Developing and Applying a Taxonomy for Mathematical Knowledge-Expertise

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1. Introduction and Background

Because proof is the arbiter of truth for the discipline of mathematics, a major goal of undergraduate mathematics programs is to assist undergraduates to develop an understanding and appreciation for proofs along with an ability to generate proofs. Not surprisingly, students' understanding of proof has been the subject of many studies.

The many misconceptions students have about what constitutes a proof were detailed and categorized by Harel and Sowder (1998). Weber (2001) found that students from secondary school through college have major difficulties with the tasks of proof construction and proof validation. In a study by Recio and Godino (2001), less than 50% of 204 beginning students at the University of Cordoba (Spain) produced a substantially correct proof of an elementary number theory statement, and 40% relied solely on empirical evidence. Selden and Selden (2002) found that, after an initial reading, eight mathematics and secondary math ed majors did no better than chance in determining if a proof was valid. However, the reflection and reconsideration ensuing from a follow-up interview subsequently produced an 81% correct determination.

In this article we present the results of a study of the evolution of students' understanding of proof as they progress through the mathematics major curriculum at a medium-sized comprehensive university. The study initially attempted to identify which courses and learning experiences promote growth in student understanding of proof, and was prompted by observing that students in a course designed for beginning mathematics majors were all too willing to be convinced by empirical evidence. To further our investigation we developed a typology of mathematical knowledge that includes six cognitive and two affective components. Then the typology was expanded into a taxonomy that describes the journey toward proficiency in each of these components, resulting in a mathematical knowledge-expertise taxonomy. Finally, we close the paper with an application of the taxonomy to an elementary but non-routine problem involving a linear system.

2. Methodology

The study was cross-sectional, taking a snapshot of students in the major during 2003-4. The initial approach utilized a survey on proof and problem solving of 50 students in math major courses that included 42 math majors and 8 computer science majors. This same survey was subsequently administered to 16 full-time mathematics faculty. In Section 3.1 we discuss the responses to a question about willingness to accept empirical evidence as proof. The responses indicated that after 4 semesters (beyond our "bridge" course) students' responses become much more similar to faculty responses; and that additional evidence would be required to gain a deeper insight into the progression of student understanding of proof.

In order to learn more about what our students think about proof, how willing they are to accept empirical evidence as proof, and whether they think a counterexample is still possible after a proof, we developed a "proof-aloud" protocol that employed think-aloud methodology with pencil and paper available for subject use. The protocol was designed to investigate student understanding of the following:

- What constitutes a proof? In particular, is empirical evidence sufficient?
- Does a proven statement apply in all cases?

- Can there be a counterexample to a proven statement?

Evidence gathered included both what subjects said and what subjects wrote on each question or task in the protocol, as well as what courses or other learning experiences helped them answer a question or complete a task.

We selected 12 mathematics majors, spanning our curriculum from first semester in the major through to six months after graduation, to participate. Students were given a mathematical situation to examine, one in which they could generate several examples, find a pattern, form a conjecture and then decide if the conjecture is true. They were asked how confident they were about their conjecture and what would make them more confident. Then they were asked to try to write down a proof for the conjecture (even if they had already made a general argument earlier in the proof-aloud) and to evaluate several "sample" proofs for correctness. Finally, students were asked to respond to several additional questions related to the statement to check whether they would apply the proven result, and if they thought a counterexample was still possible after a proof. Later on a faculty member was recruited for the proof-aloud protocol to provide an "expert" performance. In Section 3.2 we describe the results of their proof-aloud performances.

3. Results

3.1 Survey Results

The first finding related to students' acceptance of empirical proof from the survey was that after taking our bridge course, student responses to the following question become very similar to faculty responses.

Survey Question

Respond on a 5-point Likert scale ranging from Strongly Disagree to Strongly Agree: *If I see five examples where a formula holds, then I am convinced that formula is true.*

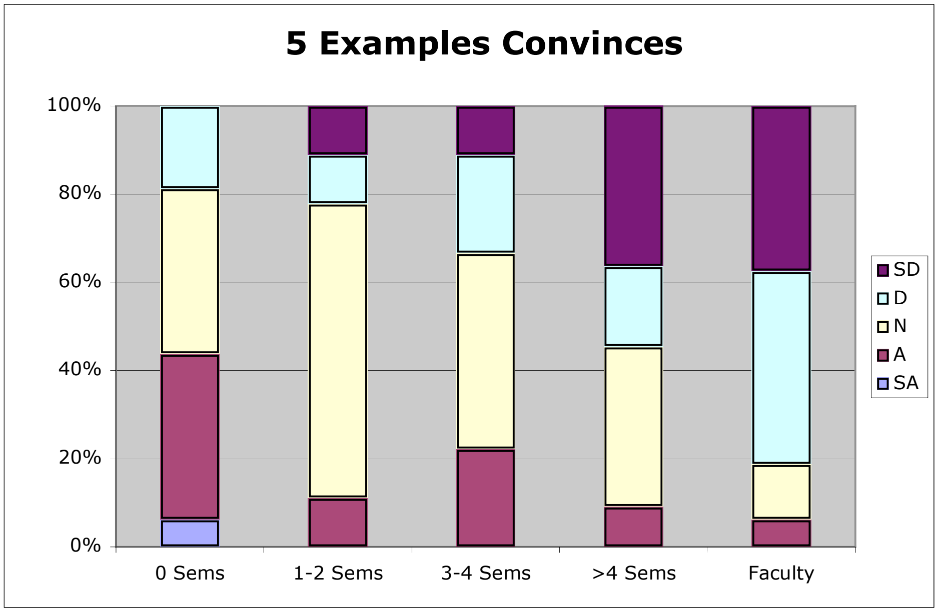

Figure 1

The percentage of students indicating a willingness to accept empirical evidence as proof (5 examples convinces me a formula is true) declined from 44% to 9% as they moved through the major (See Figure 1) with 20% of freshman and 20% of upperclassmen remaining neutral. By comparison, 1 of the 16 faculty surveyed would find 5 examples convincing. At first glance it may be disheartening that by senior year nearly 1 in 10 would still be convinced by empirical evidence, but not even 100% of the faculty gave the ‘correct response’ to “5 examples convinces me.” When asked for an explanation, the one “agreeing” faculty member stated:

> *‘Convinced’ does not mean ‘I am certain’ to me, so whenever I am testing a formula/conjecture, if it works for about 5 cases, then I try to prove that it’s true*

as the reason for his 'wrong answer.' Clearly, a Likert scale response to this type of question does not completely reveal a respondent's thinking. In the next section we present the results of the proof-aloud and the additional insights we gained from it.

3.2. The Proof-Aloud Results

The following mathematical situation was used in the proof-aloud protocol.

Please examine the statements:

For any two consecutive positive integers, the difference of their squares:

(a) is an odd number,

(b) equals the sum of the two consecutive positive integers.

What can you tell me about these statements?

This was the same statement used by Recio and Godino (2001) to investigate student proof schemes and what reasons might underlie their choices of non-deductive schemes. However, in their study proving this statement, and others, was presented as a written task to several hundred incoming freshman, whereas initially we presented it as a statement to investigate and only later gave directions to try to write a proof. Another difference was that our 12 subjects spanned our curriculum from its very beginning through just after graduation.

Recio and Godino developed a 1 to 5 numeric rubric intended to differentiate between students who relied on examples (category 2) vs. students who relied on definitions or general results to develop partially (category 4) or substantially correct (category 5) proofs. The result of applying their rubric to our students' proof-aloud performances was that all but one of our 12 students fell into the top two categories (categories 4 and 5). Moreover, the rubric, designed to distinguish between students using empirical proof schemes and students using deductive proof schemes, did not suffice for an in-depth analysis of the multifaceted work that we were able to document with the proof-aloud transcripts.

We found that, in contrast to Recio & Godino's results, all of our students realized that empirical proofs were insufficient, and all attempted to make some sort of general argument, or at least expressed concern that their argument was not general enough. However, only two of the twelve students we interviewed were beginning freshmen and thus only they were comparable to those in the Cordoba study. Of those two, one produced a very deficient proof, but did so by trying to appeal to general definitions and restating the desired conclusions.

One of our goals was to see how student's understanding of proof evolved in the major. Since, relative to proof, there are three critical courses in the curriculum, we assigned levels to students' progression in the major relative to these courses as shown in the following chart.

Level	Progression in the Major
0	Prior to an Introductory Problem Solving & Writing (IPS&W) course
I	Completed IPS&W course the preceding semester
II	Completed a "bridge" course on proofs the preceding semester
III	Completed a real analysis course the preceding semester
IV	Completed a real analysis course a year earlier
V	A graduate from the preceding year

Then various aspects of the students' performance were analyzed relative to their level in the major. As indicated in the following chart, all but one student placed in the top two of five Recio and Godino categories of performance, with 7 of 12 students placing in the top category[1].

[1] The majority of students indicated the IPS&W course as the course that contributed most to their ability to respond to the task. Even students at levels II through V were citing the IPS&W

Student\Proof	1	2	3	4	5
0	X			X	
I				X	X
II					XX
III				X	XX
IV				X	X
V					X

Consequently, we needed a different way of categorizing the proofs generated by our 12 students to separate out very definite distinctions we observed in their performances. We required a system that would allow us to describe the following:

- A Level IV student asked an insightful question about whether the order of the numbers in the subtraction mattered, and then, in worrying about that, demonstrated both uncertainty and high interest, and ultimately produced a partially correct answer.
- A Level I student exhibited advanced mathematical thinking in coming up with a novel and valid geometric interpretation but wasn't able to write down a polished proof.
- A Level III student made a poor strategic choice to use an advanced method. Then he gave up after being stuck for less than two minutes in the middle of the proof, saying he really couldn't see it, couldn't figure it out. He lacked the confidence needed to deal with the uncertainly of how to proceed.
- After finishing a well-written proof, a Level III student reflected that it probably needed refining because it didn't sound very good, but she thought it got the job done, and she wasn't interested in spending any

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course more frequently than the bridge course. The impact of the IPS&W course is confirmed by the chart.

more time on it. So lack of interest inhibited her from doing an even better job.

Confidence appeared as a factor in our expert's performance as well. Initially, he expressed trepidation at being asked to volunteer to be our "expert," especially since we had described the problem as coming from number theory, which was not his field. However, when pressed into service, he displayed a sense of assurance as he moved through the tasks: using language carefully, making a smooth and unprompted transition from investigating to proving, and writing a clear and correct proof that employed proper definitions, notation, organizational features, complete sentences, and detailed algebraic steps. In sum then, these proof-alouds gave us compelling illustrations of the various types of knowledge, strategic processing and motivation required to produce a correct and well-written proof. They showed how additional knowledge may sometimes result in poorer overall performance and that a student could exhibit both expert and novice behavior during the same task. They also indicated that affects such as confidence and interest could influence student performance.

4. The Mathematical Knowledge-Expertise Taxonomy Matrix

Our desire to describe the rich detail of these proof-aloud performances both in terms of the types of knowledge they document and the levels of expertise exhibited in different types of knowledge led us to develop a mathematical knowledge-expertise taxonomy matrix (See Figure 2). This taxonomy matrix has two dimensions. The first dimension consists of a typology of mathematical knowledge that includes six cognitive and two affective components. The six cognitive components were adapted and extended from a typology of scientific knowledge developed in response to calls for national K-12

Mathematical Knowledge-Expertise Taxonomy Matrix

Affective	Acclimation	Competence	Proficiency
Interest	Students are motivated to learn by external (often grade-oriented) reasons that lack any direct link to the field of study. Students have greater interest in concrete problems and special cases than abstract or general results.	Students are motivated by both internal (e.g., intrigued by the problem) and external reasons. Students still prefer concrete concepts to abstractions, even if the abstraction is more useful.	Students have both internal and external motivation. Internal motivation comes from an interest in the problems from the field, not just applications. Students appreciate both concrete and abstract results.
Confidence	Students are unlikely to spend more than 5 minutes on a problem if they cannot solve it. Students don't try a new approach if first approach fails. When given a derivation or proof, they want minor steps explained. They rarely complete problems requiring a combination of steps.	Students spend more time on problems. They will often spend 10 minutes on a problem before quitting and seeking external help. They may consider a second approach. They are more comfortable accepting proofs with some steps "left to the reader" if they have some experience with the missing details. They can start multi-step problems, but may have trouble completing them.	Students will spend a great deal of time on a problem and try more than one approach before going to text or instructor. Students will disbelieve answers in the back of the book if the answer disagrees with something they feel they have done correctly. Students are accustomed to filling in the details of a proof. They can solve multi-step problems.
Cognitive	**Acclimation**	**Competence**	**Proficiency**
Factual	Students start to become aware of basic facts of the topic.	Students have working knowledge of the facts of the topic, but may struggle to access the knowledge.	Students have quick access to and broad knowledge about the topic.
Procedural	Students start to become aware of basic procedures. Can begin to mimic procedures from the text.	Students have working knowledge of the main procedures. Can access them without referencing the text, but may make errors or have difficulty with more complex procedures.	Students can use procedures without reference to external sources or struggle. Students are able to fill in missing steps in procedures.
Schematic	Students begin to combine facts and procedures into packets. They use surface level features to form schema.	Students have working packets of knowledge that tie together ideas with common theme, method, and/or proof.	Students have put knowledge together in packets that correspond to common theme, method, or proof, together with an understanding of the method.
Strategic	Students use surface level features of problems to choose between schema, or they apply the most recent method.	Students choose schema to apply based on just a few heuristic strategies. Students are slow to abandon a non-productive approach.	Students choose schema to apply based on many different heuristic strategies. Students self-monitor and abandon a nonproductive approach for an alternate.
Epistemic	Students begin to understand what constitutes 'evidence' in the field. They begin to recognize that a valid proof cannot have a counterexample. They are likely to believe based on 5 examples; however, they may be skeptical.	Students are more strongly aware that a valid proof cannot have counterexamples. They use examples to decide on the truth of a statement, but require a proof for certainty.	Students recognize that proofs don't have counterexamples, are distrustful of 5 examples, see that general proofs apply to special cases, and are more likely to use "hedging" words to describe statements they suspect to be true but have not yet verified.
Social	Students will struggle to write a proof and include more algebra or computations than words. Only partial sentences will be written, even if they say full sentences. Variables will seldom be defined, and proofs lack logical connectors.	Students are likely to use an informal shorthand that can be read like sentences for writing a proof. They may employ connectors, but writing lacks clarity often due to reliance on pronouns or inappropriate use or lack of mathematical terminology.	Students in this stage write proofs with complete sentences. They use clear concise sentences and employ correct terminology. They use variables correctly.

Figure 2

science assessment (Shavelson, 2003). Richard Shavelson developed his typology of scientific knowledge, in part, to make the point that multiple choice testing will not accurately assess the complete spectrum of scientific knowledge we want our K-12 students to attain by graduation. The second dimension describes student progression toward proficiency using the language of a K-12 classroom-based theory of expertise development that arose from studies of student learning in academic domains, such as reading, history, physics, and biology (Alexander, 2003).

4.1 A Typology of Mathematical Knowledge

The descriptions provided by Shavelson for the six types of scientific knowledge – factual, procedural, schematic, strategic, epistemic, and social (R. Shavelson, personal communication, April, 2004) – were readily adapted to mathematical knowledge. For example, epistemic knowledge or how one decides if a statement is true in the discipline changed from *the scientific method* to *proof*.

Here we give general descriptions of each of the six cognitive components of mathematics learning[2]. In Section 5.1 we will provide more detailed descriptions for one particular topic (systems of linear equations). Factual knowledge is the knowledge of specific facts of a topic. In mathematics, this might be described in terms of definitions or stated theorems, and represented by such questions as "Do students know what the sum of the angle measurements of a (Euclidean) triangle is?" or "What are the steps of mathematical induction?" Without factual knowledge, students cannot proceed in

[2] We observe that these six cognitive components encompass the five strands of K-8 mathematical proficiency (conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition) delineated in *Adding It Up* (National Research Council, 2001, p. 116). We believe the taxonomy matrix can be viewed as an extension of *Adding It Up*'s intertwined strands of elementary school mathematical proficiency (p. 117) to the development of mathematical proficiency in the undergraduate mathematics major.

mathematics. However, as is true in every field, facts by themselves are of little use, and are generally not retained for very long beyond the classroom.

Procedural knowledge relates to the knowledge of how to complete the procedures related to a topic. Thus, procedural knowledge would be addressed by questions such as "Can students find the measurement of one angle of a triangle knowing the measurements of the other two angles?" or "Can students properly set up the two steps in an induction proof?" Procedural knowledge is often at the high end of what is taught in the secondary school mathematics curriculum as is noted by Stigler and Hiebert (1999) when they describe mathematics teaching in the US as being mostly "practices and procedures."

The third cognitive component, schematic knowledge, encompasses the connections between facts, procedures, methods, and underlying reasons. This echoes the “schema” of APOS theory (Dubinsky and McDonald, 2001). It can involve un-packing procedures (Ball, 2003) or be thought of as the knowledge packets described by Ma (1999). Questions about schematic knowledge in mathematics might be "Can students use the knowledge about triangles to answer questions arising in circular trigonometry?" or "Can students provide a geometric interpretation for the solution of a quadratic equation?" Work has shown (Schoenfeld, 1985) that students have different schema from faculty, and clearly an important part of mathematics learning centers around helping students develop richer schema. However, Schoenfeld has also argued that students often do not learn how or when to apply these schemas.

This brings us to the fourth component of mathematical knowledge, strategic knowledge. Strategic knowledge in mathematics involves the heuristics used to make decisions about approaching new (to the individual) problems, including which schema to apply. The component mirrors the knowledge of heuristics and how to solve problems as described by Polya (1945). The questions that one might ask about students' heuristic knowledge would include, "Can students select a useful schema to solve a nonstandard problem?" or "How do students decide which proof method is likely to be appropriate or effective for proving a particular statement?"

Epistemic knowledge refers to how one decides what is true in the discipline. In mathematics, epistemic knowledge encompasses knowledge of proof and logical argumentation, including understanding the necessity for proof, various proof techniques and implications of having proven a statement. Thus, epistemic knowledge in mathematics would correspond to the questions like "Do students recognize the role of proof in problem solving or its necessity for determining the truth of a mathematical statement?" or "Can students use their disciplinary knowledge to decide upon whether a mathematical statement is correct and/or has been correctly proven?" or "Do students understand that a proven statement cannot have counterexamples?"

Finally, social knowledge comprises the accepted ways to communicate truth or knowledge (which can vary in formality depending on the audience and purpose). This component might be described as the knowledge of discourse appropriate for the community. Thus it is the social knowledge of the discipline that allows one to carry on work jointly with others and to convince others with our arguments. In mathematics this knowledge encompasses the accepted rules of exposition in the discipline. Questions that correspond to this knowledge domain might be "Can students write proofs that follow the norms and conventions of mathematics?" or "Are students able to communicate their knowledge to their peers and those in the mathematical community in a way that will allow others to understand, accept, and use that knowledge?"

The last two components, epistemic and social knowledge, were the content core of our proof-aloud investigation; namely, how do students know something is true and how do they communicate that knowledge. The first four components encompass the knowledge and strategic processing portions of Alexander's model, which we will describe in the next section, after examining two influential affects.

Our proof-alouds indicated that in addition to these six cognitive components of knowledge, two affective components, interest and confidence, influenced students' performance. Interest plays an important role in moving students toward proficiency (Alexander, 2003) and is often cited as a motivating factor in learning (Bains, 2004; Harel, In Press). Confidence is less frequently considered, but we posit it plays a very important role in students' willingness to persist in the face of "not-knowing" (Feito, personal communication, 2004; Schoenfeld, 1985).

4.2 Describing the Journey Toward Mathematical Expertise

To describe our students' growth in understanding of proof, we turned to Alexander's (2003) Model of Domain Learning (MDL). MDL is a perspective on expertise theory that arose from studies of student learning in academic domains, such as reading, history, physics and biology (Alexander, 2003). MDL does not see someone as either a novice or an expert, but rather is concerned with the journey from novice to expert. It looks at three stages of learning. The initial stage is *acclimation* wherein the learner is orienting to a complex, unfamiliar domain. The next stage is *competence*, and the final is *proficiency/expertise.* Occupants of this latter stage are characterized by having a depth and breadth of knowledge, a mastery of methodologies, and an ability to contribute new knowledge to the field. MDL focuses on three components that play a role in the journey toward expertise in academic domains: knowledge (which roughly corresponds to factual and procedural knowledge in our typology), strategic processing (corresponding to schematic, strategic and, potentially, epistemic knowledge) and interest (an affective component). Alexander's (2003) work was applied to K-12 teaching and learning, with competence in academic domains seen as an attainable goal for most high school graduates. When MDL is applied to the academic domain of mathematics, given the specialized knowledge, advanced heuristic knowledge, and high interest of an expert mathematician, reaching the level of proficiency/expertise is unrealistic for the collegiate mathematics major. We can report that MDL has proven useful in describing performance on mathematical tasks across our typology of mathematical knowledge components. To illustrate how it applies, we will describe in more detail the three stages of learning in the epistemic and social/communication knowledge components.

A student in the Acclimation stage relative to epistemic knowledge would often depend solely on examples to test the truth of a statement, or rely on external authority to validate a claim. Thus, students at this stage would be likely to respond that to be certain of a statement they would ask a professor or look in a book. Students moving through this stage are beginning to recognize that a conjecture with a valid proof cannot have counterexamples, and they might be skeptical, at times of the validity of a statement only having seen five examples where it holds. In contrast, students at the Competence stage in epistemic knowledge will still use examples, but then will express a desire for a proof in order to be certain. These students recognize that a general proof will apply to special cases. They are also more strongly aware that a proven conjecture cannot have counterexamples, at least in specific case of proofs they have seen. Experts (at this level) in the epistemic domain recognize that truth is decided by proof, and that given a valid proof there can be no counterexample. They are quick to see that given a general argument, it can be applied to special cases. Moreover, experts refuse to take five examples as evidence of truth. In addition, in their language, experts are more likely to make distinctions between proven and conjectured statements. Experts use hedging language, words and phrases like "conjecture," "plausible," and "seem to be true," to describe statements they are validating until they arrive at a proof.

In applying these stages, one finds a lack of knowledge in one of the components might affect the ability of a student to progress in another. Surprisingly, however, an increase in knowledge in one component might hamper a student in another. For example, one of the students in our study, showed poor strategic knowledge on the proof-aloud due to a greater factual knowledge about mathematical induction as a method of proof. This notion of backwards motion in understanding is reminiscent of many other theories of learning. In particular, in Piaget's work, accommodation, that is, making room for new ideas that do not fit with previously held notions, may cause students to temporarily regress in their understanding of something. Pirie and Kieren (1989) and Dubinsky (1991) also argue that accommodating new ideas will cause students to go through a retrenchment of their understandings. In addition, the assignment of a stage of expertise is often very particular to the aspect of the task under consideration or to the person being examined. One may exhibit expert traits on some aspects and acclimating traits on others. The acclimating to expertise continuum that we describe in Figure 2 is comprised of typical college mathematics majors, rather than mathematicians as a whole.

In the social/communication knowledge component, a student in the acclimation or orienting stage will struggle to write a proof. Most will write down very few words, even though in a proof-aloud they might actually say the words and use full sentences. If they use algebraic expressions, the variables will not be explicitly defined for the reader. Their written proof will lack logical connectors, careful definitions, equal signs, and organizational labels or features such as centering equations. Students in the competence stage may write in an informal shorthand that can be read as full sentences. They may employ connectors but their writing will lack clarity, often because pronouns or non-mathematical language ("it works") are substituted for accurate mathematical terminology. By contrast, experts will write clear, concise sentences, use logical connecters and employ mathematical terminology. They will define variables and use them appropriately. Their writing will be formatted for easy reading.

5. Applying the Taxonomy

In this section we use the cognitive components of the mathematical typology dimension of taxonomy matrix to unpack the underlying mathematical knowledge in non-standard problem involving a system of two linear equations.

Problem

For what value(s) of p will the system have no, one or more than one solution?

$$3x + 6y = 12$$
$$x + py = 4$$

5.1 The Six Cognitive Components of the Problem

We now describe each of the cognitive components of the typology relative to the topic of this problem – linear systems.

Factual knowledge essential for this problem includes knowing what a linear system is and what constitutes a solution to a system of equations. Knowing important aspects of a line (slope, intercepts) and standard forms for linear equations could also be useful factual knowledge.

Procedural knowledge could be algebraic in nature, such as how to change an equation from one form to another, or how to find the slope and intercepts for each line. It could include the knowledge of graphical, substitution, elimination or matrix methods for solving a pair of equations.

Schematic knowledge for this problem could consist of one or more of the following packets. (For an expert, all three of these packets would be interconnected and incorporated into one big schema.)

- Geometric packet
  - Points on each line are solutions to a single equation
  - Points of intersection of graphs correspond to solutions to the system
  - Possible cases for intersections of two lines
- Algebraic packet
  - Interpretations of the possible outcomes (e.g., $0 = 0$, $x = 0$, or $0 = 4$) for elimination or substitution methods
- Matrix packet
  - Interpretations of the possible outcomes

Strategic knowledge refers to the decision process involved in choosing an approach to use. Students partial to geometric approaches might begin by solving for slopes and equate the two expressions, while those more comfortable with algebra might immediately begin with elimination of variables or substitution.

Epistemic knowledge for this problem requires that a student know why the answer is correct. The validity of the geometric solution can be traced back to Euclid's 5th postulate, while the validity of an algebraic or matrix solution is based on a deep understanding of the definition of "equivalent equations."

Social knowledge enables a student to communicate (orally or in writing) the solution, its process and the rationale behind it.

The taxonomy allows analysis of both student work and instructor responses on this problem. Movement from acclimation toward competence is often a result of instructor intervention.[3]

This problem was assigned in the introductory problem solving and writing class, and in examining student-generated solutions, we found it most illuminating to concentrate on the epistemic and social knowledge components in terms of expertise theory, and so we will follow that here.

Epistemic Knowledge: Students in Acclimation often produced an answer simply by writing down a list of equations with no reasons given for their final answer. On further probing, the students would simply say that when an equation deduces to 0 = 0 there are infinitely many solutions. Students in the Competence stage were more likely to try to explain why the equations showed one or infinitely many solutions, but would gloss over key ideas or be stumped when asked why they could make certain assumptions. Students

[3] The class that used this problem allowed students multiple tries at writing up their solutions to meet writing standards set out for the course (which are based on Price, 1989). Instructors generally write brief Socratic comments to guide students in their rewriting. The intent of these comments can also be categorized by the taxonomy matrix.

in the Proficiency stage showed a clear understanding of the relationship between reducing to the equation 0 = 0, and why this implies infinitely many solutions exist.

Social Knowledge: Acclimating students used few words in their solutions, and expected readers to put together the steps of a solution, while students moving to the Competence stage were more likely to attempt to indicate which previous statement they were referring to after having written a sequence of equations. Finally Proficient students wrote clear and concise arguments using correct mathematical terminology.

6. Implications for Teaching

We have shown how the Mathematical Knowledge-Expertise Taxonomy Matrix developed in our study can provide a multifaceted way of analyzing student performance on mathematical tasks. This compound way of looking at mathematical learning can assist instructors in targeting instruction to meet students' needs and move them along the path toward proficiency. It may also help with writing rubrics. Our data gives clear evidence that a lack of knowledge can inhibit acclimation and an increase in one knowledge component can cause a student to regress in others, something that teachers need to be aware of. Increased awareness of the special needs that acclimating students have is another potential result of access to the taxonomy matrix.

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