10th Conference on RUME

LEARNING PROOF: FROM TRUTH TOWARDS VALIDITY

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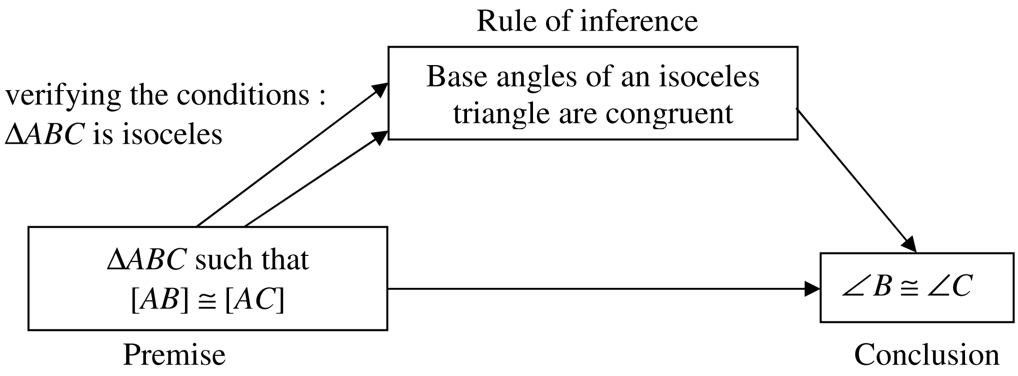
Researchers in math education have devoted numerous works to the problem of learning proof. Particularly sensitive is the specific role given to propositional logic and formal reasoning in teaching devices aimed at learning proof. For Duval (1995), the ability to validate reasoning using criteria other than group consensus or empirically induced information is a condition necessary for learning formal proof. But "... the knowledge of these rules [according to which the propositions are organized in a reasoning intrinsically linked to a language, natural or formal] does not necessarily make one aware of the valid or non-valid nature of a reasoning, any more than knowledge of grammatical rules enables a majority of students to write correctly" (op. cit., p. 212, our translation). Although recognized in the French-speaking mathematical community, the work of R. Duval is not widespread in the Anglo-Saxon community. While Dreyfus (1999) devotes a short paragraph to it in his article/summary, "Why Johnny Can't Prove", it is not even mentioned in otherwise exhaustive overviews by Hoyles (1997) or Epp (2003). In this article, we give an initial account of an experiment, which relies primarily on research orientations proposed by Duval (1995; 1991) with some differences, as stated in Tanguay (2005, § 4.1), and to which more recent considerations will be added.

1. Theoretical setting

1.1. Argumentation versus Formal Proof

The reader may refer to Duval (1991; 1995, chap. V) for a detailed characterization of argumentation and formal proof, which are, in his view, two radically different types of reasoning. Like Balacheff (1987), Duval employs the French term '*démonstration'* to refer to formal proof, establishing that a statement is true in compliance with the rules of propositional logic, by deductively chaining other statements either already proved or taken for granted as axioms. In argumentation, one tries to convince one's interlocutor by putting forward 'arguments', which are propositions combined for the purposes of mutual reinforcement or confrontation, according to two points of view: the target-statement is either true or false.

Propositions are brought into argumentation for their *contents*, obey criteria of *relevance* and are organized, as a ***discourse***, by simple *accumulation*. Formal proof has the more uncompromising structure of a (propositional) ***computation***, consisting of chained *inferences* or *deductive steps*. Each proposition in a given inference has one out of three *operative statuses*: premise (or entry proposition), rule of inference and conclusion (or inferred proposition). This status is independent of contents, as a proposition may change its status within the same proof; more often, the inferred proposition is 'recycled' as a premise for the next inference. This explains why a formal proof is referred to as a computation: it makes use of substitutions — the terms of the rules are substituted by the premises so that the conclusion may be 'detached' — and of chaining by transitivity, as in an algebraic setting.



An Inference, or Deductive Step

Duval's main thesis regarding the difficulty for students in learning formal proof, is that they don't easily grasp its specific requirements, because they perceive and treat it as if it were an argumentation. What are the reasons for this confusion?

First, linguistically, argumentation and formal proof are conveyed in the same way, using the same connectors (and, or, but, because, hence, if … then, etc.), their function nevertheless differing in each (Duval, 1992-93). Next, whether the source is teacher or textbook, because the (local) ternary structure of inferences is almost never explicitly stated in formal proofs:

- the inference is more often reduced to the binary frame of the underlying implication, with the inference rule implicit;
- when two inferences are chained, the common proposition is not repeated;
- there is no explicit verification that the premises include all terms of application of the rule;
- the theoretical status of the inference rule may not have been previously established, as is often the case with proofs using transformational geometry, etc.

Also, the global structure of more-than-one-step formal proofs is too often unintelligible to students. This global structure is seldom linear (e. g. the graphs in Annex), but even when using the two-column format, (Affirmation 1, Justification 1, Affirmation 2, Justification 2…), proofs are displayed in a linear manner, in the mode of spoken or written discourse. They are assimilated as such by young students, who have this ‘spoken way of coping with the written’ diagnosed by Duval (2001); that is, without pausing, without returning to already stated propositions, without taking a step back to get the global picture and to grasp elements of macro-organization; without any mental reorganization, by which some of the propositions separated out in the text may be brought together; in short, without reflection...

Lastly, and perhaps more fundamentally, students must detach themselves from the *epistemic semantical value* (Duval, 1995, p. 222) of the propositions, which is the degree of reliability allotted to the content — obvious, certain, plausible, unlikely, impossible… — so as to be able to acknowledge the variability of the propositions’ operative status.

1.2. Truth Value as Impediment

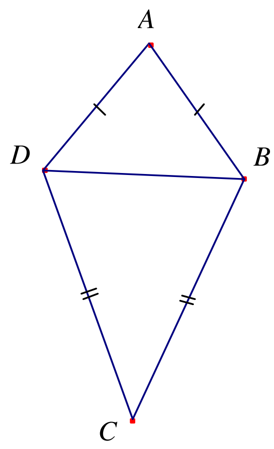
Indeed, it is often stated that in geometry, the perceptive obviousness — the widespread ‘you can tell from the diagram!’ — hinders students’ reasoning (see for instance Chazan, 1993). But how is it that so many students who have assimilated the instruction ‘you’re not allowed to rely on the diagram’, remain unable to meet the requirements of geometrical proof? In Tanguay (2005a), we have asserted that students who succeed at stemming the epistemic value of obviousness, must then face a more subtle obstacle, which we identify as *truth value as impediment*. For the sake of explanation, let’s consider Johnny[1], a 13-year-old student, who is working with the proof that *every kite has a pair of congruent opposite angles*.

$[AD] \cong [AB]$ ΔABD is isoceles
$\angle ADB \cong \angle ABD$,

$[CD] \cong [BC]$ ΔBCD is isoceles
$\angle CDB \cong \angle CBD$,

which implies that

$$\begin{aligned} \text{mes}\,\angle D &= \text{mes}\,\angle ADB + \text{mes}\,\angle CDB \\ &= \text{mes}\,\angle ABD + \text{mes}\,\angle CBD \\ &= \text{mes}\,\angle B. \end{aligned}$$



To understand the proof, Johnny has to dissociate the truth of the statement 'ΔABD is isoceles' from the truth of the statement 'ΔABD is iso-angles', which in his mind always go together. Note indeed that a student of Van Hiele level 2[2] (e. g. Crowley, 1987 or Burger and Shaughnessy, 1986) has a tendency to 'amalgamate' properties, and, when asked for a definition, to give what Van Hiele has called a 'litany of properties', that is, a list of all the properties they know which pertain to the object being defined. For instance, in defining an isosceles triangle, this student would answer, *a triangle with two congruent sides, two congruent angles, which possesses a symmetry axis*. But to grasp the structure of the proof, Johnny must understand that sides [*AB*] and [*AD*] are 'already' congruent, while the base angles are 'not yet' congruent, which, to him, is rather bewildering: how could the statement 'ΔABD is isoceles / iso-angles' be half-true??!!!

[1] With apologies to Morris Kline and Tommy Dreyfus.
[2] According to the Van Hieles, it is only when reaching level 3 that students become able to follow or produce short formal proofs, as well as longer informal jusitifications.

He becomes convinced that, as his teachers have repeated year after year, mathematics is unique among the sciences in that each statement must be either unambiguously true or false. And this sometimes in spite of his own understanding: “Squares are rectangles” is a true statement because… (Furinghetti & Paola, 1991); “Prime numbers are odd” is a false statement because… (Zazkis & Levy, 2001). He also becomes aware of the pitfalls of perceptive obviousness, particularly in geometry, but his textbook argues that ΔABD is indeed isosceles! As long as Johnny perceives the proof as an argumentation, aiming to convince that ‘ΔABD is isoceles-isoangles’ is a true statement, this truth will act as a screen and impede his reasoning.

1.3. From Truth of Propositions towards Validity of Deductive Steps

The next example will throw more light on this issue. The following research experiment is described in a collective work published under the direction of Jean Piaget. The French psychologist B. Matalon (1962) put thirty children, aged 6 to 12, to the following test. Two light bulbs, one red and one green, are hidden in a box with flaps, so that the researcher can show one light bulb while keeping the other hidden. He gives the following instructions: “These two bulbs cannot be turned on whenever and however you want. If the red light-bulb is on, then the green light-bulb is on.” The researcher then asks the child:

1. *The red light is turned on.* [He lifts the corresponding flap]. *Is the green light on or off?*
2. *The red light is turned off.* [He lifts the flap]. *Is the green light on or off?*
3. *The green light is turned on.* [He lifts the flap]. *Is the red light on or off ?*
4. *The green light is turned off.* [He lifts the flap]. *Is the red light on or off ?*

Twenty-seven of the thirty children correctly answered question 1, while fifteen correctly answered question 4 (“If the green light is turned off, the red one cannot be on”).

Respectively, only six and five children correctly answered questions 2 and 3, a correct answer, according to Matalon, being 'I can't tell'. But in analyzing the results, Matalon is very careful not to conclude that an incorrect answer is necessarily a sign that the implication 'if red is on, then green is on' is confused with its converse 'if green is on, then red is on'. Initially, he acknowledges that the answer, 'I can't tell', is very difficult to obtain from children, who, from school, are accustomed to every question having an answer. But according to him, this is not the sole or main reason for this difficulty.

> *For a mind strongly bound to the concrete, it is difficult to dissociate reality from judgment on this reality: while the subject knows well that the light is turned on or is turned off, these two states being exclusive, and hence that only one of the two answers is 'true', we are expecting an answer of an other level, which in fact focuses on the knowledge that the subject can have, and not on the condition of the equipment* (op. cit., p. 79, our translation).

Even though the context here is quite specific, we have arrived at the very core of the obstacle to which we are referring. The child stands before a hidden light and this light is either on or off. Likewise, the student has two triangles in front of him and these triangles are congruent or are not. There is nothing to help either the child or student understand that the expected answer is on another level, that we are interested in neither the truth of the statement 'the light is on' or 'the triangles are congruent', but rather in the validity of the inferred statement 'I can conclude absolutely that the light is on' or 'I can conclude (for example, by SAS) that the triangles are congruent'. Adhering to mechanisms of formal proof requires a radical shift in level, from the pragmatic to the theoretical Balacheff, 1987). That is, a fundamental re-focusing, from the truth of propositions towards the validity of deductive steps in reasoning[3].

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[3] The two thousand year long saga of the Parallel Axiom could well be seen as an epistemological shadow of the obstacle we are considering. As long as the truth of the Parallel Axiom — would two indefinitely produced parallel lines eventually meet or not? — and its'provability' have been the focus of

2. Research Hypotheses and Questions; Designing Tasks

We have adopted some of the research orientations proposed by Duval (1991): to dissociate deductive organizational tasks from heuristic tasks; to create interplay between a non-discursive representation (via a propositional graph) and a written processing of the proof. By momentarily changing register — going from written or spoken natural language register, to setting some of the work in the graphic register — we expect students to re-evaluate their understanding of proof as argumentation.

But for the student to understand how the necessity of chained propositions creates the foundations for the truth of the target-statement, the goal of the 'geometrical game', he must adhere to that 'other level', to a theoretical standpoint where the rules of the game are no longer the same. We dissociate ourselves from Duval's research orientations by assuming that students can't subscribe to this way of reaching the goal until they have, at the very least, grasped the (new) rules. We thus consider a sequence of introductory tasks, in which students work on *already constructed graphs*. These tasks consist in the reconstruction of a geometrical proof by organizing propositions, handed out higgledy-piggledy, within the empty boxes of an oriented graph (see below). Students must also assign a number to each 'inference-arrow', according to the chosen rule of inference (see Annex). Once the graph is reconstituted, they must write up the proof 'in their own words'.

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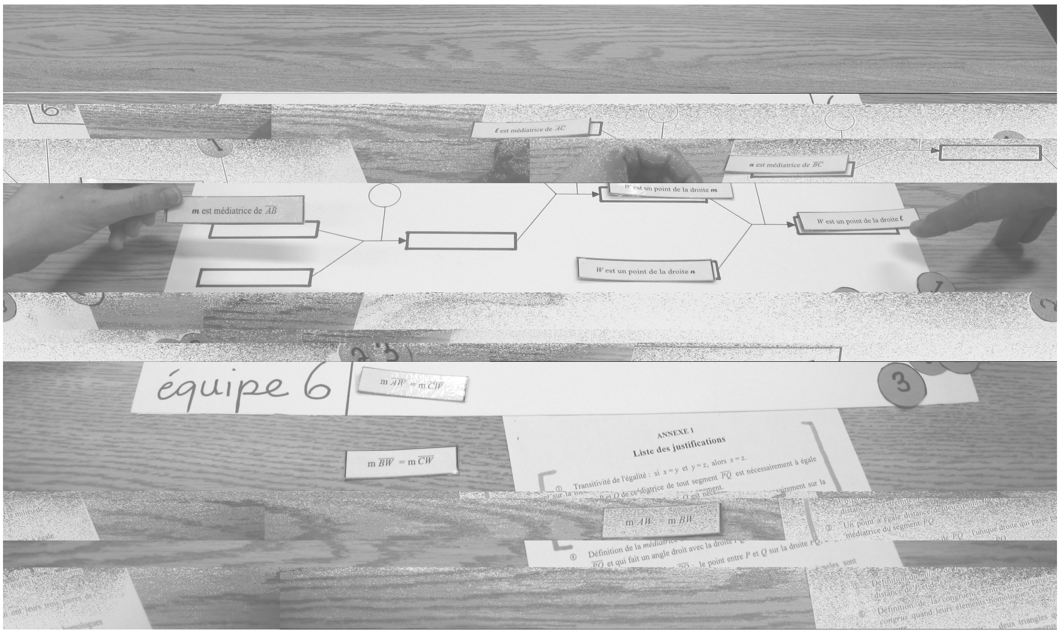
mathematicians such as Khayyam, Saccheri or Legendre, the statement itself has created short circuits and flaws in reasoning. It is only when the focus shifted toward the Parallel Axiom's operative status within different possible geometries that the problem of its independence was definitively solved.



The oriented graph on a large display

The research questions are the following:

- Are tasks such as these likely to help students in meeting the requirements of formal proof?
- Do such tasks foster this necessary re-focusing, from the pragmatic to the theoretical, from the truth of propositions towards the validity of inferences?
- To what extent do they contribute to better mastery:
    - of the deductive structure?
    - of the logical rules necessary for formal reasoning?

The target-statements dealt with in the sequence have been chosen because their proofs require subtle coordination of implications and converses (see Annex). This research is part of a larger project conducted according to the methodology of Design Research (e.g. Edelson, 2002 or Steffe & Thompson, 2000).

3. The Research Experiment

3.1. Setting and Data

The sequence of tasks (see Annex) was conducted with one grade ten class (15-16 years old) in May 2005 and two grade eleven classes (16-17 years old) in May 2006. Students worked in teams, of two or, for the most part, three. The grade ten class was allotted a fifty-five minute period for each task, one per day for three days in a row. The two grade eleven classes worked on the three tasks consecutively, during a two and a half hour time segment. The collected data consist of:

- video recordings of two teams in each class;
- photographs of the large displays (with the graphs sketched on them) ;
- researcher's notes;
- some of the written proofs (not all teams had time to finish the writing).

The data were analyzed according to a *qualitative* perspective (Patton, 1990).

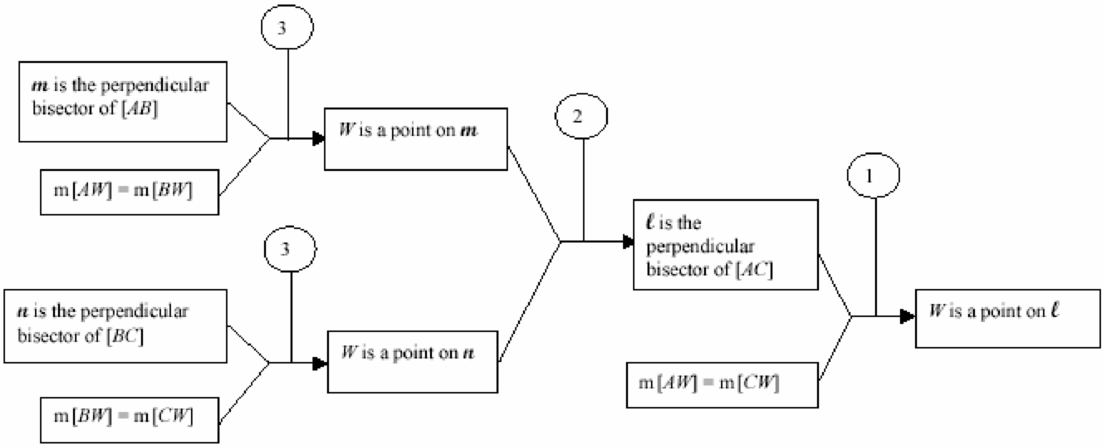
3.2. Some Excerpts from the Data and Conclusions from their Analysis

Let us first mention that except for one team doing the first task and one team doing the third task, all twenty teams managed to fulfill the reconstruction of the graphs, with, in some instances, hints from the researcher or his assistant. It is striking that, when they arrive at the correct reconstruction, the students are absolutely convinced that they got it right. They also appeared to enjoy the tasks.

1. The deductive structure, consisting of chained inferences, is neither spontaneously nor easily understood by many students.

    - Irrelevant aspects are taken into account, e.g. searching for symmetry in the graph. For instance:

Team 6, graph 15



Searching for symmetry in the graph

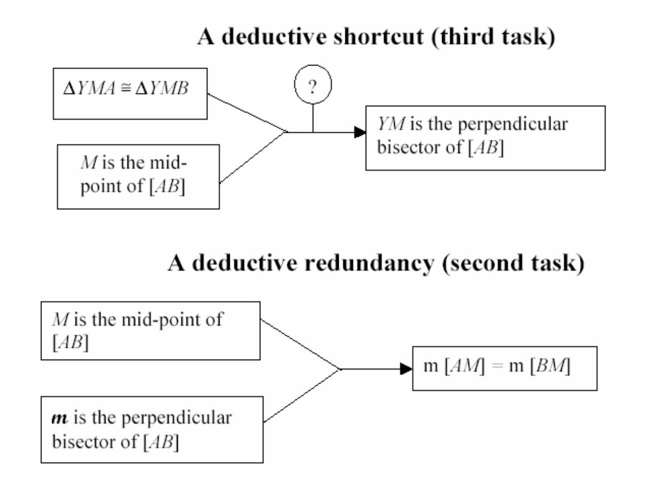
- Statements are 'flattened out', are put on the same level, as in an argumentation. For instance, even though members of Team 6 acknowledged early that 'W is a point on ℓ' was the proposition to be proved in Task 1, student Will[4] places 'ℓ is the bisector of [AC]' in the last box, stating: "It's because they are three [referring to $\boldsymbol{m}$, $\boldsymbol{n}$ and ℓ]. We have to prove that this one is the perpendicular bisector. That's it, isn't it?" It seems that from Will's point of view, showing 'W is a point on ℓ' and showing 'ℓ is the perpendicular bisector of [AC]' amounts to the same thing. He has certainly lost sight of which propositions belong to the hypothesis (the given). In analyzing the discussion between the three members of Team 6, in spite of its awkwardness, we are able to evaluate the extent to which a difficult and subtle coordination of the geometrical and logical aspects is required in order to understand organization of the proof. Will shows a clear lack of such coordination, and

[4] The names are fictitious.

dealing with one aspect at a time, is led to propose a solution such as graph 15, above. Xavier draws his own diagram, trying hard to see it in relation to Rules ② and ③. He attempts to understand the logical organization and says, in response to Will's proposal (Graph 15, above): "But you don't know that measure of [AB] equals measure of [AC]! Where does it come from if you place it on the left side [of the graph] ?" It took one half-hour and twenty different transitory states of their graph, but members of Team 6 did arrive at the correct one. Xavier then validated their solution by verbalizing aloud each inference while keeping track with his own diagram.

- The strict 'algorithmic nature' of inferences — which would enable a computer or a robot to find the inferred statement, having the rule and the premises — is not well understood. Deductive shortcuts and redundancies (see below) are indicators of this misunderstanding. Understanding the algorithmic nature of inferences — in particular, realizing that each inference is independent of the others and can be validated separately — did help several teams[5].

[5] The researcher and his assistant had decided that, when asked for validation by a team, they would isolate two inferences, one good and one bad, and ask the student to explain them in their own words. To at least one team, the researcher gave the additional explicit hint: 'You should be able to validate each deductive step independently', after which the team quickly came unblocked.



Misunderstandings of the algorithmic nature of inferences

2. Working backwards from the end to the beginning leads to better and faster achievements. Working this way, one team took less than 15 minutes to reconstruct each proof, 2 and 3. The researcher did witness two teams getting unblocked after he suggested they work this way.
3. The role played by rules of inference is underestimated. Teams who (tried to) establish the rules after the fact were generally very slow achievers, and wandered a lot before they grasped some of the proof's organizational elements. In at least two clear cases, according to the data, we have seen students getting unblocked in their reconstruction of the proof, as they began to take the rules into account and make the effort to understand their precise meaning.
4. The logical confusion between implications and converses did not emerge in the specific context of solving these tasks. For instance, no team proposed a solution for the first task in which inference rules ② and ③ would be interchanged[6]. Without being conclusive, this fact nevertheless suggests, as in Durand-Guerrier (2003),

[6] In one of the videos, we do hear a girl talking about the rules and saying: "It seems to me that ② and ③, they are alike. […] They seem to say the same thing." Unfortunately for the researcher, her two partners didn't notice or discuss the subject any further.

Dumont (1982) or Matalon as we quote him in § 1.3, that contexts and standpoints are factors of predominant influence on students' achievements involving logic and that in any case, imputing poor performance in proving to a lack of mere logical skill is an over-simplification. This issue certainly deserves further investigation.

5. From a research perspective, the written proofs are not conclusive. In the case of the grade ten class, there was not sufficient time, while the students of the two grade eleven classes didn't take the writing assignment seriously. We refer the reader to Tanguay (2005a; 2005b) for a lengthier discussion of writing assignments related to this type of tasks.

4. Conclusion

The discussion among team members (see, for instance, Xavier's response to Will's proposal, § 3.2.1) is focused by these tasks on the precise, acute elements of the deductive structure. Notwithstanding the need for more research and experimentation — for instance a comparative study measuring the impact of such activities on learning formal proof — the aforementioned observation suggests that work required by the proposed tasks contributes to students' better understanding of formal proofs' deductive structure. This involves much more than being fussy about formalism. The transfer, from what appears to be a satisfactory understanding of a proof and of how the general ideas are linked in it, to a logically written and controlled production of the proof, constitutes a fundamental leap for students and hinges on their mastery of the deductive structure. Here, we meet up with Dreyfus' concern: "Why [are] the students unable to give a decent explanation in spite of the fact that they seem to have a satisfactory understanding of the question and its answer (or of the problem and its solution)?" (1999, p. 87).

In light of this experiment, what we have called the ‘algorithmic nature of inferences’ (third item in § 3.2.1) is a crucial issue. These tasks allow students to work specifically on this issue, as a fundamental characteristic of formal proof. We suggest that instructions such as

- *the inferred statement can be algorithmically determined by a computer having the rule and the premises in its memory*; or
- *each deductive step is independent of the others and may be validated separately*

should be explicitly stated and discussed in class, during what is called by French *didacticiens* the ‘institutionalization-phase’ of such activities.

We are not saying that every geometrical proof should involve such activities. In our view, they are designed as a trigger, a ‘transitional object’ (Duval & Egret, 1989, p. 35) which should later evolve, as students are increasingly assigned components of the task: choosing and stating the rules, constructing the graph, choosing and stating propositions We are convinced that the target-statements should then be chosen in such a way that students work ‘backward’, from a relatively sophisticated statement, to less sophisticated statements, as in the sequence proposed here. The more sophisticated statement would be non-intuitively attainable, not immediately considered to be true; a statement that gives rise to a real ‘enjeu de vérité’ (Grenier & Payan, 1998). The less intuitively demanding statements would then be those arising naturally, due to the need to formally prove the initial statement. This way of functioning being the most common in mathematical research, we expect that such an approach would evoke the epistemological justification (Harel, 2007, RUME opening plenary talk) necessary to gain students’ involvement and commitment.

ANNEX : The Sequence of Three Tasks

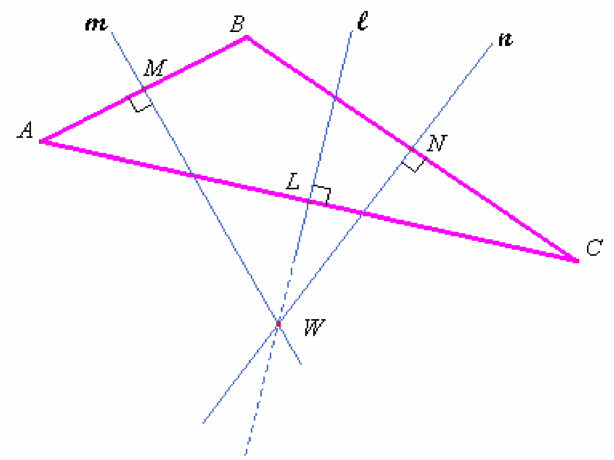
Note: In the list of justifications given to students in the first task, only justifications ①, ② and ③ appear.

List of justifications

① Transitivity of equality: if $x = y$ and $y = z$, then $x = z$.
② A point on the perpendicular bisector of any segment [*PQ*] is necessarily equidistant from the end-points *P* and *Q*.
③ A point equidistant from two points *P* and *Q* is necessarily on the perpendicular bisector of segment [*PQ*].
④ Definition of *perpendicular bisector* of [*PQ*]: the unique line passing through the mid-point of [*PQ*] and making a right angle with *PQ*.
⑤ Definition of *mid-point* of [*PQ*]: the point between *P* and *Q* on line *PQ*, equidistant from *P* and *Q*.
⑥ Definition of *congruence* between two triangles: two triangles are *congruent* when their homologous elements (sides and angles) have the same measurement.
⑦ The SSS criterion: if the three sides of a triangle have respectively same measurement as the three sides of another triangle, then the two triangles are congruent.
⑧ The SAS criterion: if two sides and the included angle of one triangle have respectively the same measurement as the two sides and included angle of another triangle, then the two triangles are congruent.
⑨ The ASA criterion: if two angles and their shared side of one triangle have respectively the same measurement as the two angles and shared side of another triangle, then the two triangles are congruent.
⑩ If point *O* is between *P* and *Q* on line *PQ*, then any point *Z* outside *PQ* determine angles $\angle ZOP$ and $\angle ZOQ$, which are supplementary.
⑪ Definition of *right angle*: an angle having the same measurement as its supplementary.

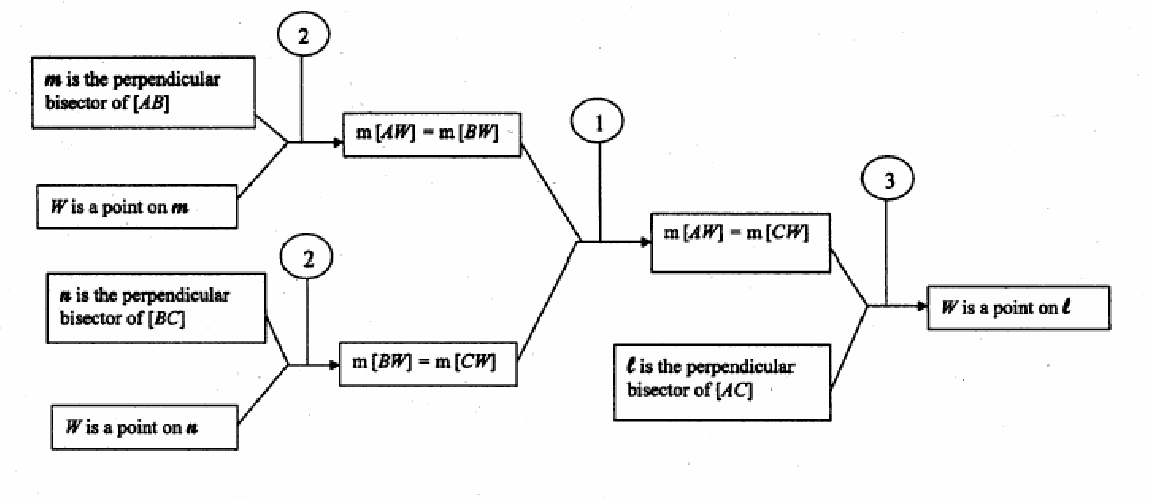
Before asking students to reconstruct the first propositional graph, the researcher presents an example of a deductive step and introduces the initial target-statement: *the three perpendicular bisectors of any triangle meet in one point*. He then gives the following explanation, while drawing the geometrical diagram on the blackboard:

> "Let ΔABC be any triangle. Let *M* be the mid-point of [*AB*], *N* the mid-point of [*BC*] and *L* the mid-point of [*AC*]. Let $\boldsymbol{m}$ be the perpendicular bisector of [*AB*], $\boldsymbol{n}$ the perpendicular bisector of [*BC*] and $\boldsymbol{\ell}$ the perpendicular bisector of [*AC*]. Being respectively perpendicular to lines *AB* and *BC,* which have point *B* in common, $\boldsymbol{m}$ and $\boldsymbol{n}$ cannot be parallel. Let *W* be their point of intersection. You must reconstruct the deductive chaining which shows that *W* is also a point of $\boldsymbol{\ell}$."



The geometrical diagram

Solution for the first task

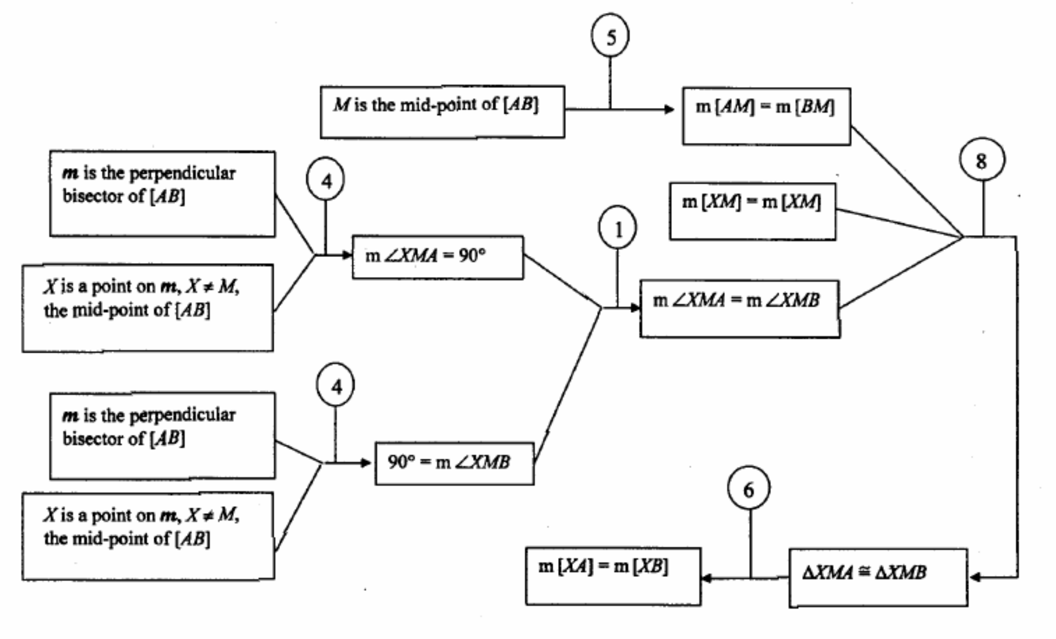


It should be mentioned that, at the start, a definition was the only information the students had regarding perpendicular bisectors. The second and third tasks on which students were later asked to work, consisted of showing each of statements ② and ③ in the list of justifications.

Before asking the students to reconstruct the second propositional graph, the researcher introduced the target-statement with the following explanation:

> "Let *M* be the mid-point of [*AB*] and ***m*** be the perpendicular bisector of [*AB*]. You must reconstruct the deductive chaining which shows that any point *X* on ***m*** is equidistant from end-points *A* and *B*. If *X* is the mid-point *M*, there is nothing to prove, as *M* is equidistant from *A* and *B* by definition of a *mid-point*. So you may consider that *X* is different from *M*."

Solution for the Second Task

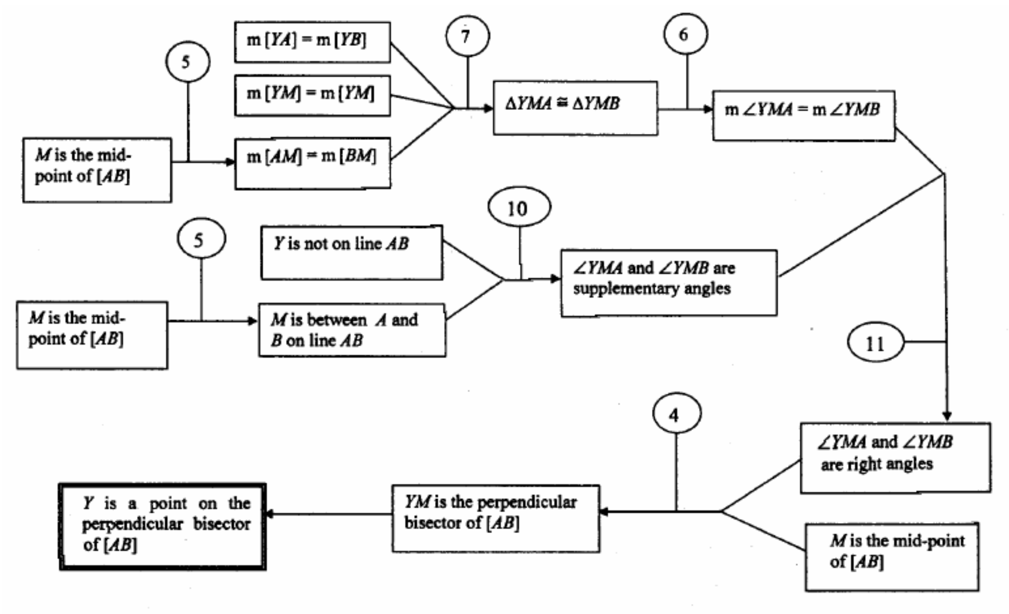


Before asking students to reconstruct the third propositional graph, the researcher introduces the target-statement with the following explanation:

> "Just as before, *M* is the mid-point of [*AB*] and ***m*** is its perpendicular bisector. You must reconstruct the deductive chaining which shows that any point *Y* at equal distance from the end-points *A* and *B* is necessarily on ***m***. You may consider that *Y* is not on line *AB*, since the only point on *AB* equidistant from *A* and *B* is the mid-point *M*, and *M* is on ***m*** by definition of a *perpendicular bisector*."

Solution for the third task

The last proposition (the bolder box) is already written in the (otherwise) empty graph.



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