USING TASK-BASED INTERVIEWS TO DISCOVER COLLEGE PHYSICS MAJORS' MATHEMATICAL THINKING AND PROBLEM-SOLVING SKILLS

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Abstract: The importance of robust mathematical problem solving skills is significant for learners in many settings, among them the physical sciences. This paper presents results from a study designed to investigate mathematical thinking of advanced physics and physics engineering students using task-based interviews. We discuss the attempt to categorize the participants' thinking into geometric, analytic, numeric or harmonic, using a modified version of a framework by Krutetskii (1976), and the difficulties that arose due to psychological and epistemological factors that influenced the students work during the interviews.

Introduction

The importance of good mathematical problem-solving skills is significant for learners in many settings, among them the physical sciences. Krutetskii (1976) differentiated among three different kinds of thinkers – *geometric, analytic* and *harmonic* – in his research involving Soviet school children. For Krutetskii, *geometric* thinkers where students who preferred reasoning from shapes, pictures and graphs while *analytic* thinkers preferred working with symbols. Krutetskii's *harmonic* thinkers moved easily between geometric and analytic representations, but this type of thinking was less common among his subjects.

Related to Krutetskii's ideas, a recurring suggestion from mathematics education literature [cf. NCTM Standards, 1989] is that emphasizing the *multiple representations* of mathematical concepts in teaching would improve student understanding of functions, calculus concepts and other mathematical ideas. The assumption was that if mathematical ideas were presented using symbolic, graphical, numerical, and verbal representations, students would be more powerful in their problem solving skills. Yet more than a decade after the first appearance of the NCTM Standards, Knuth (2000) reported finding that when high school students were given a task which could be approached either graphically or symbolically, 75 percent of the students chose to approach it symbolically, even when a geometric solution would have been more efficient. It seems that, although teachers might see these representations as connected and assume that negotiating among them is a straight-forward task, students frequently do not. Indeed there is research to support this. Janvier (1987) wrote that the translation between and among representations is more involved than we think. For instance translating from a graphical representation to a symbolic representation is not the inverse of translating from a symbolic representation to a graphical one. The first translation is often much more difficult than the second. Further, according to Hershkowitz (1998), visual thinking is not automatic. “It could be that the [mathematics education] community is making the naïve assumption that human beings are born with visual thinking abilities which are applied when needed, and therefore nothing needs to be done to nurture or develop them. (p. 33)”

Over the past few years, a group of researchers have been involved in a project designed to determine and then enhance the problem solving skills of advanced physics and physics engineering undergraduates.[1] One of the goals of the project is to encourage the development of harmonic thinking skills among these students. It is the perspective of the researchers that students can develop harmonic thinking skills through experiences that actively involve them in problem solving using geometric, analytic and numeric

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perspectives and that encourage them to reason between and among these representations. Part of this project has been to develop a research method designed to identify instances of geometric, analytic, numeric, and harmonic reasoning in students and to determine if it is possible to classify learners as geometric, analytic, numeric or harmonic.

In this paper we describe our theoretical perspective, give some background concerning the design of the research and the interview tasks, discuss results from the first interview with six upper-division college physics and physics engineering students, and then discuss what we see as psychological and epistemological obstacles that have influenced our quest to categorize our participants' thinking and problem-solving skills.

Theoretical Perspective

We frame our research questions and the analysis of our data after Krutetskii's work in analytic, geometric and harmonic reasoning among school children (1976). Although there has been an emphasis in mathematics education to develop harmonic thinking among K-12 students, our findings in pilot studies have indicated that students often do not use harmonic reasoning during mathematical problem solving and also, as Knuth's high schools students did, our students use analytic reasoning when geometric reasoning would be much more efficient. In our work we have modified Krutetskii's original framework. For the purposes of this research we define *geometric reasoning* broadly to mean thinking about and reasoning from graphs of functions and equations as well as thinking about and reasoning from geometric objects. We also classify the ability to see pictures in ones mind as geometric reasoning. In our analysis of students' work we recognize that students may be reasoning geometrically by their hand gestures as well as by sketches or graphs that they make. We define *harmonic reasoning*, differing slightly from Krutetskii’s work, as reasoning between and among a combination of analytic, geometric and numeric representations.. It is also important to emphasize that we view harmonic reasoning as a somewhat seamless or fluid movement between the various representations. In our view one would not be reasoning harmonically if he or she first approached a problem geometrically, gave up that strategy and then approached it analytically. However, the ability to approach a problem from each of the three perspectives *is* a prerequisite for harmonic thinking.

Background

A pilot study with two one-hour task-based interviews was conducted during the 2005-6 academic year. Twelve participants, most of them third-year physics majors, participated in two interviews during fall 2005 and spring 2006. We also gave tasks to a group of physics and mathematics faculty in a series of interviews conducted during 2006. The purpose of both the student and faculty interviews was to refine our list of appropriate tasks for the interviews. We realized that if we were to detect harmonic reasoning we would need robust tasks; ones that could be accomplished using multiple approaches: geometric or symbolic reasoning or possibly numeric reasoning. We also recognized that the wording or appearance of our interview tasks might influence the problem solver’s choice of strategy at least initially; thus for the two interviews we chose a variety of tasks – some that looked geometric and some that seemed more analytic – so that we could achieve some balance in the tasks. However, we wanted all tasks to require some visual or geometric thinking since one of the goals of the physics project is to enhance students’ geometric thinking as a means to gain more harmonic abilities.

As a result of our pilot interviews, two tasks (shown in Figures 1 and 2) were chosen for the first interview. Our analysis of the tasks showed us that successful completion of Problem 1 could require strategies involving geometric, analytic and/or numeric reasoning. All but one of the professors who did this task started by sketching potential solutions and reasoning geometrically about extreme cases and all of these experts used all three types of reasoning in an harmonic way. The problem appears to be geometric in nature: it is a graph which produces a geometric object, a triangle, and we are concerned with the area of that triangle. Indeed there is a very easy geometric approach to this problem that involves thinking about a minimal rectangle and picturing the hypotenuse of the triangle one is creating as it moves outside the rectangle thus adding unwanted area. However, there are also analytic aspects to the problem since the use of the words "minimal area" can cause one to recall optimization experiences from calculus, or one can think about the slope of the line through (5, 3) and approach that idea in a symbolic way.

<u>Problem 1</u>

Assume you have an x-y coordinate system and a point (5, 3). Construct a line that connects the y-axis to the x-axis, contains the point (5, 3) such that the axes and the line creates a triangle of minimal area.

Figure 1

Problem 2

How many times do the graphs of the following equations intersect?

$$y_1 = x^2$$

$$y_2 = 2^x$$

Specifically identify at least one of the points of intersection.

How can you be sure you have found all of the points of intersection?

Figure 2

Problem two suggests a graphical (geometric) approach because of the words "graphs" and "intersect," however the objects of the task are presented symbolically.
In the pilot studies we used a similar problem but asked students to find the "solutions" to a system of equations. Some students seemed to be unable to see that question as one that would require geometric thinking even though it was very difficult to solve analytically. Indeed some of our experts initially approached that version of the problem in an analytic way. We decided to give the students in the current study two questions in the first interview that had a more pointed suggestion toward a geometric approach to determine if some students would still approach the problems analytically. (This proved to be true.)

Current Study: Methods

In our current study, six students, physics and physics engineering majors, volunteered to participate in a series of three one-hour task-based interviews. The participants were chosen from a group of students enrolled in an innovative year-long junior-level physics course in which harmonic reasoning is modeled and encouraged. The first interview was conducted in summer 2007 and was designed to answer the questions:

- In what ways do physics and physics engineering majors who are in their last two years of undergraduate work display geometric, analytic, numeric and/or harmonic reasoning while doing non-standard mathematics tasks?
- To what extent can these students be categorized as geometric, analytic, numeric or harmonic problem solvers?

The first interviews were audio and video-taped and verbatim transcripts were made from the tapes.[2] Students were asked to work on large white boards while doing the tasks and were encouraged to talk about their ideas as they worked on the tasks. The students worked on two problems (shown in Figures 1 and 2) and the interviewer asked questions to clarify her understanding of the students thinking and to encourage students to verbalize their thinking. The interviewer analyzed the transcripts and viewed the video tapes, noting the students' gestures and expressions and recording what they wrote on the whiteboards. An initial coding scheme was used to identify instances of what appeared to be analytic, geometric, numerical or harmonic reasoning. This coding scheme was then expanded to capture evidence of epistemological and psychological influences on the students choices of strategies while working on the tasks.

Results

There is evidence that all six students seem to have the prerequisites for harmonic reasoning ability; they all demonstrated geometrically, analytically and numerically reasoning during the interviews. Although three of them seemed to prefer reasoning

[2] We will use the same procedures for the second and third interviews.

analytically, and the other three were more geometric or at least visual, they all showed the ability to approach problems from the other two perspectives. There was little evidence in the first interview of harmonic reasoning. Examples from two interviews follow.

Fred[3] is an older student who held a technical job before returning to college. He took algebra at a community college and, at the time of the first interview, he had just completed the calculus sequence including differential equations. He approached problem 1 visually, swiveling his pen in the air to represent different possible slopes for the line that was to connect the axes to form the triangle. His first hypothesis was that this line intersected the axes at (0, 8) and (8, 0)[4]

> Fred: Probably I'd do it more by sight, and with a ruler…. [The interviewer offers Fred a ruler.] So it looks like it's gonna intersect at, probably, close to 8 and 8. If it's this way [point above 8 on the y-axis] it's gonna add a lot more area up here, or vice versa that way [points out on the x-axis.]
>
> Int.: Do you have a way to know….?
>
> Fred: Yeah. Um, I mean, my other thought is, you know, using your y=mx+b, to try to figure out where this line is going. To be able to calculate the area. Rather than using a ruler, you know. [Later…] But I'm sure I could figure out an equation to move this up and down and find the total area. I don't remember how to do that though.

Fred worked with the problem for approximately 15 minutes but was unable to get very far with it. If he had then not worked on the second task, the researcher might have concluded that Fred was exclusively geometric or seriously hindered by impoverished

[3] "Fred" is a pseudonym, as are all the other students' names: Nancy, Pete, Paul, Mike and Ernie.
[4] This was also the initial hypothesis of two of the experts.

analytic abilities. However, on the second problem Fred, although clearly, as he says, a visual person , was able to find three intersections using both graphical and numerical methods as well as demonstrating an understanding of the contrasting rates of change of the quadratic and exponential functions.

The interview with Pete, on the other hand, provided an example of a student who seemed to prefer an analytic (symbolic) or numeric approach to the tasks. On Problem 1 he sketched the hypotenuse of the triangle that goes through (5, 3) but did not seem to reason from that representation. Instead, he immediately began talking about the slope-intercept form of the equation of that line and how he could use that to solve the problem. Eventually he used a guess-and-check method and tried various points of intersection with the axes and calculating each corresponding area. On Problem 2 his approach was clearly analytic

> Pete: I'm going to set those equal to each other. I'm just following my intuition when I'm beginning this problem…. Actually that's a system of equations.
>
> Int: Okay….
>
> Pete: I don't know how to deal with the exponent. Hmm… derivatives might play a role. [A period of five minutes follows as he tries various symbolic approaches. Finally he begins to talk about graphs.] I guess…. I should be graphing these … maybe to think more clearly.

However, he first graphed the two equations on separate axes and looked from one to the other. It was several minutes before he decided to place the equations on the same axes so he can see the possible intersection points.

Psychological and Epistemological Factors

After initial analysis of the set of interviews it became apparent that there were certain psychological and epistemological factors that were confounding the attempts to categorize the students' problem solving approaches. These factors related to how the students perceived the interviewer and what she might be expecting from them as well as their apparent epistemological misunderstandings concerning what mathematics is and what it means to do mathematics [Hammer, 2008].

The students knew that the interviewer taught mathematics, and some of them seemed to assume that she expected them to approach the tasks in a certain (often symbolic) way. For example, while doing the first task, Nancy approximates a solution and then says, "I suppose you want me to do some math here," meaning that she assumed the interviewer wanted her solve an equation or use numbers. It seems that it is possible then that students tried to approach the problems symbolically because they thought the interviewer valued that approach.

Perhaps a greater barrier to our discovering what the students were capable of doing may have been that many of them seemed to think that mathematics is about formulas and numbers, about knowing facts and methods of doing problems, and that if one does not remember a correct formula, forward progress on a task must cease. There is ample evidence of this in the interview transcripts. In several of the interviews the researcher had to urge the students to continue when they stalled because they could not "remember" how a problem had been done in the past. Following are some examples.

- As Fred struggled with the first task, he said, "Honestly, the math part is probably the hardest thing for me. I can remember some equations, like Pythagorean….is pretty much one of the only that I can name. Everybody is like, oh, you know, they use some sort of name for the equations or method or something else like that. I just kind of…huh?

- At one point during his work on the second problem Paul said, “I remember doing this, but I don’t remember enough to do it.”
- Pete makes several comments such as, “I don’t remember the last time I did a problem like this;” and “We did this in 251…and [I remember] there’s something special about e to the x, but I don’t know….” Later, “I’m just waiting for something to come to me;” and “…if I could look this up….”

Often when the students were trying to “remember” how to do a problem they did not seem to be able to reason through a situation using the valuable ideas that they did have and it is possible that we did not see the strategies that they would have used if they had not been in the interview situation.

Discussion and Conclusions

Two important factors characterized the experts’ problem solving strategies during our faculty interviews: a seamless flow of reasoning between and among various representations of the problems (harmonicity), and confidence in their solution strategies and in their final solutions. It is the goal of this project to help students adopt these characteristics in their own problem solving. We realize however that we must continue to work at making the interview setting more conducive to “realistic” problem solving so that we create a better window on the students’ actual strategoc preferences. If we cannot change their view of “what is mathematics” we can attempt to convince them to show us how to do our tasks using their developing notions of “what is physics.”

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