Scaling Up Instructional Activities: Lessons Learned from a Collaboration between a Mathematician and a Mathematics Education Researcher

Joanna Bartlo, Sean Larsen, & Elise Lockwood

Portland State University

Mathematics curricula designed to focus on conceptual rather than procedural understanding often engage teachers in activities that are different from those they have previously experienced. This means that teachers often have a difficult time making sense of tasks from such a curriculum in the manner intended by the curriculum designer (Lloyd, 1999; Wagner et al., 2007). To that end, we report on a study in which we explored how a mathematician made sense of a research-based introductory abstract algebra curriculum while implementing it for the first time.

A research mathematician, John, implemented an innovative research-based undergraduate introductory abstract algebra curriculum (Larsen, 2004). We met with John regularly for debriefing and planning sessions, collecting research data both from the classroom and our meetings. One of the most prominent themes that emerged was the complex role of mathematical content knowledge in the teaching of this type of curriculum. On the one hand, John's extensive mathematical training was crucial to his making sense of curricular tasks and student thinking. On the other hand, however, we were able to identify a number of instances in which additional kinds of mathematical knowledge could have supported a more successful implementation of the curriculum. We argue that these kinds of knowledge are instances of what Ball, Lubienski, & Mewborn (2001) refer to as mathematical knowledge for teaching. Ball, Bass, Hill, & Schilling (2005) hypothesize that mathematical knowledge for teaching includes things such as knowledge of content, knowledge of students, and knowledge of tasks.

In this paper we describe in detail two classroom episodes that illustrate the ways in which John drew upon his knowledge of mathematics to make sense of the curriculum and students' thinking, as well as the ways in which additional kinds of mathematical knowledge for teaching could have supported his efforts. We will conclude with a discussion of implications for future investigations into supporting teachers (particularly mathematicians) in implementing inquiry-based curricula.

Setting and Participants

The research-based introductory abstract algebra curricular materials used in the study were developed by the second author through a series of design experiments (Larsen, 2004; Lockwood, Bartlo, & Larsen, 2007). The goal of the curriculum is to develop the formal concepts of group theory by building on students' thinking. In its current form, the curriculum features three primary instructional sequences, one for each of the main course concepts: group, isomorphism, and quotient group. The materials actively engage students in the development of the fundamental concepts of group theory, balancing *reinvention phases* (in which students develop concepts based on their intuition, informal strategies, and prior knowledge) and *deductive phases* (in which students develop deductive proofs of important results based on formal definitions and previously established results). This approach was adopted in light of growing evidence that students' understandings of formal concepts are often disconnected from their informal understandings (Edwards & Ward, 2004; Moore, 1994; Tall, 1992; Weber & Alcock 2004).

In order to support the scaling up of the curriculum, we are currently engaged in research to 1) identify the challenges and opportunities that are likely to arise as different instructors implement the materials, 2) develop instructor support materials to meet these challenges and take advantage of these opportunities and, 3) investigate how students' learning is enhanced by the curriculum materials.

In order to gain insight into how an instructor might interact with the research-based introductory abstract algebra curriculum, we partnered with a mathematician, John. As a first step, he agreed to teach the innovative introductory abstract algebra curriculum with an eye towards helping us design instructor materials. We met regularly with John, both to discuss upcoming tasks and to hear about his experiences implementing past tasks. The weekly meetings were videotaped, and summary notes were taken during the meetings. Additionally, John sent us mid-week emails about the class sessions between our meetings. We also videotaped several select classroom sessions.

In some ways, John is a fairly typical mathematics professor. He is well-published and knowledgeable about both general undergraduate-level mathematics and his own research interests, and he tends toward a lecture-heavy teaching style. In other ways, however, John has traits that differentiate him from the average mathematics professor. For example, John is a particularly popular teacher and has won several departmental teaching awards. Furthermore, within the last three years John has been involved in a workshop sponsored by the Mathematical Association of America, on the topic of the mathematical preparation of teachers. Moreover, John participated in the design and implementation of a mathematics course for K-12 teachers as part of an NSF-funded professional development project. This project involved collaborations between mathematicians, mathematics education researchers, community college instructors, and

K-12 master teachers. So, going into the project, John had had some experience and training with teaching in an inquiry-oriented format. Therefore, John not only has ample knowledge of the content of the mathematics involved in the curriculum, but also he has more experience with and knowledge about reform-oriented teaching practices than would a typical mathematics professor.

While teaching the introductory abstract algebra course in an inquiry format for the first time, John had an undergraduate student assisting him with the class. She had experienced the innovative curriculum as a student but did not have previous experience teaching in a classroom setting. In addition to helping answer questions and facilitate group work during the class, she took notes about each class period and also attended the weekly debriefing meetings.

Results

The data corpus for this study consisted of the digitized video recordings of the weekly debrief meetings (there were 10), notes taken during the meetings, John's weekly emails, and the videotaped class sessions. We had all attended all weekly meetings and selected class sessions, and we all read John's weekly emails. This served as a preliminary phase of analysis. In the second phase of analysis, we individually watched all of the videotaped meetings and read through the meeting notes and the weekly emails. While doing this, we looked for incidents that shed light on how John made sense of the curriculum while implementing it for the first time. We also tried to pick up on any other themes that might pertain to this issue. In subsequent analysis, we discussed and reviewed these themes. In particular, we jointly re-watched some of the videos of the meetings and of relevant classroom sessions in order to identify and analyze 1) instances in which John was able to use his mathematical knowledge effectively and 2) instances in which additional kinds of mathematical knowledge would have been beneficial. We focused on instances in which we could observe John's attempts to make sense of and respond to student thinking and curricular tasks. We present our findings in the form of two case studies: The Case of Characterizing Evenness and The Case of Defining Isomorphism.

The Case of Characterizing Evenness

In this case, John encountered a surprising mathematical idea suggested by a student. By analyzing this particular incident, we learned about how John made sense of an unexpected student response. During class, when the student shared his idea, John was able to make sense of the student's idea to some extent, but was unable to capitalize on the idea and connect it with the instructional sequence. However, after class John continued to think about the student's idea, eventually connecting it to some sophisticated group theoretic concepts.

This case occurred during one of the early tasks in the sequence in which students learn about quotient groups. The basic approach of the sequence was to have students generalize the idea of parity. In particular, the students considered a Cayley table (Figure 1) showing what happens when even numbers and odd numbers are added together and explained why this table represents a group.

+	Even	Odd
Even	Even	Odd
Odd	Odd	Even

Figure 1. The table students completed when exploring evens and odds.

The first task was to try to make sense of the idea of parity in the context of the group of symmetries of a square, a group with which the students were already familiar. The students had created their own system of symbolizing symmetries and had symbolized the symmetries of a square as follows: $\{I, R, R^2, R^3, F, FR, FR^2, FR^3\}$. In their symbols the *I* is the identity element, the *F* is a flip across the vertical axis, the *R* is a rotation of 90 degrees, and the rest of the symbols are created from using *F* and *R* as generators. The students had also explored the set of even numbers and the set of odd numbers, realizing that the set containing those two elements formed a group under addition.

In the incident we will discuss, the students were looking for a way to break the symmetries of a square up into "evens" and "odds." This group has three subgroups of order 4 ($\{I, R, R^2, R^3\}$, $\{I, F, R^2, FR^2\}$, $\{I, R^2, FR, FR^3\}$), any of which can be used to partition the group into a quotient group. At this point the students were not aware that they were trying to create a quotient group, rather they were simply looking for "evens" and "odds" in D_8 (the symmetries of a square).

The Case of Characterizing Evenness begins with a group of students explaining how they see $\{I, R^2, FR, FR^3\}$ as a set of "evens." The students had created a table (Figure 2) that showed that this partition created a group.

	$\{I, R^2, FR, FR^3\}$	$\{R, R^3, F, FR^2\}$
$\{I, R^2, FR, FR^3\}$	$\{I, R^2, FR, FR^3\}$	$\{R, R^3, F, FR^2\}$
$\{R, R^3, F, FR^2\}$	$\{R, R^3, F, FR^2\}$	$\{I, R^2, FR, FR^3\}$

Figure 2. The table the students created to show "evens" and "odds" in D_8.

They then explained that the set {*I*, R^2, *FR*, FR^3} represents the evens because its elements are comprised of an even number of generators (e.g. R^2 is made of two *R*'s and *FR* is made up of two generators, an *F* and an *R*), and that {*R*, R^3, *F*, FR^2}, the remaining elements, are odd because they are comprised of an odd number of generators.

> S: It was the number of operations. In the first case you have an even number of operations, and in the second case you have an odd number of operations.
>
> J: So, if you take the generators *F* and *R,* and you look at how many of them occur in each element, I guess you could characterize these two sets, the first one (points to the subset containing {*I*, R^2, *FR*, FR^3}) as being those that are an even number of operations, those that are a product of an even number of generators. And the elements in this set (points to the set containing {*R*, R^3, *F*, FR^2}) are *R*, so one element, R^3 so a product of 3 generators, *F*, which is the product of one generator, and FR^2, that's three generators multiplied together. So that's a fair characterization of these, except for zero maybe…
>
> S: That's zero
>
> J: Which is zero of them, or *F* times *F* maybe. So I think you can make a case that that's a fair description. So does that explain then why this table should hold?
>
> S: No.

The episode ends with John asking if the explanation the students offered sheds light on why this partition creates a group that works like the even/odd group (Figure 1). However, John allowed almost no time for students to respond to his question. We interpret this as an indication that at the time it was not immediately clear to him why this kind of "local evenness" described by the student guaranteed the global parity seen in the resulting quotient group. In fact, the global parity is a consequence of the fact that both the sum of two odd numbers and the sum of two even numbers (in this case the number of generators) is an even number.

During the debrief it was clear that John's interest about this particular interaction was piqued. He had continued to think about the student's suggestion in the time between class and the debriefing meeting, and in fact during the debrief he raised the issue on his own. He explained that in the moment, during class, he had realized that the partition that the students came up with would work in the current case they had been discussing. But he had decided not to push the students on this issue.

> J: For them there was a point that was raised that was pretty cool … oh you could think about the evens and odds here as the things that are a product of an even number of generators and a product of an odd number of generators, and at first I thought that that's kind of cool, its quirky, it works in this case, I don't think it works in the triangle for example, but it works in a square, and then I was thinking I don't think it works in a pentagon, but it probably works in a hexagon.

Ultimately, John was able to make sense of the student's suggestion in terms of his knowledge of permutation groups and alternating groups. John realized that the characteristic of this subgroup that the student focused on is analogous to the defining property of the alternating subgroup of a permutation group. Permutation groups are generated by transpositions (permutations that simply swap two elements), and the alternating subgroup consists of those permutations that are generated by an even number of transpositions. So the alternating subgroup is "even" in the local sense described by the student. It is also "even" in the sense that it can be used to decompose the permutation group into a quotient group consisting of two elements ("evens" and "odds").

> J: And then I started thinking a little more, like it's a reasonable thing for them to be thinking about. Like if you went a little farther, these are symmetry groups, so they are naturally part of the symmetric group. And you always have the even permutations versus the odd permutations, and I think it's just making that split. So any time you have a subgroup of a symmetric group, you can always make the odd permutations versus the even permutation split that behave like this. So that's really what they've been looking at is symmetric groups.

When presented with a surprising student response, John turned to his mathematical content knowledge and searched for a connection between mathematics that he knows and the idea the student suggested. Ultimately, he made that connection via permutation groups. John's content knowledge, specifically his knowledge about permutation groups, allowed him to consider the student's idea and to make sense of their answer in his own terms. From this excerpt we see that John truly wanted to understand the student's answer and was able to do so by using his content knowledge. In fact, John's analysis suggests a possible path for us to pursue in the ongoing development of the curriculum (perhaps a short unit on permutation groups could draw on this idea as a starting point).

However, we also found that John missed an important opportunity during the lesson to integrate the student's idea into the instructional sequence. Given the fact that this "local evenness" described by the student could be directly connected to the "global evenness" that was the focus of the task, John could have capitalized on the opportunity provided by the student's contribution. In particular, John could have followed through on his question about whether there was such a connection.

The Case of Defining Isomorphism

This was a significant episode for us because it was particularly informative for our goal of developing instructor support materials to accompany the curriculum. In this example John deviated from the implementation plan that we had suggested, and this provided us with the opportunity to explore why he did this and what kinds of consequences arose in terms of student learning. Exploring those issues helped us to think about how to create instructor materials that would enable instructors to understand the intent of the tasks in the curriculum.

In this episode, the students' task was to define isomorphism. Recall that two groups, (G, •) and (H, *), are isomorphic if there exists a bijection, ϕ, from G to H such that that $\phi(a)\bullet\phi(b)=\phi(a * b)$ for all a and b in G. This last condition formalizes the idea that the operation must be preserved by the mapping (the operation works the same in both groups). The instructional sequence begins with having the students explore isomorphism informally. They are presented with an operation table of a group with 6 elements (Figure 3a) and are then asked if the group could be D_6 (the symmetries of a triangle, Figure 3b).

	A	B	C	D	E	G
A	B	A	D	C	G	E
B	A	B	C	D	E	G
C	G	C	B	E	D	A
D	E	D	A	G	C	B
E	D	E	G	A	B	C
G	C	G	E	B	A	D

Figure 3a. The 6 element "mystery table"

	I	R	R^2	F	FR	FR^2
I	I	R	R^2	F	FR	FR^2
R	R	R^2	I	FR^2	F	FR
R^2	R^2	I	R	FR	FR^2	F
F	F	FR	FR^2	I	R	R^2
FR	FR	FR^2	F	R^2	I	R
FR^2	FR^2	F	FR	R	R^2	I

Figure 3b. The operation table for D6

Students typically begin this task by looking for a mapping between the symbols in the mystery table and the symbols they use to represent the symmetries of a triangle. They usually begin by figuring out which element is the identity. They then determine which elements are self-inverses in order to figure out which elements in the mystery table can be flips (the rest must be the non-trivial rotations). Then some students will arbitrarily assign elements according to these rules (this has a 50% chance of success). If the students are actually trying to prove that their mapping works, those mappings that do not work will break down quickly. However, some students will use the preservation of the operation implicitly to generate their mappings. A primary purpose of this task is to have the students begin to uncover their intuitive notions about isomorphism.

In the next task the students are given a mapping ($A \leftrightarrow F$, $B \leftrightarrow I$, $C \leftrightarrow FR^2$, $D \leftrightarrow R$, $E \leftrightarrow FR$, $G \leftrightarrow R^2$) and are asked why it does not work to show that the mystery table is D_6 (i.e. why it is not an isomorphism). Although this is a bijective mapping in which the orders of the elements are preserved, the operation itself is not preserved. The students can figure this out by considering corresponding products in the two tables. For instance, *A* times *D* is *C*, but *C* does not correspond to *FR* even though *A* corresponds to *F* and *D* corresponds to *R*. When the students address why this mapping does not work, the idea of operation preservation becomes an explicit topic of discussion.

The next task also pushes the students to think more explicitly about the idea of operation preservation. In this task the generators are assigned to specific elements ($C \leftrightarrow F$ and $G \leftrightarrow R$), and the students are asked to find the rest of the correspondences given those pairings. The students use the idea that the operation must be preserved to assign the rest of the elements, making this important property more explicit in their discussions.

In the final task of this sequence the students are given an informal definition of isomorphism ("We use the term *isomorphic* to express the idea that two groups are essentially the same. So the group given by the mystery table is isomorphic to D_6") and are asked to formalize the definition (Write a definition for isomorphic. Definition: Let (G, •) and (H, *) be groups. *G* is isomorphic to *H* if…).

In the notes that accompanied the task we explained that "the preservation of the operation will be implicit in their work in the first part, more explicit in the second part (they will realize it breaks in a specific case), and very explicit in the third part (they use it to assign the remaining elements)." We also mentioned that "at this point the students should be ready to write a definition with a bit of help." To that end we gave specific suggestions on how to implement the last task, the one in which the students define isomorphism (Figure 4).

- Have them think for a couple of minutes about what is needed to finish the definition
- Then have some student share ideas. **Someone should suggest a bijective function. <u>You can use this to start the definition.</u>**
- **Ask if this condition is enough**. Someone should say no! Ask for an example that shows this is not good enough. The example in task 3 is one! It is bijective but doesn't work.
- Someone should say something in the ballpark of the answers have to work out, or the operation has to work the same. Remind them of task 4 again.
- **Have them try to express the condition they used in task 4 using function notation**. You might give them a hint to try to write an equation or equations.

Figure 4. Notes accompanying the isomorphism task (bold and underlining added)

The intention of this plan was to have students begin to explicate the processes they were using in the earlier tasks and then to work on how to notate those ideas. Although the students use operation preservation in the earlier tasks, they generally do not immediately include it in their definition. Therefore, the task is designed so the students can first share the important characteristics of the isomorphism concept that are salient to them (such as the fact that the isomorphic groups must have the same number of elements), and to begin to formulate a definition based on these characteristics. The instructional goal is then to have the students realize that additional criteria (other than, say, the existence of a bijection between the groups) are necessary to ensure that two groups are isomorphic. The students can discover this by reflecting on the earlier tasks in the sequence. Thus the stage is set for the students to focus on recognizing and articulating the remaining condition (namely the preservation of the operation)that must be met by such a bijection in order for it to show that two groups are isomorphic. This can be a difficult task, since formally expressing the operation preservation property is not easy for students, even when they are aware that this property is necessary.

Rather than following the suggested plan, however, John and his teaching assistant developed their own implementation plan on the fly. They had the students write the entire definition at once on transparencies. However, this meant that the students focused mainly on the fact that the two groups have to have the same number of elements to be isomorphic, and they were not pushed to think about what their definition was missing. Therefore, students that did not immediately think to include operation preservation were not pushed to do so. Furthermore, few of the students thought about how to express operation preservation, and those that did were not given the advice to use function notation or to write equations.

We suspect that the reason the instructors deviated from the plan was two-fold. First of all, the instructional materials for each day include a lot of information, and it was a tall order to expect that John could have taken in all of that information. In fact, it was not uncommon for the curriculum designer to tell John that a particular suggestion or part of a task could be changed or ignored. John and his teaching assistant had learned, then, that some of the suggestions included were more important than others. Thus, it may have been difficult for them to know that this specific information related to defining isomorphism was particularly essential.

Secondly, it is important to note that this task was expressly designed in light of literature on students' difficulty with isomorphism (Leron, Hazzan, & Zazkis,1995) and in light of prior work by Larsen (2004). Specifically, the students in Larsen's early teaching experiments had struggled with defining isomorphism, and thus the isomorphism-defining task had been tailored to help future students overcome these struggles. In particular, the task was designed to help students recognize the need to address all of the important aspects of the definition and to focus their attention on the difficult parts of symbolizing those characteristics. Instructional interventions were included to help create opportunities for each of those things to happen and to provide support for students in the process.

It is important to note, however, that John was not aware of Larsen's experience with his students, nor was he explicitly aware of the literature on isomorphism. He thus underestimated the difficulties that students would have with this definition. When teaching in a traditional format, the students are often given the operation preservation condition, and they are given it in function notation. So it did not occur to John that the students would not immediately attend to the operation preservation part of the definition, nor did he predict that function notation would not be the students' default way of symbolizing the definition. He also did not anticipate how hard it would be for students to symbolize the operation preservation aspect of the definition once they were aware of it.

Apparently, as a consequence of the modified implementation of the task, the definitions the students wrote focused mainly on the bijective portion of the isomorphism definition. Since the students were not pushed to address the operation preservation part explicitly, it was generally left out of their definitions. Additionally, the students who did attend to operation preservation did not formulate this property in the standard way (using function notation).

Uncomfortable with the students' understanding of the definition, John spent part of the following class period trying to connect the students' informal notions of isomorphism to the formal definition via a mini-lecture. This differed from the intention of the task in that John was attempting to *connect* students' informal reasoning to the formal definition, whereas the intention of the task was to *build up* the formal ideas from students' intuition (Gravemeijer, Cobb, Bowers, & Whitenack 2000). Consequently, the students had to try to reconcile their informal reasoning with the formal definition rather than formalizing their intuitive notions to create a formal definition.

Although most of the students did not focus on the operation preservation aspect of the definition, a few of them did. However, when it was attended to, it was not written in a conventional format (Figure 5, for example). Therefore, it was difficult for John to recognize that they were attending to it. He did say that "you could argue that some of them were correct." However, the fact that he included that qualification suggests that he had a hard time seeing them as such.

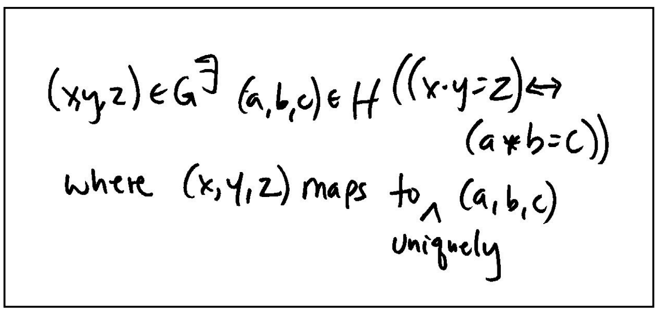

Figure 5. An example of a student attending to operation preservation in an unconventional way.

Discussion and Conclusion

The two episodes that we have described illustrate a number of significant issues regarding John's experience implementing the innovative introductory abstract algebra curriculum. These issues suggest a number of productive avenues for further research and have implications for the design of instructor materials to accompany the curriculum.

Mathematical knowledge for teaching is said to include things such as knowledge of content, knowledge of students, and knowledge of tasks (Ball, Bass, Hill, & Schilling, 2005). This type of knowledge is presumed to help instructors explain terms and concepts, interpret students' statements and solutions, as well as make sense of curricular treatments of topics (Hill, Rowan, & Ball, 2005). The two cases that we described above illustrate the importance of this type of mathematical knowledge.

The Case of Characterizing Evenness illustrated the importance of John's mathematical content knowledge in making sense of student thinking. This case also illustrates the distinction between what is traditionally considered to be mathematical content knowledge and mathematical knowledge for teaching (Ball, Lubienski, & Mewborn, 2001). In particular, while John's significant content knowledge helped him make sense of the student's response, this knowledge did not support him in making the most of the student's contribution. However, we also saw John' understanding of the intention of the task grow as he continued to make sense of the student's idea.

The Case of Defining Isomorphism illustrated another aspect of mathematical knowledge for teaching. It seems likely that the root of John's difficulty implementing this task was that he was unaware of the difficulties that students have with the isomorphism concept (and the function concept as well). Not being aware of these difficulties (in particular how difficult it would be for students to become explicitly aware of the operation preserving property and then to formalize this property), John failed to recognize the significance of the two activities that followed the Mystery Table task, and also failed to see the significance of the implementation approach suggested for the Defining Isomorphism task.

In terms of developing instructor materials, our study suggests (among other things) that it is important to engage instructors in making sense of student thinking as they prepare to implement the curriculum. As instructors work to interpret samples of student thinking in response to tasks from the curriculum, they will engage their own knowledge of mathematics and likely draw on this content knowledge to actively make sense of the curriculum in a new way (one that is connected to student thinking). The activity of making sense of student thinking may also stimulate an interest in the research literature on teaching and learning, thus linking selected findings from the literature to specific samples of student work could be a powerful way to increase the instructor's mathematical knowledge for teaching.

Finally, we note that our project may contribute to the research on teachers' knowledge for teaching in unique and important ways. Because in our reserach we are working with teachers who are exceptionally well prepared in terms of their knowledge of mathematics, we will have the opportunity to tease out the aspects of mathematical knowledge of teaching that are distinct from what has been traditionally referred to as mathematical content knowledge. Thus our project can help to identify aspects of teachers' mathematical development that are unlikely to be supported by traditional training in mathematics, setting the stage for the design of relevant mathematics education experiences for teachers.

References

Ball, D. L., Lubienski, S.T., & Mewborn, D. S. (2001). Research on teaching mathematics: The unsolved problem of teachers' mathematical knowledge. In V. Richardson (Ed.), *Handbook of Research on Teaching* (pp. 433-456). New York: MacMillan.

Ball, H., Bass, H., Hill, H., and Schilling, S. (2005). Developing measures of mathematical knowledge for teaching. Presented at the Teachers Development Group Leadership Seminar. Portland, OR.

Edwards, B. S., & Ward, M. B. (2004). Surprises from mathematics education research: Student (mis)use of mathematical definitions. *Mathematical Association of America Monthly*, 111, 411-424.

Gravemeijer, K. P. E., Cobb, P., Bowers, J., & Whitenack, J. (2000). Symbolizing, modeling, and instructional design. In: P.Cobb, E. Yackel, & K. McClain (Eds.), *Symbolizing, Communicating and Mathematizing: Perspectives on Discourse, Tools, and Instructional Design* (pp. 225–273). Hillsdale, NJ: Erlbaum.

Hill, H.C., Rowan, B., & Ball, D. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal, 42* (2), 371-406.

Larsen, S. (2004). Supporting the guided reinvention of the concepts of group and isomorphism: A developmental research project (Doctoral dissertation, Arizona State University, 2004) Dissertation Abstracts International, B 65/02, 781.

Leron, U., Hazzan, O., and Zazkis, R. (1995). Learning group isomorphisms: A crossroad of many concepts. *Educational Studies in Mathematics*, 29, 153-174.

Lloyd, G. (1999). Two teachers' conceptions of a reform-oriented curriculum: Implications for mathematics teacher development. *Journal of Mathematics Teacher Education 2,* 227–252.

Lockwood, E., Bartlo, J., Larsen, S. (2007). Learning how to support students' reinvention of the quotient group concept: A preliminary report. Presented at the Conference on Research in Undergraduate Mathematics Education. San Diego, CA.

Moore, R.C. (1994). Making the transition to formal proof. *Education Studies in Mathematics, 27*, 249-266.

Tall, D. (1992). The Transition to Advanced Mathematical Thinking: Functions, Limits, Infinity, and Proof, in Grouws D.A. (ed.) *Handbook of Research on Mathematics Teaching and Learning*, Macmillan, New York, 495– 511.

Wagner, J. F., Speer, N., and Rossa, B. (2007). Beyond mathematical content knowledge: A mathematician's knowledge needed for teaching an inquiry-oriented differential equations course. *Journal of Mathematical Behavior*, 26(3), 247-266.

Weber, K., and Larsen, S. (in press). Teaching and learning abstract algebra. In M. Carlson & C. Rasmussen (Eds.) *Making the Connection: Research and Teaching in Undergraduate Mathematics*. Mathematical Association of America.

Weber, K. and Alcock, L. (2004). Semantic and syntactic proof productions. Educational Studies in Mathematics, 56, 209-234.