A Framework to Describe the Solution Process for Related Rates Problems in Calculus

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Abstract

Related rates problems are a source of difficulty for many calculus students. There has been little research on the role of the mental model when solving these problems. Three mathematicians were observed solving three related rates problems. From the examination of their solutions, a framework for the solution process emerged. The framework is based on five phases: draw a diagram, construct a functional relationship, relate the rates, solve for the unknown rate, and check the answer for reasonability. Each phase can be described by the content knowledge the problem solver accesses, the mental model that is developed, and the solution artifacts that are generated.

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Three mathematicians were observed solving three related rates problems. This data was analyzed to develop a framework for solving related rates problems. It was found that the mathematicians identified the problem type as a "related rates problem" and then engaged in a series of phases to generate pieces of their solution. These phases were identified as: draw and label a diagram, construct a meaningful functional relationship, relate the rates, solve for the unknown rate, and check the answer for reasonability. To complete each phase and construct a piece of the solution, the mathematicians built and refined a mental model of the problem situation. The framework captured how the solution to the problem emerged from the mathematicians' thinking as they solved to related rates problems.

Background

Related rates problems emerged historically as a means to reform the manner in which calculus was taught. Rev. William Ritchie (1790-1837) was a pioneer in mathematics education; it was his desire to write a reform calculus text that was more accessible to ordinary, non-university students (Austin, Barry, & Berman, 2000). In Ritchie's text, related rates problems were meant to be fundamental, explanatory problems that illustrate the power of calculus. However, in today's calculus classroom, related rates problems are often given only a cursory treatment and are sometimes even omitted from the curriculum. Most traditional calculus textbooks provide the student with a list of steps that should be followed to solve such problems. For example, at the beginning of the section on related rates in his calculus textbook, Stewart (1991) stated:

> In a related rates problem the idea is to compute the rate of change of one quantity in terms of the rate of change of another quantity (which may be more easily measured).

> The procedure is to find an equation that relates the two quantities and then use the Chain Rule to differentiate both sides with respect to time. (p. 160)

Stewart's description implies that these problems are easily solved in two quick steps. Two pages later, he outlines a strategy for solving related rates problems that involves seven steps:

1) Read the problem carefully.
2) Draw a diagram if possible.
3) Introduce notation. Assign symbols to all quantities that are functions of time.
4) Express the given information and the required rate in terms of derivatives.
5) Write an equation that relates the various quantities of the problem. If necessary, use the geometry of the situation to eliminate one of the variables by substitution.
6) Use the Chain Rule to differentiate both sides of the equation with respect to *t*.
7) Substitute the given information into the resulting equation and solve for the unknown quantity. (p. 162)

Martin (2000) conducted a study investigating students' difficulties with geometric related rates problems. In attempting to understand students' difficulties with these problems, Martin broke down the procedure for solving them into seven steps which are closely related to those presented by Stewart (1991). She classified these steps as either conceptual or procedural as outlined in Table 1 below. In her study, Martin found that the problems that appeared to be the easiest for students were the ones that required only the selection of the appropriate geometric formula, differentiation, substitution, and algebraic manipulation. The most difficult questions were those that required Step 7, solving an auxiliary problem. She also indicated that the conceptual steps are more difficult for students than the procedural ones. However, Martin concluded that students' poor performance on these types of problems was linked to difficulties with both procedural and conceptual understandings.

Table 1: *Martin's Step Classification*

Step	**Description**	**Classification**
1	Sketch the situation and label variables	Conceptual
2	Summarize the problem and identify given and requested information	Conceptual
3	Identify the relevant geometric formula	Procedural
4	Implicitly differentiate the geometric equation	Procedural
5	Substitute specific values and solve	Procedural
6	Interpret and report results	Conceptual
7	Solve an auxiliary problem (e.g. solve a similar triangles problem before attempting to use the volume-of-a-cone formula to relate the variables)	Varies

The research to date suggests that students have a procedural approach to solving related rates problems (Clark, Cordero, Cottrill, Czarnocha, Devries, St. John, Tolias, & Vidakovic, 1997; Martin, 1996, 2000; White & Mitchelmore, 1996). It has also been reported that students' difficulties appear to stem from their misconceptions about variable, function, and derivative – particularly the chain rule (Carlson, 1998; Clark et al., 1997; Engelke, 2004; White & Mitchelmore, 1996). The ability to engage in covariational reasoning allows a problem solver to construct a mental model of the problem situation that may be manipulated to understand how the system works (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Saldanha & Thompson, 1998;

Simon, 1996). The ability to engage in transformational and covariational reasoning appears to be critical for success when solving related rates problems (Engelke, 2004). The purpose of this study was to develop a framework to describe how a mental model for a related rates problem developed during the solution process.

To describe how a mental model emerged during the solution process, Carlson and Bloom's (2005) multidimensional problem solving framework provided an initial structure with which to examine the data. Their framework describes four phases of the problem solving process: orienting, planning, executing, and checking. They observed that in each phase the problem solver accessed resources and heuristics. In addition, the problem solvers cycled through the phases of plan, execute and check throughout the problem solving process. Using these ideas, it was the researcher's goal to identify distinct phases and the related content knowledge and heuristics that were used when solving a related rates problem. It was also desired to describe how content knowledge and heuristics informed the development of the problem solver's mental model of the problem situation.

The Study

Three mathematicians were asked to solve three related rates problems in a think aloud problem solving session which lasted about 45 minutes. Each mathematician was given a sheet of paper with the three problems printed on it and asked to work through the problem aloud and to attempt to verbalize everything they were thinking. The interviewer asked clarifying questions about their statements when additional information was needed. Each session was videotaped and transcribed for analysis. Discourse analysis was used to analyze the mathematicians' solution processes. Video recordings captured the problem solver's gestures which were particularly important to be able to identify how the problem solver used and referenced information he had written as part of the solution process. Particular attention was given to how they talked about what they were imagining and how they talked about the diagrams they drew and modified throughout the solution process. Because it is not possible to see what an individual is imagining in his mind, these are the data that allow one to best conjecture how the mind is interpreting and modeling the problem situation. In watching the video, it became apparent that what the mathematicians wrote down played an integral part of their solution process. Each written piece of written information was coded with the transcript and called an artifact. As the data analysis progressed, solution artifacts were defined to be written pieces of information that the problem solver used as an additional resource during the problem solving process to further develop his mental model of the problem situation.

Results

Five phases were identified in the related rates problem solving process: draw a diagram, construct a functional relationship, relate the rates, solve for the unknown rate, and check the answer for reasonability. During the first three phases, the mathematicians were observed talking about the changing quantities in the problem situation while referencing and modifying their diagrams. From these statements, conjectures about how their mental models of the problem situation developed and a framework to describe this process emerged. Due to space constraints, we will examine just one mathematician's solution to one problem.

One of the related rates problems the mathematicians were asked to solve required the problem solver to create a powerful mental model as the geometric figure in the problem situation does not have a commonly known volume formula. The problem was stated as follows: Coffee is poured at a uniform rate of 20 cm^3/sec into a cup whose inside is shaped like a truncated cone. If the upper and lower radii of the cup are 4 cm and 2 cm, respectively, and the height of the cup is 6 cm, how fast will the coffee level be rising when the coffee is halfway up the cup?

After reading the question, each of the mathematicians drew a diagram of the problem situation. This diagram was then usually labeled with variables and constants. However, before drawing the diagram, there is evidence that the mental model may have already been revised one or more times. For example, Adam began the solution process with an incorrect interpretation of the problem statement as see in the transcript below (transcript lines have been numbered to facilitate the discussion):

Adam:

1. All right, well this is my kind of problem. Coffee is poured at a uniform rate of 20 cubic cm per second into a cup whose inside is shaped like a truncated cone.
2. Oh, this must be one of those horrible, uh, um hotel things, right? They have the little plastic deals with the little conical cups and that always signals bad coffee. So, cause they always make it much too weak, you know. So there you are in this windowless ballroom, drinking out of this flimsy [hand motion]…yeah, I know what you're talking about. All right, a truncated cone.
3. All right, upper and lower radii of the cup are 4 and 2.
4. Um, this makes a difference. I don't know what the upper and lower radii means.
5. Do you mean is the cup 4 inches wide? so it's a shallow cup? or a tall narrow cup?

INT: The cone has actually been truncated so the bottom doesn't actually exist.

Adam: Oh, oh, oh, I see. I'm still thinking about those horrible hotel cones. All right, so it's truncated like this. All right, gotcha, like that. [artifact-diagram]

Observe in Line 2 above that Adam's initial interpretation of the problem situation is of a whole cone and is related to his experience with coffee cups at conference hotels. In Lines 3 and 4, the phrase upper and lower radii appear to be in discordance with his current mental image. After the interviewer explained the truncated aspect of the problem, Adam adjusted his mental image and drew an accurate diagram of the problem situation. Thus, we have observed Adam's mental model develop from an inaccurate interpretation of the problem statement to an accurate interpretation. In addition to the verbal statements that Adam made, he drew a diagram. Adam's diagram of the problem situation was the first solution artifact that he created. This diagram was further modified to include the labeling of constants and variables as Adam completed the first phase of the problem solving framework: draw a diagram.

Adam then shifted his focus to the second phase of the framework, construct a meaningful functional relationship:

Adam:

1. I hope I'm getting these answers right. It's kind of embarrassing; you show this at my post tenure review, and I'll get fired.
2. Let's see. Ok, so now, we're going to pour coffee into this thing, and we want to know how fast the coffee is rising when the coffee is halfway up. So, this means we have to figure out first of all what the volume of this thing is. So what's the volume? The volume is, um, is like one half pi *r* squared *h*, or something like that [artifact-equation]
3. cause this is where you could give them the whole deal. There's a factor of 2 pi about the, you know. Oh, there's a factor of 2 pi here cause you do the little shells, and you rotate it around, right? Yeah. [hand motion] I don't remember it, Nicole. pi *r* squared *h*. Does that sound right? I will go on that assumption here. You know, I'm terrible at memorizing formulas.

Adam restated that he wanted to find how fast the coffee is rising when the coffee is half way up the cup which caused him to identify that he needed to know what the volume of the cup is. (Line 2) Thus, it would appear that the next step for Adam is to determine a formula for the volume of a cone. The volume of a cone formula is not readily accessible as part of Adam's content knowledge. In an attempt to reconstruct the volume of a cone formula, Adam appears to have accessed his knowledge of integral calculus and the shell method. (Line 3) Adam then spent a considerable amount of time trying to reconstruct the volume of a cone formula using fundamental calculus concepts.

After Adam decided on the formula for the volume of a cone, he proceeded:

Adam:

1. I'm glad it's not a timed test. All right. So, now we're going to slice this off. So, we have a cone. [artifact-modified diagram] Um, all right. So, now we have to figure out how much this is, and so we have to figure out the volume of the missing piece. So, what I'm going to do here is find the volume of the whole thing minus the volume of

the missing piece. If I took this thing down to the tip and then that will give me the volume of the remainder.
2. Maybe there's an easier way to do this problem. That's why we always have example 9, cause somebody else has actually thought about it, and so we can follow the pattern.
3. All right, let's see, uh is this going to get me anywhere? [check] As the radius decreases linearly, it goes from 4 cm to 2 cm in a distance of 6. Um, so that radius is going to be decreasing uniformly with height. So, um, if we went down another 6 cm, we'd have zero radius. So that would be the tip. So the whole the missing piece here would be a cone of height of 12 cm. So the volume of the missing, of the bottom is one half times pi times *r* squared, which is 2 squared times the *h,* which is 6. Um, all right. So this is going to be 3 times 4 times pi, I think, right. [artifact-volume of missing bottom]
4. And then the volume of the, the volume of the whole thing, that's the volume that's in the bottom. The volume of the complete cone is one half times pi times uh 4 squared times 12. Um, so this is 6 times 16 is 96 pi. [artifact-volume of complete cone]
5. All right, the volume of the cup is the difference between the two. So we are 86, 84 pi [artifact-volume of cup] that's a pain, let's see [25:31]

Adam found the volume of the actual coffee cup by computing the difference between the volume of the entire cone and the volume of the piece of the cone that does not really exist. As in the trough problem, these calculations appear to allow Adam to better understand what is happening in the problem situation.

After completing these calculations, Adam reconsidered how he should be thinking about the problem situation.

Adam:

1. All right now, ok. So now actually what I need to think about here is, um, should I think about height from the, um, the imaginary tip here? Or should I, should I start thinking about height from here on up? And I think about height from the bottom of the imaginary tip and hope that I made the correct adjustment when I get over here
2. Um, cause all you want to know is the rate of the change in the coffee. So whether I measure from here or from down here, um, that doesn't make a whole lot of difference. So, that's important, ok.
3. Let's see. Um, yep, ok. Now in this case, since we have a cone, the volume depends upon the height. So in this particular case then, the radius is, um, so the radius… We're going down here what, 2 cm in the space of 6 cm. So it's changing at the rate of um one third cm per cm, so the slope is a third. So that means that this is one half times pi times one third *h* [artifact-volume equation]
4. Ahh, I'm looking back to worry about whether to measure from my imaginary tip, or whether I'm going to start measuring from um the actual bottom. Um, what do I want

to do here? This is much, boy this is, there must be a slicker way of doing this in the book cause it would be too hard for the students to be talking about, well, measuring from the tip or here or whatever. Um, so what should I think about? I'm going to go with the imaginary tip. So this is one third *h,* zero is here, 6 is here, and 12 is here. [artifact-modified diagram]

Adam was concerned about how he should measure the height of the cone. (Line 1) This lead him to the revelation, "cause all you want to know is the rate of the change in the coffee so whether I measure from here or from down here um that doesn't make a whole lot of difference so that's important," and decided that he would measure from the imaginary tip of the cone. (Lines 2, 4)

In the middle of deciding how he should measure the height, Adam also noted that the volume depends on the height. (Line 3) To relate the volume and height, he observed that, "so the radius we're going down here what 2 cm in the space of 6 cm so it's changing at the rate of um one third cm per cm so the slope is a third so that means that this is one half times pi times one third *h*" and modified his volume equation so that it was in terms of height. He constructed a linear relationship between the radius and the height of the cone and used composition to express the volume in terms of height. Thus, it would appear that his content knowledge of functions was accessed to allow him to relate the volume and the height of the cone. Adam also commented that an example in the book would likely be helpful and provide a "slicker" way to solve the problem. (Line 4) However, this resource was not available to him in this session.

He continued:

Adam:

1. So this is what I would do in my office, and then I would think of a slicker way of doing it. Then it would all look nice for the students. That's why we always look like geniuses in front of the black board, cause ok.
2. Um, so do I have to worry about my offset here? Maybe not, ahh. I'm worrying about what this offset means here. I'd have to offset the bottom, ehh. I'd have to worry about, see what I'd have to worry about is this, uhh, maybe. Let's see. I have to remember to subtract 12 pi from everything. I don't know. What would I do?

3. If I were to think out loud about this, I would say, ok my guess is it doesn't matter. See, because if you had a, if you had a really truly conical coffee cup, not one that was lopped off here, you could just as well imagine. So instead of worrying about, so what I would tell the students is, I would say, well instead of worrying about this kind of cup, um, think about this kind of cup. Just imagine that you already filled it up with coffee to here, um, because who cares about the total volume. All you want to know is how fast the coffee is rising when you're at this volume. So you're, it doesn't really [hand motion] So it's the same answer. Getting the students to see that is perhaps one of the things that distinguish the expert like me from them.
4. Of course, I could be getting all the answers wrong, and you'll be laughing at me later on in the cutting room. Look at this professor who can't even do the problem. Ok, all right. So, I'm going to ditch this stuff. I'm just going to look at this one. [indicates diagram in middle of paper]
5. And um, all right. So now, um, in the cup, all right. So this, for that reason I'm just going to look at this long cup. And so now, I'm going to measure height from the tip of the cup. So this is *h,* and here is *h* equal to zero, and here is going to be *h* equal to 12. [artifact-labeled diagram] All right. So halfway up the cup then is going to be, I'm going to worry about the point where *h* equals 9, because that's the point where the height in my imaginary cup corresponds to halfway up the actual cup. [artifact-modified labeled diagram]

As Adam continued to struggle with how to measure the height of the cone, he appeared to have a revelation about how the problem situation worked. (Line 2) He then stated how he would it explain it to students, "well instead of worrying about this kind of cup um think about this kind of cup just imagine that you already filled it up with coffee to here um because who cares about the total volume all you want to know is how fast the coffee is rising when you're at this volume." (Line 3) He realized that it did not matter if the cone had been truncated. Adam's mental model appears to have undergone several changes as he developed his formula for the volume of the cone and what each variable meant in terms of the problem situation. It would seem that each time he played out the situation in his mind he gained a little insight until he finally understood how the situation worked. It is at this point that Adam completed his step of construct an algebraic relationship between the variables. An important part of completing this step was that he understood exactly what the relationship between the variables was. (Line 3) Thus, his algebraic equation had meaning in the context of the problem.

Adam then proceeded to simplify his volume equation as is seen in the following excerpt.

Adam:

All right, um, ok. So we're going from a radius of 4 at the top to a radius of zero at the bottom over, over a space of 12 cm. So that means that the radius changes, uh, as the height over three, and that's what we're going to go with. So this is one half pi. Let's see. This is a kind of small piece of paper. So, all right, so this is one half pi times the radius, which is one third *h,* all squared, times *h.* [artifact-volume equation $V = \frac{1}{2}\pi\left(\frac{1}{3}h\right)^2 h$] Um, so we get a ninth times a half. So this is 18. So this is pi over 18 times *h* cubed. That's the volume of my reshaped cone here, [artifact-volume equation $V = \frac{1}{18}\pi h^3$]

After simplifying his volume equation, Adam continued to the next phase of relate the rates:

Adam:

And, um, we're interested in *dh dt.* So *dv dt* is *dv dh* times *dh dt.* [artifact-general chain rule form equation $\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$] Um, and that means that *dv dt* is 20 cm per second cubed, cm per second, and *dv dh,* uhh is what it's going to be, pi over 9 times *h* squared times *dh dt.* [artifact-differentiated equation $20 = \frac{\pi}{9}h^2 \cdot \frac{dh}{dt}$]

Adam identified $\frac{dh}{dt}$ as the rate he wanted to find and then expressed a relationship between the rates using the chain rule. He did not operate on the volume equation he constructed. Rather, he expressed a relationship between the rates using the chain rule and then associated each rate with a numeric value or a function.

Adam completed the final phase of the framework, check the answer for reasonability, by performing a unit analysis:

Adam:

(1) All right. So when you're halfway up the cup, that's when *h* is equal to 9. So we have pi over 9, times 9 squared, times *dh dt* equals 20 [artifact-substituted in values into differentiated equation]

(2) And this is all going to be cubic cm per second because you have 9 squared, 9 square cm here, and we have cm per second here. So, our units are consistent. [check]

(3) Um, all right. So this 9 cancels one of these 9s. So I have 9 pi times *dh dt,* and so I would say then that the height *dh dt* is 20 over 9 pi, uh, cm per second [artifact-solution]

(4) Looks right. So, the reason I'm worried about the units here… let's see. We have 20 on the left, here we have cubic cm per second, over here we have squared cm, and we have cm per second. Um, so when I do the division here, I have a one over cm squared so over here, and therefore my answer here is in terms of cm per second, which is consistent with the idea of the height rising. [check] Um, so that's the answer, I think.

After he substituted the known values into his differentiated equation (Line 1), Adam quickly obtained an answer to the problem. (Line 3) He performed a unit analysis to check his answer for reasonability. (Lines 2, 4)

In Adam's solution to the coffee cup problem, he frequently referenced and interacted with the numerous diagrams he drew. This suggests that Adam is regularly manipulating and modifying his mental model of the problem situation.

An interesting aspect of Adam's solution is that even though he expressed the volume equation in terms of height, he did not operate on the function he constructed to relate the rates. Instead, when he carried out the relate the rates step, he wrote out the chain rule as $\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$. He then proceeded to associate each rate, such as $\frac{dV}{dh}$, with either the appropriate numeric value or function. He did this when solving the other problems, too. In contrast, the other mathematicians in the study thought of each variable as a function of time and implicitly differentiated their function with respect to time.

In solving the coffee cup problem, each phase generated solution artifacts. Adam drew and labeled diagrams that were frequently accessed and used to further develop his ideas about the problem situation. He also constructed and combined algebraic relationships as needed to relate the volume and the height. After relating the volume and the height, he differentiated his algebraic relationship with respect to time using the chain rule and known values were substituted into this equation. Adam then performed algebraic manipulations to yield a solution which was checked for reasonability.

Conclusions

While a textbook may provide a list of about seven steps to follow when solving a related rates problem, most mathematicians appear to condense this to four or five steps which may be described as phases. These phases include: draw and label a diagram, construct a meaningful functional relationship, relate the rates, solve for the unknown rate, and check the answer for reasonability. The completion of each of these phases is influenced by the problem solver's ability to access mathematical content knowledge to construct a meaningful mental model of the problem situation. As this content knowledge is accessed and the mental model is developed, the problem solver creates solution artifacts (written pieces of information that result from analyzing the problem) to represent his understanding of the problem. While a problem solver may write down many things, the solution artifacts are the written pieces of information that become an additional resource for the problem solver and frequently help to advance the solution process. Table 2 below describes what solution artifacts are likely generated at each phase of the solution process. It also describes what content knowledge (geometry, function, derivative, etc.) and related heuristics may be accessed to develop the mental model and create solution artifacts. It should be noted that these phases are not necessarily hierarchical. While it is likely that one will proceed in this order, it is possible that certain aspects of one phase may be carried out during another phase.

Table 2: *A Framework for the Solution Process for Related Rates Problems*

Phase	Solution Artifacts	Mental Model	Content Knowledge and Related Heuristics
Draw a diagram	• Diagram that accurately represents the problem situation • Diagram that has been labeled with constants and variables • Other diagrams representing different perspectives of the problem statement	• Describe what one is imagining or picturing in one's mind • Anticipate relationships that may exist • Attend to the nature of the changing quantities o Attend to the direction of the change in the variables o Attend to the amount of change in the variables o Attend to the average rate of change in the variables o Attend to continuous changes in the variables	• Restate the problem or parts of the problem in one's own words • Geometry o Ask or consider, "What is a ______?" o Accurately interpret terminology o Ask or consider, "Which perspective of the geometric shape will provide the most information?" o Ask or consider, "Do I need to draw more than one perspective of the problem situation?" o One diagram may represent any of the possible states of the problem situation • Variable o Label constants and variables appropriately
Construct Meaningful Functional Relationships	• Algebraic equation (s) to relate the variables in the diagram	• Imagine the problem situation changing • Identify useful relationships between variables • Modify the mental model to determine which variables need to be related	• Understanding the nature of functional relationships o Relate the variables representing the known rate and the unknown rate o Eliminate variables if possible o Use a diagram labeled with variables and constants to identify relationships o Understand the role of the independent variable and the dependent variable in a functional relationship o Understand what relationship between the independent and dependent variables is determined by the phrase "in terms of" o Understand that function composition (or substitution) allows one to construct a new function from two or more smaller functions, eliminating one or more variables
Relate the Rates	• Differentiated algebraic equation • Chain rule equation		• Understanding the nature of rate of change o Understand what relationship between the independent and dependent variables is determined by the phrase "with respect to" o Interpret from the given rate that time is the independent variable in the functional relationship o Imagine each variable in the functional relationship as a function of time o Differentiate the functional relationship "with respect to" time o Perform differentiation operations on an implicitly defined function

Phase	Solution Artifacts	Mental Model	Content Knowledge and Related Heuristics
Solve for the Unknown Rate	• Algebraic manipulations of the differentiated equation		• Algebraic Knowledge o Substitute in known values for variables o Apply algebraic operations to the equation to calculate the unknown rate
Check the Answer for Reasonability	• Notation of units • Other calculations	• Ask "Is this answer reasonable?" • Manipulate the mental model o Attend to the amount and direction of change in the variables o Compare the answer to another known quantity such as the average rate of change	• Measurement units • Perform a unit analysis, i.e. check that the units work out or match up

The first phase of draw and label a diagram is the same first step Martin (2000) identified. Draw a diagram was the second step listed in Stewart's (1991) calculus text book, after read the problem carefully. Stewart's third step, introduce notation by assigning symbols to all quantities that are functions of time, is also partly captured in this phase. Is it necessary to be thinking about the changing quantities as functions of time at this point in the problem solving process? The mathematicians appeared to think about the quantities as changing when they assigned them variables, but specifically thinking about them as functions of time usually occurred later in the problem solving process (usually in the relate the rates phase).

The second phase of construct a functional relationship has aspects of Martin's third and seventh step (identify the relevant geometric formula and solve an auxiliary problem if necessary) and Stewart's fifth step (write an equation that relates the various quantities in the problem and use the geometry of the situation to eliminate variables if necessary). The mathematicians identified the given rate and the unknown rate to determine that they needed "one variable as a function of another variable." Thus, they used their content knowledge of geometry to identify appropriate relationships and eliminate variables if necessary by applying function composition.

The third phase of relate the rates captures Martin's fourth step (implicitly differentiate with respect to time) and Stewart's fourth and sixth steps (express the given and unknown rates in terms of derivatives and use the chain rule to differentiate with respect to time, respectively). For the mathematicians, it was because they were able to think about the variables in the functional relationship as functions of time that allowed them to successfully apply the chain rule or implicit differentiation. The mathematicians used the phrase "with respect to" to restate both the known and unknown rates in the problem. It seemed that this was an extremely powerful tool for the mathematicians as the phrase appeared to help them identify the independent and dependent variables in the problem. Thus, they were able to compute the rate of change of each variable "with respect to time" using the chain rule.

The fourth phase of solve for the unknown rate captures Martin's fifth step (substitute in values and solve) and Stewart's seventh step (substitute in values and solve). This step did not cause any difficulty for the mathematicians aside from a few algebraic errors.

The final phase of check the answer for reasonability may be considered part of Martin's sixth step, interpret and report results. Stewart does not direct the problem solver to check his solution. This step was an inherent part of the mathematicians' solution process, and they also checked the results of many of their steps along the way. Performing checks after each step had been carried out allowed the mathematicians to catch errors before moving on in most cases.

Figure 1 below illustrates the cyclic nature of the problem solving process. It begins with the problem solver reading and interpreting the problem statement. As part of the interpretation, the problem solver begins to access the resources available to him. The result of interpreting part of the problem situation is a solution artifact.

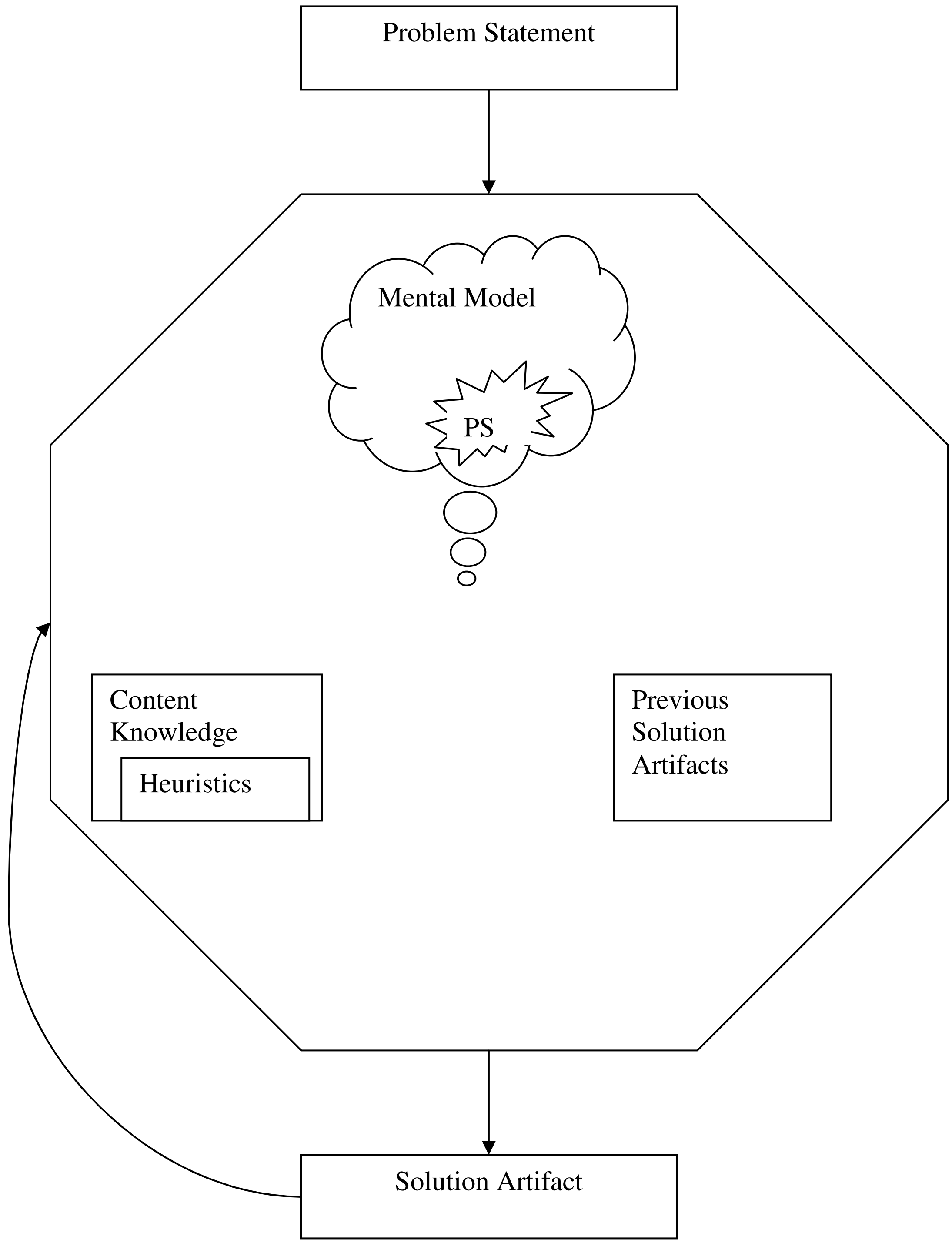

Figure1: The Solution Process for Related Rates Problems

The top box labeled problem statement represents the physical statement of the problem situation on a piece of paper or in a book. The process of finding a solution to a given problem begins when an individual reads the problem statement. The octagon in the middle of Figure 1 represents what may be happening in the individual's mind as he constructs each piece of his solution. It represents the collection of resources available to the problem solver. Each of these resources may be accessed individually or in various combinations. As the individual reads the problem, he begins interpreting the words and constructing a mental model of the problem situation. It is not possible to see inside a person's mind and what he may be visualizing, but it is possible to make conjectures about his mental model based on statements he makes. Thus, the mental model of the problem situation is represented by a thought cloud in Figure 1. An integral part of the mental model is the individual's interpretation of the problem statement and this is represented in Figure 1 by the PS in the star shape. It would appear that the problem solver holds much of the problem statement information in his memory (either accurately or inaccurately), but may occasionally return to the original problem statement to check the accuracy of his mental model. Reading the problem statement also likely causes the problem solver to access additional problem solving heuristics, some of which may be specific to related rates problems and others more general. Furthermore, mathematical content knowledge may be called upon. The mind of the problem solver accesses heuristics and content knowledge and manipulates the mental model to produce a solution artifact. The third box in Figure 1 represents a written piece of the solution, a solution artifact, which is the result of the interactions in middle section of the framework. Solution artifacts appear to be generated as a result of completing one phase of the solution process. The problem solver then cycles back to the problem statement (either the original or the one represented by his mental model) and goes through the process again to obtain the next solution artifact. As each solution artifact is generated, it becomes an additional resource for the problem solver. It should also be noted that after a solution artifact has been constructed, the problem solver may choose to check that part of his solution process. This cycle continues until the problem solver reaches a solution.

Content knowledge of geometry, variable, function, and derivative is necessary at different points in the problem solving process. Knowledge of geometry is necessary to correctly interpret the words in the problem, draw an accurate diagram, and choose an appropriate relationship between the variables. Knowledge of the concept of function is necessary to apply composition to eliminate unnecessary variables and successfully apply the chain rule. Knowledge of the concept of derivative is necessary to successfully compute derivatives.

Heuristics that the problem solver chooses to employ appear to be closely related to content knowledge that has been accessed recently. For example, reading "how fast" in the problem statement may cause one to rely on the heuristic that "how fast" indicates we are talking about a rate. Many of the heuristics employed are likely related to how one interprets phrases from the problem statement and are difficult to discern from content knowledge. Thus, heuristics are indicated by a box that overlaps with content knowledge.

This framework could be used to guide the development of teaching materials for related rates problems. Alternatively, it may be used to analyze any problem solver's solution process and highlight any misconceptions or difficulties. In my own work, I used the framework to analyze the results of a related rates teaching experiment. By using the descriptions in the framework, I was able to compare and contrast how the students and mathematicians approached related rates problems. The results of the analysis using the framework allowed me to discover a fundamentally different approach to solving related rates problems based on a deep understanding of the chain rule that needs to be further investigated.

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