From test cases to special cases:
Four undergraduates unpack a formula for combinations

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Abstract

Through a case study of four elementary education undergraduates, we seek new analytic constructs that might help make clearer how arguments discovered or tested in quite special cases might come to support assertions that are understood to hold in general. We began analysis from a particular standpoint: to focus fundamentally on learners' representations and on how the learners reason from them. We have found it helpful to distinguish two perspectives to guide the subsequent analysis. On the one hand, we direct detailed attention to how learners reason, most especially on how they organize the logic of their arguments. On the other hand we seek to understand the learners' representations based on the way they structure them, and through the ways such structures might be reshaped or reframed over time. Based on this analysis the student subjects offer further evidence of the depth and power that students often viewed as less mathematically inclined can demonstrate in learning situations that engage them deeply.

Introduction

We often focus, when we plan instruction, on how we might imagine learners' thinking to proceed. Thus we offer tasks or problems to our students with specific learning goals in mind, assess learners' progress in relation to the goals that we have set, and make further choices based on what we may have learned about the learners' understanding in relation to our own. In particular, each time we pose a problem for investigation, we invite learners to shape their explorations and their discourse in response to *us*. Yet also, at least on good days, we would like our students to explore and to invent based on their *own* perceptions and experience. In such cases, learners may pose new tasks for themselves, where, often responding to each other rather than to us, they may challenge us, as teachers, to explore and to invent, sometimes from the very depths of our experience.

In this paper, we discuss an extended exploration by four elementary education undergraduates, whose thinking led us to reflect deeply on the mathematics they explored. On their own initiative, the students sought to justify a formula that they had memorized in high school (without proof) for the number of combinations of n objects taken k at a time, written as follows:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \qquad (1)$$

They did not work primarily to prove the formula, because they had already done so, from a different perspective, one week earlier. In the work at hand, these students chose to test a new approach. This second way of thinking—especially its boldness and originality—surprised us and hence led us to look more deeply at the data that we had. As cognitive investigators, we felt impelled to reconsider and reframe our prior understanding of the mathematics that we brought to our analysis. Indeed, as teachers and task designers, we sought to understand, based on whatever data that we were able to obtain, how these students, in their *own* terms, might have understood the exploration they conducted.

Setting, data sources

The four subjects for this study were members of a focus group in an experimental section of the second semester of our university's two-semester mathematics content course for elementary education undergraduates, taught by the first author in 1997. In most class sessions (two hours, twice a week) the work and conversation of the focus group were videotaped, with one camera, operated by Carole Sullivan, a graduate research assistant. Some work we report builds on and extends her MA thesis (Sullivan, 1998).

To support analysis, we build a composite narrative (Speiser, Walter & Maher, 2003; Speiser & Walter, in press) from three strands of data. Videotape of student conversation and completed presentations on March 5 provides the central strand. We surround it with two further segments: videotape of a prior full-class discussion on February 17, and students' journal entries that reflect on each main data segment.

We captured student discussion with a high-definition videotape recorder. For group work on March 5, we placed the camera about 5 feet from the student subjects. To capture audio, we taped two foam-mounted cardioid condenser microphones to the students' table, wired to separate recording channels through a mixer. For video, we favored a wide-angle setting, with frequent zooms to capture notebook entries, diagrams, and actions with objects. For the full-class discussion on February 17, we placed the camera toward a front corner of the room for students' gestures and facial expressions, and used one boom-mounted shotgun microphone to record, especially, the focus group. We recorded each class period in full, on one two-hour high-definition tape.

Background

In our main data segment the camera followed Rachael, Jill, Margie, and Holly, the four members of our focus group, through the entire the March 5 session. Both here and in the previous semester, the students used block towers, with two colors available, to present combinations (Maher & Martino, 1996; Maher & Speiser, 1997; Speiser, 1997). In the data we report, the blocks are red and blue. A combination of n objects taken k at a time will correspond to a tower n blocks high with exactly k red blocks, where k might range from 0 to n. In the work that we report, the student subjects might also write numbers, as indices or labels, on the blocks.

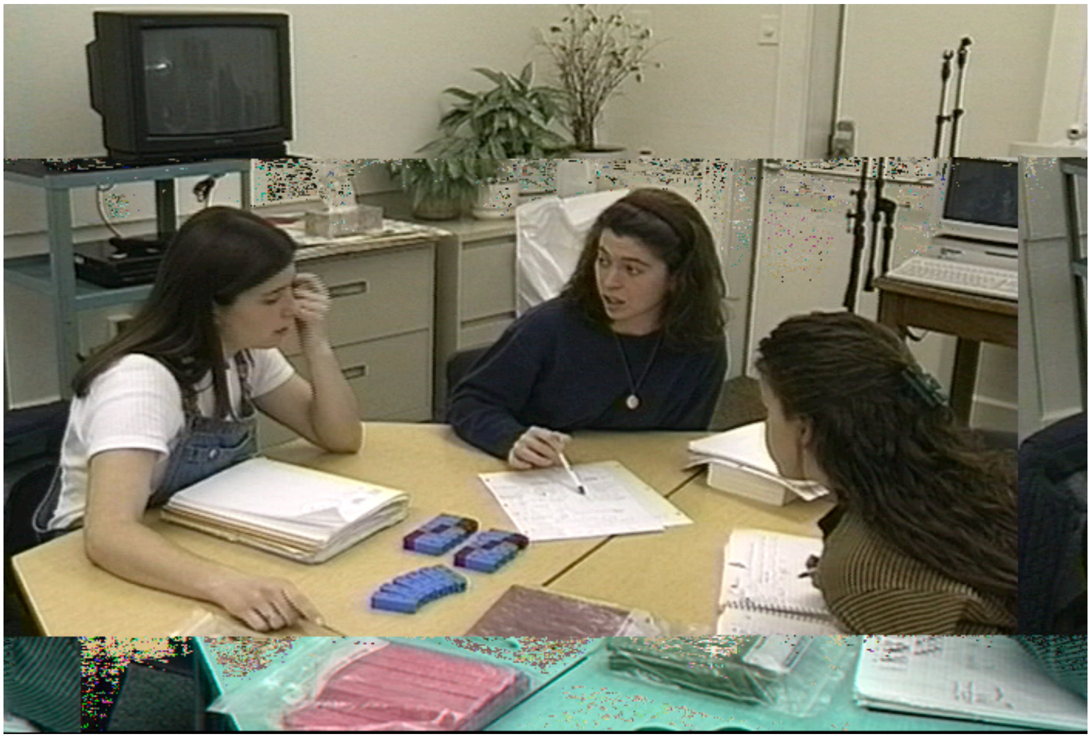

Figure 1. Rachael, Jill, and Margie. (For Holly, see Figure 5 below.)

A month later, in her journal, Rachel recalled the March 5 session in a sequence of reflections that will help us set the stage for the main data segment.

> *Rachael*. We saw that the formula worked but we didn't know all of the reasons behind it. So we set out to discover the meaning. I thought it had something to do with the fact that you cancel out the same numbers on the top and bottom, but it turns out it didn't have a great role. By the end we had found what each *n* and *k* meant and why they were in each position.

The students' goal will be to unpack equation (1) factorial by factorial. In Figure 1, from the first minute of the main data segment we will analyze, we see cancellation marks across some of the number labels. The latter appear, enlarged, in Figure 2. As Rachael suggests, such traces of prior explorations play no part what will follow.

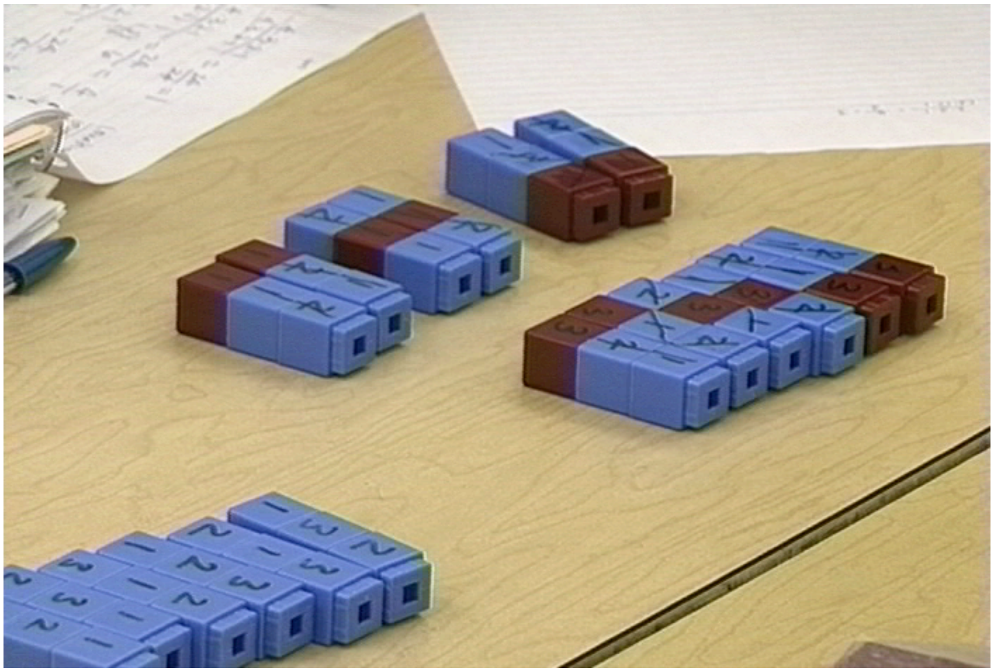

Figure 2. Marked towers.

Factorials count permutations. To account for each factorial in equation (1), the corresponding permutations need to be made visible. Hence the number labels on the blocks.[1]

Rachael. We found that it had much to do with unique and un-unique towers.

When numbered blocks in any given tower permute, several distinct numbered towers might represent a single combination. To describe such duplications, the group, on March 5, proposed a terminology that we, too, will adopt in our discussion. We now explain this terminology, as illustrated by the sets of numbered towers shown in Figure 2.

[1] Once permutations had clearly come in play, the third author suggested that the students number their blocks with an erasable marker to keep track of each permuted block. This suggestion, by the third author in the previous class session, was the only teacher intervention in these students' exploration of equation (1).

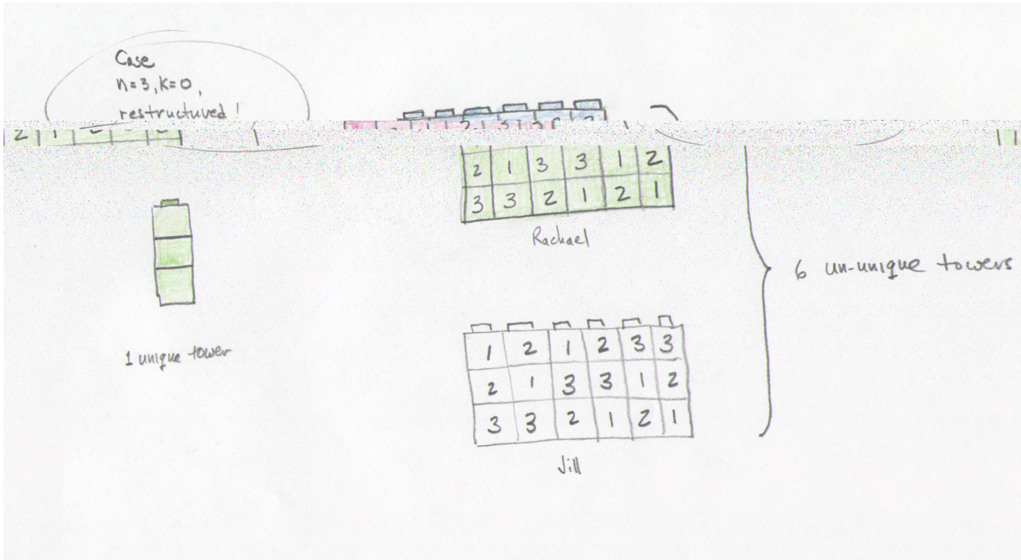

Figure 3. The case $n = 3$, $k = 0$. The six un-unique towers at right have been arranged by Jill and Rachael to present the numerator of Equation (1). The six blue towers in the upper center form Set 1. Jill preferred to think of these towers as colorless, as shown in the lower center. (Drawing by first author, from notes in 2005. Source: videotape.)

In Figure 3, we see six numbered towers. These constitute what we will call Set 1. In these towers, every block is blue. Hence any tower in this set will represent the unique combination for $n = 3$, $k = 0$. The positions of the number labels indicate the permutations of the given blocks. There are $3! = 6$ such permutations. These six marked towers are called *un-unique,* to emphasize the different permutations they display. Despite these differences, these towers represent a single combination. Hence the students will at times refer to them as *duplicates*. In this case, the six marked towers duplicate a single *unique tower*, the unmarked tower that displays the common combination that each marked tower, regardless of its marks, presents.

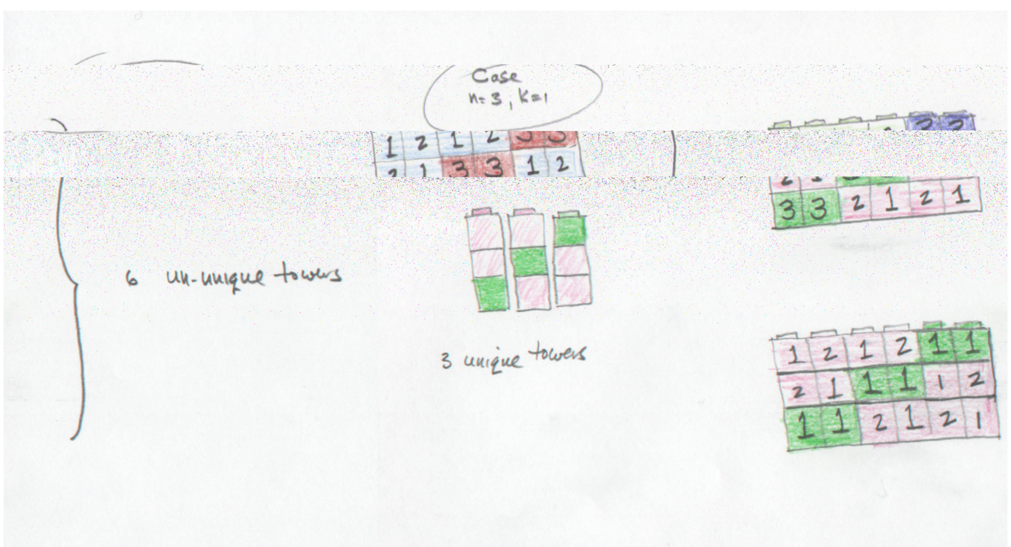

Figure 4. The case $n = 3$, $k = 1$. Sets 2 and 3, as built by Jill and Rachael, are shown at right. Each set has six un-unique towers, arranged in pairs of duplicates. In Set 2, each red block is labeled 3. In Set 3, each red block is labeled 1. (Drawing by first author, from notes in 2005. Source: videotape.)

Figure 4, displays Set 2 and Set 3. These present the case $n = 3$, $k = 1$. The focus group constructed them explicitly as variations of Set 1, first by replacing the single blue block labeled 3 in each tower of Set 1 with a red block labeled 3, then by changing the label on each red block from 3 to 1.[2] In both sets here we find 3 unique towers (as equation (1) predicts), each presented by two duplicates placed side by side. As for the special cases treated here, so for any n and k.

We have described the sets of blocks above, together with the words our students chose to talk about them, at the very least, to make the conversation in our main data segment more accessible. But there is more. Our analysis will emphasize the centrality of the development and use, by the *learners*, of personal as well as standard representations as anchors for mathematical reasoning. Such representations as words, pictures, gestures, and physical actions have indeed emerged in a range of recent studies as central for the development of explanations, lines of reasoning, and proofs (Davis, 1984; Maher & Martino, 1996; diSessa & Sherin, 2000; Speiser & Walter, 1997, 2000, 2004; Speiser, Walter & Lewis, 2004). Rachael, indeed, touches on this aspect in her journal.

> *Rachael*. As we figured all this out we used the blocks, which were very helpful because they gave a visual picture. I think that will be very important for children.

Rachael's first sentence refers, four weeks later, to the central March 5 data segment we propose to analyze. Her last sentence suggests what the experience she writes about, in retrospect, might mean for her.

Analytic starting points

We concentrate analysis on the mathematics that the subjects demonstrate, on their explanations of that mathematics, and on what the evidence we have permits us to infer about their thinking.

Problem solving. In this study, mathematics will be something that one *does*. Specifically, one solves a problem. We approach our students' problem solving from two perspectives. On the one hand, we take what we will call a *logical* perspective, where we focus on the learners' reasoning. We shall build especially from the approach to problem solving urged by Davis (1984), who placed central emphasis on representations. For us a representation is a presentation, either to help convey specific details of one's thinking to oneself, or to facilitate communication with others. From the logical perspective, learners' explorations are seen to proceed from the construction of one or more presentations of the problem situation to solutions to be justified through reasoning based how the given presentations have been structured. On the other hand, we can take what we will call a *psychological* perspective. Here we build on prior research about students' discourse and presentations (Speiser & Walter 1997, 2000, 2004; Speiser, Walter & Maher, 2003) where we have emphasized especially how students choose to present their thinking to themselves and others. Here we work first in the context of the group's collaboration on the problem right at hand, but then widen the context to consider further data, when available, that may help us to connect a given subject's work to long-term goals, values and past experience.

Mathematics. In practice, students often do not think as we hypothesize or wish, nor do they often tend to speak and write as we expect them to. For this reason, we might need to take what we will call a *mathematical* perspective, to examine our own understanding critically, especially when learners catch us by surprise. Given the universal tendency to favor one's own way of thinking, we would like to entertain explicitly the possibility that our preferred perspectives could contribute analytic bias. To be precise, such bias might constitute an imposition, by researchers, of a symbolic or interpretive context that the learners

[2] By design, the students' markers were erasable.

do not share. In this spirit, we can agree with Ellen Langer (1989, p.37): “A context is a premature cognitive commitment, a mindset,” and propose *systematically* to reconsider and if need be to reshape our understanding of the mathematics that we work with, when our students’ thinking does not fit specific images or processes that we have favored in the past. Nonetheless, we hasten to make clear that we prefer not to identify such reshaped understandings with the thinking of the learners that we seek to explicate. Indeed, without important differences between our own perspectives and the ways of thinking of the learners that we work with, we might recognize few opportunities to work productively as teachers and designers of instruction.

Guiding questions

From the perspectives that we sketched above, we pose two initial guiding (or perhaps leading) questions to our data.

1. How did these learners understand, present and communicate their reasoning? Here we concentrate on data from March 5. From the logical perspective, given the representations that these students had already built, how did they understand these representations, and how did they reason from their understanding? Further, from the psychological perspective, how did these students’ lines of reasoning take shape in their March 5 conversation?

2. What might it mean, for the students we consider here, to learn through solving problems? Here, to set the March 5 data in a larger context, we include further student data, drawn from a whole-classroom conversation, two weeks before the March 5 conversation, where members of the focus group reflected on the learning process as they saw it.

An earlier case study by the first two authors (Speiser & Walter, 2000) emphasized the need, for many learners, to directly challenge a prevailing discourse of rote, instrumental mathematics learning, by inventing and defending mathematics of their own, in ways that emphasize conceptual clarity and analytic rigor. The present study builds on this perspective, especially to test new analytical perspectives that facilitate more detailed discussion of learners’ emerging mathematics in relation to our own.

Data

In Jill’s notebook, we see the following equation.

$$\binom{3}{1} = \frac{3 \cdot 2 \cdot 1}{(2 \cdot 1) \cdot 1} = \frac{6}{2} = 3 \qquad (2)$$

05:24	*Jill*. We figured out what the equation means.
Tape 1	*Rachael*. Yeah.
	Jill. We wanted to find out, we wanted to see it on the blocks. So we did an equation with the blocks.

On the table we find three sets of six 3-high towers, as described above. Set 1 rests nearest Jill, Set 2 lies halfway between Jill and Rachael, while Set 3 rests nearest Rachael.

In the transcript just above, the meaning Jill refers to should emerge from how these students, in their conversation, use specific permutations of their blocks to read equation (2).

Several minutes earlier, Jill identified Set 1, with all blocks blue, as a presentation of the numerator $3 \cdot 2 \cdot 1 = 3!$ of the first fraction of (2). In response to Jill, Rachael noted that Set 2, also with labels 1, 2 and 3, presents the same numerator, but with a difference: Rachael's use of color (red vs. blue) to direct attention to three pairs of duplicates. At the same time, both said they saw Set 3 (with labels 1 and 2 on blue blocks as before, but now with 1 instead of 3 on the unique red block) as more directly linked, through its modified notation, to the denominator.

They begin with the numerator. This represents the permutations of three blocks, regardless of the way these blocks may have been colored. To make sense of Jill and Rachael's further conversation, we adopt the logical perspective first. Because the towers of each set on the table present the numerator, attention turns to the denominator, to unpack it as a count of duplicates. In the case at hand, each unique tower appears twice.

07:25 *Jill*. We realized that the six on the top means there's six total towers that are un-unique, which are these. [Picks up Rachael's Set 2 and smacks it down.] Un-unique towers. And then [touches twice one matched pair in Rachael's Set 3] we wanted to find out how many towers will have [points to the equation in her notes for C(3,1)] two doubles.
Rachael. Well. Yeah. But can you *just say* that?

Here it may be helpful to contrast reports of empirical observation from arguments for logical necessity.[3] To establish (2), based on any given set of towers on the table, it is not enough simply to *notice* that each unique tower comes has been presented by pair of duplicates. One also needs to *show* that each such tower *has to be* presented by a pair of duplicates. From this perspective, we take Rachael's question just above to indicate a need to shift attention to potential *explanations* for a structure that has been observed.

07:50 *Jill*. We'll have, how many? Well, I dunno. We know that there will be three unique towers.
Rachael. Uh, huh. But, OK, like, 'cause you're not gonna say six with two doubles every time.
Jill. No, you're gonna say whatever the bottom number is.

The six marked towers, with two duplicates for each unique tower, are special features of the special case under examination. In other cases, the number of marked towers and the expected number of duplicates (the numerator and denominator of the first fraction of (1), respectively) will vary. Hence, to support an argument for (1), an explanation for the pairs of duplicates in (2) needs to exemplify a way of thinking that could hold, at least potentially, in general. In other words, we may note a shift in emphasis from *checking* one or several special cases to *predicting* what must hold in general.

We now consider further student data. One week before, based on their interpretation for the numerator, $n!$, of its first fraction, the group reduced their stated problem, to prove (1), to showing that the denominator, namely $(n - k)!\, k!$, counts duplicates for any unique tower.

15:50 *Jill (to Margie)*. Because we have the total number of un-unique towers, thank you [Margie laughs] … [inaudible] …that you have to get rid of the duplicates that aren't unique to find the one unique.
Margie. Right. But then I have to explain it.
Rachael. Is that why you cancel, to get rid of the un-unique?

[3] Source: theoretical memos. Axis: problem solving, logical perspective.

Jill. Yes, to get rid of the duplicates to find the un-unique [corrects herself] to get rid of the *un-unique* to find the unique.

Here, given the need to address (1) for any *n* and *k*, we choose to view the special case at hand, in Jill's assertions just above, as a concrete means to present a way of thinking she expects to work in general. Margie, who has listened quietly so far, affirms Jill's emphasis on explanation. Consistent with a need for generality, attention shifts, for several minutes, to a new test case, C(4,1), for which *no* block models have been built so far. Hence the students can now move directly, anchored to several test cases, to unpack equation (1).

From the logical perspective, based on the students' work so far, we propose two guiding themes:

1. *A strategy for presentation*. Present combinations as block towers. Connect each term of (1) to suitable marked of towers of height n, then argue for (1) based on the way such towers can be grouped in sets of duplicates.

2. *A potential proof outline*. To prove (1), make sense its first fraction. Unpack the numerator as a count of un-unique towers via the permutations of their blocks; interpret the denominator as a count of duplicates, also via permutations, this time of blocks of each color taken separately. From these observations, as these students know, equation (1) will follow easily.

To interpret the denominator as a count of duplicates, the group returns to C(3,1), where they have three block models on the table. Referring to Set 2 for the numerator and Set 3 for the denominator, Jill identifies the color red with *k*. (Set 1, unused, rests to one side.)
Then Rachael responds, asserting that specific models, in particular the ones in front of them, might *no longer be needed* to predict how many duplicates appear.

What might Rachael mean? To explain, Rachael invites the other members of the group to consider any given 3-high tower with exactly one red block. Here she gestures toward Set 3, where the two blue blocks of each tower have labels 1 and 2 while the unique red block has the label 1. Turning to the denominator of the first fraction of equation (2), she observes that the first factor, 2!, can be understood to count the permutations of the two blue blocks, while the second factor, 1!, can be seen to count the single permutation of the one red block. In this way, the factors of the denominator of (2), hence by extension those of (3) and (1), can be now read as counts of permutations, of blue blocks and red blocks respectively, and this reading holds for any *n* and *k*, not just the case at hand.

Perhaps to anchor Rachael's last key step more clearly to the three block presentations on the table, Jill and Rachael now explicitly connect and then contrast these presentations, each viewed as a set of towers *together with a reading* of that set of towers.

21:49 *Jill*. But you can think of it as, I like to think of it [places the towers of Set 1 on top of the corresponding towers of Set 2.] like this. Because that's how you're thinking of it. Because all, all [picks up Set 1 and places it to one side] of the same color…

When Jill places her Set 1 on top of Rachael's Set 2, the number labels (1 through 3) on each tower of Set 1 match the corresponding numbers of Set 2 exactly. In terms of number labels, then, both sets are the same. In this sense, the two sets of towers both present the numerator of equation (2). But what about the *denominator*? For that, Set 2 might seem more natural. Nonetheless, Jill prefers Set 1.

21:58 *Rachael*. I like to think of it [points to Set 2] like this.
Jill. When I see the reds in there, it messes me up.

Rachael. Oh, really.
Jill. 'Cause when I see this [indicates Set 1], that's how it is, really.
Rachael. Well, 'cause, I don't think so, but that's OK. When there's one red…
Jill. No, that's OK.

In particular, for the denominator, Jill and Rachael will now reason from the same *structure*, their number labels, while remaining anchored to *distinct* block presentations that differ only in their use of color.

22:15 *Rachael*. Because I think if, 'cause it would mess me more up if these [points to the towers with the red and blue blocks] were all blue and this was red, 'cause I'm like what the in the heck…
Jill. Mm-hm. [Pause.] Just 'cause in my mind I think of a three-high tower, how many are there? [Pause.] Without any colors. I mean you add one color…
Rachael. Yeah. That's true.
Jill. When you add the color, to it…
Rachael. Yeah.
Jill. …there are still that many towers.
Rachael. But you see, I think of it as, there's this many towers [touches Set 2] with color…
Jill. Even though [smiles, body language affirms Rachael] color doesn't matter.
Rachael. That's right.
Jill. And [points to Set 1, all-blue] that's what that is.

Because Margie and Holly, not Jill and Rachael, will present to the whole class, we expect the former to make certain that they can present the argument so far in their own words. Hence we would expect the conversation, from here on, to revisit fundamentals, perhaps to help negotiate a common repertoire of models and interpretations.

43:28 *Margie*. What am I supposed to say about unique? I don't get that.
Jill. OK. How many un-unique towers. Because, if you have three blocks, and you…
Margie. Oh! [Covers her mouth with one hand.]
Jill. …and you build [faces Margie, cradles 3 blue blocks between her hands, then tosses them and catches them eight times] how many towers can you build with three blocks?
Margie. Six.
Jill. Six. But how many *unique* towers can you build? [Faces Margie, snaps the three blue blocks together to form all-blue three-high tower.] Three-high towers. [Thrusts the all-blue tower forward, raises her eyebrows, then looks suddenly at Holly.]
Rachael. Whoa!
Holly. Wait. That's right. One.

Tossing the three blocks randomly suggests the permutations that produce, in this case, six duplicates for exactly one unique tower. To do this, she views her three blue blocks, first, as distinct but "colorless" objects (as in her language use above) whose six permutations can be counted. But then, in effect, Jill invites Holly to "add color" and in doing so ignore the differences between her blocks. Based on this second way of structuring, Holly now envisages six copies of the *same* (unique) blue tower.

Next, Margie takes up the case of three-high towers with exactly one red block. In this case there are six (un-unique) towers, structured as three pairs of duplicates.

45:21 *Margie*. The red doesn't matter when it's, when you're talking about the number of towers you can make, it's just talking about how many.

Jill. OK, so color doesn't matter?
Margie. No.
Jill. OK.
Margie. Color doesn't matter when you're *making* [laughs] the towers.

Psychologically, the conversation seems to center on how to make clear, in the coming presentation to the class, the fundamental correspondence between each given set of distinct marked (un-unique) duplicates and the (unique) unmarked tower that corresponds to them. In the next exchange, returning to the case of all-blue towers, Margie constructs a new presentation, this time of the correspondence, central for these students' argument, between marked and unmarked towers. At this point, one three-high all-blue tower from Set 1 rests directly in front of Margie. The remaining towers of Set 1, side by side with numbers showing, appear to Margie's right.

Logically, Margie sees random permutations as a way of *constructing* numbered (un-unique) towers. They arise at random, in all possible ways. Her use of the word *making* above suggests that she would rather emphasize what happens to the towers *after* they are made.

45:34 *Jill*. But then [touches the all-blue tower in front of Margie] what about this one? You said there were six. I only see one.
Margie. [Takes the tower that Jill touched and returns it to Set 1, which she picks up.] When you number them…
Jill. So those are, so those aren't unique?
Margie. No, 'cause they're all the same thing. [Stacks the six all-blue towers of Set 1 vertically.] When you turn them over [returns the towers side-by-side to the table, but now with numbers facing down] they're all the same color.
Jill. So, what does unique mean?
Rachael. That's good.
Jill. When you turn them over, they're the same. So they're all the same, there's only one?
Margie. Right [turning two towers, first with numbers up, then down] there's only one unique tower.
Jill. Oh.
Margie. They're all the same.
Jill. So, do that. That was a smart thing to do.

Based on these last exchanges, from our psychological perspective, we suggest two further themes:

3. *A reference to construction*. Margie makes clear that that she *constructs* the n-high towers (for any k) by permuting n blocks, numbered 1 through n, of which exactly k are red and the remaining $n - k$ are blue. The resulting towers are un-unique.

4. *A new presentation*. With the case $k = 0$ as a first example, Margie turns her all-blue numbered (un-unique) towers over so that the numbers do not show. Hence, in this case, we see six duplicates of a single *unique* tower.

Margie's representation emphasizes not how towers might be made, but rather how marked (un-unique) towers can map naturally to unmarked (unique) towers, concretely by forgetting or ignoring their number labels. We can view this correspondence as an *in vivo* theoretical construct. Duplicates arise when different un-unique tower map to the same unique tower.

From the logical perspective, Margie, with support from Jill, has put forward a new way to *show concretely*, for any k, how the $n!$ un-unique towers of height n with exactly k red blocks can be organized directly into disjoint sets of duplicates. As we have seen, only the product of $(n - k)!$ and $k!$ still requires an explanation. Holly, without rehearsal, will address this final detail in the full-class presentation. There, once she has identified the given factors as suitable counts of permutations, Holly asks Margie for a second red block, to shift attention to a new concrete test case, $n = 4$, $k = 2$, where blocks of each color must permute to produce the duplicates. To begin, Holly holds up such a tower and then permutes its blocks, while keeping fixed the order of its colors. (See figure 5.)

Figure 5. The case $n = 4$, $k = 2$: Holly permutes blocks while keeping fixed the order of their colors.

06:59 tape 2 — *Holly*. This is like, kind of an easier way to understand this, is because you can switch these two reds around. At the same time [exchanging the red blocks] I'm gonna switch these two reds, but then [exchanging the blue blocks] I can also switch these two blues at the same time. So that's just showing that, the reason that I'm going to multiply the places for the blues times the places for the reds is that I can change them at the same time, and that's what that is… And when I multiply them together I get the number of repeats for every unique tower.

In the case $n = 4$, $k = 2$, the product in question will reduce to 2×2. Holly's independent actions on the red and blue blocks, supported by her choice of words ("that I can change them at the same time") suggest strongly that she understands the product of these factors as a count of ordered, independent choices. This further step concludes the argument for (1).

Analysis

Recall the first guiding question that we posed above: *How did these learners understand, present and communicate their reasoning?* To address it we will seek more clearly to conceptualize the ways these students worked with presentations and test case experiments to build their arguments. Hence, based on the brief thematic annotations that surround specific small-scale data items just above, we will now construct what we have called composite narratives (Speiser & Walter, 1997, 2000, 2004) to make qualitative sense of how the learners built their argument. In this way, our students' motivations for key choices, most especially, can more clearly motivate the analytic constructs we propose below to explicate their reasoning. We begin analysis from two starting points that we have emphasized above: *problem solving* and *mathematics*. Each leads to distinct but closely intertwined conceptual developments.

1. Problem solving. Here we take the psychological and logical perspectives.

Viewed psychologically, as we have seen, each representation has a story: its initial use to treat a special case, and then its later use, in a new sense, to anchor arguments and explanations that can hold in general. From the psychological perspective, our analysis provides a narrative through which the uses and interpretations of a given presentation may be seen to change, according to specific purposes and motivations. Such shifts of use and meaning, in our view, reflect qualitative changes in the thinking of the learners.

For example, Set 2 entered the group's discussion as a to present the un-unique towers in the specific test case $n = 3$, $k = 1$. There, Rachael built Set 2 to present the numerator of the first fraction of equation (2), and, through the way she structured it (in pairs of duplicates), Rachael could connect it to the denominator. Jill connected Set 2 to her preferred Set 1, through the number labels, which, tower by corresponding tower, were the same. Both connected Set 2 to Set 3, where the one red block was labeled 1 instead of 3, and read Set 3 as a presentation of the division indicated by the given fraction. Later in the conversation, this test case ($n = 3$, $k = 1$) became a special case, to anchor an argument, for any n and k, to justify equation (1).

Viewed logically, each representation has been built, interpreted and structured to support deductive reasoning.

Indeed, consider Set 2. According to our reading of the data, the six towers of Set 2 were built and understood to present the six permutations of three colored blocks. These blocks were numbered 1 through 3 to help distinguish them. To connect to combinations of three objects taken one at a time, each tower displays exactly one red block. For each tower, the labels 1 and 2 appear on blue blocks, while 3 appears on the unique red block. If we ignore the number labels, each tower presents a combination of three objects taken one at a time, namely the combination indicated by the position of the one red block with label 3. Further, the six towers of Set 2 have been set in order, so that each unique tower (read: *combination*) appears as a pair of duplicates placed side by side. Based on this structure, our students see the six towers of Set 2 partitioned as three pairs of duplicates. Based on a standard model for division (counting groups of equal size) we therefore find that there are $6/2 = 3$ combinations of three objects taken one at a time. This argument, seen logically, looks like a sequence of reductions. Indeed, let's trace it through again, this time with a view to how each step contributes progress toward the stated goal. The problem was to prove equation (1). The students' first step was interpretive: to read the numerator, $n!$, as a count of permutations of n blocks. This step reduced the problem to explaining the denominator as a count of duplicates for any given unique tower. The next step is also interpretive: to read the denominator as a product of two factors, $k!$ and $(n - k)!$, so as to identify these factors, for any given unique tower, as counts of permutations of its k red and $n - k$ blue blocks, respectively. This second step reduces the students' problem to showing, simply, that the product of these factors counts each given tower's duplicates. This last step completes the argument. In this way, we can view the students' actions as successive steps to reach their goal.

Recall that each reduction but the last can be anchored to the towers of Set 2, or indeed (equivalently) to any of the three models on the table, in particular to the special case $n = 3$, $k = 1$. But the last reduction, as we know, was justified by Holly through a *new* test case: $n = 4$, $k = 2$, where both colors need to be permuted independently. This shift of models provides, as we have seen, strong evidence that, by this point in the emerging *logic*, the students have come to see Set 2, now in the background, as just one special instance for an argument that the students understand as general. In this way we can see the shift of models and interpretations, which the psychological perspective helped make visible in terms of motivations that proceed from how the students understood the logic.

In this way, the students' use of three related but contrasting block models might appear more clearly motivated. Indeed, for the numerator, Set 1 was seen to be sufficient. Sets 2 and 3 came into play to help make sense, first of the denominator (perhaps by making unique towers and their duplicates more visible) and then (as with the shift of number labels from Set 2 to Set 3) to make sense of the denominator's factors. For the product, as we saw, Holly introduced a further model, in effect to enrich the structure. In each shift, what came to be understood through working with a given model, namely the reduction that it corresponds to, can be seen to carry through to the next model and the reduction step to which it corresponds.

From the point of view of problem solving, with Set 2 in the foreground as an illustration, we can thus, based on explicit evidence, understand more clearly how and, most especially, *why* these students chose to work across their different presentations.

So far, we have focused on the students' concrete presentations, to trace the way such presentations have been used to anchor or illustrate emerging lines of reasoning. From the starting point of problem solving, based on our work so far, we propose three analytic constructs to describe what we have found: *multiple presentation, proof through structuring, emergent generality*. We now explain these terms in more detail, and connect them to relevant prior work.

- *Multiple presentation*: no specific chosen "master" presentation for a given problem context, but instead a collection of linked presentations, each built to emphasize key features needed for a given logic step.
- *Proof through structuring*: first structuring a given presentation, then arguing based on the given structure.
- *Emergent generality*: movement from experiments in particular *test* cases, toward general predictions anchored to selected *special* cases.

Specifically, we have already emphasized a kind of flexibility, indeed a *purposeful* flexibility, in how the students built and worked with presentations. We see this flexibility especially in these students' willingness to validate and connect alternative, contrasting presentations. Further, through the ways the learners map between their presentations of a given case, we recognize a common structure for that case, a structure that our students reason from *across* their distinct presentations. In particular, through the way these students *use* contrasting but related presentations, we can safely view their thinking as conceptual, not merely procedural. But we can argue for still more. We have seen in detail, based on how our students structured and then reasoned through specific cases, the reframing of prior test cases as special cases, that is, as concrete anchors or specific metaphors for arguments that are understood to hold in general. By moving from experiments in test cases to general conclusions evidenced by reasoning in special cases, we conclude that these learners passed from understanding what *had* happened to infer, in general, what *has to* happen. In this way, our analysis suggests a clear progression of attention and interpretation: the development of multiple, connected, presentations in test cases under exploration; the development of experimental lines of reasoning, based on how suitable test cases have been structured; an emerging recognition that strategies for structuring and reasoning that hold in key test cases, through further reflection, can be understood to hold in general.

2. *Mathematics*. We need to treat explicitly our own emerging understanding of our students' mathematics. When we began (as mathematicians, with backgrounds in contemporary research) to formulate our own interpretations of our students' conversations, it seemed natural to assimilate our students' thinking to familiar schemas that we held already. When we tried, we found immediate discrepancies. We address these now.

To highlight important differences between our students' thinking, as we came to understand it, and our own evolving understanding, it should help first to set in place a common presentational and logical background, in effect a common language for analysis, across two contexts: our students' thinking and our own. The common background, specifically, includes a presentation, a series of evolving structures for that presentation, and an emerging line of reasoning, in general, that anchors to the given structures.

The presentation is a set T of towers that were never built but nonetheless imagined. These towers are n blocks high, each block distinguished from the others. On this basis, we find exactly $n!$ such towers, one for each way to set n given blocks in order. These are the *un-unique towers* of height n. So presented, T corresponds to the numerator of the first fraction in equation (1), viewed as a count of permutations of n objects.

Now we begin to structure T. Choose any k from 0 to n. Now select k blocks from the given set of n blocks, and color these k blocks red. There remain $n – k$ blocks in the set; color these blue. Each tower in T now has k red blocks and $n – k$ blue blocks, and all possible combinations will appear in T. We say two towers in T *give the same unique* tower if their colors match, in order, from bottom to top. In general, more than one un-unique tower will give the same unique tower, because different orderings of the n blocks may show the same sequence of colors. When this happens, we call different but matching un-unique towers *duplicates*. In this way, we view T as a collection of distinct (necessarily nonintersecting) sets of duplicates. Each such set of duplicates gives a different unique tower, and each of these presents, as we have seen, a unique combination of n objects taken k at a time. Structured in this way, the sets of duplicates in T (in effect the unique towers) are counted by the left side of equation (1).

Given this first level of structure, we can begin to reason. Consider one given set of duplicates.[4] Each tower in this set displays exactly the same combination (in order) of red and blue blocks, but the corresponding un-unique towers will be different. With our students, we now pose two questions: (a) For each given unique tower, how many such duplicates are there? And (b) for different unique towers with will the number of duplicates remain the same, or might it change?

To respond, we structure further. To build a given unique tower, the k positions, from top to bottom, to be occupied by red blocks, for example, do not change. What changes is the choice, in order, of the red blocks that will fill them. We find exactly $k!$ such choices, one for each permutation of the k red blocks. Similarly, we find $(n – k)!$ choices for the blue blocks. In this way, each un-unique duplicate in the given set is determined by the choice of two distinguished permutations, one of the red blocks, the other of the blue blocks. These choices are independent, in the sense that choices made for one do not constrain the choices for the other. In this way, we can view each set of duplicates as indexed by the (ordered) pairs of permutations that determine them.

Reasoning further, we can now answer questions (a) and (b). Because the two choices of permutations are independent, the number of such choices can be given by the product $n! \cdot (n – k)!$. In particular, the number of duplicates we find will be the same for *any* unique tower. Further, because the sets of

[4] Throughout the discussion that follows, we will hold n and k fixed, while the towers vary.

duplicates have equal size, the number of such sets be must be given by the quotient $n! / (n! \cdot (n-k)!)$. In this way, (1) follows, by means of the standard model for division as a count of groups.

We might describe this argument as a presentation of the *emerged* (as distinguished from emerging) generality that we attribute to our students' thinking, based on our interpretation of the transcript. It is general in several senses. First, n and k are arbitrary. So are the structures and the reasoning that they support to verify the connections between the given sets of towers and the terms and operations of equation (1). Second, the perception and selection of key features to support the given lines of reasoning, in effect the central thrust or motivation of the learners' underlying strategy, also can be seen to hold in general, across all possible special cases. In keeping with this generality, we chose not to bring any specific or concrete block presentations (analogous to Sets 1, 2 or 3) into play as anchors for the postulated reasoning we gave above.

To locate potential differences between the ways we, as researchers, and our student subjects worked with generality, at least in the present context, we now propose two further analytic constructs.

- *Seeing the general in the particular:* here a presentation anchored to a given, concrete case, demonstrates or advocates a strategy for structuring and reasoning that holds, and will be understood to hold, in general.
- *Seeing the particular in the general:* here a presentation, anchored to a given, special case, illustrates a strategy already known or claimed to hold in general, by demonstrating how that strategy applies concretely in the special case.

These constructs describe related but contrasting ways of using concrete presentations in a mathematical discussion. We have argued that the generality that we have made explicit just above has been presented by our students (perhaps most especially by Holly) through a sequence of test cases seen as special cases. Hence these students, based on our analysis so far, invite their audience to recognize the general in the particular.

In contrast, at least at first, we did not. Instead, as mathematicians, we saw equation (1) as a special case of an abstract, standard model that did not directly specialize to what we thought our students did, even in the common context we proposed above.

In the standard model, the group G of permutations of n distinguishable blocks acts on the set T of $n!$ un-unique towers of height n by permuting blocks.[5] With this action, T is called a G-space, and its elements (the towers) are called points. We suppose further that k of the n blocks are red, and that the remaining $n - k$ blocks are blue. Hence each element of T represents a unique tower with exactly k red blocks. As we have seen, each such unique tower can have duplicates. Write U for the set of unique towers; these correspond to combinations. Our goal will be to count the unique towers. By virtue of its action on T, G also acts on U. According to the standard model, we would fix one element of U, say u_0. As G acts, permuting blocks, each element of U appears. We view u_0 as a base-point, a kind of origin in U. In the standard model, we say that U is the orbit of u_0, or equivalently that G is transitive on U. We denote by H the stabilizer of u_0. By definition, H is the subgroup of G consisting of those permutations that hold u_0 fixed, that is, preserve the sequence of colors that defines u_0 while exchanging blocks. By a key result (Jacobson, 1974, p.74), the points of U correspond naturally to the cosets of H, and hence the number of elements $|U|$ is given by the quotient $|G| / |H|$. We know that $|G| = n!$, and, as Holly has argued (anchored to the special case $n = 4$, $k = 2$), we have $|H| = k!(n-k)!$. Therefore, in the standard model, we have can prove equation (1).

[5] More technically, we view G as acting on the left.

As we thought about the student data, we began to see discrepancies between the way the standard model's proof strategy worked out, in detail, for the case of towers, and the way our students worked. In particular, they never, in our view, fixed a base point, like u_0, or considered anything like cosets. Instead, as we have seen, each specific unique tower they considered serves just as well as any other unique tower to anchor the reasoning they demonstrate. To support this claim, consider Margie's action of turning towers over, where each marked, hence un-unique tower t in T mapped naturally to its unmarked, hence unique corresponding tower u in U.

To be precise, denote by p the mapping T —> U which sends each un-unique tower to the unique tower (or combination) given by its color sequence. Clearly p is compatible with the actions of G on T and U, and does not depend on any special choices. For any unique tower u, write T_u for the subset of all un-unique towers t that map to u. These form the set of duplicates for u. As the unique tower u varies in U, every set of duplicates appears. Further, Holly's counting argument applies equally to all such sets. Thus any, hence every, subset T_u will have exactly $k!\ (n-k)!$ elements, the duplicates for u. Therefore, as Holly showed, the number of unique towers must be given by the quotient in equation (1).

In contrast, in the coset model, H and hence its cosets *depended on an arbitrary choice*: the selection of the base-point u_0. Indeed, the cosets will be *different* when we modify the choice of base-point, hence of H. However, as we saw above, the map p: T —>U, the sets of duplicates T_u, and Holly's crucial counting argument, do not depend on *any* arbitrary choice. These observations led us to prefer a second standard but perhaps more abstract model, which we feel more closely corresponds to what our students did.

In this second model, we suppose given an abstract group G, two G-sets, denoted T and U, and consider an arbitrary G-map p: T—> U, a map of sets compatible with the given actions. For any given u in U, write S_u for the stabilizer of u, the subgroup of G of elements whose action holds u fixed, and denote by T_u the inverse image $p^{-1}(u)$. In this more abstract setting, a key basic result[6] asserts the following. For any g in G and u in U, conjugation by g defines an isomorphism $S_u \longrightarrow S_{gu}$ between the corresponding stabilizers.[7] To deduce equation (1), we specialize to towers and permutations, and then argue just as Holly did. Specifically, for any unique tower u in U, the number of its duplicates $|T_u|$ equals the count $|S_u|$ of permutations that will hold its colors fixed. Therefore this count of permutations, namely $k!(n-k)!$, does not depend on u, hence counts the duplicates for any u. As we have seen, this proves equation (1).

From what we have called the logical perspective, our students established the general equation (1) by successive reduction, and addressed key steps in general by means of carefully constructed special cases. Hence, in our terms, they saw the general in the special. In contrast, we saw the special in the general, because we argued as above by progressive specialization. Indeed, even the case our students saw as general (combinations, seen as n-high towers with exactly k red blocks) was special for us, because we started from results about abstract group actions. Even at that level of abstraction, to capture our students' thinking in a language we found natural, we discovered that we had to change our minds, and hence our understanding.

[6] We found it difficult give a textbook reference for this well-known statement, a well-known basic research tool. It follows quite directly from the basic definitions, so we leave the details for those readers likely to enjoy them.
[7] Further, translation by g defines an isomorphism $T_u \longrightarrow T_{gu}$ of the corresponding sets of duplicates, but we do not use this observation here. Its proof is also quite direct.

Discussion

We begin the present section with several statements by the members of our focus group, in a classroom conversation on February 17, two weeks before the data we have analyzed. As in the previous analysis, we consider student data that can help us to address our second guiding question: *What might it mean, for the students we consider here, to learn through solving problems?*

The February 17 conversation focuses on task-based group collaboration as a way of learning mathematics. Such collaboration contrasts sharply with prior experience these students had as mathematics learners. To begin, Holly describes how she had learned to function as a mathematics student.

> *Holly*. I learned how to take tests, and I know what the teacher wants, and I do it. I reciprocate that on the test. That's the way most of my math classes have been.

Holly does not prefer the social practice that she calls *reciprocation*.

> *Jill*. In our group, we've always got two or three different opinions, *strong*, and it's, it's not wrong, and so, and we're not finding one answer, we've got two or three different solutions, and that's why we can come up with what we do. 'Cause everywhere else, in all the other classes I have right now, you know, what's the right answer? And nothing else. And if you get anything else, you can't pursue it, you're not allowed to.

Jill views the classroom as a social system where a play of differing ideas might be encouraged, or alternatively channeled or suppressed. She offers evidence against suppression. For Jill, productive thinking builds from exploration of potentially conflicting alternatives *by learners,* in problem settings rich enough to impel learners to collaborate across important differences. In particular, Jill identifies *productive roles* for difference, doubt, and even conflict.

> *Jill*. You don't really know where your solution's going to be, that's one key… we're just given a little teeny question. In other math classes, you know exactly what your solution's going to look like, and what form it's got to be in.

Alternatives must be considered when the challenge (triggered by the task design) is to *find* and then present productive ways of thinking. In contrast, when the form of a solution has been given in advance, the learners, in Holly's terms, are expected to reciprocate instead of think.

> *Holly*. I think you allow them to think in different ways by not giving them all the answers. It's just in the way the teacher presents it, and the way that what they outline what they want you to do.

On the basis of the reasoning that Holly indicates, these students posed their task (to prove equation (1) based on a new interpretation of its terms) to provoke productive exploration of significant alternative approaches. But they did more than just explore and then select. As we have seen, they *connected* and *preserved* a range of presentations, built by different members of the group, in the central presentations we have followed.

Perhaps Jill and Holly, in the last two excerpts, emphasize the way a teacher poses tasks in order to make clear, in concrete practice, how a classroom based on exploration represents a very different *practice of instruction* than a classroom centered on reciprocation. While both students begin with descriptions of concrete practice, they both do so to highlight the different kinds of *thinking* that can take place in the different kinds of classrooms. Our analysis of their group's work one week later makes clear, we feel, the nature of the thinking that they advocate, and hence sets their claims in direct correspondence with their actions.

What we think today, and how we come to think it, will soon, reshaped, leave traces in memory that might help us think tomorrow. In this spirit, Jill and Holly focus on their own experience of learning mathematics, through personal experience with two contrasted social practices: prior classrooms, which we might describe as cultures of reciprocation, and their more recent experience as problem solvers at the time they spoke, which we might describe, in contrast, as a culture of collaboration.

As we have learned from recent work in neuroscience (Squire & Kandel, 2000; Freeman, 2000), we cannot remember sequences of events divorced from what those events have come to *mean* for us. Our second guiding question therefore focuses on memory as well as meaning. We remember not just what we learned, but, intertwined with that, the way we learned it. To unpack a memory, much like the data we have probed above, we must pose initial guiding questions and be ready to reshape them as we go along. For example: Who presents? Who explores? How do alternatives emerge? What fates might such alternatives receive? And perhaps most of all: Who sets the agenda? Why?

At this point in the students' conversation, Rachael offers us a memory.

> *Rachael.* When I was in elementary school in math, my teacher would be like, OK, this is the way that we're going to do it, but if you can think of another way, that's great, you know, come and show me, maybe I'll show the class. But I didn't know how to think of it another way.

If we have been what we remember, then we become, at least to some extent, what we choose now to undertake. Taken on this sense, Rachael's remarks suggest what might be possible for her. As a child, she "didn't know how to think of it another way", but now, she suggests, she does know. One week later Rachael and her group did not just take a teacher's "little bit" and fly with it, but instead chose their *own* point of departure, designed their *own* rich task, and made us, the designated teachers, join *them* as fellow learners.

In this process, what might we, the researchers, have learned from how our students framed the central issues? In effect, Jill, Holly, Rachael and Margie have conducted an extended *learning experiment*, where the data we have analyzed provide a key event.[8] Their experiment, as further data indicate, supports drastic conclusions. Consider, for example, Holly's journal entry, written later the same day as her remarks above.

> I would like to write about a few of the things I have felt in this class and feel that it is important to pass on to my students with a similar classroom atmosphere.
>
> 1) I feel that my opinion is valued and my contributions are worthwhile. Thus, I feel like sharing.
> 2) I feel that there is an equal opportunity for everyone to succeed, regardless of how they think.
> 3) I feel that I am making significant discoveries in mathematics.
> 4) I feel like I am really thinking for myself, instead of letting a book do it for me.
> 5) I feel that I am learning valuable skills in mathematics, in social relationships, and teacher preparation.
> 6) I feel excited to learn and come to class.
> 7) I feel that the teachers treat me as their equal and they have my respect.

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[8] Speiser, Walter and Maher (2003).

8) I feel that the class is more unified and accepting of ideas.
9) I feel that I grow from the insights of others.
10) I feel that the class members build upon each other's strengths.

In previous remarks, Holly saw the culture of reciprocation as an obstacle to learning. Here, through personal experience in a contrasting culture of collaboration, she shapes her observations carefully in terms of implications for her future teaching. Through what we have called a learning experiment, as Holly makes quite clear through her selection of key observations, these students may have sought, through their shared enterprise, to test new ways to understand themselves as not just as learners but as future teachers.

References

Davis, R. B. (1984). *Learning mathematics: The cognitive science approach to mathematics education*. Norwood, NJ: Ablex.

diSessa, A. A., & Sherin, B.L. (2000). Meta-representation: An introduction. *Journal of Mathematical Behavior, 19*, 385-398.

Freeman, W. J. (2000). *How brains make up their minds*. New York: Columbia University Press.

Jacobson, N. (1974). *Basic algebra I*. San Francisco: W.H. Freeman.

Langer, E. (1989). *Mindfulness*. Reading, MA: Addison-Wesley.

Maher, C.A. & Martino, A. (1996). The development of the idea of mathematical proof: A 5-year case study. *Journal for Research in Mathematics Education, 27* (2), 194-214.

Maher, C.A. & Speiser, R. (1997) How far can you go with block towers? *Journal of Mathematical Behavior*, 16, 125-132.

Speiser, B. (1997). Block Towers and Binomials, *Journal of Mathematical Behavior, 16*, 113-124.

Speiser, B., & Walter, C. (1997). Performing algebra: Emergent discourse in a fifth-grade classroom. *Journal of Mathematical Behavior, 16*, 39-49.

Speiser, B., & Walter, C. (2000). *Five women build a number system*. Stamford CN: Ablex Publishing.

Speiser, B., & Walter, C. (2004). *Placenticeras*: The evolution of a task. *Journal of Mathematical Behavior, 23,* 259-269.

Speiser, B., & Walter, C. (in press). Rethinking change and rate of change. In B.E. Reynolds (Series Ed.) & M. Carlson & C. Rasmussen (Vol. Eds.) *MAA Notes. Making the Connection: Research and Practice in Undergraduate Mathematics Education*. Washington DC: Mathematical Association of America.

Speiser, B., Walter, C., & Lewis, T. L. (2004). Talking through a method. *For the Learning of Mathematics*, *24*(3), 40-45.

Speiser, B., Walter, C., & Maher, C.A. (2003). Representing motion: An experiment in learning. *Journal of Mathematical Behavior, 22,* 1-35.

Squire, L. R., & Kandel, E. R. (2000). *Memory: From mind to molecules*. New York: Scientific American Library.

Sullivan, C. (1998). Making sense of combinations. Unpublished master's thesis, Brigham Young University, Provo, UT.