Examining Trigonometric Representations as a Source of Student Difficulties

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Research examining cognitive difficulties and mathematics achievement shows that many students experience difficulty transitioning from secondary school to college mathematics (Hoyles, Newman, & Kent, 2001; Marcus, Fukawa-Connelly, Conklin, & Fey, 2007). Smith and Star (2007) contend that little attention has been devoted to investigating mathematical proficiencies students develop in different programs (p. 30). And while instructors are encouraged to reflect on their teaching practices and investigate effective classroom strategies to support successful student transitions they are often unclear about how to help students develop mathematical skills necessary for particular programs

There has been much discussion regarding students' inability to successfully transfer fundamental mathematical knowledge to the college mathematics classroom with the majority of the current literature describing how algebra, in particular, is a source of misconceptions. In fact, little attention has been given to trigonometry and the various ways that it has been represented in classroom teaching (Davis, 2005). Therefore, an examination of difficulties students experience learning trigonometric representations and how they are taught in secondary school and college is warranted.

Defining Representations

When students learn to use representations as tools, they are preparing for the kinds of activities that are common among mathematicians, scientists, engineers, and others who use mathematics in their professional work. (Greeno & Hall, 1997, p. 363) Representations are considered as objects (nouns) and actions (verbs) by a number of researchers (cf., Font, Godino, & D'Amore, 2007; Goldin, 2003; Greeno & Hall, 1997; NCTM, 2000; Pape & Tchoshanov, 2001). Goldin (2003) interconnects this dualism providing the definition of representations used in this article:

> A representation is a configuration of signs, characters, icons, or objects that can somehow stand for, or "represent" something else. According to the nature of the representing relationship, the term *represent* can be interpreted in many ways, including the following (the list is not exhaustive): correspond to, denote, depict, embody, encode, evoke, label, mean, produce, refer to, suggest, or symbolize (italics in the original; p. 276).

Goldin's (2003) representational system begins with primitive characters or *signs*. Signs are combined through established rules and practices into *configurations* accepted by the mathematics community. Configurations evolve, becoming increasingly complex or altered based on the relationship with other configurations. They can also combine uniquely to form a unified model for a representation. Additional structure is added to the representation by rules, sometimes ambiguous, which allow movement from one configuration to another within the system "establishing a kind of network structure" (Goldin, 1998, p. 144). The final representational system is characterized by a higher structure which relates the configurations meaningfully and provides meaning to the signs and configurations themselves (Goldin, 2003, p. 276).

There are three stages in the development of systems of representations interlinking the three components (Goldin, 2003, p. 279). In the inventive/semiotic stage, the learner assigns meaning to internal configurations based on previously learned representations. This meaning may be understood as being singular and rigid, without reference to other meanings not yet learned. This initial meaning of the representation is held for "a long time psychologically"

(Goldin, 2003, p. 279) before its real meaning with respect to its role in the representation is accepted or its role in other representational systems is understood and accepted. The second phase involves a period of structural development whereby relationships are built on the framework of the prior one. In the third phase, the autonomous phase, the representation is fully detached from previously held meanings and it begins to function flexibly with new meanings. As the new system becomes more powerful in the learning process, new interpretations of other representational systems are possible.

Representational objects are categorized as internal and external (c.f., Goldin, 2003; Pape & Tchoshanov, 2001; Smith, 2003). Cognitive representation is the representational process of internalizing external representations (Pape & Tchoshanov, 2001, p. 126) into abstractions or personal images of mathematical concepts gained through experience. This process takes place in the "zone of internal and external representations" where students are engaged in socially through meaningful learning activities, which provide opportunities to establish relationships and justifications for learning representations (Pape & Tchoshanov, 2001, p. 126). Smith (2003), in particular, underscores the value of children creating their own representations "internally" and reminds teachers that "each child follow[s] an individual path of creation and invest[s] his or her representation with specific meaning" (Smith, 2003, p. 271).

Cognitive development of mathematical representations focuses on how representations become internalized, incorporated into conceptual understanding, and become understood as accepted practice by the mathematics community (Goldin, 2003; Pape & Tchoshanov, 2001). Learning the practice of representation involves learning how to construct and interpret representations through "complex practices of communication and reasoning in which the representations are used" (Greeno & Hall, 1997, p. 361). According to Goldin, representations are not developed through algebraic manipulation, complex calculations, routine applications, inferential procedures, theorems and proofs and their applications (p. 282); representational systems rely on metaphors, imagery, visualization, and affect particularly when used in problem-solving (p. 282).

Research suggests that using multiple representations, in both teaching and learning, supports the development of mathematical understanding (Choike, 2000; Davis, 2005; Font, Godino, & D'Amore, 2007; Goldin, 1998, 2003; Pape & Tchoshanov, 2001; Schultz & Waters, 2000). Activities that build on previously learned representations facilitate the construction of new mathematical objects and the development of deeper understanding. For instance, in regard to trigonometry, Calzada and Scariano (2006) found that secondary school teaching that begins with familiar algebraic and geometric representations provided an easier base for student understanding (p. 450). The new representations add to a student's cognitive repertoire of tools that can be called upon in mathematical problem solving situations (Schultz & Waters, 2000). Students explore, design, and manipulate new representations as they internally construct mathematical concepts. Being able to mathematically connect different representations or generate new representations of the same object is a strong indicator of the college student's mathematical knowledge and ability (AMATYC, 2006, p. 5; Kessel & Linn, 1996). A key role for teachers then is to design learning activities incorporating multiple representations to support student learning (AMATYC, 1995; Davis, 2005).

Deep mathematical understanding is demonstrated when students are able to (a) move between multiple representations and (b) call upon representations that are most beneficial to solve problems or check solutions (NCTM, 2000; Pritchard and Simpson, 1999). According to NCTM (2000) traditional representations that have long-standing presence in the mathematics curriculum are diagrams, graphical displays, and symbolic expressions (p. 67). But evolving technology (e.g., graphing calculators, geometry and statistics dynamic software, and motion sensors) now allows students to work with all three. Learners can create idiosyncratic representations, develop institutional representations, and link both forms to obtain deeper understanding of mathematical concepts (NCTM, 2000, p. 67). The AMATYC *Standards for Intellectual Development* (2006) refer to students learning through modeling, linking multiple representations, and, selecting, using, and translating among numerical, graphical, symbolic, and verbal representations to organize and solve problems (p. 5).

Linking Representations to Hypothetical Learning Trajectories

Mathematics concept development has been defined within a hypothetical learning trajectory model (Clements, 2007; Clements & Sarama, 2004; Simon & Tzur, 2004). A hypothetical learning trajectory, according to Clements (2004) involves descriptions of the learner's thinking and learning in:

> a specific mathematical domain and a related, conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through a developmental progression of levels of thinking, created with the intent of supporting children's achievement of specific goals in that mathematical domain (p. 83).

Goldin (2003) supports the role of representations in learning mathematics and does not view learning representations solely through memorizing external representations or computational exercises (p. 278). Student learning is built through a series of well-crafted learning activities that reflect student cognitive development. The learning activities help students internalize the beginning representations needed to build subsequent representations. In fact, teaching resources have specifically linked multiple representations to a particular concept being taught. For example, Schultz and Waters (2000) discuss five different representations for solving an independent system of linear equations: concrete, tabular, graphical, algebraic, and matrices (pp. 449-450). Therefore, learning trigonometric representations can be viewed as a mapping to hypothetical learning trajectories.

The curriculum framework for a course may be determined by a mathematics department or government policy documents; however, the teacher analyzes this framework and makes decisions regarding his/her teaching based on student intellectual and emotional development, and students' current mathematical knowledge (Simon & Tzur, 2004, p. 96); that is, the teacher builds a hypothetical learning trajectory to encompass the mathematical concept expectations.

A hypothetical learning trajectory is defined by its three components: "the learning goal, the developmental progressions of thinking and learning, and [a] sequence of instructional tasks" (Clements & Sarama, 2004, p. 85). A key learning goal for the college technology student is: learning trigonometry. The developmental progressions of thinking and learning refer to the development of mathematical representations. The learning tasks are developed with (a) an understanding of the student's cognitive development and prior knowledge, (b) the concept being learned, and (c) the representation that appropriately objectifies the concept through its primitive signs and configurations as described by Goldin (2003). In designing these learning tasks, the teacher can employ static, dynamic, interactive, and recording media when teaching representations. The media have the potential to engage the students in a process of learning a representation identified in the learning task, thereby influencing the learner's ability to bridge internal and external representations. In this way, learning trigonometric representations is supported by a hypothetical learning trajectory for trigonometry. And, the learning trajectory for learning trigonometry is supported through learning and applying trigonometric representations. The connection between a hypothetical learning trajectory and a model for a system of representations is proposed in Figure 1, which illustrates how learning representational structure is mirrored with a learning pathway for representations.

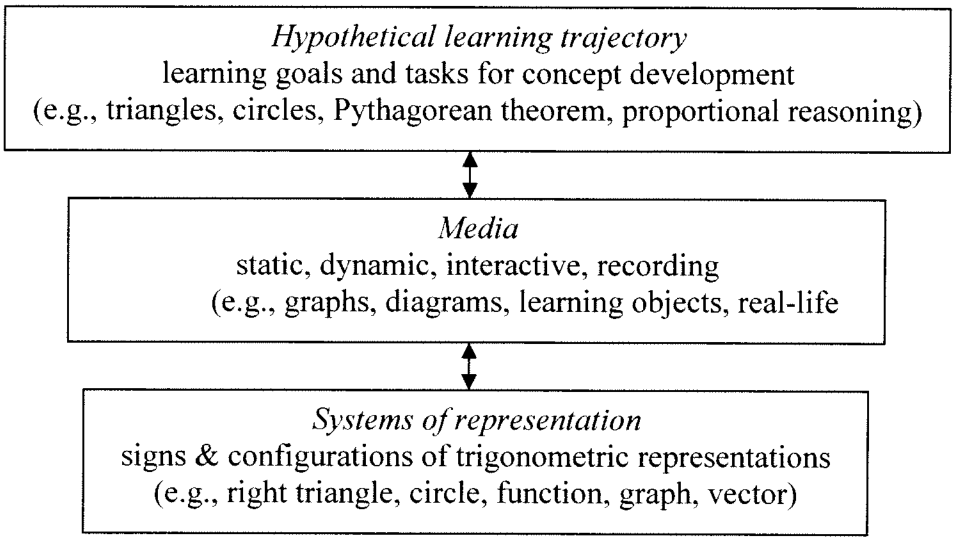

Figure 1. Proposed relationship between systems of representations and hypothetical learning trajectories

Sources of Student Difficulties

Difficulties that students have with learning trigonometry are often connected to difficulties in learning mathematics in general. Mathematical difficulties can be categorized as cognitive conflicts, learning obstacles (epistemological, cognitive, or didactic), semiotic conflicts, and mathematical discontinuities. As students advance their mathematical thinking, *cognitive conflicts* arise when they choose, often unconsciously, between previously held ideas and newly acquired ones (Tall, 1992, p. 495). *Learning obstacles* include *epistemological obstacles* that arise from the complexity of mathematics knowledge, *cognitive obstacles* that are caused by an inability to conceptualize or apply the mathematics, and *didactic obstacles*, which result from teaching practices (Selden & Selden, 2001, p. 240-241). At other times, students experience a "discordance, disparity or mismatch between the meanings attributed to the same expression" resulting in a *semiotic conflict* (Godino, Batanero, & Roa, 2005, p. 7). In addition, students experience a *mathematical discontinuity* when prior notions of how they are to think and do mathematics differ from current expectations (Smith & Star, 2007, p. 18). Student challenges are exacerbated when there is a mismatch in curriculum between educational panels.

Each of these difficulties complicates learning, particularly when a student's career path evolves to require more complex levels of mathematical thinking.

Difficulties Learning Trigonometric Representations

Student difficulties arise specifically when learning trigonometry with its own unique challenges. An examination of college curriculum documents reveals recommending teaching five key trigonometric representations. The first three are the right triangle, the unit circle, and the function representation. The second two develops from a layering process which interweaves the first three representations into additional forms also taught at the college level. The right triangle representation is superimposed on the unit circle to form the vector representation; the unit circle unwraps and combines with the function form to result in the trigonometric graphic representation. Students experiencing cognitive conflicts may lack understanding of the right triangle, the unit circle, and/or the function representations, their signs, configurations, and/or their supporting network structures, and are at risk for misunderstanding and applying the vector and trigonometric graph representations.

Additional lack of understanding may come in the form of cognitive obstacles. Goldin (2003) recognizes that students can experience cognitive and epistemological obstacles at various stages of their representational system (p. 279). In each stage, personal interpretations based on prior learning experiences and an inability to structure connections between configurations can interfere with the development of higher order representational systems. According to Goldin (2003), "These interpretations pose cognitive obstacles because they require abandonment of the initial, semiotic connections on which the system was constructed" (p. 279).

Semiotic conflicts can develop for students in a number of different ways when learning representations. One key difficultly results when students do not understand the signs and configurations or the connecting mathematical structures linking these forms to create a representation. As a result, representations can be disconnected from mathematical concepts (Font, Godino, & D'Amore, 2007, p. 4; Goldin, 2003, p. 279).

Another source of difficulty may be found in the learning trajectory for trigonometry from secondary school to college. Mathematical discontinuities are evident when students struggle with prior notions and expectations of how to think and do mathematics (Smith & Star, 2007). For instance, Smith and Star address the issue of curriculum mismatch, where students move from a traditional algebraic-based curriculum to a reform curriculum that incorporates multiple representations and vice-versa, as another source of student difficulties. The difficulties students experience transferring knowledge of mathematical concepts from one educational domain to another (e.g., secondary school to college) is another example.

Conclusion

Students may experience a variety of difficulties when learning trigonometric representations – cognitive conflicts, learning obstacles (cognitive obstacles, epistemological obstacles, and didactic obstacles), semiotic conflicts, and/or mathematical discontinuities. Student difficulties may also arise when teaching representations is not mapped using a hypothetical learning trajectory from secondary school to college learning environments. It is possible that the mathematical discontinuities as described by Smith and Star (2007) refer only to traditional and reform classroom teaching approaches and do not include all three features of a hypothetical learning trajectory – the learning goal, the cognitive developmental progressions, and sequenced instructional tasks. In this context, it is possible that student difficulties may arise when the hypothetical learning trajectory for a mathematical concept is not seamless from secondary school to college studies, resulting in interrupted mathematical transitions for college students.

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