Calculus Students' Difficulties in Using Variables as Changing Quantities

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Introduction

The study of calculus requires an ability to understand algebraic variables as generalized numbers and as functionally-related varying quantities. Research indicates that students who have difficulty using algebraic variables in these advanced ways have difficulty in their performance in calculus (Carlson, Jacobs, Coe, Larson & Hsu, 2002; Gray, Loud & Sokolowski, 2005; Jacobs, 2002; White & Mitchelmore, 1996). In their investigation of students' understanding of the concept of derivative, White and Mitchelmore (1996) concluded that their subjects' difficulties with the derivative were attributable to an "under-developed concept of variable" (p. 91). In a previous report (Gray et al., 2005), we found that the cohort of students who had shown a high level of understanding of the variable as a varying quantity (Level 4), according to their scores on an algebra test (Sokolowski, 1997) adapted from a large-scale British study (Hart, Brown, Kerslake, Küchemann, & Ruddock, 1985; Küchemann, 1981) was the only group of students with a mean final grade of B- in Calculus I when compared with groups of students who had achieved only a basic (Levels 1 or 2) or a moderate (Level 3) level of understanding the variable. The mean grade of students in Levels 1 through 3 was D+ (Gray et al., 2005). These results are shown in Figure 1 below. Of the 174 calculus students in the study, 102 tested into Level 4, 52 in Level 3, and a total of 18 in Levels 1 and 2 combined. A more complete description of these levels of understanding as determined by the Algebra Test is provided in the Theoretical Framework for this paper.

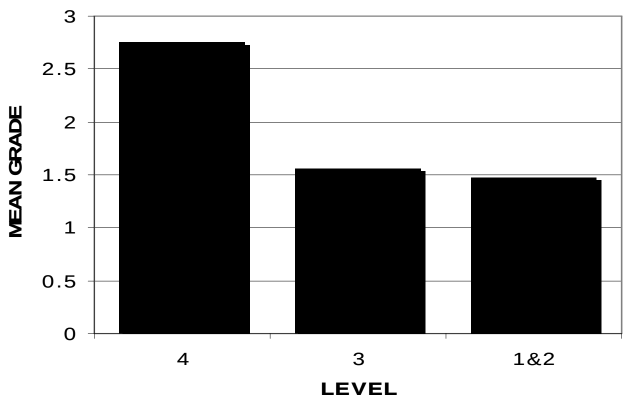

Figure 1. Level of Understanding of Variable vs. Mean Final Grade in Calculus I.

It must be noted that it is *mean* grades that are represented in Figure 1. There were many students who had achieved Level 4 status with respect to their understanding of the variable, but who did not earn high grades in their calculus course. However, no students who had achieved only Levels 1 or 2 understanding succeeded in calculus. Understanding variables as varying quantities is not a sufficient condition for student success, but it may be a necessary condition. In an effort to understand what types of difficulties students exhibit in this regard, this paper describes students' approaches to particular test items that showed poor success rates and required the use or interpretation of variables as generalized numbers or varying quantities.

Theoretical Framework

An operational – structural framework was used to categorize and describe students' responses to a selection of five problems from the Algebra Test that utilize variables as generalized numbers or varying quantities. Many researchers have written about algebraic concepts in this way (Dubinsky, 1991; Dubinsky & Harel, 1992; Jacobs, 2002; Kieran,1992; Sfard & Linchevski, 1994; Stacey & MacGregor, 2000; Tall, Gray, Bin Ali, Crowley, DeMarois, McGowan, et al., 2000; Trigueros & Ursini, 2003).

Sfard and Linchevski (1994) speak of operations, or processes, as a series of actions that ultimately is reified into a cognitive structure, or object, upon which new processes can be performed in order to form even more highly refined objects. They describe an operational viewpoint as one which takes students from arithmetic to “algebra of a fixed value” (p. 102), in which students see an algebraic expression as a sequence of commands to perform operations. Jacobs (2002) writes of students using a *calculational* approach, in which her AP/BC calculus subjects treated the “variable as a tool for solving an equation or finding an unknown value” (p. 203). Stacey and MacGregor (2000) use the term *arithmetic thinking* to describe this same kind of thinking in terms of operations that must be performed in order to arrive at a numerical answer.

According to Sfard and Linchevski (1994), a structural viewpoint would be manifested in “functional algebra” (p. 108), in which students see an algebraic expression as a reified object on which other operations can be performed. Jacobs (2002) describes students with a *conceptual* view as those who recognize that variables express mathematical relationships. She states that this conceptual view is “characterized by a concern for how a variable relates to its domain and how two or more variables relate to each other” (p. 204). Stacey and MacGregor (2000) refer to this as *algebraic thinking*, which allows students to view variables and expressions as structures of general representation. There is no urgency to find the numerical referents for the variables. Several researchers have developed their own models of interpreting their undergraduate subjects' actions based on these various interpretations of operational-structural cognitive development (Carlson et al., 2002; Jacobs, 2002; Trigueros & Ursini, 2003).

For this paper, the terms *arithmetic thinking* and *algebraic thinking* will be used to describe the operational or structural thinking, respectively, that was observed in students' work. The use of these terms most closely aligns with Stacey and MacGregor's descriptions, above. To clarify this terminology more specifically with regard to students' understandings and use of the algebraic variable, Küchemann's (1981) well-documented analyses of his Chelsea Diagnostic Algebra Test items (Hart et al., 1985) were utilized. Küchemann developed four hierarchical categories of students' understandings of algebraic variables, based on six uses of variables.

Arithmetic thinking would be roughly equivalent to Küchemann's Levels 1 and 2, very elementary ways of understanding and using the variable. As students first learn about algebra, they tend to *evaluate* variables, *ignore* them, or use them as *labels*. For example, in order to solve a simple equation such as $x + 3 = 5$, a student could easily find that the answer is $x = 2$ by replacing the x with values until the correct value is obtained; he could simply *evaluate* the variable. An example of a problem that requires only that a student ignore the variables would be: If $a + b = 43$, then $a + b + 2 = ?$ Here, the student need only recognize that 2 must also be added to the 43; the student need not actually work with the variables; they are essentially *ignored*. And anyone who has taken an elementary algebra course probably would recognize the use of variables as *labels* in an expression such as $5a$, for which the interpretation of "5 apples" is common, rather than the correct interpretation of "5 times the *number* of apples." This interpretation of variables as labels or objects, rather than as indicators of quantities, has proved to be a very resilient interpretation. When students must model word problems with algebraic equations, they often make errors based on an inappropriate use of variable as label (Gray et al., 2005).

Algebraic thinking would be roughly equivalent to Küchemann's Levels 3 and 4, more sophisticated ways of understanding and using the variable. Level 3 understanding appears to be a bridge between arithmetic and algebraic thinking and entails a fourth use of the variable, that of a *specific unknown*. When one solves an equation such as $3x + 5 = 4 - 2x$, one is seeking the *specific unknown*, represented by the variable, that will make the equation a true statement. The two most advanced uses of the variable are as a *generalized number* and as a true *varying quantity*, which are generally indicators of Küchemann's Level 4 understanding. Variables are used as *generalized numbers* when they are used to represent entire sets of numbers, such as when they are used to represent the commutative property of addition for Real numbers: $a + b = b + a$. Finally, in a function such as $y = mx + b$, two variables, x and y, are *dependently varying* with each other, while the other two variables, m and b, are not considered varying quantities, but are really parameters that may be used as specific unknowns in this case. It is these advanced uses of variables that are at once the most crucially foundational for understanding the major concepts of calculus and the most difficult for students to fathom.

Research Questions

The research questions that guided this study were as follows:

1. What are the success rates of entering calculus students on questions that use variables as generalized numbers or varying quantities?
2. What types of responses do students make to questions that use variables in these ways?

3. Is there a relationship between students' course grades and their performance on questions that use variables in these ways?

Method

An adaptation (Sokolowski, 1997) of the Chelsea Diagnostic Algebra Test (Hart et al., 1985) was administered on the first day of class to 174 Calculus I students at two private four-year colleges in New England. For this report, a selection of five questions was analyzed because their responses were likely to illustrate algebraic thinking and their success rates were relatively low. Success rates were recorded for each of the five questions. All responses were recorded and grouped according to whether they appeared to be illustrative of arithmetic or algebraic ways of thinking about the problem. In addition, a graph was prepared that illustrates the relationship between these Calculus I students' mean final grades and their performance on these five questions.

Results

In order to answer research questions 1 and 2, the success rates and types of responses on the five selected test items are described here. For each item, the most common errors are described, along with categorizations of the responses and errors according to an arithmetic – algebraic framework. The problems are numbered as they were on the Algebra Test.

Problem 3

"Which is greater, $2n$ or $n + 2$? Explain"

Correct response: "It depends. If $n < 2$, $n + 2$ is greater; if $n = 2$, $2n = n + 2$; if $n > 2$, $2n$ is greater."

This problem involves the comparison of two expressions, both using the same variable. There is a need to think of the variable as taking on a range of values while making this comparison. Seventy-nine of the 174 students (46%) answered this question correctly. Less than one-fifth of these correct answers were good general explanations as described above. About four-fifths of the correct answers used two or three numerical examples to support the conclusion that "it depends." Although these answers were technically correct, they indicate a tendency toward arithmetic thinking as opposed to the more general algebraic thinking implied by the simultaneous comparison of a range of values used to produce the general response above.

Of the 95 incorrect answers, four-fifths of the errors were "2n" with an explanation that "multiplication makes numbers larger" or with one or more numerical examples for support. This persistent general view that multiplication makes numbers larger than addition appears to indicate that these students are thinking only of natural numbers as the referents of the variable here.

In general, most students used natural number calculations or inequalities such as $n \leq 1$ or $n \geq 3$ that appeared to indicate that they thought of the referents of this variable as being natural numbers. Approximately nine percent of all responses included negative integers, but there were very few indications of a Real number domain.

Problem 16

"What can you say about c if $c + d = 10$ and c is less than d?"

Correct response: "$c < 5$"

In this problem, two variables are co-varying and must be compared in the context of one function under the constraint of an inequality. Only 57 students (33%) answered this problem correctly. The need to think of a set of values co-varying with a second set of values and the ability to extend these sets to a context of Real numbers are illustrative of algebraic thinking in this problem.

The incorrect responses include solving for c: $c = 10 - d$ or $c < 10 - d$, or giving a single number or list of numbers. These responses indicate an arithmetic view of the problem, a general inability to recognize the relationship between c and d in a more algebraic way. Many of these incorrect responses suggest that students were thinking only of integers between 0 and 10. It was decided not to allow a systematic list to serve as a correct response because the sample was a group of calculus students that should be aware of the Real number domain of these equations and inequalities. In addition, there were many errors in the use of mathematical symbols, most especially an inappropriate use of the inequality sign. For example, responses such as $0 < c < 4$ or $c \leq 4$ were used in ways that appeared to signal only integers, rather than Real numbers.

Problem 19a

"$a = b + 3$ What happens to a if b is increased by 2?"

Correct response: "a increases by 2"

Two variables are co-varying in an additive functional relationship in this problem. Sixty-eight percent (118/174) of the responses to this problem were correct. Although this does not appear to fit the criterion for inclusion into these analyses of problems with high error rates, the problem statement is so closely aligned with the problem that follows it on the Algebra Test that its results help to inform those of the subsequent problem.

Of the 56 incorrect responses, 14 were simply, “increases.” This could be considered a partially correct response, since the problem statement is somewhat vague. The remaining incorrect responses, however, indicated arithmetic thinking. Typical responses in this category were $a = b + 5$ or $a = (b + 2) + 3$ or $a + 2$. Students were adding 2 to the *b*, but were keeping the number separate from the variable. They were really not increasing the value of *b* by 2, but were carrying out their perception of the procedure of adding 2 to the *b*. These responses and others, such as $2b + 3$ or $a = 5$, appear to “apply” the 2 to the existing value of *b* or *a* and do not seem to take into account the fact that the two variables are varying.

<u>Problem 19b</u>

“$f = 3g + 1$ What happens to *f* if *g* is increased by 2?”

Correct response: “*f* increases by 6”

Here, two variables are co-varying in a multiplicative and additive functional relationship. Although this problem is quite similar to the previous problem, the results are almost exactly the inverse. Only 53 students (30%) responded correctly to this problem. Of these correct responses, it is unclear how many actually would be considered examples of algebraic thinking because several of them were the result of students having generated a table of values and then generalizing the results. Although the generation of a table of values appears to indicate arithmetic, point-by-point, thinking, the act of generalizing the results to produce an answer appears to have the seeds of algebraic thinking.

Twenty-five of the 121 incorrect responses were "increases." As above, these could be considered as partially correct. The most common incorrect responses indicated that *f* "increases by 7," "increases by 2," or by some other number. These and other responses that indicate an arithmetic view were $f = 7$, $f = 5g + 1$, or $f = 6g + 1$. Here, as in #19a, students appear to have applied the 2 to the variable by having performed some sort of calculation with the given numerical coefficient or by having substituted the 2 for *g*. These are all elementary, arithmetic ways to view this problem. The aspect of one variable changing in response to changes in the other variable seems to be lost.

Several responses stated that "*f* doubles." This does not take into account the algebraic description of the relationship between *f* and *g*. It also suggests that "increased by 2" was being translated as "doubling" by these students.

<u>Problem 21</u>

"If the equation $(x+1)^3 + x = 349$ is true when $x = 6$, then what value of *x* will make this equation, $(5x+1)^3 + 5x = 349$ true?"

Correct response: "6/5" or "1.2"

Only one variable is used in this problem. It is used as a specific unknown in both equations, but its value is different in the context of the second equation; thus it is necessary to see the structural similarity between the two equations and to compare the value of the variable in one equation with that in the other.

Only 27% (47) of the students responded correctly to this problem. The vast majority of these (32/47) showed just the answer with no other marks. This may be an indication that these students could see the structural relationship, $5x = 6$, and simply solved this mentally.

Of the 127 incorrect responses, 55 were blank. Although this problem was near the end of the test, most of these students completed the following few problems, so the blanks do not appear to signal a lack of time. These students appear not to have known where to begin on this problem. An additional 40 of the incorrect responses showed that these students attempted to expand the cubic expression or tried other numeric or algebraic manipulations. This indicates an operational or arithmetic approach to the problem. Some other typical incorrect numerical answers were 30, 1, 2, 3, 4, or 6. Since none of these had other marks on the test paper, they may have simply been guesses, but appear to indicate arithmetic, calculational approaches to the problem, rather than seeing the structure of the expressions and equating 5*x* and 6.

In order to answer the third research question, a graph (Figure 2) was produced that illustrates the relationship between Calculus I students' mean final grades and their performance when entering calculus on these five test items that use variables as generalized numbers or varying quantities.

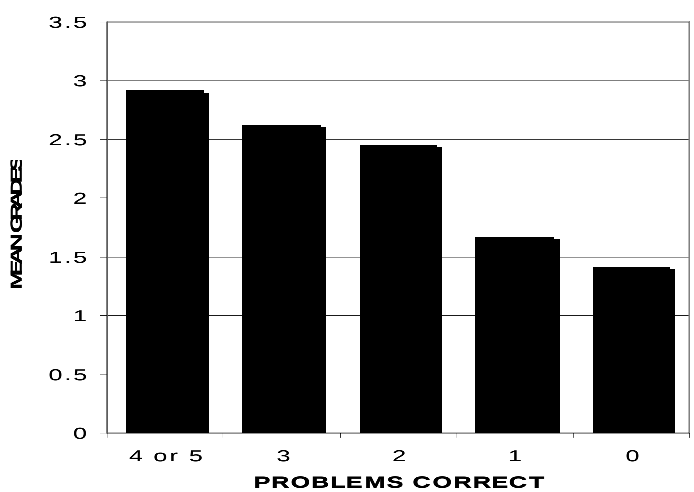

Figure 2. Number of Problems Correct vs. Mean Final Grade in Calculus I.

The numbers of students who fell into each of the categories of the bar graph, from left to right, are as follows: 4 or 5 problems correct, 31 students; 3 problems correct, 31 students; 2 problems correct, 47 students; 1 problem correct, 37 students; no problems correct, 28 students. The graph shows how students' success rates on these five Algebra Test items are related to their mean final grades in calculus.

Conclusions

The majority of these calculus students had difficulty using variables as generalized numbers and varying quantities. In general, the incorrect responses to these five questions suggest that students frequently used an arithmetic approach when an algebraic approach was needed. They tested one or more integer values to compare quantities in several problems when they should have considered and incorporated into their answers a broader range of values. For example, in Problem #19b, many students ignored or evaluated the independent variable in a point-by-point or static way. In addition, they unsuccessfully used symbolic manipulation to solve Problem #21, a problem that required a more structural, algebraic approach.

These results add to the body of evidence that calculus students have difficulty using variables as varying quantities. Trigueros and Ursini (2003) found that there was strong evidence among the 164 first year undergraduates in their study that the students did not understand the concept of variable when it is was used as a varying quantity in relational situations. Jacobs (2002) found in her study of AP/BC Calculus students that those students who had what she described as a "calculational view" (p. 205) of the variable were likely to have difficulty understanding the calculus concepts of limit and derivative. When her students worked with functions, they seemed to have an understanding of the variable as a varying quantity, but this view was not sufficiently robust as to translate to the contexts demanded by the study of calculus.

For all five of the selected Algebra Test items in this study, the over-whelming written evidence indicates that incorrect responses to these problems were the result of a very elementary, arithmetic way of thinking, not the more advanced, algebraic approach required in calculus. Students' success rates on these five test items were related to their mean final grades in calculus. The groups of students who answered only one or none of the items correctly had mean grades in the D range, while those who answered 4 or 5 items correctly had mean grades approaching B. Although there are myriad factors that contribute to students' success in calculus, students' difficulties using variables as varying quantities undoubtedly adds to their difficulties in understanding the concepts of calculus.

This study also revealed numerous examples of questionable use of mathematical symbolism, particularly related to inequality notation. Students did not realize that the inequality sign should be used to refer only to sets of Real numbers unless otherwise indicated. Evidence from responses to Problems #3 and #16, in particular, indicated that, when working with situations of inequality, these students rarely appeared to consider Real number domains for the variables. Their seemingly default thinking about referents of variables as being integers or natural numbers in inequalities could impact their understanding of limits in calculus.

Instruction in courses prior to calculus should include explicit attention to these issues of symbolism, domain, and the many different uses of variables. When students are attempting to learn the concepts of limit, derivative, integral, and the connections among these concepts crytallized in the Fundamental Theorem of Calculus, they must possess a robust and flexible view of all aspects of the changing quantities that form the foundation of their mathematical studies.

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