Modeling Perspectives in Linear Algebra: A Look at Eigen-Thinking

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Linear algebra poses a number of significant challenges for students that need to be better understood in order to improve instruction and student understanding. At the time the tenth conference on Research in Undergraduate Mathematics Education, we had just begun a study intended to explore these challenges. Our preliminary report was given in a "working group session" format, in which we brought together participants to examine and discuss the potential for a specific modeling task designed to develop, explore, and reveal students' thinking about ideas relating to eigenvalues, eigenvectors, and eigenspaces. This brief report provides background information on our study and summarizes the discussion from the working group session.

The task we presented at the working group session is a "Model-Eliciting Activity," from Lesh's Models and Modeling approach, in which students work on "real-life" tasks that are designed to require them to invent, refine, and generalize powerful mathematical constructs (Lesh & Doerr, 2003). The solutions to these tasks take the form of complex artifacts that are testable and reusable and that reveal students' thinking. These artifacts may take on a variety of forms, but they often involve the development and description of a general solution process to the problem situation. Students work on these tasks collaboratively in groups to facilitate the problem solving process and they are encouraged to explain their thinking, to listen to the thinking of their fellow group member, and to indicate agreement or disagreement (with reasons) with others' thinking. These specific social norms pertaining to explanation and justification (Yackel & Cobb, 1996) contribute to the research teams' ability to document students' thinking.

The research questions we hope to answer by placing students in this problem situation and analyzing their work are: (1) What are the strategies that linear algebra students invent to reason about situations in which there is a need to explore and modify long term systemic behavior? And (2) How do those strategies differ when they have already been exposed to tools and strategies that may support (or constrain) their thinking in such situations?

Background

This study is situated within a larger research program, whose broad goals are to build theory about (1) how introductory linear algebra students learn and understand fundamental concepts such as vectors, vector spaces, linear (in)dependence, bases, generators, rank, and dimension and (2) how teachers can proactively support student learning and understanding of these concepts. According to Sierpinska (2000), geometric, arithmetic, and structural reasoning are fundamentally important in learning and understanding the core ideas of linear algebra. Many students encounter particular difficulties engaging in structural reasoning and/or shifting between the three types of reasoning. We bring to bear on these problems two compatible perspectives on the role and function of models in bridging the informal and formal aspects of mathematics. In particular we draw on the Models and Modeling perspective described by Lesh and Doerr (2003) and the instructional design theory of Realistic Mathematics Education (RME) (Gravemeijer, 1999).

Central to RME is the design of instructional sequences that challenge learners to organize key subject matter at one level to produce new understanding at a higher level. In this process, referred to as mathematizing, graphs, algorithms, and definitions become useful tools when students build them from the bottom up through a process of suitably guided reinvention (Rasmussen, Zandieh, King, & Teppo, 2005). The mathematization process is embodied in the core heuristics of guided reinvention and emergent models. Guided reinvention speaks to the need to locate instructional starting points that are experientially real to students and that take into account students' current mathematical ways of knowing. One aspect of the reinvention principle entails examination of students' informal solution strategies and interpretations that might suggest pathways by which more formal mathematical practices might be developed. The heuristic of emergent models highlights the need for instructional sequences to be a connected, long-term series of problems in which students create and elaborate symbolic models of their informal mathematical activity (Gravemeijer, 1999). The term model is meant to be an overarching idea, one that encompasses students' evolving activity with a chain of symbols, such as number tables, algorithms, graphs, and analytic expressions. From the perspective of RME, there is not one model, but a series of models where students first develop *models of* their mathematical activity in an experientially real task setting, which later becomes *models for* reasoning about mathematical relationships, and the creation of a new mathematical reality (Rasmussen, Zandieh, King, & Teppo, 2005).

Research Design

The methodological approach for the larger study falls under the genre of "design-based research" (Cobb, 2000; Design-Based Research Collective, 2003). The study will take place in an intact introduction to linear algebra course taught by one of the research team members. Data sources will include videorecordings of each class sessions, individual problem solving interviews, written assessments, and copies of all student work produced during the semester. This type of research involves daily debriefing sessions and weekly planning meetings with all project team members. The research design for the portion of the study that focuses on Eigen-thinking is given in Table 1.

	First Interview	Second Interview	Eigen Unit	Third Interview
Group A	Practice MEA	Eigen MEA	Eigen Unit	Individual Interviews
Group B	Individual Interviews	Practice MEA	Eigen Unit	Eigen MEA

Table 1. Research Design

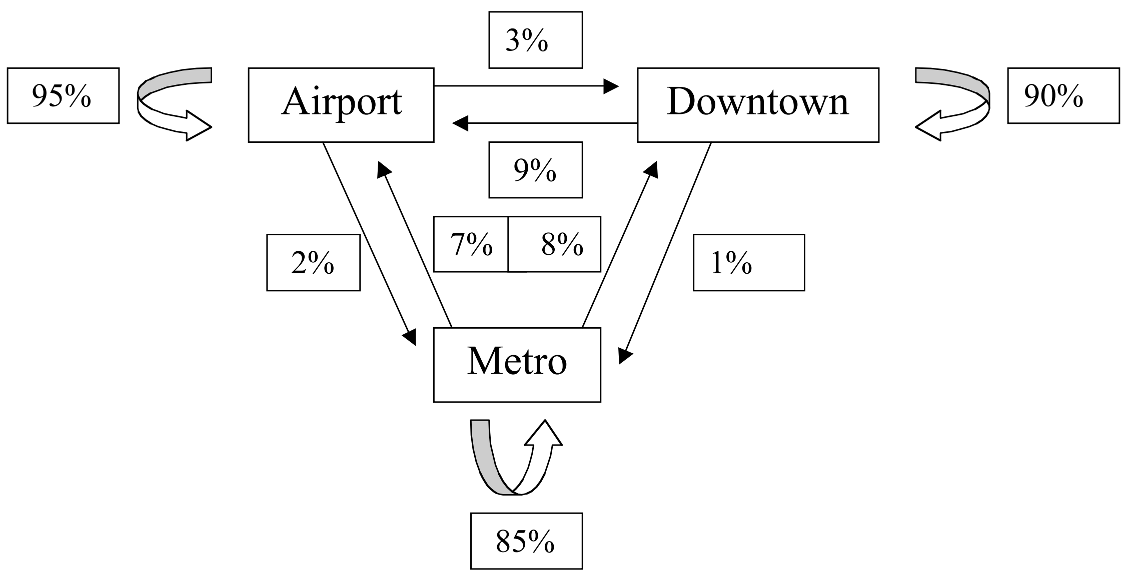
Students in our linear algebra class have had no previous experiences with MEA type tasks. We therefore designed the eigen-unit study so that Group A and Group B (each consisting of approximately nine students) would engage in a "practice" MEA before working on the eigenvector MEA. We also staggered Group A and Group B's work on the eigenvector MEA so that we will gain information on how students' strategies might differ before and after instruction on eigenvectors and eigenvalues. We will also conduct individual problem solving interviews with one of the two groups of students in order to gain more detailed information on their understandings of eignevalues and eigenvectors.

The following problem was given to participants at the working group session. Approximately 15 minutes was devoted to working on the problem (students would typically have 60 minutes), followed by discussion and suggestions for improving the task.

Eigenvector MEA

The Mertz car rental company has three locations in a large west coast city: at the Airport, Downtown, and off the Metro. The company has been doing exceptionally well during the last year, and the management believes that this success has been in part due to their policy of allowing customers to return a rental to any of the three locations, regardless of which location the vehicle was rented from. Unfortunately, this policy has created something of a logistical nightmare for the company, as they have started to have problems with too many vehicles at the Airport and not enough at Metro. Currently, the company reshuffles the cars at the end of each week so that there are 500 cars at the Airport, 250 Downtown, and 200 at Metro. There is always more demand at each location than can be met.

Each week, about 95% of the vehicles rented from the Airport location are returned at the Airport location, about 3% rented at the Airport are returned Downtown, and about 2% of the cars rented from the Airport location are returned at the Metro location. The diagram below indicates the analogous statistics for the Downtown and Metro locations.



The management of Mertz has hired your team of consultants to help them build a better understanding of the distribution of cars, how that distribution changes over time, and how the company might most efficiently manage its resources to meet demand and optimize profit. In order to do this, the management has given you three tasks. First, you are to project the long term distribution of the cars, assuming that there are initially 500 cars at the Airport, 250 Downtown, and 200 at Metro. Second, you are to determine whether changing the initial distribution of the cars will change the long-term distribution. Finally, you are to develop a proposal for a business plan that will address the distribution problem. In your plan, use the management's projection of a demand for 550 cars at airport, 275 Downtown, and 225 at Metro. The management would like your scheme to work for other cities where their chain operates. For this reason, it is important that you provide them with a description of your process for determining the long-term distribution of the vehicles as well as the rationale for your business plan that is detailed and general enough so that your procedure can be adapted and used in other cities with other numbers of locations and redistribution rates.

Summary of Working Group Session Discussion

Feedback from the working group yielded insight into possible student strategies as well as suggestions about how the activity might be leveraged instructionally. Three possible student strategies and interpretations identified were: (1) creation and iteration of a coefficient matrix that aligns with a classic Markov chain treatment of the problem, (2) creation of a system of linear equations with an attempt to find an initial distribution that will leave the cars distributed according to the projected demand after a single week, and (3) interpretation of the redistribution statistics as rates of change.

Some in the group suggested that one way to leverage the instructional value of this task might be to have a class discussion before the students begin working. During this discussion, students would be asked to predict the long-term behavior of the system. This could help the students focus their thinking on the task of predicting long-term systemic behavior, which could then be leveraged in the future for thinking about steady state behavior. Considerations such as these could help students develop meaningful and powerful ways of thinking about eigenspaces and the associated eigenvectors and eigenvalues.

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