The Importance of the Concept of Function for Developing Understanding of First-Order Differential Equations in Multiple Representations

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Modern ordinary differential equations [ODEs] curricula (e.g. Blanchard, Devaney, & Hall, 2006) differ vastly from their traditional counterparts. Traditional curricula emphasize solving ODEs with analytical solution methods; procedures applied to algebraic equations produce algebraic families of solution functions. Modern ODE curricula emphasize multiple solution methods. Traditional analytic methods are complemented by numerical and qualitative methods that produce approximate numerical and graphical solutions. Numerical solutions are computationally demanding, but can be very accurate. Qualitative solutions share the essential characteristics of algebraic solutions but lack their precision. Modern technology has made qualitative and numerical analyses accessible to undergraduates and as a result the set of solvable ODEs extends beyond the proper subset to which analytical methods apply. An important consequence of the modern approach is the prominence of graphical representations. Research on students' understanding of ODEs (Habre, 2000; Rasmussen, 1999, 2001) suggest students' understanding the relationship between equations and graphs is a cognitive obstacle to learning ODEs in a modern curriculum.

In this paper Sfard's theory of reification is extended to describe the cognitive development of FODEs. This extension of the theory of reification is based on data from sharply contrasting case studies of two students, Rich and Hassan (pseudonyms), solving problems in clinical interviews. The theory of reification explains for the differences observed.

The Theory of Reification

The theory of reification (Sfard, 1987, 1988, 1991, 1992, 1994; Sfard & Linchevski, 1994) describes how concepts come into existence from a cognitive perspective. The theory is based on the fact that many mathematical concepts are conceived in two complementary ways, operationally and structurally. Operational conceptions are "about *processes, algorithms, and actions* rather than about objects" (emphasis in original, Sfard, 1991, p. 4), in contrast to structural conceptions where mathematical entities are conceived as objects, wholes, or as the result of a process instead of the process itself.

Reification theory posits that mathematical objects are formed from mathematical processes in a three-phase progression - interiorization, condensation, and reification. The most important step, and most difficult to achieve, is reification. Reification is “an ontological shift – a sudden ability to see something familiar in a totally new light” (Sfard, 1991, p. 19); what was previously only a process can now be seen as an object also.

Reification is the most important phase because of cognitive change that accompanies it. Cognitive compression, or reification, is extremely important to thinking because it circumvents the limitations of working memory. When an entity is conceived solely as a process the steps of the process are activated in memory, but after reification the steps of the process become secondary conceptions allowing the object to become “one step in some other mental process.” The operational information is “filed away,” or “squeezed… into a compact whole” (Sfard, 1991, p. 26).

In the context of the theory of reification, referring to a mathematical entity as an object necessarily implies it has formed as the result of reification of some process (Sfard & Thompson, 1994). The distinction between an object and an entity is very subtle, “the later will refer to the way of handling information and will mean not much more than ‘integrated whole’ while the former will convey an ontological message and will thus be more restricted in its scope” (Sfard, 1992, p. 60). When an entity is manipulated in an object-like manner but has not evolved from reification it is referred to as a pseudo-object. Pseudo-structural conceptions are significant because the mental compression of the process has not occurred.

Three essential characteristics of structural conceptions distinguish them from pseudo-structural conceptions:

1. “being able to recognize the same concept under many different disguises” (Sfard, 1992, p. 76),
2. “various representations of the concept [are] semantically unified by this abstract, purely imaginary construct” (Sfard, 1991, p. 20), and
3. “being able to recognize the idea ‘at a glance’ and to manipulate it as a whole without going into details” (Ibid, p. 4).

Learning in general, and constructing objects in particular, is not as linear and hierarchical as the reification theory suggests, but the "model does seem to present a *prevailing* tendency" (emphasis in original, Sfard, 1991, p. 23).

The FODE concept lends itself nicely to analysis through the lens of the theory of reification. My analysis assumes a larger grain size than Zandieh (2000), although both types have their value. The value of a look at FODEs from a smaller grain size is discussed in the conclusion.

FODEs Through the Lens of The Theory of Reification

Solving a FODE can be considered as a process, see Figure 1(a), and thus conceived operationally and structurally. A FODE serves as the input upon which a solution technique acts. The result of this process is a family of solution functions. The stages of development from operational to structural for FODEs are illustrated in Figure 1(b). The first stage of the transition, interiorization, is of particular importance as students begin the study of FODEs.

Interiorization begins when processes are performed on lower level objects, but what if structural conceptions of the necessary lower level concept have not developed? In fact, such an occurrence is likely:

> On the one hand, without an attempt at the higher-level interiorization, the reification will not occur; on the other hand, existence of objects on which the higher level processes are performed seem indispensable for the interiorization. In other words: *the lower-level reification and the higher-level reification are prerequisite for each other* (emphasis in original, Sfard, 1991, p. 31).

The result is "the vicious cycle of reification."

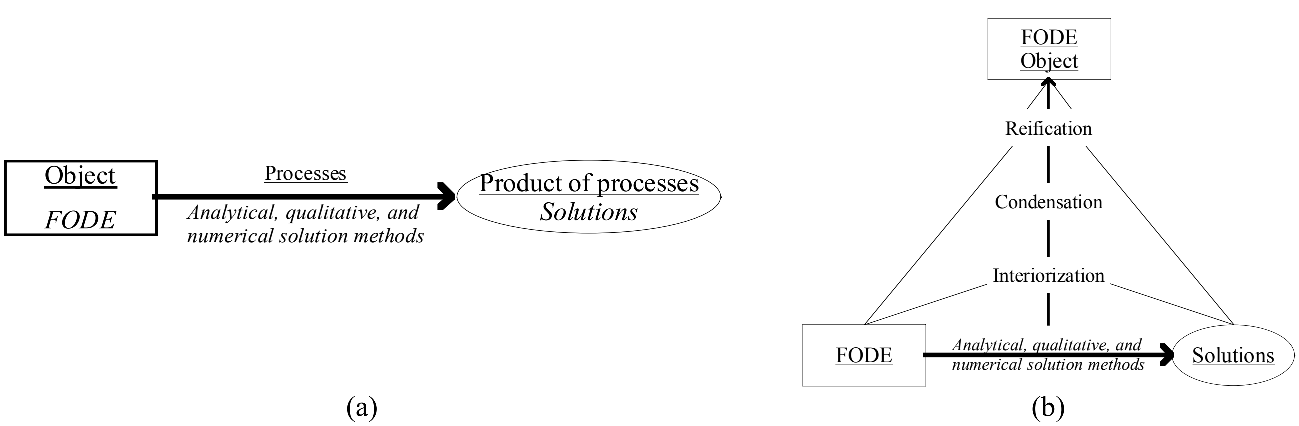
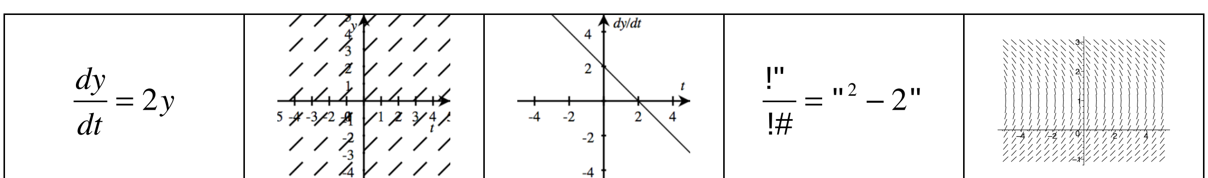


Figure 1: Illustrations of (a) the process of solving and (b) reification stages of solving FODEs

This cycle implies that learning a new concept necessarily includes a period of uncertainty, of pseudo-structural understanding, as learners perform processes on entities they have not yet reified. For FODEs this raises the question, what is the nature of the object that serves as the basis for interiorization of the process of solving FODEs? More specifically, what does it mean to consider the input ODE as an object? The data collected in this study indicate that it is ability to conceive FODEs structurally as functions. Details of the analysis follow after an overview of the methodology and presentation of the results.

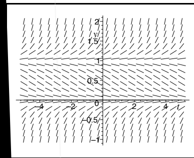
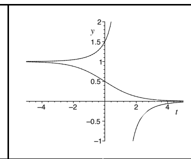
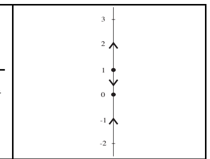
Methodology

The primary data source was a series of three task-based clinical interviews. Each interview had primary tasks that were the same type, but the types varied across the interviews. The first interview contained a series of 5 prompts, i.e. a graph or an equation on an otherwise blank sheet of paper was given to the participant who was prompted to tell everything they could about it; see Figure 2. The philosophy behind this task was to see what meaning the symbol had for the participant absent of any contextual clues. The protocol was to probe only the things they said, not introduce any concepts or ideas that might provide alternative cues. For example, participants were not asked, “Can you solve that?” if they did not make a reference to solving or solution. In the second interview a collection of 20 FODEs and functions were given to the participant on individual cards with the directions to sort them into appropriate groups explaining their rationale as they went. The examples were designed so that specific mathematical relationships existed, as shown in Figure 3. The protocol for the second interview was similar to the first, questions were asked only about participant’s statements and their rationales for forming groups. Participants were asked to continue sorting until it seemed they could no longer form any new groups. The third interview contained a series of 4 nonroutine problems, one example is shown in Figure 4.

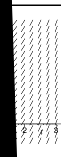
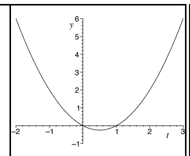


(a)Prompt 1 (b) Prompt 2 (c) Prompt 3 (d) Prompt 4 (e) Prompt 5

Figure 2: Prompts from interview 1

	$y(t)=1$		$y(t)=\dfrac{y_0}{y_0-(y_0-1)e^t}$	
E	N	O	P	T

Representations of $dy/dt = y^2 - y$ and its solutions

	$\dfrac{dy}{dt}=2t-1$	$y=t^2-t$	$y=t^2-t+2$	
	D	F	H	M

b) Representations of $dy/dt = 2t-1$ and its solutions

The study took place at a large state university in the northeastern United States. Participants were sought from a differential equations class taught by a professor that had been recently recognized by the State University System for excellence in teaching. The class met for lecture thrice weekly for 50 minutes and for an additional 50 minute recitation. The recitation, run by the professor and a graduate student, consisted of investigations usually worked on collaboratively and often with the aid of the mathematical software Maple. The researcher attended each class and recitation, taking field notes and making observations of the classroom behaviors of the participants. FODEs were the curricular focus during the first three and a half weeks of the semester; and were tested on an in-class examination during week 5. The 4-interview sequence was timed to correspond to the schedule of the class. The preliminary interview was conducted during the second week and the first FODEs interview was the third week after the unit on FODEs was complete. The second FODE interview was during week 5, the week before the exam, and the final interview was during week 6, the week after the exam. Each interview was video-taped and audio taped. Complete transcripts were made for each interview. Copies of the participants' classwork, homework, and exam were obtained to supplement interview data.

Results

Over the first two interviews, the FODE prompts and the card sorting task, patterns emerged in both Hassan's and Rich's work. Each student consistently made connections between equations and graphs of both FODEs and functions, but their motivations to do so varied significantly.

In the first and second interviews, Hassan's primary concern was finding solutions. Each of the FODE prompts activated in him an association with solutions to FODEs, and in every case he found solutions. Hassan showed a preference for equations, but if he could not apply analytical methods (e.g. separation of variables) he found solutions qualitatively using a graph of the FODE.

Hassan's work in the card-sorting task was very similar to his work on the prompts. Connections between FODEs and their solutions were the primary rationale for the groups he formed. Hassan moved easily from graphs to equations and vice-versa, and considered them as "the same," either because they were in the "same general solution of [a] differential equation" or they were "the same differential equation." Because his work in the first two interviews allowed insight into his thinking about solutions, his work on the tasks in interview 3 are not included in the analysis except to say his work their was consistent with his previous work.

Rich's work in the first two interviews provided no evidence about his understanding of solutions of FODEs, he neither mentioned solution nor used the idea implicitly. Instead Rich's focus was finding graphs for equations and vice-versa. His work was procedural in nature, inconsistent, and plagued with errors and uncertainty. He was often unsure whether or not he had arrived at the correct equation or graph. This was consistent with his work in interview 3 where he was cued to find solutions.

Hassan

Episodes from Hassan's work in interview 1 and 2 are analyzed in this section. The episodes come from his work on one prompt and one of the groups he formed in the sorting task. The examples illustrate how Hassan thinks about FODEs represented as graphs and equations, and their connections to solutions. They also show the underlying consistency in Hassan's thought processes about relationships and his use of multiple representations.

An example of Hassan's work with the prompts

The first FODE interview with Hassan took place during the third week of the semester, after the 7th class session. The class lectures up to this point had covered modeling cooling liquids, checking solutions, separation of variables, direction fields, and Euler's method. This episode comes from his work on the third prompt (shown in Figure 2). This episode was chosen because in it he makes two errors and his work in resolving these errors highlights important aspects of his thinking. For the purposes of discussion Hassan's work is broken down into three sections. The sections emerged naturally in his work as he shifted focus to different aspects of the problem. In the first section Hassan decides to find a solution for the differential equation and debates how to go about it. In the second section Hassan recognizes a conflict in his reasoning about the nature of the solution he obtains and resolves the issue. In the third section he forms definite conclusions about the shape by solving the equation using a second solution method.

The dialogue presented is not a complete transcript of his work on this task, but it illustrates the important aspects of his work. The chronological order of the sections is accurate. The diagrams in Figure 6 and Figure 7 illustrate the structure and process of Hassan's thinking. The meanings of these diagrams are elaborated below.

How to solve this ODE

Hassan was presented with the prompt (third prompt, see Figure 2) and the following dialogue proceeded.

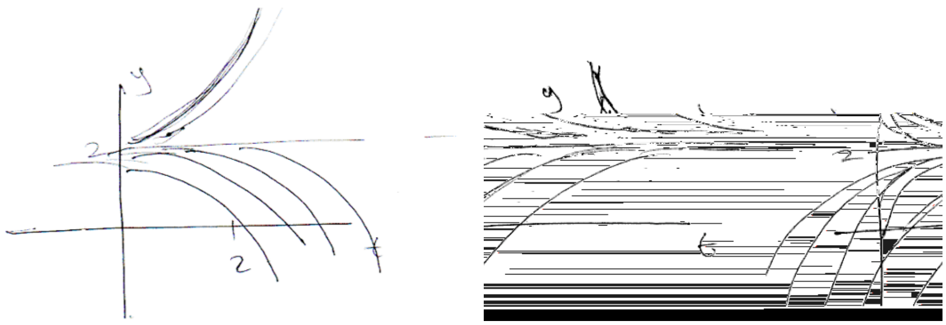
(1) J: Tell me everything you can about it.

(2) H: Minus 1, plus 2. [*Writes equation* $\frac{dy}{dt} = -y + 2$.]

(3) J: Where did you get those numbers from?

(4) H: From the line, …the slope [is -1], …and *b* is 2. And so if you want to solve that [using separation of variables] you can. No you can't. Yes you can, no you can't. [*Pause.*]

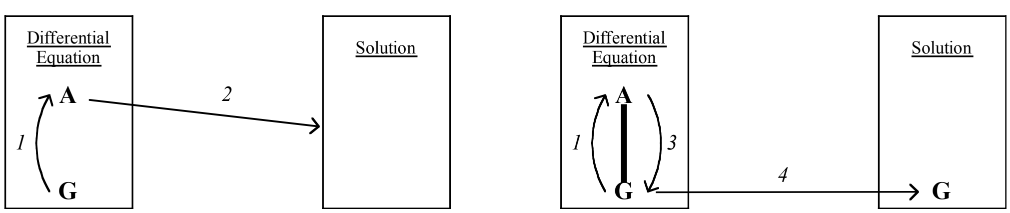
(5) H: Let's go like this. [*Hassan began to make the sketch shown in* Figure 5(a).]



(a) (b)

Figure 5: Hassan's solution sketches for Prompt 3

Hassan made 3 important decisions in this episode. First, Hassan wrote an equation for the graph using the slope and intercept (transcript lines 2 – 4). This connection is illustrated in Figure 6a with the arrow labeled 1 from the graph [**G**] to the algebraic equation [**A**] of the FODE. Next, Hassan decides to solve the equation (line(4)), shown as arrow 2 in Figure 6a. At this point he engages in a debate with himself trying to decide if he can solve the equation using separation of variables (line (4)). His conclusion, for the time being, was that he could not (end of line (4)), and thus he made his third decision to use the graph he was given to sketch the solutions (line (5)); these decisions are represented by the arrows labeled 3 and 4 in Figure 6b.



(a) (b)

Figure 6: Structure-process diagram illustrating Hassan's initial work on prompt 3

There is an important subtlety in this first episode in the manner Hassan uses the equation and graph. His immediate response to the graphical prompt was to write an equation. Then, he began to think about solving the equation. As he debates whether he can "solve that" he is referring specifically to the equation; he is not sure whether he can solve it using separation of variables. Then he switches back to the graph and says, "Let's go like this." then proceeds to solve the differential equation qualitatively using the graph. The importance of this is his implicit recognition that he could "solve that" using either the equation or the graph. This equivalence is represented in Figure 6b by the bold line linking **A** and **G** in the set of ODEs.

What type of functions are these solutions? Part 1

Hassan went on to sketch "all possible solutions" based on his interpretations of the FODE graph. His initial sketches were backwards in the sense that the solutions above the equilibrium solution $y = 2$ were increasing and below $y = 2$ they were decreasing, see Figure 5a. Before realizing his error he stated that the graphs below $y = 2$ are "all going to be parabolas; they are going to be going down in a parabolic kind of way." Before he said anything further he realized that he had drawn the graphs incorrectly and said, "Okay, this is wrong. I just realized that. …Because when *y* is greater than 2 its going down, and when *y* is less then two it is going up." He explained that solutions above the line $y = 2$ are decreasing because $dy/dt < 0$, and that solutions below $y = 2$ were increasing because $dy/dt > 0$.

Hassan's comment about the solutions being parabolic is incorrect, as he comes to realize. This characterization would be true if $dy/dt = f(t)$, but in this example the FODE is autonomous $dy/dt = f(y)$.

When Hassan completed his second sketch, Figure 5b, the following dialogue took place:

(6) J: Okay, you said parabolic, does that still apply or was that only for the first graphs?

(7) H: …I guess you can't say it's going to be parabolic. It's just I feel like it would be because… [the graph of the FODE] is a line. The next term, I mean when you anti-differentiate, it's going to be y^2, you know what I mean?

(8) J: Um-hum.

(9) H: But that doesn't make any sense either because my solutions are asymptotic. Well I guess you cannot say because it is not separable.

In this episode a conflict emerges for Hassan. On one hand the solutions he drew are asymptotic to the line $y = 2$. On the other hand, he believes the solutions should be parabolic because the FODE is linear. He does not resolve the conflict and says, "Well I guess you cannot say because it's not separable." This statement reveals an important aspect of his understanding. If the equation is separable he can resolve the conflict by classifying the solution equation he would obtain by solving it analytically. Hassan's reasoning is illustrated as connections between representation types in Figure 7a. The solution graph is informed by the solution equation he would get from solving the algebraic FODE. Arrow 5 is dashed because he has not yet made the necessary computations. This shows his understanding of the relationship between qualitative and analytical solutions.

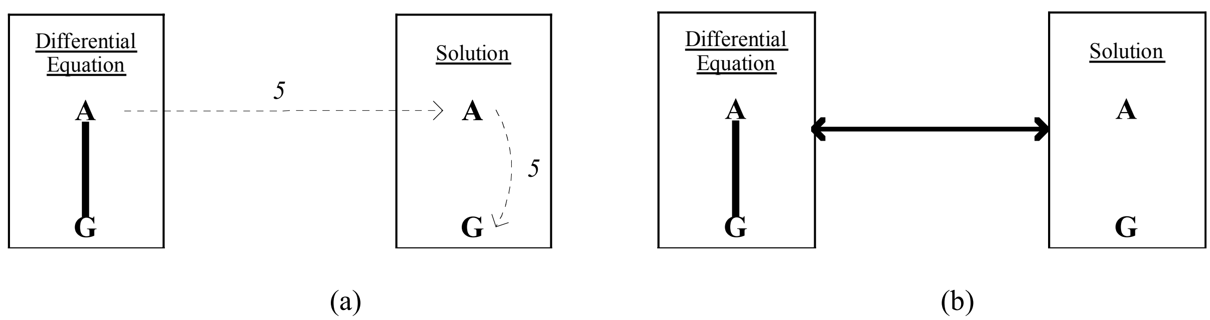


Figure 7: Structure-process diagram illustrating Hassan's reasoning about solution behavior

Another important aspect of this episode is Hassan's anti-derivative reasoning. As the dialogue continues he realizes why his reasoning is incorrect and is able to say with certainty the solutions are not parabolic.

(10) H: …I don't even know what I'm trying to graph any more. I was just trying to figure out like where the parabola would come in. Can you give me a hint?

(11) J: I'm really not sure… where you are trying to go with this right now so I'm not sure I can.

(12) H: I just feel like… *Pause*.

(13) J: If I feel like I understand what you said, you think it's going to be parabolic because you said this is a line.

(14) H: Um hum. Well *y* vs. *t* would be parabolic but it is not, it looks like this [*points to Figure 5b*]. It doesn't, it does?

(15) J: I think you've established your graph. I think you've managed to confuse yourself a little bit but you established your graph based on [the FODE graph].

(16) H: Yeah well, there's no parabola. *Pause.*

(17) H: The derivative of a parabola is a straight line, which is what [the prompt] is. When I anti-differentiate I should get *Abrupt stop*. *Pause.*

(18) H: No this isn't, this isn't, this isn't, never mind. [The prompt] isn't *dy/dt* vs. *t*, oh so, yeah. If [the horizontal axis on the prompt] was *t* it would be a parabola, because it's *y* it looks like my graph. There we go.

As Hassan finished this statement he seemed triumphant; he dropped his pen on the desk and sat back in his chair, confident he had figured the source of his conflict.

At the outset of this episode Hassan was confused and could not reconcile his conflict (line (10) and line (14)). In order to help him refocus, the interviewer (Author) rephrased the shared understanding from earlier in the interview (line (15)). In response to this, Hassan restated the source of his conflict (line (16)). He then went through his reasoning again, but he first stated it in terms of the relationship a solution has to a differential equation, i.e. the FODE is the derivative of the solution. In line (17) he says, “The derivative of a parabola is a straight line, which is what [the prompt] is.” He then begins to restate his anti-derivative reasoning but comes to a sudden stop before he finishes, this is the moment he realizes his error. He recognized that the variables were incorrect, for his anti-derivative reasoning to work the prompt would have had to express *dy/dt* as a function of *t* not *y*. Notice back in line (14) he focused in on the variables noting, “*y* vs. *t* would be parabolic.” So its likely that when he went through his reasoning about the derivative he realized that the of the derivative would also have to be a function of *t* which caused him abruptly stop what he was saying.

This episode illustrates Hassan’s reflexive understanding of the relationship between FODEs and their solutions. This is illustrated by arrow 6 in Figure 7b connecting the set representing solution to the set representing differential equation. On the one hand Hassan conceives FODEs as equations to be solved, on the other he conceives them as derivatives of a family of functions.

What type of functions are these solutions? Part 2

Now that Hassan is certain that the solutions are not parabolic, the question remains if the shape can be classified.

(19) J: What shape do you think it is?

(20) H: I’d have to say exponential.

(21) J: Why would you say exponential?

(22) H: It’s the only thing I know that can go to an asymptote like that. Well it’s not the only thing, is it? $1/y$ does. …I really don’t know how you’d find out what shape that is unless you found out what $y!t)$ is and …you can’t just do it. You can’t do it analytically; it’s not separable. Wait, ohhhh, I should just divide the entire thing by y + 2.

In line (22) Hassan realizes he can indeed find the solution analytically. His surprise was reflected in his speech; he said "oh" in a drawn out manner as if he had made a discovery. Following up on this discovery he found the correct solution equation and verified his hypothesis that the solutions are exponential.

An example from Hassan's card sort

The card-sorting task took place in the second interview during the fourth week of the semester. Hassan formed the group shown in Figure 8, although initially the group only contained D and F. Upon adding H he said, "Because [H and F] are part of the general solution of the differential equation [D]." He reinforced this connection later saying, "these two [F and H] are solutions to this [D]" as he mentally performed separation of variables.

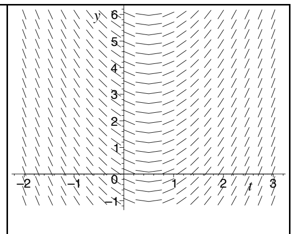
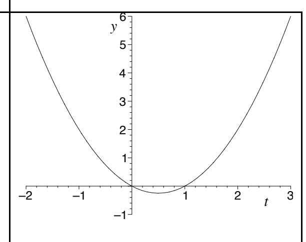
$\frac{dy}{dt} = 2t - 1$		
D	C	
$y = t^2 - t + 2$	! = "^{2} − 2"	
H	F	M

Figure 8: Hassan's sort of cards D, C, H, F, and M

Hassan then added cards C and M to the group [H, F, and D] and describes a "hierarchy:"

(23) J: Okay, so you're putting C and M with the group you just had of D, F, and H.

(24) H: They're in groups, but they have a hierarchy.

(25) J: Okay.

(26) H: This [D] goes first, and then this [C]. Actually these two go like this [*C and D at the top of the hierarchy*]. And then, these two [F, H] go under them and this [M] would go with… $t^2 - t$.

(27) J: Why the hierarchy?

(28) H: Well this is a differential equation [*points at C and D*] and like it has this many solutions to it [*pointing up and down C*]. …H, F, and M are two different solutions to that.

(29) J: So this [C] is a differential equation?

(30) H: Yeah, both of these. [*Points to C and D*].

The structure of the hierarchy is differential equations above solutions.

Hassan's statements give further insight into his thinking about the graphs and equations. First, describing the hierarchy in line (28), he points at the top row and says, "this is a differential equation" referring to both D and C. The singular use of differential equation indicates that each was a representative of the same ODE. Second, he focuses on C, moves his finger up and down the graph and says, "[the ODE] has this many solutions to it." So, whereas he sees many solutions in C, he continues and says, "H, F, and M are two solutions." This is not a slip of the tongue, he arranged F and M so they were overlapping and refers to them as one solution and to H as another. Again his singular reference to a graph and equation suggest each is a representative of the same thing.

The nature of Hassan's understanding

Taken together, these episodes suggest a structure to Hassan's understanding that is reflected in the diagram shown in Figure 9. The connection between differential equation and solution is the overarching structure of his work throughout the interviews as illustrated by these examples. Throughout the interviews he used graphs and equations interchangeably and these two examples provide evidence he thought of graphs and equations as representing the same abstract object.

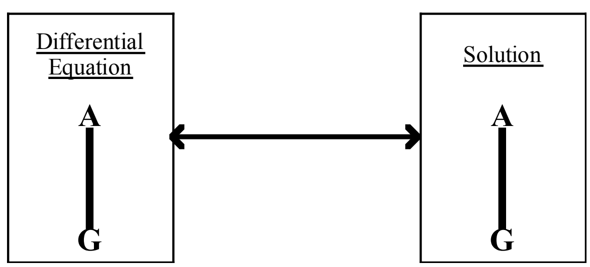


Figure 9: The structure of Hassan's understanding

Rich

Results from Rich's work in the interviews are analyzed in this section. Like Hassan, Rich's work across the interviews was consistent, but the picture of Rich's understanding that emerges is very different than that of

Hassan. Whereas Hassan tended to conceive the given entities as differential equations that had solutions, the notion of solution did not arise in either of the first two interviews with Rich. This raises the question, if Rich did not conceive the FODEs as equations to be solved, how did he conceive them? The evidence suggests Rich conceived them as functions.

Three examples of Rich's work are discussed below, two from the first interview and one from the third. A shared feature of each example is Rich's use of a procedure from his existing knowledge of functions to find the equation for a graph or vice-versa. The examples illustrate some of the technical difficulties Rich had throughout the interviews when connecting graphs and equations. The also illustrate his uncertainty about the connections he makes. The episode from the third interview shows how these difficulties play out in the context of his work to solve a FODE and give some insight into his understanding the concept of solution.

Episode 1 – Does $y = mx + b$ *Apply?*

The first episode comes from Rich's work the third prompt in the first interview (graph of $dy/dt = -y + 2$). As he worked to write an equation for the graph the following dialogue took place:

(1) R: I don't know, I don't think I can [write an equation for the graph] right now.

(2) J: What are you thinking?

(3) R: I'm thinking maybe I could use the slope, you know, rise over run. …[The] rise is 2 run is 2 so the slope is 1.

(4) J: Okay.

(5) R: That's what I am thinking…. I don't know if I should [apply]… the mx + b. But there's no x in [the graph] so I can't use that. So I am thinking if, since y is on [the horizontal axis]…, let me just try this.

Rich wrote down $dy/dt = y + 2$ *and changed it to* $dy/dt = -y + 2$ *realizing his error with the slope.*

(6) R: I am unsure if I should use that just cause it seems so. Pause.

(7) J: What do you mean by that? Unsure about what?

(8) R: I am unsure that I should be using $my + b$.

(9) J: Oh, okay, okay.

(10) R: But I am just because it is my first idea I am just going with it.

At the outset Rich is uncertain if he can write an equation for the graph (line(1)). As he continues, he explains the reason for his uncertainty (lines (3) – (5)). Rich knows that linear equations have the general form $y = mx + b$ where m is the slope and b is the y-intercept, but in this example the variables are different and "there is no x." In the generic linear equation y is the dependent variable, yet in this example y is the abscissa (the presumptive "x-axis). Rich decided to just to go ahead with his first idea (lines (5) & (10)). He chooses to ignore the factors that are causing him to hesitate and he wrote the equation using the variables as they appeared in the graph. Even so, he remains uncertain about his equation (line (8)).

This was the only time in the interviews that Rich was obviously hesitant to apply his general understanding. The exact reason that Rich is hesitant is unclear. There are at least two possible explanations. It could be that simply having y on the x-axis is the cause, but it also could be something deeper like the underlying implicit relationship between dy/dt and y. Whatever the cause, he decides to ignore it. In doing so he chooses to accept y in place of x (the independent variable) and dy/dt in place of y (the dependent variable); in effect he treats the graph as a function.

Episode 2 – Does that parabola open up or down?

Responding to the prompt $dy/dt = y^2 - 2y$ in the first interview Rich said, "This isn't a straight line… because this is a y^2. The first thing I would do, just because… I always do, is set it equal to 0. …It is a parabola." Rich confidently factored $y^2 - 2y$, set each factor equal to 0, and solved the equations to find $y = 0$ and $y = 2$. Using this information he sketched an appropriate graph of the parabola. He continued saying, "Nothing else comes to mind," but just before the next prompt was introduced he said, "Now that I am thinking… it could open up downwards. …That's what's going through my head. …But, it looks to me like it would go up because the y^2 is positive." Although Rich's gives a correct reason for the parabola to open upwards, he was not certain about this conclusion.

Episode 3 – Reasoning about the solution of a FODE

This final episode comes from Rich's work on the nonroutine task in interview 3 shown in Figure 4. This example is significant because the problem requires Rich to reason about the solution of a FODE. His work on the task is consistent with his work in the previous episodes. In fact, Rich uses the same procedure (in reverse order) to find the equation for a "parabola" that he used in the previous example. Further, he has the same uncertainty about the orientation of the graph.

Rich began, "Sketch a graph of the solution of the initial value problem; so we want to take dy/dt and make it $y(t)$ is equal to something, and $y(0)=3\ldots$ So, somehow in my mind we have to figure out a way to get $y(t)$ before we can draw an initial value problem like that." His approach is illustrated in the diagram in Figure 10. Working from the graph he first needed to find the equation of the FODE. Then he needed to "take dy/dt and make it $y(t)$ is equal to something," and then use this equation for $y(t)$ to sketch the requested solution. Rich could not find solutions directly from the graph, he said, "I am sure you can, …but I can't do that so I always look for the equation."

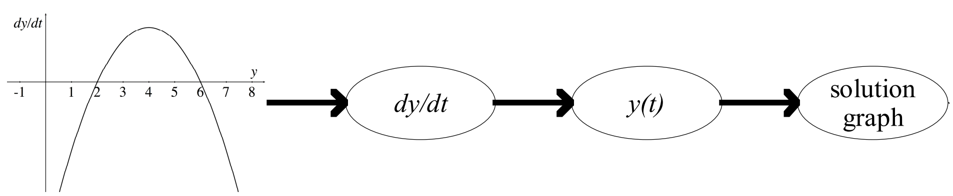


Figure 10: Diagram of Rich's reasoning for the nonroutine task

Rich's work to find the equation from the graph, shown in Figure 11, is noteworthy because of the steps involved. He followed the same exact procedure, in reverse, that he used three weeks earlier in his work to sketch the graph of the prompt $dy/dt = y^2 - 2y$. This is significant because it shows how important the steps are for him to make the connection. He continued, questioning whether the equation he generated matched the given graph saying, "wouldn't [$dy/dt = y^2 - 8y + 12$] be a parabola opening upwards?" He then worked for 4 minutes to try to resolve this conflict. Ultimately, he settled on the correct equation $dy/dt = -\left(y^2 - 8y + 12\right)$ but he was very unsure. He said, "I am going to keep it for now, …but I am not sure." Rich continued to work along the path he had described, but was not able to solve this problem. He was stymied in his attempt to use separation of variables.

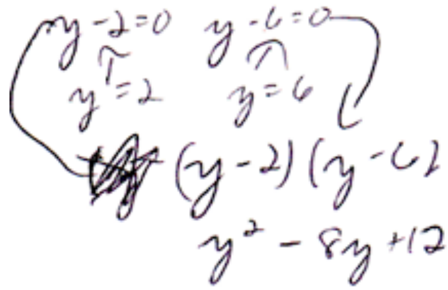


Figure 11: Rich's work to find an equation for the graph given in the task

The Nature of Rich's Understanding

In the first two episodes Rich applies familiar procedures to make connections between the graphs and their equations, but nothing in his work suggests he conceives them as differential equations that can be solved. He gives no indication that dy/dt is the derivative of the function y, or that y may represent a function. These procedures are familiar to him because they are standard ways to make connections between different representations of functions. In interview 3 the problems specifically asked for solutions and although he showed some understanding of how to go about getting a solution, he was unable to make the connections necessary to find a solution.

Discussion

The analysis of the interviews reveals patterns in both participants' thinking about FODEs, but the characteristics of the patterns are very different. In a broad sense the difference is captured by Skemp's (1978) distinction between instrumental and relational understanding, i.e. "rules without reasons" and "knowing what to do and why," respectively, but this distinction is too coarse to provide insight. Analyzing the results through the lens of the theory of reification provides a finer description and shows that the participants understanding of the concept of function accounts for the differences observed.

The model shown in Figure 1 illustrates the development of the concept of FODE from operational conceptions to structural conceptions. According to the theory of reification, operational conceptions of FODEs develop from lower level structural conceptions. In the initial description of this model the question was posed what is the nature of the object that serves as the basis for interiorization of the process of solving FODEs? The preceding analysis of Hassan and Rich's work offers insight.

Rich's work revealed his conceptions of FODEs as functions. The same implicit evidence is seen in Hassan's work, but Hassan addressed the issue directly in his work on the card-sort. Making the connection between FODE $\frac{dy}{dt} = y^2 - y$ and its graph Hassan explained, "*dy/dt* is just like it's own variable. … It is a function of itself but I just don't think about that. …So $\frac{dy}{dt} = y^2 - y$ is like $y = x^2 - x$ and that's what that graph looks like." The question can be raised whether or not this characterization of his thinking applies more generally that the situation he described, the evidence suggests it can. Looking back at his work on the linear prompt described earlier, Hassan found the equation for the graph by noting the slope of the graph was -1 and "*b* is 2". So as was the case for Rich, Hassan applied general knowledge about linear functions to find the equation. Similar examples are evidenced throughout his work.

Hassan's view of FODEs as functions adds to the structure of his understanding described earlier. In addition to conceptions of FODEs as equations to be solved, and as derivatives, at times he conceives them as functions. This added structure is shown in Figure 12. The differences between Rich and Hassan's conceptions of FODEs as functions can now be addressed from the perspective of the theory of reification. This analysis reveals that Hassan's conceptions are structural and Rich's are operational.

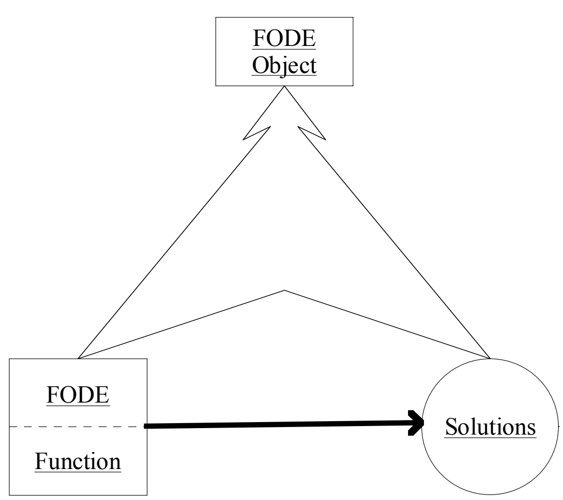


Figure 12: Structure of Hassan's understanding modified to include conception as function

The first characteristic of structural understanding is "being able to recognize the same concept under many different disguises" (Sfard, 1992, p. 76). A FODE is a function in disguise and both Rich and Hassan display the ability to conceive it as such. They both use this view to find equations for graphs and vice-versa.

The second characteristic of structural understanding is "various representations of the concept [are] semantically unified by this abstract, purely imaginary construct" (Sfard, 1991, p. 20). The evidence that Hassan conceives functions as "abstract objects" is the evidence that supported the claim that he saw graphs and equations as different representations of the "same thing." This is exact type evidence Sfard uses to make a case for abstract objects in her discussion of abstract objects with Thompson (Sfard & Thompson, 1994). Sfard argues that "being able to make smooth transitions between different representations means that there is something that unifies these representations… it is neither the formula, nor the graph – it's an abstract being" (p. 13). Rich's work does not share the same characteristics; he was unable to make changes among representations smoothly and was generally uncertain if he had done so correctly.

The third characteristic of structural understanding is "being able to recognize the idea 'at a glance' and to manipulate it as a whole without going into details" (Sfard, 1991, p. 4). For Hassan the details are in the background, they are secondary conceptions, but the same is not true for Rich. Rich's work shows that the steps were important to perform.

Hassan's structural view of functions distinguishes his work from Rich's. The fact that Rich did not discuss the notion of solution in the first two interviews is consistent with reification theory's model of cognitive growth. In order to begin interiorizing the process of solving the input ODE must be conceived structurally. Recall Sfard's (1991) description of interiorization:

> At the stage of interiorization learner gets acquainted with the processes which will eventually give rise to a new concept… These processes are operations performed on lower level mathematical objects. Gradually the learner becomes more skilled at performing these processes. …we would say that a process has been interiorized if it can be 'carried out through [mental] representations', and in order to be considered, analyzed and compared it needs no longer to be actually performed. (pp. 18-19)

Hassan's work shows that he has interiorized the concept. Recall he naturally formed groups based the concept of solution without the need to make calculations, but the calculations were essential for Rich and ultimately got in his way. The hierarchical importance of the concept of function for the development of FODEs is captured in the diagram in Figure 13.

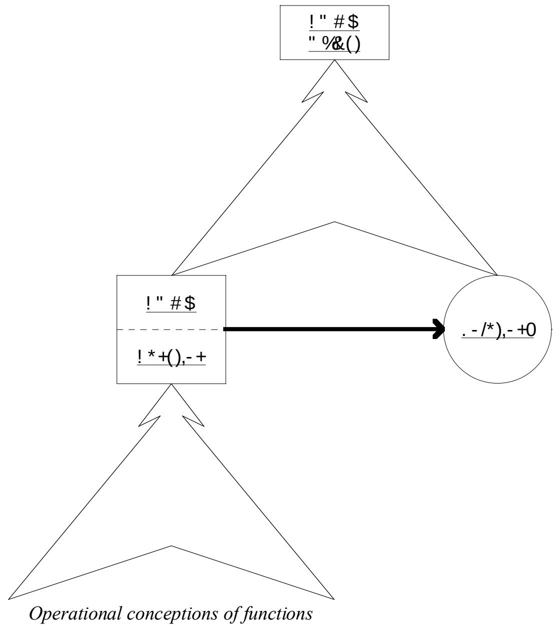


Figure 13: Reification diagram illustrating the dependence of FODE processes on functions

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