Student Understanding of Complex Numbers

M. Elizabeth Conner
San Diego State University

Chris Rasmussen
San Diego State University

Michelle Zandieh
Arizona State University

Michael Smith
San Diego State University

The purpose of this preliminary report is to (1) describe shifts in student conceptions of complex number over the course of a two and a half week instructional unit, and (2) discuss data that challenge some of our assumptions about how student conceptions develop around the complex plane. Our initial motivation for this study arose as a direct consequence of our experience teaching prospective high school mathematics teachers. Although these students are exposed to complex numbers as early as intermediate high school algebra, and continue to encounter them sporadically throughout their undergraduate education, student conception of complex number often fails to extend much past $i = \sqrt{-1}$. This is particularly unfortunate in light of the fact that many of these students will become the teachers who will be introducing complex numbers to the next generation of students. An additional motivation for this study was the fact that a literature review did not reveal any empirical studies about student learning of complex number.

Background Literature

Although we did not find any empirical studies, we did find several sources that addressed the historical development of complex number, and hypothesized some of the cognitive milestones that might be necessary for the development of an understanding of complex number. We next summarize a few of the insights that we gained from these readings.

Sfard (1991) takes a process/object view of the development of complex number, similar to the way in which other numbers were developed. For example, when subtracting, say, five from three, the *process* — subtraction — led to the development of an *object* — negative number. Each expansion of our concept of number can be explained in a similar manner, with division motivating the need for rationals, taking square roots motivating the need for irrationals, and, later, the taking of roots of negative numbers motivating the need for complex numbers.

Penrose (2004) notes that historically mathematicians defined $i = \sqrt{-1}$ and augmented it with two real parts, a and b, to create sums of the form $a + bi$. He explains that although the sums can be treated as a pair of numbers, acceptance of complex numbers as a new category of number is dependent on being able to conceptualize $a + bi$ as a single entity.

Fauconnier and Turner (2002) claim that complex numbers were not thought of as numbers in their own right until conceptual blending occurred between two mental spaces: The real numbers, with arithmetic operations, and vectors in the Cartesian plane with magnitudes and directions. This blend creates the notion of complex number as both a number and a vector.

Lakoff and Nuñez (2000) use related conceptual blending ideas, but also employ a series of metaphors for number and number operation to build their description of the conceptual development of complex numbers. The metaphor for negative numbers was particularly interesting to us. They write, "…negative numbers are conceptualized by the cognitive, spatial rotation operation via the metaphor Multiplication by –1 is rotation." By blending this multiplication-as-rotation metaphor with the Cartesian plane, the rotation-plane blend is realized. Then the complex plane is the rotation-plane blend combined with the notion that $i = \sqrt{-1}$ in the sense that if rotation by 180 degrees is multiplication by -1, then rotation by 90 degrees must be $\sqrt{-1}$ which must be the (0,1) point on the plane.

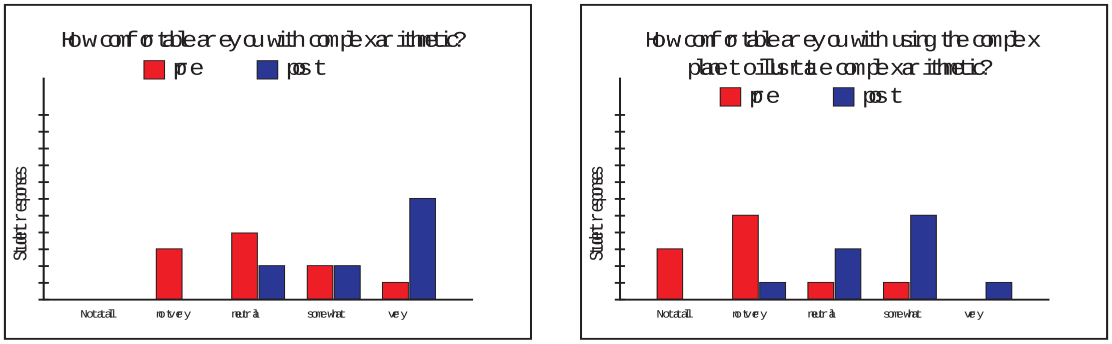
Method

We conducted a classroom teaching experiment (Cobb, 2000) during the last three weeks of *Mathematics for High School Teaching,* a capstone course for prospective secondary school teachers. There were thirteen students enrolled in the course, ten of whom agreed to participate in pre-and-post interviews. Data was also collected in the form of in-class videorecordings, in-class assessments, homework assignments and student responses to two items placed on the final exam. The design of our instructional sequence was based in part on our interpretation of the aforementioned readings. In addition, we were informed by the instructional design theory of realistic mathematics education (RME) (Gravemeijer, 1999).

We began the unit by considering the movement of dynamically changing roots of a quadratic equation as one of the parameters of the quadratic was varied, and using the complex plane as a tool to investigate that motion. After students began to view complex numbers as points in the complex plane, they were asked to invent ways to geometrically add and multiply complex numbers. Finally, students engaged in tasks where they were asked to view complex arithmetic as a method of achieving transformations on other complex numbers in the complex plane. In terms of the emergent model heuristic from RME, we conjectured that the complex plane would initially serve as a model-of complex roots to quadratics, and later serve as a model-for transformations on the complex plane.

Results

The pre-instruction interviews revealed that all ten of the students were able to perform addition and multiplication of complex numbers, and expressed a moderate comfort level with basic complex arithmetic (addition and multiplication). This comfort level increased with instruction, as shown in Figure 1(a).



(a) (b)

Figure 1. Student familiarity with complex arithmetic

In addition, only three students expressed any prior knowledge of the complex plane in the pre-instruction interview. When presented with a complex plane and shown how to plot points, however, eight of the ten students successfully used the plane to perform addition. Of these eight, five used what was essentially vector addition, while three more added component-wise along the axes. None of the ten students had a model for multiplication. They expressed a far lower comfort level with using the plane to demonstrate complex arithmetic than they had with simple calculation. This deficit persisted through the unit of study, but improvement was seen. For example, prior to instruction, only one student was even somewhat comfortable using the complex plane to demonstrate arithmetic. In comparison, as shown in Figure 1(b), six out of ten students were somewhat or very comfortable using the complex plane to demonstrate arithmetic.

We were also interested in the extent to which students interpreted the word "imaginary" literally, so we included a question that asked students to agree or disagree with the statement, "Complex numbers are not really numbers." They were asked to choose a number from 1 (disagree strongly) to five (agree strongly). At the outset of the study, nine students disagreed, four strongly, and one neither agreed nor disagreed (see Figure 2).

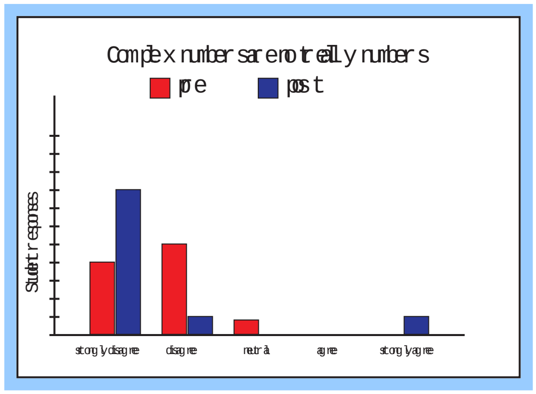


Figure 2. Complex numbers are not really numbers.

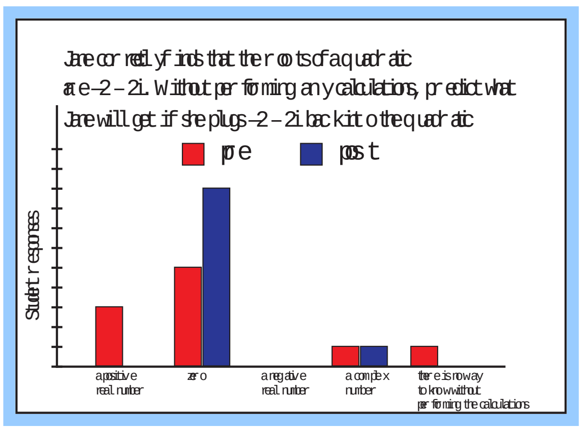
By the end of the instructional sequence, all but two of the students disagreed strongly with the statement “Complex numbers are not really numbers.”

One of our questions in the pre-instruction interview was intended to gain insight into student thinking about complex roots of quadratics. The results took us a bit by surprise. The students, after using the quadratic formula to find the complex roots of a pictured parabola, were asked what they would get if they plugged one of those roots in for *x*. Many of the students said they would get either some sort of complex number, or that there was no way to predict what they would get without performing the computation. Only one of the ten students expressed that the result of the computation would be zero. In response to these interview results, we included the following question as part of the pre-instruction written assessment administered to all 13 students in the class:

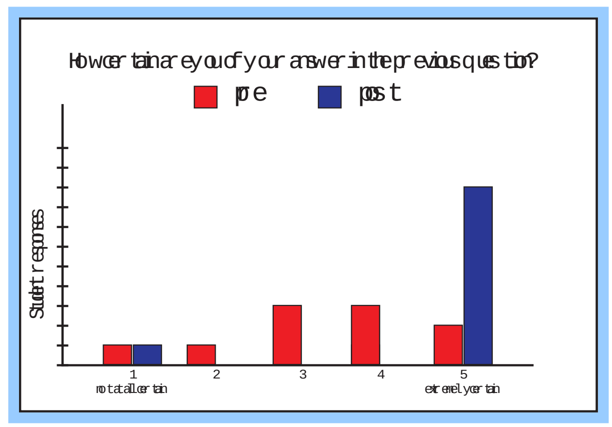
The Parabola Question: Jane correctly finds the roots of the quadratic $x^2 + 4x + 8$ to be $-2 \pm 2i$. Without performing any calculations, predict what Jane will get if she plugs $-2 + 2i$ back into the quadratic.

(a) A positive real number
(b) Zero
(c) A negative real number
(d) A complex number
(e) There is no way to know for certain without performing the calculation.

The pre-instruction data from the in class assessment (Figure 3(a)) already show growth in comparison to the intake interviews, where students seemed far more unconvinced that complex roots — even ones that they had found themselves — would behave as other roots do. Part of this change may be attributable to the fact that the students had just completed a unit on quadratics immediately prior to the intake assessment — a unit that was still in progress during the intake interviews. In addition, participating in the interview provided occasion for some of the students to rethink some of their initial responses.



(a)



(b)

Figure 3. Parabola question.

Although we see some movement in students' willingness to trust that the results they get when working with complex numbers, results we suspect would parallel their expectations when working with real numbers, the students' answers to the follow-up question — which asked students how certain they were of their answer to the previous question — suggests that they are still reluctant to commit to this position. Although we did not ask, students might also be reluctant to commit to this position if the roots were real. In any case, by the close of instruction, all but one student stated that the result of plugging the complex root back into the quadratic would give a result of zero, and nine of the ten students were extremely certain of their response (See Figure 3(b)).

Another area of student thinking that interested us was whether students thought of a complex number as a single thing or a pair of things. We included the following question in an effort to better understand student thinking on this issue:

> Jason says that he thinks of the number 2 + 6*i* in terms of two different parts; the “2” and the “6*i*”. Sharilyn, however, says she thinks of 2 + 6*i* as a single number, “2 + 6*i*” rather than in terms of two different parts. Do you think about 2 + 6*i* like Jason does, like Sharilyn does, or a different way? Please explain.

Of the ten students interviewed, two thought of a complex number as a single thing while eight viewed it as a pair of things. We saw movement over the course of instruction, with three students thinking of a complex number mostly as a single thing, three thinking of it mostly as a pair of things, and three who volunteered that they could think about complex numbers as single things or pairs of things, depending on the context in which they were encountered.

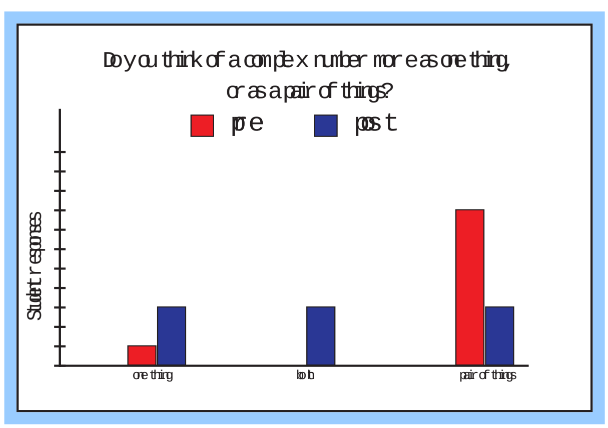


Figure 4. Complex number as one thing or two.

In light of our reading of Lakoff and Nuñez’ (2000) analysis of the complex plane as a conceptual blend that includes the Cartesian plane with a metaphor for multiplication by (–1) as a 180° rotation, we were intrigued by the possibility of being able to leverage students' multiplication-rotation metaphors in order to develop the concept of the complex plane. We asked several interview questions that were designed to elicit any multiplication-by-negative-one-as-rotation ideas that they might hold, including the following question: How can you use the number line to demonstrate why $7(-1) = -7$?

We were surprised to find that no students mentioned rotation as a way of explaining why $7(-1) = -7$. The overwhelming student response to the multiplication by negative one question was to iterate (-1). When asked how they could think about $7(-1)$ starting at 7 (rather than iterating (-1) seven times), only two students said they could think of $7(-1)$ as a reflection, and rotation was not mentioned by any of the ten interviewees. When we conducted the post-instruction interviews, we found that some students now mentioned rotation as one way to think about multiplication by (-1) on the number line. When faced with the same task of explaining why $7(-1) = -7$ in the exit interviews, three of the students now mentioned rotation as a possibility, two said that it goes "in other direction," one used translation, sliding the length from zero to seven to the left on the number line, one iterated, one reflected, and one had no way to think about multiplication on the number line.

Conclusion

The issues that interested us most when viewing this pre-and-post data were students' perceived legitimacy of complex numbers, as demonstrated by the parabola problem and students' responses to whether or not complex numbers are really numbers; the changes over the course of instruction in students' ability to think of complex number as a single thing rather than a pair of things; and the apparent ease with which students found a way to use the complex plane in order to represent addition of complex numbers geometrically.

The results from the 7(–1) question surprised us, and are leading us to question whether the multiplication by (–1) as rotation metaphor is truly a component of the conceptual blend of the complex plane, or whether this metaphor arises as a consequence of experience with the complex plane (or other experience with multiplication of polar coordinates).

References

Cobb, P. (2000). Conducting teaching experiments in collaboration with teachers. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 307–334). Mahwah, NJ: Lawrence Erlbaum Associates.

Fauconnier, G. & Turner, M. (2002). *The way we think: Conceptual blending and the mind's hidden complexities*. New York, New York: Basic Books.

Gravemeijer, K. (1999). How emergent models may foster the constitution of formal mathematics. *Mathematical Thinking and Learning, 1,* 155–177.

Lakoff, G. & Nuñez, R. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being.* New York, New York: Basic Books.

Penrose, R. (2004). *The road to reality*. New York, New York: Alfred Knopf Publishing.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, *22*, 1-36.