Student use of mathematical reasoning in quasi-empirical investigations using dynamic geometry software

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Abstract

The Geometer's Sketchpad (GSP) can be used to discover and investigate the validity of geometrical conjectures. When GSP is used to investigate the validity of a conjecture, the investigation is empirical, and not deductive, in character. Still, investigations should be rooted within the investigator's mathematical understanding of the statement being tested. This note discusses a set of interviews that provides evidence that GSP investigations can suffer from some of the same difficulties novice proof writers experience in creating proofs. The literature on the use of definitions proved to be illuminating in analyzing their investigations.

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Instructional technology has opened a door for mathematics students and faculty to discover and investigate mathematical assertions via non-deductive means. In order to be effective, though, a dynamic geometry software (DGS) investigation of a geometrical statement must be designed logically and rooted in a mathematical understanding of the assertion being investigated. There is certainly a possibility that novice proof writers could experience difficulties in designing DGS investigations similar to the ones they experience in creating proofs.

Loosely speaking, the Geometer's Sketchpad (GSP) provides a software model of Euclidean geometry. Each of Euclid's axioms are embedded into the basic commands available to the user. One key feature of GSP is the ability to construct a figure and to then maintain geometrical relationships while moving or 'dragging' the points used as the basis for the construction. While it is clear that a GSP investigation is not the same as a formal deductive proof, GSP investigations can be used as the basis of conjectures and lead to the discovery of counter-examples.

Literature

Charles Pierce (1932) identified three basic methods of making inferences: deduction, induction, and hypothesis or abduction. Abduction or hypothesis is associated with the making of a conjecture and an induction is the inference of a general rule from the analysis of cases. Michael de Villiers (2004) refers to any non-deductive method of investigation based on experimental, intuitive, inductive, or analogical reasoning as quasi-empirical and argues that quasi-empirical techniques have played a significant role in the development of mathematics.

A number of researchers have investigated topics related to the topics addressed in this note. There is an extensive literature on DGS investigations, e.g., Govender and de Villiers (2002); Hadas, Hershkowitcz, and Schwarz (2000); Healy and Hoyles (2001); Hölzl (1996); Hollerbrand (2002); Laborde (2000); Leung and Lopez-Real (2002); Marrades, R., & Gutiérrez (2000); and Mariotti (2000). A model of discovery and demonstration, in which abduction plays a key role, is applied to DGS investigations in Arzarello, Olivera, Paola, and Robutti (2002). Investigations into the validity of a mathematical conjecture need to correctly identify the premise and conclusion of the statement and the use of quantifiers. The literature suggests that novice proof writers have difficulties in these areas, e.g., Dubinsky (1991); Dubinsky, Elterman, and Gong (1988); A. Selden and J. Selden (2003); J. Selden and A. Selden (1995); and Smith (1940). There is already a substantial literature concerning the mathematical definitions and their use by students just starting advanced mathematics, e.g., Edwards and Ward, M. (2004); Gray, Pinto, Pitta, and Tall (1999); Hazzan and Leron, U. (1996); Moore (1994); Smith (2004); Tall & Vinner, (1981); and Vinner (1991).

Alcock and Simpson (2002) have developed a framework for analyzing the use of definitions. Mathematical definitions precisely describe all of the objects in a given category and no objects in the category enjoy a special status. When testing the validity of an assertion for members of the category, one needs to ultimately rely only on the properties set forth in the definition. This differs from a prototypical strategy for developing an understanding of a category of objects where one is introduced to the category via examples (and non-examples), which are used to build an understanding of the general class of objects described by the definition. One obstacle to mathematical reasoning is the tendency to reason by using a mathematical definition to build a prototypical understanding of the category associated with the definition, and then argue based on the prototypical understanding.

The authors noted the use of prototypical reasoning in an earlier study (Connor, Moss, and Grover, 2007). In that study, four of the six participants organized their investigation of a conjecture regarding triangles by examining different types (e.g., right, equilateral, isosceles, obtuse, and so on) of triangles. Chazan (1991) made some similar observations when working with a group of secondary school geometry students.

Description of the Study

In this study, the participants were asked to explore the correctness of a given statement using GSP. In that the premise and conclusion of the statement were given at the outset, one form of an exploration would be an inductive investigation using dragging to create a variety of examples.

The six participants in the study were all either junior or senior preservice secondary school teachers from a medium-sized comprehensive university in the midwestern United States. Each participant taken a ‘transitions to proof’ course; were enrolled in the second course of a two-quarter sequence on Euclidean and non-Euclidean geometry where the class had ready access to GSP; and had demonstrated good reasoning skills, good GSP skills, or both. Neither of the authors was the instructor of the course and the interviewer had not had any of the participants in a previous class.

The interviews were conducted during spring, 2006. During each interview, a paper document listing a set of mathematical statements and appropriate definitions was provided to the participant. A computer with GSP launched was also available. The interviews were video taped and the video output of the participant's GSP work was also recorded on video tape. Each interview lasted between 45 and 75 minutes.

Each participant was asked to explore the validity of three statements in an individual semi-structured task-based interview. The first statement to be investigated was:

- Statement 1: If the incenter and circumcenter of a triangle coincide, then the triangle is equilateral.

Statement 1 had also been used by the researchers in a set of interviews conducted in spring 2003 (Connor, et. al., 2007). After investigating Statement 1, the participants were asked to perform two investigations incorporating two unfamiliar definitions.

- A figure *ABCD* is called a *frame* if *A*, *B*, *C*, and *D* are four distinct coplanar points, no three of which are collinear and $ABCD = \overline{AB} \cup \overline{BC} \cup \overline{CD} \cup \overline{DA}$.

- A point *M* is called the *minimum distance point* of the frame *ABCD* provided that $MA + MB + MC + MD \leq PA + PB + PC + PD$ for any point *P*.

The participants were then asked to use GSP to investigate the correctness of the following conjectures using the above definitions:

- Statement 2: Let *ABCD* be a frame and *M* be a minimum distance point for *ABCD*. Then *M* must lie on one (or both) of the line segments $\overline{AC}$ or $\overline{BD}$.

- Statement 3**:** Let *ABCD* be a frame. If *E,F,G* and *H* are the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ respectively, then the frame *EFGH* is a parallelogram.

Statements 2 and 3 of the 2006 interviews were designed to explore the participant's use of prototypical reasoning. In keeping with a typical mathematical practice of using a familiar word (e.g., set, function, line) for a mathematical object, the word *frame* was used to describe the union of four distinct line segments as given in the definition. Note that a frame can be either a convex or concave quadrilateral (i.e., the sides meet only at the endpoint of a segment) or be “crossed” in the sense that two sides intersect at a point which is not an endpoint of the segment. The second statement is true for convex or concave quadrilaterals, but not crossed frames. The third statement is true as long as the midpoints form a frame. The researchers speculated that the participants would tend to investigate convex or concave quadrilaterals while ignoring crossed frames and that this would constitute evidence of prototypical reasoning.

Observations

It was observed that the participants had some difficulties in correctly identifying the given and conclusions of a statement and in incorporating the universal quantifier into their investigations. Five of the six interviewees in the 2003 interviews exchanged the role of the premise and conclusion during their investigation of Statement 1. In the 2006 interviews, initially all six correctly identified the premise and conclusion and, as the task pressed on, four of six addressed the converse in their investigation.

The participants would often come to a conclusion after considering only one example. In the 2003 interviews, four of the six participants made a final inference of the validity of Statement 1 based on studying one example. In the 2006 interviews, four of the participants came to an initial conclusion based on one example in their investigations of Statements 1 and 2 and two came to a conclusion after investigating a single example for Statement 3.

In the interviews conducted for this study, two of the participants adopted a prototypical strategy in their investigations of Statement 1. In Statement 2, all six of the participants based their initial conclusions on an analysis of convex quadrilaterals. In four cases, a single convex quadrilateral was used to come to an initial conclusion. In the two other cases, the participants used the dragging feature to explore a number of frames but restricted these frames to convex quadrilaterals.

After the participants came to their initial conclusion regarding Statement 2, they were asked to investigate the validity of Statement 3. In this statement, three of the participants initially restricted their investigations to convex quadrilaterals; in two cases the conclusion was based on a single frame and in one case on investigation based on dragging the vertices to look at a number of convex quadrilaterals. During this portion of the interview, three of the participants were prompted to consider concave quadrilaterals and then crossed frames.

The participants were then asked to reconsider their conclusions for statement 2. Four of the participants now included concave quadrilaterals and crossed frames in their investigation before coming to a conclusion. Only one participant came to the conclusion that statement 2 was false. In two of the investigations, this was because the investigation only covered non-crossed frames. In one case the participant argued that crossed frames were not frames and also offered an abstract argument as to why the result was valid for non-crossed frames.

Two of the participants felt that the definition of frames or that Statement 2 could be rewritten so as to make it a valid statement. This behavior supports an analysis of mathematical investigations developed by Arzarello, Andriano, Olivero, and Robutti (1998).

Conclusion

The considerations of this note suggest that prototypical reasoning could limit both induction and abduction, and hence limit both the process of discovery and verification.

It would seem natural to believe that DGS would be used by the participants to create an object from a definition and, through dragging, build a comprehensive understanding of the objects characterized by the definition. These interviews suggest, though, that DGS investigations can be constrained by prototypes and also that active instruction may be needed to help novice users to push their understanding of the objects characterized in a given definition and to seek unfamiliar objects satisfying the criteria of a given definition.

These problems may not be inherent to GSP and could appear in other uses of instructional technology. As many of the difficulties appeared to be rooted not so much in the DGS software but in the underlying strategies employed for investigating the statement, the same sorts of difficulties could also manifest themselves in the use of other programs and calculators to do quasi-empirical investigations. It may be that, even in the presence of instructional technology, developing an understanding of the use and nature of mathematical definitions remains to be a key instructional challenge.

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