A Framework for Developing Algebraic Understanding and Procedural Skill: An Initial Assessment

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Abstract

This study examined the effectiveness of instruction based upon a Framework designed to promote deep procedural knowledge, which presumably facilitates recall and aids future learning. The matched-pairs design paired six college algebra instructors according to teaching experience. Students' SAT / ACT scores established the equivalence of treatment and control groups. Data consisted of classroom observations, homework samples, common hour exams, procedural understanding assessments, and interviews with treatment instructors. An ANCOVA revealed that treatment group students scored significantly higher than control group students on procedural understanding. Moreover, although treatment students were assigned fewer drill questions, there were no significant declines in procedural skill. Overall, students possessing procedural understanding exhibited greater skill, regardless of instructional approach. Interviews revealed implementation issues surrounding Framework-based instruction.

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Mathematics instruction often seeks to equip students with a well-organized tool chest of procedures that they can use to solve problems (Carpenter & Lehrer, 1999; Hiebert & Carpenter, 1992). Unfortunately, this goal is not often attained. Instead, many students develop cluttered collections of procedures that they do not understand and cannot apply. Moreover, knowledge that is not understood is easily forgotten (Hiebert & LeFevre, 1986), which many believe is the underlying reason for the high percentage of college freshmen enrolled in remedial mathematics courses, even among those who have completed several years of high school math (see, for example: Kranz, 2004; Schultz, 2000). Too many students fail to develop deep knowledge of procedures despite having learned to compute with them efficiently. At one time, they were able to perform well enough on tests of familiar procedural skills, but their knowledge was fragile, inflexible, and soon forgotten.

There is growing research support for designing classroom instruction that focuses on developing deep knowledge about mathematics procedures. Conceptual versus procedural knowledge is increasingly viewed as a false dichotomy (Burke, Erickson, Lott, & Obert, 2001; Star, 2005; Wu, 1999); rather, research suggests that conceptual and procedural knowledge support one another, with increases in one type of knowledge leading to advances in the other in a hand-over-hand fashion (Rittle-Johnson, Siegler, & Alibali, 2001). When instruction is focused only on skillful execution, students develop automated procedural knowledge that is not strongly connected to any conceptual knowledge network (Star, 2000). Without those connections, procedures are not executed “intelligently,” and systematic errors persist (Star, 2002). Understanding can be achieved, however, if students are given opportunities to develop appropriate relationships, extend and apply what they know, reflect on their experiences, and make mathematical knowledge their own (Carpenter & Lehrer, 1999).

When mathematical knowledge is understood, that knowledge is more easily remembered (Carpenter & Lehrer, 1999; Hiebert & Carpenter, 1992; Van Hiele, 1986) and more readily applied in a variety of situations (Hiebert & Carpenter, 1992; Kieran, 1992). Moreover, when a unit of knowledge is part of a well-connected network of mathematical understandings, parts of the network can facilitate recall (and even recreation) of other parts. Finally, when knowledge is understood it becomes easier to incorporate new knowledge into existing networks, so that current understanding facilitates future learning (Hiebert & Carpenter, 1992). It is therefore important to develop teaching methods that help students develop mathematical understanding.

The purpose of this study was to test an instructional framework designed to help students develop a deeper understanding of algebraic procedures. We believe this framework can lead students to develop knowledge of algebraic procedures that extends beyond simply knowing how to perform them. During one semester of college algebra, participating students from three sections were asked to predict or estimate answers in advance; discuss the goals of procedures; explain why (and when) certain procedures were effective; compare, contrast, and discriminate between related procedures, and learn how to choose among them; identify other areas of mathematics where the procedures would be useful; and perform other tasks designed to develop deeper procedural knowledge. By embedding such questions into the lectures, homework, and assessments used in college algebra, students in the treatment groups were exposed to questions (and answers) that presumably lead to richer, more connected views of mathematics.

Theoretical Background

National and state standards for mathematics education routinely call for students to "understand" the mathematics they are learning. For example, the Algebra Standard of the *Principles and Standards for School Mathematics* (*PSSM*; NCTM, 2000) recommends that all students should:

- *understand* relations and functions and select, convert flexibly among, and use various representations for them, …
- *understand* and perform transformations such as arithmetically combining, composing, and inverting commonly used functions, …
- *understand* and compare the properties of classes of functions, including exponential, polynomial, rational, logarithmic, and periodic functions (p. 296, emphasis added).

PSSM subsequently illustrates (via examples) how such understanding might emerge, but it does not define the term precisely. NCTM's *Navigating Through Algebra in Grades 9-12* (Burke et al., 2001), which was published shortly after *PSSM*, offers a list of six guidelines that define the "multi-modal nature of mathematical understanding (p. 31)." When these guidelines are used to help direct instruction, "conceptual versus procedural understanding becomes a false dichotomy and greater fluency … can be achieved" (Burke et al., 2001, p. 32).

As we investigated how these guidelines could impact classroom instruction, we found it useful to re-express them as a series of student-centered questions that we refer to as the *Framework for Procedural Understanding*. Expressed in this way, the Framework is a device that teachers can use to develop lessons, examples, problems, and assessments that are aligned with principles of learning mathematics procedures with understanding. The Framework is a flexible tool that can add depth to the lessons teachers already use. Its flexibility and portability make it easy to incorporate into any lesson and any course that has a focus on procedures.

Our adaptation of the Framework for Procedural Understanding is presented in Figure 1. We recommend having a specific procedure in mind while reading through the Framework questions for the first time. The procedure may be mathematical (e.g. using the quadratic formula or applying the quotient rule for differentiation) or non-mathematical (e.g. operating a microscope, using a GPS unit, or rotating the tires on your car). In each case, the Framework provides a clear definition of what might constitute "understanding" of the given procedure.

Figure 1 – The Framework for Procedural Understanding

1. (a) What is the goal of the procedure?
   (b) What sort of outcome should I expect?
2. (a) How do I execute the procedure?
   (b) What are some other procedures I could use instead?
3. Why is the procedure effective and valid?
4. What connections or contextual features could I use to verify my results?
5. When is this the "best" procedure to use?
6. What can I use this procedure to do?

It is worth noting that the conceptual underpinnings of the procedure (item 3) represent just one facet of procedural understanding. For instance, one could understand how to operate a microscope or navigate using a GPS unit without knowing about optics or how the GPS satellites pinpoint locations, but that information would certainly add to the user's understanding and would likely enhance their ability to use it. Similarly, one may know how to quickly identify the number of real solutions to a quadratic equation (item 1b, predicting the outcome) by examining the discriminant, even if they do not understand the full derivation of the quadratic formula. Each item in the Framework represents one aspect of what it means to understand a procedure, including simply carrying it out (item 2a). The goal of using the Framework is to increase the depth and connectedness of students' procedural knowledge.

For the present study, we used the Framework in several ways. First, we used it to operationally define procedural understanding: students who understand a procedure should be able to answer Framework-oriented questions, and vice-versa. We also used the Framework to guide instruction by incorporating Framework-oriented questions into daily lectures. Third, we selected Framework-oriented homework questions from among the bank of Writing in Mathematics questions available in each section of the Blitzer (2004) algebra text. Finally, we used the Framework as a guide for writing quiz questions that would reinforce the ideas introduced in lectures and homework assignments. In this way, students participated in a modified instructional program with an increased focus on procedural understanding, which was expected to lead to deeper, longer lasting, and more flexible knowledge of those procedures.

Methods

Research Questions

This research project implemented an instructional treatment designed to promote procedural understanding in college algebra and assessed its impact on students' procedural understanding and procedural skill. Specifically, the research questions for the study were:

1. Were there significant differences in students' performance on tests of procedural skill in college algebra between treatment and control students?
2. Were there significant differences in students' understanding of college algebra procedures between treatment and control students?
3. What were the treatment instructors' perceptions of the overall effectiveness of the instructional treatment?

Sample and Population

The study employed a quasi-experimental design using six intact sections of college algebra at Montana State University in Bozeman, a research-intensive land grant university located in the northern Rocky Mountains. College algebra is a three credit hour course offered every semester, and it is intended to provide "further development of algebraic skills through the study of linear, quadratic, polynomial, exponential, and logarithmic functions" (Montana State University, 2004). The Blitzer (2004) *College Algebra* text was the required textbook for the course. Because college algebra does not satisfy the university's core mathematics requirement, students who enroll in college algebra typically do so in order to prepare themselves for future mathematics or statistics coursework. These students are typically first-year college students who have not satisfied the prerequisites necessary for pre-calculus, calculus, or introductory statistics.

To enroll in college algebra, students are expected to have earned an SAT Math score of at least 530, an ACT Math score of at least 23, or satisfactory completion of a previous course in intermediate algebra. Students who do not meet one of those requirements but wish to enroll in college algebra are required to pass the math placement exam that is administered by the mathematics department at the start of each semester. Students' ACT or SAT scores were available for 85% of the students who enrolled, and we converted them into standardized *z*-scores and used them as a measure of prior math ability. For those students for whom both scores were available, we used the greater of the two standardized *z*-scores.

One week before the beginning of the semester, we recruited six college algebra instructors (out of 11 total) who had prior teaching experience, and all agreed to participate. In order to control for teaching experience, we grouped the six instructors into three matched pairs (Highly Experienced, Moderately Experienced, Less Experienced). One of the two Moderately Experienced instructors requested assignment to the control condition (citing time constraints), and the request was granted. We randomly assigned the treatment condition for the other two pairs.

The course supervisor for College Algebra, who was in charge of writing the common hour exams used to assess procedural skill, was included as one of the two Highly Experienced instructors; he was randomly assigned to the control condition. The other Highly Experienced instructor was an adjunct instructor with recent high school teaching experience. The Moderately Experienced instructors were both graduate teaching assistants (GTAs) seeking master's degrees in mathematics; they each had more than two semesters of prior teaching experience. The Less Experienced instructors were also GTAs seeking master's degrees, and both possessed only one semester of prior teaching experience.

Instructional Treatment

Three sections of college algebra received specialized instructional treatment based upon the Framework for Procedural Understanding. The treatment included Framework-oriented lecture content, homework tasks, and quiz items. Each of these items is discussed in turn below.

Lecture content. To prepare treatment instructors to incorporate the Framework into instruction, one researcher met with the three treatment group instructors every Monday morning of the semester to discuss the lessons for the coming week. Each meeting was centered around a packet of instructor notes developed by the researchers. In addition to the weekly homework set and the course supervisor's objectives and lecture comments (which were sent to all instructors), each packet included examples designed to demonstrate for the treatment instructors how the Framework might be used to deepen lecture content along one or two dimensions for each lesson. We encouraged the instructors to use these examples or to develop their own Framework-oriented content to implement in the coming week. Each packet concluded with a selection of Framework-oriented quiz questions, and instructors selected one or two questions to include in their weekly quiz. In this way, we encouraged instructors to adopt a more Framework-oriented manner of teaching, while ultimately leaving most of the day-to-day instructional decisions with the individual instructors.

Homework. Homework assignments for the treatment group were designed to reinforce the importance of Framework-oriented ideas. At the start of the semester, students from all sections were given a list of homework questions for the semester. The treatment students received a modified list that included an 18% smaller subset of the drill exercises assigned to other groups, with several additional Writing in Mathematics questions added per section (3.3 questions, on average). We identified these writing questions as closely aligned with the Framework-oriented objectives. Treatment instructors directed their students to complete the Writing questions before the other assigned problems in order to emphasize their importance and ensure that the questions would not be overlooked. Even with the added Writing questions, treatment students were assigned 8% fewer problems than controls.

Quizzes. In order to further reinforce the importance of Framework-oriented ideas, each weekly quiz in the treatment section included one or two Framework-oriented questions. These questions, which were provided for the treatment instructors in the weekly packet of instructor notes, often mimicked (and occasionally were direct copies of) one or more of the Writing in Mathematics homework problems assigned during the preceding week.

Data Sources

Procedural skill. Procedural skill was assessed using three common hour exams and one final exam, which were written by the course supervisor (a control instructor) without direct knowledge of the specific nature of the instructional treatment. The course supervisor agreed to write the exams with the intention of assessing computational skill. All college algebra instructors shared in the grading process, including those inexperienced instructors who did not otherwise participate in the study. Each instructor was responsible for grading a small subset of the exam questions across all sections to ensure consistency in grading.

Procedural understanding. Procedural understanding was assessed for students in all six sections using a series of Writing Tasks given as a series of three two-question quizzes at the end of the semester. The researchers wrote these six tasks so that each question would reflect one of the six Framework objectives (note: Framework items 1a / 1b and 2a / 2b were combined for these assessments). Content was taken from recent coursework whenever possible. Students were told in advance that these six tasks would be combined into a single quiz grade, and that they would receive full credit on each task as long as they made an honest effort to complete it. The tasks were photocopied by the researchers prior to their being returned to students. Later, task-specific scoring rubrics were developed for scoring these tasks, and one researcher (blinded to treatment condition) assigned a score ranging from "0 – No understanding" to "3 – High degree of understanding." We tested the scores for stability over time and for inter-rater reliability by having the researcher and a trained peer re-code 20% of the responses approximately four weeks after the initial coding. Correlations were high for both stability over time (r = .868) and between raters (r = .805). We concluded that the scoring rubric was a reliable instrument for assessing procedural understanding.

Fidelity of implementation. Classroom observations were conducted of all six sections during each of weeks two, six, and 10 of the 15-week semester. Trained observers worked in pairs to observe each lesson, independently tallying Framework-oriented lesson events. Immediately after observing the lesson, each observer reviewed his or her tally and independently assigned a holistic score for each of the eight dimensions of the Framework (1a, 1b, 2a, and 2b were considered distinct dimensions). A score of 0 indicated that evidence of the dimension was *Absent* during the lesson; 2 indicated *Infrequent* occurrences, 4 represented

Frequent occurrences, and 6 represented *Pervasive* occurrences. Scores for each pair of reviewers were averaged to compute the final scores for each lesson.

As a secondary measure of fidelity of implementation, homework samples were collected from all treatment sections during Weeks 3, 7, and 11. The course supervisor required all college algebra instructors to collect homework every week, so this phase of data collection was transparent to the treatment students. In addition to examining completion rates, one researcher coded students' answers to the Writing questions using a holistic scoring rubric. The coding scheme was designed primarily for the purpose of assessing the degree of attention and effort students put forth on the Writing in Mathematics homework tasks. Responses were identified as either *Terse* or *Verbose*, and as either *Incorrect / Incomplete* or *Essentially Correct*. We re-coded nine of the ratings to establish intra-rater reliability and obtained an agreement rate of $\kappa = .767$, which we interpreted as a substantial rate of agreement (Landis & Koch, 1977).

Results

Initial Group Equivalence

Across all sections, the mean standardized SAT / ACT scores were 0.02 for controls and 0.18 for treatment. Cohen's d was computed as a measure of effect size, and the difference ($d = 0.30$) represented a small effect in favor of the treatment students (Gliner & Morgan, 2000). Levene's test for equality of variances was not significant ($p = .27$), so equal variances were assumed for an independent samples t-test. The difference in SAT / ACT scores was not significant ($\alpha = 0.05$, $t(117) = 1.63$, $p = .11$), so we concluded that the treatment and control groups did not differ significantly in their pre-requisite mathematics knowledge.

Instructional Emphasis on Understanding

Table 1 summarizes the emphasis placed on item 2a (performing the procedure) and on the other Framework items for treatment and control groups. It is clear that treatment and control instructors emphasized item 2a far more than other objectives. The fact that the common hour exams were explicitly designed to assess procedural skill helps explain this tendency. Importantly, however, the three instructors who were coached to include Framework-oriented content in their lessons managed to implement a more balanced set of lessons than their control group counterparts.

Additional analysis of the classroom observation data revealed that on the three days observed, the Highly Experienced and Moderately Experienced instructors each successfully implemented more balanced lessons than their matched control instructors. Unlike the more experienced instructors, however, the Less Experienced treatment instructor did not incorporate appreciable levels of Framework-oriented instruction on the days observed. From our observations, we concluded that these results most likely reflect a relative lack of confidence in the presence of the observers; it is unclear whether this instructor had more success on days when observers were not present.

Table 1 – Classroom observations revealed high emphasis on Objective 2a

	Objective 2a (% coded "6 – Pervasive")	All other objectives (% coded higher than "2 – Infrequent")
Control	94%	9%
Treatment	72%	18%

Homework Completion Rates

Not unexpectedly, homework completion rates declined slightly across all sections as the semester progressed (Table 2), although the decline was relatively minor for the Less

Experienced instructor. No clear trends in verboseness were observed over time, and the aggregate results for each instructor are summarized in Table 3. We concluded that although completion rates declined over time, the students who continued to complete the Writing problems maintained their level of effort. Interestingly, the Less Experienced instructor's students completed more of the Writing questions (Table 2), and they completed them more "verbosely" than students in other sections (Table 3). Overall, if the observation data is correct and the Less Experienced instructor's lessons were in fact less focused on Framework-oriented instruction than other instructors' were, it appears the students made up for it through more focused practice on Framework-oriented homework outside of class.

Table 2 – Writing Task Completion Rates over Time

	Week 3	Week 7	Week 11
Highly Experienced	87%	74%	52%
Moderately Experienced	81%	77%	73%
Less Experienced	86%	86%	79%

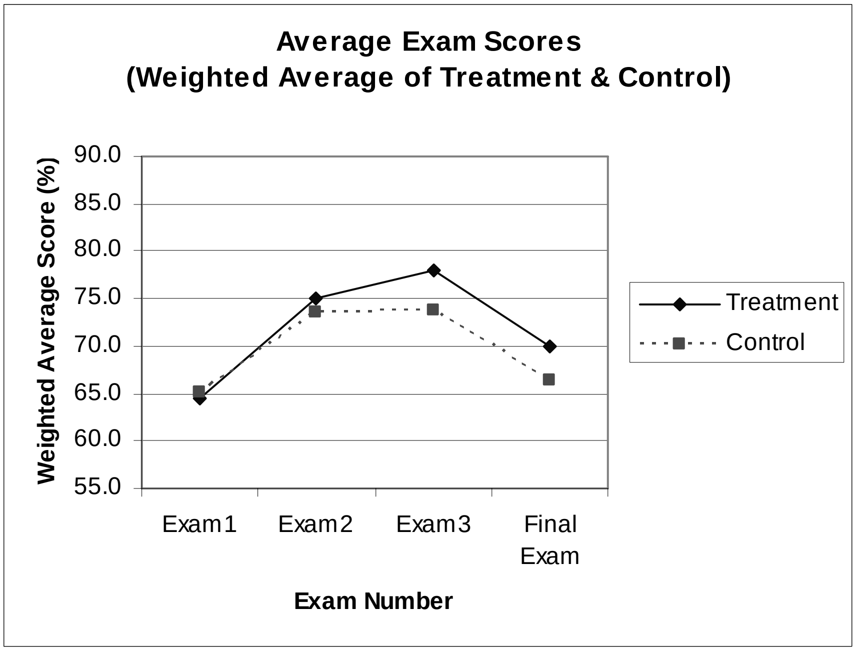
Table 3 – Writing Task Verboseness by Instructor

	Verbose	Terse	None
Highly Experienced	34%	51%	15%
Moderately Experienced	44%	45%	11%
Less Experienced	56%	32%	11%

Research Question 1 – Comparable Procedural Skill

Figure 2 compares the group means on the four common hour exams. Visually, we can see evidence that the treatment students scored slightly higher on the common hour exams, particularly those given later in the semester. The high variability among exam scores, however, prevented the mean differences from approaching the usual standards for statistical significance ($\alpha = .05$) except on Exam 3, where ANCOVA revealed a significant main effect of treatment condition ($F(1, 110) = 5.357$, $p = .023$) even when ACT/SAT was used as a covariate to adjust for the slight initial group differences. Overall, while treatment students performed slightly better than control students on the common hour exams and the difference grew as the semester progressed, we concluded that there was little statistically reliable evidence to suggest that the treatment led to significant differences in procedural skill between treatment and control groups.

Figure 2 – Average Exam Scores Over Time



It is worth noting that the instructional treatment was designed to produce deeper understanding, not improve procedural skill. In fact, treatment students were assigned approximately 18% fewer drill exercises over the course of the semester and were assigned 8% fewer problems overall. That the reduction in drill exercises did not hurt students' performance on the skill-oriented exams is an important result. It is consistent with results from cognitive psychology that have clearly demonstrated a sort of "law of diminishing returns" for practicing a new skill, whereby students eventually reach a point beyond which additional practice produces relatively little gain (e.g., see Bower, 2000). Developing understanding takes time *and* reflection (Carpenter & Lehrer, 1999), and the results of this study suggest that one way the requisite time might be obtained is by reducing the numbers of drill exercises that students are asked to complete.

Research Question 2 – Differences in Procedural Understanding

On average, the treatment students scored nearly a half point higher than control students on the journal tasks designed to assess students' Framework-oriented procedural understanding (Figure 3). Approximately 56% of treatment students' responses were coded at or above Level 2, meaning the response revealed either a *moderate* or a *high* degree of understanding. By comparison, only 37% of control students' responses were classified at or above Level 2 (Table 4). These results suggest that the treatment was effective at increasing students' understanding of the procedures tested. Cohen's *d* was calculated as a measure of effect size, and the difference ($d = 0.85$) was interpreted as a large effect (Gliner & Morgan, 2000). Independent sample *t*-tests revealed the difference was statistically significant ($t(105) = 4.317$, $p < .001$), and follow-up ANCOVA tests revealed a significant main effect of treatment condition ($F(1, 103) = 14.589$, $p < .001$) after accounting for variability associated with attendance rates, ACT/SAT scores, and instructor experience. We concluded that the combination of Framework-oriented instruction, homework tasks, and quiz questions was effective at helping treatment students develop comparatively greater procedural understanding than control students.

Figure 3 – Distribution of Average Journal Task Scores

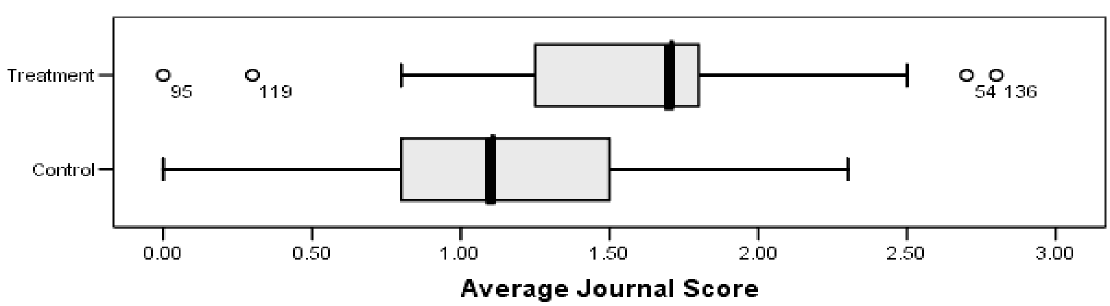


Table 4 – Frequencies of Coded Levels of Understanding on Journal Tasks

Coded Level of Understanding	Treatment	Control
3 – High	18%	6%
2 – Moderate	38%	31%
1 – Low	28%	35%
0 – None	17%	28%

Research Question 3 – Instructor Perception

We met with the three treatment instructors at the end of the semester to gain their perspective on the Framework-based instruction. The instructors each noted that it was not until "sometime after the first exam" that they began to feel comfortable incorporating the procedural understanding questions in their lectures. Reviewing the results of the common hour exams, we observed that the gains in procedural skill in favor of treatment students did not begin to manifest until Exam 2, and the gap eventually reached statistical significance by Exam 3. The instructors also noted that their students would "kind of lean forward in their seats" when the lesson turned toward actually performing a procedure. These college algebra students expected a skills-oriented mode of instruction, and including Framework-oriented questions on the weekly quizzes was important for helping change this expectation. We drew two conclusions from this data: training is needed before instructors are ready to teach for procedural understanding, and persistence is needed during implementation and assessment if we are to change students' perception of what it means to learn mathematics procedures.

Discussion and Conclusion

The instructional treatment employed in the present study was used as a supplement to an existing skills-oriented curriculum, yet it still produced large and statistically significant gains on the tests of procedural understanding without associated declines (and perhaps some improvement) on the tests of procedural skill. Moreover, these gains were realized despite an

18% reduction in the number of drill exercises assigned. Strictly interpreted, the improvements on the procedural understanding tasks reveal that instruction focused on Framework-oriented content can lead students to develop Framework-oriented knowledge (which we have associated with procedural understanding), and that students are willing to take such knowledge to heart. It also demonstrates, however, that understanding does not necessarily develop naturally in the absence of an explicit instructional emphasis. Future research is needed to determine whether the development of such knowledge shares the characteristic signs of understanding, including flexibility, longevity, and robustness.

Our work with the Framework has led us to believe it has a promising role to play in curriculum development. Its flexibility allows it to be adapted for use in almost any setting where procedural knowledge is discussed. With positive results emerging from our relatively modest implementation, we are confident that the benefits would be expanded if the Framework-oriented examples were embedded into the curriculum. Curriculum developers and textbook authors should note that we found it easy to build the Framework-oriented questions into existing skill-oriented examples. In fact, many of the examples used in the treatment were simply adaptations of the examples provided in the Blitzer (2004) algebra text. We developed additional examples as we reflected on the eight Framework questions and asked ourselves which questions would be the most appropriate for the procedures under consideration. The Framework-oriented questions were used to augment skills-oriented questions in the curriculum with the intention of producing more balanced instruction that emphasized both understanding and skill. In this way, skill-oriented questions became vehicles for helping students develop their own procedural understanding. Seamlessly building the Framework into an existing curriculum and examining its effectiveness is a natural next step that can help more students develop their understanding (and hopefully increase their retention) of mathematics procedures.

One treatment instructor had previously been a high school math teacher, and she suggested that implementing the Framework-oriented instruction would have been easier in high school. In high school, she said, more time is available for addressing the Framework-oriented questions used to promote the growth of procedural understanding. We are in the process of expanding our implementation through high school teacher training and ongoing support during the school year. It is our hope that the Framework-oriented instruction will promote longer lasting knowledge that can begin to address the high rate of mathematics remediation in colleges across the country.

It is important to acknowledge the role the active homework component played in this study. The Less Experienced treatment instructor apparently had less success implementing the Framework-oriented lessons in her classroom (at least on the days observed). Nonetheless, her students consistently outperformed the students in other sections on both the skill and understanding assessments. Her students were not exceptional in terms of prior math ability (as measured by SAT / ACT). Rather, our analysis suggests the most likely explanation is that her students completed more of the Writing questions and answered them more verbosely than students in other sections. The implication is that we must not ignore the active homework component in implementations designed to develop students' understanding.

Finally, we point out that there was a correlation between understanding and skill. Students who performed well on the three midterm exams tended to score higher on the written assessments of procedural understanding, and those who scored well on procedural understanding tended to score higher on the final exam. Moreover, when predictors were entered in steps in our regression analysis, we found that including treatment condition did not significantly improve the predictive power of the model (r^2 increased from .471 to .473 when treatment condition was added to a model already containing SAT / ACT, attendance rate, and average journal score). The fact that treatment condition was not a significant predictor in the model is important: the treatment was successful in improving students' procedural understanding, but other approaches that similarly advance students' procedural understanding should be expected to produce corresponding advances in procedural skill. While this study was not originally designed to assess the relationship between understanding and skill, the links that have been demonstrated are consistent with findings that suggest the two knowledge types develop in an iterative fashion (Rittle-Johnson & Koedinger, 2002; Rittle-Johnson et al., 2001).

In closing, the results of the present study are especially pertinent to those who have come to realize the important role that understanding plays in making knowledge robust, flexible, and long lasting. This study has demonstrated that students enrolled in a remedial college algebra course can develop deeper procedural knowledge when instruction and practice focus on questions designed to elicit understanding. Moreover, these benefits can be realized without reductions in procedural skill, even when many skill-oriented homework problems are replaced with questions that promote reflection and understanding. As existing models of mathematical knowledge continue to be refined through research, reflection, and practice, new instructional models must also be developed. We believe the Framework for Procedural Understanding is one promising approach for helping students develop deeper understanding of mathematics procedures.

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