Eigenvalues and Eigenvectors:

Formal, Symbolic and Embodied Thinking

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Many beginning university students struggle with the new approaches to mathematics that they find in their courses due to a shift in presentation of mathematical ideas, from a procedural approach to concept definitions and deductive derivations, with ideas building upon each other in quick succession. This paper highlights this situation by considering some conceptual processes and difficulties students find in learning about eigenvalues and eigenvectors. We use the theoretical framework of Tall's three worlds of mathematics, along with perspectives from process-object and representational theory. The results of the study describe the thinking about these concepts of groups by first and second year university students, and in particular the obstacles they faced, and the emerging links some were constructing between parts of their concept images formed from the embodied, symbolic and formal worlds. We also identify some fundamental problems with student understanding of the definition of eigenvectors that lead to problems using it, and some of the concepts underlying the difficulties.

Background

The first year university student who has no prior understanding of linear algebra has a long way to go before being able to grasp the full picture. The course appears to them very intense, with ideas and definitions introduced very rapidly, with little connection to what they already know and can do from school mathematics. Since linear algebra is often the first course students encounter that is based on systematically built mathematical theory the course is highly demanding cognitively, and it can be a frustrating experience for both teachers and students. While some believe that the course is taught too early, Dubinsky's (1997) view is that students can develop their conceptual understanding by doing problems and making mental constructions of mathematical objects and procedures.

Two inseparable sources of difficulties with the linear algebra course, identified by Dorier and Sierpinska (2001, p. 256), are “the nature of linear algebra itself (conceptual difficulties) and the kind of thinking required for the understanding of linear algebra (cognitive difficulties)”. Dorier *et al*. (2000) claim that while many students fear linear algebra because of its abstract, esoteric nature, many teachers also suffer because of the abstruse reasoning involved. Historically, many of the concepts of linear algebra found their final form after several iterations of applications of linear techniques, and with little apparent unification. Hence, it is not surprising that many students have real difficulties with definitions of such concepts. An important principle enunciated by Skemp (1971, p. 32), and illustrating one major problem with definitions, is that “Concepts of a higher order than those which a person already has cannot be communicated to him by a definition, but only by arranging for him to encounter a suitable collection of examples.” The concepts may be presented through a definition in natural language, which may have embedded symbolism, or be linked to a symbolic presentation. These definitions are considered to be fundamental as a starting point for concept formation and deductive reasoning in advanced mathematics (Vinner, 1991; Zaslavsky & Shir, 2005).

A developing theory by Tall (2004a, b), extending some of the action, process, object, schema (APOS) ideas of Dubinsky (Dubinsky, 1991; Dubinsky, & McDonald, 2001), proposes that learners of mathematics can benefit from experiencing the results of actions in an embodied world, and processes in a symbolic world (or stages), before being able to live in the world of formal mathematics. This theoretical position suggests that it would assist university students if they were presented with embodied aspects of concepts, and associated actions, wherever possible. Extending his idea of an embodied manner of learning about differential equations (DE’s) (Tall, 1998) in which an *enactive* approach builds an *embodied* notion of the solution to a DE before introducing algebraic notions, Tall (Tall, 2004a, b) has recently developed these ideas into the beginnings of a theory of the cognitive development of mathematical concepts. He describes learning taking place in three worlds: the embodied; the symbolic; and the formal. The embodied is where we make use of physical attributes of concepts, combined with our sensual experiences to build mental conceptions. The symbolic world is where the symbolic representations of concepts are acted upon, or manipulated, where it is possible to "switch effortlessly from processes to *do* mathematics, to concepts to *think* about." (Tall, 2004a, p. 30). Movement from the embodied world to the symbolic changes the focus of learning from changes in physical meaning to the properties of the symbols and the relationships between them. The formal world is where properties of objects are formalized as axioms, and learning comprises the building and proving of theorems by logical deduction from the axioms. This theory of three worlds extends the theoretical perspective provided by Dubinsky and others (Cottrill, Dubinsky, Nichols, Schwingendorf, Thomas, & Vidakovic, 1996; Dubinsky, & McDonald, 2001). They have described how actions become interiorised as processes that in turn may be *encapsulated* as objects, forming part of a schema. This is especially important when concepts are presented in a symbolic representation, as they often are in linear algebra. Gray and Tall (Gray & Tall, 1994) have discussed the idea of *procepts,* where these symbolic forms need to be perceivable by students (and experts) in a dual manner, as either process or concept, depending on the context. This ability, which forms part of the mathematical versatility described by Thomas (2006) is essential to make progress in mathematical thinking in the symbolic world.

In linear algebra too it has been recognised by Hillel (2000) and Sierpinska (2000) that conceptual difficulties are often linked to its three kinds of description or representation: the general theory; the specific theory of R^n and the geometry of n-space. Forming the links between these abstract, algebraic and geometric levels, or representations, is the basis of many student problems, and there is a need to make explicit links between them. Addressing multiple representations of concepts is important since students require an ability to establish meaningful links between representational forms, referred to as *representational fluency* (Lesh, 1999). This notion forms part of *representational versatility* (Thomas & Hong, 2001; Thomas, 2006), which includes a) addressing the links between representations of the same concept, b) the need for both conceptual and procedural interactions with any given representation, and c) the power of visualization in the use of representations. Such understanding is so important that it has been suggested that 'a central goal' of mathematics education should be to increase the power of students' representations (Greer & Harel, 1998, p. 22). One reason for this strong emphasis is that, according to Lesh (2000, p. 74), the idea of representational fluency is "at the heart of what it means to 'understand' many of the more important underlying mathematical constructs".

When eigenvalues and eigenvectors are introduced to students, the formal world concept definition may be given in words, but since it has an embedded symbolic form the student is soon into symbolic world manipulations of algebraic and matrix representations, e.g. transforming $Ax = \lambda x$ to $(A - \lambda I)x = 0$. In this way the strong visual, or embodied metaphorical, image of eigenvectors can be obscured by the strength of the formal and symbolic thrust. However, using an enactive, embodied approach first could give a feeling for what eigenvalues, and their associated eigenvectors are, and how they relate to the algebraic representation. This was an area considered in this research, along with the idea that the symbolic world thinking inherent in the manipulation process might be obscuring understanding of the concepts of eigenvalue and eigenvector. For example, explanations of what an eigenvector is may start with it as an object and then explain the effect of performing actions upon it; applying a transformation to it and multiplying it by a scalar. However, to find the eigenvector one must first find its associated eigenvalue, holding in obeyance any action to be performed upon the eigenvector until it's found. Hence the specific research questions addressed in this study were:

- How do students think about concepts of eigenvector and eigenvalue?

- Is their thinking influenced by embodied, or geometric notions?
- What is the role of definitions in structuring their thinking?
- How do they cope with the potential process-object obstacle in the eigenvector definition?

Method

The research project comprised a case study of three groups of students from Auckland University. Group A comprised 10 students from Maths 108, a first year computation-to-abstraction course covering both calculus and elementary linear algebra. Of these students six (numbered 1-6 below) had attended the first-named author's lectures, while the rest attended other streams. The teacher-researcher tried to emphasize a geometric, embodied approach and she also took the students for two tutorials in a computer laboratory, showing them how to use Maple for linear algebra, and highlighting the visual aspects in the lectures. In group B were 70 stage one mathematics students taking a core mathematics paper (Maths 152), covering both calculus an linear algebra, and designed for mathematics majors, and hence having more advanced topics than 108. The final group, C, comprised 42 students from the second year Maths 208 course. These students had all studied Maths 108 or its equivalent.

The group A students sat a test designed to assess student understanding of the concepts of eigenvectors and eigenvalue, and their ability to carry out the process of finding them for a given 2x2 matrix (see Figure 1). The group B students completed a test designed to assess student understanding of linear algebra, including eigenvectors, in each of the geometric, matrix and algebraic representations. Group C students sat a linear algebra test on the concept of eigenvalues and eigenvectors (see Figure 1) also examining students' geometric, matrix and algebraic understanding rather than simply their procedural abilities.

Maths 108 Questions

1. Describe the definitions of eigenvalues and eigenvectors in your own words.

2. Let $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$ be a 2×2 matrix. What are the eigenvalues and eigenvectors of matrix A?

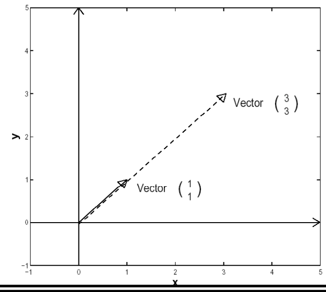
   [This is a worked example in the course manual.]

3. Suppose A is a matrix representing a transformation and: $A\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 3\begin{pmatrix} 2 \\ 3 \end{pmatrix}$. What does this tell us about the 3? Describe this geometrically.

4. How can we decide whether a given vector is an eigenvector of a matrix? Explain this in your own words.

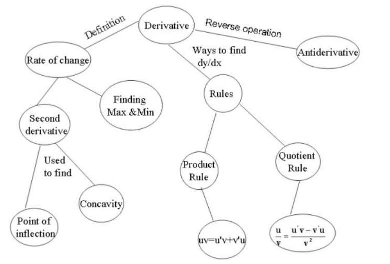
8. How many different eigenvectors are there associated with a given eigenvalue?

11. Describe what the following diagram may represent as best as you can.

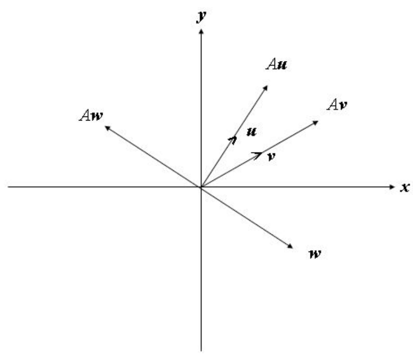


Maths 208 Questions

1. Define the notion of eigenvalues and eigenvectors in your own words.
2. Can $\begin{bmatrix} 3 \\ -4 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$ both be eigenvectors for a given matrix? Explain your answer.
3. Concepts maps are often a good way of learning about a new concept. Here is a concept map for the derivative of a function. Draw one below for eigenvectors and eigenvalues.



4. If A is a 2 by 2 matrix, explain why the picture below is not possible.



5. If $Ax = \lambda x$ put in all the necessary steps in order to show that $(A-\lambda I)\, x = 0$.
6. (a) What do these all have in common? Explain.

$$\begin{pmatrix} -1 & -2 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \begin{pmatrix} -1 & -2 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ -3 \end{pmatrix} \qquad \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \end{pmatrix}$$

   (b) Fill in values a,b,c,d that do not follow the above pattern. $\begin{pmatrix} -1 & -2 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}$

Some formatting changed.

Figure 1. The Maths 108 and 208 test questions.

This compares with courses designed in such a way that students are able to pass them simply by knowing routine processes, and not necessarily understanding ideas. Only the questions relating to eigenvalue and eigenvector are discussed in this paper.

Results

Embodied, geometric conceptions

We were aware that approaches to eigenvalue and eigenvector often ignore the embodied, geometric aspects of learning this topic, especially in our university courses, but wondered if the students could be encouraged to think this way. For the first year students, meeting linear algebra for the first time, there were some indications in the test responses that they were using the embodied world to help build their thinking about eigenvalues and eigenvectors. For example question one asked them to describe the terms in their own words. Three of the students (1, 6, 8) mentioned the idea of the 'direction' of a vector. Although student 1 did not use it correctly, the other two had a clearer embodied aspect to their concept image of eigenvector.

> Student 6: After transformation the direction of eigenvectors will not change.
> Student 8: Eigenvector is a vector which does not change its direction when multiplied (or transformed) by a particular matrix. An eigenvector can change in length, but not in direction.

We see that student 8 has also added the embodied notion of change of length to her thinking. While the other students who answered the question referred to the procedural, symbolic manipulations in their answers, two of them (5 and 9) had formed a mental model of the structure of this (see Figure 2).

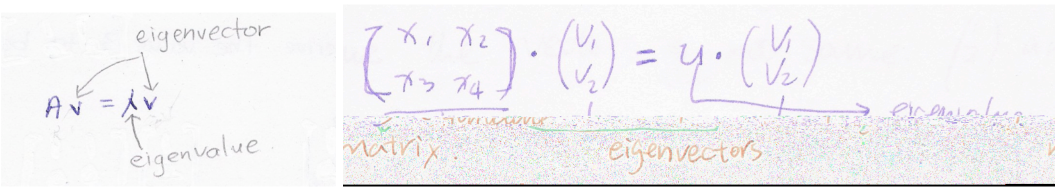


Figure 2. Students 5 and 9 use a structural model of the algebra.

In the answers to question 3 we also saw examples of the embodied nature of the students' thinking. Student 1 explains that "the eigenvalue changes the vector's direction. ie more steep.", using the embodied notions of 'change of direction' and 'steepness'. Student 4 also said that “3 is not an eigenvalue of the equation. Hence it changes the direction of the original vector.” Student 6 had similar recourse to the embodied idea of change of direction of the vector, drawing the picture in Figure 3.

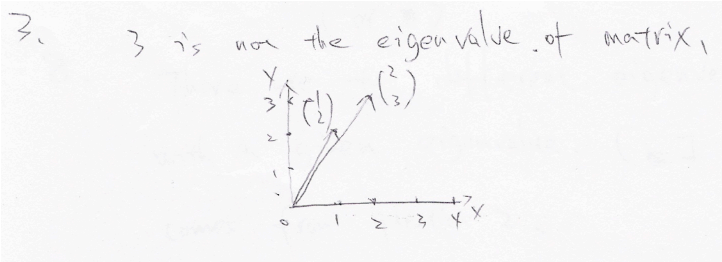


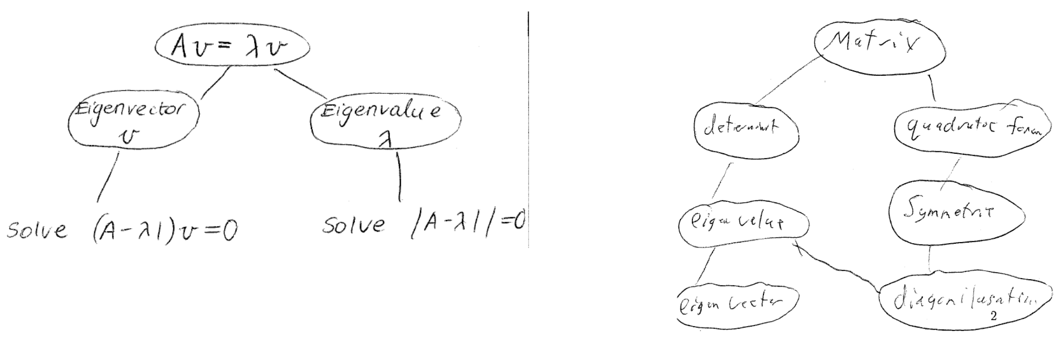
Figure 3. Student 6’s embodied notions of ‘change of direction’ and ‘steepness’.

Question 4 saw students 4 and 8 also refer to the idea of direction to decide on whether a vector is an eigenvector. In question 11, all the students except 7 and 10 linked the diagram to a vector (1,1), an eigenvalue of 3 and a final vector (3,3). In doing so they again used embodied terms such as “being stretched” (1), “it makes eigenvector longer” (3), and “stretch the length of (1,1)” (5). It seems that the researcher’s students did make more use of embodied ideas than the others.

The section on eigenvectors and eigenvalues in the second year, Maths 208, coursebook does not contain a single diagram, and thus totally ignores the embodied aspects of learning this topic. Hence we expected that students would not have these ideas. An embodied concept we investigated was whether the students had abstracted and assimilated to their eigenvector schema the geometric idea that when an eigenvector is multiplied by the transformation matrix it ends up in the same direction (but not necessarily the same sense). To answer question 4 (Figure 1) in their test the students had to see geometrically that each of the three vectors satisfied the eigenvector definition, link this to the data from the matrix size, and use this inter-representational reasoning to see there is a contradiction, since a 2x2 matrix can not have three independent eigenvectors. Of the 42 students 14 were unable to answer the question at all, and only 6 gave a correct explanation of why the diagram was not possible. These included student E, who wrote “The picture above implies A has 3 eigenvectors of different directions (but if A is 2x2, it has a maximum of 2 eigenvectors of different directions.)", student F who said "If A is 2x2 matrix, it can have maximum of 2 linearly independent vectors in its basis. Therefore one of A_w, A_u and A_v must be impossible." and student V "Diagram shows 3 eigenvalues/eigenvectors a 2x2 matrix should have only 2." Others (brackets contain the student code) were unable to relate the picture to the concept of eigenvector, and instead a number seemed to relate it to the basis for a space, which may have been the place where they had seen a similar diagram. They wrote comments such as "Since there are only 3 vectors it will generate a space." (Q), "because you don't need that many vectors to span the plane" (U), "Maybe too many dimensions?" (AC), "$\Rightarrow$ linearly dependent" (AE) and "It's got way to [sic] many vectors in it." (AD). Some appeared confused and wrote, for example "The picture shows scalar multiplication which should not occur in a 2x2 matrix." (L), "Because *w* is in a different direction." (W) and "because the vectors are on different planes." (X).

We asked question 2 in order to see if the students' understanding of eigenvectors was limited to the algebraic and matrix (vector) representations or whether they used embodied, visual explanations in their answers. Of the 42 students, 14 correctly answered the question, while 4 could not write anything. However, of those who were correct 13 used only an algebraic or matrix procedural explanation, often involving multiples, such as "Yes $\begin{bmatrix}-3\\4\end{bmatrix}^{\times(-1)}=\begin{bmatrix}3\\-4\end{bmatrix}$" (T) "Yes they are merely a factor of –1 of each other" (A), "Yes, since the eigenvalues are –1 and 1" (Q) and "Yes. Eigenvectors of a given eigenvalue is any multiple of any given eigenvector." (E). Only very occasionally was a geometric comment made, such as "Yes as $\begin{bmatrix}-3\\4\end{bmatrix}$ & $\begin{bmatrix}3\\-4\end{bmatrix}$ are multiples of each other in the opposite directions" (C). Some confusion again showed through between different eigenvectors and multiples of an eigenvector, with 7 students commenting on eigenvectors having to be independent "No, because there is a linear relationship between them." (K) and "No, eigenvectors of a matrix should be linearly independent.” (L). Of course, this does not mean that the students answering in a non-geometric manner were not able to think geometrically. However, it does imply that this embodied, visual mode of thinking is definitely not at the forefront of their approach when the question is presented in a matrix format.

This lack of a link to a geometric perspective was certainly confirmed by the definitions in question 1 and the concept maps drawn in question 3. The first question asked students to relate their understanding of the formal definition, but without repeating it. Of the question 1 responses 16 did not write anything (or wrote ‘no idea’), 17 gave a procedural response based on the equation $Ax = \lambda x$ or ‘multiples’ of a vector, and only two made any mention of geometric idea, either correctly stating for the vector space R^n that “A matrix ‘A’ when multiplied by a vector ‘*v*’, the resulting vector has the same direction as the original vector ‘*v*’.”(E) or wrongly saying that they “generate a plane” (T). In addition there was one vaguely conceptual answer and 6 that we were unable to categorise. In question 3 not a single student put anything even remotely linked to geometry in their concept map for eigenvector and eigenvalue. 18 of the students did not draw anything at all, and of the remainder 21 drew a procedural map and 3 a conceptual one, with few, or no, action verbs. Figure 4 gives a typical example of the procedural concept maps, and one of the rarer concept ones. Student E sees the actions in the solving process as the only relevant detail, while student G presents only concepts, including a link to one from another part of the course.



Student E Student G

Figure 4. Procedural and conceptual concept maps.

Procedural, process and structural conceptions

Group B, comprising more advanced first year students, were asked, among other things, to give in their own words, a definition of eigenvector. Of the 60% (42) who responded, 43% gave one of the following procedurally-based or symbolic world answers: a vector that when multiplied by a particular matrix will equal a multiple of itself; or $Ax = \lambda x$ where x is the eigenvector. A further example of this emphasis on procedures was the fact that 9% of the students answered by indicating that an eigenvector is constructed from an eigenvalue. One student commented that he couldn't explain but only knew 'how to calculate' (see Figure 5). This is evidence that such students' thinking is firmly based in the symbolic world.

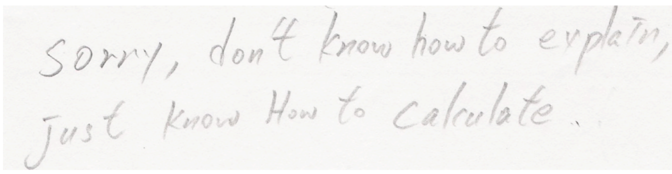


Figure 5. A student's response for a definition of eigenvector.

Figure 6 again shows this predilection for symbolic world procedures in the response of another student who instead of defining, tried to illustrate how to find the eigenvectors.

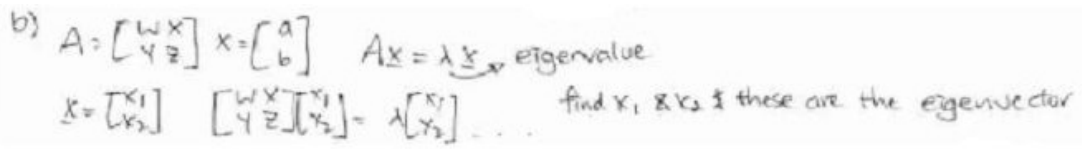


Figure 6. A student's procedural response when asked for a definition of eigenvector.

It appears that many students realise that definitions are important and valuable, stating for example that "I think it's quite important to understand the definitions, otherwise you are completely lost further on", however they find them too difficult ("I think one of the hardest things to understand is the definitions...If you don't know the definitions, then you can't answer the question"), and they often believe that they don't need to know them to solve problems. A view typified by the statement "I can solve linear algebra questions even though I don't know the exact definition". Generally definitions arise from formal world thinking and many of these university students are not yet thinking in this way so the full value of the definitions often eludes them.

A second question for the group B students asked them "If a linear transformation is represented by a matrix Q, and a vector P exists such that $QP = 3P$, what does the 3 tell us about this transformation?" It was interesting in this question, which was presented using algebraic and symbolic representation of vectors and matrices, that while 25.7% did not answer this question, many did, with the most common responses, '3 is the eigenvalue' (57.7%), '3 times longer', 'expanded by 3', 'Q= 3', '3 is invertible', and 'QP is in the same direction as P but 3 times longer'. Here again we see embodied constructs such as 'longer', 'expanded' and 'same direction' coming to the fore. Thus, while some of these answers are not correct, a good percentage of students managed to interpret the symbols in question 8 well, in spite of the words 'linear transformation' that may have hindered some students. It may be that an algebraic context links better with the symbolic world procedural emphasis that we see many students prefer, even when the question is conceptual in nature. This is further confirmed by responses to the question in Figure 7. Here only 11% of the students mentioned either word eigenvectors or eigenvalues. Hence, although they may be familiar with the notion of eigenvector, and most likely can solve standard problems relating to the topic, they were not able to relate to them in a different, geometric, representation.

Which concept in linear algebra does this diagram refer to?

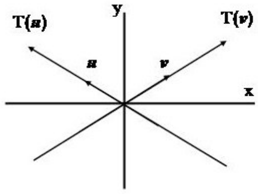


Figure 7. A question on eigenvector employing a graphical representation.

As we see above, students whose thinking is in the symbolic world often prefer a procedural approach, but they also often have difficulties with this too. Of the ten students we considered in detail from group A, 5 were able to find correctly both the eigenvalues and eigenvectors for the matrix $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$ in question 2. Of the others, two (students 2 and 9) found the eigenvalues but were unable to find the corresponding eigenvectors, and three (students 1, 3 and 10) were unable to make any progress. Student 3 wrote "I just forgot conseption [sic] of it" and student 10 "I would like to do this with the help of Maple."

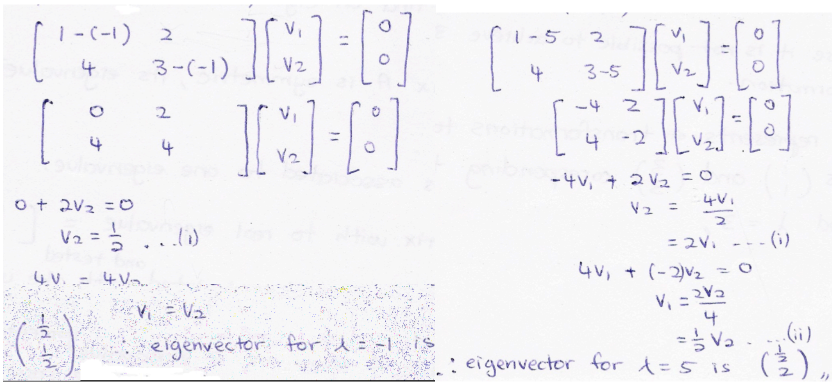


Figure 8. Student 9's working to find the eigenvectors.

Student 9's working as she tries to find the eigenvectors is shown in Figure 8. The arithmetic and symbolic manipulation here contains a number of errors (1–(–1)=0; $0+2v_2=0 \Rightarrow v_2=1/2$; $4v_1+4v_2=0 \Rightarrow 4v_1=4v_2$; and $v_2=2v_1 \Rightarrow$eigenvector is (0.5, 2)), showing a weakness in such symbolic manipulation, rather than in the understanding of the conceptual process. Student 2 made a similar manipulation error, moving from writing the matrix $\begin{pmatrix} -4 & 2 \\ 4 & -2 \end{pmatrix}$, without a vector, say, $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, or an '=0', to an incorrect vector $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

The fact that the working in the final stages of the process of finding the eigenvector caused some problems is not surprising when we look at the course manual. We can see from Figure 9 that the final steps in the method are not delineated, but are presumably left for the student to complete.

$$\begin{pmatrix} 1-5 & 2 \\ 4 & 3-5 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$\begin{pmatrix} -4 & 2 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

One non-zero solution to this system is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Figure 9. Part of the course manual's method for finding an eigenvector.

As we can see, they sometimes find this a problem. This omission proves, we think, to be even more costly in terms of conceptual understanding. Questions 4 and 8 in the test addressed the conceptual nature of the eigenvector by considering two of its properties. A student with a structural, or object, perspective of eigenvector might be expected to describe whether a vector is an eigenvector or not, without resorting to a procedural calculation (Q4), and to say that any scalar multiple of the eigenvector will also be an eigenvector (Q8). Students 2, 4, 5, 6, 7, and 8 correctly found the eigenvectors from the procedure. Of these three (5, 6 and 7) gave a procedural response to question 4, referring to key aspects in the symbolic world, rather than giving an object-oriented answer.

Student 5: First let the matrix times the vector…If the answer equal to $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ *or* n *times* $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ *then* $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ *is an eigenvector of the matrix.*

Student 6: Using the equation Ab=λb *to confirm the relation.*

Student 7: Use the formular [sic] Av=λv.

In contrast, the others gave replies based on the word definition, showing some move away from the need only to employ the symbolic world towards inclusion of some embodied thinking, with concepts such as ‘direction’ and ‘expanded’:

Student 4: When a given vector multiply with a matrix, if the direction of the vector doesn’t change, only expanded or shrinked [sic] we can say the given vector is the eigenvector.

Student 8: When its direction isn’t changed when it’s multiplied by the matrix.

In their responses to question 8, students 1, 2, 4, 5, and 9 stated that there is only one eigenvector associated with each eigenvalue. Stating, for example, “Each instance of an eigenvalue has one and only one eigenvector associated with it.” (student 2) and “One eigenvector is associated to one eigenvalue.” (student 9). However, students 3 and 7 said that there were an infinite number, writing “I think that for any eigenvalue can be infinitely [sic] number of eigenvector because [blank].” (4) and “infinity” (7).

As mentioned above (see Figure 9) the course manual did not put in all the details at the end of the method to find the eigenvector. 2 students (2 and 6,) followed this pattern and tried to write down the vectors from the matrix form of $(A-\lambda I)x=0$, one succeeding (6) and the other not (2). Others (students 7, 8), went further, and were sometimes unsuccessful due to manipulation errors (7, 9—see Figure 6), or managed it correctly (5, 8). In the case of student 5 this was accomplished using v_1 and v_2, but giving them the values 1 and 2 at a crucial point. However, only one student (4) managed to write the eigenvectors in the form $\begin{pmatrix} v_1 \\ 2v_1 \end{pmatrix} = v_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} v_1 \\ -v_1 \end{pmatrix} = v_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ before getting the eigenvectors correct. Unfortunately, this student was not among the two who were able to say that there are an infinite number of possible eigenvectors. This step of seeing that any scalar multiple v_1 of the vector satisfies the equation may be a direct consequence of understanding this last step in the symbolic world manipulation, missing from the manual.

A process-object problem

One of the early tensions in eigenvector study that we have uncovered in our work involves the basic equation $Ax = \lambda x$ used in defining the concept. Here the two sides of the equation represent quite different mathematical processes, but each has to be encapsulated to give equivalent mathematical objects. In this case the left hand side is the process of multiplying (on the left) a vector (or matrix) by a matrix, while the right hand side is the process of multiplying a vector by a scalar. Yet in each case the final object is the same vector.

A teaching presentation of the transformation of this equation to the form $(A-\lambda I)x=0$, required in order to carry out the process of finding the eigenvalue λ, may tend not to make explicit the change from λ to λI, from a scalar to a matrix. The section of the first year course manual of group A where this is done is shown in Figure 10a. We see that the problem is skated over and in the line moving from $Ax = \lambda x$ to $(A-\lambda I)x=0$ the comment is simply made “note the use of *I* here.” In Figure 10b we see the example of student 6 who explicitly replaces the 5 and –1 in 5***b*** and –1***b*** on the right of the equation with the matrices

$5I$ and $-I$. This no doubt helped him with equating the objects constructed from the processes, but thinking of $Ax = \lambda x$ as $Ax = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} x$ may be an obstacle to understanding how the definition of an eigenvector relates to the algebraic representation. We note that he has not used the version in the manual but needed to make the processes the same in the first line. Also, on obtaining the solution student 6 has employed embodied thinking to draw a graph of the solution set.

$$\begin{aligned} A\mathbf{v} &= \lambda \mathbf{v} \\ A\mathbf{v} &= \lambda \mathbf{v} \\ A\mathbf{v} - \lambda \mathbf{v} &= \mathbf{0} \\ (A - \lambda I)\mathbf{v} &= \mathbf{0} \qquad \text{[note the use of I here]} \\ B\mathbf{v} &= \mathbf{0} \qquad \text{[where } B = A - \lambda I\text{].} \end{aligned}$$

a. b.

Figure 10. The course manual dealing with the two processes and Student 6's solution to the problem.

Is the above a problem that soon dissipates, or does it still exist with students in their second year? To answer this we asked the second year group, C, about the same issue (Q5 in Figure1). Moreover, we noticed that the coursebook for Maths 208 (group C) also glossed over the steps required to go from $Ax = \lambda x$ to $(A - \lambda I)x = 0$. Figure 11 shows the section of this coursebook where it is presented. On the surface there seems a subtle change of object from a scalar λ to a matrix λI, but the nature of I in terms of the field it belongs to is not discussed. The bottom part of the figure shows the corresponding section from the textbook (Anton & Busby, 2003) used, and here a small step is inserted, showing $Ax = \lambda Ix$, but it is not emphasised that it comes from $\lambda(Ix)$. This has the effect of changing the process on the right hand side to one very similar to that on the left, namely multiplication of a vector (or matrix) by a matrix (and then by a scalar afterwards).

Definition 4.1. *Given a square $n \times n$ matrix A, we can sometimes find a non-zero vector $\mathbf{v} \in \mathbb{R}^n$ and a corresponding scalar λ, such that*

$$\boxed{A\mathbf{v} = \lambda\mathbf{v}.} \tag{4.1}$$

We call any non-zero vector $\mathbf{v}$ *which satisfies (4.1) an* **eigenvector** *of A, and the corresponding scalar λ an* **eigenvalue.**

The matrix equation (4.1) can be rewritten

$$(A - \lambda I)\mathbf{v} = \mathbf{0}. \tag{4.2}$$

The most direct way of finding the eigenvalues of an $n \times n$ matrix A is to rewrite the equation $A\mathbf{x} = \lambda\mathbf{x}$ as $A\mathbf{x} = \lambda I\mathbf{x}$, or equivalently, as

$$(\lambda I - A)\mathbf{x} = \mathbf{0} \tag{4}$$

Figure 11. The coursebook and textbook explanations of the move from $Ax = \lambda x$ to $(A - \lambda I)x = 0$.

We wanted to know how the student perspective on this equation-changing would influence their ability to perform the task, and hence the question. In the event it proved to be quite revealing. It was clear that 12 of the students did not understand what the *I* was, where it came from, and why it was there. We see in Figure 12 that this affected their ability to complete the relatively simple three-line transformation of the equations. These three students, A, K and S, either ignored the identity matrix (S) or simply inserted it in the final line (A and K).

$Ax = \lambda x$
$Ax - \lambda x = 0$
$(A - \lambda)x = 0$

Student S

$Ax = \lambda x$
$Ax - \lambda x = 0$
$(A - \lambda I)x = 0$

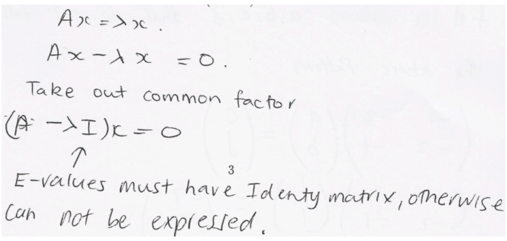
Student A

$Ax = \lambda x$
$Ax - \lambda x = 0.$
$(A - \lambda)x = 0.$
$(A - \lambda I)x = 0$

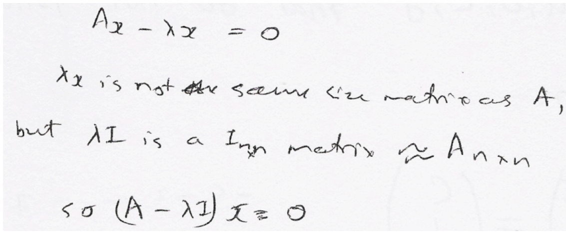
Student K

Figure 12. Working of students A, K and S on question 5.

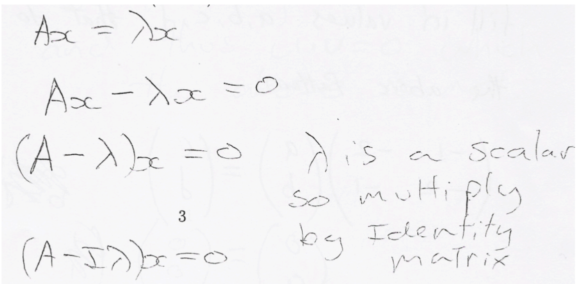
Some evidence of what was causing the difficulty was found in the explanations of other students. Figure 13 shows the work of four more students, C, J, L and P. Here students C and J are finding it difficult to explain why the λ seems to become λI. Student J tries to explain, with little understanding, that “E[igen]-values must have Identity matrix, otherwise can not be expressed.” and hence the *I* has to be inserted. However, students L and P have both decided that $A - \lambda$ cannot be accomplished (“can’t work”) since they are of different types—“*A* is a matrix λ is a number”—and so it is necessary to “multiply [λ] by the identity matrix” to solve this problem, and P almost correctly performs this. On the other hand, student C is clearly struggling with the idea that the order of λx will not be the same as that of A, but is happier that λI is also an nxn matrix. To overcome the difficulty he has focussed on the input objects on each side of the equation that are operated on, rather than the object produced by the process, and the processes are still causing cognitive conflict.



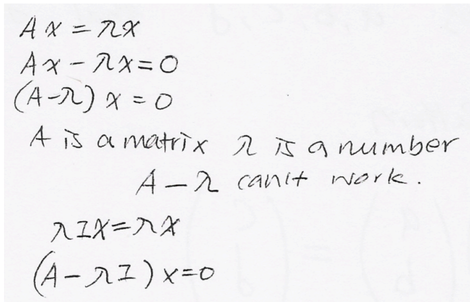
Student J



Student C



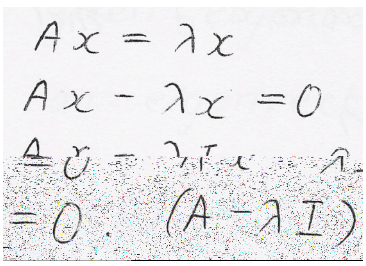
Student L



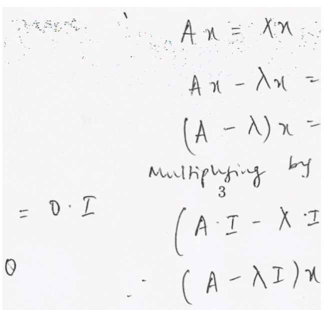
Student P

Figure 13. Working of students C, J, L and P on question 5.

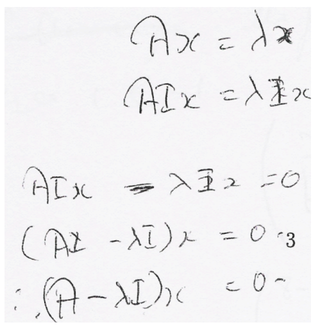
There were 5 students who either performed an operation that they were used to and understood, namely multiplying the equation through by I (and assuming associativity) or replaced x by Ix. Both x's were replaced immediately by student B, but student Q, like P, multiplied by I only when there was clearly a problem with the $A–\lambda$ (see Figure 14), and may not have fully understood. On the other hand, student E chose to follow the textbook and replaced x by Ix.



Student E



Student Q



Student B

Figure 14. Different strategies from students B, E and Q on question 5.

The above scenario demonstrates that paying attention to the surface features of manipulations in the symbolic world, looking at, in the words of Mason (1992, 1995), has led some of these students to a situation where they are faced with a cognitive obstacle due to an apparent contradiction, namely subtracting a scalar from a matrix. In order to cope with such a problem it is necessary to look deeper, to look though (Mason, 1992, 1995) the symbolism at the conceptual meaning of the objects (c.f Ainley, *et al.*, 2002). For example, in the above one needs to see the I as the identity in the field of matrices rather than real numbers, and to appreciate that $Ix=x$, for all x, is a property of this object. Even the 'successful' students above have not clearly identified in their working the precise nature of the identity I that they have used. In general, the difficulties described above show that while the symbolic world thinking stresses manipulation of algebraic symbols it is important for students to have a clear understanding of what it is that those symbols represent and how they interact with one another in order to build understanding of why the manipulations are acceptable or not.

Conclusions

This study suggests that students tend to think about the concepts of eigenvector and eigenvalue in a primarily symbolic way. They prefer to think of linear algebra as the application of a set of procedures, which if learned will enable them to solve given problems, rather than to think about concepts. While they may know that formal ideas such as those presented in definitions are important, they do not like them, and do not seem to learn them or quote them. Not surprisingly, this means that they do not understand the meaning of definitions and are unable to apply them even in simple situations.

Our data also appear to show some aspects that seem important for the teaching of eigenvectors and eigenvalues in linear algebra. One is the importance of explicitly presenting complete procedures for finding eigenvectors, and of linking these to conceptual ideas such as the number of possible eigenvectors. Another is that students seem reasonably confident with the algebraic and matrix procedures, but the vast majority had no geometric, embodied view of eigenvectors or eigenvalues, and could not reason on the relationship between a diagram and eigenvectors, to their detriment. This is not surprising since our coursebooks did not present such a view, and it appears that the lecturers did not do so either. In addition, we found some limited evidence to suggest that students who received encouragement to think in an embodied way about eigenvectors found it a useful adjunct to the procedural calculations they carry out in the symbolic world. This may be because these manipulations in the matrix and algebraic domains cause some conflict with understanding the natural language definition of eigenvalues and eigenvectors, and that an embodied approach may mediate initial understanding. Hence, since embodied notions of mathematics are regularly employed at all levels of mathematical thinking it is something that teachers should consider putting in place. This is in agreement with the suggestion of Harel (2000), who, while cautioning that some students persist in seeing a geometric object as the actual mathematical object and not as a representation of it, maintains that "In elementary linear algebra, there should be one world–R^n–at least during the early period of the course." (p. 185). When asked whether they thought computers should be used in linear algebra lectures the majority of students agreed that such work was beneficial, and this may provide a way to introduce this geometric thinking.

Another key finding of this study is that the two different processes in $Ax = \lambda x$ may be preventing understanding of ideas such as the algebraic progression, in the symbolic world, from $Ax = \lambda x$ to $(A - \lambda I)x = 0$. Many students do not perceive this as straightforward. The different processes in the first equation may prevent students knowing what identity the I refers to, and the teaching may tend to move the focus of attention to λI rather than Ix. Since students seem to lack the understanding of how the second equation is obtained from the first, this implies a need to make it explicit in teaching, explaining that the identity being used in the process is an nxn matrix, and it is the x that is being multiplied by this identity. This will also solve the process problem with the first equation, if it's done immediately.

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