INFINITE MAGNITUDE VS INFINITE REPRESENTATION: INTUITIONS OF "INFINITE NUMBERS"

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Abstract. This report explores students' naïve conceptions of infinity as they compared the number of points on line segments of different lengths. Their innovative resolutions to tensions that arose between intuitions and properties of infinity are addressed. Attempting to make sense of such properties, students reduced the level of abstraction of tasks by analysing a single number rather than infinitely many. In particular, confusion between the infinite magnitude of points and the infinite amount of digits in the decimal representation of numbers was observed. Furthermore, students struggled to draw a connection between real numbers and their representation on a number line.

The research presented in this paper is part of a broader study that investigates changes in students' conceptions of infinity as personal reflection, instruction, and intuitions are combined. It strives to uncover naïve interpretations of a concept that has puzzled and intrigued minds for centuries. The counter-intuitive and abstract nature of infinity provides a particularly interesting avenue for investigation. Moreover, infinity is a concept steeped in personal convictions that may stem from religion or philosophy. Consequently mathematical arguments might not be sufficient to convince an individual of properties that even Cantor saw but could not believe.

It has been well established that intuitions are persistent, especially when dealing with counter-intuitive concepts (Fischbein, 1987). As a learner's mathematical background increases, so too does his or her reliance on systematic, logical schemes (Fischbein, Tirosh, and Hess, 1979). Thus an individual might feel ill equipped to deal with the mathematical anomalies that arise with infinity. This study presented undergraduate students with a geometric representation of infinity, and observed how those students responded to contradictory or inconsistent results that they themselves discovered. A benefit of a geometric approach is that it provided a context for investigating infinity without the necessity of introducing unfamiliar symbolic representations or terminology, such as in a set theoretic approach. Students were able to reflect on and develop their ideas by considering familiar and accessible objects, and with minimal instruction.

THEORETICAL BACKGROUND

Students' reasoning concerning the counterintuitive nature of infinity and cardinalities of different infinite sets have been a popular foci of current research (see among others: Dreyfus and Tsamir 2004, 2002; Fischbein, Tirosh, and Hess, 1979; Tall 2001; Tsamir, 1999, 2001; Tsamir and Dreyfus, 2002; Tsamir and Tirosh, 1999). To the best of my knowledge, only a few studies examine students' conceptions with regard to infinity in a geometrical context (see Dreyfus and Tsamir, 2004; Fischbein, Tirosh, and Melamed, 1981; Tall, 1980; Tirosh, 1999; Tsamir and Tirosh, 1996). Tsamir and Tirosh (1996), for instance, reported that when infinite sets were represented in a geometrical context, such as line segments of different lengths or squares of different perimeters, students were more likely to recognize one-to-one correspondence than when similar sets were represented numerically. In an earlier study, Fischbein et al. (1979) observed students' intuitive decisions when considering the infinite divisibility of a line segment. They concluded that, despite mathematical training, intuitions, such as the finite possible divisions of a "limited" line segment, remained unchanged. These conclusions supported the claim of Fischbein et al. (1981) that an intuitive leap is necessary to establish meaning about infinity.

In my research, I build on several theoretical perspectives. The first framework introduced is Tall's (1980), which interprets intuitions that extrapolate experiences with finite measurements. In Hazzan's (1999) perspective, the use of familiar procedures to make sense of unfamiliar problems is an attempt to reduce the level of abstraction of certain concepts. Hazzan described reducing abstraction as a way for students to "cope with new concepts" and make them "mentally accessible, so that they [the students] would be able to think with them and handle them cognitively" (1999, p.75). Furthermore, she suggested that such an attempt to reduce the level of abstraction is indicative of a process conception. Process and object conceptions of infinity are characterized by APOS theory, another of the theoretical perspectives to which I refer. Dubinsky, Weller, McDonald, and Brown (2005) proposed that process and object conceptions of infinity correspond, respectively, to an understanding of potential and actual infinity. Extending on these topics, my study examines students' naïve responses to tasks such as considering the number of points "missing" from the shorter of two line segments.

Tall's "Measuring Infinity"

As indicated, much of current research on infinity in mathematics education focuses on students' understanding of cardinal infinity. Tall (1980) suggested an alternative framework for interpreting intuitions of infinity that instead extrapolates measuring properties of numbers. Many of our everyday experiences with measurement and comparison associate "longer" with "more." For example, a longer inseam on a pair of pants corresponds to more material. Likewise, a longer distance to travel corresponds to more steps one must walk. Tall (1980) proposed extrapolating this notion can lead to an intuition of infinities of "different sizes," but one that is contrary to cardinal infinity.

A measuring intuition of infinity coincides with the notion that although any line segment has infinitely many points, the longer of two line segments will have a "larger" infinite number of points. Tall (1980) called this notion "measuring infinity" and suggested it is a reasonable, and indeed natural, interpretation of infinite quantities, especially when dealing with measurable entities such as line segments and points. With this interpretation, if a line segment has $\aleph$ many points, then a segment twice as long has $2\aleph$ many points. Conversely, properties of cardinal infinity assert that any two line segments have the same number of points, $\aleph$, regardless of length. One of the definitions of an infinite set is that it can be put into a one-to-one correspondence with one of its proper subsets. Thus, although a shorter line segment might be viewed as a subset of a longer one, they nevertheless contain the same number of points. Certainly, cardinal infinity does admit different infinite magnitudes – the natural numbers, for example, have cardinality less than that of the real numbers.

While at first, a measuring interpretation of infinity may seem at odds with cardinal infinity, they are linked via properties such as $\aleph = 2\aleph$. Moreover, Tall's measuring infinity is consistent with non-standard analysis, a branch of mathematics concerned with properties of the superreals, a field extension of the real numbers that includes infinitesimals as well as infinitely large numbers. Recognizing properties of transfinite numbers such as the equality mentioned above might be a fundamental aspect of encapsulating infinity, as explained in APOS theory in the next section.

APOS Analysis of Infinity

Dubinsky et al. (2005) proposed an APOS analysis of conceptions of cardinal infinity. They suggested that interiorising infinity to a process corresponds to an understanding of potential infinity, while encapsulating to an object corresponds to actual infinity. For instance, potential infinity could be described by the process of, say, creating as many points as desired on a line segment to account for their infinite number. Whereas actual infinity would describe the infinite number of points on a line segment as a complete entity. Dubinsky et al. suggested encapsulation occurs once one is able to think of infinite quantities "as objects to which actions and processes (e.g., arithmetic operations, comparison of sets) could be applied" (2005, p.346). Dubinsky et al. also suggested that encapsulation of infinity entails "a radical shift in the nature of one's conceptualisation" (2005, p.347) and might be quite difficult to achieve. This theoretical perspective, as well as Tall's (1980) "measuring infinity," will be used throughout the study to interpret students' intuitions, and their attempts to reduce the level of abstraction of properties of infinity.

SETTING AND METHODOLOGY

The participants of this study were 24 first-year undergraduate university students enrolled in a foundations course in analytic and quantitative reasoning. This introductory mathematics course was designed as an upgrade for students who lacked a sufficient mathematical background for level one courses. The course met twice a week for two-hour sessions over 13 weeks. Students had no prior experience investigating properties of infinity in a mathematical context and none were in a mathematics program.

Data collection relied primarily on a series of written questionnaires intended to elicit students' naïve conceptions of infinity. One of the aims of this study was to determine what sort of connection, if any, participants made between a geometrical representation of infinity and a numerical one. In other words, the question of whether students were associating points on a line with values on a number line was considered. The rationale behind administering a series of questionnaires throughout the span of several weeks was to determine if and in what ways students' ideas may change as a result of personal reflection. In order to avoid swaying students' responses, very little instruction was provided and it was made clear that there was no one "right" answer being sought.

The questionnaires were designed in such a way as to provide students with an opportunity to reflect and build on their previous ideas. Tasks were developed based on students' responses and aimed to unravel some of their shared conceptions. Certain questions recalled students' previous responses and presented them with a slight twist. The rationale for this was to confront students with some of the counterintuitive properties of actual infinity that they unearthed. Questions also took the form of presenting students with an argument that claimed to be from one of their peers. Students were then asked to assess and discuss the ideas involved. The basis for this style of question was to avoid presenting an authoritative position. It was imperative to this study that students' responses were not affected by seemingly correct solutions. The students addressed each issue based on its appeal to their own naïve ideas.

Due to the nature of the tasks, details concerning specific questions are developed in the following section. The primary focus of this paper is on students' responses to two questions in particular. The first question (Q1) that is analyzed in detail confronted students with an idiosyncrasy of infinite quantities and asked for an explanation. Of particular interest was the response of one participant, Lily. Her attempt to formulate an argument that was consistent with her experiences and intuitions prompted a follow up to this questionnaire. In Q2 students were asked to respond to Lily's argument as well as to a variation of it.

RESULTS AND ANALYSIS

Q0, Lily, and her classmates

From the early stages of the study, a clear lack of connection between points on a real number line and numerical values was observed. One of the first questionnaires administered (Q0) was intended to establish some preliminary concepts as well as introduce students to the style of tasks in the study. Students were asked to identify the number of points on line segments and speculate on the relationship between number of points and length. Their responses revealed interesting conceptions concerning the distinction between points and values for those points. For instance, 70% of participants indicated that points were either the places that a line segment starts and ends, or else they were markers that partition a line segment into equal units. Conversely, these students recognized and were able to justify the infinite number of possible partitions of a line segment, as well as the infinite number of rational numbers between any two numbers. These initial responses indicated misconceptions about the geometry of points, and prompted the only instance in the study of instructor intervention. A brief description of points and lines complemented a class discussion that addressed questions such as, if points are only indicators of the start and end of a line segment, then what lies in between?

Lily's responses to the early questionnaires were characteristic of these conceptions. Though not particularly confident in her mathematical abilities, Lily was a thoughtful student who was eager to share and reflect on her conceptions of infinity. Various changes in her responses were observed throughout the course of the study. For instance, in Q0, Lily stated that the length of a line segment was equal to its number of points:

> There are a total of 3 points on the line segment… I think I know this because [the question] states that [the line segment is] 3 units long; therefore, I just divided the line segment into 3 parts.

During the class discussion that followed, she reasoned that it was possible to divide a line segment "into many different partitions" and concluded there must be "an infinite of points on any line segment."

Once a shared understanding, to use the term loosely, of the infinite magnitude of a line segment was established, a connection between point and number began to develop. During students' attempts to justify the infinite number of points, the notion of point size came about. Lily remarked, "In a line, there can be many points present because the size of the points have no limit. It could be an extremely big point or a microscopic size point." Conceptions of point size might develop because "[p]hysical points have size when they are marked with the stroke of a pen" (Tall, 1980, p.272). Alternatively, some students' ideas of point size stemmed from an association that students were making between point and number. This perspective was exemplified in Dylan's statement:

> 0, 1, 2 those would be big points, or you could have 0, 0.5, 1, 1.5, then those would be smaller points. And you could go smaller or bigger depending on what you want to do.

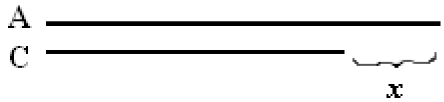
Thus, a microscopic point might be associated with, say, the number 0.00…001, whereas "big points" were associated with whole numbers.

This association between point size and numeric value, although different from the conventional one, was a connection nevertheless. Moreover, it seemed to indicate a change in students' conceptions. However, subsequent questionnaires revealed that students' point-number correspondence was flawed and inconsistent if it was made at all.

The questionnaire directly preceding Q1 expanded on the relationship between points and line segments in two ways. First, students were asked to compare the number of points on line segments of different lengths, and then students were asked to reflect on the number of points "missing" from the shorter of the two segments. Then, in order to probe students' rationale when comparing the number of points on line segments of different lengths, Q1 presented their conclusions with a slight twist.

Q1 and analysis of Lily's response

In an effort to investigate conceptions of what it may mean for a line segment to have infinitely many points, students were asked to reflect on their previous arguments concerning the number of "extra" points on the longer of two line segments.

> **Q1**. On a previous question, you reasoned that two line segments A and C both have infinitely many points.
>
> 
>
>
> Suppose that the length of A is equal to the length of C + *x*, where *x* is some number greater than zero. You also previously suggested that the segment with length *x* has infinitely many points. That is, the ∞ points on A minus the ∞ points on C leaves an ∞ number of points on the segment with length *x*. Put another way,
>
> $$\infty - \infty = \infty.$$
>
> Do you agree with this statement? Please explain.

Of the various responses to this question, Lily's stood out. In her response, she disagreed with the possibility that ∞ - ∞ = ∞. She wrote:

> I disagree with this statement. For example, π is an infinite (on going) number. If we subtract π – π the answer is 0, NOT ∞. But, if there is a restriction that says we can't subtract by the same number it could still be an infinite number, but just a smaller value. For example, π – 2π = -π, is still an infinite number, only negative.

Lily reasoned that since π is an "infinite (on going) number" and π – π = 0, then the difference ∞ - ∞ must also be 0. In Lily's conception, an "infinite number" appears to be a number that has an infinite decimal representation. Her objection to Q1 seems to stem from confusion between an infinite magnitude, such as the number of points on a line segment, and the infinite number of digits in the decimal representation of π. Her use of π to justify claims about infinite magnitudes suggests a disconnect between points on a line segment and real numbers. Not only did she overlook the particular value of π itself, but she also failed to distinguish the differences between acting on one specific element as opposed to infinitely many.

Lily's generalization of properties of π to draw conclusions about the entire set of points is likely an attempt to reduce the level of abstraction of dealing with an infinite number of elements. The use of one number to explain properties of infinitely many coincides with Hazzan's (1999) observation that students will try to reduce the level of abstraction of a set by examining one of its elements rather than all of them. It is possible that addressing the entire set of points on a line segment as an entity itself may not be feasible at this stage of Lily's concept formation. Her use of the qualifier "on going" to describe her notion of an "infinite number" is further evidence that she maintains a process conception of infinity.

Another interesting aspect of Lily's response was her use of "restrictions." She proposed that the difference of two "infinite numbers" might be another "infinite number" if there are appropriate restrictions placed on the quantities. By restricting the "values of infinity" she reasoned that it is possible to attain "an infinite number, it [will] just be a smaller value." For instance, she noted that a line segment with "missing points" may still have infinitely many points, just fewer than the longer segment. This idea is consistent with an intuition of measuring infinity (Tall, 1980). Also, Lily's response is consistent with the observation that students' conceptions of infinity tend to arise by reflecting on their knowledge of finite concepts and extending these familiar properties to the infinite case (Dubinsky et al. 2005; Dreyfus and Tsamir 2004; Tall 2001; Fischbein 2001; Fischbein, Tirosh and Hess 1979). The use of familiar concepts and procedures to describe the unfamiliar properties of infinity is yet another example of Hazzan's (1999) "reducing abstraction". In this case, Lily applies the familiar procedure of subtraction not to the transfinite number $\aleph$, but to the real number π, thereby reducing the level of abstraction of working with the infinite number of points on a line segment.

Q2 and Lily's classmates

Lily's confusion between an infinite number of elements and an infinite number of digits in one particular element provoked my curiosity. The question of whether other students shared Lily's ideas naturally arose. Thus, a follow up questionnaire (Q2) recalled Q1, presented Lily's argument verbatim, as well as a similar one, and asked students to elaborate on whether or not they agreed with the arguments.

> Q2. Recall Q1.
> Student X: [Lily's response as quoted above]
> Student Y: I disagree with this statement. You can subtract two infinite numbers and NOT end up with ∞. For example, 1/3 is an infinite number, but 1/3 – 1/3 = 0, NOT ∞. Also, 4/6 and 1/6 are both infinite (on going) numbers, but if we subtract 4/6 – 1/6 = 3/6 = ½ = 0.5, which is not an infinite number. But sometimes it's possible to subtract two infinite numbers and get an infinite number. For example, 1/3 – 1/6 = 1/6, which is infinite and smaller than 1/3. So, sometimes ∞ - ∞ = ∞, but usually not.

Interestingly, most participants agreed with at least one of the arguments above. The failure to distinguish between infinite magnitude and infinite decimal representation was shared by 22 of the 24 participants in this study. Two distinct interpretations of "infinite numbers" were observed. For the students who agreed with both arguments, confusion between infinite magnitude and infinite decimal representation was broad: they ignored the finite magnitude of both rational and irrational numbers. For instance, Jack wrote:

> 4/6 and 1/6 are both infinite (on going) numbers but when subtracting them your result is ½ which is not infinite. This proves that an infinite number subtracting by another infinite number is not always another infinite number. As a result the statement ∞ - ∞ = ∞ is not true because sometimes the result is infinite but a different value and other times the result is not infinite.

Again it is clear that the differences between a specific (finite) value and an infinite quantity are being neglected. Also, this response highlights the common notion that infinity has no specific value. In particular, Jack seems to use the ∞ symbol to represent numbers of different magnitude. This and similar responses revealed that students were not only extrapolating their experiences with finite quantities, but they were using them explicitly (though perhaps unknowingly) to justify their intuitions of infinity.

Conversely, there were students who recognized rational numbers as finite quantities but confused irrational numbers with infinite quantities:

> 4/6 and 1/6 are not infinite numbers. They are both ongoing but have a set pattern; the definition on infinite is a number with ongoing decimal digits that have no set pattern such as π.

Students who agreed with Lily's argument but disagreed with Student Y associated rational numbers with points on a number line but did not make the same association with irrational numbers. This interpretation was exemplified in Rosemary's response to Q2:

> $\pi - \pi = 0$ that is correct because one is taking away the same amount of points from what they initially began with will give 0, but in the line segment question, the amount of points in x (which is ∞ amount) is much less than the amount of points in A and C. Which because of this, I agree with Student X's second statement of how there should be restrictions. In this case, points in *x* are less than points in A or C. Student Y states: 1/3 – 1/6 = 1/6 (which is an ∞ number) but 4/6 – 1/6 = 3/6 (which is only 0.5 and not an ∞ number). Well, when we represent these numbers on a number line *[drew two line segments, one from 0 to 1/6 and one from 0 to ½, and labelled the segments A and B, respectively]* then won't both line segments have ∞ points? (But of course segment B will have more than segment A)

Rosemary was a high-achiever who had consistently expressed the opinion that line segments had infinitely many points. She had realized prior to Q1 that her arguments supported the counterintuitive $\infty - \infty = \infty$, and after reflecting, rationalized the expression by invoking a measuring intuition. In her response to Q1, she claimed that while any line segment will have infinitely many points, a longer segment would have a larger infinite number of points. She also alleged that subtracting an infinite quantity from another (albeit "larger") infinite quantity would leave "a lot of points… extending into infinity," and "it will take forever" to count them. These last two statements pertain to a notion of potential infinity, and suggest a process conception.

In her response to Q2, Rosemary related Lily's notion of restrictions to her own measuring conception. Placing restrictions on the symbol used to represent the infinite number of points on each line segment accommodated the possibility that a longer line segment will have a greater number of points. Like Lily, Rosemary used π to reduce the level of abstraction of $\infty - \infty = \infty$. As she stated, "taking away the same amount of points […] will give 0" just as $\pi - \pi = 0$.

Rosemary also reiterated her thoughts regarding measuring infinity when she addressed Student Y's argument. In this case, however, she did not use the rational numbers analogously with infinite quantities, as she had used π. Although Rosemary stated that 1/6 was an "infinite number," she observed its specific value on the number line. Similarly, she remarked that though ½ was not infinite itself, when represented on a number line there were still infinitely many points between 0 and ½. This distinct handling of rational and irrational numbers suggests a misconception about real numbers: whereas rational numbers were associated with points, irrational numbers were not. Furthermore, Rosemary seemed to use the words "infinite number" in two different ways: to represent a number with infinitely many (nonzero) digits in a decimal representation, as well as to represent the number of points in a line segment. It would be interesting to see if Rosemary's measuring conception would be so resilient had she not applied the same terminology to two distinct notions.

CONCLUSION

This paper presents some naïve conceptions of undergraduate students concerning infinity, and attempts to interpret their understanding as those ideas evolved. During their attempts to merge intuition with formal mathematics, students discovered some features of cardinal infinity that were at odds with their personal experiences and logical schemata. As they struggled to make sense of the material, students revealed interesting ways to cope with the abstract concepts. Students' attempts to make material more comprehensible suggested their ideas of infinity developed in part from misconceptions concerning the magnitude of numbers that have an infinite decimal representation. Furthermore, these conceptions contributed to a disconnect between geometric and numeric representations of infinity.

The confusion between the infinite magnitude of points on a line segment and the infinite decimal representation of particular numbers is indeed an obstacle to students' understanding of certain mathematical concepts. Not only does it hinder an appreciation or even recognition of properties of actual (cardinal) infinity, but it also demonstrates a shortcoming in the conception of number. The use of a finite quantity to explain phenomena of infinite ones misguides students' intuitions and ultimately their understanding. While "measuring infinity" may indeed have a distinguished place in mathematics research, intuitions that rely on numbers, or merely *a number*, are clearly hazardous to the progress of mathematical reasoning about infinity. The various attempts to reduce the level of abstraction of infinitely many points by considering properties of a single point have, in the cases discussed here, revealed an intuition of infinity that may be at odds with future instruction on limits and set theory.

Certainly, the importance of establishing an apt understanding of number, magnitude, and infinite quantities that is consistent with the mathematically accepted notions is not trivial. It has been well established that when formal notions are counterintuitive, primary, inaccurate intuitions tend to persist (see, among others, Fischbein et al., 1979). Furthermore, individuals may adapt their formal knowledge in order to maintain consistent intuitions (Fischbein, 1987). Fischbein et al. (1981) stressed that intuitive interpretations are active during our attempts to solve, understand, or create in mathematics, so it is clear that for the sake of advancing mathematical understanding, adequate intuitions must be developed.

This study opens the door for further investigation of some issues that may be taken for granted, such as the relationship between magnitude and representation, and the connection between points on a line and numbers. Future research will attend to the prior experiences and constructions that might have impacted students' naïve perceptions of infinity as well as their approaches to resolving the questions addressed in this study.

References

1. Dreyfus, T. and P. Tsamir (2004). Ben's consolidation of knowledge structures about infinite sets. Journal of Mathematical Behavior, 23, 271–300.
2. Dubinsky, E. and M. A. McDonald (2001). 'APOS: A constructivist theory of learning in undergraduate mathematics education research', in Derek Holton, et al. (eds.), The Teaching and Learning of Mathematics at University Level: An ICMI Study, Kluwer Academic Publishers, Dordrecht, Netherlands, pp. 273–280.

3. Dubinsky, E., K. Weller, M. A. McDonald, and A. Brown (2005). Some historical issues and paradoxes regarding the concept of infinity: an APOS-based analysis: Part 1. Educational Studies in Mathematics, 58, 335–359.
4. Fischbein, E. (2001). Tacit models of infinity. Educational Studies in Mathematics, 48, 309–329.
5. Fischbein, E. (1999). Intuitions and schemata in mathematical reasoning. Educational Studies in mathematics, 38, 11-50.
6. Fischbein, E. (1987). Intuitions in Science and Mathematics: An Educational Approach. Dordrecht, The Netherlands: Reidel.
7. Fischbein, E., D. Tirosh, and U. Melamed (1981). Is it possible to measure the intuitive acceptance of a mathematical statement? Educational Studies in Mathematics, 10, 3-40.
8. Fischbein, E., D. Tirosh, and P. Hess (1979). The intuition of infinity. Educational Studies in Mathematics, 10, 3–40.
9. Hazzan, O. (1999). Reducing Abstraction Level When Learning Abstract Algebra Concepts. Educational Studies in Mathematics, 40, 71-90.
10. Tall, D. O. (2001). Natural and formal infinities. Educational Studies in Mathematics 48, 199–238.
11. Tall, D. O. (1980). The notion of infinite measuring numbers and its relevance in the intuition of infinity. Educational Studies in Mathematics, 11, 271–284.
12. Tall, D. (1980a). Mathematical intuition, with special reference to limiting processes. Proceedings of the Fourth International Conference for the Psychology of Mathematics Education, Berkeley, 170-176.

13. Tall, D. and D. Tirosh (2001). Infinity – The never-ending struggle. *Educational Studies in Mathematics*, 48, 129–136
14. Tirosh, D. (1999). Finite and infinite sets. *International Journal of Mathematical Education in Science and Technology*, 30, 341-349.
15. Tirosh, D. and P. Tsamir (1996). The role of representations in students' intuitive thinking about infinity. *Journal of Mathematical Education in Science and Technology*, 27, 33–40.
16. Tsamir, P. (2001). When "the same" is not perceived as such: the case of infinite sets. *Educational Studies in Mathematics,* 48, 289-307.
17. Tsamir, P. (1999). The transition from the comparison of finite sets to the comparison of infinite sets: teaching prospective teachers. *Educational Studies in Mathematics*, 38, 209–234.
18. Tsamir, P., T. Dreyfus (2002). Comparing infinite sets — a process of abstraction The case of Ben. *Journal of Mathematical Behavior*, 21, 1–23.
19. Tsamir, P. and D. Tirosh (1999). Consistency and representations: The case of actual infinity. *Journal for Research in Mathematics Education,* 30, 213–219.