Assessments that Improve Proof-Writing Skills:

Students' Perceptions

Hortensia Soto-Johnson

Casey Dalton

Nissa Yestness

University of Northern Colorado

ABSTRACT

In this qualitative study, we summarize students' descriptions of what and how assessments helped improve their proof-writing skills. To investigate these questions, we administered an open-ended questionnaire and conducted interviews with students who completed the second semester of an undergraduate abstract algebra course. We transcribed and coded data from both the interviews and the questionnaires using grounded theory methodologies as described by Strauss and Corbin (1998). The three themes that emerged as primary contributors to the improvement of students' proof-writing skills are: practicing writing proofs, observing proof presentations, and receiving feedback on their proofs. Students reported that in-class proof presentations provided an opportunity to engage in all three themes, and homework provided opportunities to practice writing proofs and receive feedback on proofs. Our results indicate that courses designed to improve students' proof-writing skills should embrace assessments consisting of these three themes.

Introduction

In an attempt to assist students with their proof-writing, we incorporated several diverse assessments into an undergraduate two-semester sequence of abstract algebra. The assessments consisted of homework, tests, worksheets, proof presentations, oral interviews, and projects. In this qualitative study we investigate how these diverse assessments help students develop and improve their proof-writing skills. The purpose of this research is two-fold. First, we determine what students reported as being beneficial in improving their proof-writing skills. Second, we identify the assessments possessing these qualities as described by the participants.

Literature Review

Research related to proofs is rich and abundant, yet there is much to learn about how we can best assist students with proof-writing. Overall research suggests students' understanding of proof and syntactic knowledge of the facts are not enough for students to become competent proof writers (Weber, 2001), students have difficulty validating proofs (Selden & Selden, 2003), and students do not seem to progress beyond the empirical proof scheme (Fischbein & Kedem, 1982; Fujita & Jones, 2003; Vinner, 1983). Suggestions for improving students' proof-writing skills include but are not limited to: lecturing (Wahlberg, 1997), reducing abstraction level through examples or advance organizers (Hazzan, 1999; Scandura & Wells, 1967; Sowder & Harel, 2003), presenting proofs in class (Freedman, 1983; Reisel, 1982), using diagrams or technology (Gibson, 1998; Hadas, Hershkowitz, & Schwarz, 2000; Mariotti, 2000), and capitalizing on learning situations of uncertainty (Zaslavsky, 2005).

Creating new teaching techniques to assist students with their proof-writing skills is not a new mission of mathematicians and mathematics educators. Over twenty years ago Reisel (1982) and his colleagues created a course designed to teach proof construction and criticism through student presentations. Such a course helps students improve their proof-writing and confidence level in writing and presenting proofs. Gibson (1998) found the use of diagrams beneficial for understanding information, judging the truthfulness of a statement, discovering ideas, and writing ideas. Students who used diagrams were able to construct their proofs by utilizing the diagram as a skeleton of the proof. Sowder and Harel (2003) delineate the importance of modeling the work needed to construct a proof and creating a curriculum which supports proof understanding, proof production and proof appreciation for all students. By making proofs more tangible, weaker students have an opportunity to grow in their proof understanding and abilities.

Upon completion of a lengthy summary on the perspectives of proof, Harel and Sowder (in press) leave the reader with many open questions related to the teaching and learning of proofs. Factors used in the research of the teaching and learning of proofs can be classified as mathematical and historical-espistemological, cognitive, and instructional-socio-cultural. Under each factor Harel and Sowder give a number of research questions related to the study of learning and teaching of proofs. Under the heading of instructional-socio-cultural factors are questions such as: *What classroom environment is conducive to the development of the concept of proof with students? What form of interactions among the students and between the students and the teacher can foster students' conception of proof? What mathematical activities can enhance students'*

conceptions of proof? Martin, McCrone, Bower, and Dindyal (2005) provide partial answers to these questions by investigating a high school geometry class.

Martin et al (2005) speak to the importance of the teacher-student relationship in the teaching and learning of proof. They stress the significance of posing open-ended tasks, engaging students in classroom dialogue, analyzing students' arguments, coaching students, and creating a learning environment where students actively engage in classroom discourse in the development of students' understanding of proofs. Their research like others (Anderson, Greeno, Reder, & Simon, 2000; Dubinsky, Dautermann, Leron, & Zazkis, 1994; Edwards & Brenton, 1999; Steinbring, 2005) describes the role of social interaction, either through student-teacher or student-student, in the learning of proof. In general, they show students are more likely and comfortable to make conjectures in a classroom where students were encouraged to voice their opinions. They also stress the importance of learning to write proofs by having multiple opportunities to write proofs. In this study, we describe what and how alternative assessments facilitated proof-writing skills as documented by the students.

Theoretical Perspective

Assessments should promote valid inferences, allow for multiple measures of performance, measure what is worth learning, and support every student's opportunity to learn important mathematics (Steen, 1999). In an effort to implement assessments relevant to the literature and in line with Steen's criteria, we used homework, exams, oral interviews, projects, worksheets, and presentations to evaluate students' understanding of the content. We also used these assessments to develop and enhance students' proof-writing skills. The six assessments promote valid inferences, allow for multiple measures

of performance, measure what is worth learning, and support every student's opportunity to learn important mathematics. The continuous cycle assessments are homework, exams, and presentations, while the projects, worksheets, and presentations serve as an open process. Social constructivism guided the implementation process of the assessments, which assumes that "the terms by which the world is understood are social artifacts, products of historically situated interchanges among people" (Gergen, 1985, p.267). We believe social interaction influences students' understanding of abstract algebra and enhances students' ability to construct proofs. The researchers' belief in the necessity of social interaction influenced the choice of assessments and the social nature of these assignments. The worksheets, projects and presentations required collaborative work and interaction among the students in and out of the classroom. The difficulty of the homework and specificity in which it was graded promoted collaborative work.

Method

The Course & Participants

The first-named author, who was the instructor of both courses, used the text *Abstract Algebra: A First Undergraduate Course*, by Hillman and Alexanderson (1994). The first semester centered on group theory and the second semester focused on rings and fields. Although students successfully completed calculus I, II, and III, discrete mathematics, and linear algebra, before enrolling in the first semester course, this is the first course where they are introduced to formal proof-writing. Successful completion of the first course was a requirement for the second semester course. Eight male and twelve female students ($N = 20$) who completed the abstract algebra sequence participated in this study.

The students were primarily preservice secondary teachers; a few students intended to pursue graduate school.

Assessments Implemented

Soto-Johnson, Dalton, and Yestness (2007*)* provide a rich description of the assessments, but it is not necessary for this report. In order to replicate the study, we provide a brief description of the assessments.

The teacher graded all assessments for correctness and clarity. She assigned homework daily, collected and graded it on a weekly basis, and returned it to the students the following class period. The instructor encouraged students to work together on homework and provided solutions to the required exercises in the hope that students would assess their own work. Frequently, the solutions distributed came from students in order for students to examine other proofs.

The primary purpose of the exams was to assess students' mastery of the content in a timed setting. The exams throughout the year included in-class, take-home, and oral interview components. Students did not work together on take-home components; this is one of the few assessments where the instructor prohibited collaboration. The oral interviews were also individual assessments, which served as an opportunity for students to communicate their knowledge orally. Two to three weeks after the midterm oral the instructor gave students a list of potential questions for the final oral.

During both semesters the instructor implemented projects designed to assist students with discovering abstract algebra ideas, connecting abstract algebra and the high school mathematics they will teach in the future, and studying abstract algebra applications. The students investigated the topic with little assistance from the instructor. The groups

turned in one written copy of their project, and presented their project. Students frequently completed worksheets intended to connect several complex abstract algebra concepts through in-class group work. In the second semester, presentations served as a channel for students to communicate mathematics orally and to learn presentation and proof techniques from one another.

Research Instruments

The data for analysis came from semi-structured interviews (Patton, 2002) with the 20 students enrolled in the second semester of an undergraduate abstract algebra course (see Appendix I) and from a written questionnaire (see Appendix II). We audio-taped and transcribed all interviews. Using a grounded theory approach (Strauss & Corbin, 1998) we analyzed the interviews and questionnaires allowing themes to emerge. Two researchers performed the coding and theme searching to provide validity to the research and to improve the quality of research by providing a setting to openly discuss ideas and confirm findings.

Results

Students' Perceptions of Their Proof-writing

In the interview, we asked the students if they thought their proof-writing had improved since the beginning of the course. All but two of the participants answered yes, and the ones that answered affirmatively, did so enthusiastically:

> Very much so - Jayden
>
> Definitely – Melissa
>
> I think so, greatly - Lauren

Of the two students that responded no, one student mentioned she had stopped growing after the first semester of this year-long course. These results are not surprising given this is the first course where students write formal proofs.

> Last semester yes, this semester no, I definitely kind of hit a wall. I grew so much last semester. – Madison

Emergent Themes

Overall, our students felt their proof-writing improved and identified classroom assessments, which they believed contributed to this improvement. Three themes emerged from the data as contributors to developing students' proof-writing skills. Participants claimed observing others do proofs, practicing writing proofs, and receiving feedback on proofs as the most beneficial components of the course, which assisted with their proof-writing. We explore each of the three themes in more detail below.

Observing: Seventeen of the twenty students reported observing others present proofs helped them improve their own proof-writing. They discuss observing their peers and the teacher and speak to the usefulness of the text. Table 1 contains a summary of these results. A noteworthy result is that more students credit the improvement in their proof-writing skills to watching their peers present proofs rather than watching the teacher present a proof.

Table 1. **Observing and Proof-Writing**

Watching Peers do Proofs	16
Watching Teacher do Proofs	10
Reading Proofs in Text	3

Participants distinguish between peer proofs and teacher proofs. In elaborating on the benefits of watching their peers, participants often reported that it was helpful to see other people's ideas for proofs, and the different proving styles of their classmates.

> I think it is helpful to see how other people do their presentations and kind of take bits and pieces. I like how she writes proofs or he writes proofs, seeing other people's ideas so that you can make your own better. - Lauren

> I got to watch the way that classmates went through proofs, we each have our own individual styles and that gave me other options of proof-writing techniques. – Agustin

One student went as far as saying that it was nice to see proofs beyond those of the instructor's.

> I saw how everybody else looked at it, rather than just how you did it, the book did it, or I did it. It was nice to see the different aspects of how people looked at problems. – Sarah

In contrast, in their discussion about the value of watching the instructor write a proof, the students mention the benefits of seeing a quality proof. The participants seem to learn quality writing from their instructor, but their peers provide distinct prove-writing styles.

> From the lecture, I was able to see what a quality proof looked like and then I was able to take the examples and apply the same ideas into my proof-writing. – Victoria

> Seeing how the teacher did proofs in class established the "good taste" element in proof-writing. – Agustin

Practice: There seems to be a universal understanding that for one to develop a skill they need to practice that skill. It is, as the common saying goes, practice makes perfect. All the students who participated in the study remarked on the necessity and value of practice as an activity that helped them to become a better proof writer.

> All the practice I had helped me to become better. - Victoria

> The biggest thing is just practicing. The more I write proofs the better I get at writing them. - Amparo

The students categorized practice into two camps: quantity and quality (see Table 2). Homework gives students ample practice, but the students viewed the presentations as an opportunity for quality practice.

Table 2. **Practice and Proof-Writing**

Homework Practice	18
Proof Presentation Practice	17
All Assessments Practice	5

In referencing practice in conjunction with homework students referred to the sheer quantity of proofs required to complete the homework as most helpful in improving their proof-writing skills. The repetition of writing similar proofs helped students to master a technique or to make decisions about proof techniques for different situations.

> Homework gave me great practice. By repeating proof-writing skills in the homework, I learned how to write better proofs. - Logan

> Homework assignments gave me a lot of practice writing proofs and exploring what types of proofs are best in certain situations, and therefore was the most helpful part of the class because it gave me the practice I needed. - Miguel

The type of practice communicated by the students in their proof presentations dialogue was qualitatively different from the practice revealed in the homework discourse. The students described practice in this context as working on writing clear and concise proofs. This was an opportunity to practice writing quality proofs. Lauren defined this type of practice as “perfect practice”.

> The presentations helped the most because you had to make sure it was absolutely right, and put things clear and concisely so others can understand. - Gabby

> I am a big believer in practice makes perfect, and perfect practice is the best way to do it, and so the more you practice writing out those proofs and you can make sure that they are elegant and that they are pretty and that they flow, the better that you will get at it. - Lauren

Some students (N = 5) stated all the assessments i.e. tests, homework, proof presentations, projects, oral interviews, and worksheets provided opportunities to practice the skill of proof-writing and thus were helpful. Each assessment contributed something new to improving their proof-writing skills.

Everything was helpful. The biggest thing is just practicing. - Amparo

Feedback: Fourteen of the twenty students reported that feedback on their proofs was imperative for improving their own proof-writing (see Table 3). Students mentioned the value of receiving feedback from both the teacher and their peers. As opposed to the observing theme, the students referenced feedback from the teacher more often then feedback from their peers, and once again, the description of the feedback was qualitatively different.

Table 3. **Feedback and Proof-Writing**

Feedback from Teacher	12
Feedback from Peers	9

Although, students discussed the value of receiving feedback from the teacher as helpful for developing their proof-writing, it was typically in the context of proofs completed for the homework assignments. George comments on the usefulness of the

consistent and continuous feedback. They also reference how the strict grading motivated them to improve their proof-writing.

> As much as I hated the fact that our homework was graded every week, always having feedback on the little mistakes we were making in our proofs was nice. Homework was my biggest help in writing proofs because we got constant feedback each week on our proofs. – George

> Homework definitely helped the most because the grading was so strict, it really forced me to write elegant proofs. – Angelina

On the other hand, when students mentioned feedback from their peers, it was typically in the context of proof presentations, and described as supportive and helpful. Caroline expressed the time she spent preparing with a classmate for a proof presentation.

> Watching their presentations on the proofs, and being able to talk to students throughout the week helped. Everybody in the class was always really supportive and they provided helpful comments that I wasn't really writing everything down and that helped. – Agustin

> Having us do these presentations where we are supporting one another and doing evaluations. – Jayden

> I spent time with a classmate just focusing on my proof-writing style and technique. – Caroline

Discussion

Our findings suggest that students may be able to improve their proof-writing by observing others do proofs, practicing writing proofs on their own, and receiving frequent and meticulous feedback on their proofs. These findings support what social cognitive theory states about learning complex skills,

> *Most complex skills are learned through a combination of modeling, guided practice, and corrective feedback.* (Schunk, 2004)

Figure 1 illustrates a model of how different assessments contributed to developing proof-writing skills for our students. The model exemplifies the relationship between the

assessments and the factors that contributed most to proof-writing as portrayed by the students. Presenting proofs allowed students to engage with all three themes, while homework provided opportunities to engage with two of the themes.

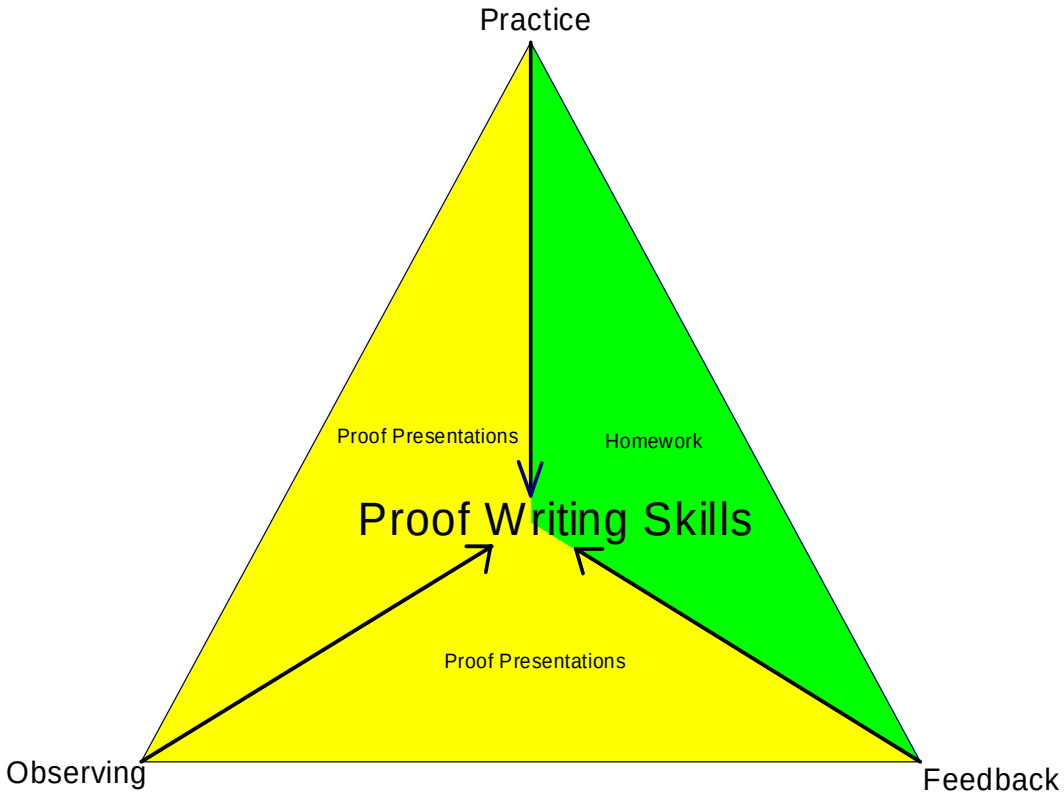

Figure 1: Model for Developing Proof –Writing Skills

Since proof presentations allow students to engage with all three themes, we recommend incorporating proof presentations as an assessment if improving students' proof-writing is a course objective. This suggestion is further grounded on the fact that more students (N = 16) mentioned observing classmates than observing the teacher (N = 10) as being helpful in improving their proof-writing; and that findings by Shunck and Hanson (1985) suggest self-efficacy and achievement were raised more by observing a peer model than by observing a teacher model. Furthermore, because proof presentations are the core element of a modified Moore Method course, one might consider using such an approach

in a transition to advanced mathematics course (where the main objective is to help students become familiar with and write proofs). Previous research on this topic also suggests that proof presentations increase proof-writing skills and confidence in proof (Reisel, 1982; Freedman, 1983; Myers, 2000). Our results support the findings of these prior studies.

Our results are novel in the fact that students stress the significance of practice and feedback for improving their proof-writing skills. Students need ample opportunities to practice communicating their proofs, either through written or oral work. Although our students found the most value in writing their proofs through the homework and in conveying their proofs during the student presentations, students also had opportunities to communicate proofs during the oral interviews. These oral interviews gave students another venue to practice articulating their proofs. Our findings infer that these "practice" opportunities must allow for critical evaluations in a safe environment, where students learn to appreciate and value the feedback. The instructor evaluated all of the assessments, but students remarked that both student and teacher comments contributed to improving their proof-writing skills. A combination of critical comments from a teacher and gentle comments from peers gives students a sense of what a reader wants and needs in a proof.

We provide a few answers to Harel and Sowder's (in press) questions: *What classroom environment is conducive to the development of the concept of proof with students? What form of interactions among the students and between the students and the teacher can foster students' conception of proof? What mathematical activities can enhance students' conceptions of proof?* Through the students' voices we learned practice is needed in both quantity and quality for improving proof-writing skills.

Observing both their teacher and their peers allows students to recognize quality proofs and to absorb new proof techniques. Receiving constructive yet critical feedback from the instructor and supportive feedback from peers allows students to identify their mistakes and provides multiple resources for developing their proof-writing skills.

LIMITATIONS AND FURTHER RESEARCH

A limitation of this study is that the primary researcher was the instructor of the course and conducted the student interviews. Although the study lacked anonymity, students openly provided positive and negative feedback during the interviews and on the questionnaire. Given oral interviews served as a venue for students to articulate proofs, we recommend further research to determine the significance these interviews in developing proof-writing skills. We suggest investigating other assessments, which may contribute to the development of proof-writing, such as one on one mentoring between students.

REFERENCES

Anderson, J. R., Greeno, J. G., Reder, L. M., & Simon, H. A. (2000). Perspectives on learning, thinking, and activity. *Educational Researcher* 29(4), 11-13.

Dubinsky, E. & Leron, U. (1994). *Learning abstract algebra with ISETL*. New York: Springer-Verlag.

Edwards, G. E. & Brenton, L. (1999). An attempt to foster students' construction of knowledge during a semester course in abstract algebra. *The College Mathematics Journal* 30(2), 120-128.

Fischbein, E. & Kedem, I. (1982). Proof and certitude in the development of mathematical thinking. *Proceedings of the Sixth International Conference of the Psychology of Mathematics Education*, 128-131.

Freedman, H. (1983). A way of teaching abstract algebra. *The American Mathematical Monthly* 90 (9), 641-644.

Fujita, T. & Jones, K. (2003). Critical review of geometry in current textbooks in lower secondary school in Japan and the UK. Paper presented at the 27th annual meeting of the International Group for Psychology of Mathematics Education, Honolulu.

Gergen, K. J. (1985). The social constructionist movement in modern psychology. *American Psychologist 40*, 266-275.

Gibson, D. (1998). Students' use of diagrams to develop proofs in and introductory analysis course. *Conference Board of the Mathematical Sciences: Issues in Mathematics Education 7*, 284-305.

Hadas, N., Hershkowitz, R., & Schwarz, B. B. (2000). The role of contradiction and uncertainty in promoting the need to prove in dynamic geometry environments.

Educational Studies in Mathematics 44, 127-150.

Harel, G. & Sowder, L. (in press). Toward comprehensive perspectives on the learning and teaching of proof. *Second Handbook of Research on Mathematics Teaching*, National Council of Teachers of Mathematics.

Hazzan, O. (1999). Reducing abstraction level when learning abstract algebra concepts. *Educational Studies in Mathematics* 40(1), 71-90.

Hillman A. P. & Alexanderson G. L. (1994). *Abstract Algebra A First Undergraduate Course* (5th ed.). Prospect Heights, IL: Waveland Press

Leron, U. & Dubinsky, E. (1995). An abstract algebra story. *American Mathematical Monthly* 102 (3), 227-242.

Marriotti, M. A. (2000). Introduction to proof: The mediation of a dynamic software environment. *Educational Studies in Mathematics* 44, 25-43.

Martin, T. S., McCrone, S. M. S., Bower, M. L. W., & Dindyal, J. (2005). The interplay of teacher and student actions in the teaching and learning of geometric proof. *Educational Studies in Mathematics* 60, 95-124.

Myers, N. C. (2000). An oral-oral intensive abstract algebra course. *PRIMUS* 10 (3), 193-205.

Patton, M. Q. (2002). *Qualitative Research and Evaluation Methods* (3rd ed.). Thousand Oaks, CA: SAGE.

Reisel, R. B. (1982). How to construct and analyze proofs – a seminar course. *The American Mathematical Monthly* 89(7), 490-492.

Scandura, J. M., & Wells, J. N. (1967). Advance organizers in learning abstract Mathematics. *American Educational Research Journal* 4(3), 295-301.

Schunk, D. H. (2004). *Learning Theories: An Educational Perspective* (4th ed.). Upper Saddle River, NJ: Pearson.

Schunk, D. H. & Hanson, A. R. (1985). Peer models: Influence on children's beliefs and learning. *Journal of Educational Psychology, 81*, 431-434.

Selden, A. & Selden, J. (2003). Validations of proofs considered as texts: Can undergraduates tell whether an argument proves a theorem? *Journal for Research in Mathematics Education* 34 (1), 4-36.

Sowder, L., & Harel, G. (2003). Case studies of mathematics majors proof understanding, production, and appreciation. *Canadian Journal of Science, Mathematics, and Technology Education* 3, 251-267.

Soto-Johnson, H., Dalton, C. & Yestness, N. (2007). Assessment of Mathematics Alternative Assessments. Manuscript submitted for publication.

Steen, L. A. (1999). Assessing assessments. *Assessment in Undergraduate Mathematics: MAA Notes # 49*, 1-6. Washington, DC: MAA.

Steinbring, H. (2005). Analyzing mathematical teaching-learning situations – the interplay of communicational and epistemological constraints. *Educational Studies in Mathematics* 59, 313-324.

Strauss, A. & Corbin, J. (1998). *Basics of qualitative research: Techniques and procedures for developing grounded theory.* Thousand Oaks, CA: SAGE.

Vinner, S. (1983). The notion of proof-some aspects of students' views at the senior high Level. *Proceedings of the Seventh International Conference of the Psychology of Mathematics Education*, 289-294.

Wahlberg, M. (1997). Lecturing at the "Bored". *The American Mathematical Monthly* 104(6), 551-556.

Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic Knowledge. *Educational Studies in Mathematics* 48, 101-119.

Zaslavsky, O.: 2005, '*Seizing opportunity to create uncertainty in learning mathematics*', Educational Studies in Mathematics 60, 297-321.

APPENDIX I

Interview Questions

1. Did you learn from the presentations? Why or why not? Discuss from the standpoint of a presenter and evaluator.
2. Did you learn from the projects? Discuss in terms of the presentations and presenting?
3. What were the advantages of working on a project as an exam?
4. What were the disadvantages of working on a project as an exam?
5. Do you feel that it was valuable to try to read mathematics on your own as part of the exam #3 project? Why or why not?
6. What assessments do you feel reflected your knowledge of the material best? Why?
7. What assessments do you feel do not reflect your knowledge of abstract algebra? Why do you feel this way?
8. What assessments did you feel were the most challenging? Why?
9. What assessments did you enjoy the most? Why?
10. Did you enjoy having a practice midterm oral? Why or why not?
11. How do you feel about having an oral component to the final? Explain.
12. Have your feeling towards the oral component changed from last semester? If so, how? If not, why not?
13. Do you feel that your proof –writing skills have improved over the last two semesters? What do you attribute this too?
14. Is there any thing that you would like to share with me about the assessments that have been used in the abstract algebra class?
15. Did you feel that there was a sense of community during this and last semester? Why or why not?
16. What do you feel contributed to this?

APPENDIX II

Questionnaire

For each of the assessments state what you liked and disliked about the assessment and explain your answers.

1. Homework Liked: Why?
2. Homework Disliked: Why?
3. Tests Liked: Why?
4. Tests Disliked: Why?
5. Oral Interviews Liked: Why?
6. Oral Interviews Disliked: Why?
7. Worksheets Liked: Why?
8. Worksheets Disliked: Why?
9. Projects Liked: Why?
10. Projects Disliked: Why?
11. Presentations Liked: Why?
12. Presentations Disliked: Why?
13. What classroom activities helped you the most with improving your proof-writing skills? Please explain your answer.