling problems and that could diminish the types of mistakes typically made on these problems. We have considered several visualization types, including node-link diagrams, tables, and timeline diagrams, by solving different word problems using these visualizations. After reviewing a number of different types of problems and student work, we found that node-link diagrams provided a possible basis for a robust visualization scheme to represent the conceptual ideas in a wide range of word problems. We are currently developing a prototype of an interactive visualization web tool (shown in Figure 1) based on our investigation of students work on a composition constraint pattern problem. The tool is intended to guide the students by letting them interact with the question (given in written form at the top) by allowing them to form node-link diagrams that represent their conceptual understanding of the problem. This is to help students understand the flow of the problem and identify the constraints available in the question.

At this point, our data collection has only included quantitative data from textbooks and student work samples. This data has provided us with an informative view of students' experiences in modeling in linear programming, but it is incomplete. In the next stages of our project, we will conduct qualitative interviews to further understand difficulties in modeling, begin to test our prototype of the visualization tool with students, and expand the tool to include additional feedback capabilities and to handle several different constraint pattern problems.

Questions for Consideration

Our team would be interested in discussing the following questions during the conference:

1. What approaches should we use to investigate the underlying causes of the mistakes that many students make in order to inform the design of our visualization tool?
2. How can we most effectively study the usefulness of the tool with students?
3. How can we ensure that the skills that the students learn through our tool (if any) are generalizable beyond the types of problems for which the tool is designed?
4. What are best practices for incorporating an interactive learning tool into a traditional lecture-driven course? What would make such a tool appealing to other instructors?

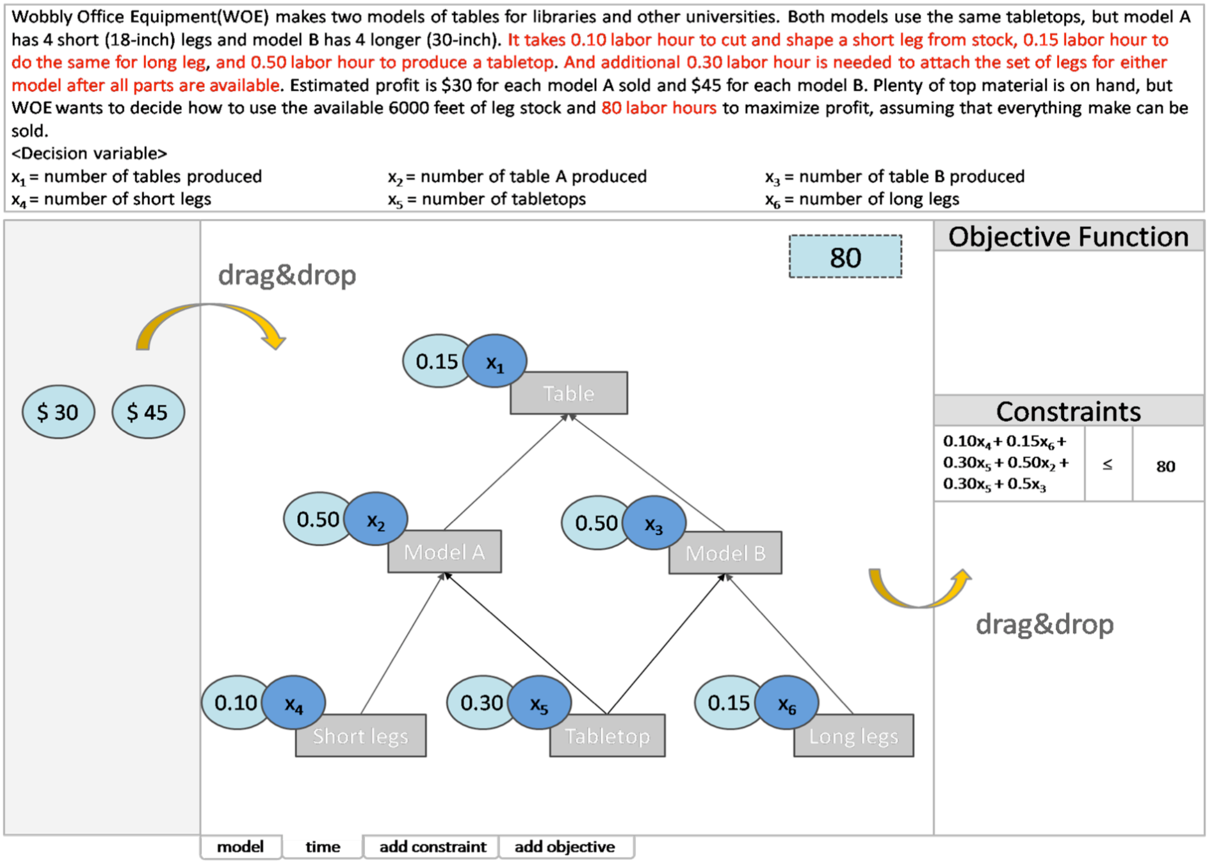

Figure 1. Screenshot of the visualization tool prototype

References

Arcavi, A. (1994). Symbol sense: Informal sense-making in formal mathematics. For the Learning of Mathematics, 14, 24–35.

Clement, J. (1982). Algebra word problem solutions: Thought processes underlying a common misconception. Journal for Research in Mathematics Education, 13, 16–30.

Clement, J., Lochhead, J., & Monk, G. (1981). Translation difficulties in learning mathematics. American Mathematical Monthly, 88, 286–290.

Doerr, H. M., & English, L. D. (2003). A modeling perspective on students' mathematical reasoning about data. Journal for Research in Mathematics Education, 34, 110–136.

Dreyfus, T. (1994). The role of cognitive tools in mathematics education. In R. Biehler, R. Scholz, R. Strasser, & B. Winkelmann (Eds.) Didactics of mathematics as a scientific discipline. Dordrecht: Kluwer.

Hayes, J. R., & Simon, H. A. (1976). The understanding process: problem isomorphs. Cognitive Psychology, 8, 165–190.

Hillier, F. S., Lieberman, G. J. (1995). *Introduction to Mathematical Programming*. 2nd edition. McGraw-Hill.

Hillier, F. S., Lieberman, G. J. (2001) *Introduction to Operations Research*. 7th edition. McGraw-Hill.

Janvier, C. (1987). Translation processes in mathematics education. In C. Janvier (Ed.) Problems of representation in the teaching and learning of mathematics, (pp. 27–32). Hillsdale, NJ: Lawrence Erlbaum.

Jonassen, D. H. (2000). Toward a design theory of problem solving. Educational Technology, Research and Development, 48, 63–85.

Koedinger, K. R. (1994). Emergent properties and structural constraints: advantages of diagrammatic representations for reasoning and learning. In B. Chandrasekaran, & H. Simon (Eds.) Reasoning with diagrammatic representations, (pp. 151–156). Menlo Park, CA: AAAI Press.

Larkin, J., & Simon, H. (1987). Why a diagram is (sometimes) worth ten thousand words. Cognitive Science, 11, 65–69.

Lucangelli, D., Tressoldi, P. E., & Cendron, M. (1998). Cognitive and metacognitive abilities involved in the solution of mathematical word problems: Validation of a comprehensive model. Contemporary Educational Psychology, 23, 257–275.

Rardin, R. L. (1997). *Optimization in Operations Research*. Prentice Hall, Upper Saddle River, NJ.

Riley, M. S., & Greeno, J. G. (1988). Developmental analysis of understanding language about quantities and solving problems. Cognition and Instruction, 5, 49–101.

Sokol, J. (2005). Teaching O. R. modeling. OR/MS Today, December 2005.

Srinivasan, G. (2007). *Operations Research: Principles and Applications*. Prentice Hall of India.

Winston, W. L. (1994) *Operations Research: Applications and Algorithms*. 3rd edition. Dux bury Press.

Student Approaches and Difficulties in Understanding and Using of Vectors

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Abstract

A configuration of vector representations based on multiple representation, cognitive development, and mathematical conceptualization, to serve as a new unifying framework for studying undergraduate student approaches and difficulties in understanding and using of vectors is proposed. Using this configuration, the study will explore 5 important transitions, 'physics to mathematics', 'arithmetic to algebraic', 'analytic to synthetic', 'geometric to symbolic', 'concrete to abstract', and corresponding student difficulties along epistemological and ontological axes. As a part of validation of the framework, a study on undergraduate students' approaches and difficulties in understanding and using of vectors with both quantitative and qualitative methods will be introduced, and we will see how useful this new framework is to analyze student approaches and difficulties in understanding and using of vectors.

Keywords: Vector, Representation, Vector Representation, Undergraduate Mathematics Education

Introduction

Undergraduate students usually experienced vectors in school physics and school mathematics. When students study undergraduate mathematics, they see vectors again in multivariable calculus, linear algebra, abstract algebra and geometry courses. (Figure 1) Some students see vectors in introductory physics or engineering courses while they are studying vectors in mathematics. Although undergraduate students' experiences with the concept of a vector varied, students still have difficulties in understanding and using vectors in various situations. In this research, we are going to explore the following: (1) constructing a framework to analyze student approaches and difficulties in understanding and using of vectors, (2) classifying approaches and difficulties, (3) seeing how much one approach prevails over the others in student thinking and in school and undergraduate mathematics curricula, and (4) locating the sources of student difficulties.

Root of a Theoretical Framework

Most of the studies about multiple representations are centered on the concept of a function (Janvier, 1987; NCTM, 2000). Unlike the representations of a function, vector representations have a hierarchy and are strongly dependent on the contexts of given questions. To grasp what student approaches and difficulties are in understanding and using of vector representations, many different contexts and different levels of sophistication should be considered. Many mathematics teachers and professors already knew student difficulties from their experience of teaching. However, those difficulties are not classified systematically and they are very scattered and isolated. As Tall (1992) mentioned, "the idea of looking for difficulties, then teaching to reduce or avoid them, is a somewhat negative metaphor for education. It is a physician metaphor - look for the illness and try to cure it. Far better is a positive attitude developing a theory of cognitive development aimed at an improved form of learning." To have more *positive attitude*, we need to have more deeper understanding of student approaches and difficulties on vector representations to the level of the theory of cognitive development.

Most studies about vectors are from physics point of views related with physical quantities and by physics educators. J. Aguirre and Erickson (1984); J. M. Aguirre (1988); Knight (1995); Nguyen and Meltzer (2003) are just a few of them. Some studies such as Watson and Tall (2002); Watson, Spyrou, and Tall (2003) attempted to analyze student approaches and difficulties on vector representations with more mathematical point of views. However, their studies cover only secondary level mathematics and the transition from physical thinking to mathematical thinking. This brings up a necessity of the new framework for investigating vector concepts that can cover vectors in more advanced and wider levels of undergraduate mathematics as well as in physics and secondary level mathematics.

Student approaches and difficulties in learning and using of vectors in undergraduate mathematics are very complex issues which have not yet definitely resolved. Dorier (2002) brought up these issues and analyzed them with a series of research. However this book placed the focus at linear algebra so that vectors in geometry were covered very briefly. Linear algebra courses are just one of the fields that requires the concept of vectors frequently, but most studies on the concept of a vector so far are regarded as parts of bigger topic research on linear algebra (Dorier, 2002; Harel, 1989; Dorier & Sierpinska, 2001).

Lesh, Post, and Behr (1987) pointed out five outer representations including real world object representation, concrete representation, arithmetic symbol representation, spoken-language representation and picture or graphic representation. Among them, the last three are more abstract and at a higher level of representations for mathematical problem solving (Johnson, 1998; Kaput, 1987). However, in most cases, picture representation is not geometric enough to show geometric structure, and graphical representation does not reflect synthetic geometry point of views but rather analytic.

The problem of vector representations lies not only on the multiple representations but also on the translations. Sfard and Thompson (1994); Yerushalmy (1997) are based on the assumption that students ability to understand mathematical concepts depends on their ability to make translations among several modes of representations. Tall, Thomas, Davis, Gray, and Simpson (1999) analyzed several theories of these. These transitions are referred to as “encapsulation” by Dubinsky (1991) and “reification” by Sfard (1991). The proposed framework tries to reflects this idea of encapsulation or reification not just in symbolic modes of representation but also in geometric modes of representation that has not been studied much along with algebra view point (Meissner, 2001b, 2001a; Meissner, Tall, et al., 2006).

Construction of a Framework to Analyze

In this research, a configuration of vector representations based on multiple representations, cognitive development, and mathematical conceptualization, to serve as a new unifying framework for studying student approaches and difficulties in understanding and using of vectors is proposed. Using this configuration, the study will explore five important transitions, ‘physics to mathematics’, ‘arithmetic to algebraic’, ‘analytic to synthetic’, ‘geometric to symbolic’, ‘concrete to abstract’, and corresponding student difficulties along epistemological and ontological axes. (See figure 2.) As Zandieh (2000) stated in her study on the framework for the concept of a function, “The framework is not meant to explain how or why students learn as they do, nor to predict a learning trajectory. Rather the framework is a ‘map of the territory,’ a tool of a certain grain size that we, as teachers, researchers and curriculum developers, can yield as we organize our thinking about teaching and learning the concept...”, this new framework serves as a ‘map of the territory’. Therefore, with this new framework, we will classify approaches and difficulties, see how much one approach prevails over the others in student thinking and in mathematics curricula, and locate the sources of student difficulties.

Comparison with Other Frameworks

This new framework has some important features. First, it suggests that the interplay between ontological aspect and epistemological aspect is critical in understanding and using of vectors and the key transitions between representations require both ontological and epistemological aspects of understanding simultaneously. Second, it can distinguish and put greater emphasis on difference between analytic geometric representations of vectors and synthetic geometric representations of vectors. It can also distinguish and put greater emphasis on difference between physical representations of vectors and mathematical representations of vectors. Furthermore, it distinguishes/shows/embeds/connects parallel developments of symbolic representations and geometric representations along with cognitive development theories such as reification, or APOS theory. And finally it systematizes the transitions between various representations of vectors.

Research Questions

This research focuses on specific issues arising when representations for vectors are utilized in undergraduate mathematics instruction: (1) What student approaches and difficulties can be identified in understanding and using of vectors?, and (2) How is the students understanding and using of vectors similar to and different from vectors as seen in the written curricula? By proposing the configuration of vector representations, related with the above issues of vector representations, we hypothesize that students difficulties lies on ontological and epistemological jumps in the configuration of vector representations, and students have more difficulties in geometric representations of vectors than symbolic representations at some levels. Hence, the following will be the research questions that we will investigate in this study:(1) What is the theory that explains the process of undergraduate students understanding and using of vectors?, (2) Do students tend to use particular vector representations more?, (3) Do students tend to use vector representations in particular developmental order?, (4) Do different representations of vectors constitute different entities that may not convey the expected vector concepts?

Questions for Discussion

(1) What are the better ways of validating this framework both qualitatively and quantitatively?

(2) Can we think of any philosophical considerations on the framework? (wording issues such as ontological, epistemological, etc.)

(3) What are the views from mathematicians, mathematics educators, physicists?

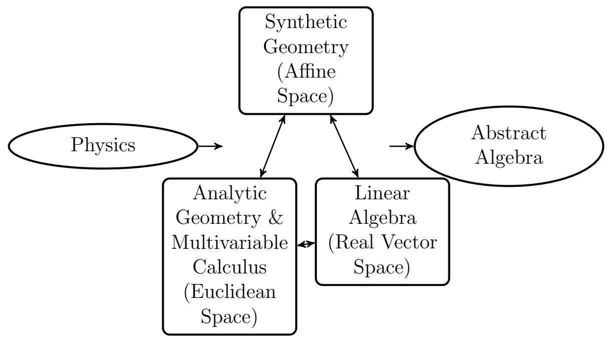

Figure 1. Vectors in Undergraduate Mathematics Curricula

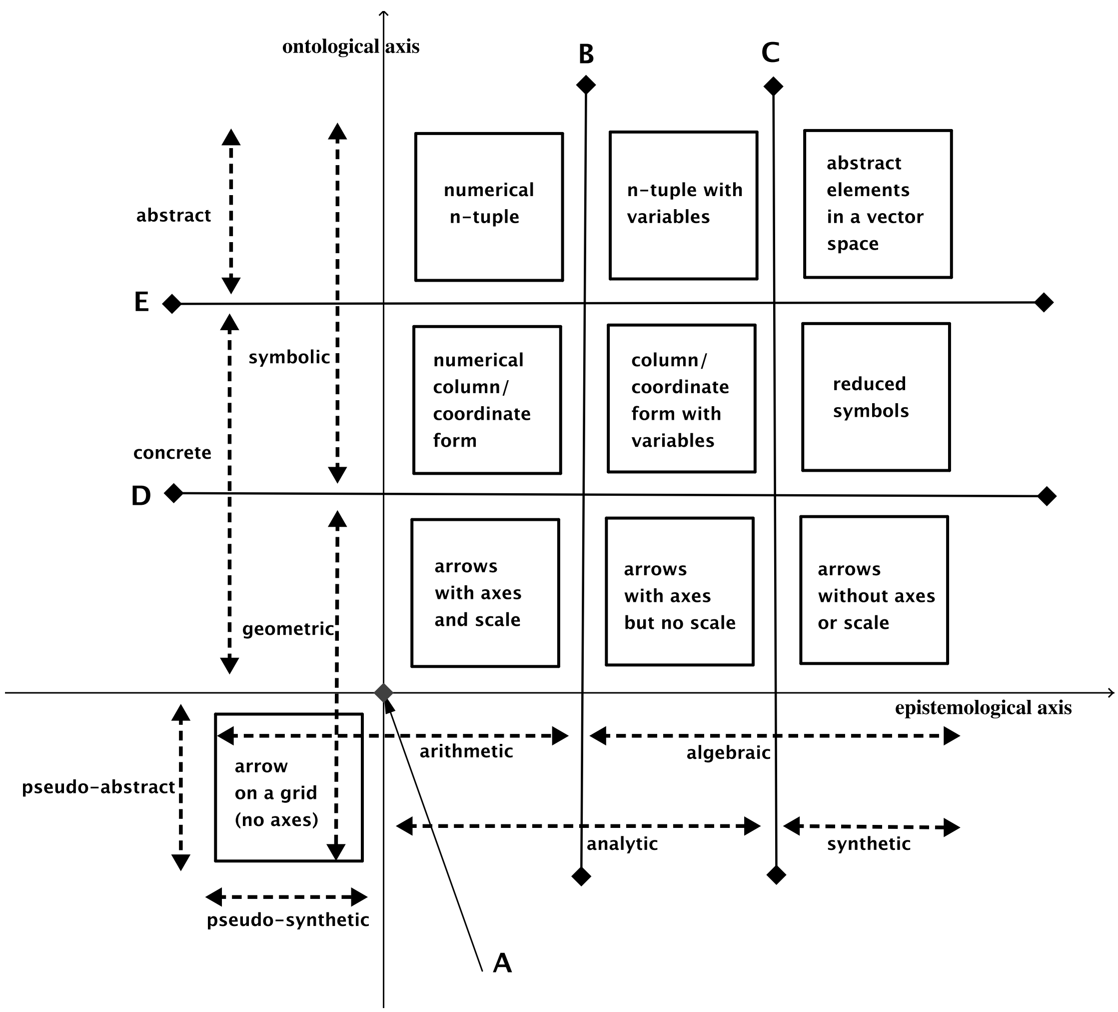

Figure 2. The Configuration of Vector Representations

References

Aguirre, J., & Erickson, G. (1984). Students' conceptions about the vector characteristics of three physics concepts. *Journal of Research in Science Teaching*, *21*(5), 439–457. Available from http://dx.doi.org/10.1002/tea.3660210502

Aguirre, J. M. (1988, April). Student preconceptions about vector kinematics. *The Physics Teacher*, *26*(4), 212–216.

Dorier, J. (2002). *On the teaching of linear algebra*. Available from http://dx.doi.org.proxy1.cl.msu.edu/10.1007/0-306-47224-4

Dorier, J., & Sierpinska, A. (2001). Research into the teaching and learning of linear algebra. *The Teaching and Learning of Mathematics at University Level*, 255–273.

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. *Advanced mathematical thinking*, 95–126.

Harel, G. (1989). Learning and teaching linear algebra: Difficulties and an alternative approach to visualizing concepts and processes. *Focus on Learning Problems in Mathematics*.

Janvier, C. (1987). *Problems of representation in the teaching and learning of mathematics*. L. Erlbaum Associates.

Johnson, S. (1998). What's in a representation, why do we care, and what does it mean? examining evidence from psychology. *Automation in Construction*, *8*(1), 15–24.

Kaput, J. J. (1987). Representation systems and mathematics. *Problems of representation in the teaching and learning of mathematics*, 19–26.

Knight, R. D. (1995). The vector knowledge of beginning physics students. *Physics Teacher*, *33*, 74–74.

Lesh, R., Post, T., & Behr, M. (1987). Representations and translations among representations in mathematics learning and problem solving. *Problems of representation in the teaching and learning of mathematics*, 33–40.

Meissner, H. (2001a). Encapsulation of a process in geometry. *PME CONFERENCE*, *3*, 3–359.

Meissner, H. (2001b). Procepts in Geometry. *European Research in Mathematics Education II*, 58.

Meissner, H., Tall, D., et al. (2006). Proceptual Thinking in Geometry. *RETIREMENT AS PROCESS AND CONCEPT*, 165.

NCTM. (2000). Principles and standards for school mathematics. *Reston, Va.: National Council of Teachers of Mathematics*.

Nguyen, N., & Meltzer, D. E. (2003, June). Initial understanding of vector concepts among students in introductory physics courses. *American Journal of Physics*, *71*(6), 630–638. Available from http://link.aip.org/link/?AJP/71/630/1

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational studies in mathematics*, *22*(1), 1–36.

Sfard, A., & Thompson, P. (1994). Problems of reification: Representations and mathematical objects. In *Proceedings of the xvi annual meeting of the international group for the psychology of mathematics education–north america* (pp. 1–32).

Tall, D. (1992). Students' difficulties in calculus. *Plenary presentation in Working Group 3, ICME*, 1–8.

Tall, D., Thomas, M., Davis, G., Gray, E., & Simpson, A. (1999). What is the object of the encapsulation of a process? *The Journal of Mathematical Behavior*, *18*(2), 223–241.

Watson, A., Spyrou, P., & Tall, D. O. (2003). The relationship between physical embodiment and mathematical symbolism: The concept of vector. *The Mediterranean Journal of Mathematics Education*, *1*(2), 73–97.

Watson, A., & Tall, D. (2002). Embodied action, effect and symbol in mathematical growth. In *Pme conference* (Vol. 4, pp. 4–369).

Yerushalmy, M. (1997). Designing representations: Reasoning about functions of two variables. *Journal for Research in Mathematics Education*, *28*(4), 431–466.

Zandieh, M. (2000). A theoretical framework for analyzing student understanding of the concept of derivative. *Research in collegiate mathematics education IV*, *8*, 103–126.

A Systemic Functional Linguistics Analysis of Mathematical Symbolism and Language in Beginning Algebra Textbooks

Elaine Lande

Abstract: I propose the use of systemic functional linguistics (SFL) as a tool to better understand how mathematical ideas are conveyed through multiple semiotic resources. To demonstrate the tools that SFL offers, mathematical symbols and written language in college beginning algebra textbooks will be examined. I argue that using SFL to research how mathematical content is communicated to undergraduate students can expose important nuances that may otherwise go unnoticed.
Key Words: Beginning Algebra, Language and Mathematics, Mathematical Symbolism, Systemic Functional Linguistics, Textbooks

How well do college beginning algebra textbooks integrate mathematical symbolism and language? To answer this question I will use systemic functional linguistics (SFL) to link the mathematical symbolism to language. To make this connection, two topics will be focused on: the use of the hyphen, as both an operator for subtraction and modifier for the opposite; and the simplification of algebraic expressions. These topics were chosen not only because many students in these courses struggle with them, but because their simplicity can reveal how SFL can be used as an aid for researcher to tease out subtle distinctions in a subject matter that is so clear in their minds that they might otherwise be overlooked.

This research is extended from the work of Kay O'Halloran (i.e. 2000; 2005) which looks at the multisemiotic nature of mathematics through the systemic functional linguistic perspective. My research differs from much of the linguistic research in mathematics education (i.e. Herbel-Eisenmann, 2007; Mesa & Change, 2010; Wagner & Herbal-Eisenmann, 2008) in that it looks at the linguistic nature of the mathematical symbols alongside the use of language with a focus on content.

The choice to examine college beginning algebra textbooks comes from of the lack of research in teaching and learning in college development mathematics (Stigler, Givvin, Thompson, 2010) despite the need, and the potential role of the textbook.

Developmental mathematics

The number of college students needing developmental mathematics is larger than many realize. More than one out of five students entering college are required to take a developmental mathematics course and in two-year public institutions more than one out of every three students needs to take at least one developmental mathematics course (NCES, 2003). Developmental college mathematics courses include arithmetic, beginning algebra, and intermediate algebra, and are labeled developmental or remedial because it is expected that students would have acquired this knowledge in high school or earlier. Compounding this issue, the a majority of students (70%) taking developmental mathematics courses need more than one attempt to pass these courses (Attewell, et al., 2006).

While community colleges are open-access[1] institutions which offer individuals a means of upward social mobility (Cohen & Brawer, 2008), developmental education plays an important role within the colleges by increasing access and equity for underprepared students (Perin & Charron, 2006). Developmental mathematics courses (or the equivalent knowledge) are required for future college courses in science, technology, and engineering, and for those students who intent to transfer—over 90% of four-year colleges have a quantitative component to their general education requirements to obtain bachelor's degree (Lutzer, Rodi, Kirkman & Maxwell, 2007, pg. 64).

Textbooks

I have chosen to examine textbooks because they are a resource for both the student and teacher. The textbook represents part of the intended curriculum that is a source of potential learning for the student (Mesa, 2004) and support for the teacher (Newton & Newton, 2006). While the resources a part-time faculty member or even a full-time faculty member has available varies by individual and college, the textbook is one resource that is always available. As a result the textbook may even guide the content and methods of a course.

There are over 60 textbooks available for beginning algebra at the college level published in a number of different formats[2]. The textbooks I have chosen are based on their use (by number of students and number of colleges) in Michigan community colleges. Within each textbook the sections on subtraction of numbers, negative numbers, order of operations, combining like terms, and simplification of expressions will be identified for analysis.

SFL Tools

So far, I have obtained 5 textbooks and am exploring which SFL tools will be most useful. To date I have identified several SFL tool to use.

To explore the use of the hyphen as subtraction and the opposite of, analyzing cohesion of the text through reference chains, conjunctions, and lexical chains will show how the text uses and develops the hyphen as a symbol and what language accompanies its use. This analysis will make explicit and highlight the two uses of the hyphen, particularly if they are interchanged and how they are related. An important aspect of how I want to explore the cohesion of the text is to analyze not only the mathematical symbolism and the written language, but also how they interact. In my initial examination of the texts, there is often a crossing of the two meanings of the hyphen with out explanation and there are occasions when the mathematical symbols have one meaning but the words express or imply the other.

To explore the order of operations, combining like terms and simplifying expressions looking at rank-shift, in addition to the cohesion of the text is of interest. The notion of rank-shift is of particular interest for these topics because of how lexically dense and highly embedded mathematical symbolism can be and using written or spoken language to describe this can be very difficult.

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[1] Jackson Community College, Jackson, MI has recently moved away from this trend. Any student scoring below a certain grade level in reading, writing, or mathematics will not be admitted to the college.

[2] Only textbooks with different author/title combinations were counted. Textbooks of different editions, packaging (i.e. with solutions manual, online access), bindings or combined course (i.e. beginning and intermediate algebra) were not counted. This was brief survey was done using Amazon and four publisher websites. The publishers were chosen based on those books found in the Amazon search and were Pearson Higher Education, McGraw Hill Higher Education, Cengage Higher Education, and Kendal/Hunt Publishing.

Further Research

The textbook is simply one form of mathematical discourse to be examined. Further research could explore using SFL to parallel the spoken and written language and mathematical symbolism in lectures. It could also be used to compare how these semiotic resources are used by teachers and students.

In presenting my preliminary findings, I will give examples of the different SFL tools I have used and show findings of interest. I seek general input and interpretations of my work so far, what others see as the potential for using these tools, and suggestions for further steps

Questions

How could the SFL tools presented be useful in exploring higher level undergraduate mathematics?

Are there other SFL tools that could be used to further explore the connections in mathematical content between the different semiotic resources?

Does SFL seem like an appropriate tool to explore the similarities and differences between how instructors and undergraduate students use the different semiotic resources? How could research of this sort contribute to our understanding of the differences in how mathematicians and novices (undergraduate students) think about mathematics?

How can these SFL tools be used to better understand how undergraduate students make sense of and make connections between the different semiotic resources?

References

Attewell, P., Lavin, D., Dominia, T., & Levey, T. (2006) New evidence on college remediation. *The Journal of Higher Education*, 77(5):886-924.

Cohen A. M. and Brawer, F. B. (2008). *The American community college*. San Francisco: Josey-Bass.

Herbel-Eisenmann, B (2007). From intended curriculum to written curriculum: examining the "voice" of a mathematics textbook. *Journal for Research in Mathematics Education*, 38(4):344-369.

Lutzer, D. J., Rodi, S. B., Kirkman, E. E., & Maxwell, J. W. (2007). *Statistical abstract of undergraduate programs in the mathematical sciences in the United States: Fall 2005 CBMS Survey*. Washington, DC: American Mathematical Society.

Mesa, V. (2004). Characterizing practices associated with functions in middle school textbooks: An empirical approach. *Educational Studies in Mathematics, 56*:255- 286.

Mesa, V., & Chang, P. (2010). The Language of Engagement in Two Highly Interactive Undergraduate Mathematics Classrooms*Linguistics and Education,* 21:83-100.

National Center for Education Statistics. (2003). *Remedial Education at Degree-Granting Postsecondary Institutions in Fall 2000* (NCES 2004-010). Washington, DC: Parsad, B., Lewis, L., Greene, B.

Newton, D. P., & Newton, L. D. (2006). Could elementary mathematics textbooks help give attention to reasons in the classroom? *Educational Studies in Mathematics,* 64:69-84.

O'Halloran, K. L. (2000). Classroom discourse in mathematics: A multisemiotic analysis. *Linguistics and Education,* 10(3):359-388.

O'Halloran, K. L. (2005). *Mathematical discourse: language, symbolism and visual images*. Continuum: New York.

Perin, D & Charron, K. (2006). Lights just click on everyday. In Bailey, T. & Morest, V. (Eds.), *Defending the Community College Agenda* (pp. 155-194). Baltimore, MD: The Johns Hopkins University Press.

Stigler, J., Givvin, K., Thompson, B. (2010). What community college developmental mathematics students understand about mathematics. *MathAMATYC Educator*, 1(3):4-16.

Wagner, W., Herbel-Eisenmann, B., (2008). “Just don’t”: The suppression and invitation of dialogue in the mathematics classroom, 67:143-157

Student Use of Set-Oriented Thinking in Combinatorial Problem Solving

Elise Lockwood and Steve Strand

This study seeks to contribute to research on the teaching and learning of combinatorics at the undergraduate level. In particular, the authors draw upon a distinction characterized in combinatorial texts between set-oriented and process-oriented definitions of basic counting principles. The aim of the study is to situate the dichotomy of set-oriented versus process-oriented thinking within the domain-specific combinatorial problem-solving activity of students. The authors interviewed post-secondary students as they solved counting problems and examined alternative solutions. Data was analyzed using grounded theory, and a number of preliminary themes were developed. The primary theme reported in this study is that students showed a strong tendency to utilize set-oriented thinking during the problem-solving phase that Carlson & Bloom (2005) refer to as checking, especially when they engaged in the evaluation of alternative solutions.

Keywords: combinatorics, counting, problem-solving, grounded theory

Introduction and Motivation. In spite of the seemingly elementary nature of "counting," students tend to experience a great deal of difficulty as they encounter increasingly complex counting problems. These difficulties are well-documented in the mathematics education research literature (Batanero, Navarro-Pelayo, & Godino, 1997; English, 2005; Kavousian, 2006). Also well-established is the relevance of combinatorics in the K-12 and undergraduate curricula (Batanero, Navarro-Pelayo, et al., 1997; English, 1991; NCTM, 2000), particularly because of its applications in probability and computer science. English (1993) emphasizes the value in studying combinatorics education, noting that "the domain of combinatorics is a particularly fertile field for research in mathematics education" (p. 451). Attempts have been made to improve the implementation of combinatorial topics in the classroom (Kenney & Hirsch, 1991; NCTM, 2000), but in spite of such efforts, students overwhelmingly struggle with understanding the concepts that underpin this growing field. Batanero, Godino and Navarro-Pelayo (1997, p. 182) make the following claim:

> …[C]ombinatorics is a field that most pupils find very difficult. Two fundamental steps for making the learning of this subject easier are understanding the nature of pupils' mistakes when solving combinatorial problems and identifying the variables that might influence this difficulty.

This call by Batanero, Godino et al. acknowledges the difficulties described above, and it also highlights a need for a deeper look at students' ways of thinking that will help researchers comprehend the nature of their mistakes.

This preliminary report stems from the first author's doctoral dissertation work, which examines two particular ways of combinatorial thinking. The aim of the study is to situate the dichotomy of set-oriented versus process-oriented thinking within students' domain-specific combinatorial problem-solving activity. Combinatorics textbooks (e.g., Brualdi, 2004; Tucker, 2002) tend to formulate two foundational counting principles – the addition and multiplication principles – in one of two different ways: either they employ *set-theoretic* language or they describe them using *process-oriented* language. For example, as found in Tucker (p. 170, emphasis in original) the exact statements of each principle are:

> <u>The addition principle</u>: If there are r_1 different objects in the first set, r_2 different objects in the second set, …, and r_m different objects in the mth set, and *if the different sets are disjoint*, then the number of ways to select an object from one of the m sets is $r_1 + r_2 + ... + r_m$.
>
> <u>The multiplication principle</u>: Suppose a procedure can be broken into m successive (ordered) stages, with r_1 different outcomes in the first stage, r_2 different outcomes in the second stage,

…, and r_m different outcomes in the mth stage. If the number of outcomes at each stage is independent of the choices in the previous stages and *if the composite outcomes are all distinct,* then the total procedure has $r_1 \times r_2 \times ... \times r_m$ different composite outcomes.

In Tucker's definition of the addition principle, his language involves sets explicitly. The definition reflects a fundamental conception of counting as the enumeration of the number of objects in a set. In his definition of the multiplication principle, however, counting is framed as the completion of a task consisting of successive stages. Other authors reflect this distinction as well; some (e.g., Brualdi, 2004; Rosen, 2007) include two different definitions of each principle, one in terms of sets, and the other in terms of processes.

This dichotomy in the way mathematicians present these basic principles suggests that a relevant distinction could be manifested in student approaches to counting problems. The literature does not address this issue – only a handful of studies in combinatorics education (English, 1991; Hadar & Hadass, 1981) refer to the distinction between sets and processes at all, but no study has explicitly addressed this phenomenon and its potential bearing on students' counting. Noticing this distinction between sets and processes has led the authors to study whether these two formulations indicate any differences in the ways students think about and approach counting problems.

Design and methodology. In designing the study, based on her experiences, the first author suspected that students may draw more heavily on set-oriented thinking when asked to justify whether an answer is right or wrong. Therefore, in an attempt to narrow the scope, she purposefully put students in situations in which they had to evaluate alternative solutions (thus engaging in error detection and correction). Furthermore, for efficiency, she focused on counting problems that are commonly susceptible to errors – problems that have incorrect solutions that frequently seem correct to students. Problems were drawn from Martin (2001) and Tucker (2002).

In the study reported here, eight students were interviewed individually in two 60-90 minute, videotaped sessions. The students were drawn from an upper-division mathematics courses at a large urban university and included mathematics majors, computer science majors, and post-baccalaureate students. In order to accomplish the goals above, the general interview protocol was as follows. In Interview 1, the subjects were given five to seven counting problems and were instructed to solve them as they naturally would (some talked with the author during this time, others were silent). Then, they were asked to explain their thought process and were posed questions about their work. At no point in either interview were they told whether or not a given answer was correct. In Interview 2, students were given alternative answers to the same problems they had solved in Interview 1. They were asked to evaluate the new answers, explore how the new answer compared to their original answer, and determine which answer they thought was correct.

The videotape of each interview was viewed repeatedly and transcribed. The methodological framework of grounded theory (Strauss & Corbin, 1998; Auerbach & Silverstein, 2003) was implemented in order to code the data. Coding consisted of the initial identification of repeated ideas (Auerbach & Silverstein, Ibid) and phenomena, which were then consolidated into themes related to the set/process distinction. The authors also drew upon Carlson & Bloom's (2005) problem solving cycle (which consists of four major stages: orienting, planning, executing, and checking) for analysis purposes. Coding thus took place along two dimensions: based on phenomena that the authors observed and categorized, and according to the problem-solving stages put forth by Carlson & Bloom.

Results. Preliminary analysis of the data indicates promising themes about the occurrences of set- and process-oriented thinking as students solve counting problems. In some contexts, there does indeed seem to be some correlation between the types of counting activity students carry out as they draw upon certain kinds of thinking. The primary theme reported in this study is that students show a strong tendency toward set-oriented thinking during the checking problem-solving phase, particularly when they engage in the evaluation of alternative solutions. Specifically, there is evidence of students categorizing an answer as incorrect by identifying a particular object that was counted more than once. Additionally, there are cases in which students identified two different answers as the same when they evaluated the process – it was not until they adopted a set-oriented perspective that they could explain the different numerical results.

While not all of these findings may be explored in this proposal, an example of student work on one problem is discussed below. The Test Questions problem states *A student must answer five out of ten questions on a test, including at least two of the first five questions. How many subsets of five questions can be answered?* One solution to this problem utilizes a case breakdown, based on whether exactly two, three, four, or five of the first five questions are answered, yielding $\binom{5}{2}\cdot\binom{5}{3}+\binom{5}{3}\cdot\binom{5}{2}+\binom{5}{4}\cdot\binom{5}{1}+\binom{5}{5}\cdot\binom{5}{0}$. A common incorrect answer is $\binom{5}{2}\cdot\binom{8}{3}$, which is obtained by first choosing two of the first five questions to answer, and then choosing any three of the eight remaining questions to answer. The rationale behind such a solution is that the "at least two" constraint is satisfied in the first step, and thus any remaining choice of three problems will still satisfy the constraint of the problem. The trouble with this strategy, however, is that some of the possible outcomes can be counted more than once. For example, in utilizing this strategy, suppose problems 1 and 2 are chosen as the first step. Then, in the next step, problems 3, 7 and 8 are chosen as the second step. Thus, the subset of five questions to be answered is $\{1, 2, 3, 7, 8\}$. However, this subset could be found in a different way using the same counting strategy, namely, by first choosing problems 1 and 3, and then choosing problems 2, 7, 8. Thus, the expression $\binom{5}{2}\cdot\binom{8}{3}$ actually counts some solutions more than once and is therefore incorrect. If the students solved this problem correctly initially, they were given the common incorrect answer in Interview 2. If they first arrived at an incorrect answer, they were asked to examine the correct solution.

Don was a student who displayed both set and process-oriented thinking at various times. In the excerpt below, while working on the Test Questions problem Don decides that the incorrect expression is too big. In justifying this belief, he appeals to two sets of questions generated by the incorrect attempt that are in fact the same. That is, in order to show that $\binom{5}{2}\cdot\binom{8}{3}$ is incorrect, he identifies a particular set of questions (a1, a2, a3, a9, a10) that is counted more than once.

> Don: And so let's see I have, we'll call them a1 and a2 [*he writes down the numbers as he's talking*], and then I also have a3, a9, and a10. But then, add up all these combinations again, you know next time I might have a1, and a3, and then a2 and a9 and a10, and, so this is the same, and this [the incorrect attempt] um, perhaps doesn't account for that.

Don's response reflects that, on some level, he was able to view the counting process as the enumeration of objects – he is counting objects in a set, and he identified one object that was counted too many times. Set-oriented thinking was his chosen way of justifying to himself that the incorrect attempt was too big.

Conclusion. The study described here suggests that students draw upon set-oriented thinking during particular moments in their combinatorial problem solving. These findings stand to inform current understandings of student thinking about counting, offering a meaningful contribution to the field of mathematics education. Subsequent research will include an additional round of data collection based on the preliminary themes identified in this study. The authors will continue to examine the data and make connections among themes that emerge in the new data set, coordinating new and existing themes appropriately. The first author also hopes to conduct further studies that address similar issues, perhaps investigating ways in which combinatorial mathematicians view the domain specific set versus process dichotomy.

Questions:

1) Is the distinction between sets and processes, as it relates to combinatorics, a relationship that teachers of combinatorics have noticed?
2) In what ways can the process-oriented thinking, as specified here, be related to other themes in mathematics education (such as students' notions of functions)?
3) For what other areas of combinatorics and discrete mathematics might this distinction be relevant, and in what ways?
4) What would a domain-specific problem-solving framework for combinatorics look like?

References

Auerbach, C. & Silverstein, L. B. (2003). *Qualitative data: An introduction to coding and analysis.* New York: New York University Press.

Batanero, C., Godino, J., & Navarro-Pelayo, V. (1997). Combinatorial reasoning and its assessment. In I. Gal & J. B. Garfield (Eds.), *The Assessment Challenge in Statistics Education* (pp. 239-252): IOS Press.

Batanero, C., Navarro-Pelayo, V, & Godino, J. D. (1997). Effect of the implicit combinatorial model on combinatorial reasoning in secondary school pupils. *Educational Studies in Mathematics, 32,* 181-199.

Brualdi, R. A. (2004). *Introductory Combinatorics* (4th ed.). Upper Saddle River, New Jersey: Pearson Prentice Hall.

Carlson, M & Bloom, I. (2005) The cyclic nature of problem solving: An emergent multidimensional problem-solving framework. *Educational Studies in Mathematics, 58,* 45-75.

English, L. D. (1991). Young children's combinatorics strategies. *Educational Studies in Mathematics, 22,* 451-474.

English, L.D. (2005). Combinatorics and the development of children's combinatorial reasoning. In G.A. Jones (Ed.), *Exploring probability in school: Challenges for teaching and learning.* Kluwer, Academic Publishers, 40, 121-141.

Hadar, N., & Hadass, R. (1981). The road to solve combinatorial problems is strewn with pitfalls. *Educational Studies in Mathematics, 12*, 435-443.

Kenney, M. J., & Hirsch, C. R. (Eds.). (1991). *Discrete mathematics across the curriculum, K-12: 1991 Yearbook.* Reston, VA: National Council of Teachers of Mathematics.

Kavousian, S. (2008) *Enquiries into undergraduate students' understanding of combinatorial structures.* Unpublished doctoral dissertation, Simon Fraser University – Vancouver, BC.

Martin, G. E. (2001). *The Art of Enumerative Combinatorics.* New York: Springer.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: Author.

Rosen, K. H. (2007). *Discrete Mathematics and Its Applications* (6th ed.). New York: McGraw Hill.

Strauss, A. & Corbin, J. (1998). *Basics of Qualitative Research: Techniques and Procedures for Developing Grounded Theory* (2nd ed.). Thousand Oaks, California: Sage Publications, Inc.

Tucker, A. (2002). *Applied Combinatorics* (4th ed.). New York: John Wiley & Sons.

Conceptual Writing and Its Impact on Performance and Attitude

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There is an abundance of recommendations and articles that extol the virtues of writing in the mathematics classroom. The National Council of Teachers of Mathematics encouraged mathematical communication in its 1989 *Standards for School Mathematics* and again in its update of the *Standards* (2000). The Mathematical Association of America (2004) underscored the need for developing communication skills in mathematics and was joined by a host of other countries that encouraged writing (Ntenza, 2006). Yet, with twenty years of advocacy by researchers and policy-makers, very few students have experience with mathematical writing when they come to college (Borasi & Rose, 1989; Ntenza, 2006, Pugalee, 2004).

In an earlier study, the authors of this preliminary research report modeled their writing assignments after a writing heuristic that used concept mapping, resources, and refinement (Keys, Hand, Prain, & Collins, 1999). Our experimental groups did eight graded writing assignments, each composed of two parts; an initial intuitive piece and a subsequent theme. The writing assignments were oriented toward concept(s) that were currently being taught in the course. The intuitive piece was designed to be a structured concept mapping that would allow students to reflect on the vocabulary, relate the concept to prior knowledge, consult resources, provide examples and counter-examples, and identify areas of confusion. Then the students wrote a theme that was designed to answer specific questions related to the concept explored in the intuitive piece. Each theme assignment was also presented to the control group as an in-class exploration with the discussion ultimately leading to a solution. Thus, all students in the study discussed the theme's concepts; the students in the control group did homework problems related to the concepts, while the students in the experimental group did fewer problems but wrote about the concepts.

We found the writing groups improved more than their control group counterparts numerically on a post test; but the difference was not significant overall except in the case of the lower level mathematics class. Furthermore, the authors found that within the groups who wrote about mathematical concepts: 1) females had more negative attitudes about communicating mathematically than males, and 2) students who were the most diligent in their writing about concepts had significantly more negative attitudes about their ability to do mathematics which seemed to correspond with the adage, "The more I learn, the less I know,"

While the concepts were related to those taught in the courses, the construct of the questions on the pre- and post-tests were wholly different from the in-class conceptual writing assignments. We concluded:

> In providing structure for the assignments and aiding in the planning stage (Flower & Hayes, 1980) we hoped to encourage students to produce well-thought out, carefully revised pieces but it may be that they would be more invested if the assignment was of an argumentative nature where students are asked to defend divergent views. Motivation in argumentative writing has been shown to be a key factor in cognitive gains. (Voss, Wiley, & Sandak, 1999) It may be possible that brief conceptual pieces that allow for arguing a position would be more effective. (Authors, submitted for publication)

The authors believe that more focused writing is key to conceptual understanding and are now in the process of piloting particular writing assignment questions. For example, one writing assignment question will be motivational in its challenge to argue a position on why students tend to miss particular concepts within a problem. Another writing assignment question will be direct in asking students to explain what concepts are important in the context of a given problem. When we have determined the type of question that elicits the most effective response, we will replicate our study on a larger scale using focused conceptual writing. Our research questions will be the same as our previous study.

- Do students in a course that requires writing do better than students in the same course that requires no writing?
- Do students who write about concepts regularly improve more on a visual skills assessment than their counterparts who did not write about concepts on a regular basis?
- If students write regularly about concepts in a mathematics class, does their attitude toward mathematics change?
- Do students believe the writing assignments help them understand the material better?

The Literature

Although much has been written about the benefits of writing in the classroom, the results are mixed from the relatively few studies that compare the learning of students who write with students who do not write. One such study was done by Pugalee in which one group of high school algebra students provided written descriptions of problem solving processes and the other provided verbal descriptions. He concluded, "Students who wrote descriptions of their thinking were significantly more successful in the problem solving tasks ($p<.05$) than students who verbalized their thinking" (Pugalee, 2004, p. 27). Two comparative studies from the early 1990's on college algebra classes found that students who wrote did better on algebra skills exams (Guckin, 1992; Youngberg, 1990). Porter and Masingila (2000) collected data from two sections of calculus. One section of students wrote about their activities and the other did not. Categorizing errors from in-class and final examinations to assess procedural and conceptual understanding, they found no significant difference between the writing and the non-writing groups. Their conclusion was that the nature of the activities may be more important than the writing itself. There is also very little research about attitudes toward writing in mathematics courses and the impact writing has on attitude toward mathematics and most of the evidence is anecdotal.

Theoretical perspective

A great deal of research exists on the cognitive processes involved in the act of writing. A classical model was developed by Flower and Hayes (1981) and confirmed by many later studies.(Alamargot and Chanquoy, 2001) Successful writers are seen as going through recursive stages of planning, then translating , and finally revising and reviewing. The planning stage entails generating ideas using memory and resources, organizing the ideas and setting goals. Our experimental treatment, having students write about mathematical concepts that had been discussed, aided students in the planning stage. The assignments were selected as a possible tool for bridging the gap between Vygotski's 'potential concepts,' so named because they indicate an ability to solve problems that stem from a familiar context or routine algorithm and the distinctly higher, more flexible, conceptual ability to solve complex, non-routine problems independently

(Sierpinska, 1992). The possibility that students could demonstrate conceptual understanding by being able to apply what was learned to a visual skills assessment that was not directly related to the course should not be construed as a permanent and all-encompassing ability to solve problems independently. However, if it could be demonstrated that students who write about mathematics show evidence of significant gains over their non-writing counterparts on such an instrument, then the persuasiveness of using writing as an instructional device would be more compelling.

Research methodology

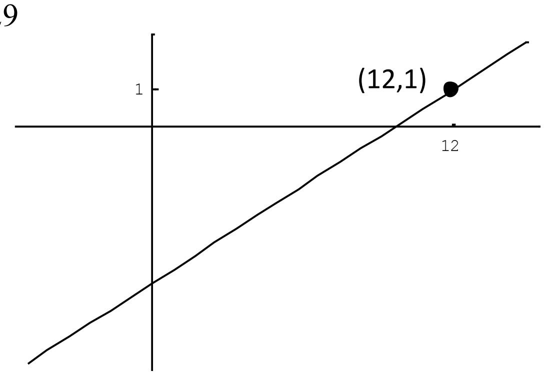
Our previous study was small with participation by two faculty members and their students in two Finite Mathematics classes and two Applied Calculus classes. We plan to conduct the next study with a larger group involving college-level mathematics courses from other universities in the area. We will have an experimental and control group and will change the treatment to more focused conceptual writing. There will be a pre-test and a post-test using a visual skills assessment of 10 or more items that concentrate on four areas; Cartesian Connection, Slope, Function Notation, and Monotonicity. This assessment was designed to measure the comfort level of students entering calculus with the Cartesian Connection and Basic Principles of Graphing (Van Dyke & White, 2004).

As an example, see Figure 1 below. We put the point very close to the equation and used coordinates that do not call for burdensome arithmetic. If the test takers understand the Cartesian Connection, they should be able to easily provide the correct answer to the question.

Figure 1

Consider the following equation along with its graph.

3x-7y=29



Choose one of the following statements.

a) An obvious solution in integers to the equation 3x-7y=29 is_______.

b) I see no obvious solution in integers to the equation 3x-7y=29.

Our conjecture is that conceptual writing is not routinely practiced at the grade-levels where it can quite possibly have the most impact; middle school through high school. If the indicators for using conceptual writing are positive, there will be more impetus for encouraging this pedagogy early and regularly. This, in turn, may help with the phenomenon that we observed in our students' expressed belief that writing in math was not appropriate. Both students and teachers stand to gain if writing becomes a regular part of the mathematics curriculum.

References

Alamargot, D., & Chanquoy, L. (2001). *Through the Models of Writing*. The Netherlands: Kluwer Academic Publishers.

Borasi, R., & Rose, B. J. (1989). Journal Writing and Mathematics Instruction. *Educational Studies in Mathematics , 20*, 347-365.

Flower, L., & Hayes, J. R. (1981) A Cognitive Process Theory of Writing. *College Composition and Communication* 32(4), 365-387Guckin, A. (1992). The Role of Mathematics Informal Focussed Writing in College Mathematics Instruction (Doctoral Dissertation, University of Minnesota, 1992). *Dissertation Abstracts International , 53* (5), 1435A.

Keys, C. W., Hand, B., Prain, V., & Collins, S. (1999). Using the Science Writing Heuristic as a Tool for Learning from Laboratory Investigations in Secondary Science. *Journal of Research in Science Teaching , 36* (10), 1065-1084.

MAA. (2004). *Undergraduate Programs and Courses in the Mathematical Sciences: CUPM Curriculum Guide 2004.* Mathematical Association of America, Committee on the Undergraduate Program in Mathematics of The mathematical Association of America. Washington, D. C.: Mathematical Association of America.

National Council of Teachers of Mathematics. (2000). *Principles and Standards for School Mathematics.* Reston, Virginia: National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics. (2000). *Principles and Standards for School Mathematics.* Reston, VA: National Council of Teachers of Mathematics.

Ntenza, S. P. (2006). Investigating Forms of Children's Writing in Grade 7 Mathematics Classrooms. *Educational Studies in Mathematics , 61* (3), 321-345.

Porter, M., & Masingila, J. O. (2000). Examining the Effects of Writing on Conceptual and Procedural Knowledge in Calculus. *Educational Studies in Mathematics , 42*, 165-177.

Pugalee, D. K. (2004). A Comparison of Verbal and Written Descriptions of Students' Problem Solving Processes. *Educational Studies in Mathematics* , 27-47.

Sierpinska, A. (1992). The Development of Concepts According to Vygotski. (J. Schmittau, Ed.) *Focus on Learning Problems in Mathematics , 15* (2&3), 87-107.

Van Dyke, F., & White, A. (2004). Examining Students' Reluctance to Use Graphs. *Mathematics Teacher , 98*, 110-117.

Voss, J. F., Wiley, J., & Sandak, R. (1999). Reasoning in the Construction of Argumentative Texts. In J. Andriessen, & P. Coirier, (Eds.), *Foundations of Argumentative Text Processing.* (pp. 29 – 41). The Netherlands: Amsterdam University Press.

Youngberg, S. A. (1990). The Effect of Writing Assignments on Student Achievement in a College Level Elementary Algebra Class (Doctoral Dissertation, University of Iowa, 1989). *Dissertation Abstracts International , 50* (9), 2819A.

An Exploration of the Transition to Graduate School in Mathematics

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In recent years, researchers have given much attention to the new mathematics graduate student as a mathematics instructor. In contrast, this study explores the academic side of the transition to graduate school in mathematics—the struggles students face, the expectations they must meet, and the strategies they use to deal with this new chapter in their academic experience. This talk will look at preliminary results and analysis from a qualitative study designed to explore these aspects of the transition to graduate school in mathematics from a post-positivist perspective. In order to explore the transition as fully as possible, interview data from a varied sample of graduate students and faculty members at one university are being incorporated to gain multiple perspectives on the transition experience. Potential implications for graduate recruitment, retention, and program protocols in mathematics will be discussed.

Keywords: graduate students, academic transition, semi-structured interview, case study

Research Problem

Students entering graduate programs in mathematics often experience an "abrupt change of status" during this transition (Bozeman & Hughes, 1999, p. 347). While their undergraduate records may be exemplary, these students often have trouble adjusting to the rigorous new environment of graduate school in mathematics. Setbacks, such as insufficient prerequisite knowledge or an inability to discern or meet a professor's expectations, may generate diminished self-esteem or even a desire to drop out of the program.

These issues impact departments as well: Students' struggles with the transition to graduate mathematics may negatively affect program recruitment as admissions committees are less likely to admit applicants with similar backgrounds in the future. Retention is also impacted across the discipline as promising students may incorrectly assume they lack mathematical ability and leave the field forever. Finally, these struggles can affect the representation of women and minorities in such programs, as these groups are less likely to find the support structures they need to survive graduate school (Bozeman & Hughes, 1999).

Literature. Several researchers have explored issues related to this transition. For instance, Duffin and Simpson's (2006) interviews with Ph.D. students explored the transition from undergraduate to graduate work in mathematics in the United Kingdom's educational system. The authors concluded that both undergraduate and graduate education could be modified to smooth this transition for different types of learners. Marilyn Carlson (1999) explored the problem-solving behaviors and mathematical beliefs of mathematics graduate students who were considered "successful" in their programs. Persistence, high levels of confidence, and the presence of a mentor during key periods of mathematical development (often as early as high school) all played a role in these students' "success." While much of the work in these studies was done with students already securely in a graduate program, they may still have implications for the transition to graduate school: Encouraging new graduate students to develop the good habits of thriving students in their program may help smooth the transition into graduate school in mathematics.

In 2002, Herzig qualitatively examined persistence specific to graduate school in mathematics by conducting a case study of one mathematics department. She interviewed both current students in the doctoral program and some who had left the program, as well as faculty members in this department, to investigate factors influencing doctoral student persistence and attrition. Herzig found that legitimate peripheral participation both in departmental life and in the field itself encouraged persistence in a doctoral mathematics program.

Useful work has also been done in recent years regarding the transition from secondary to tertiary mathematics as colleges and universities have tried to narrow the gap among various groups of incoming college freshmen. For instance, Selden (2005) discussed this transition to collegiate mathematics, noting that new college students must often reconceptualize ideas from previous mathematical training (such as the idea of a tangent line) in order to incorporate them into the new, demanding educational structure they have encountered. As another example, Kajander and Lovric (2005) detail McMaster University's efforts to address this transition through surveys of students' mathematical backgrounds, course redesign, and provision of a departmental review manual to enable students' voluntary preparation for their mathematics courses. They noted that students' motivation, ability to delve beyond surface learning, and secondary school preparation in mathematics were all key to the transition process. Transferring the ideas from these two studies to the transition to graduate school in mathematics identifies several relevant issues in this transition process: undergraduate preparation, ability to both reconceptualize prior knowledge and dig deeply into new mathematical material, and a "bridge" review process prior to graduate work.

Research questions. Building on the aforementioned work, I seek to establish a clear picture of what happens during the transition to graduate school in mathematics in the United States so that further research can be done on the impact of various aspects of or changes to this process. Accordingly, the purpose of this study is to explore the academic transition to graduate school in mathematics—the struggles students face, the expectations they must meet, and the strategies they use to deal with this new chapter in their academic experience. In particular, I am seeking answers to the following exploratory research questions: What happens during the academic transition from undergraduate student to graduate student in mathematics? How do professors' expectations of new graduate students' mathematical knowledge affect students' success? How do new graduate students in mathematics adjust to the rigors of graduate school and/or compensate for prior knowledge deficiencies? How do attitudes, beliefs, and relationships play a role in the success of new graduate students in mathematics? I hope that this research will provide a more accurate picture of graduate student preparation for and experiences in graduate school in mathematics; then, we can work to modify resources for prospective and current graduate students accordingly to help make the transition as smooth as possible.

Research Design

I have chosen to conduct an exploratory single-case study to delve into the in-depth meanings of one mathematics department's experiences with the transition to teaching. To fully explore these experiences, I am following Herzig (2002) and conducting interviews with both graduate students and faculty members in this department. In keeping with a post-positivist stance, I am striving for a rigorous, scientific approach to my research (Creswell, 2007) while allowing a place for intuition (Crotty, 1998). Furthermore, the "truth" derived from participants' experiences will not be absolute, but my data can still show how participants describe and perceive aspects of their transition experiences. These data can also help construct a valuable portrait of this experience to aid admissions, advising, graduate student life, and other aspects of mathematics departments' and universities' preparation for and support of new graduate students in mathematics.

Both graduate student and faculty semi-structured interviews are centered around the research questions given above, with probing questions included as needed. The student interviews are allowing me to ask specific questions about my participants' experiences surrounding the transition to graduate school, while the faculty interviews will provide a new perspective on the same aspects of the transition experience. Graduate student interviewees are being selected from among those who had taken core courses in the Ph.D. track at this university and who were interested in participating in the study (as indicated in a brief online survey). Based on these survey responses, maximal variation sampling (Creswell, 2007) ensures that selected participants vary along characteristics such as gender, year of program entry, year core courses were taken, and degree sought to avoid highlighting issues specific to any particular subgroup. Faculty interviewees will be selected from those who had recently held positions related to graduate students—such as Chair, Associate Chair, Graduate Director, or core course instructor—and who are willing to participate. As interviews and transcription are completed, I will use an open coding procedure (Strauss & Corbin, 1990) to build a structure to this transition that is grounded in participants' views (Creswell, 2007). Preliminary codes will be merged to identify themes in the data; this analytic inductive process (LeCompte & Preissle, 1993) should help me discover areas of this academic transition that are impacting students' success, as well as recruitment, retention, and other areas.

Results and Implications

As of the submission of this proposal, data collection is still ongoing, so any statement of results would be premature. However, based on the literature, preliminary data collection, and personal experience, I expect to identify themes in the transition experience that have implications for recruitment, retention, and other graduate program protocols and policies. Also, the generation of a clear portrait of the transition experience can help inform future research questions and methods in this area. Preliminary results and implications based on student interview data will be complete in time for the conference. I hope to generate a conversation with other researchers interested in this topic to help me refine the themes and conclusions I am drawing from my data.

Questions

During the presentation of this preliminary research report, the following questions will be posed to audience members to generate discussion useful to the continuation of analysis and to other future work in this area:

- What other data might be useful to help complete a picture of this transition experience in this department?
- What other implications might the results hold for recruitment, retention, departmental policy, or other aspects of the graduate student experience?
- What other pieces of literature, frameworks, or research contacts might be relevant to or helpful for my work?
- Based on your experience, what things should I have considered (or reconsidered) in conducting this study? That is, what general feedback can you give for current or future work?

References

Bozeman, S. T., & Hughes, R. J. (1999). Smoothing the transition to graduate education. *Notices of the AMS, 46*, 347-348.

Carlson, M. P. (1999). The mathematical behavior of six successful mathematics graduate students: Influences leading to mathematical success. *Educational Studies in Mathematics, 40*, 237-258.

Creswell, J. W. (2007). *Qualitative inquiry and research design: Choosing among five approaches* (2nd ed.). Thousand Oaks, CA: Sage.

Crotty, M. (1998). *The foundations of social research: Meaning and perspective in the research process*. Thousand Oaks, CA: Sage.

Duffin, J., & Simpson, A. (2006). The transition to independent graduate studies in mathematics. In F. Hitt, G. Harel, & A. Selden (Eds.), *Research in Collegiate Mathematics Education VI* (pp. 233-246). Washington, DC: American Mathematical Society.

Herzig, A. H. (2002). Where have all the students gone? Participation of doctoral students in authentic mathematical activity as a necessary condition for persistence toward the Ph.D. *Educational Studies in Mathematics, 50*, 177-212.

Kajander, A., & Lovric, M. (2005). Transition from secondary to tertiary mathematics: McMaster University experience. *International Journal of Mathematical Education in Science and Technology, 36*, 149-160.

LeCompte, M. D. & Preissle, J. (1993). *Ethnography and qualitative design in educational research*. San Diego: Academic Press.

Selden, A. (2005). New developments and trends in tertiary mathematics education: Or, more of the same? *International Journal of Mathematical Education in Science and Technology, 36*, 131-147.

Strauss, A., & Corbin, J. (1990). *Basics of qualitative research: Grounded theory procedures and techniques.* Newbury Park, CA: Sage.

Inquiry-Based and Didactic Instruction in a Computer-Assisted Context

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One direction taken by course reform over the past few years has been the development of sophisticated computer-assisted instruction. This approach has been applied to large-enrollment service courses in mathematics, including algebra. Elementary algebra is typically taken by under-graduate students who do not place into a credit-bearing course. Traditionally, the goal of such a developmental algebra course has been to enhance students' "algebra skills," for example, dealing procedurally with rational numbers and expressions. Higher-order thinking may be largely absent. Alternately, one might focus on developing quantitative reasoning and communications skills, rather than, or in addition to, training to acquire a set of specific algebraic skills (Wiggins, 1989; Blais, 1988). Our position is that incorporating an inquiry-based component, either together with, or in place of, a didactic component, into a computer-assisted instructional environment may enhance student learning. Two previous studies in the literature bear this out (Mayer, 2009, 2010).

Fundamental Question. We compare three treatments in a quasi-experimental design: (1) two weekly inquiry-based class meetings, (2) two weekly lecture meetings, and (3) one of each meeting weekly. The computer-assisted component is the same for all treatments. Our hypothesis is that, of the three treatments, the one affording the most inquiry-based involvement to the students will differentially benefit the students in terms of mathematical content knowledge, reasoning and problem-solving ability, and communications.

Prior Research. Prior to the two most recent studies (Mayer, 2009, 2010), the methodology of simultaneously comparing different pedagogies within one semester, had few direct comparisons in the literature (Doorn, 2007). Some studies have compared different pedagogies over a longer time frame (Gautreau, 1997; Hoellwarth, 2005). The results of the quasi-experimental studies in (Mayer, 2009) of a finite mathematic course, and in (Mayer, 2010) of an elementary algebra course showed in both cases that students in the inquiry-based treatment did significantly better ($p<0.05$) comparing pre-test and post-test performance in the areas of problem identification, problem-solving, and explanation. Moreover, students, regardless of treatment, performed similarly (no statistically significant differences) when compared on the basis of course test scores. Outcomes of the two studies differed in gain in accuracy, pre- to post-test: in the finite mathematics study, there was no significant difference between treatments; in the elementary algebra study there was a significant difference between treatments in favor of the inquiry-based treatment. A limitation of both studies by Mayer was that accuracy was assessed on a small set of open-ended problems. The previous studies also did not test a blend of inquiry-based and traditional class meetings in a single treatment (Marrongelle, 2008).

Research Methodology. Our methodology is quasi-experimental in that it seeks to remove from consideration as many confounding factors as possible, and to assign treatment on as random a basis as possible, constrained only by students being able to choose the time slot in which they take the course. All students involved in the courses have identical computer-assisted instruction provided in a mathematics learning laboratory. 86% of the grade in the course is determined by evaluation in the computer-assisted context (online homework and supervised online quizzes and tests). The remaining 14% of the grade is determined by one of three pedagogical treatments, described below. Students registered for one of three time periods in the Fall 2010 semester schedule, a 9:00 AM, 10:00 AM or noon time slot, for three days a week (MWF), for their 50 minute class meetings and 50 minute required lab meeting. Students in each time slot were randomly assigned to one of the three treatments for the semester. Three instructors agreed to participate in the experiment. Each instructor teaches in three time slots. In one slot the instructor administers the twice-weekly inquiry-based treatment, in another time slot, the twice weekly lecture treatment, and in a third time slot, the blended treatment. The three instructors consist of a full professor, a regular full-time instructor, and a graduate student with prior teaching experience. All instructors had previous experience in both didactic and inquiry-based teaching, and in computer-assisted instruction. A graduate teaching assistant works with each instructor in the inquiry-based meetings, and in evaluating written student work product from such meetings. Each instructor also meets with each class in the mathematics computer lab. The computer lab meeting for all treatments occurs on Wednesday.

The three pedagogies to be compared are: (1) two sessions weekly of inquiry-based group work (random, weekly changing, groups of four) without prior instruction, on problems intended to motivate the topics to be covered in computer-assisted instruction; (2) two sessions weekly of traditional summary lecture with teacher-presented examples on the topics to be covered in computer-assisted instruction, and (3) a blend of treatments (1) and (2), with one weekly meeting traditional lecture, and one weekly meeting inquiry-based group work. In the inquiry-based treatments, each student turns in each class meeting a written report on his/her investigation and solution of the problem(s) posed in that class period. This report is evaluated based upon the same rubric as the open-ended items on the pre/post-test. Students are aware of the rubric and receive written feedback consistent with the rubric. In the lecture treatment, the instructor gives a traditional lecture on the upcoming material. All instructors operate from the same outline of topics for each lecture. The 14% (140 of 1000 points) of the final grade determined by the class meetings differs among the three treatments as follows: (1) 5 points are earned for each of the two weekly reports on the group work; (2) 5 points are earned for attendance at each class meeting; (3) 5 points are earned for the one weekly report on the group work meeting, and 5 points are earned for attendance at the lecture meeting.

The research is underway in Fall, 2010. Data to be gathered includes (1) course grades and test scores, (2) pre-test and post-test of content knowledge based upon a test which incorporates three open-ended problems, evaluated on rubric dimensions of conceptual understanding, evidence of problem-solving, and adequacy of explanation (3) pre-test and post-test of content knowledge based upon a test consisting of 25 objective questions, (3) focus groups selected from each of the nine class sections, (4) student course evaluations using the online IDEA system (IDEA, 2010), and (5) RTOP observations of the instructors in each of the nine class sections (RTOP, 2010; Sawada, 2002). The above data will be gathered and analyzed and will form the basis of the proposed preliminary report. Data and preliminary analysis will be available by December 15, 2010 should this be needed by the committee reviewing proposals.

A limitation of the studies by Mayer (2009, 2010) is that the pre/post-test consisted of only three or four open-ended problems which made a reliable evaluation of accuracy gains, if any, problematic. The pre/post-test in the study described herein consists of two parts: (A) three open-ended problems, evaluated by a rubric as described above, and (B) 25 objective questions which have been validated for testing algebraic content knowledge in previous studies. A battery of the previously validated (for content) objective questions was piloted in Summer 2010 on students in the same course, and item analysis was used to select the items for the pre/post-test in this study. As a result of the more careful test design, we expect that differential gains in accuracy between treatments, if present, will be more detectable than in the two earlier studies cited.

Questions that we pose to ourselves and the audience are as follows:

- Will all treatments result in similar course grades and course test scores?
- Will all treatments result in similar gains in accuracy on the objective pre/post-test?
- Will the inquiry-based and blended treatments result in differentially improved conceptual understanding, problem-solving ability, and mathematics communications skills, as assessed by the open-ended pre/post-test?
- Do students perceive any value in the inquiry-based components of the treatment?

We expect this research to inform our teaching of elementary algebra. Moreover, we expect to extend this study in subsequent years to credit courses such as intermediate algebra, college algebra, and pre-calculus (Oehtrman, 2008).

References

Blais, D. M. (1988), Constructivism -- a Theoretical Revolution for Algebra. Mathematics Teacher, Nov. 1988: 49--56.

Doorn, D. and O'Brien, M. (2007), Assessing the Gains from Concept Mapping in Introductory Statistics. International Journal for the Scholarship of Teaching and Learning 1(2); online at \url{http://academics.georgiasouthern.edu/ijsotl/v1n2/articles/doorn/index.htm}.

Gautreau, R. and Novemsky, L. (1997), Concepts First-a Small Group Approach to Physics Learning. Am. J. Phys. 65(5): 418-429.

Hoellwarth, C., Moelter, M. J., and Knight, R.D. (2005), A direct comparison of conceptual learning and problem solving ability in traditional and studio style classrooms. Am. J. Phys. 73(5): 459-463.

IDEA (2010), The IDEA Center. \url{http://www.theideacenter.org/our-services/student-ratings-instruction/0030-idea-online}.

Marrongelle, K. and Rasmussen, C. (2008), Meeting New Teaching Challenges: Teaching Strategies that Mediate Between All Lecture and All Student Discovery. In M. Carlson and C. Rasmussen (Eds.), Making the Connection: Research to Practice in Undergraduate Mathematics, pp. 167-178. Mathematical Association of America, Washington, DC.

Mayer, J.C., et al. (2009), Incorporating Inquiry-Based Class Sessions with Computer Assisted Instruction. Electronic Proceedings of 2009 SIGMAA Conference on Research in Undergraduate Mathematics Education, Raleigh, NC, February, 2009.

Mayer, J.C., et al. (2010), Blending Inquiry-Based and Computer Assisted Instruction in a Basic Algebra Course: a Quasi-Experimental Study, Electronic Proceedings of Conference on Research in Undergraduate Mathematics Education, Raleigh, NC, February, 2010.

Oerhtman, M., Carlson, M., and Thompson, P. (2008), Foundational Reasoning Abilities that Promote Coherence in Students' Function Understanding. In M. Carlson and C. Rasmussen (Eds.), Making the Connection: Research to Practice in Undergraduate Mathematics, pp. 65-80. Mathematical Association of America, Washington, DC.

RTOP (2010), Reformed Teaching Observation Protocol. Developed by ACEPT Project (Arizona State University). \url{http://physicsed.buffalostate.edu/AZTEC/RTOP/RTOP_full/}

Sawada, D., Piburn, M., Judson, E., Turley, J., Falconer, K., Benford, R. & Bloom, I. (2002). Measuring reform practices in science and mathematics classrooms: The Reformed Teaching Observation Protocol. School Science and Mathematics, 102(6), 245-253.

Wiggins, G. (1989), The Futility of Trying to Teach Everything of Importance. Educational Leadership, Nov. 1989: 71-78.

Do Leron's structured proofs improve proof comprehension?

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Abstract: In undergraduate mathematics courses, proofs are regularly employed to convey mathematics to students. However, research has shown that students find proofs to be difficult to comprehend. Some mathematicians and mathematics educators attribute this confusion to the formal and linear style in which proofs are generally written. To address this difficulty, Leron (1983) suggested an alternative format for presenting proofs, named *structured proofs*, designed to enable students to perceive the main ideas of the proof without getting lost in its logical details. However, we are not aware of any empirical evidence that such format actually helps students comprehend proofs. In this presentation we report preliminary results of a study that employs a recent model of proof comprehension to assess the extent to which Leron's format help students comprehend proofs.

1. Introduction

In advanced mathematics courses, proofs are a primary way that teachers and textbooks convey mathematics to students. However, researchers note that students find proofs to be confusing or pointless (e.g., Harel, 1998; Porteous, 1986; Rowland, 2001) and undergraduates cannot distinguish a valid proof from an invalid argument (Selden & Selden, 2003; Weber, 2009). Some mathematicians and mathematics educators attribute students' difficulties in understanding proofs to the formal and linear style in which proofs are written (e.g., Thurston, 1994; Rowland, 2001).

To address this difficulty, several mathematics educators have suggested alternative formats for presenting proofs, such as using generic proofs (e.g., Rowland, 2001), e-proofs (Alcock, 2009), explanatory proofs emphasizing informal argumentation (e.g., Hanna, 1990; Hersh, 1993), and structured proofs (Leron, 1983). These suggestions have an obvious appeal; if changing the format of a proof can increase students' understanding of its content, then these alternative proof formats provide a practical way to improve the effectiveness of lectures and textbooks in advanced mathematics courses. However, we are not aware of any empirical evidence that suggests any of the proposed formats above actually increase students' understandings of the proofs they read or observe. In fact, at least one study suggests the opposite. When Roy, Alcock, and Inglis (2010) attempted to see if Alcock's (2009) e-proofs improve students' comprehension of proofs in a pilot study, they found that students who studied an e-proof surprisingly performed significantly *worse* on a post-test than students who studied the same proof from a lecture or textbook.

The goal of this study is to examine the extent to which Leron's (1983) structured proofs will improve student understanding. Leron (1983) suggests that linear proofs limit students' understanding because this format masks the overarching structure of the proof and the methods and variables introduced in a linear proof appear to come out of thin air. He suggests instead organizing a proof into levels with Level 1 providing a summary of the main ideas of the proof (without going into detail at his level into how they will be implemented), Level 2 giving a summary of how each of the main ideas is implemented, and successively lower levels filling in more of the details of the proof. In some cases, an "elevator" between levels provides an informal rationale for why the proof is proceeding the way that it is. This format enables the reader to perceive the main ideas of the proof without getting lost in its logical details, but still allows the reader to read about or verify these logical details if he or she desires to do so.

Although several mathematics education researchers cite Leron's structured proofs as a possible way to improve proof presentation (e.g., Alibert & Thomas, 1991; Hersh, 1993; Movshovits-Hadar, 1988), we are not aware of any empirical evidence that such proofs will help students. Indeed, in an exploratory study, Cairns and Gow (2003) present theoretical difficulties that students may encounter with a structure proof and illustrate how some students experience these difficulties based on interviews with three students. They concluded a structured proof "is not *a fortiori* the best presentation for proofs" (p. 186).

2. Theoretical perspective

Our model and means of assessing proof comprehension is based on Mejia-Ramos et al's (2010) presentation at last year's RUME conference. This model posits that students' proof comprehension can be measured along six dimensions: (a) understanding of terms and statements in the proof, (b) ability to cite justifications for statements in the proof, (c) the logical structure of the proof, (d) the high-level ideas of the proof, (e) the method used in the proof, and (f) how the proof relates to examples or informal images. Our assessment of students' proof comprehension was based on this model.

3. Methods

For this study, we recruited two groups of six students. Each participant met individually with one of the co-authors of this paper. The participants were asked to study a proof and were told they would be asked a series of questions about the proof. After they studied the proof to their satisfaction, they returned the proof to the interviewer. The participants were asked on a scale of 1 through 5 how well they understood the proof, with a 5 indicating they understood the proof completely. They were then asked an open-ended question about the proof (e.g., "How was the fact that $f'(x)>0$ used in the proof?") followed by a multiple-choice question of the same item. After they answered all the questions, the proof was returned and they were permitted to change their answers. This process was repeated with a second proof. The assessment questions were based on the model of Mejia-Ramos et al (2010).

Participants in the first group (Group A) first studied a linear presentation of a proof of the assertion "The only solution to the equation $x^3 + 5x = x^2 + \sin x$" (from here on Proof 1). They then studied a structured proof of the statement "There are infinitely many primes of the form 4k+3". Participants in the second group (Group B) studied a structured version of Proof 1 and a linear version of Proof 2. The structured and linear versions of Proof 2 were taken with minor modifications from Leron (1983). If participants read a structured proof, they were also asked about their opinions of the proof, what (if anything) they found positive about it, and what (if anything) they found negative about it.

Our analysis focuses on: (a) how well participants felt they understood the proofs they read, (b) participants' performance on the open-ended questions that they answered immediately after reading the proof (without having the proof to refer to), and (c) participants' comments on the benefits and drawbacks of structured proofs.

4. Results

For Proof 1, Group A (who received the linear version of the proof) appeared to perform better than Group B (who received the structured version) on the assessment items. On average, they answered 5 of the 8 assessment questions correctly (63%) while the students in Group B answered only 2.33 questions correctly (29%). Group A and Group B reported nearly equal levels of understanding Proof 1 (4.17 vs. 4.00).

For Proof 2, Group A (who received the structured version of this proof) performed slightly better than Group B (who received the linear version). They answered 2.5 of the 7 assessment questions correctly (36%) while Group B answered 2 questions correctly (29%). Group B reported a higher level of understanding than Group A for Proof 2 (3.83 vs. 2.33).

Combining across proofs, participants studying the linear proofs reported a mean understanding of 4.00 and answered an average of 7 of the 15 assessment questions correctly (47%), while students studying structured proofs reported a mean understanding of 3.13 and answered 4.83 out of 15 assessment questions correctly (32%).

Among the 12 participants, two reacted positively to the structured proof format, citing that it made explicit the goals of proof and the relationships between its different parts. The remaining 10 participants cited drawbacks with the approach, with some claiming they found it generally confusing.

5. Discussion

In summary, this study did not find evidence that structured proofs improved students' comprehension of proofs. When the participants read a structured proof as opposed to a linear proof, they reported less understanding and performed worse on the assessment questions. Only two participants cited more benefits of structured proofs than drawbacks, with the remaining participants citing that the difficulties in following the structured proofs hindered their understanding.

Of course, it is imperative to note our study does **not** demonstrate that structured proofs are ineffective as the design of our study could be criticized on several grounds. Most importantly, our sample size was limited and we cannot infer that the results of our study would not change if we expanded our sample. We also note that there are threats to the construct validity of our study. In mathematics classes, students are not given a short period of time to read a proof and then are given a test on it; they are often given a proof and expected to study it for a longer time over several days. Finally students' difficulty may have been due to the novelty of the structured format. Perhaps giving students more exposure to structured proofs, or instruction on reading them, may have improved their performance.

On the other hand, Roy, Alcock, and Inglis (2010) illustrate how a theoretically motivated alternative proof presentation format can, in some cases, decrease students' understanding of the proof. We note that our results about students' difficulties with structured proofs are consistent with the findings of Cairns and Gow (2003). Finally, we also note there are no empirical studies that offer any evidence that structured proofs do improve understanding.

We are not arguing such studies cannot be done, but we believe they would take careful thought to design, and would likely include instruction for students on how a structured proof should be read. We contend such studies are necessary if structured proofs are to continue to be proposed as a means of increasing students' proof comprehension, both because claims of this type in mathematics education should require empirical support and because a study of this type can offer practical pedagogical direction for teachers who wish to incorporate pedagogical proofs in their own classrooms.

6. Questions for audience

Under what conditions might we see the benefits of structured proofs? What type of evidence would be required to convince the community that structured proofs (or, more generally, any pedagogical suggestion) might not be effective?

References

Alcock, L. (2009). E-proofs: Student experience of on-line resources to aid understanding of mathematical proofs. In *Proceedings of the 12th Conference for Research in Undergraduate Mathematics Education*. Available for download from: http://sigmaa.maa.org/rume/crume2009/proceedings.html

Alibert, D. and Thomas, M. (1991). Research on mathematical proof. In D. Tall (Ed.) *Advanced Mathematical Thinking*. Dordrecht: Kluwer.

Cairns, P. & Gow, J. (2003). A theoretical analysis of hierarchical proofs. In A. Asperti, B. Buchberger & J.H. Davenport (Eds.) *Mathematical knowledge management. Proceedings of MKM 2003*. LNCS 2594. Springer.

Hanna, G. (1990). Some Pedagogical Aspects of Proof. *Interchange*, 21(1), 6-13.

Harel, G. (1998). Two dual assertions: The first on learning and the second on teaching (or vice versa). *American Mathematical Monthly*, 105, 497-507.

Hersh, R. (1993). Proving is convincing and explaining. *Educational Studies in Mathematics*, *24*, 389-399.

Leron, U. (1983). Structuring mathematical proofs. *American Mathematical Monthly*, 90(3), 174-184.

Mejia-Ramos, J.P., Weber, K., Fuller, E., Samkoff, A., Rhoads, K., and Search, R. (2010). Modeling the comprehension of proofs in undergraduate mathematics. In *Proceedings of the 13th Conference for Research in Undergraduate Mathematics Education*. Available for download from: http://sigmaa.maa.org/rume/crume2010/Abstracts2010.htm

Movshovits-Hadar, N. (1988). Stimulating presentation of theorems followed by responsive proofs. *For the Learning of Mathematics*, 8(2), 12-19, 30.

Porteous, K. (1986). Children's appreciation of the significance of proof. In *Proceedings of the Tenth International Conference for the Psychology of Mathematics Education*, 392-397. London, England.

Rowland, T. (2001). Generic proofs in number theory. In S. Campbell & R. Zazkis (eds.) *Learning and teaching number theory: Research in cognition and instruction*. Westport, CT: Ablex Publishing.

Roy, S., Alcock, L., & Inglis, M. (2010). Supporting proof comprehension: A comparison study of three forms of comprehension. In *Proceedings of the 13th Conference for Research in Undergraduate Mathematics Education*. Available for download from: http://sigmaa.maa.org/rume/crume2010/Abstracts2010.htm

Selden, A. & Selden, J. (2003). Validations of proofs written as texts: Can undergraduates tell whether an argument proves a theorem? *Journal for Research in Mathematics Education*, 34(1), 4-36.

Thurston, W.P. (1994). On proof and progress in mathematics, *Bulletin of the American Mathematical Society,* 30, 161-177.

Weber, K. (2009). Mathematics majors' evaluation of mathematical arguments and their conception of proof. In *Proceedings of the 12th Conference for Research in Undergraduate Mathematics Education*. Available for download from: http://sigmaa.maa.org/rume/crume2009/proceedings.html

Teaching Approaches of Community College Mathematics Faculty: Do Teaching Conceptions and Approaches Relate to Classroom Practices?

Vilma Mesa and Sergio Celis

Abstract

In this study we compare teaching approaches of 14 community college mathematics instructors with their classroom questioning and their classroom non-mathematical discursive interactions. The teaching approaches were drawn from interviews and the application of an analytical framework derived from the higher education literature. The questioning and the non-mathematical discursive interactions were characterized using transcripts of classroom observations and the application of an analytical framework derived from the mathematics education and higher education literature. From the interviews, we found a wide range of espoused teaching approaches, although the majority of instructors favored instructor-centered approaches. From the observations, we found that these instructors ask a large amount of questions, a sizable proportion of which generate opportunities for students to engage with authentic mathematical knowledge. Also, we found that these espoused teaching approaches are related to observed non-mathematical discursive interactions.

Keywords: classroom research, community college, mathematics teaching

Teaching Approaches of Community College Mathematics Faculty: Do Teaching Conceptions and Approaches Relate to Classroom Practices?

Preliminary Research Report

This study explores the consistence between community colleges mathematics instructors' descriptions about their teaching and a subset of their classroom practices. With this paper we seek to create a bridge between the literature on teaching that exists in higher education and in mathematics education. We believe that it is important to understand how the same phenomenon—instruction—is conceived in each field and seek commonalities and ways to bring these traditions together.

Our investigation started with an analysis of prominent frameworks that have been used in higher education to characterize instruction. In this field the research has focused on "teaching conceptions" and "teaching approaches" (Prosser & Trigwell, 1999; Kember & Kwan, 2000) suggesting that approaches that have a student focus can be more effective than 'traditional' approaches that tend to be instructor- or content-centered (Kember & Gow, 1994; Kember, 1997; Prosser & Trigwell, 1999; Åkerlind, 2003). Most of the studies about teaching approaches are based on analysis of students' perceptions about their learning processes or considering instructors' orientations to teach, obtained mainly through interviews and inventories (Meyer & Eley, 2006; Ashwin, 2009). It is less common to find literature that illustrates teaching approaches based on actual interaction between instructors and students (Ashwin, 2009), and far less common to find studies that look at specific disciplines, such as mathematics.

The literature on teaching in mathematics education is, in contrast more extensive, and is mostly based on in-depth analysis of classroom observations; interviews and inventories are usually subsidiaries to what happens in the classroom. In mathematics education, an important focus has been on the quality of the interactions between teacher, the students, and the mathematical content (Cohen, Raudenbush, & Ball, 2003). Earlier studies of what instructors say in interviews and what they actually do, framed under the agendas of reform promoted by the National Council of Teachers of Mathematics [NCTM] (1989, 2000), highlighted discrepancies that pointed at instructors' difficulties in implementing reform (Cohen, 1990). When looking at instruction in undergraduate mathematics, there are very few of these accounts; the most prominent come from studies with teaching assistants (Speer, 2005; Speer, Gutman, & Murphy, 2005), or with mathematicians (Nardi, Jaworksi, & Hegedus, 2005; Speer & Wagner, 2009; Stephan & Rasmussen, 2002).

We sought to combine these two traditions in an analysis of teaching practices of a group of 14 community colleges mathematics instructors. We sought to investigate, using the frameworks from higher education and from mathematics education, what their teaching approaches were by looking at both, their descriptions of teaching stated during interviews, and their enactments of those approaches in their classrooms. In looking at the work in the classroom, we focused on the mathematical questions that were posed and on other discursive interactions between instructors and the students that were not necessarily mathematical in nature. We wanted to determine the extent to which there was consistency between what instructors declared to be their teaching approaches and what we observed in the classroom. Because of the extensive body of literature in mathematics education, we anticipated discrepancies between the two analyses.

Methods

The data come from a larger study that seeks to characterize community college mathematics instruction. We used interviews and class observations of 14 mathematics instructors (six full-time) at a large suburban community college in the Midwest. Although the observations come from a wide range of courses, half of them were trigonometry courses. The instructors volunteered to participate in the study. The instructors were interviewed prior to the observations to obtain their views about teaching and learning, awareness of context, and institutional support for instruction. The instructors were observed at least three times during the term in which they were teaching. The classes were audio recorded and extensive field notes were collected. Pseudonyms were assigned to each instructor.

To analyze the interviews and non-mathematical discursive interactions of the classroom observations, we used a framework to characterize teaching approaches derived from higher education. The classroom observations were further analyzed using a framework developed for the study that characterizes the questions posed in the classroom. Reliability in using these frameworks to code the data ranged from 69% to 93%.

Teaching approaches framework: We created a six-category framework combining three different perspectives on teaching approaches. For purposes of the comparisons studied in this report, we mainly focus on Grubb and colleagues' (1999) three approaches to teaching at community colleges, "Traditional," "Meaning Making," and "Student Support." The "Traditional" approach would be the most common in community colleges, and among its more frequent actions are controlling time, making reference to higher math courses, and covering the material. The "meaning making" approach can be associated to many names in the literature, such as "progressive," "constructivist," or "student-centered" (p. 31). This approach emphasizes that students are able to construct meaning for themselves through strategies such as seat- and group-work or connecting the content with real context. The "student support" approach seeks to empower students and to increase their autonomy and self-confidence. In this approach, mastering the subject content is secondary.

Questioning framework: This framework emerged as a synthesis of frameworks that analyze interaction in mathematics classrooms (Nathan & Kim, 2009; Truxaw & DeFranco, 2008; Wells, 1993; Wells & Arauz 2006). With this framework we sought to characterize the opportunities that students have to express their thinking about doing mathematics and to contribute mathematics that is new to the class (Mesa & Lande, 2010) and we focused on questioning strategies. In particular we describe two types of questions, routine and novel. To answer a routine question (e.g., "what is the common denominator here"), students are expected to know the answer to or know how to procedurally figure out the answer. Novel questions (e.g., "under what conditions would the orbit [of the satellite] have been hyperbolic?") require the students to give an opinion or to connect different pieces of knowledge in order to provide an answer that is not already known. Novel questions represent opportunities for the students to engage in mathematical work and to obtain meaning of what they are learning.

We sought then to contrast instructors' declared approaches with their interaction in the classroom, in terms of the types of questions they asked and their non-mathematical discursive interactions. We anticipated seeing instructors distributed along the spectrum of teaching approaches, and expected to see an association between the types of questions and the declared teaching approaches, with teachers espousing a 'meaning-making' or a 'student support' focus asking more novel questions, and with teachers espousing a more traditional approach asking more routine questions.

Results

Table 1 shows the comparison between the coding approaches and the percentage of instructor and student questions. The three first columns after the instructors' names represent the coding approaches drawn from the interviews. We considered an instructor holding one of the approaches when more than 10 percent of the codes fell in that category. As a result, we found four groups of instructors. In the first group are four instructors (Evan, Ernest, Emmet, and Elijah) that only hold a Traditional approach to teaching. In addition to a traditional approach, three instructors (Elliot, Edwina, and Elrod) exhibit a Meaning Making approach. A third group (Elizabeth, Edward, Earl, and Emily) holds all the three approaches. Finally, three instructors (Elena, Erin, and Erik) exhibit only Meaning Making and Student Support approaches, excluding a Traditional approach to teaching. Table 1 is organized from the more instructor-centered to the more student-centered instructors. The next three columns present the same three approaches but reflected in non-mathematical discursive interactions. So far, eight instructors have been coded. The shaded circles represent fifths of the relative proportion of the number of non-mathematical discursive interactions classified into one approach out of the total of discursive interactions coded for each instructor. For instance, in the case of Elrod, he exhibits 53% (40 to 59% range) of Traditional strategies, 29% of Meaning Making (20 to 39% range), and 18% of Student Support (0 to 19% range). These preliminary results show certain association between the declared approaches and the non-mathematical discursive interactions observed in the classrooms. Traditional instructors tended to use more traditional discursive interactions, such as following the book and covering the material, whereas instructors at the bottom of Table 1 used more Meaning Making and Student Support discursive interactions, such as making connection to real context and praising students.

Regarding classroom questioning, first, it is important to notice the large number of questions that instructors asked in these classrooms: on average instructors asked 90 questions per period (85 min long), with four instructors asking less than half of those per class. Other data (not included in Table 1) reveal that students ask 17 questions on average, which is consistent with a Traditional approach, in which the instructor holds the authority for managing interaction. From Table 1 we see a less clear pattern regarding the proportion of novel questions and instructors' declared approaches, although it appears that the Student Support group of instructors ask relatively fewer novel questions than instructors in the other groups.

In this preliminary analysis we found alignment between instructors' approaches to teaching derived from the interviews and their non-mathematical classroom discursive interactions, which was anticipated by the higher education literature. However, when looking at the proportions of novel questions asked, we do not see an association between the number of novel questions and the espoused teaching approaches. Our math education frameworks would have predicted larger proportions in the meaning making and student support categories. Our preliminary results hint at gaps in both traditions in analyzing instruction. For higher education researchers, instructors' espoused concepts and approaches to teaching are related to what instructors do in classroom, but they do not necessarily relate to mathematics content and mathematics learning opportunities. On the other hand, for mathematics researches is important to notice that although instructors' espoused theories might not be not related to mathematics content, they have an effect in the classroom. To understand the extent to which these non-mathematical discursive interactions of instruction have an effect in college math education we require further research.

Table 1: Comparison between teaching approaches and classroom practices [a]

Instructor	Traditional	Meaning Making	Student Support	Traditional classroom[b]	Meaning Making classroom[b]	Student Support classroom[b]	Total Instructor Questions per class period[c]	Instructor Novel Questions[d]
Evan	X			n.a.	n.a.	n.a.	46	26%
Ernest	X			n.a.	n.a.	n.a.	99	27%
Emmet	X			◑	◔	○	44	33%
Elijah	X			◕	○	◔	90	12%
Elliot	X	X		n.a.	n.a.	n.a.	89	16%
Edwina	X	X		n.a.	n.a.	n.a.	17	12%
Elrod	X	X		◑	◔	○	109	43%
Elizabeth	X	X	X	n.a.	n.a.	n.a.	73	35%
Edward	X	X	X	n.a.	n.a.	n.a.	85	28%
Earl	X	X	X	◑	◔	○	123	21%
Emily	X	X	X	○	◔	◑	92	14%
Elena		X	X	◑	◔	◔	176	16%
Erin		X	X	○	◑	◔	148	18%
Erik		X	X	○	◔	◑	163	3%

○ 0-19% ◔ 20-39% ◑ 40-59% ◕ 60-79% ● 80-100%

Notes: a. Because some of the instructors were observed in more than three times, the percentages of questions were obtained from the average of the amount of questions that instructors asked during each observed period. b. These categories represent the percentage by fifths of non-mathematical discursive interactions in the observed classrooms. c. A class period corresponds to 85 minutes. d. These percentages represent the number of novel questions out of routine and novel questions asked by each instructor. n.a.: not available for this preliminary report.

References

Ashwin, P. (2009). *Analysing teaching-learning interactions in higher education: Accounting for structure and agency*. London: Continuum International Pub. Group.

Cohen, D. K. (1990). A revolution in one classroom: The case of Mrs. Oublier. *Educational Evaluation and Policy Analysis, 12*, 327-345.

Cohen, D. K., Raudenbush, S. W., & Ball, D. L. (2003). Resources, instruction, and research. *Educational Evaluation and Policy Analysis, 25*, 119-142.

Creswell, J. W. (2005). *Educational Research: Planning, conducting, and evaluating quantitative and qualitativ research* (3rd ed.). Upper Saddle River, NJ: Pearson.

Gregory, J., & Jones, R. (2009). "Maintaining competence": A grounded theory typology of approaches to teaching in higher education. *Higher Education: The International Journal of Higher Education and Educational Planning, 57*(6), 769-785.

Grubb, W. N. & Associates. (1999). *Honored but invisible: An inside look at teaching in community colleges*. New York: Routledge.

Kane, R., Sandretto, S., & Heath, C. Telling half the story: A critical review of research on the teaching beliefs and practices of university academics. *Review of Educational Research, 72*(2), 177-228.

Kember, D. (1997). A reconceptualisation of the research into university academics' conceptions of teaching. *Learning and Instruction, 7*(3), 255-275.

Kember, D., & Gow, L. (1994). Orientations to teaching and their effect on the quality of student learning. *The Journal of Higher Education, 65*(1), 58.

Kember, D., & Kwan, K. (2000). Lecturers' approaches to teaching and their relationship to conceptions of good teaching. *Instructional Science, 28*(5-62), 469-490.

Menges, R., & Austin, A. (2001). Teaching in higher education. In V. Richardson (Ed.), *Handbook of Research on Teaching* (4th ed., pp. 1122-1156). Washington DC: American Educational Research Association.

Mesa, V., & Lande, E. (2010). *"Does that make sense?" Exploring community college instructors' questions in college mathematics courses* (Manuscript in preparation). Ann Arbor: University of Michigan.

Meyer, J. H. F., & Eley, M. G. (2006). The approaches to teaching inventory: A critique of its development and applicability. *British Journal of Educational Psychology, 76*(3), 633-649.

Nardi, E., Jaworksi, B., & Hegedus, S. (2005). A spectrum of pedagogical awareness for undergraduate mathematics: From 'tricks' to 'techniques'. *Journal for Research in Mathematics Education, 36*(4), 384-316.

Nathan, M. J., & Kim, S. (2009). Regulation of teacher elicitations in the mathematics classroom. *Cognition and Instruction, 27*(2), 91-120.

National Council of Teachers of Mathematics [NCTM]. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics [NCTM]. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

Pratt, D. (2005). Personal philosophies of teaching: A false promise? *Academe, 91*(1), 32.

Prosser, M., & Trigwell, K. (1999). *Understanding learning and teaching: The experience in higher education*. Buckingham [England]: Society for Research into Higher Education & Open University Press.

Speer, N. (2005). Issues of methods and theory in the study of mathematics teachers' professed and attributed beliefs. *Educational Studies in Mathematics, 58*, 361-391.

Speer, N., & Wagner, J. F. (2009). Knowledge needed by a teacher to provide analytic scaffolding during undergraduate mathematics classrooms discussions. *Journal for Research in Mathematics Education, 40*, 530-562.

Speer, N., Gutman, T., & Murphy, T. J. (2005). Mathematics teaching assistant preparation and development. *College Teaching, 53*, 75-80.

Stephan, M., & Rasmussen, C. L. (2002). Classroom mathematical practices in differential equations. *Journal of Mathematical Behavior, 21*, 459-490.

Truxaw, M. P., & DeFranco, T. C. (2008). Mapping mathematics classroom discourse and its implications for models of teaching. *Journal for Research in Mathematics Education, 39*(5), 489-525.

Wells, G. (1993). Reevaluating the IRF sequence: A proposal for the articulation of theories of activity and discourse for the analysis of teaching and learning in the classroom. *Linguistics and Education, 5*, 1-37.

Wells, G., & Arauz , R. M. (2006). Dialogue in the classroom. *The Journal of the Learning Sciences, 15*(3), 379-428.

Using Animations of Teaching to Probe the Didactical Contract in Community College Trigonometry Classes

Vilma Mesa and Patricio Herbst

Representations of teaching can be seen not only as cases of practice but also as probes on the rationality that practitioners use as they teach (Herbst & Chazan, 2003). Herbst and Chazan have developed a new kind of representation of teaching—animations of classroom scenarios, deliberately designed to probe some of the unspoken norms of classroom practice. Herbst and Miyakawa (2008) provided some details of how those animations are produced to be prototypes of models of instructional situations: Instructional situations are identified and modeled by hypothesizing the norms or tacit responsibilities of classroom participants in a situation, then scenarios are created that fulfill some of those norms but breach others; finally those scenarios are prototyped in a cartoon animation. Herbst, Nachlieli, and Chazan (in press) have shown how such animations can elicit data that informs about the rationality of teaching.

We describe how we applied those ideas in designing a research instrument that would be used to elicit community college teachers' practical rationality apropos of the knowledge management demands when solving problems on the board. We use the situation of 'finding vales of trigonometric functions' as context for this inquiry into the rationality that sustain larger contractual norms. The animations are meant to be representations of trigonometry teaching that occurs in a hypothetical community college that is similar to other large community colleges in the United States. Trigonometry is one of the mathematical domains conventionally taught in community colleges, either as a separate course or incorporated into other courses that are prerequisites to calculus (Lutzer, Rodi, Kirkman, & Maxwell, 2007). The course can be perceived as a skills- and knowledge-building course, in which the purpose is to ensure that students demonstrate competence in solving standard problems of trigonometry and familiarity with the definition and properties of the trigonometric functions. In the college where we collected the intact classroom data, the course has a guiding textbook and a master syllabus that outlines the knowledge for which students and instructors are held accountable.

The core question that we want to answer with the tool that we designed is: How much and what kinds of student participation do instructors perceive as feasible to handle when they work through examples at the board in a trigonometry class?

Identifying Key Norms of the Trigonometry Contract

The following describes our observations of the didactical contract in community college Trigonometry courses. The instructor is responsible for presenting the material and solving examples on the board. Students are responsible for doing homework, showing up for class, asking questions whenever they do not understand something, taking tests, and participating in class as demanded by the instructor. Students work under the assumption that their teachers are there to help them gain competence with the material and in general expect that their teachers press them for doing challenging work and believe that they are capable of doing what it takes to be successful (Mesa, 2010). The instructors are aware of the multiple demands that their students have on their time due to work and family responsibilities and have learned to not take it personally when students stop coming to their class (Grubb & Associates, 1999; Seidman, 1985). Instructors are also aware of the "holes" that students have in their mathematical preparation that hinder their opportunities to learn the content. They are also conscious that they have limited amount of time to ensure students' development of competence with the material.

When examining intact lessons' excerpts in which exemplification occurs, we have noticed the following:

- Instructors rarely ask questions regarding the plausibility or correctness of a response or a final solution to a problem;
- Instructors engage the students by asking questions about how to apply known procedures but they rarely, if ever, ask them to decide what procedure to apply;
- Instructors offer as examples problems that admit only one solution.

We hypothesize that these observations respond to contractual norms, that is, to tacit rules of the didactical contract. The instructional situation that has been chosen as context to explore the normative nature of those observations deals with solving the following problem:

> Using Fundamental Identities, find the exact values of the remaining trig functions given
>
> $$\sin x = -\frac{4}{5} \quad \cot x = \frac{-3}{4}$$

The original transcript of the class where the solution of this problem takes place illustrates what we believe are norms regarding exchange, division of labor, and organization of time. Exchange norms refer to what needs to be done, what it counts as, and what is not done; division of labor norms indicate who has the responsibility to do what, and norms of organization of time establish when things need to be done and how long they take. This particular problem calls for making a decision regarding the quadrant where the angle would be located, which permits the determination of the appropriate sign for the value of cosine of the angle, which is used to derive the value for the secant of the angle. As the solution unfolds, asking for justification of the steps or whether the answers make sense is not done; it appears that there is no explicitly assigned responsibility for justifying steps in the process and that the instructor alone determines how the solution unfolds; we also believe that the swiftness with which the problem is solved is related to the need of conveying the idea that problems are easy and that the homework won't take long. To test whether these are reasonable hypothesis, we created alternative scenarios in which some of the hypothesized norms are breached and we seek input from instructors regarding those breaches.

Consider the following scenario:

Teacher: So we know sine and cotangent, **what do you think we should do now?**

Male1: We can draw the unit circle and put these ratios in…

M2: We could draw the graphs of sin and cot and see what *x* gives us those values…

Female1: Nah, I think it is simpler than that. We could use that thing about the quadrants and the signs of the functions…

F2: We could use a circle with radius 5 and, then sin -5 over 4 is saying that the opposite is -4… so the angle must be somewhere here [on Quadrants 3 or 4], then the adjacent is….

M1: The point must be (3, -4) because of the cotangent; that's quadrant four.

M2: **We could flip sine *x* to get -5/4 for cosecant**
F2: **and tangent would be flipping cotangent…**
M2: **so cosine is… is three over five.**
F1: **and cosecant is just flipping that one. We're done, we got them all.**
T: **that's OK, but,** …

This scenario is meant to address the issue of control over the solution process, with students answering the problem using 'old' rather than the current material ('fundamental [trig] identities'). We anticipate that teachers won't see it feasible to relinquish control for two reasons. First, teachers perceive students as expecting the instructor to be in control, showing how things are done, and with the responsibility of explaining the content; in principle students are perceived as capable of negative reactions to what other students have to say, because they do not see their peers as having authority of knowledge to do that (Cox, 2009). Second, there is too much material to cover and a very efficient way to handle it in reasonable time is for the instructor to illustrate the process so students can mimic it later (Grubb & Associates, 1999). In this scenario, the students have 'solved' the problem but it is of less value or import, because it does not use the content of the unit. The instructor will need to validate the solution given by the students or to reject it as inadequate for the expected use of the new content. Thus, if the teacher gives control of the solution to the students he or she risks loosing control of the exchange value of the problem/solution. In either case, we hypothesize, the instructors would make sure that in addition to the proposed solution, the students would also see how the new content is used.

With scenarios such as these we expect to be able to uncover the resources instructors have at their disposal for making decisions regarding how to manage similar situations. They would either align with or distance from the teacher in the scenario and in that process they would make explicit what they do that the animated teacher does not. The information that we gather in this way, will allow us to map out community college instructors' rationality in teaching trigonometry with examples, as we test these animations with groups of faculty.

During our presentation in the conference we want to share a preview of the animation illustrated above and get input from the audience regarding its use as a research tool; if available—the animations are being produced now—we will share preliminary data illustrating teacher's reactions to the animations, and how the analysis allows us to formulate more specific conjectures regarding the reasons for the level of student participation that instructors perceive as feasible to handle when they work through examples at the board in a trigonometry class.

Answering this question is fundamental to understand the extent to which calls for reform of undergraduate college math classes (Blair, 2006), in which students play a more significant role in the construction of knowledge, can be effectively carried out at the community college level.

References

Blair, R. (Ed.). (2006). *Beyond crossroads: Implementing mathematics standards in the first two years of college*. Memphis, TN: American Mathematical Association of Two Year Colleges.

Brousseau, G. (1997). *Theory of didactic situations in mathematics* (N. Balacheff, Trans.). Dordrecht, Netherlands: Kluwer.

Cox, R. D. (2009). *The college fear factor: How students and professors misunderstand one another*. Boston, MA: Harvard University Press.

Grubb, N. W., & Associates. (1999). *Honored but invisible: An inside look at teaching in community colleges*. New York: Routledge.

Herbst, P., & Chazan, D. (2003). *ThEMaT: Thought Experiments in Mathematics Teaching*: NSF Research grant proposal: ESI-0353285.

Herbst, P., & Miyakawa, T. (2008). When, how, and why prove theorems: A methdology to study the perspective of geometry teachers. *ZDM The International Journal on Mathematics Education, 30*, 469-486.

Herbst, P., Nachlieli, T., & Chazan, D. (in press). Studying the practical rationality of mathematics teaching: What goes into "installing" a theorem in geometry? *Cognition and Instruction*.

Lutzer, D. J., Rodi, S. B., Kirkman, E. E., & Maxwell, J. W. (2007). *Statistical abstract of undergraduate programs in the mathematical sciences in the United States: Fall 2005 CBMS Survey*. Washington, DC: American Mathematical Society.

Mesa, V. (2010). *Achievement goal orientation of community college mathematics students and the misalignment of instructors' perceptions* (Manuscript under review). Ann Arbor, MI: University of Michigan.

Seidman, E. (1985). *In the words of the faculty: Perspectives on improving teaching and educational quality in community colleges*. San Francisco, CA: Jossey-Bass.

Mathematics Faculty's Efforts to Improve the Teaching of Undergraduate Mathematics

Susana Miller

In recent years, much attention has been given to the pre-service preparation and professional development of mathematics teachers at the elementary, middle, and high school levels. Researchers have concluded that strong content knowledge is not enough to insure effective teaching. Yet, many colleges require little to no professional development for their mathematics faculty. Without supports similar to those provided to K-12 teachers, how do college mathematics faculty members develop and improve their teaching of undergraduate mathematics? A department-wide survey and follow-up interviews were used to investigate if and how the mathematics faculty at one research university have acquired and honed skills for teaching undergraduate mathematics. Preliminary analyses of this data will be presented, and feedback for future directions will be solicited. Understanding if and how mathematics faculty currently seek supports for improving their teaching can inform the design of future professional development programs for college mathematics faculty.

Keywords: professional development, undergraduate mathematics instruction, teaching resources, mixed methods research

Learning and making sense of mathematics is a complex psychological, cognitive, and social process. Research suggests that mathematics content knowledge is not sufficient for teaching, even at the earliest stages of schooling (Hill, Rowan, & Ball,, 2005). Darling Hammond argues that "teachers who have more preparation for teaching are more confident and successful with students than those that have had little or none" (pg. 167, 2000). Why should college-level mathematics instruction be any different? New faculty members face the same challenges of developing, testing, and honing their teaching skills; more experienced faculty members may need to adapt their current skills to accommodate a new generation of learners who may have graduated from reformed and technology rich high school classrooms. Understanding if and how college mathematics faculty members pursue various supports when coping with these challenges can serve an important role in the design of future professional development materials. It is possible that without external supports some college mathematics faculty learn from their own teaching by planning, executing, reflecting on and revising lessons, a method similar to that described by Hiebert, Morris, Berk and Jansen (2007).

According to Lortie's notion of the *apprenticeship of observation* (1975), teachers develop beliefs, ideas, and images of the work of teaching as they observe their own teachers teach during their many years as school and university students. Analysis of interview data from a pilot study I conducted last year indicated that the same was true for many mathematics college instructors. Without formal training in education, it is not surprising that faculty members often rely on their own experiences as students in undergraduate and graduate mathematics courses to build a vision of how college mathematics instruction should or should not look. This can be problematic because faculty with advanced degrees in mathematics may not have ever experienced the struggles their undergraduate students often encounter in "elementary" undergraduate mathematics classes.

My research investigates if, how, and where mathematics faculty find supports for developing and honing their skills for teaching undergraduate mathematics, and which faculty members are most likely to seek out this type of support. In particular, this research study was designed to pursue the following research questions, working with a population of mathematics faculty members at one research university: (1) What efforts, if any, do mathematics faculty at employ to improve their teaching of undergraduate mathematics?, (2) What ideas do mathematics faculty members have about what it means to improve one's teaching, and what do they take as evidence that one's teaching has improved?, and (3) What demographic trends, if any, exist among faculty members who report interest in improving their teaching? These questions lead to a mixed methods approach which is exploratory rather than evaluative in nature. In this study, I use a combination of surveys and interviews to understand what resources mathematics faculty at one research university have explored and which resources they have found most useful. In the first phase of the study, I sent an email invitation to all faculty in the mathematics department at one research university to participate in an online-survey. The survey consisted of four parts. The first part consisted of items designed to collect demographic data from the survey participants, such as their current position in the department, their years of teaching experience, and their education background. The second part consisted of eight Likert scale questions about their beliefs about teaching undergraduate mathematics courses and about improving teaching. The third section of the survey contained a few free response questions about their efforts and opportunities to learn about and improve their teaching. The final section asked participants if they would be willing to participate in a follow-up survey, and if so, to provide their contact information. At this point in time, I have sent the email invitation to the mathematics faculty members in the department to complete the online survey. Another e-mail will be sent approximately two weeks from now as a gentle reminder to those who have not completed the survey.

The second phase of the study consists of conducting follow-up audio-taped interviews with at least eight of the faculty members who completed the survey and agree to be interviewed. The faculty who are interviewed will be chosen to best represent the overall population of those who responded to the survey. The follow up interview will include three parts. The first set of interview questions provides an opportunity for the participant to reflect on and share information about their own teaching practice and their efforts to improve their teaching. The second part of the interview asks the participant to read and analyze a brief written vignette from a hypothetical undergraduate mathematics class. The third and final part of the interview provides an opportunity for me to follow up on specific responses the participant provided on the survey. The interviews will be completed no later than December of this year. Thus, by the time of the conference I will have gathered and conducted at least a preliminary analysis of all of my data.

The survey and interview data will be analyzed to explore trends observed through an initial review of the data. The majority of the analysis will be qualitative in nature, but some simple quantitative analyses may be performed to indicate frequency of particular types of responses or the mean and standard deviation of certain categories of responses to certain items. I will aim to develop group-level, sub-group level, and individual-level claims from the analyses.

For example, I hope to disaggregate participants' responses according to specific demographic features such as the number of years of teaching experience or amount of formal training in education. This will highlight major themes in the responses including which resources and strategies for improving teaching are most frequently mentioned and which strategies and resources faculty report as most helpful. The exploratory nature of this research makes it difficult to provide more specific details about the analysis.

Depending on the final response rate to the survey, I may choose to proceed beyond this study in multiple ways. Ideally I will have a large response and a rich data set which I can use as the basis of my dissertation research. There are several other options I am considering which I can pursue whether my data is as rich as I anticipate or not. One option would be to use the information from this survey to construct a more targeted and detailed survey to be used with the mathematics department at another research university. Another option would be to focus in on the practice of one faculty member as he/she endeavors to institute changes to improve his/her teaching practice. I also could repeat this data collection and analysis at a teaching-focused college or university and/or at a community college and then compare and contrast the responses.

Discussion Questions

- Based on the preliminary analysis provided, what additional queries would you have about trends in the data? What story about the data would you like to hear?
- If I were to conduct an additional interview with one or more of my participants, what kinds of questions should I ask? Which of my participants might be a good choice for targeted case studies?
- What journals might be a good fit to publish this research in? Would the results of my research would be useful to practitioners?
- How do the mathematics faculty members in attendance feel about the goals, methods, and results of this study? Do they relate to it? Object to it? Find it surprising or typical?
- How might the findings from this study and follow-up studies inform the development of future professional development programs for college mathematics faculty?

References

Darling-Hammond, L. (2000). How teacher education matters. *Journal of Teacher Education, 51*(3), 166–173.

Hiebert, J., Morris, A. K., Berk, D., Jansen, A. (2007). Preparing teachers to learn from teaching. *Journal of Teacher Education, 58*(1), 1–15.

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal, 42*(2), 371–406.

Lortie, D.C. (1975). Schoolteacher. Chicago. Chicago University Press.

Mathematicians' Pedagogical Thoughts and Practices in Proof Presentation

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Abstract:

Little is known about how mathematicians present proofs in undergraduate courses. This descriptive study uses ethnographic methods to explore proof presentations at a large comprehensive research university in the Midwest. We will investigate three research questions: What pedagogical moves do mathematics faculty members make when presenting proofs in a traditional undergraduate classroom? What do mathematics faculty members contemplate as they plan lectures that include proof presentations? To what degree and in what ways do faculty members engage students when presenting proofs? To pursue these questions, four faculty members who were teaching proof-based mathematics courses were interviewed and 6-7 observations of each classroom were conducted throughout the course of the semester. The data were analyzed to identify some of the pedagogical content tools that were used, to develop an observation instrument, and to understand how mathematicians think about the pedagogy of proof presentation.

Keywords: proof presentation, pedagogical content tools, teaching proof, ethnographic methods

Literature Review:

It has been well documented that students struggle with mathematical proof (Grassl & Mingus, 2007; Larsen, 2009; Larsen & Zandieh, 2008; Selden & Selden, 2003). The transition from computational mathematics to formal mathematics is a dramatic shift (Tall, 1997). Undergraduate level proof-based mathematics courses have been studied by mathematics educators for the past few decades. This research mostly comes in two flavors: investigating student thinking (Knuth, 2002; Larsen, 2009; Simpson & Stehlikova, 2006; Healey & Hoyles, 2000; Almeida, 2000; Selden & Selden, 2003) and developmental research projects (Gravemeijer, 1994) that focus on developing innovative ways to teach proof (Leron and Dubinski,1992; Larsen, 2009; Weber, 2006). These studies shed light on teaching and learning in the context of mathematical proof, but it is often difficult to translate these findings into widespread changes in teaching.

It is generally acknowledged that lecture is the norm in most university classrooms. The lecture style has been criticized by many, especially by those who propose alternative, more interactive teaching methods (Leron & Dubinski, 1995; Leron, 1985; Larsen, 2009). Leron (1985) called for a divergence from a linear proof presentation method in favor of "heuristic" presentations, which give the audience a better idea of how the ideas were constructed. The "pure telling" lecture-style format has generally been contrasted with inquiry-oriented teaching (Rasmussen & Marrongelle, 2006), but personal experience tells us that many instructors are somewhere in between those two extremes. Little is known about how variations within the lecture style of proof presentation affect student understanding.

There are very few research projects directed at what is currently going on in a traditional university classroom. In the area of geoscience education (Markley, Miller, Kneeshaw & Herbert, 2009), a study was done to study the relationship between instructors' conceptions and practice in the classroom. There were interviews with the faculty members about their perceptions of teaching and learning, and then there were observations of their classrooms. The observation data focused on how the instructor interacted with students and whether or not the classrooms were student centered. In mathematics education, a recent study addressed the issue of proof presentation by interviewing nine mathematics faculty members to explore their pedagogical decisions concerning proof presentations (Weber, 2010). Fukawa-Connelly (2010) observed a mathematics faculty member over the course of a semester in a traditionally taught abstract algebra course. He analyzed classroom dialogue through the lens of pedagogical content tools, looking for instances in which the faculty member 'modeled mathematical behaviors.' This study gives an existence proof that university mathematics professors do not always use a "pure telling" method of proof presentation.

While some studies are beginning to address proof presentation, much more work needs to be done. Most of these proof-based courses are taught by working mathematicians, who are likely unfamiliar with current mathematics education research. Though an instructor identifies himself as traditional, he may still make efforts to involve and engage students in proof construction, but may not be familiar enough with the language of mathematics education to describe his pedagogical moves. This study will combine faculty interviews with classroom observations to explore not only how mathematics faculty members think about presenting proof, but also what they do in practice.

This study has several goals; one is to investigate how the faculty members' pedagogical ideas about proof presentations manifest themselves in the classroom. Another aim is to analyze the nuances of traditional teaching methods in regard to proof presentation, and to identify some of the tools that mathematics faculty members currently use to help students understand proof and write their own proofs. A final goal is to develop an observation instrument to simplify data collection and analysis. The video data will be useful both to develop an observation instrument and to minimize validity concerns, since the instrument is in the developmental stages.

Research Questions:

What pedagogical moves do mathematics faculty members make when presenting proofs in a traditional undergraduate classroom? What do mathematics faculty members contemplate as they plan lectures that include proof presentations? To what degree and in what ways do faculty members engage students when presenting proofs?

Methodology:

Since the teaching and learning of mathematics can be viewed as an enculturation process, we will view the data through an interpretivist lens, which "looks for culturally derived and historically situated interpretations of the social real world" (Crotty, 1998, p. 72). The instructor is viewed as an expert in the discourse on mathematical proof, trying to help the "newcomers" enter into the discourse community (Sfard, 2008). This discourse can be analyzed through symbolic interactionism, because the language and other communicative tools that the professor is using to help the students understand will be studied (Crotty, 1998). Since the classroom is studied as a culture, pragmatism will be our theoretical perspective (Morgan, 2007), which is generally associated with the ethnographic methods that will be used.

The first phase will be semi-structured interviews with four faculty members at a large comprehensive research university in the Midwest. These faculty members are currently teaching undergraduate level math courses that emphasize mathematical proof. The interviews will address what the instructors do when they present proofs in class, why they make those choices, and what they do to help students understand their presentation of proofs in class. The interview data will be analyzed for emergent themes.

Throughout the semester, 6-7 observations of each classroom will be conducted and analyzed in detail. Three of the participants agreed to allow the observations to be video-taped, and for the fourth, we will analyze field notes collected with an observation instrument. Though much of the data analysis will be qualitative, some of the qualitative observation data can be quantified (Chi, 2007) to more easily see the trends that occur. The researcher is developing an observation instrument to collect data about proof presentations. The first draft of the instrument was based solely on the researcher's experience as an observer and as a student in proof based mathematics courses. The themes from the interviews will be used to modify the observation instrument, and as the observations occur throughout the semester, the observation instrument will evolve. Because the instrument is not in its final form, video data is crucial, because the researcher may need to go back to look at earlier observations.

Before the final data analysis, there will be an additional interview with the faculty members for a member check. At this time, the participants will be able to see the themes and trends that have emerged, and they will have the opportunity to give an insider's perspective into the data. Since the researchers are constructing their own knowledge about how proofs are presented in class (VonGlassersfeld, 1996), the input of the participants will be a valuable resource for data analysis.

Applications to Further Research:

As we work to describe how faculty members present proofs in class and what they think about the pedagogy of proof presentation, we hope to be able to identify more pedagogical content tools (Rasmussen & Marrongelle, 2006) that they use to train students in reading and writing proof, and to help students enter into the culture of mathematical proof. Once we are able to identify some of these tools, we hope to be able to design some studies that can investigate their value. The recent work of Mejia-Ramos, Weber, Fuller, Samkoff, Search, & Rhoads, (2010), has designed a model for proof comprehension with six different dimensions that can be assessed by a quiz. Future research will combine their method of assessment with the results of this study to evaluate the efficacy of different methods of proof presentation in a traditionally taught proof-based course.

Questions:

Do you have any suggestions about how to analyze the data from the classroom where I was not video-taping? Should that data be thrown out entirely?

Are there suggestions for the observation instrument? Have any of you used an observation instrument in the past?

We plan to design a study to evaluate the pedagogical content tools we have identified. Any suggestions about study design?

References:

Almeida, D. (2000). A survey of mathematics undergraduates' interaction with proof: Some implications for mathematics education. *International Journal of Mathematics Education in Science and Technology, 31*(6), 869-890.

Chi, M. (2007). Quantifying qualitative analyses of verbal data: A practical guide. *The Journal of the Learning Sciences, 6*(3) 271-315.

Crotty, M. (1998). *The Foundations of social research: Meaning and perspective in the research process.* Thousand Oaks, CA: Sage Publications USA.

Fukawa-Connelly, T. (2010). Modeling mathematical behaviors; Making sense of traditional teachers of advanced mathematics courses pedagogical moves. *Proceedings of the 13th Annual Conference on Research in Undergraduate Mathematics Education*.

Grassl, R., & Mingus, T. T. Y. (2007). Team teaching and cooperative groups in abstract algebra: Nurturing a new generation of confident mathematics teachers. *International Journal of Mathematical Education in Science and Technology, 38*(5), 581-597.

Gravemeijer, K. (1994). Educational development and developmental research in mathematics education. *Journal for Research in Mathematics Education 25*(5), 443-471.

Healey, L., & Hoyles, C. (2000). A study of proof conceptions in algebra. *Journal for Research in Mathematics Education, 31,* 396-428.

Knuth, E. (2002). Teachers' conceptions of proof in the context of secondary school mathematics. *Journal of Mathematics Teacher Education, 5*, 61-88.

Larsen, S. (2009) Reinventing the concepts of group and isomorphism: The case of Jessica and Sandra. *The Journal of Mathematical Behavior, 28*(2-3), 119-137.

Larsen, S., & Zandieh, M. (2008). Proofs and refutations in the undergraduate mathematics classroom. *Educational Studies in Mathematics, 67*(3), 205-216.

Leron, U. (1985). Heuristic presentations: The role of structuring. *For the Learning of Mathematics 5*(3), 7-13.

Leron, U. and Dubinski, E. (1992). An abstract algebra story. *American Mathematical Monthly.* 102, 227-242.

Markley, C., Miller, H., Kneeshaw, T., & Herbert, B. (2009). The relationship between instructors' conceptions of geosciences learning and classroom practice at a research university. *Journal of Geoscience Education. 57*(4), 264-274.

Mejia-Ramos, J., Weber, K., Fuller, E., Samkoff, A., Search, R., Rhoads, K., (2010). Modeling the comprehension of proof in undergraduate mathematics. *Proceedings of the 13th Annual Conference on Research in Undergraduate Mathematics Education*.

Morgan, D. L. (2007). Paradigms lost and pragmatism regained: Methodological implications of combining qualitative and quantitative methods. *Journal of Mixed Methods Research, 1*(1), 48-76.

Rasmussen, C. & Marrongelle, K. (2006). Pedagogical content tools: Integrating student reasoning and mathematics in instruction. *Journal for Research in Mathematics Education 37*(5), 388-420.

Selden, A., & Selden, J. (2003). Validations of proofs considered as texts: Can undergraduates tell whether an argument proves a theorem? *Journal for Research in Mathematics Education, 34(1),* 4-36.

Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing*. Cambridge. New York, NY.

Simpson, A. & Stehlikova, N. (2006). Apprehending mathematical structure: A case study of coming to understand a commutative ring. *Educational Studies in Mathematics.* 61, 347-371.

Tall, D. (1997). From school to university: The transition from elementary to advanced mathematical thinking. Presented at the Australasian Bridging Conference in Mathematics at Auckland University, New Zealand on July 13th, 1997.

Von Glassersfeld, E. (1996). Aspects of radical constructivism and its educational recommendations. In Steffe, L., Nesher, P., Cobb, P., Goldin, G., & Greer, B. (eds) *Theories of Mathematical Learning*. Lawrence Erlbaum Associates, Mahwah, NJ. p. 307-313.

Weber, K. (2006). Investigating and teaching the processes used to construct proofs. In Hitt, F., Harel, G., and Selden, A. (eds) *Research in Collegiate Mathematics Education IV.* American Mathematical Society.

Weber, K. (2010). The pedagogical practice of mathematicians with respect to proof. *Proceedings of the 13th Annual Conference on Research in Undergraduate Mathematics Education*.

Geometric Constructions to Activate Inductive and Deductive Thinking Among Secondary Teachers

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For this preliminary research report, I have two goals: a) to present initial findings from the pilot study, and b) to use feedback from the session to design a more robust follow-up study.

The following research question formed the basis of the pilot study:

> To what extent can students improve their abilities in geometric reasoning and proof through learning experiences that combine dynamic geometry software and traditional compass and straightedge constructions?

Students were provided an inquiry-oriented, construction-based experience dealing with Euclidean geometry topics. Researchers hoped to demonstrate that such an experience can gain increase students' ability to write deductive proofs. A learning environment was created that involved extensive work with constructions using traditional compass and straightedge techniques as well as with dynamic geometry software. A major piece of the work was a rigorous program of "deconstructions" whereby participants gave written and oral validations of each construction. A pre-test/post test consisting of formal, written proofs served as one assessment instrument.

Building on the work of Hollebrands (2003; 2004) and Galindo (1998), written response analysis was used to determine students' understandings and attainment of reasoning abilities related to transformational geometry. I propose in the follow-up study to use a mixed-method approach, adding task-based interviews.
A case study methodology, as described by Gall, Borg, and Gall (1996) and Lincoln and Guba (1985) will be particularly helpful in producing a detailed description of a phenomenon. In this instance, the specific phenomenon under study is geometric reasoning and proof, situated within the context of a technological learning environment. Since part of the goal of the research is to describe the interaction of traditional and modern construction techniques as they related to concept development, case study is particularly useful (Yin, 1994).

In addition, a quantitative analysis will be utilized. The construct of "normalized gain," initially described by Hake (1998) is appropriate in this setting. Widely duplicated (e.g. Fagen, Crouch & Mazure, 2002), normalized gain has been shown to provide meaningful data on changes in student achievement. A small number of students in the pilot completed a pre-test/posttest inventory (Usiskin, 1982) designed to assess how students' levels of geometric reasoning is affected by particular, targeted instruction. This inventory is supported by a large reliability and validity database. Preliminary results will be reported, although the overarching goal is to administer such an instrument to a larger number of participants in the future.

This proposal is for a work in progress. Data from the pilot study have been collected, but not yet fully analyzed. I anticipate analyzing these data shortly. A follow-up study is planned for late spring or summer of 2011.

The need for studying geometric learning is great. Clements (2003) points out in compelling detail the generally poor performance of students in the field of geometry. Other researchers have found limited success in improving particular aspects of student work. Pandiscio (2002) found that participants can fundamentally alter their view of proof as it relates to inductive reasoning through targeted instruction. Hiebert (2003) describes how students "learn what they have the opportunity to learn," citing the success of focused outcomes aligned with particular instructional settings. Other studies have determined that proportional and conceptual reasoning can be enhance through curricular modifications (Ben-Chaim, et. al., 1998). Battista and Clements (1995) posit that alternatives to axiomatic approaches may lead to greater success in students' proof and reasoning abilities, and cite Geometer's Sketchpad as viable platform. With the development of GeoGebra (Hohenwarter, 2002) we now have a freely available, open source software that combines many features of dynamic geometry software and computer algebra systems into a single package. Hohenwarter and Jones (2007) have posited great potential for helping students to visualize mathematics through GeoGebra. Based on this, a rationale exists to examine the stated research questions.

Preliminary questions for the audience:
1. Does the essential research question have enough merit to justify a follow-up study?
2. Does anyone know of empirical evidence to support the intuitive suggestion that by deliberately strengthening inductive reasoning, students will increase their proficiency at formal geometric deduction?
3. How do I address the concern that it is unlikely to have more than 10-12 students participate in a follow-up study that requires a substantial commitment of time on the part of the students?
4. Are there well-designed instruments to measure student's ability at writing formal geometric proofs?
5. How likely is it that task-based interviews will reveal student reasoning about geometric ideas

Brief References:

Battista, M.T., & Clements, D.H. (1995). Geometry and proof. *The Mathematics Teacher, 88*(1), 48-54.

Ben-Chaim, D, Fey,, J.T., Fitzgerald, W.M., Benedetto, C., & Miller, J. (1998). Proportional reasoning among 7th grade students with different curricular experiences. *Educational Studies in Mathematics, 36*(3), 247-273.

Clements, D. H. (2003). Teaching and learning geometry. In J. Kilpatrick, W.G. Martin & D. Schifter (Eds.) *Research Companion to Principles and Standards for School Mathematics* (pp. 151-178). Reston, VA: National Council of Teachers of Mathematics.

Fagen, A.P., Crouch, C.H., & Mazur, E. (2002). Peer instruction: Results from a range of classrooms. *The Physics Teacher*, *40*(5).

Galindo, E. (1998). Assessing justification and proof in geometry classes taught using dynamic software. *The Mathematics Teacher*, *91*(1), 76-82.

Gall, M.D., Borg, W.R., & Gall, J.P. (1996). *Educational research: An introduction. White Plains,* New York: Longman Publishers.

Hake, R.R. (1998). Interactive-engagement versus traditional methods: A six-thousand-student survey of mechanics test data for introductory physics courses. *American Journal of Physics. 66*(1), 64-74.

Hiebert, J. (2003). *What research says about the NCTM standards*. In J. Kilpatrick, W.G. Martin & D. Schifter (Eds.) *Research Companion to Principles and Standards for School Mathematics* (pp. 5-23). Reston, VA: National Council of Teachers of Mathematics.

Hill, H.C., & Ball, D. L. (2004). Learning mathematics for teaching: Results from California's mathematics professional development institutes. *Journal for Research in Mathematics Education*, *35*(5), 330-351.

Hohenwarter, M. (2002). *GeoGebra – Ein software system fur dynamische geometrie and algebra der ebene*. Masters thesis. University of Salzburg.
Hohenwarter

Hohenwarter, M., & Jones, K. (2007). Ways of linking geometry and algebra: The case of *GeoGebra*. In D. Kuchemann (Ed.) *Proceedings of the British Society for Research into Learning Mathematics, 27*(3), 126-131.

Hollebrands, K. F. (2003). High school students' understandings of geometric transformations in the context of a technological environment. *Journal of Mathematical Behavior*, *22*(1), 55-72.

Hollebrands, K. F. (2004). High School Students' intuitive understandings of geometric transformations. *The Mathematics Teacher*, *97*(3), 207-214.

Lincoln, Y.S., & Guba, E.G. (1985). *Naturalistic Inquiry*. Beverly Hills, CA: Sage.

Pandiscio, E.A. (2002). Exploring the link between preservice teachers' conception of proof and the use of dynamic geometry software. *School Science and Mathematics, 102*(5), 216-221.

Usiskin, Z. (1982). *Van Hiele levels of achievement in secondary school geometry*. Chicago, IL: The University of Chicago Cognitive Development and Achievement in Secondary School Geometry Project.

Yin, R.K. (1994). *Case study research: Design and methods (2nd ed.)*. Thousand Oaks, CA: Sage Publications.

Title: The Internal Disciplinarian: Who is in Control?
Preliminary Research Report.

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Key words: Professional development, lecture research, decisions

Abstract
A group of mathematicians and mathematics educators are collaborating in the fine-grained examination of selected ‘slices’ of video recordings of lectures drawing on Schoenfeld’s KOG framework of teaching-in-context. We seek to examine ways in which this model can be extended to examine university lecturing. In the process we have identified a number of lecturer behaviours There are times when, in what appears to be an internal dialogue, lecturing decisions are driven by the mathematician within the lecturer despite the pre-stated intentions of the lecturer to be a teacher.

Introduction
In contrast with the manner in which a school teacher’s Knowledge, Orientation and Goals (KOGs) determine their decision making, we present evidence that for mathematicians this decision making is additionally complicated by an inner argument between the lecturer-as-mathematician and the lecturer-as-teacher. Are there conflicting orientations and goals active in the decision moment? The way in which the decisions play out is thus a function not only of the lecturer’s knowledge of mathematics but of the way they work mathematically.

Research base
This paper reports on an aspect of a project that explores how Schoenfeld’s KOGs may be used to direct lecturers’ attention to aspects of their decision-making in the lecture theatre as a professional development activity. The project is informed by research concerning how a teacher’s knowledge, orientations or beliefs and goals impact on their teaching practice (Schoenfeld, 2007; Ball, Bass & Hill, 2004; Shulman, 1986; Speer, Smith & Horvath, 2010; Torner, Tolka, Rosken & Sriraman, 2010) However, these are studies of teacher practice in primary and secondary schools and similar work at the college level is ‘virtually non-existent’ (Speer et al, 2010, p 99). The project is also designed to build on the effectiveness of communities of practice (Lave & Wenger, 1999) and a culture of enquiring conversation (Rowland, 2000) for professional development. The project is described in more detail in Barton, Oates, Paterson and Thomas (to be published).

Structure of project and data collection
A group of four mathematicians and four mathematics educators are collaborating in the fine-grained examination and discussion of lecturer actions in video recordings of lectures (Kazemi, Franke, Lampert, 2009; Prushiek, McCarty, & Mcintyre, 2001). The theoretical approach draws on Schoenfeld’s theory of teaching-in-context (Schoenfeld, 2002). The data for each lecture consists of videotape, an observer record, and a written lecturer-KOG (a statement by the lecturer of the knowledge used, orientation held, and goals, both specific goals intended for the lecture and more

general educational goals). A section of the lecture is chosen for discussion by the lecturer and is transcribed along with all discussion of the lecture.

The project aims to examine ways in which Schoenfeld's model can be used and extended to examine university lecturing and to support the professional development of lecturers (Van Ort, Woodtli, Hazard, 1991)) In the process we have identified a number of lecturer behaviours one of which is discussed in this paper.

Observations and Discussion
Schoenfeld (2007) argues that

> Teaching, …… depends on a large skill and knowledge base … its practice involves a significant amount of routine activity punctuated by occasional and at times unplanned but critically important decision making – decision making that can determine the success or failure of the effort. (p 33)

We have observed a number of instances of what appears to be an inner argument, or regulation by an inner voice, in the lecturer's communication with the class. In subsequent group discussion it has become clear that many lecturers are aware of this.

The example below is from a lecture to a general education first year course in which the students are introduced to the Fibonacci sequence and the golden ratio. In her personal KOG, written before the lecture, the lecturer stated:

> Knowledge I need includes: knowledge of the subject material, knowledge of the levels of the students.
>
> Orientation: I see this whole course partially as an exercise in 'public understanding of mathematics', and so try to treat the lectures as such – rarely going into much depth mathematically, and trying to keep everyone engaged and interested.
>
> Goals: for the students to appreciate the appearance of Fibonacci numbers in nature. To keep all the students engaged throughout the lecture.

In the lecture, once they have found the sequence of numbers, a recursive formula for the sequence, and arranged them in a table she says to the students:

> ML1: Then compute the ratio as you go down for each one. So for instance I have got 1divided by 1 is 1, 2 divided by 1 is 2 and this next one will be 3 divided by 2 which is 1.5. If you have a calculator you can calculate what they are otherwise you can leave them as fractions and I'll write down what they are in decimal notation."

After they have worked them out she continues:

> ML1: What's 5 divided by 3?
> Student: 1.6
> ML1: 1.6 recurring so I'll put 1.667
> (*looks at it a brief moment)* **dot dot dot**"

As a group we examined what caused her to pause and decided it appeared that she felt that the fact 5/3 was a recurring decimal had to be acknowledged. It led to the following exchange:

> ML2: I do this kind of thing all the time, I think it's really distracting because you've gone

> out and tried to make your big point and then you get all flustered over some detail and you say oh sorry you know, you have to get it right and the students go "what the hell is going on and now I'm completely confused because it sounded really simple."
> ML1: So should you just ignore that corner and just hope that it's not noticed but then is that bad because you've somehow told them something incorrect?

We saw another instance of the lecturer's need for rigor later in the lecture. Note that nowhere in her written KOG does she mention rigor, on the contrary she says, "rarely going into much depth mathematically".

She is proving that the value for f_{n+1} divided by f_n is the golden ratio, ψ. There is some literal hand-waving as it is established that the values oscillate about 1.6 something and then she says

> OK suppose you want to compute what this number actually is
> And it seems to be converging – **and it does actually converge** (*who is she reassuring*?)
> So you know that f_{n+1} is bigger than f_n so this is going to be a number that is bigger than 1
> Right? (*sounds as if she hears herself and adds this*) **Or equal to 1**.
> So ..If I am thinking about what this ratio becomes as n gets really, really big
> **So, for any specific n these 2 things are going to be different**
> Right?
> Because for one thing it was 1.6 and for the next one it was 1.625
> So for any specific n it's going to be different

In this interlude we see and hear her spending a lot of time emphasising that for particular values of *n* the values of f_n and f_{n+1} are different and under what circumstances they are justified in making the approximation:

> But as *n* gets bigger and bigger and bigger these 2 things are going to get closer and closer together
> As long as *n* is big enough
> So we will assume that we are in a place where *n* is big enough then **we can make this approximation**

The highlighted language in this excerpt shows her need to be mathematically explicit; hand waving will not do even in a class that is 'an exercise in public understanding of mathematics.'

Further examples seem to indicate that the manner in which the inner argument manifests appears to differ depending on the research field of the mathematician. A pure mathematician in the group spent a long time disentangling notation to ensure that a proof held together effectively even while he had stated that he believed the students were capable of deriving it for themselves. When discussing these actions he spoke of "KOG dissonance" to refer to his actions in contradiction of his stated intentions. An applied mathematician had a similarly mathematician-inspired interaction with a Matlab generated display that did not show what he knew it should show about the number of bifurcations in a logistic equations.

Conclusion

It is not our argument that schoolteachers do not have an inner mathematical voice but we contend that in their case their motivation is the elucidation of the content so that the students can understand it better. In the case of the research mathematicians in a university environment we argue that they are concerned that the mathematics be appropriately presented, or at least not be misrepresented, because their relationship to it and the way in which they work with it demands this. There is evidence that there are times when, in what appears to be an internal dialogue, decisions are driven by either the mathematician, or the teacher, within the lecturer persona. We suggest that these two personae may have different orientations and goals, affecting the decisions reached at critical points and consequently influencing student learning. This teacher-mathematician interplay might prove to be a productive construct to work with in the professional development of lecturers.

Questions:
Q1 As a mathematics lecturer are you aware of this tension? Have you caught yourself listening to an inner voice in the middle of a lecture?
Q2 Do you think it would be useful for your lecturing to consider this phenomenon explicitly as part of your professional development?
Q3 Do you have alternative suggestions for aspects of your lecturing to focus upon in professional development sessions?

References:

Ball, D., Bass, H., & Hill, H. (2004). *Knowing and using mathematical knowledge in teaching: Learning what matters.* Paper presented at the South African Association for Research in Mathematics, Science and Technology Education. Durban.

Barton, B., Oates, G., Paterson, J. &Thomas, M. (accepted for publication) The Datum Project. Community for Undergraduate Learning in Mathematics Newsletter, Auckland, New Zealand

Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. Cambridge: Cambridge University Press.

Kazemi, E., Franke, M. & Lampert, M. (2009) *Developing Pedagogies in Teacher Education to Support Novice Teachers' Ability to Enact Ambitious Instruction* In Roberta Hunter, Brenda Bicknell, Tim Burgess (Eds), Crossing Divides, Proceedings of the 32nd Annual Conference of The Mathematics Education Research Group of Australasia, Vol 1, pp. 11-29. Palmerston North, New Zealand: Mathematics Education Research Group of Australasia

Prushiek, J., McCarty, B., & Mcintyre, S. (2001). *Transforming professional development for preservice, inservice and university teachers through a collaborative capstone experience.* Education, 121 (4), 704-712.

Rowland, S. (2000) *The Enquiring University Teacher.* Open University Press, Philadelphia, PA

Schoenfeld, A. H. (1998). Toward a theory of teaching-in-context. *Issues in Education, 4*, 1-94.

Schoenfeld, A. H. (2002). A highly interactive discourse structure. In J. Brophy (Ed.), *Social Constructivist Teaching: Its Affordances and Constraints* (Volume 9 of the series Advances in Research on Teaching, pp. 131–169). Amsterdam: JAI Press.

Schoenfeld, A. (2007) *A theory of teaching and its applications. In Commemoration of Guenter Toerner's 60th Birthday* The Montana Mathematics Enthusiast**,** ISSN 1551-3440, Monograph 3, The Montana Council of Teachers of Mathematics, pp.33-38

Shulman, L. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14.

Speer, N.M., Smith, J.P & Horvath, A. (2010) *Collegiate mathematics teaching: An unexamined practice.* Journal of Mathematical Behavior , 29. pp 99–114

Torner, G., Rolka, K., Rosken, B. & Sriraman, B. (2010) Understanding a Teacher's Actions in the Classroom b Applying Schoenfeld's Theory Teaching-in-Context: Reflecting on Goals and Beliefs. In B. Sriraman and L. English (Eds.) *Theories of Mathematics Education. Seeking the New Frontiers* (pp. 401-420) Heidelberg/Dordrecht/London/New York: Springer

Van Ort S, Woodtli A, Hazard ME. (1991) *Microteaching: developing tomorrow's teachers.* Nurse Education, *(16)*

Mathematical Knowledge for Teaching: Exemplary High School Teachers' Views

Kathryn Rhoads

1. Introduction and Research Questions

Mathematical knowledge for teaching (MKT) has been a central topic of recent research in mathematics education (e.g., Ball, Thames, & Phelps, 2008; Ma, 1999). However, most of this research has focused on elementary or middle school mathematics. Few researchers have investigated the specific content knowledge needed to teach high school mathematics. Although most high school math teachers complete an undergraduate major in mathematics, some researchers have argued that high school teachers should receive content preparation specific to teaching (e.g., Moriera & David, 2008). In order to determine exactly what this content preparation should be, more research is needed on high school mathematics teacher knowledge.

Ball, et al. (2008) suggested building a theory of teacher knowledge by beginning with classroom episodes and observations of effective teaching in order to analyze the knowledge teachers need for such endeavors. This study also begins with effective teaching to build theory, but rather than observing teachers, it seeks the perspectives of exemplary high school mathematics teachers. These teachers' voices can provide critical knowledge associated with practice that researchers may not recognize or understand (Cochran-Smith & Donnell, 2006).

In this proposal, we present an exploratory study on exemplary high school teachers' views on the subject matter components of MKT. *Subject matter components* are aspects of mathematical knowledge that are not necessarily pedagogical (Ball, et al., 2008). The following research questions guided this study: (a) What subject matter components of MKT do exemplary high school teachers believe are important in their practice? (b) When and how do these teachers believe that their MKT developed?

2. Related Literature and Theoretical Framework

This study draws on the model of Mathematical Knowledge for Teaching proposed by Ball, Bass, and colleagues (e.g., Ball et al., 2008). In this model, MKT is comprised of subject matter knowledge and pedagogical content knowledge. Both types of knowledge are specific to mathematics content, but pedagogical content knowledge is knowledge of mathematics pedagogy, and subject matter knowledge is knowledge of content that is not necessarily pedagogical. The purpose of this study is to explore aspects of the latter.

Research focusing specifically on the subject matter components of MKT has explored how teachers understand particular concepts. Even (1990) synthesized research on teacher knowledge and conjectured that there are six components of a teacher's understanding of a particular concept. These are knowing, with regards to the concept, (a) essential features, (b) different representations, (c) multiple perspectives and applications, (d) unique characteristics, (e) a basic repertiore of examples, and (f) a conceptual understanding. Even added that teachers must also have knowledge of mathematics as a discipline. Other similar frameworks have been developed in the content strand of geometry (Chinnapan & Lawson, 2005) and for both mathematics and science teaching (Kennedy, 1998). The goal of this study is to further explore the elements of teachers' subject matter knowledge by talking to exemplary teachers rather than from systematically reviewing the research literature.

Other researchers have been interested in secondary mathematics teachers' perspectives on the use of advanced mathematical knowledge (i.e., undergraduate-level matheamtics) in their teaching. Zazkis and Leikin (2010) surveyed 52 secondary matheamtics teachers and found that, even for teachers who claimed to use their advanced mathematical knowledge often, they could rarely cite a specific example of the use of this knowledge. Recognizing the fact that teachers may have difficulty describing their subject matter knowledge and its use, this study takes into account both teachers' explicit statements of their subject matter knowledge as well as elements of their subject matter knowledge that are revealed through analysis of written lesson plans.

3. Methodology

3.1. Participants. Eleven high school mathematics teachers from one state participated in the study. These teachers received at least one of three prestigious honors in the state: Between 2000 and 2010, these teachers (a) were state or national finalists for the Presidential Award for Mathematics and Science Teaching (NSF, 2009), (b) were named County Teacher of the Year in their county, or (c) were National Board Certified Teachers in Adolescent and Early Adulthood Mathematics (NBPTS, 2010). The 27 teachers in the state who met these criteria were invited to participate, and 11 teachers accepted the invitation. Of these 11 teachers, four received the Presidential Award for Mathematics and Science Teaching, three were named County Teacher of the Year, and seven were National Board Certified. (Some teachers met more than one criterion.)

Eight of the participants taught at public schools and three taught at private schools or vocational schools. The eight teachers at public schools were well distributed among a range of schools in terms of socioeconomic status and student success rates. Similar statistics for the private schools were unavailable.

3.2. Data collection. Two sources of data were obtained for this study: (a) lesson plans and (b) interviews. Researchers have argued that MKT may be tacit (e.g., Zazkis & Leikin, 2010). Hence, lesson plans were used as stimuli during interviews in order to help teachers recall and discuss aspects of their content knowledge (Meade & McMeniman, 1992). Each participant was asked to submit one lesson plan from a traditional high school course (i.e., not college-level courses such as AP Statistics or AP Calculus). Lesson plans were obtained approximately one week before the interviews in order to tailor interview questions to the lesson where appropriate.

The main data source was individual interviews with participants. Interviews were semi-structured and lasted approximately one hour. Participants were asked about their background in mathematics education, the specialized content knowledge that went into the lesson that they shared, and general aspects of their mathematical knowledge as it related to their practice.

3.3. Data analysis. All interviews were audiotaped and fully transcribed for analysis. A grounded theory approach to analysis was used in the style of Strauss and Corbin (1990). After listening to the interviews and reading through the transcripts, initial codes were assigned to episodes that pointed to teachers' specialized content knowledge. Like codes were organized to form categories using the constant-comparative method (Strauss & Corbin, 1990). Each transcript was then revisited individually. Categories were refined and new codes and categories were formed when appropriate.

Next, lesson plans were revisited. Elements of the lesson plan which pointed to MKT were coded according to the categories developed from interview analysis. In most cases, the analysis of lesson plans supported interview data. In cases where the lesson plans provided disconfirming evidence, codes and categories were revised to accommodate the data from the lesson plans or led to proposed explanations for why the disconfirming evidence existed (Creswell, 2007).

4. Results

4.1. Essential aspects of subject matter knowledge. The results of our preliminary analysis include five aspects of subject matter knowledge that the exemplary teachers in this study found to be essential.

First, teachers believed that connections between mathematical ideas were important for teaching. Teachers claimed to be able to help students see basic mathematical ideas within a complicated mathematical concept and connect different mathematical concepts to help students understand mathematics more completely. In addition, they were able to connect the topics they were teaching to higher-level courses, such as calculus or non-Euclidean geometry.

Second, teachers believed that knowing the key examples of a concept was an important piece of MKT. When presenting mathematical concepts to their students, these teachers considered all cases of a concept or challenged their students to consider for which cases the concept would hold. In addition, the teachers had a flexible knowledge of cases of mathematical concepts so that they could create interesting and intriguing examples when necessary.

Third, understanding where mathematical concepts could be applied was an important piece of MKT for these teachers. They were knowledgeable of applications of the concepts they were teaching that were relevant to everyday life (and hence the students they were teaching). Fourth, teachers were aware of many techniques for problem-solving that were sometimes unusual or unique. Fifth, teachers recognized several representations of a concept and understood the ways in which a concept could be interpreted through these representations.

4.2. Development of MKT. Teachers also discussed ways in which they believed their MKT developed. These were through (a) formal courses, (b) professional experiences, and (c) personal experiences. Although some teachers spoke of individual courses as being influential to their thinking, most teachers did not cite formal coursework as a main source of MKT. Teachers overwhelmingly felt that their experience teaching a variety of courses and working with a variety of students helped them to develop MKT. In addition, several teachers mentioned that they individually sought to improve their practice through reading, conducting research, or applying for National Board Certification, and these activities helped develop MKT.

5. Significance

The teachers in this study appeared to understand the mathematics that they were teaching in a deep way. Many of the elements of their subject matter knowledge aligned with Even's (1990) framework for understanding of a concept, but the teachers' emphasis on connections between mathematical topics is important to note. Teachers overwhelmingly indicated that their MKT was developed through practice, not formal coursework. Hence, these findings can inform design of undergraduate mathematics courses for teachers. An important open question is whether courses that aim to develop these elements of subject matter knowledge are more productive for future teachers than traditional undergraduate mathematics courses.

6. Questions for Discussion

What aspects of MKT might be tacit? What research methods might help in exposing this knowledge? Which aspects of MKT (if any) might be specific to high school teaching? What might mathematics courses for teachers look like if they were to help teachers gain a depth of understanding? What specific parts of undergraduate-level mathematics are relevant to teachers?

References

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*, 389-408.

Chinnappan, M., & Lawson, M. J. (2005). A framework for analysis of teachers' geometric content knowledge and geometric knowledge for teaching. *Journal of Mathematics Teacher Education, 8*, 197-221.

Cochran-Smith, M., & Donnell, K. (2006). Practioner inquiry: Blurring the boundaries of research and practice. In J. L. Green, G. Camilli, & P. B. Elmore (Eds.), *Handbook of complementary methods in education research* (pp. 503-518). Mahwah, NJ: Lawrence Erlbaum Associates.

Creswell, J. W. (2007). *Qualitative inquiry and research design: Choosing among five approaches*. Thousand Oaks, CA: Sage.

Even, R. (1990). Subject matter knowledge for teaching and the case of functions. *Educational Studies in Mathematics, 21*, 521-544.

Kennedy, M. M. (1998). Education reform and subject matter knowledge. *Journal of Research in Science Teaching, 35* (3), 249-263.

Ma, L. (1999). *Knowing and teaching elementary mathematics*. Mahwah, NJ: Lawrence Earlbaum Associates.

Meade, P., & McMeniman, M. (1992). Stimulated recall--An effective methodology for examining successful teaching in science. *Australian Educational Researcher, 19* (3), 1-18.

Moreira, P. C., & David, M. M. (2008). Academic mathematics and mathematical knowledge needed in school teaching practice: Some conflicting elements. *Journal of Mathematics Teacher Education, 11*, 23-40.

National Board for Professional Teaching Standards. (2010). *Mathematics/ Adolescence and young adulthood*. Retrieved from National Board for Professional Teaching Standards: http://www.nbpts.org/for_candidates/certificate_areas1?ID=3&x=42&y=9

National Science Foundation. (2009). *Presidential Awards for Excellence in Mathematics and Science Teaching.* Retrieved from http://www.paemst.org/controllers/home.ctc?method=view

Strauss, A., & Corbin, J. (1990). *Basics of qualitative research.* Newbury Park, CA: Sage.

Zazkis, R., & Leikin, R. (2010). Advanced mathematical knowledge in teaching practice: Perceptions of secondary matheamtics teachers. *Mathematical Thinking and Learning, 12* (4), 263-281.

Analysis of Undergraduate Students' Cognitive Processes When Writing Proofs about Inequalities

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The purpose of this presentation is to discuss undergraduate students' cognitive processes when they attempt to write proofs about inequalities involving absolute values. We employ the theory of conceptual blending to analyze the cognitive process behind the students' final proof of inequalities. Two undergraduate students from transition-to-proof courses participated in the study. Although the instruction about inequalities was given graphically, the students recruited algebraic ideas mainly when they attempted to construct a proof for the inequality. We illustrate how students apply the algebraic ideas and proving structures for their mental activity in their proving activity.

Keywords: proof construction, inequalities, absolute values, conceptual blending,

Introduction and Research Questions

The purpose of this presentation is to focus on undergraduate students' cognitive processes when they attempt to write a proof about an inequality. An understanding of inequalities plays an important role in comparing two quantities and identifying quantitative relationships between them. Research in mathematics education has paid little attention to students' ways of thinking and their difficulties with inequalities although some research reports that students encounter difficulties understanding the meaning of inequalities and their solutions (Tsamir & Almog, 2001; Tsamir & Bazzini, 2004; Vaiyavutjamai & Clements, 2006). In this presentation, we discuss the following research questions:

1. What were the students' key mathematical ideas and proving frames used when proving inequalities?
2. What are the students' cognitive processes behind their final proofs of inequalities involving absolute values?

The research literature indicates that undergraduate students struggle with proof writing (e.g. Selden & Selden, 2008). Students tend to structure their proofs in the chronological order of their thought process instead of reorganizing it with proper implications (Dreyfus, 1999). Also, students have difficulty with utilizing conceptual ideas strategically to generate their proof (Weber, 2001). Therefore, students' challenges are related to how to structure a proof, construct a key idea, and strategically use their key idea in their proof structure (Zandieh, Knapp, & Roh, 2008). This research adds to that literature by describing students' cognitive process when proving inequalities involving absolute values, which have not been much addressed in previous work.

Theoretical Framework

We employ Fauconnier and Turner's (2002) theory of conceptual blending to analyze our data. This theory postulates the existence of a subconscious process in which an individual combines elements of current knowledge in order to build new knowledge. An individual may use their knowledge to form one or more mental spaces (referred to as *input spaces*), each of which involves an array of elements and their relationships to one another. Some elements of one input space may be matched with similar elements of another (we refer this as *cross matching* in this study). Such a cognitive process entails the blending of two or more input spaces to form a new mental space (called the *blended space*) as follows (Zandieh et al., 2008): Once the individual considers elements in each input space as important, he or she is *mapping* them into the blended space. As he or she organizes information in the blended space, they are *completing the blend*. This may be done by the use of knowledge outside of the input spaces (called a *conceptual frame*) to organize the blended space. Following this is called *running the blend*, which is a simulation or manipulation of the information in order to make inferences. In this presentation we will illustrate how students are constructing input spaces, cross-matching elements between the two input spaces, mapping from the input spaces to the blended space, applying a conceptual frame to complete the blend, and running the blend. We extend the idea of using conceptual blending to understand students' cognitive process in proofs to inequalities involving the absolute value.

Research Methodology

Data for this study comes from a teaching experiment (Steffe & Thompson, 2000) conducted at a southwestern university in the USA. The teaching experiment involved two undergraduates who were enrolled in different sections of a transition to proof course at the time. Both were strong students in their transition-to-proof courses and neither had instruction in real analysis before this. As a research team, we (identified as Instructor and TA in this study) met to design tasks prior to the teaching sessions, and team-taught during the teaching sessions. The tasks were also served to gauge students' reasoning and their understanding of topics. The data include transcripts of videotapes from the teaching sessions, photo-copies of students' written proofs and scratch work, and student reflections. In their reflections, the students reported aspects of the task or topic they found most interesting or challenging.

In this presentation we focus on the first session with the students, in which they were asked to prove an inequality involving absolute values. The session began with Instructor introducing the definition of the absolute value function and its graph. Instructor then presented properties of absolute values including the Triangle Inequality: For any $a, b \in \mathbb{R}$, $|a+b| \le |a| + |b|$. The students used several values of a and b to make sense of the properties of absolute values. Instructor left the definition and theorems out for the students and told them that they may refer to it while working on problems with TA. TA then led the student discussion about how to construct a proof for the exercise statement: "Let $a, b, c \in \mathbb{R}$, then $|a-b| \le |a-c| + |c-b|$."

For our data analysis, we identified each student's key mathematical ideas and their proving frame when proving the inequality. In terms of the theory of conceptual blending, we then identified how each student formed inputs and cross-matched elements from one space to another. We examined how the student is mapping the cross-matched elements into to the blended space, and how the student uses proving frames and his key mathematics ideas as he is completing the blend and running the blend, respectively.

Results and Discussions

When the students attempted to construct a proof for the inequality, their key mathematical ideas were mainly algebraic although the instruction about inequalities was given graphically. Accordingly, we identified three algebraic ideas used by the students: The first one was observed when a student considered an element of one input space to be identical to the corresponding element of the other input space. The corresponding elements were therefore considered as equal so he could "replace" one with another. The second algebraic idea was "substitution", in which a student introduced a new object and substituted it with an element in an input space (e.g., substitute a variable x for the variable a in the triangle inequality $|a+b|\le|a|+|b|$). The third one was referred to "zero-trick" in which a student added in something equal to zero (e.g., adding "$-c+c$" to $a-b$).

We also found that students set up a conditional statement of form a conditional implies a conditional statement $(p\to q)\Rightarrow(r\to s)$, and manipulated premises and conclusions p, q, r, and s. They then framed their proof in terms of what is called a Conditional Implies a Conditional Frame (CICF) as Zandieh et al. (2008): a student assumes r, then induces p. Applying $p\to q$, he concludes s (Zandieh et al., 2008). However, there was some variation in recruiting CICF in proving the inequality. In particular, a student assumed all of all of p, q, and r, then induced s.

Example. We identified six episodes through our data coding procedure based on students' conceptual frames. Usually their conceptual frame consisted of one key algebraic idea and one proving frame. Here, we illustrate how conceptual blending can be used to describe the cognitive process in the second episode. In this episode, a student Jon stated: "What if we substitute this like: $a=a-c$, $b=c-b$? […] Suppose we have $a=a-c$. Can we do this?" He wrote "$b=c-b$. $a-b=a-c\ -\ c-b$" then crossed out c's in "$a-c\ -\ c-b$" to induce $a-b$. One might note that Jon actually wrote $a-b=a-c\ -\ c-b$, and crossing out the c's, he stated that he will have a-b. This calculation is incorrect. However, since he says that the c's will cancel, it is probable that he meant $(a-c)+(c-b)=\ a-b$. The analysis below reflects this conjecture.

Analysis. We characterize Jon's conceptual frame in the second episode as a combination of "replacing" and CICF, and describe his conceptual blending as follows: To begin, he identified the exercise statement as one of his input spaces, say *Input A*. He used the statement of the Triangular Inequality as his strategic knowledge (Weber, 2001) to create another input space, say *Input B*. He then *cross-matched* $|a|$, $|b|$, and $|a+b|$ in Input A with $|a-c|$, $|c-b|$, and $|a-b|$ in Input B, respectively. Identifying the elements he viewed as important ($a-c$, $c-b$, and $a-b$ from Input A, and a, b, and $a+b$ from Input B), Jon was *mapping* the cross-matched elements into his blended space. Then Jon was *completing his blend* by recruiting his conceptual frame: he decided that he would begin with $a-c$ and $c-b$, and manipulate these elements by "replacing" to construct $a-b$. Finally, Jon was *running the blend* in four steps. First, he "replaced" $a-c$ and $c-b$ from Input A with a and b, respectively. Thus, he had constructed a and b in Input B. Second, he created $a+b$ in Input B by adding these two elements. Third, by "replacing" again, he constructed $(a-c)+(c-b)$. Finally, his fourth step is to simplify this to eliminate the c's and construct $a-b$ (See Figure 1).

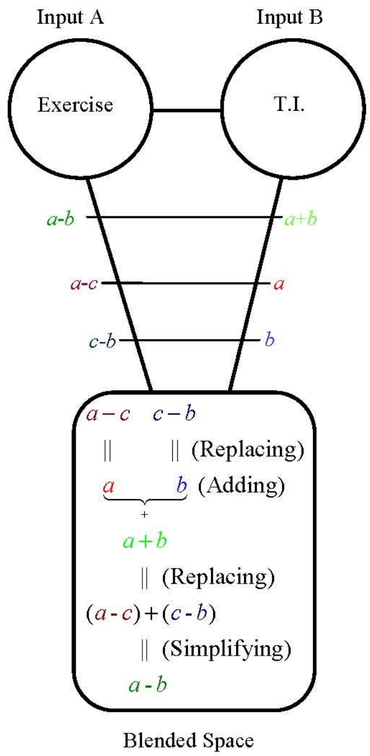

Figure 1: Jon's blending in the second episode

We found that the theory of conceptual blending accounts for students' cognitive processes behind their reasoning in proving inequalities involving absolute values. In particular, it sheds light on why and how students come to ignore the inequality when they prove or solve problems about inequalities. In fact, the students did not map the inequalities and the absolute value symbol into blended spaces, and hence they were not integrated in the blended spaces. In addition, logical structures in the input spaces were often dropped from the process of mapping to the blend, and as a consequence implication structures were obscured in the blended space. (e.g., conditionals $p \to q$ in input spaces were treated as p and q in the blended space.) Finally, the students also carried out algebraic ideas improperly while they recruited these ideas as their conceptual frames. (e.g., when running the blend, Jon recruited his key algebraic idea and hence identified the cross-matched elements into his blended space instead of using a proper substitution.)

Discussion Questions

1. What are the areas of research that are related to the proving of inequalities, but which are not considered in this study?
2. What are alternative frameworks for analyzing students' cognitive process while writing proofs and how are they going to be useful to explore our research question?

References

Dreyfus, T. (1999). Why Johnny can't prove. *Educational Studies in Mathematics*, *38*, 85-109.

Fauconnier, G., & Turner, M. (2002). *The way we think: Conceptual blending and the mind's hidden complexities*. New York: Basic Books.

Ramen, M. (2003). Key Ideas: What are they and how can they help us understand how people view proof? *Educational Studies in Mathematics*, *52*, 319-325.

Selden, A., & Selden, J. (2008). Overcoming students' difficulties in learning to understand and construct proofs. In M. Carlson & C. Rasmussen (Eds.), *MAA Notes. Making the connection: Research and teaching in undergraduate mathematics education* (pp. 95-110). Washington, DC: MAA.

Steffe, L. & Thompson, P.(2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. E. Kelly (Eds.), *Research design in mathematics and science education* (pp. 267-307). Hillsdale, NJ: Erbaum.

Tsamir, P., & Almog, N. (2001). Students' strategies and difficulties: The case of algebraic inequalities. *International Journal of Mathematical Education in Science and Technology*, *32*, 513-524.

Tsamir, P., & Bazzini, L. (2004). Consistencies and inconsistencies in students' solution to algebraic 'single-value' inequalities. *International Journal of Mathematical Education in Science and Technology*, *35*, 793-812.

Vaiyavutjamai, P. & Clements, M. (2006). Effects of classroom instruction on student performance on, and understanding of, linear equations and linear inequalities. *Mathematical Thinking and Learning*, *8*, 113-147.

Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics*, *48*, 101-119.

Zandieh, M., Knapp, J., & Roh, K. (2008). When students prove statements of the form $(P \to Q) \Rightarrow (R \to S)$. *Proceedings of the 11th Conference of the SIGMAA on RUME*, http://mathed.asu.edu/crume2008/Proceedings/RUME_paper_2008.pdf, San Diego, CA.

Where is the Logic in Proofs?

Preliminary Research Report

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Often university mathematics departments teach some formal logic early in a transition-to-proof course in preparation for teaching undergraduate students to construct proofs. Logic, in some form, does seem to play a crucial role in constructing proofs. Yet, this study of 43 student-constructed proofs of theorems about sets, functions, real analysis, abstract algebra, and topology, found that only 1.7% of proof lines involved logic beyond common sense reasoning. Where is the logic? How much of it is just common sense? Does proving involve forms of deductive reasoning that are logic-like, but are not immediately derivable from predicate or propositional calculus? Also, can the needed logic be taught in context while teaching proof-construction instead of first teaching it in an abstract, disembodied way? Through a theoretical framework emerging from a line-by-line analysis of proofs and task-based interviews with students, I try to shed light on these questions.

Keywords: Logic, transition-to-proof courses, analysis of proofs, task-based interviews

To obtain a Masters or Ph.D. in Mathematics, one must be able to construct original proofs. This process of proof construction is usually explicitly taught, if at all, to undergraduates in a transition-to-proof or "bridge" course. At the beginning of such courses, teachers often include some formal logic, but how it should be taught is not so clear. Epp (2003) stated that, "I believe in presenting logic in a manner that continually links it to language and to both real world and mathematical subject matter" (p. 895). However, some mathematics education researchers maintain that there is a danger in relating logic too closely to the real world: "The example of 'mother and sweets'[1] episode, for instance, which is 'logically wrong' but, on the other hand, compatible with norms of argumentation in everyday discourse, expresses the sizeable discrepancy between formal thinking and natural thinking…" (Ayalon & Even, 2008, p. 245).

There are also those who do not think that logic needs to be explicitly introduced. For example, Hanna and de Villiers (2008) stated, "It remains unclear what benefit comes from teaching formal logic to students or to prospective teachers, particularly because mathematicians have readily admitted that they seldom use formal logic in their research" (p. 311). Selden and Selden (2009) claimed that "Logic does not occur within proofs as often as one might expect … [but] [w]here logic does occur within proofs, it plays an important role" (p. 347). Taken together,

[1] The scenario is that the mother says to the child, "If you don't eat, you won't get any sweets" and the child responds by saying, "I ate, so I deserve some sweets." (Ayalon & Even, 2008)

these views suggest that it would be useful for mathematics education researchers to further examine the role of logic and logic-like reasoning within proofs.

In this paper, I begin to answer the question, "Where is the logic in students' proofs?" by first searching for uses of logic in a line-by-line analysis of 43 student-constructed proofs in various areas of mathematics, and then examining the actions of the proving process in search of additional uses of logic. This research was done in conjunction with a course, "Understanding and Constructing Proofs", at a large Southwestern state university, giving Masters and Ph.D.'s in mathematics. Students in the course were first-year mathematics graduate students along with a few undergraduates. Topics covered included sets, functions, real analysis, algebra, and topology. The 43 proofs analyzed were all of the student-constructed proofs in the course. The professors verified all of these as correct. For example, some theorems that were proved by the students included: "The product of two continuous functions is continuous"; "Every semigroup has at most one minimal ideal"; and "Every compact, Hausdorff topological space is regular".

In the process of coding the lines of the proofs, a theoretical framework emerged. Twenty-three categories were developed and used to code the lines. Here I will describe just four categories: informal inference, formal logic, interior reference, and use of definition; and the others will be in the research report. *Informal inference* is a category that refers to a line of a proof that depends on common sense reasoning. I view informal inference as being logic-like, as it seems that when one uses common sense, one does so automatically and does not consciously bring to mind formal logic. For example, given $a \in A$ and $A \subseteq B$, one gets $a \in B$ as a common sense conclusion, which need not call on formal logic such as Modus Ponens. By *formal logic,* in this report I mean conscious use of predicate and propositional calculus beyond common sense. *Interior reference* is the category for a line in the proof that uses a previous line as a warrant for a conclusion. For example, if there were a line indicating $x \in A$ earlier in the proof, then subsequently stating "Since $x \in A$…" later in the proof would be an interior reference. Lastly, *use of definition* or *definition of* refers to when a line in the proof calls on the definition of a mathematical term. For example, consider the line "Since $x \in A$ or $x \in B$, then $x \in A \cup B$." The conclusion "then $x \in A \cup B$" is implicitly calling on the definition of union.

In the line-by-line analysis of the proofs, 14% of the 630 lines were informal inference, and less than 2% of the lines were formal logic, such as Modus Tollens and DeMorgan's laws. In fact, collecting all the logic-like categories together, I found that only 18% of the lines were logic-like. These *logic-like* categories included induction cases, induction hypothesis, induction conclusion, contradiction hypothesis, contradiction conclusion, informal inference, and formal logic. If only 18% of the lines were logic-like, what were the rest of the lines of the proofs like? I found that 21% of the lines were use of definition, 15% were interior reference, and 13% were categorized as *assumption*, meaning that the proof-writer introduced a new object into the proof. Thus, use of definition, interior reference, and assumption accounted for 49% of the lines in the analyzed proofs. While most of the lines of a proof may aid reasoning, they are not themselves logic-like. Also, in a randomly selected line, there is about a 98% chance that there is no formal logic.

Is logic that might not appear in the finished proofs called on by the actions of the proving process? To begin to answer this question, five proofs were selected and the possible actions a student might take in the proving process were hypothesized and analyzed. There were also task-based interviews with three students who had taken the "Understanding and Constructing Proofs" course one year earlier to observe their actions while proving one of the theorems. A one-page set of notes was given to the students (excerpted from the course notes they had used), starting with the definition of a semigroup, and ending with the theorem to prove, "Every semigroup has at most one minimal ideal." The students were videoed while they thought aloud and attempted to prove the theorem at the blackboard. An interesting result was that these students took three different approaches to the proof, including voicing different concept images for concept definitions. For example, in the notes there was a definition of a "minimal ideal of a semigroup", and one student considered Venn diagrams while reflecting on the definition, while the other two students stated in a subsequent debriefing that they had not thought of using a diagram.

Another result was that the actions hypothesized for the proof construction did not match the actual actions of the interviewed students. For example, I had hypothesized that the students would write the first line or assumptions, leave a space, and then would write the last line of what was to be proved (as they had been encouraged to do in the earlier "Understanding and Constructing Proofs" course). This is a proving technique (Downs & Mamona-Downs, 2005) that is not often taught. While all three interviewed students wrote "Let S be a semigroup" almost immediately at the beginning of their proofs, only one student wrote the conclusion after playing a bit with the algebra of a semigroup. An analysis of the proof actions in another student's interview revealed that she wanted to understand and write definitions on scratch work before attempting the proof. She then attempted to comprehend what a minimal ideal is, because she had previously assumed A and B were minimal ideals and intended to arrive at the conclusion $A = B$. She then used the definition of minimal ideal to claim (without justification) that either $A = B$ or $A \cap B = \emptyset$. After using a theorem listed in the notes, she concluded $A = B$, which in her mind finished the proof. Most of the above mentioned actions (e.g., assuming two minimal ideals, deriving a conclusion, and using modus ponens with a theorem) are examples of logic-like actions in the proving process.

An implication for teaching that arises out of this study is that it might be useful for teachers to explicitly attend to students' logic-like actions in the proving process. Also, because formal logic occurs fairly rarely, one could teach it in context as the need arises. In addition, it would be good to explicitly help students to learn how to read and understand definitions, and when to introduce mathematical objects into a proof, because these together with interior reference constituted 49% of the lines analyzed. Some interesting questions arise from this study:

How many beginning graduate students need a course specifically devoted to improving their proving skills? Can one identify a range of logic-like actions that students most often need to use in constructing proofs? Would a structural analysis of proofs, in contrast to a line-by-line analysis, yield different results? In particular, is it reasonable to regard certain structures in a proof as logic-like? For example, knowing one can prove P *or* Q by supposing *not* P and arriving at Q has the effect of using logic. So is it reasonable to regard *not* $P \ldots then\ Q$ as a logic-like structure in a proof of P *or* Q?

References:

Epp, Susanna S. (2003). The role of logic in teaching proof. *American Mathematical Monthly*, *110* (10), 886-899.

Ayalon, M., & Even, R. (2008). Views of mathematics educators on the role of mathematics learning in the development of deductive reasoning. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano, & A. Sepúlveda (Eds.), *Proceedings of the Joint Meeting of PME 32 and PME-NA XXX,* Vol. 2 (pp. 113-120). México: Cinvestav-UMSHN.

Hanna, G., & de Villiers, M. (2008). ICMI Study 19: Proof and Proving in Mathematics Education (Discussion Document). *ZDM-The International Journal of Mathematics Education*, *40(2)*, 329-336.

Selden, J., & Selden, A. (2009).Teaching proof by coordinating aspects of proofs with students' abilities. In D. A. Stylianou, M. L. Blanton, & E. J. Knuth (Eds.), *Teaching and learning proof across the grades: A K-16 perspective* (pp. 339-354). New York, NY: Taylor & Francis.

Downs, M. L. N., & Mamona-Downs, J. (2005). The proof language as a regulator of rigor in proof, and its effect on student behavior. In M. Bosch (Ed.), *Proceedings of CERME 4, Working Group 14: Advanced Mathematical Thinking* (pp. 1748-1757). Sant Féliu de Gix, Spain: FUNDEMI JQS – Universitat Ramon Llull. Available online at http://ermeweb.free.fr/CERME/CERME4_WG14.pdf .

Reading Online Mathematics Textbooks

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Introduction

Many would agree that reading is critical for gaining understanding within a discipline, and that students will not reap the full benefits of their studies if they skim through (or worse yet, ignore) their reading assignments. Even in quantitative disciplines such as mathematics, teachers may assign readings from the textbook with the intent of having students come to class more prepared and giving them exposure to more material than can be taught in the time allotted to class meetings. However, few teachers would be so naïve as to believe that students actually read the text, and often complain about the unpreparedness of the students for instruction. On their part, students complain about how hard it is to read mathematics textbooks, perhaps because they lack appropriate reading strategies that might remedy the situation. Indeed, even first-year undergraduate who are good general readers do not read mathematics textbooks well (Shepherd, Selden & Selden, 2009).

One solution to the problem of getting students to read mathematics texts effectively, despite their deeply instilled poor reading habits, is to harness technology. Online mathematics textbooks are a fairly recent (and increasingly popular) addition to the available set of instructional resources. In contrast to physical textbooks, online texts have affordances for interactive and responsive engagement. In particular, online texts can include activities that foster effective reading through embedded tasks that provide feedback and hints. The purpose of this project is to begin to understand how readers interact with an online mathematics textbook in a quasi-authentic setting, and to study the effects of some scaffolded online activities intended to help students monitor their comprehension of what is read.

Literature & Theoretical Perspective

Reading involves both decoding and comprehension. On the comprehension side of the coin, research has identified several strategies that good readers employ as they engage with a text (Flood & Lapp, 1990; Palincsar & Brown, 1984; Pressley & Afflerbach, 1995). Of course, these strategies depend on the individual reader, the reader's goals, and the material being read. Mathematics textbooks, in particular, are "closed texts" in the sense that they seek to elicit a well-defined, "precise" response that is not open to differing interpretations from readers (Weinberg & Wiesner, in press). Yet, many students have not been taught how to read their mathematics textbooks, and do not read them as intended. For instance, authors of mathematics texts include expository material to help students develop a deeper understanding of the mathematical concepts. Yet, despite the fact that an overwhelming percentage of students claim to read their mathematics textbooks for understanding, few students report attempts at reading the expository sections (Weinberg, in press). Our research addresses how students who *are* making an attempt to read their textbooks engage in this process, and how they might be better supported in their endeavors.

Our theoretical perspective is aligned with the view that reading is an active process of meaning-making in which knowledge of language and the world are used to construct and negotiate interpretations of texts (Flood & Lapp, 1990; Palincsar & Brown, 1984; Rosenblatt, 1994). In helping students navigate mathematics texts, we advocate reading strategies that stem from the Constructively Responsive Reading framework (CRR) that was developed in reading comprehension research (Pressley & Afflerbach, 1995). These strategies are intended to help students maximize their construction of knowledge from texts. In addition, we place an emphasis on *cautious reading* (Shepherd, Selden & Selden, under review) that helps students minimize inappropriate interpretations of their mathematics texts by detecting and correcting errors, misunderstandings, and confusions. Taken together, CRR-based strategies and cautious reading advocate encouraging students to carefully read expository text and check the correspondence between the inferences they have drawn and the author's intent, and discouraging students from forging ahead without carrying out and evaluating their performance on tasks provided by the authors.

Research Methods

The participants are 30 students enrolled in sections of a redesigned precalculus course at a large southwestern university. The course uses an online text, *Precalculus: Pathways to Calculus,* which was developed at Arizona State University and was designed to foster students' ability to reason conceptually about functions and quantity (Carlson & Oehrtman, 2009). Students were recruited to volunteer for participation in seven Study Hall sessions once weekly of approximately 1.5 hours each. Approximately half of the students received reading instruction prior to their participation in the research project. This reading instruction consisted of reading guides stepping them through how to read each of the first several sections of the online course text, and a 40-minute one-on-one reading session of one section of the text with the researcher/instructor that was carried out about 1/3 of the way through the semester. During the Study Hall sessions, students were asked to complete their current reading assignment on the computers provided. In order to investigate authentic student reading habits as closely as possible, nonintrusive screen capture software was used to measure activities such as scrolling, latency, and browsing. In addition, prior to and following their reading of the text at each Study Hall session, students completed short mathematical assessments based on the relevant text material. Other data sources included brief surveys addressing reading habits, and, for most students, admissions testing scores (SAT/ACT) as a control for mathematical and reading preparedness. Finally, half of the participants from each reading instruction group (received/did not receive) were randomly assigned during the final four Study Hall sessions to a version of the text in which questions with pop-down solutions (e.g., hidden answers) were replaced with scaffolded tasks that provided students with right/wrong feedback and sequences of hints[1] (see Figure 1).

[1] The authoring tools for these activities were developed by the Open Learning Initiative at Carnegie Mellon University.

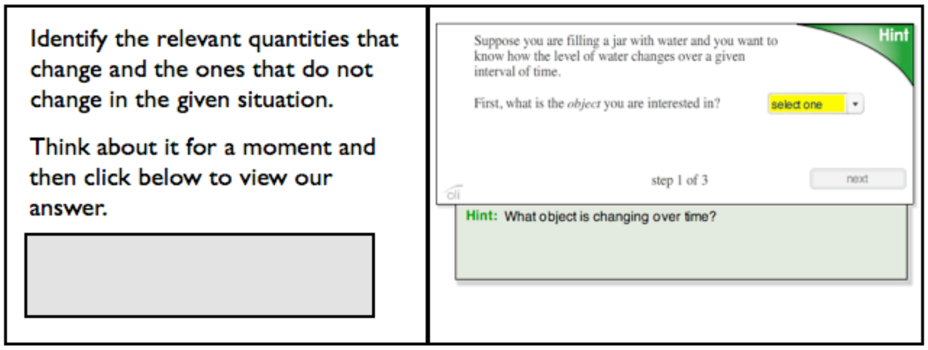

Figure 1. Example of pop-down activity (left) and scaffolded task with hint sequence (right).

Implications for Further Research & Teaching

This research project is a preliminary step for identifying and constructing activities that promote effective reading strategies and that can be embedded in online mathematics textbooks. At this stage, we are restricting our activities to multiple-choice questions. There is a need for research that identifies statistically valid response choices that capture common student errors and ways of thinking so that appropriate sequences of hints can be designed. For instance, certain incorrect answers might be best addressed by posing hints that promote cognitive conflict with that particular way of reasoning.

We would also like to explore how students who rely on embedded scaffolded tasks to read their textbooks effectively can be graduated to the adoption of their own reading strategies that are consistent with reading for understanding. To address this issue, both the timing and manner in which the activities are faded need to be investigated.

Finally, this research has implications for how teachers can connect with the reading aspect of their students' instruction. At present, in order to check whether students have completed a reading assignment, many teachers resort to giving quizzes during (valuable) class time on the relevant material. Online texts can be designed to capture and log student actions, and so provide indicators of whether (and how) students are completing their reading assignments.

Summary

At the heart of our project is the goal of helping students become more effective readers of introductory level mathematics texts. In order to achieve this goal, we are harnessing the affordances of technology, and exploring the ways that activities can be embedded within online textbooks. Although the goal of these activities is to foster reading with understanding, we do not anticipate that they will produce "cautious readers." Instead, our much more modest hope is that we can help students turn over a new page in the way they interact with their textbooks.

Discussion Questions

1. Traditional texts: We chose a text with a large amount of exposition and in which examples function as checks of understanding rather than as analogies, which is the case in traditional precalculus texts. Since online versions of traditional textbooks are also becoming more popular, how might we support students reading these texts?
2. Fading: How might readers be weaned from having to engage in embedded activities in order to read effectively to adopting their own strategies for reading with understanding?

References

Carlson, M., & Oehrtman, M. (2009). *Precalculus: Pathways to Calculus*. Retrieved with permission from https://www.rationalreasoning.net/pdp/

Flood, J., & Lapp, D. (1990). Reading comprehension instruction for at-risk students: Research-based practices that can make a difference. *Journal of Reading, 33,* 490-496.

Palincsar, A. S., & Brown, A. L. (1984). Reciprocal teaching of comprehension-fostering and comprehension-monitoring activities. *Cognition and Instruction, 1,* 117-175.

Pressley, M., & Afflerbach, P. (1995). *Verbal protocols of reading: The nature of constructively responsive reading.* Hillsdale, NJ: Lawrence Erlbaum Assoc.

Rosenblatt, L. M. (1994). The transactional theory of reading and writing. In Ruddell, Ruddell, & Singer (Eds.), *Theoretical models and processes of reading* (4th ed., pp. 1057-1092). Newark: International Reading Association.

Shepherd, M., Selden, A., & Selden, J. (2009). *Difficulties first-year university students have in reading their mathematics textbook.* Technical Report No. 2009-1. Tennessee Technological University, Cookeville, TN.

Weinberg, A. (in press). Undergraduate students' self-reported use of mathematics textbooks. *PRIMUS.*

Weinberg, A., & Wiesner, E. (in press). Understanding mathematics textbooks through reader-oriented theory. *Educational Studies in Mathematics*.

Calculus from a virtual navigation problem

Olga Shipulina

Calculus appeared from the real world application, has a real world context, and is fundamentally a dynamic conception; this is why the framework of Realistic Mathematics Education (RME) should be the most efficient approach to teaching and learning calculus. The current study is devoted to investigation of the computer simulated bodily path optimization calculus. I adapted the conception of 'tacit intuitive model' for the particular calculus task of path optimizations. My hypothesis is that tacit mental modeling takes place with the allocentric frame of reference. I designed a paradigm in the Second Life virtual environment which allows simulating the navigational task of path optimization with two different mediums and with voluntary choice between allocentric/egocentric views. The reinventing the calculus problem of path optimization from the virtual navigation and its mathematizing would give a powerful intuitive link between the everyday real world problem and its symbolic arithmetic.

Key words: calculus, virtual navigation, egocentric/allocentric view, tacit intuitive model, Realistic Mathematics Education

Introduction

In the late 1980s the 'Calculus Reform Movement' began in the USA. The Calculus Consortium at Harvard (CCH) was funded by the National Science Foundation to redesign the Calculus curriculum with a view of making Calculus more applied, relevant, and more understandable for a wider range of students.

The didactical goal of the present study is to help learners to 'unearth' (Torkildsen, 2006) a calculus path optimization problem from the real world navigation problem simulated in the Second Life (SL) virtual reality. The learners would reinvent the calculus problem by controlling computer simulated body movements with either egocentric or allocentric views. The egocentric view provides the perception of 'being' within the virtual environment and seeing objects from the 'first person' view. The allocentric view is provided when the learner's avatar is present in the environment and the learner controls the avatar navigation: in this case the virtual reality objects are spatially related to the avatar. This enactive computer paradigm would allow the learners to explore mathematical ideas being engaged immediately into the optimal navigation problem. Since the designed virtual environment contains two different mediums, the task of path optimization should involve the intuitive anticipation of speed difference in different mediums: when being on land and when being in water; thus, the intuitively planned optimal path will be based on this speed difference anticipation. After a few trials of virtual navigating, the learner should reinvent the calculus path optimization problem and should try to mathematize it. When the problem has been mathematized the learner can connect and compare the intuitive understanding of the problem with its symbolic arithmetic. According to Tall (1991), "by providing a suitably powerful context, intuition naturally leads into the rigor of mathematical proof" (p.20). Since this paradigm gives a strong link between the everyday real world problem and its symbolic formal representation, it strongly relates to the theoretical framework of RME (Freudenthal, 1991; Freudenthal, 1973; Freudenthal, 1968).

The research goal of the study is to explore how egocentric and allocentric frames of references relate to different phases of optimal path problem solving, which, in turn, would provide better understanding of mental processes during the particular calculus problem solving.

Theoretical perspective and related literature

Navigation consists of two aspects: a topographic aspect and a procedural aspect that represents the trip itself (Besthoz, 2000). The topographic aspect is connected with a construction of cognitive map; the procedural aspect is connected with actual movements. Both topographical and procedural navigations include spatial orientation (ibid). Virtual navigation differs from real navigation: it doesn't involve vestibular, translation, or locomotor memory which, according to Berthoz (2000), is inherent to real space body navigation. In virtual environment the visual system plays the main and crucial role.

The new virtual paradigm of optimal path navigation is intrinsically of enactive nature. Tall (1997), in his turn, asserted that "the calculus concepts are starting from enactive experiences as an intuitive basis" (p.4). So, the optimal path virtual navigation paradigm is in accordance with his schematization of building of the concepts of calculus.

On the other hand, the computer simulation of body movements expressed either by an egocentric view of 'being' in the environment or by an allocentric view trough controlling the avatar navigation, provides an explicit perception of 'bodily' navigation which can be expressed in terms of embodiment. Tall (2007) categorizes mathematical thinking into three intertwined worlds: the *conceptual-embodied*, the *proceptual-symbolic* and the *axiomatic-formal*. He considers such categorization particularly appropriate in the calculus. According to Tall (2007), the conceptual-embodied world of mathematics is based on perception of and reflection on properties of objects. For the particular dynamic tasks of optimal navigation and taking into account the dynamic nature of calculus, I modify a conceptual-embodied world into '*procedural- conceptual-embodied'* world, reflecting embodied dynamism of body movement. This extended world is based not only on perception of and reflection on properties of objects, but also on an active body experience in its dynamism such as change of body position, speed, and acceleration.

For the navigational type of tasks, we first, mentally simulate the trajectory, and then we compare the actual movement with the predicted movement (Berthoz, 2000). For the mental simulation stage of navigation I adopted the conception of '*tacit intuitive model*' introduced by Fischbein (1989). The common characteristics of the tacit intuitive models are that they have structural entity; they are of practical and behavioral nature; they are mental, intuitive, and primitive; they are representable in terms of action; they are autonomous entity with their own rules; they are not perceived consciously by an individual. The important characteristic of the intuitive mental model is its robustness and its capacity to survive long after it no longer corresponds to the formal knowledge (ibid). For the case of optimal path navigation the last characteristic should be omitted and the tacit intuitive model should be modified. As Cazzato, Basso, Cutini, & Bisiacchi (2010) pointed out: people produce incomplete plans at the beginning of a route and continuously make decisions along the trajectory of navigation. So, the tacit intuitive model should be modified into a more flexible conception, reflecting dynamism and procedural nature of continuous adjustment according to the model's effectiveness. The term of *'tacit dynamics simulation'* would reflect both the procedural embodied world, on the basis of which the kind of tacit model is constructed, and flexibility and procedural character of such intuitive modeling.

The hypotheses of the research are: 1) *the tacit dynamics simulation* of finding the optimal path takes place with allocentric frame of reference even when the environment is viewed egocentrically; 2) the topographic phase of navigation also takes place with allocentric frame of reference, even if the virtual environment is viewed egocentrically; 3) the procedural phase of navigation can involve both frames of references in parallel, which is in accordance with Burgess's (2006) assertion.

Experimental design and methodology

The designed in SL virtual environment paradigm contains a big water pool with a platform, located at B (Figure 1). The paradigm is related directly to the calculus problem of finding the optimal path from an initial position A to the platform B under the condition that available paths must transverse two different mediums, involving different rates of speed (Figure 1).

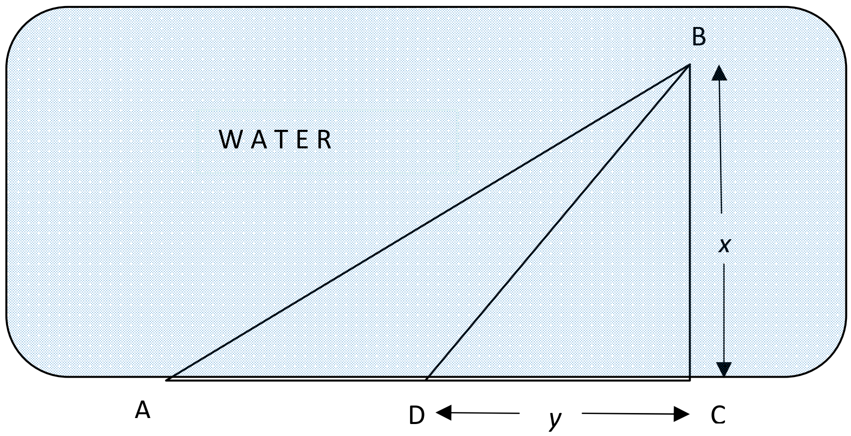

Figure 1: Paths to the platform (adopted from Pennings, 2003)

There are three phases in the experimental paradigm: 1) the exploration phase which allows the participant to learn how to control the avatar, and how to interchange between egocentric and allocentric views; 2) the topographic phase of staying on the platform and memorizing its location with the egocentric view; 3) the procedural phase of reaching the invisible platform from the beach position A as fast as possible; the participant can choose between the egocentric and the allocentric views; 4) repeating the topographic and the procedural phases with changed location of the platform B; 5) the problem mathematizing phase.

The last phase 5) includes the following reasoning. Let T(y) represents the time of reaching the platform. Let the participant decides to get into water at D, which is of y meters from C. Let z represents the entire distance from A to C; r is the running/ walking speed on land; s is the speed in water. To minimize $T(y)$ means that $T'(y)$=0, then

$$T(y) = \frac{z-y}{r} + \frac{\sqrt{x^2+y^2}}{s}; \quad T'(y)=0, \text{ which gives } y = \frac{x}{\sqrt{\frac{r}{s}+1}\sqrt{\frac{r}{s}-1}}$$

The learners can see from the formula that since r and s are fixed, y is proportional to x. They can compare this result with their virtual navigation based on their intuitive mental simulation.
The measurements to be analyzed include: distance between B and C for every changed location of platform B, distance between A and D, choice of view (allocentric or egocentric) during the procedural phase of navigation, and after experiment interview data, which include the following questions: a) What view did you choose (allocentric or egocentric) and why ? b) What did you have in mind choosing your particular path to the hidden platform? c) How mathematics describing the process corresponds to your intuitively simulated optimal path?

Conclusion

This research is aimed to "un-earth" calculus from a virtual optimal path navigation problem. The study can have an important learning effect due to enactive nature of revealing the innate capacities of tacit intuitive simulation of optimal path. Mathematizing the problem has a certain didactical value as a particular case of RME. Choice of view at the procedural stage of navigation should serve as an indirect indication of what frame of reference is utilized while constructing cognitive map and simulating mentally the optimal path. The offered study can have an important learning effect from the viewpoint of developing intuitive understanding of the calculus problem due to active participation of learners in reinventing it from the real life situation.

Questions to the audience:

1) The SL virtual environment implies the same speeds in water and on land. What is better: to program different speeds in different mediums or let learners reveal themselves after a few trials that the speeds in the SL are the same and let learners explore this special case mathematically?
2) To what extent voluntary choice between egocentric and allocentric views in the SL virtual environment reflects corresponding frames of references as the brain encodings?
3) Do the number and quality of distant cues influence the path choice?

References

Berthoz, A. (2000). *The Brain's Sense of Movement.* Cambridge: Harvard University Press.

Burgess, N. (2006). Spatial memory: How egocentric and allocentric combine. *Trends in Cognitive Sciences*, *10* (*12*), 551–557.

Cazzato, V., Basso, D., Cutini, S., & Bisiacchi, P. (2010). Gender differences in visuospatial planning: An eye movements study. *Behavioral Brain Research, 206*, 177-183.

Fischbein, E. (1989). Tacit models and mathematical reasoning. *For the Learning of Mathematics, 9*, 9-14.

Freudenthal, H. (1991). *Revising mathematics education*. Dordrecht, The Netherlands: Kluwer Academic.

Freudenthal, H. (1973). *Mathematics as an Educational Task*. Dordrecht, The Netherlands: Riedel Publishing Company.

Freudenthal, H. (1968). Why to teach mathematics so as to be useful? *Educational Studies in Mathematics, 1*, 3–8.

Pennings, T. J. (2003). Do dogs know calculus? *The College Mathematics Journal*, *34*(*3*), 178-182

Tall, D. O. (1991). Intuition and rigour: the role of visualization in the calculus. In S. Cunningham & W. S. Zimmermann (Eds.),*Visualization in Teaching and Learning Mathematics,*105-119, Washington DC: MAA.

Tall, D. O. (1997) Functions and Calculus. In A. J. Bishop et al (Eds.), *International Handbook of Mathematics Education*, 289–325, Dordrecht: Kluwer.

Tall, D. O. (2007). Embodiment, symbolism and formalism in undergraduate mathematics education. *Plenary at 10th Conference of the Special Interest Group of the Mathematical Association of America on Research in Undergraduate Mathematics Education*, Feb 22–27, 2007, San Diego, California, USA. [Available from electronic proceedings http://cresmet.asu.edu/crume2007/eproc.html. Downloaded 10 October 2007].

Torkildsen, O.E. (2006). *Mathematical Archaeology on Pupils' Mathematical Texts. Unearthing of Mathematical Structures*. *Dissertation for the Degree of Dr. Philos*. Oslo: University of Oslo, Faculty of Education.

Construct Analysis of Complex Variables: Hypotheses and Historical Perspectives

Hortensia Soto-Johnson and Michael Oehrtman

Quantitative reasoning combined with gestures, visual representations, or mental images has been at the center of much research in the field of mathematics education. In this report we extend these studies to include complex numbers and complex variables. We provide a construct analysis for the teaching and learning of complex variables, which includes a description of existing frameworks that hypothesize about how students can best comprehend the arithmetic operations of complex numbers. In order to test these conjectures, we interviewed mathematicians, physicists, and electrical engineers to explore how they perceive complex variables content. Through phenomenolgogical and microethnography analysis methods we found how these experts integrate perceptuo-motor activity and metaphors into their descriptions.

Keywords: Complex variables, Operational components, Perceptuo-motor activity, Structural components

The study of learning about numbers and their arithmetic operations is one of the best-developed fields in mathematics education research. The literature goes beyond the four basic operations to include composing and decomposing of whole numbers (Kilpatrick, Swafford, & Findell, 2001), verbal number competencies (Baroody, Benson, & Lai, 2003), ordering and comparing (Brannon, 2002), modeling and visual representations of the operations of whole numbers (Sowder, 1992). The research is not limited to whole numbers; rational numbers are part of the extensive literature related to number sense (Steffe & Olive, 2010). The studies on rational numbers entail investigating students' ability to create word problems that require division or multiplication of two fractions as well as students' visual representations of multiplication and division of two fractions. Studies of quantitative reasoning have elaborated the role of forming a mental image of the measurable attributes in a situation and conceiving of the relevant operations and relationships among these quantities (Thompson, 1994; Moore, Carlson, & Oehrtman, 2009). A natural extension to these studies is to investigate similar characteristics in the teaching and learning of complex numbers.

The main purpose of this preliminary report is to share a construct analysis for understanding complex numbers and variables. A secondary purpose is to describe how experts such as mathematicians, engineers, and physicists conceive of complex variables in both contextual and purely mathematical problems. Methodologically, we explore their use of geometric representations, gestures, verbiage, and symbolism to support their reasoning and convey their understanding. Our construct analysis is based on the few existing pieces of literature that hypothesize about students' understanding of complex numbers and a couple of studies that begin to provide empirical evidence about students' perspective for adding and multiplying complex numbers. We also include a description on how historical perspectives may influence the teaching and learning of complex variables.

Sfard (1999) argued for the need for students to become more flexible in moving between operational and structural conceptions of complex numbers. She encouraged viewing the operational and structural components of complex numbers as complementary pieces rather than as dichotomous. Researchers and instructors could support this perspective by integrating the two representations, for example through geometric illustrations of the operations since "visualization, … makes abstract ideas more tangible, and encourages treating them almost as if they were material entities" (Sfard p. 6). In order to transition from an operational to a structural perspective of complex numbers, Sfard posed three stages that students must navigate in order to develop their understanding of complex numbers.

The first stage is *interiorization*, which occurs when a process is performed on a familiar object. For the case of complex numbers Sfard claimed students who are just becoming proficient in using square roots, would be at the interiorization stage. *Condensation* is the second stage and it occurs when the learner is able to view a process as a whole without the tedious details. For example, students may continue to view 5+2i as a shorthand for certain procedures, but they would still be able to use this symbol in multi-step algorithms. The third stage, *reification*, is achieved when the learner has the ability to view a novel entity as an object-like whole. Learners who are at this stage would recognize 5+2i as a legitimate object that is an element of a well-defined set. According to Sfard (1999) this stage occurs as an "instantaneous leap" much like an "aha moment." Although the stages presented by Sfard are insightful, they do not provide empirical evidence that students actually follow these stages in learning complex variables. Furthermore, Sfard's analysis is restricted to introductory-level conceptions about complex numbers. We intend to elaborate the interplay between more advanced applications and learners' evolving conception of complex numbers/variables.

Lakoff and Núñez (2000) also offered a framework for the conceptual development of complex numbers. Their framework entails a conceptual blend of the real number line, the Cartesian plane, and rotations combined with the use of metaphor for number and number operations. Similar to historical descriptions, they portrayed multiplication of a real number x by -1 as a rotation of 180° to obtain $-x$. Thus, multiplying a number by i is equivalent to rotating by 90° counterclockwise. The beauty of this description is that it works mathematically, but empirical evidence suggests students do not view multiplying a number by -1 as a rotation of 180°, rather they perceive it as a reflection (Conner, Rasmussen, Zandieh, & Smith, 2007). This might be explained by the fact that students are focused on the real number line rather than the Cartesian plane.

In a more recent study, Nemirovsky, Rasmussen, Sweeney, and Wawro (in press) described the results of a teaching experiment with prospective secondary teachers enrolled in a capstone course. The goal of the teaching experiment was to create an instructional sequence that allowed students to create and discover the conceptual meaning behind adding and multiplying complex numbers. In this phenomenological study, the researchers incorporated microethnography to portray students' body activities over short time periods. These depictions included language use, gaze, gestures, posture, facial expressions, tone of voice, etc. As a result of their study, the researchers found:

1. mathematical conceptualization of adding and multiplying complex numbers was communicated through and comprised of perceptuo-motor activity, and

2. perceptuo-motor activity situated by the learning environment and the setting influenced the learning about the structural components behind adding and multiplying complex numbers.

This study is the first to provide hypotheses about how students make sense of arithmetic operations of complex numbers with supporting empirical data. As such it may provide insight into how best to introduce complex numbers to students besides as a mechanism for solving $x^2 + 1 = 0$. Incorporating reconstructed historical pieces of the development of complex numbers may also engender structural understanding of this area of mathematics (Glas, 1998).

Historically, the introduction and initial development of complex numbers was purely algebraic to resolve the issue of finding the real solution to certain cubic equations. Even after the square root of negative numbers was introduced, mathematicians such as Cardan found such numbers to be *sophistic* because they could not attach a physical meaning to these numbers (Nahin, 1998). These mathematicians tended to ignore the conceptual difficulties of these numbers and proceeded to apply the procedures "mechanically" (Glas, 1998, p. 368). It was Wallis who first dedicated much of his career attempting to represent the square root of a negative number through geometric constructions. Although, Wallis made progress his work was not convincing to other mathematicians or himself. It was more than a hundred years later that Wessel introduced the interpretation of placing $i = \sqrt{-1}$ at a unit distance from the origin on an axis perpendicular to the real number line to form the complex plane and that multiplying by i geometrically represents a rotation of 90º counterclockwise. This representation allowed mathematicians to begin to think about complex numbers as vectors, which in turn led to geometric representations of the arithmetic operations of complex numbers. These models were essential for mathematicians to prove that extended theories of complex numbers (i.e., quaternions, Cauchy-Riemann equations) are consistent and preserve the structure of the complex number system. Such historical developments may provide insights into how "concepts and theories can be best brought to light" for students (Glas, 1998, p. 377).

From the literature and personal reflection we hypothesize a framework in which learners may gain a better understanding of complex numbers and complex valued functions if they have opportunities to visualize arithmetic operations of two complex numbers, complex valued solutions to a quadratic equation, mappings of complex-valued function, poles, geometrical illustrations of theorems, etc. In order to better prepare ourselves to conduct teaching experiments that corroborate this hypothesis, we began our investigation by interviewing "experts." We used phenomenological methods with microethnography to synthesize their responses and to describe how they integrate perceptuo-motor activity and metaphors. We have chosen to interview experts since a goal of this research program is to eventually build a theory that describes how students understand the structural components of complex variables beyond the arithmetic operations of two complex numbers. Our hope is that this framework informed by experts' perceptions will help inform our future research.

As part of our presentation, we will show video clips of our interviewees so that the audience has an opportunity to confirm or argue against our interpretations. Questions for our audience are:

1. How can the interview questions be improved?
2. Is there another framework for data interpretation besides microethnography that might be more appropriate?
3. What impact might social constructivism have on student responses that are not evident in expert responses?

4. How might recognizing the structural component of complex numbers and/ or complex variables contribute to understanding the abstract facets of this mathematical domain?

References

Baroody, A. J., Benson, A. P., & Lai, M. I. (2003). *Early number and arithmetic sense: A summary of three studies*.

Brannon, E. M. (2002). The development of ordinal numerical knowledge in infancy. *Cognition, 83*, 223-240.

Clements, D. H., & Battista, M. T. (1992). Geometry and spatial reasoning. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp.420-464). New York: Macmillan.

Conner, M. E., Rasmussen, C., Zanieh, M., & Smith, M. (2007). Mathematical knowledge for teaching: The case of complex numbers. *Proceedings for the Tenth Special Interest Group of the Mathematical Association of America on Research in Undergraduate Mathematics Education*

Glas, E. (1998). Fallibalism and the use of history in mathematics education. *Science and Education, 7*, 361-379.

Kilpatrick, J., Swafford, J., & Findell, B. (2001). *Adding it up: Helping children learn mathematics*. Washington, D.C.: National Academy Press.

Lakoff, G. & Núñez, R. E. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. New York, New York: Basic Books.

Moore, K., Carlson, M., & Oehrtman, M. (2009). The Role of Quantitative Reasoning in Solving Applied Precalculus Problems, *Proceedings of the Twelfth Conference on Research in Undergraduate Mathematics Education*, 26 pages, Web publication at http://rume.org/crume2009/Moore1_LONG.pdf.

Nahin, P. J. (1998). *An imaginary tale: The story of* $\sqrt{-1}$. Princeton, New Jersey: Princeton University Press.

Nemirovky, R., Rasmussen, C., Sweeney, G., & Wawro, M. (in press). When the classroom floor becomes the complex plane: Addition and multiplication as ways of bodily navigation. *Journal for the Learning Sciences*.

Sfard, A. (1999). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies*, 22, 1-36.

Sowder, J. T. (1992). Making sense of numbers in school mathematics. In G. Lenhardt, R. Putnam, & R. A., Hattrup (Eds.), *Analysis of arithmetic for mathematics teaching*. Mahwah, NJ: Erlbaum.

Steffe, J. P. & Olive, J. (2010). *Children's fractional knowledge*. New York, New York:

Springer.

Thompson, P. W. (1994). The Development of Speed and its Relationship to Concepts of Rate. In G. Harel & J. Confrey (Eds.), *The Development of Multiplicative Reasoning in the learning of Mathematics.* Albany, NY: SUNY Press, 179-234.

Spanning set: an analysis of mental constructions of undergraduate students

María Trigueros, Asuman Oktaç, Darly Kú

Abstract

In this study we use APOS theory to propose a genetic decomposition for the concept of spanning set in Linear Algebra. We give examples of interviews that were conducted with a group of university students who were taking an analytic geometry course and their analysis in relation to our genetic decomposition. We also comment on the nature of difficulties that students experience in constructing this notion. One of the results that are obtained in this research that is in line with previous results reported in the literature is the difficulty in distinguishing a spanning set from a basis. Another aspect is that students have varying levels of difficulty when working with different types of vector spaces. As was expected, the concept of linear combination plays a very important role in the understanding of the notion of spanning.

Keywords: Spanning set, APOS Theory

Introduction and research objective

In an earlier study about the construction of the concept of basis in Linear Algebra (Kú, Trigueros and Oktaç, 2008) we observed the difficulties that students have with the concept of spanning set and the coordination of the underlying process with the process related to linear independence. These difficulties seemed to interfere in a serious manner with the construction of an object conception of basis of a vector space. As a result we decided to carry out research in order to look at these concepts separately, so that we could offer an explanation about the construction of each concept and related problems.

Some literature published previously touch certain issues related to the learning of spanning sets focusing on task design, cognitive difficulties and suggestions for teaching (Nardi, 1997; Ball et al., 1998; Dorier et al., 2000; Rogalski, 2000). What we are interested in with this research is to offer a viable path that students may follow in order to construct this concept as well as explaining the nature of related difficulties while learning it. Informed by our theoretical analysis and empirical data, we also focus on making pedagogical suggestions.

Theoretical framework and methodology

APOS theory has been used successfully in explaining the construction of several concepts in undergraduate mathematics curriculum. Its use with Linear Algebra concepts is more recent (Roa-Fuentes and Oktaç, 2010; Parraguez and Oktaç; 2009; Trigueros, Oktaç and Manzanero, 2007). We continue with this line of research and study the mental constructions and mechanisms involved in the learning of spanning sets.

The steps that are followed in APOS-related methodology are given in Asiala, Brown, DeVries, Dubinsky, Mathews and Thomas (1996). In line with this methodology, our research starts with a theoretical analysis which consists in a genetic decomposition as a possible way to construct the concept of spanning set. This is done in terms of mental constructions (actions, processes, objects, schemas) and mechanisms (interiorization, coordination, encapsulation, assimilation) that students might employ when learning this concept. We then designed an interview that consists in 7 questions, in order to test the viability of our genetic decomposition. This instrument was applied to a group of 11 undergraduate students who were taking an analytic geometry course at a Mexican university. These interviews are analyzed according to our theoretical framework (we are at this stage of our research). We will revisit the preliminary genetic decomposition and make the necessary modifications. Finally we hope to make some suggestions as to the didactical strategies to be employed, in order to facilitate the construction of this concept.

In our design of the interview questions we took into account different aspects of a spanning set. We asked questions of the type whether a certain set spans a given vector space, but we also asked the construction type of questions, namely given a vector space identifying possible spanning sets for it. We also asked the students to compare the vector spaces generated by different spanning sets. By dealing with different aspects of the concept of spanning set in this manner, we hope to shed light on where the difficulties lie and verifying the mental constructions involved in its learning.

Some results

One of the results that are obtained in this research that is in line with previous results reported in the literature (Nardi, 1997) is the difficulty in distinguishing a spanning set from a basis. Another aspect is that students have varying levels of difficulty when working with different types of vector spaces. In particular, when the vector space is not R^n, the interpretation of a spanning set becomes problematic. On the other hand, as was expected, we confirmed that the concept of linear combination plays a very important role in the understanding of the notion of spanning. We are also exploring the connections that students seem to make among the concepts of linear independence/dependence, basis, linear combination, dimension, spanning set and generated vector space. Our analysis so far indicates that it will be necessary to make certain modifications in the preliminary genetic decomposition, but the general model is in line with data.

References

Asiala, M., Brown, A., Devries, D.J., Dubinsky, E., Mathews, D., Thomas, K. (1996). A framework for research and curriculum development in undergraduate mathematics education. In J. Kaput, A. H. Schoenfeld, E. Dubinsky (Ed.s) *Research in collegiate mathematics education*. Vol. 2. Providence, RI: American Mathematical Society. pp. 1-32.

Ball, G., Stephenson, B., Smith, G., Wood, L., Coupland, M. and Crawford, K. (1998). Creating a diversity of mathematical experiences for tertiary students. *International Journal of Mathematical Education in Science and Technology*, **29**(6), 827-841

Dorier, J. L., Robert, A., Robinet, R. and Rogalski, M. (2000). The Obstacle of Formalism in Linear Algebra. A Variety of Studies From 1987 Until 1995. In J.-L. Dorier (ed.), On the Teaching of Linear Algebra. Dordrecht : Kluwer, pp. 85-124.

Kú, D., Trigueros, M. and Oktaç, A. (2008). Comprensión del concepto de base de un espacio vectorial desde el punto de vista de la teoría APOE. *Revista Educación Matemática*. 20(2), 65-89.

Nardi, E. (1997). El encuentro del matemático principiante con la abstracción matemática: Una imagen conceptual de los conjuntos generadores en el análisis vectorial. *Educación Matemática*. **9**(1), 47-60.

Parraguez, M. and Oktaç, A. (2010). Construction of the Vector Space Concept from the Viewpoint of APOS Theory. *Linear Algebra and its Applications*, 432, 2112-2124.

Roa-Fuentes, S. and Oktaç, A. (2010). Construcción de una descomposición genética: Análisis teórico del concepto transformación lineal. *Revista Latinoamericana de Investigación en Matemática Educativa*, 13 (1), 89 – 112.

Rogalski, M. (2000). The Teaching Experimented in Lille. In J.-L. Dorier (ed.), On the Teaching of Linear Algebra. Dordrecht : Kluwer, pp. 133-149.

Trigueros, M., Oktaç, A. and Manzanero, L. (2007). Understanding of Systems of Equations in Linear Algebra. Demetra Pitta – Pantazi & George Philippou (Eds.), *Proceedings of the 5th Congress of the European Society for Research in Mathematics Education, CERME* (pp. 2359-2368).

The Construction of Limit Proofs in Free, Open, Online, Help Forums

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Abstract: Free, open, online, help forums are found on public websites and allow students to post queries from their course assignments that can be responded to asynchronously by anonymous others. Several of these forums are tailored to helping students with mathematics assignments from various courses, and Calculus, in particular, is a heavily trafficked area. Students use the forums when they have reached an impasse, either in constructing or understanding a solution to an exercise that they have encountered, or to seek verification of their own reasoning. The queries posted by students include both computational tasks as well as proof constructions. In this project, we examine threads on limit proofs for single-variable functions from two popular online forums. Our goal is twofold: to characterize the help students are receiving as they wrestle with using the formal definition of limit, and to compare the construction of proof to other tasks in online forums.

Keywords: computer-mediated discourse; limits; online help; student understanding of proof

Introduction

As students tackle assignments or struggle to understand their coursework, they seek access to a large and varied set of resources. One such resource that has emerged fairly recently and appears quite popular is found on the Internet. Free, open, online, help forums are located on public websites and allow students to post queries from their course assignments or materials that can be responded to asynchronously by anonymous volunteers. Help forums exist for many subject areas and grade levels, with mathematics, in general, (and Calculus, in particular) receiving much traffic. Students use the forums to post questions from exercises and examples that they encounter in their material classrooms. These include both computational or procedural tasks, as well as proofs. It is the help seeking for the construction and comprehension of proofs in the forums that has drawn our attention. Of the many proof types that surface on the forums, we focus on limit proofs for single-variable functions. This type of proof serves as students' first introduction to rigorous proof in Calculus, but is challenging and often poorly understood. Our goals are to characterize the help that students are receiving as they wrestle with examples and exercises of proofs using the formal definition of limit, and to compare help seeking on proof to that on other types of exercises in online forums.

Literature & Theoretical Perspective

In contrast to being cheat sites, there is evidence that many forums profess the intent to assist students rather than to do their exercises for them (van de Sande & Leinhardt, 2007). In such forums, students, instead of simply publishing the problem statement, are more prone to demonstrate understanding of the exercise (e.g., what they have tried) and contribute to the construction of the solution (e.g., by responding to helpers). Students also take more responsibility for initiating resolution by communicating how the interaction was helpful. In terms of the helpers, in some forums, they exhibit a strong sense of community, as they correct one another, work collectively to help individual students, and engage in collegial banter. The students and helpers who participate in these forums are using technology to engage with one another on the mathematics that students are expected to perform. Proof is an example of one such activity that students face.

However, students have little experience knowing what counts as mathematical proof and how these differ from common sense proofs. Unlike the logical deduction used by mathematicians, undergraduates consider specific cases to constitute proof (Harel & Sowder, 1998). Even for students who possess an accurate conception of mathematical proofs, proof writing is challenging (Selden & Selden, 2008). One of the main difficulties occurs when students structure their proofs in terms of the chronological order of their thought process instead of rearranging them with careful consideration on proper implications (Dreyfus, 1999). It is also a challenge for students to transform informally written statements into mathematically formally structured ones in calculus (Selden & Selden, 1995). In addition, students have difficulty in relating conceptual ideas that can help them generate their proofs. Knowing formal definitions or theorems is insufficient for students to construct a proof (Weber, 2001). Research calls attention to personal heuristic knowledge of mathematical concepts (Roh, in press), and the connection of heuristic ideas about mathematical concepts to proof construction (Raman & Weber, 2006; Raman & Zandieh, 2009).

We hold that help seeking is an important strategy that can be instrumental in the development of autonomous skill and ability (Nelson-LeGall, 1985). In terms of constructing limit proofs, this ability requires not only deductive reasoning, but also abductive reasoning to extract a pattern from observations, measurements, or events in a holistic manner, and inductive reasoning to identify or synthesize common regularities across several events.

Research Methods

Our general methodology is one that has been applied in the context of open, online, help forum research (van de Sande & Leinhardt, 2007). It involves selecting an online forum(s) as a research site, searching the archives using keywords (such as “epsilon” and “prove”), and selecting threads from the search that match a particular target (here, limit proofs for single-variable functions). We have adopted this observational methodology for ethical reasons and because of the exploratory nature of the research.

For this project, we selected two online mathematics help forums, located at www.freemathhelp.com (FMH) and www.sosmath.com (SOS). Both forums allow any member to contribute (as a helper) to any ongoing thread (as opposed to restricting the set of helpers or assigning incoming queries to one particular helper). This participation structure allows multiple helpers to be involved and interact in a given thread. In addition, member status in both forums depends only on the number of individual threads to which one has contributed (as opposed to depending on others’ ratings of one’s contributions). Finally, both forums are functional and active in the sense of daily postings and membership, and have existed for about a decade.

Our search and selection process in the current FMH and SOS archives netted 73 threads involving limit proofs for single-variable functions: FMH, n=19, dated 8/30/05-6/13/10 and SOH, n=48, dated 6/1/03-5/26/10.

Preliminary Results

We analyze the types of query in terms of (Q1) functions (linear or nonlinear functions), (Q2) layers of quantification (e.g., in finding a value of δ for a given value of ε, or for any ε), and (Q3) student need (how to prove or why we prove what we prove). We also analyze the types of help in terms of (H1) representations (algebraic or graphical approach) and their connections, (H2) reasoning promoted by helpers (inductive, abductive, or deductive reasoning), and (H3) pedagogical moves (lecturing, scaffolding, or hinting). Finally, we analyze levels of resolution from the student perspective.

Example: On September 30, 2009, a student pomar posted a problem on SOS "*Use the given graph of f(x) = √x to find a number δ that fulfills the following condition. If absolute value(x-4) < delta then absolute value(sqrt(x)) - 2) < 0.4, δ =*" with a message "I do not know how to solve this problem can you please show the steps to solve this problem and explain why you did what you did." Pomar received 3 responses from two different helpers, Jack and measurable, as follows:

- Jack: "It's easy enough to do directly: If $|x\text{-}4|<\delta$, then $|\sqrt{x}+2|\ |\sqrt{x}\text{-}2|<\delta$." (September 30, 2009)
- measurable: "It is obvious that $\delta = 10^{-10}$ would do." (October 1, 2009)
- Jack: "Expanding on my earlier answer... If $|\sqrt{x}\text{-}2|<0.4$, then $-0.4<\backslash\sqrt{x} -2<0.4$, so $1.6<\backslash\sqrt{x} <2.4$. Now, add 2 to get $3.6 <\sqrt{x} +2<4.4$. The part I care about is the upper bound of 4.4. Since we are adding and $\sqrt{x} \geq 0$, then $\sqrt{x} +2<4.4$ is equivalent to $|\backslash\sqrt{x} +2|<4.4$, so $|\sqrt{x} -2|<0.4$ => $|x\text{-}4| = |\sqrt{x} -2|\ |\sqrt{x} +2| < (0.4)(4.4) =1.76$. Thus $\delta =1.76$ or any other smaller number will do, such as measurable's answer." (October 1, 2009)

We claim that this exchange exhibits the following characteristics:

- (H1) Focus on algebraic manipulations without connections to graphical meaning (Even the graph that the student mentions is never addressed.)
- (H2) Improper use of abductive reasoning (Jack) or inductive reasoning (measurable)
- (H3) Restricted range of pedagogical moves (Scaffolding and hinting are not in evidence.)
- Low level of resolution from perspective of student (Student need to know how and why we construct δ that way is not addressed.)

Implications for Further Research

Students are using open, online help forums to seek advice on comprehending and constructing proofs requiring the formal definition of limit. Our preliminary findings suggest that the help they are receiving from this resource is unsatisfactory based on our understanding of proof construction, the formal definition of limit, and student reasoning. This work points to the need to either develop new, or modify existing, theory-based approaches to teaching limit proofs in response to student queries in an online forum environment. Furthermore, the major limitation of our methodology, namely that using observations alone severely restricts the analysis of student and helper thinking, calls for experimental studies that link activity on the forum to mathematical understanding and performance.

Discussion Questions

1. Can you suggest relevant analytical frameworks for our observational research project?
2. How do you envision a trajectory from an observational to an experimental research program?

References

Dreyfus, T. (1999). Why Johnny can't prove. *Educational Studies in Mathematics*, *38*, 85-109.

Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. *Research in Collegiate Mathematics Education* III, 234-283.

Nelson-Le Gall. S. (1985). Help-seeking behavior in learning. *Review of Research in Education, 12,* 55-90.

Raman, S. M. & Weber, K. (2006). Key ideas and insights in the context of three high school geometry proofs. *Mathematics Teachers*, *99*, 644-649.

Raman, S. M., & Zandieh, M. (2009). The case of Brandon: the dual nature of key ideas in the classroom. *Nordic Studies in Mathematics Education, 14*, 29-48.

Roh, K. (in press). College students' reflective activity in advanced mathematics. To appear in *Proceedings of the 32nd Conference of the North American Chapter of the International Group for the Psychology of Mathematics Education, Columbus, OH, Ohio State Univ.*

Selden, A., & Selden, J. (2008). Overcoming students' difficulties in learning to understand and construct proofs. In M. Carlson & C. Rasmussen (Eds.), *MAA Notes. Making the connection: Research and teaching in undergraduate mathematics education* (pp. 95-110). Washington, DC: MAA.

Selden J. & Selden A. (1995). Unpacking the logic of mathematical statements. *Educational Studies in Mathematics, 29(2),* 123-151.

van de Sande, C., & Leinhardt, G. (2007). Online tutoring in the Calculus: Beyond the limit of the limit. *Éducation et Didactique, 1(2),* 115-154.

Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics*, *48*, 101-119.

The van Hiele Theory Through the Discursive Lens: Prospective Teachers' Geometric Discourses

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Abstract
This project investigates changes in prospective elementary and middle school teachers' van Hieles levels, and in their geometric discourses, on classifying, defining and constructing proofs with geometric figures, resulting from their participation in a university geometry course. The project uses the van Hiele Geometry Test from the Cognitive Development and Achievement in Secondary School Geometry (CDASSG) project, in a pretest and posttest, to predict prospective teachers' van Hiele levels (Usiskin, 1982), and also uses Sfard's (2008) framework to analyze these same prospective teachers' geometric discourses based on in-depth individual interviews. Additionally, the project produces a translation of van Hiele levels into a detailed model that describes students' levels of geometric thinking in discursive terms. The discussion will focus on studying college students' reasoning and methods of proof regarding geometric figures in Euclidean geometry.

Keywords: prospective teachers, Euclidean geometry, mathematical discourse, the van Hiele Theory

Over the past decade, there has been an increasing push in the mathematics education research community to study students' reasoning and understanding in the teaching and learning of mathematics, and to examine issues emphasizing the use of vocabulary and terminology in the mathematics classroom. In response, this project investigates the changes in prospective teachers' levels of geometric thinking, and the development of their geometric discourses, in the classification of quadrilaterals.

Theoretical Framework

In Sfard's (2008) *Thinking as Communicating: Human Development, the Growth of Discourses, and Mathematizing,* she introduces her commognitive framework, a systematic approach to analyzing the discursive features of mathematical thinking, including word use, visual mediators, routines, and endorsed narratives. To examine thinking about geometry, this project connects Sfard's analytic framework to another, namely the van Hiele theory (see van Hiele, 1959/1985). The van Hiele theory describes the development of students' five levels of thinking in geometry. The levels 1 to 5 are described as visual, descriptive, theoretical, formal logic and rigor. In addition, this project produces, on the basis of theoretical understandings and of empirical data, a detailed model, namely, *the Development of Geometric Discourse*. This model translates the five van Hiele levels into five discursive stages of geometric discourses with respect to word use, visual mediators, routines, and endorsed narratives at each van Hiele level.

Three overarching questions guide the project: (1) How do prospective teachers' familiarities with basic geometric shapes, abilities to formulate conjectures, and abilities to derive geometry propositions from other geometry propositions change as a result of their participation in a university geometry course? (2) What are the changes in prospective teachers' geometric thinking with regard to the van Hiele Levels? (3) How do the findings help to revise the proposed model of geometric discourse development?

Method

Guided by these research questions, the process of inquiry includes the data collection and analysis of a pretest and posttest, each of which is followed by the collection and analysis of interview data. Seventy-four college students who enrolled in a college mathematics content course for elementary and middle school teachers participated in the pretest and posttest. Twenty-one of these 74 students participated in the interviews. Data for this project comes from three resources: (1) Written responses to the van Hiele Geometry Test (see Usiskin, 1982) (from pretest and posttest), (2) Transcripts (from two in-depth interviews, the first interview conducted right after pretest, and the second right after posttest), (3) Other written artifacts (students' written statements, and answer sheets to the tasks during the interviews).

Data collection takes place in four phases: (1) the pretest is administered to all students during class time in the first week of the semester, (2) Student volunteers are interviewed a week after they participae in the pretest, (3) All students participate in the posttest at the end of the semester, (4) Students who participated in the interviews at the beginning of the semester are interviewed once more. All tests are collected and analyzed. All interviews are video and audio recorded. All interview data are transcribed and analyzed.

Results

Preliminary results suggest that most students in the project have moved one or two van Hiele levels, and the majority of the students' levels of geometry thinking are at van Hiele levels 2 or 3 after their participation in a college geometry course. However, when comparing a student's written response in the van Hiele Geometry Test with his/her interview response, it appears that the van Hiele level of a student determined by the written test is not always coherent with expected geometric discourse at the given level described in the model of the *Development of Geometric Discourse*. For example, after being assigned to van Hiele level 3 based on his/her written response in the van Hiele Geometry Test, the student is interviewed. Analysis of the student's geometric discourse with respect to his/her word use, routines, visual mediators and endorsed narratives shows that the student's van Hiele level is at level 2 instead of level 3. This result does not indicate that the van Hiele Geometry Test is inaccurate in determining students' van Hiele levels, but rather suggests that using a discursive lens to analyze students' geometric discourses at each van Hiele level provides additional information about the student's levels of geometric thinking, and detects information which has been missed in the van Hiele Geometry Test.

Educational Significance

The project provides a better understanding of what prospective teachers know about geometric figures such as triangles, quadrilaterals and their properties, and of prospective teachers' abilities in mathematical reasoning, conjecturing, and proving. The project also sheds light on prospective teachers' use of mathematical terminologies and definitions related to triangles and quadrilaterals, through their geometric discourses. This information about prospective teachers' competencies in geometry helps to improve the teacher preparation program with regard to their mathematical content knowledge. Additionally, the project produces, on the basis of theoretical understandings and empirical data, a detailed model, the *Development of Geometric Discourse*. This model helps to identify additional information that are missed, or not clearly presented, in the general description of van Hiele levels through the analyses of geometric discourse. In practice, distinguishing students' levels of geometric thinking helps to recognize obstacles faced by students, and provides information for instructors teaching prospective teachers, to help improve classroom interactions and instructions.

I am interested in feedback from the audience about (1) the issues of students' abilities in reasoning and proof in introductory undergraduate geometry courses; (2) students' ways of using mathematical terminologies, definitions and propositions in mathematics classrooms; and (3) comments on the model, *the Development of Geometry Discourses.*

Reference

Sfard, A. (2008). *Thinking as Communicating: Human development, the growth of discourses, and mathematizing*. New York: Cambridge.

Senk, S. L. (1989). Van Hiele levels and achievement in writing geometry proofs. *Journal for Research in Mathematics Education, 20*(3), 309-321.

Usiskin, Z. (1982). *Van Hiele levels and achievement in secondary school geometry* (Final report of the Cognitive Development and Achievement in Secondary School Geometry Project: ERIC Document Reproduction Service No. ED 220 288). Chicago: University of Chicago.

Van Hiele, P. M. (1959/1985). The child's thought and geometry. In D. Fuys, D. Geddes & R. Tischler (Eds.), *English translation of selected writings of Dina van Hiele-Geldof and Pierre M. Van Hiele* (pp. 243-252). Brooklyn, NY: Brooklyn College, School of Education.

Van Hiele, P. M. (1986). *Structure and insight: A theory of mathematics education*. Orlando, FL: Academic Press.

Title: Function Composition and the Chain Rule in Calculus

Preliminary Report

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The chain rule is a calculus concept that causes difficulties for many students. While several studies focus on other aspects of calculus, there is little research that focuses specifically on the chain rule. To address this gap in the research, we are studying how students use and interpret the chain rule while working in an online homework environment. We are particularly interested in answering three questions: 1) What characterizes student's understanding of composition of functions? 2) What characterizes student's understanding of chain rule? and 3) To what extent do students' understanding of composition of functions play a role in their understanding and ability to use chain rule in calculus?

Keywords: Calculus, precalculus, procedural knowledge, conceptual knowledge, technology

Literature Review

Studies have indicated that success in calculus is likely linked to a robust understanding of the concept of function (Carlson, Oehrtman, & Engelke, 2010; Ferrini-Mundy & Gaudard, 1992). Unfortunately, many students enter calculus with a weak understanding of the concept of function. Carlson (1998) investigated and described what is required for students to gain a mature understanding of the concept of function and concluded that a mature concept of function is slow to develop, even in strong students. Studies have also show that function composition is particularly problematic for students (Engelke, Oehrtman, & Carlson, 2005).

There have been a number of studies that focus on what it means to understand the concept of derivative (Asiala, Cottrill, Dubinsky, & Schwingendorf, 1997; Ferrini-Mundy & Gaudard, 1992; Orton, 1983; Zandieh, 2000). "It is known that some students are introduced to differentiation as a rule to be applied without much attempt to reveal the reasons for and justifications of the procedure." (Orton, 1983, p. 242) In fact, many first semester calculus students earn a passing grade without ever achieving a conceptual understanding of the derivative. Students are adept at using rules to find the derivative function and using this result to compute the desired answer. When asked about the chain rule, most students will simply provide an example of what it is rather than explain how it works (Clark et al., 1997; Cottrill, 1999). The literature related to studies in calculus provide evidence that students develop more procedural understanding than conceptual in differentiation. However, there is a gap in the studies investigating the characteristics of student's understanding of composition of functions and the chain rule. We aim to provide a description of the possible relationships among these understandings.

Theoretical Perspective

Star (2005) carefully examines the existing literature on procedural and conceptual understanding in mathematics education and points out the necessity to develop broader frameworks to investigate both procedural and conceptual knowledge and understanding. Since the publication of Hiebert's book (1986) on conceptual and procedural knowledge many studies have used the definitions and the framework (Rittle-Johnson, Siegler & Alibali, 2001). Hiebert and Lefevre state that conceptual knowledge is "characterized most clearly as knowledge that is rich in relationships" whereas procedural knowledge "consists of rules and procedures for solving mathematical problem" (1986, p.3, p.7). Star (2005) suggests broadening these definitions in order to provide more in depth analysis of both procedural and conceptual understanding. He criticizes earlier research studies not providing in depth analysis of these concepts but rather focusing on the order of them: "Which comes first: procedural or conceptual knowledge?"

This study employs Star's (2005) approach toward procedural and conceptual knowledge and understanding to map out the students' understanding of composition of functions and the chain rule. We aim to describe possible characteristics of students' surface and deep procedural knowledge and understanding of composition of functions and the chain rule by examining student work.

Methodology

The 41 students in this study are first semester calculus students enrolled at a large Midwestern University who regularly take online quizzes using tablet computers. Student work on several function composition and chain rule problems was collected using a modified online homework system and digital ink. The system records and replays, in real-time, the work each student did to complete the problem. Students had three opportunities to submit each problem. The system also collected how they modified the problem, enabling us to focus on students whose initial work was incorrect and to identify the steps they thought needed to be fixed in order to answer the problem correctly.

Student ability to complete precalculus tasks, including the chain rule, was measured during the first four weeks of the semester. Students were given pre-tests including both the Pre Calculus Concept Assessment (PCA) Instrument (Carlson et al., 2010) and an 84-item precalculus assessment focusing on procedural knowledge. Students were allowed to practice items from the procedural assessment during the subsequent two weeks, and completed a post-test involving the PCA and the procedural assessment during the fourth week. This data helps identify strengths and weaknesses in various students and as baseline data for our analysis of the student work gathered by the online homework system.

Results

Students completed function composition and chain rule problems during the first week and seventh week quizzes. For the purposes of this study, students are classified as strong, average, or weak according to their ability to work with function composition as measured by their initial and final scores on the PCA. The study is still collecting and analyzing data, some of which is shared below.

The students were given the following problem during the first week of the semester: *The graph of y=f(x) and y=g(x) are shown below. Calculate f(f(-2) and f(g(2)).* Figure 1 shows a *work map* of Student 1 and Student 2, selected from the strong and weak groups. Vertical bars on the work map indicate when the student was drawing, graphing, erasing, adding images, navigating between problems, and submitting correct or incorrect answers for a problem. Figure 2 shows part of the work done by each student. From our initial observation of Student 1 and Student 2 work, we noticed the procedurally more proficient student (Student 1) was capable of using a graphical representation of a function to find the composition of function. This could be a possible characteristic needed for a deep procedural knowledge of a composition of function.

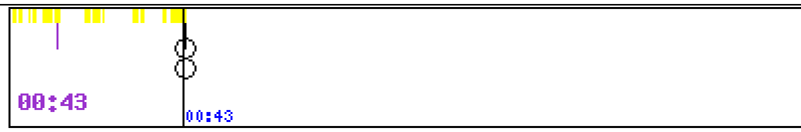
Student 1:	
Student 2:	

Figure 1: Work maps of Student 1 and Student 2 on Function Composition Problem.

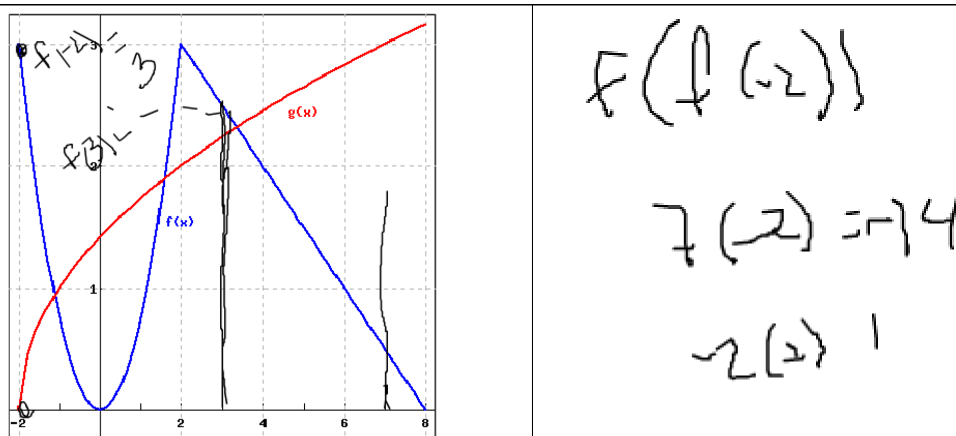
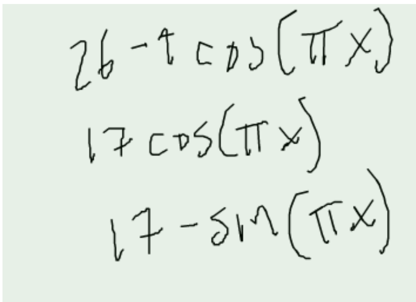
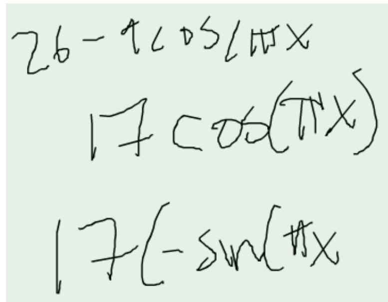
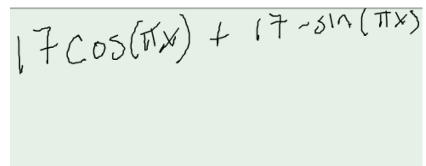


Figure 2: (Partial) work of Student 1 and Student 2 on Function Composition Problem.

During the seventh week of the semester, students were asked: *Find the derivative of R(x) = 26 – 6 cos(πx).* Initial observations show that students who consistently scored high on the function composition problems on the PCA correctly apply the chain rule in this simple case, while students from the moderate and weak groups typically failed to recognize that differentiating *cos(πx)* required the use of the chain rule. Replay of student work shows many students struggling with function manipulations involving signs, addition, and multiplication. For instance, Figure 3 shows three separate attempts by Student 2 to solve the problem. In each case, the student struggled with the application of an incorrect procedure but failed to address the use of the chain rule.

In this study we would like to investigate these cases further in depth to elicit more features of surface and deep procedural knowledge and understanding. Also, we plan to include contextual problems to investigate student's conceptual understanding further.

		
Work before first attempt: *17(-sin(pi x))*	Work before second attempt: *17 - sin(pi x)*	Work before third attempt: *17cos(pi x) + 26 - 9 - sin(pi x)*
Figure 3: Student 2 work on solving a simple chain rule problem		

Questions for the audience:

- What would you like the technology to be able to do?
- How would you envision using the work map?
- What does a work map tell us about student's procedural and conceptual knowledge?
- We are currently looking at this as further refining some of the procedural/conceptual frameworks that have come before now. Is there a better theoretical perspective that we could be working with?

References

Asiala, M., Cottrill, J., Dubinsky, E., & Schwingendorf, K. (1997). The Development of Students' Graphical Understanding of the Derivative. *Journal of Mathematical Behavior, 16*(4), 399-431.

Carlson, M. (1998). A Cross-Sectional Investigation of the Development of the Function Concept. *Research in Collegiate Mathematics Education III, Conference Board of the Mathematical Sciences, Issues in Mathematics Education, 7*, 114-163.

Carlson, M., Oehrtman, M., & Engelke, N. (2010). The Precalculus Concept Assessment: A Tool for Assessing Reasoning Abilities and Understandings of Precalculus Level Students. *Cognition and Instruction, 28*(2), 113-145.

Clark, J. M., Cordero, F., Cottrill, J., Czarnocha, B., Devries, D. J., St. John, D., et al. (1997). Constructing a Schema: The Case of the Chain Rule? *Journal of Mathematical Behavior, 16*(4), 345-364.

Cottrill, J. (1999). *Students' understanding of the concept of chain rule in first year calculus and the relation to their understanding of composition of functions.* Unpublished Doctoral Dissertation, Purdue University.

Engelke, N., Oehrtman, M., & Carlson, M. (2005, October 20-23). *Composition of Functions: Precalculus Students' Understandings.* Paper presented at the 27th Annual Conference of the North American Chapter of the International Group for the Psychology of Mathematics Education, Roanoke, VA.

Ferrini-Mundy, J., & Gaudard, M. (1992). Secondary School Calculus: Preparation or Pitfall in the Study of College Calculus? *Journal for Research in Mathematics Education, 23*(1), 56-71.

Hiebert, J. (1986). Conceptual and procedural knowledge: The case of mathematics. Hillsdale, NJ: Lawrence Erlbaum.

Hiebert, J., & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.), Conceptual and procedural knowledge: The case of mathematics (pp. 1-27). Hillsdale, NJ: Lawrence Erlbaum Associates.

Orton, A. (1983). Students' Understanding of Differentiation. *Educational Studies in Mathematics, 14*, 235-250.

Rittle-Johnson, B., Siegler, R. S., & Alibali, M. W. (2001). Developing conceptual understanding and procedural skill in mathematics: An iterative process. *Journal of Educational Psychology, 93,* 346–362.

Star, J. R. (2005). Reconceptualizing procedural knowledge. *Journal for Research in Mathematics Education, 36,* 404 – 411.

Zandieh, M. (2000). A Theoretical Framework for Analyzing Student Understanding of the Concept of Derivative. In E. Dubinsky, Schoenfeld, A., Kaput, J. (Ed.), *Research in Collegiate Mathematics Education, IV* (Vol. 8, pp. 103-127). Providence, RI: American Mathematical Society.

Effective Strategies That Successful Mathematics Majors Use to Read and Comprehend Proofs

Keith Weber and Aron Samkoff

Abstract. Proof is a dominant means of conveying mathematics to undergraduates in their advanced mathematics courses, yet research suggests that students learn little from the proofs they read and find proofs to be confusing and pointless. In this presentation, we examine the behavior of two successful mathematics majors as they studied six proofs to identify productive proof comprehensive strategies. Prior to reading a proof, these students would attempt to understand the theorem by rephrasing and trying to determine why it was true. While reading a proof, these students would partition the proof into sections, attend to the proof framework being employed, and illustrate confusing aspects of the proof with examples. Implications and limitations of this study will be discussed.

Keywords: Proof, proof reading, proof comprehension.

1. Introduction

In advanced mathematics courses, much of students' time is spent observing their professor present proofs of theorems during course lectures and reading proofs in their textbooks. The implicit assumption underlying this practice is that students can effectively learn mathematics by studying proofs. However, many researchers in mathematics education question this assumption, noting that undergraduate mathematics majors often find the process of reading proofs to be confusing and pointless (e.g., Harel, 1998; Rowland, 2001) and students often do not develop an adequate understanding of a proof after reading it (e.g., Conradie & Frith, 2000). One area that has received little attention in mathematics education research is *how* students should read and study a proof to foster comprehension. The present study seeks to address this void in the literature by describing the proof reading strategies of two successful mathematics majors.

2. Related literature

In the mathematics education research literature, there is a great deal of research on mathematical proof. In analyzing this research, Mejia-Ramos and Inglis (2009) observed that the large majority of empirical studies on proof in mathematics education concerned students' construction of proofs rather than their reading of proofs. Mejia-Ramos and Inglis further noted that most studies focusing on students' reading of proofs analyzed the way students evaluated mathematical arguments; these studies, for instance, asked students if they found an argument to be convincing or if they thought the argument would qualify as a proof. There were few studies that concerned students' comprehension of proofs. As a main goal of presenting proofs to students in their advanced mathematics courses is to increase their understanding of mathematics, the lack of research into this area represents an important void in the literature.

3. Theoretical perspective

In the reading comprehension literature, it is widely accepted that the meaning that an individual obtains from a text is based on three factors: the individual, the text, and the way the individual interacts with the text (e.g., Alexander & Fox, 2004). While reading comprehension can be improved by improving the background knowledge of the reader or the quality of the text, it is also worthwhile to improve the ways that individuals interact with the text. A common approach to conducting research in this area is to identify strategies that effective readers use to comprehend text and to then instruct less successful readers on how to use these strategies (e.g., Palinscar & Brown, 1984; Chi et al, 1994). The study reported here is consistent with this research paradigm.

4. Methods

Two students, with the pseudonyms Kevin and Tim, from a large state university agreed to participate in this study. Both students were mathematics majors in their senior year; these students were also both simultaneously enrolled in a secondary mathematics teacher preparation program. They were invited to participate in this study because they performed well in their mathematics education courses, they were articulate, and they had successfully participated in mathematics education research studies in the past.

The participants met as a pair with the first author of this paper for two 2-hour videotaped task-based interviews. The participants were initially given a proof. They were asked to "think out loud" as they read and studied the proof. They were told to study the proof until they felt they understood it and informed they would be asked questions to assess their comprehension after they read the proof and they would not have the proof to refer to while they answered these questions. This process was repeated for each of the six proofs. Each proof was chosen so that it was of moderate length (between 4 and 20 lines), was based on calculus or basic number theory (to insure Kevin and Tim had an adequate background knowledge to comprehend the proof), and employed a novel technique.

As noted in the introduction, there are few research articles on proof comprehension (Mejia-Ramos & Inglis, 2009) and we are not aware of any research on the strategies that students should use to read proofs for comprehension. Consequently, we did not have any pre-existing categories in mind when analyzing this data and opted to use an open coding scheme in the style of Strauss and Corbin (1990).

In a first pass through the data, we independently noted each attempt that Kevin and Tim made to make sense of the theorem statement or the proof and provided a summary of the students' behavior. (Here, "attempt" was construed broadly to mean anything beyond a literal reading of the text). After these summaries were produced, the authors met to discuss their findings.

From here, it was noted that Kevin and Tim's proof reading could be divided into four phases: (a) studying the theorem, (b) reading the proof, (c) re-reading and summarizing the proof, and (d) critically evaluating the proof.Within each phase, similar proof-reading attempts were grouped together to form categories of the proof reading strategies that Kevin and Tim employed. After categories were named and defined, we again independently viewed the videotape, coding for each instance of the proof reading strategies. We then compared notes and discussed disagreements until they were resolved. Most disagreements were the result of oversight on one of our parts. After our coding, Kevin and Tim were again interviewed about whether the strategies we observed were commonly used and why they engaged in those proof-reading strategies.

4. Results

Kevin and Tim spent considerable time studying the proofs, with the time spent on each proof ranging between 3 minutes and 16 minutes, with an average of 7 minutes and 20 seconds per proof. We note this is significantly longer than other studies on undergraduates' proof reading (e.g., Selden & Selden, 2003; Weber, 2009). It is not clear if Kevin and Tim spent more time reading the proof was due to them being unusually thoughtful and deliberate or because of the task design (they were given an assessment test after reading each proof).

Kevin and Tim averaged nearly three minutes studying the theorem *prior* to reading its proof, in one case spending nearly six minutes studying the theorem. This finding suggests that in examining that strategies for proof comprehension should not focus only on how students interact with proofs, but also the things they do to understand theorems.

Kevin and Tim would attempt to understand the theorem by rephrasing the theorem and by attempting to see why the theorem was true for themselves before reading the proof. The latter was done for each of the six theorem-proof pairs that Kevin and Tim read. They cited numerous benefits to trying to see why a theorem was true, both in terms of understanding the proof and motivating the need to read the proof.

When reading the proof, Kevin and Tim would explicitly attend to the proof framework (in the sense of Selden and Selden, 1995) employed (for example, by once engaging in a lengthy process where they verified that the assumptions and conclusions of the proof actually satisfied the framework for proof by contraposition), partition the proof into sections to verify it, and check problematic assertions in the proof with examples.

After reading the proof, Kevin and Tim would sometimes re-read the proof, summarizing the proof based on its high-level ideas. In other cases, Tim would point out assertions within the proof that appeared to be inconsistent (either with other assertions or his own mathematical understanding), at which point he and Kevin would resolve these apparent inconsistencies together.

5. Significance

This presentation outlines strategies that the two successful mathematics majors used to effectively comprehend six proofs. Clearly, due to limitations of the study (in particular, only using two students and six proofs), no definitive claims can be made. The purpose of this presentation is to make a contribution to the literature by suggesting strategies that other students can be taught to use to improve proof comprehension.

6. Questions for the audience

+ This study will be replicated with another pair of successful students. How can this study be modified to elicit more proof reading strategies?
+ What other types of methodologies can be used to investigate the successful proof reading strategies of these students?
+ What types of classroom environments might foster the use of these strategies?

References

Alexander, P.A. & Fox, E. (2004). A historical perspective on reading research and practice. In R.B. Ruddell & N.J. Unrau (eds.) *Theoretical models and processes of reading*. Newark, DE: International Reading Association.

Chi, M. T. H., de Leeuw, N., Chiu M-H., & Lavancher C. (1994). Eliciting self-explanations improves understanding. *Cognitive Science, 18,* 439-477.

Conradie, J., & Frith, J. (2000). Comprehension tests in mathematics. *Educational Studies in Mathematics, 42*, 225-235.

Harel, G. (1998). Two dual assertions: The first on learning and the second on teaching (or vice versa). *American Mathematical Monthly*, 105, 497-507.

Mejia-Ramos, J. P. & Inglis, M. (2009). Argumentative and proving activities in mathematics education research. In F.-L. Lin, F.-J. Hsieh, G. Hanna & M. de Villiers (Eds.), *Proceedings of the ICMI Study 19 conference: Proof and Proving in Mathematics Education* (Vol. 2, pp. 88-93), Taipei, Taiwan.

Palinscar, A.S. & Brown, A.L. (1984). Reciprocal teaching of comprehension-fostering and monitoring activities. *Cognition and Instruction*, 1(2), 117-175.

Rowland, T. (2001). Generic proofs in number theory. In S. Campbell & R. Zazkis (eds.) *Learning and teaching number theory: Research in cognition and instruction*. Westport, CT: Ablex Publishing.

Selden, J. and Selden, A. (1995). Unpacking the logic of mathematical statements. *Educational Studies in Mathematics*, 29(2), 123-151.

Strauss, A. and Corbin, J. (1990). *Basics of qualitative research: Grounded theory procedures and techniques*. London: SAGE Productions.

Weber, K. (2009). Mathematics majors' evaluation of mathematical arguments and their conception of proof. In *Proceedings of the 12th Conference for Research in Undergraduate Mathematics Education*. Available for download from: http://sigmaa.maa.org/rume/crume2009/proceedings.html

Student Understanding of Integration in the Context and Notation of Thermodynamics: Concepts, Representations, and Transfer

Preliminary Research Report

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Students are expected to apply the mathematics learned in their mathematics courses to concepts and problems in physics. Little empirical research has investigated how readily students are able to "transfer" their mathematical knowledge and skills from their mathematics classes to other courses. In physics education research (PER), few studies have distinguished between difficulties students have with physics concepts and those with either the mathematics concepts, application of those concepts, or the representations used to connect the math and the physics. We report on empirical studies of student conceptual difficulties with (single-variable) integration on mathematics questions that are analogous to canonical questions in thermodynamics. We interpret our results considering the representations used as well as the lens of knowledge transfer, with attention to how students solve problems involving the same mathematical principles in the differing contexts of their physics and mathematics classes.

Keywords: Physics, integrals, conceptual understanding, representations, transfer

Mathematics is a vital part of how physics concepts are represented (e.g. equations, graphs and diagrams), manipulated and how problems are solved, from the introductory to the upper level. It allows students to simplify the analysis of complex problems by representing complicated conceptual physics problems as a relatively simple relationship between variables. Appropriate interpretation of these representations requires recognition of the connections between the physics and the mathematics built into the representation and subsequent application of the related mathematical concepts (Redish, 2005). Students are expected to apply the mathematics learned in their mathematics courses to concepts and problems in physics. Despite the fact that students are expected to carry out such interdisciplinary study as a matter of course, little empirical research has investigated how readily students are able to "transfer" their mathematical knowledge and skills from their mathematics classes to other courses.

With many physics areas, specific mathematical concepts are required for a complete understanding and appreciation of the physics. Meltzer (2002) has shown a link between mathematical acumen and success in an algebra-based physics class. Tuminaro and Redish (2007) have combined the frameworks of resources (Hammer, 1996), epistemic games, and framing to analyze student use of mathematics in physics. To date, however, there have only been a few PER studies exploring physics students' difficulties with calculus concepts (Pollock et al., 2007; Rebello et al., 2007, Black and Wittmann, 2009).

Our work aims to identify the extent to which mathematical knowledge and understanding affects physics conceptual knowledge, specifically in the context of upper-level thermal physics. We have two main research questions: What difficulties do advanced-level undergraduate students have when learning thermal physics concepts? To what extent does students' mathematical knowledge and understanding influence their responses to physics questions?

The empirical framework that guides descriptions of student reasoning in our research is that of *specific difficulties* (Heron, 2003). We start with targeted, context-dependent results and then generalize across contexts, seeking larger patterns of student responses in our data. Our emphasis is on gathering and interpreting empirical data that can act as a foundation for future studies on reasoning in physics and for curriculum development to address specific difficulties. The more cognitive framework, ideally suited for the study of knowledge transfer and the context-sensitivity of mathematical knowledge, is that of *transfer in pieces* (Wagner, 2006).

A key feature of thermal physics is the reliance on many ways of thinking about integration. In physics in general, the idea of an integral is tied closely with a graphical interpretation as the area under the curve.

Previous findings on student understanding of integral calculus concepts in mathematics education research indicate that students do not possess the necessary knowledge to allow them to successfully complete problems involving concepts of integration, especially with regard to considering the integral as the area under the curve. The literature in mathematics education repeatedly documents the lack of student understanding of the relationship between a definite integral and the area under the curve (Orton, 1983; Vinner, 1989; Thompson, 1994; Grundmeier, 2006). These include student difficulties with recognition of integrals as limits of (Riemann) sums (Orton 1983, Sealey, 2006); student confusion about the concept of "negative area" for integrals of curves that fall below the *x*-axis either conceptually (Bezuidenhout and Olivier, 2000), computationally (Orton, 1983; Rasslan & Tall, 2002) or both (Hall, 2010). Thompson and Silverman (2008) showed that the reliance on *area under curve* reasoning may limit applicability of the conception of integrals.

In our current research, we seek to isolate mathematical difficulties that may underlie observed physics difficulties by asking physics questions that are completely stripped of their content, and focus on the calculus concepts under investigation (e.g., integration and partial differentiation). We call these *physicsless physics questions,* since they typically use notation that is more consistent with representations used in physics rather than following the conventions of mathematics (Christensen & Thompson, 2010).

We administered the physics questions as well as the analogous physicsless questions to the students in our thermodynamics course. Data have been obtained from written responses to ungraded free-response questions, and interviews have been conducted on physics students at various levels. Student responses to the physics questions were compared to reported categories in the literature (Loverude et al. 2002, Meltzer 2004). Responses to the physicsless questions were analyzed for patterns and categorized; the categories were then compared to those from the (paired) physics questions. The results from the paired physics and math questions among physics students show that some of the difficulties that arise when comparing thermodynamic work based on a pressure-volume (*P-V*) diagram may be attributed to difficulties with the *mathematical* aspect of the diagram, in particular with the correct application of an understanding of integrals, rather than physics conceptual difficulties (e.g., treating work as an equilibrium state function). These results suggest that some students aren't necessarily attributing state function properties to work so much as failing to recognize the same variable as two different functions during integration.

To further explore the question of transfer closer to the source of the concept of integration (and area under the curve), we asked the physicsless integral questions near the end of a multivariable calculus class to over 150 calculus students from 3 different semesters. These results from the multivariable calculus course suggest that the observed mathematical difficulties are not just with transfer of math knowledge to physics contexts. Some of these difficulties seem to have origins in the understanding of the math concepts themselves.

We have recently extended this work to vary the features of the representation(s) used in the math-based question, with the goal of exploring the extent to which students are using features of the representation – either tacitly or explicitly – to interpret the question being asked.

In one case, different formats of the basic question discussed (Figures 1a and b) were administered near the end of a few recitation sections of calculus classes, both introductory integral calculus (Calculus II) and multivariable calculus (Calculus III). The questions asked students to compare the absolute value of the definite integrals for two functions, *f(y)* and *g(y)* on

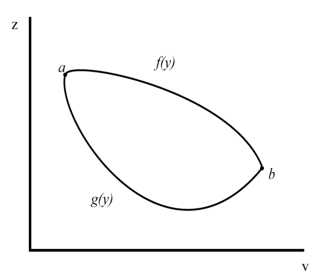

Figure 1(a)

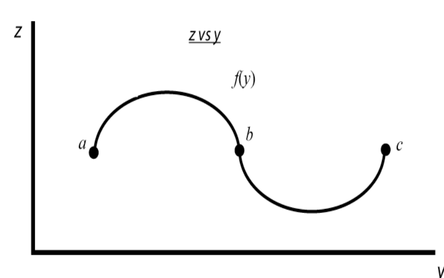

Figure 1(b)

a qualitative graph.

The reasoning given with answers was not always clear, but they could roughly be put into a number of categories: area under the curve, position of the function, curvature of the curve, slope of the line. Just over 50% explicitly used the word "area" in their reasoning. While there is evidence of these categories of reasoning, it does not provide definitive proof that students have a concept image that is consistent with their explanations. One interesting result is that calculus 2 students outperformed calculus 3 students on the question, with nearly all of calculus 2 students giving the correct answer and 60% of calculus 3 students successfully able to compare the absolute values for the integrals. Building on these results, clinical interviews are being carried out to explore the nature of students' concept image (Vinner, 1989) of the integral in variations of the same question.

Furthermore, we are interpreting many of the above results through the lens of knowledge transfer, with attention to how students solve problems involving the same mathematical principles in the differing contexts of their physics and mathematics classes. Existing data are being examined through a *transfer-in-pieces* framework, initially to consider how student performance may be influenced by the different representational forms used in the domains of mathematics and physics. This analysis complements existing analyses in ways that suggest further research tasks. This research angle focuses on how undergraduate students come to see contextually different problems or situations as "alike," in that they demonstrate instances of the same mathematical principle. This perspective complements and expands the initial research questions about how students recognize and understand the relationship between the concepts of physics and the mathematics that underlies them.

Questions for audience

What are the implications for the teaching of these topics, both in mathematics and in physics?

We intend to continue research along this vein of the mathematics-physics connections. We recognize that our questions aren't presented in rigorous mathematical notation; how reasonable are our questions in the opinion of mathematicians, given their relevance to the way physicists use and apply the mathematics?

Even if students come out of their mathematics classes with a good *mathematical* understanding of the principles at stake, it is reasonable to expect that transferring their mathematical knowledge into a physics context should be unproblematic?

References

Bezuidenhout, J., and Olivier, A. (2000). Students' conceptions of the integral. In *Proceedings of the Conference of the International Group for the Psychology of Mathematics Education* (PME) (24th, Hiroshima, Japan, July 23-27, 2000) **2,** 73-80.

Black, K.E, & Wittmann, M.C. (2009). Procedural Resource Creation in Intermediate Mechanics. In *2009 Phys. Educ. Res. Conf.,* M. Sabella, C. Henderson C. Singh, eds., AIP Conference Proceedings **1179**, 97-100.

Christensen, W.M., and Thompson, J.R. (2010). Investigating Student Understanding of Physics Concepts and the Underlying Calculus Concepts in Thermodynamics. In *Proceedings of the 13th Annual Conference on Research in Undergraduate Mathematics Education* (Mathematical Association of America).

Grundmeier, T.A., Hansen, J., and Sousa, E. (2006). An exploration of definition and procedural fluency in integral calculus. *PRIMUS: Problems, Resources and Issues in Mathematics Undergraduate Studies*, **16**, 178-191.

Hall, Jr., W.L. (2010). Student Misconceptions of the Language of Calculus: Definite and Indefinite Integrals. In *Proceedings of the 13th Annual Conference on Research in Undergraduate Mathematics Education* (Mathematical Association of America).

Hammer, D. (1996). Misconceptions or p-prims: How might alternative perspectives of cognitive structures influence instructional perceptions and intentions? *Journal of the Learning Sciences,* **5**, 97-127.

Heron, P.R.L. (2003). Empirical investigations of learning and teaching, Part I: examining and interpreting student thinking. Proceedings of Enrico Fermi Summer School on Physics Education Research. Italian Physical Society, 341-350.

Loverude, M.E., Kautz, C.H., and Heron, P.R.L. (2002). Student understanding of the first law of thermodynamics: Relating work to the adiabatic compression of an ideal gas. *American Journal of Physics*, **70**, 137-148.

Meltzer, D.E. (2002). The relationship between mathematics preparation and conceptual learning gains in physics: a possible 'hidden variable' in diagnostic pretest scores. *American Journal of Physics,* **70,** 1259-1268.

Meltzer, D.E. (2004). Investigation of students' reasoning regarding heat, work, and the first law of thermodynamics in an introductory calculus-based general physics course. *American Journal of Physics*, **72**, 1432-1446.

Orton, A. (1983). Student's understanding of integration. *Educational Studies in Mathematics*, **14**, 1-18.

Pollock, E.B., Thompson, J.R., and Mountcastle, D.B. (2007). Student understanding of the physics and mathematics of process variables in P-*V* diagrams. In *2007 Phys. Educ. Res. Conf.,* L. Hsu, C. Henderson, L. McCullough, eds., AIP Conference Proceedings, **951**, 168-171.

Rasslan, S. & Tall, D. (2002). Definitions and images for the definite integral concept. In Anne D. Cockburn & Elena Nardi (Eds), *Proceedings of the 26th Conference of the International Group for the Psychology of Mathematics Education*, (Norwich, UK),

4, 89–96.

Rebello, N.S., Cui, L., Bennett, A.G., Zollman, D.A. and Ozimek, D.J. (2007). Transfer of Learning in Problem Solving in the Context of Mathematics & Physics. In *Learning to Solve Complex Scientific Problems,* David Jonassen, ed., Lawrence Earlbaum Associates, New York, 2007.

Redish, E. F. (2006). Problem Solving and the Use of Math in Physics Courses. In *Proceedings of the Conference, World View on Physics Education in 2005: Focusing on Change*, Delhi, August 21-26, 2005. Available at http://www.physics.umd.edu/perg/papers/redish/IndiaMath.pdf

Sealey, V. L. (2006). *Student Understanding of Definite Integrals, Riemann sums and Area Under a Curve: What is Necessary and Sufficient?* Paper presented at the Twenty-Eighth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Merida, Mexico.

Thompson, P.W. (1994). Images of rate and operational understating of the fundamental theorem of calculus. *Educational Studies in Mathematics*, **26**, 229-274.

Thompson, P. W., & Silverman, J. (2008). The concept of accumulation in calculus. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics* (pp. 43-52). Washington, DC: Mathematical Association of America.

Tuminaro, J., and Redish, E.F. (2007). Elements of a cognitive model of physics problem solving: Epistemic games. *Physical Review Special Topics - Physics Education Research* 3, 020101.

Vinner, S. (1989). The avoidance of visual considerations in calculus students. *Journal of Mathematical Behavior*, **11**, 149-156.

Wagner, J. F. (2006). Transfer in Pieces. *Cognition and Instruction,* **24**(1), 1-71.

Ian Whitacre and Susan D. Nickerson

Abstract

We report on results of the implementation of a local instruction theory for number sense development in a course for prospective elementary teachers. Students involved in an earlier teaching experiment developed improved number sense, particularly in the form of flexible mental computation. The previous research was informed by a conjectured local instruction theory and informed the refinement and elaboration of that local instruction theory. The present study concerns a recent iteration of the classroom teaching experiment, in which the local instruction theory guided instructional planning. In the recent iteration, the local instruction theory was extended from the whole-number portion of the course to the rational-number portion. Envisioned learning routes that were developed in the context of mental computation and estimation were applied to reasoning about fraction size. In this way, the application of the local instruction theory was extended from whole-number sense to rational-number sense.

Keywords: Local instruction theory, number sense, prospective teachers, rational number

We report on results of the implementation of a local instruction theory for number sense development in a mathematics content course for prospective elementary teachers. Our previous research showed that students involved in an earlier teaching experiment developed improved number sense, particularly in the form of flexible mental computation (Whitacre, 2007). The previous research was informed by a conjectured local instruction theory and informed the refinement and elaboration of that local instruction theory (Nickerson & Whitacre, 2010). The present study concerns a recent iteration of the classroom teaching experiment, in which the local instruction theory guided instructional planning. In the recent iteration, the local instruction theory was extended from the whole-number portion of the course to the rational-number portion. In particular, envisioned learning routes that were developed in the context of mental computation and estimation were applied to reasoning about fraction size. In this way, the application of the local instruction theory was extended from whole-number sense to rational-number sense.

Instruction

The course was intended to foster the development of number sense. In particular, the broad instructional intent was for students to come to act in a *mathematical environment* in which the properties of numbers and operations afforded a variety of calculative strategies, as opposed to one in which mathematical operations map directly to particular algorithms (Greeno, 1991). The mathematics content course is the first course in a four-course sequence. There are multiple sections of the course, and a common final exam is used. Topics in the curriculum include quantitative reasoning, place value, meanings for operations, children's thinking, standard and alternative algorithms, representations of rational numbers, and operations involving fractions. We have adapted the course in such a way as to engage students in authentic computational reasoning throughout the semester. This includes such activities as mental computation, estimation, and reasoning about the relative magnitudes of fractions and decimals.

By authentic, we mean that students encounter the need for computational reasoning in the process of their work on larger tasks. For example, in the process of solving a story problem, computations naturally arise, and students are expected to solve these mentally. Particular students' strategies are discussed by the class, and a set of established strategies gradually builds.

Local Instruction Theory

A *local instruction theory* refers to "the description of, and rationale for, the envisioned learning route as it relates to a set of instructional activities for a specific topic" (Gravemeijer, 2004, p. 107). Note that a local instruction theory (LIT) is distinct from a hypothetical learning trajectory (HLT). Gravemeijer (1999) identifies two key distinctions between these related constructs: (1) an LIT tends to describe an instructional sequence of longer duration; (2) an HLT is situated in a particular classroom, whereas an LIT is not.

We have described elsewhere our local instruction theory for the development of number sense (Nickerson & Whitacre, 2010). Here, we briefly list the three major goals around which this LIT is organized: (1) Students capitalize on opportunities to use number-sensible strategies; (2) Students develop a repertoire of number-sensible strategies; (3) Students develop the ability to reason with models. In the proposed session, we focus primarily on the second of these goals.

Design Research

We conduct design research in the form of classroom teaching experiments, which are reflexively related to theory building (Cobb & Bowers, 1999). In this case, our LIT was developed and refined in the context of a classroom teaching experiment in a course for prospective elementary teachers. The previous experiment focused on mental computation and estimation, primarily with whole numbers. In the present iteration, the LIT guides instruction throughout the same content course, including instruction concerning rational numbers. We describe how the LIT has guided instructional planning for a sequence concerning reasoning about fraction size. By the time of the conference, we will be able to report on how this sequence played out in the classroom.

Applying the LIT to Reasoning about Fraction Size

We focus on the goal of students developing a repertoire of number-sensible strategies, particularly those strategies involved in comparing fractions and otherwise reasoning about fraction size. The framework of Smith (1995) informed our thinking about these strategies and influenced the planning of the instructional sequence. Our pilot and pre-instruction interviews, as well as the findings of other researchers, such as Newton (2008), suggested that these students would come to the course with standard algorithms for comparing fractions. Smith describes the strategies of converting to a common denominator or converting to a decimal as belonging to the Transform Perspective. We also expected students to come to instruction with a meaning for fractions as indicating a number of parts of a whole (e.g., so many slices of a pie). In Smith's terms, this is an example of the Parts Perspective.

Tasks were designed and sequenced so as to begin with students' current ways of reasoning and to provide opportunities for reasoning about fraction size in new ways. In particular, we sought to engage students in reasoning about fraction size from Smith's Reference Point and Components perspectives. The Reference Point Perspective involves reasoning about fraction size on the basis of proximity to reference numbers, or *benchmarks* (Parker & Leinhardt, 1995). The Components Perspective involves making comparisons within or between the two fractions, as in coordinating multiplicative comparisons of numerators and denominators. The design and sequencing of tasks involved consideration of these perspectives relative to number choices, contexts, and anticipated student reasoning, and guided by the envisioned learning routes described in our LIT. Although Smith (1995) does not describe the perspectives or particular strategies belonging to his framework in a hierarchical way, we view the Reference Point and Components perspectives as generally more sophisticated categories of reasoning about fraction size. Thus, in our instructional sequence, we aim for these more sophisticated strategies to be used by students and established for the class by mathematical argumentation.

Questions for Discussion

By the time of the conference, we will be prepared to report on how the instructional sequence played out, although formal analysis of collective activity will be ongoing. Questions for discussion specific to the fraction-size sequence concern productive models for reasoning about fraction size, as well as our conjectured trajectory of strategies. We welcome suggestions concerning tasks and ordering of tasks in the instructional sequence. More broadly, we are interested in discussion local instruction theories, their relationship to hypothetical learning trajectories, and the notion of extending an LIT to new topics or implementing it in new contexts.

References

Cobb, P., & Bowers, J. (1999). Cognitive and situated learning perspectives. *Educational Researcher, 28* (2), 4–15.

Gravemeijer, K. (1999). How emergent models may foster the constitution of formal mathematics. *Mathematical Thinking and Learning, 1,* 155-177.

Gravemeijer, K. (2004). Local instruction theories as a means of support for teachers in reform mathematics education. *Mathematical Thinking and Learning, 6*, 105–128.

Greeno, J. (1991). Number sense as situated knowing in a conceptual domain. *Journal for Research in Mathematics Education, 22*, 170–218.

Newton, K. J. (2008). An extensive analysis of preservice elementary teachers' knowledge of fractions. *American Educational Research Journal, 45,* 1080-1110.

Nickerson, S. D., & Whitacre, I. (2010). A local instruction theory for the development of number sense. *Mathematical Thinking and Learning*, 12, 227–252.

Parker, M., & Leinhardt, G. (1995). Percent: a privileged proportion. *Review of Educational Research, 65*, 421-481.

Smith, J. P., III. (1995). Competent reasoning with rational numbers. *Cognition and Instruction, 13,* 3-50.

Whitacre, I. (2007). *Preservice teachers' number sensible mental computation strategies*. Proceedings of the Tenth Special Interest Group of the Mathematical Association of America on Research in Undergraduate Mathematics Education. Retrieved from http://sigmaa.man.org/rume/crume2007/papers/whitacre.pdf

Diana White and Angie Hodge

Background and perspectives

The supply of qualified, competent mathematics majors entering secondary teaching professions is not keeping pace with demand (Liu et al., 2008; National Research Council, 2002). According to Ingersoll and Perda (2009), the problem is more than a number game. Part of the problem resides in the fact that teachers are not happy with the profession once they are out in the fields, causing the number of teaching leaving the profession to be greater than the number of teachers entering the profession. This is especially the case in low-income areas where they are 77% more likely to be taught by out of field teachers compared to students from high socioeconomic backgrounds (Ingersoll, 2003). Attrition is also a compounding factor as recent research reveals 20 to 30% of teachers have left the profession within the first five years (Darling-Hammond, 2001).

The NSF Robert Noyce Scholars Program seeks to help address these issues through a scholarship program aimed at recruiting and retaining secondary mathematics teachers for high needs school districts. Districts experiencing high teacher turnover rates, teachers teaching in areas outside of their content area, and high poverty rates all meet the condition of high needs for this program.

Specifically, this preliminary report discusses how the Rocky Mountain Noyce Scholars Program (RM-NSP), a five-year National Science Foundation scholarship grant for undergraduate pre-service secondary mathematics teachers, aims to target undergraduate education as part of the solution. The central tenant is that if we recruit teachers who are strong in their content area and also dedicated to serving students in high needs school districts that the attrition rates may decrease. If we also help prepare and support our teachers well (both in the areas of mathematics and pedagogy), they will hopefully have high job satisfaction rates and be effective classroom teachers.

Adhering to these ideals in the undergraduate education of pre-service secondary mathematics teachers, the RM-NSP has been a catalyst for revision of the undergraduate secondary mathematics teacher preparation program. We discuss one component of this revision in detail in conjunction with preliminary results from the first year of the program. In our presentation, we will seek advice from our audience members on future research steps and data collection to strengthen the preliminary results.

Program, participants, and context

A novel component of this revision is a college internship experience for the pre-service teachers. The idea for this project was adapted from Hodge's (in press) idea of a Teaching Algebra Seminar. This was part of the goal of having the pre-service teachers experience teaching not only at the level they eventually plan to teach at, the secondary level, but also at the level beyond, namely introductory college courses. We thought that this experience would provide them exposure to common algebra misconceptions, experience facilitating guided problem-solving groups, deeper involvement with the university and the Department of Mathematical and Statistical Sciences, a deeper knowledge of the algebraic and geometric reasoning skills students need for success in trigonometry, and an exploration of historical importance of trigonometry.

During this internship, they act as recitation instructors for a section of college trigonometry. This is an optional, but highly attended, one-hour session before each hour of lecture. It should be noted that prior to this program, recitations were not held for college trigonometry classes at this university. During the pilot semester of this, 6 pre-service teachers participated. They were grouped into threes based on their individual class schedules and placed into two recitation sections. Both sections had the same instructor for the lecture portion of class – a full time instructor from the Department of Mathematical and Statistical Sciences. During the pilot, the pre-service teachers did not receive any credit for the experience. Rather, it was a condition of accepting the scholarship, and for two of them, a volunteer experience.

Duties of the recitation instructors included: attending lectures by the instructor, leading associated recitation sections, designing activities, handouts, and review activities, working with students on an individual basis, grading and providing feedback on student work, facilitating technology-driven application problems and attending a weekly seminar on teaching trigonometry.

Data collection and data analysis

Since this was a pilot study, some of our outcomes were unexpected and data collection went beyond what we had anticipated. We were attempting to gather data on the impact of the recitation program on the pre-service teachers. However, early in the semester the instructor began noticing changes in the performance of the trigonometry students compared with previous semester. We monitored these changes throughout the semester. The instructor also began to observe changes in his approach to the class, and began to consciously reflect on this. In future semesters, we would like to pursue these effects in more depth and with more intentionality of research design. We will seek input from the audience on how to best research the input on the trigonometry students and on the instructor's changes in practice.

As we began to notice changes, we began to collect data in multiple ways. First, we collected scores of the trigonometry students on each of their three exams and the final and compared theses to prior semesters of trigonometry taught by the same instructor. Once we noticed the significant increase in performance on the first exam, we began to intentionally gather data on the remaining exams for comparison purposes. The instructor also began to pay more attention to how the experience was affecting his teaching. The external evaluator of the grant interviewed him about this experience and its impact on him. A large part of our desire to give this talk is to solicit audience input on how to more methodically pursue our investigation of this trigonometry internship experience.

Preliminary results of programmatic change

One of the first outcomes of this that we noticed was that scores by the trigonometry students were considerably higher than they had been in past semesters. Specifically, the median score on each exam increased by 8-10%. This is a comparison among college trigonometry classes taught by the same instructor at the same university. The instructor considered the exams very similar each semester, with the Spring 2010 exam actually more difficult. We will be having a group of "experts" look over the exams to see which ones they would consider more difficult.

Additionally, the class has three technology-driven application problems. The application problems require the students to synthesize their knowledge of the material and apply mathematics to real world scenarios. The percentage of students who completed these went up considerably with the implementation of the trigonometry recitation sections, from approximately a 75% completion rate to a 100% completion rate

In addition to impacting both the trigonometry students and the pre-service teachers, we found that this class had a significant impact on the instructor. In an interview with the external evaluator, he reported a renewed focus on conceptual understanding of the material, consistent feedback and insight from recitation instructors, a deeper understanding of individual trigonometry student's strengths and weaknesses, and a greater understanding of the historical background of trigonometry and its connections with other fields.

Discussion and implications

Some of the noted impacts could be explained simply by the fact that students who have more support in classes can be expected to perform better. However, the magnitude of these changes seems to indicate that something more is occurring. There is walk-in tutoring available for all of the lower level math courses in the Math Education Resource Center, staffed by undergraduate and graduate math students, and there are recitation sessions available for other lower level math courses, specifically College Algebra and Calculus I. Neither of these support structures seems to have the same impact on student achievement.

Some questions that arise and may be good for discussion include:

1. How can we better analyze what is leading to this improvement in student achievement in trigonometry?
2. Can this internship be leveraged to impact student achievement more broadly in the lower level math classes?
3. Does the impact on the instructor transfer to increased student achievement in his other lower level classes? If so, how could this be measured?
4. How might we isolate the components of this internship that are central to the increase in student achievement?

References

Darling-Hammond, L. (2001). The challenge our schools face. *Educational Leadership, 58,* 12-17.

Ingersoll, R. M., & Perda, D. (2009). The mathematics and science teacher shortage: Facts and myth. Available at http://www.cpre.org/images/stories/cpre_pdfs/math%20science%20shortage%20paper%20march%202009%20final.pdf

Ingersoll, R. (2003). *Is there really a teacher shortage?* Philadelphia, PA, and Seattle, WA: Consortium for Policy Research in Education, University of Pennsylvania, and the Center for the Study of Teaching and Policy, University of Washington.

Liu, E., Rosenstein, J., Swann, A., Khalil, D. 2008. When Districts Encounter Teacher Shortages? The Challenges of Recruiting and Retaining Math Teachers in Urban Districts. *Leadership and Policy in Schools.* 7(3): 296-323.

National Research Council. (2002). *Learning and understanding: Improving advanced study of mathematics and science in U.S. schools*. Washington, DC: National Academies Press.

Redefining Integral: Preparing for a New Approach to Undergraduate Calculus

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Abstract: This study is a pilot to a larger design research project that aims to explore an alternative approach to teaching a Calculus I course. Central to this approach is the introduction of the integral first, utilizing a non-standard definition, but which is equivalent to the standard definition. This is immediately followed by the introduction of derivative. This approach allows methods of derivation and integration, which are analogs of one another to be introduced in close succession, allowing the relationships between these methods to be a major theme of the course. The alternative definition of integral is the focus of this study. I present preliminary results of a teaching experiment that explores how students develop an understanding of this alternative definition of integral and how these understandings relate to prerequisite notions, such as area and arithmetic mean.

Keywords: Calculus, arithmetic mean, new methodology, teaching experiment.

Introduction

In the late 1980s and early 1990s the NSF responded to growing concerns regarding student success in Calculus by moving considerable resources into Calculus curricular reform efforts. The National Research Council (1991) followed suit issuing a challenge to "revitalize undergraduate mathematics." In spite of efforts which these initiatives sprouted student success remains much the same today as it did in the early 90's (Seymour, 2006). The larger research project – to which this study is a pilot – aims to work toward addressing this issue by developing and implementing a set of innovative course materials for a first course in Calculus and by studying the impact these have on student learning, disposition, and retention.

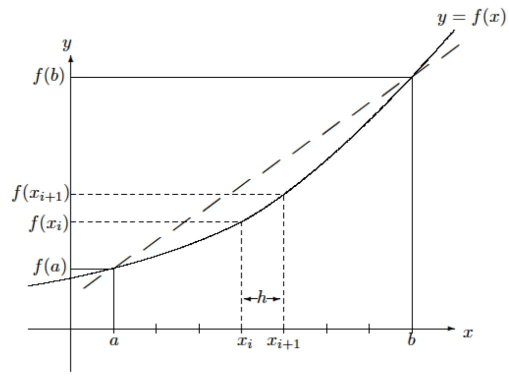
The Definition of Integral

The definition of integral utilized in the proposed reformed Calculus curriculum does not rely on the notion of Riemann sum. Instead, the definition relies on the mean of a uniform sample of *n* heights of a function:

Uniform Sampling: $x_{i-1} - x_i = h = \frac{b-a}{n}$, where $x_i = a + ih,\ i = 1,\ 2,\ \ldots,\ n$

Sample Data: $y_i = f(x_i),\ i = 1,\ 2,\ \ldots,\ n$

Statistical Mean: $\bar{y}(n) = \frac{1}{n}\sum_{i=1}^{n} y_i$



Definition of integral**:** $I \equiv \lim_{n\to\infty} \bar{y}(n)(b-a)$ is the integral of $f(x)$ over the interval $[a, b]$. This is written as $I \equiv \int_a^b f(x)dx$.

The Teaching Experiment

In order to explore how students come to interpret and understand this definition, a teaching experiment (Steffe & Thompson, 2000) was conducted with a group of students who typically do not take Calculus, pre-service elementary school teachers. This choice of participants exaggerates the potential struggles, misinterpretations and conceptual hurdles, providing a richer picture of what Calculus students are likely to encounter. Since pre-service elementary teachers likely have no prior exposure to Calculus, interference from previous instruction is limited.

The teaching experiment took place over four 50-minute sessions, conducted biweekly with a group of four students. The instructional sequence used student concepts and reasoning as the starting point, from which more complex and formal reasoning developed. Developing and understanding of the relationship between mean and area was the goal of the first session. Students were prompted to develop a standardized technique for finding the area of a display of uniform width columns. This served as a catalyst for gaining an understanding of why the method of multiplying the mean height of the columns by the width of the figure yields the area. The next session focused on estimating the area of more rounded figures utilizing a sample of their heights, which was obtained manually using a ruler. These figures were then replaced by functions in the subsequent session, where instead of finding heights manually the function values are utilized. Formal notation was then introduced. The final session explored students' conceptions of what happens as the sample size is increased. This culminated in the definition of integration. These sessions were video-taped and preliminary analysis will be presented.

The Questionnaire on Understanding Mean

An understanding of the alternative definition of integral relies heavily on a non-procedural understanding of mean. In the definition, the mean of a sample of heights of a function is abstract—not tied to any specific function. Hence, in order to understand the definition, students must be able to consider the abstract properties of mean. To explore understandings of mean amongst potential Calculus students, a questionnaire was designed and administered with pre-Calculus students. It included questions that could be approached both using procedural methods and non-procedural understandings of mean. For example, one such question shows two triangles and asks about the mean measure of their six angles. Another question involves a circle that is divided into several sectors. Students were asked to determine the average measure of the area of these sectors. In both problems the specific sizes of the angles are given, but are not necessary in order to find the solution. To account for students that possess conceptual understanding but prefer procedural solutions to demonstrate their method, as well as for students that develop conceptual insight after completing a procedural solution, students are prompted for an additional solution after each of the tasks. The results of this questionnaire will be presented.

Synopsis

This pilot study focuses on how students' develop an understanding of the alternative definition of integral and the hurdles they encounter along the way. Additionally, students' conceptions of mean, as well as how those conceptions relate to area, are explored. This additional focus is included in order to paint a better picture—one which includes an exploration of the kind of relevant tools students bring into the new Calculus curriculum. In the session participants will have the opportunity to reflect on and discuss the following questions related to the study:

- What student difficulties have you observed/experienced with the conventional definition of integral? What difficulties do you foresee (or have experienced) with the alternative definition of integral?
- What lens/framework do you suggest for the in-depth analysis of the teaching experiment?

References

National Research Council (2001). *Adding it Up: Helping Children Learn Mathematics*. National Academy Press: Washington, DC.

Seymour, E. (2006). Testimony offered by Elaine Seymour, PhD., University of Colorado at Boulder, to the Research Subcommittee of the Committee on Science of the U.S. House of Representatives Hearing on *Undergraduate Science, Math and Engineering Education: What's Working?* Wednesday, March 15, 2006.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. E. Kelly (Eds.), *Research design in mathematics and science education* (pp. 267-307). Hillsdale, NJ: Erlbaum.