Speaking with Meaning in a College Algebra Course

By

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Abstract

The purpose of this research is to describe the emergence of the sociomathematical norm of *speaking with meaning* (Carlson, Clark, & Moore, In Press), delineate how a college algebra instructor helped enable this emergence, and demonstrate how it can be used as a tool to make inferences about students' progress and shortcomings in the classroom. *Speaking with meaning* has the dual nature of being both a sociomathematical norm regarding what constitutes sufficient mathematical participation as well as being a tool that can be used in the classroom to elicit such participation. Our analysis showed that attention by the teacher to student responses helped enable the students to speak more meaningfully. In the case of this college algebra course the students were able to explain functions in terms of inputs and outputs. Thus, in this class, to *speak with meaning* about functions means to couch responses about functions in terms of input and output. Our analysis also uncovered that students were unable to speak about changing rates of change meaningfully (i.e. a normative way of speaking about changing rates of change in the classroom) which led to probing from the instructor. As a result of this questioning, it was revealed that when dealing with two quantities changing in tandem, the students had difficulty with identifying and remaining attentive to the quantities changing in tandem.

Introduction

Students across the country continue to have difficulty in college algebra, a course required by many disciplines besides mathematics and science (need reference). While the topics covered in a college algebra course may vary from one University to the next, the underlying concepts necessary to succeed in Calculus are what are most valuable in the course. Research has shown that when students have a deep conceptual understanding of the mathematics, these understandings become powerful tools for success (Carlson, 1998; Monk, 1992; A. Thompson et al., 1994). Thus, within the classroom setting teachers need a way to determine if students are able to develop a deep understanding.

Speaking with meaning provides a tool for instructors to use to determine whether or not their students are developing this conceptual understanding in a classroom setting (Carlson, Clark, & Moore, In Press). By requiring students to *speak with meaning*, students are asked to respond to mathematical questions in a conceptual manner, thus allowing the correctness of their reasoning to be judged by the audience (in this case the teacher and the rest of the class). When students are observed giving answers that are not considered mathematically sufficient, *speaking with meaning* becomes a tool that the teacher and other students in the class can use to encourage a student to respond in a conceptual manner.

The focus of our research is on a College Algebra course in which contextual activities were used and covariational reasoning (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002) was an underlying focus throughout. Presented is an investigation into how the teacher was able to influence speaking with meaning within the classroom and how this created learning opportunities for the students. Also explored is an instance in which the students were unable to speak meaningfully about a topic and how further analysis was necessary to determine their underlying misconceptions about recognizing quantities and their inability to quantify in certain situations. As we describe this activity, we will make note of teacher moves that helped enable this activity and created opportunities for collective reflection by the students.

Theoretical Perspective

Recent research has shown that incoming undergraduates often have very weak understandings of the concept of function, understandings that are central to success in a college algebra classroom (Carlson, 1998). Central to a strong understanding of the concept of function is the fostering of a process view of functions (Carlson, Oehrtman, & Thompson, in press). A process view of function (e.g., understanding function as a dynamic situation of two covarying quantities) is crucial to understanding concepts developed in calculus. It is recommended that in order to promote a process view of functions, students should “explain basic function facts in terms of input and output.” In order to promote this type of discussion, the teacher in the classroom under investigation has continually attempted to promote and model this focus, which has been described as *speaking with meaning* relative to function “facts.”

Speaking with meaning implies that conceptually based descriptions are given when providing an explanation, describing an approach to solving a problem or having a discussion in the classroom. An individual who is *speaking with meaning* may be observed referencing quantities and relationships (instead of just numbers and procedures) and provides a rationale for various steps in a solution approach.

Speaking with meaning has the dual nature of being both a sociomathematical norm and a tool for enabling it as normative. Sociomathematical norms refer to normative behaviors that are specific to mathematics, such as understanding what constitutes an acceptable mathematical solution, and emerge from what counts as acceptable mathematical behavior in the classroom (Yackel & Cobb, 1996). With regards to its normative aspect, it refers to what counts as sufficient mathematical participation in the classroom. Teachers and students can also use the phrase to encourage *speaking with meaning,* in which case it becomes a tool for use within the classroom. It brings clarity to what kind of participation is expected in the classroom.

In order to deal with the type of classroom discourse that constitutes conditions for possible mathematical learning, Cobb, et. Al. (1997) introduce the concepts of *reflective discourse* and *collective reflection*. Reflective discourse refers to the objects of discussion that emerge from the discourse that takes place between the students and teachers. Once these explicit objects are created, they become items which can be reflected upon. This communal reflection is then referred to as collective reflection.

It is important to note here the distinction between reflective discourse and Piaget's reflective abstraction, which is a psychological construct referring to the process by which an individual reorganizes their own mathematical activity (1972). Reflective discourse denotes the opportunity for this reorganization, but in order to make inferences about students' individual reflective abstractions and conceptual reorganizations would require deep psychological analysis.

Reflective discourse gives us a way to account for student contributions to the shifts that occur in the classroom discourse. As students make contributions to the classroom norms, they create objects which can be reflected upon and create opportunities to form a deeper understanding of mathematical concepts. It is important that the teacher initiates shifts that enable the students to create these objects. Our stance is that once these objects are created the goal is to come to a collective reflection which can be observed by the students promoting and acting within the classroom norms.

It is well documented that even high performing precalculus and calculus students have weak understandings of the function concept (Carlson, 1998). Studies have also revealed that the ability to reason covariationally (e.g., the image of formulas and graphs representing the varying magnitude of two quantities as they change in tandem) is critical for understanding functions and central concepts of calculus (Carlson, 1998; Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; P. W. Thompson, 1994a; Zandieh, 2000) and differential equations (Rasmussen, 2001).

Covariational reasoning is defined as the "cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (Carlson et al., 2002; Carlson et al., in press). According to Thompson,

> Once students are adept at imagining expressions being evaluated continually as they "run rapidly" over a continuum, the groundwork has been laid for them to reflect on a set of possible inputs in relation to the set of corresponding outputs (1994a).

Here, the reflection on the set of possible inputs in relation to the set of corresponding outputs is related to considering rates of change.

Methods

This study took place in a College Algebra class in a large southwestern University. Due to low success rates in College Algebra, several of the sections were redesigned to incorporate curriculum and teaching techniques that research has shown as successful. The course was designed to improve curriculum and instruction by incorporating active learning in each class session with a focus on research-based conceptual activities designed to promote student interaction and understanding. Course design focused on deep conceptual understanding of a few key topics, rather than a large number of procedures and rules: covariation, function, function composition and inverses, average rate of change, proportionality and linearity, polynomial functions, exponential growth and logarithms.

In order to determine the effectiveness of these redesigned courses, they were videotaped for analysis. For our purposes we focused on one particular class to provide evidence of how speaking with meaning emerged in the classroom and how the teacher could use it to gauge the students’ understandings as a class. In order to do so, the video tape was analyzed for instances in which the students were contributing to the norm of speaking with meaning. In other words, instances where students were creating sufficient mathematical arguments for their claims and able to use the mathematical terms associated with concepts were identified and analyzed. When instances were noted in the video, they were transcribed for further analysis.

In addition to identifying instances of students contributing to the norm, we were also interested in instances in which students were operating outside the norm. In this situation, this means the students were unable to speak with meaning about a particular topic. We were able to identify these situations because students had difficulty relating the context of the problem to the mathematics. When this occurred, it was difficult to determine why the students were having so much difficulty. Thus individual interviews were also conducted and analyzed to determine student thinking on several covariational activities. These interviews were videotaped, transcribed and analyzed to help reveal student misconceptions on these activities.

Results

Within the classroom, it is the instructor’s role to help initiate and sustain *speaking with meaning*. With regard to the function concept, this involves using input and output language. In the beginning of the semester, the instructor explicitly focused on both modeling and promoting *speaking with meaning* relative to the input-output process of a function. This included introducing input and output language during a classroom discussion, probing of students when input and output language was

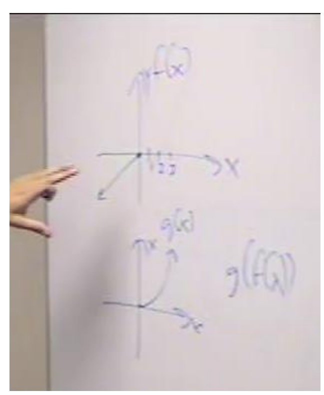

Figure 1

not used, and curriculum focused on incorporating the use and language of functions as an input-output process.

By the sixth week of the class, *speaking with meaning* about the input-output process of functions appeared to be normative. The following excerpt occurred during the sixth week and focused on the topic of function composition. During this excerpt, the class was focused on the graphs of two functions (figure 1) and whether or not the two functions could be composed as $g(f(x))$. After establishing the domain and range of each of the functions presented, the following interaction occurred.

Instructor:	Okay, now can we compose those functions?
Student 2:	No.
Student 1:	No.
Instructor:	Why do I hear "No"?
Student 2:	'Cuz whatever output you get can't be an input of, I mean whatever output you get of *f*(x) can't be an input of *g*(x).
Instructor:	Right, if we look at, let's say, x equals a negative value as our input for *g*(*f*(x)), what's the output of *f*?
Student 2:	Negative.
Instructor:	It's negative, right? Now can we, so that output should become the…
Students:	Input of *g*.
Instructor:	The input of *g*, but can that happen?
Student 2:	No.
Student 3:	No, 'cuz its positive.

During this interaction, multiple students mentioned that the two functions could not be composed as $g(f(x))$ (lines 2-3). When asked why this was the case (line 4), Student 2 used the input-output relationship of function composition, a difficult understanding for students to develop, to describe that output values of $f(x)$ could not become input values of $g(x)$ (lines 5-6). Thus, we observed a student *speaking with meaning* without prompting, and hence contributing to the establishment of the norm.

As the interaction continued, multiple students described the process under question in terms of the output of $f(x)$ and the input of $g(x)$. It appears that the students were engaged in reflective discourse, thus creating an opportunity for learning. After this interaction, the instructor shifted the discussion to focusing on the domain and range of the two functions and their relation to the input and output of a function, after which the following interaction occurred.

Instructor:	So, what's our output for *f*, range or domain?
Student 4:	Domain, Oh, I'm sorry, output is range.
Instructor:	Output is range, right? [Circles range on board]
Students:	Right, correct.
Instructor:	Now, but in composition, our output for *f* becomes the…
Student 5:	Input for *g*.
Instructor:	The input for *g*. But, what happens between that domain [Circles domain on board] and … and the input for *g* is that *g*'s domain, right? [Students speaking inaudibly].
Instructor:	They don't match, right?
Student 2:	So, is the range, is the range of *f*(x), if the range of *f*(x) is not in the domain of *g*(x)it can't be a, it can't be a composition.

Again, we see the instructor and class focused on speaking in terms of input and output (line 1), but now with a focus on the domain and range of the functions. Students appear to have made the connection between domain and range and their relation to the input and output (lines 2 & 4); the students appear to be reflecting on and reorganizing their understandings of domain and range. Thus, the discourse supported and sustained collective reflection.

The instructor then shifted the discourse to focus on the domain and range of the functions in relation to the function composition under discussion (line 5). During this discussion, the *instructor* led the class to focus on whether the output of $f(x)$ matches the input for $g(x)$ (line 10). It is then Student 2 that spontaneously makes the claim “So…if the range of $f(x)$ is not in the domain of $g(x)$ it can’t be a, it can’t be a composition” (lines 11-12) It appears that the domain, or inputs, and range, or outputs, had become objects that were reflected upon and reorganized by the student. Also, it is important to note that if no students had been able to respond in this manner, this shift of discourse would not have occurred. It is therefore reasonable to say that an individual student contributed to the development of the discourse that supported and sustained collective reflection.

The above interactions first reveal a student, without prompting, *speaking with meaning* relative to the input-output process of functions (Excerpt 1), thus creating a shift in the discourse. The discussion that ensued created opportunities for the students to deepen their understandings of domain and range in the context of the composition of functions. In this case, the outputs of $f(x)$ and inputs of $g(x)$ became objects upon which the students could possibly reflect in order to strengthen their understandings of domain and range and their relation to function composition. Furthermore, it was both the student and teacher initiated shifts in the discourse that enabled this reflection.

In the previous example, we noted how a student *speaking with meaning* contributed the opportunity for students to reorganize their understanding of domain and range in the context of function composition. Of importance was that a student speaking with meaning *initiated* this learning opportunity. In the following example, we present a case where reflective discourse appeared to occur between the teacher and students until *speaking with meaning* occurred. This contribution enabled an opportunity for collective reflection.

The topic of focus in the classroom was considering how the height of a stack of books varied as the number of books (of height 2.5 centimeters) in the stack varied. The emphasis was on both the relationship of the input and outputs of the function that defined this situation and the relationship of corresponding amounts of change of input and output. The question proposed before the following interaction asked the students to interpret what the ratio of the change of the height of the stack (output) to the change of number of books (input) represented.

Instructor: What did we come up with?

Students: About 2.5.

Instructor: 2.5. Does that make sense?

Students: Yes…

Instructor: Why? [Calls on Student 2].

Student 2: Because it’s proportional to the first, what you do first, at the beginning when you change the height…[inaudible]…instead of the 17, you put the amount of books that you need.

Instructor: Okay, and let’s not, let’s not even look at the proportionality right now. What does this ratio represent? What’s the top unit? [Some students respond]. Height of…just the unit is height of books, right? And this

12		is…?
13	Students:	Number of books.
14	Instructor:	So, the height of books is measured in, in the number of books, so it’s
15		just books, right? So… [Writes cm/book].
16	Student 6:	So, every time it changes by 2.5, which means there’s a constant rate of
17		change, right?
18	Instructor:	Right, there’s…she said, so explain more what’s constant rate of change
19		mean? When you say…
20	Student 6:	Well… every…as you add each book, you’re adding…umm, an
21		identical, umm, umm number of centimeters, which means that each
22		incremental change is identical to the incremental change before it and
23		the incremental change after it…therefore it’s constant.

During this interaction, we first observe the students responding with the correct ratio of 2.5 (line 2). However, this does not fully address the question proposed of interpreting the ratio. When the instructor probed for further explanations (lines 3 and 5), a student then attempted to connect the ratio to proportionality (lines 6-8). Although this is a correct connection and a possible instance of collective reflection, the teacher then made the move to shift the class away from proportionality and move into a discussion about what the ratio represented in terms of the quantities of the situation (lines 9-12).

As the discussion proceeded, a student then questioned whether there was a constant rate of change (lines 16-17). When asked to explain what she meant by a constant rate of change (line 18-19), the student appeared to have constructed an image that a constant rate of change (in this situation) implies that for each additional book (the input of the function), the height of the stack increases by an identical amount (the output of the function). She then continued to discuss constant rate of change in terms of incremental changes (lines 20-23), that is, she was *speaking in meaning* relative to a constant rate of change. It appeared that the student re-organized what the ratio under discussion represented. That is, it appears that the student had re-conceived the ratio from a static state to a generalized rate, an instance of collective reflection creating a possible learning opportunity for the students and contributing to the norm of *speaking with meaning*.

In our final example we look at what can happen when students do not appear to be able to engage in collective reflection. In analyzing the discourse we note that the students were unable to speak with meaning about a situation represented graphically involving changing rates of change. In the classroom setting it was difficult to ascertain why the students were unable to speak with meaning in this context. It was not until the students were individually interviewed that light was shed on the nature of their misunderstandings.

In the following interaction, the students were asked to develop graphs that related the volume of water put into a container using syringes and the height of the water in the container. In this problem, volume of water is treated as the input and the height of the water in the bottle is treated in the output. Specifically, the students were asked to produce a graph that described a cup that at first widened and then remained a constant diameter (Figure 2). The initial student produced graph (Figure 2) led to the following discussion.

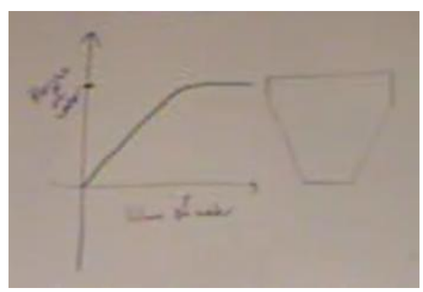

Figure 2

1	Instructor:	What's gonna go on here (referring to figure 2)?
2	Student 7:	It's going to increase at a decreasing rate, and then when it hits the,

3		uhh, bottom of where it flattens off, it’s just gonna be the steady rate,
4		the constant rate.

Here, a student gave a correct response by describing the graph as increasing at a decreasing rate (lines 2-4). However, the student was not speaking with meaning, as he did not give a description that focused on the quantities of the situation (e.g., what it meant to increase at a decreasing rate in terms of height and volume). In an attempt to illicit a response that focused on the way quantities involved in the problem were changing, the instructor continued questioning the class.

1	Instructor:	Okay, so on this one, he said it’ll be increasing at a decreasing rate
2		first, why?
3	Student 8:	It’s getting wider.
4	Instructor:	‘Cuz it’s getting wider – which means?
5	Student 8:	It’s uhh… more space to fill up.
6	Instructor:	More space to fill up, or, as we put cups in - as we put syringes in,
7		what happens to the change in height?
8	Student 8:	Decreases.
9	Instructor:	The changes in height decrease. So, how can we denote that on the
10		graph?
11	[Long Pause]	
12	Student 8:	Make more slope? Umm..
13	Instructor:	Well is it more slope?
14	Student 9:	It’d be less slope.
15	Instructor:	It would be less slope. What does slope represent?

16	Student 9:	The rate.
17	Student 8:	It’s getting wider, then it should be more…
18	Instructor:	Let me – I’ll draw more slope first [draws graph increasing at an
19		increasing rate, figure 3]. So, there’s more slope.
20	Student 8:	No…not like that. Like, slope that’s down.
21	Instructor:	Well, you said more slope.
22	Student 8:	The line should be closer to the x-axis.
23	Instructor:	What’s that?
24	Student 8:	The line should be closer to the x-axis.
25	Instructor:	What do you mean, closer to the x-axis?
26	Student 8:	Like, instead of it being like this (mimics increasing, concave up
27		graph with hand), it should be like that (mimic increasing, concave
28		down graph with hand).
29	Student 7:	It should go out further to your right.
30	Instructor:	So, are you saying…like that [draws increasing at a decreasing rate].

The instructor made the move to focus the class on the increasing at a decreasing rate section in an attempt to promote speaking with meaning (lines 1-2). The instructor led the class to describe that the change in height of water decreased for each syringe of water put in (lines 10-12). However, when probed to relate this to the graph (lines 13-14), the class was unable to speak with meaning about the slope of the graph (lines 16, 24, 26, 28, 30-31). Throughout the above discussion, it remained unclear what was meant, relative to the contextual situation, by “more slope,” “less slope,” and “closer to the x-axis.”

Immediately after the excerpt above, the student that originally created the graph expressed that he believed the portion of the graph that was increasing at a decreasing rate should reach the same height of the cup. Despite the fact that the students were unable to discuss what was going on in the graph in terms of the quantities involved, the teacher acted on this move to shift the focus on the cylindrical portion of the bottle, for which the student had drawn a horizontal line. This was due to the fact that although the students were unable to speak meaningfully about the graph, that portion of the graph was still correct. Due to time constraints the teacher found it necessary to have a discussion about the section of graph that was incorrect, in this case the horizontal part, which would indicate that the height was not increasing as water was being poured into the cylindrical part of the bottle.

Student 8:	No, no, because eventually it levels off, and it stays level at the same time…it stays the same, right…the same time…so you need…
Instructor:	So, you’re saying it stays level.
Student 8:	Yeah.
Instructor:	Like that? [draws horizontal line for remainder of graph].
Student 8:	No, eventually, yeah, it flattens out, because, because there’s a certain point where it doesn’t…it doesn’t become any wider or any narrower.
Student 9:	It’s still increasing. It might be increasing at a constant rate, but…
Instructor:	Right, but the height…what’s the height of the water doing? So, you’re saying it should stop right here. It should level off?
[Inaudible student responses].	
Instructor:	What’s the…what’s the water doing?
[More talking].	

14	Instructor:	So, is the line gonna be like that?
15	Student 10:	There should still be a slope to it, but it should be a very gradual
16		slope.
17	Student 11:	And straight.
18	Instructor:	So, what should the slope be? [Calls on Student 3].
19	Student 9:	It should be constant up to the top.
20	Instructor:	It should be. So, [draws constant increasing slope], is it gonna be like
21		that?
22	Student 7:	That’s what I was thinking.
23	Instructor:	Is it gonna be like that?
24	Student 8:	Yeah it is.
25	Student 5:	Yeah.
26	Instructor:	Why is it gonna be like that?
27	Student 12:	Because if the width stays the same…but the height and volume both
28		increase

This interaction revealed the struggles of the students to speak about how the cylindrical portion of the bottle and the graph are related. First, a student believed that the graph should be horizontal because the cup does not become wider or narrower (lines 6-7). As the discussion continued, a student offered up the conjecture that the graph may increase at a constant rate (line 8). The instructor then focused the discussion back to the student who felt the graph should be horizontal (lines 9-10). A student then determined that there should be a “gradual slope” (lines 15-16), and later described this as constant (line 19), but was unable to speak with meaning by using the context of the problem about why it was going to be a constant slope (lines 27-28).

Because of the struggles of the students to articulate the covariation of volume and height of water in the bottle, the teacher then shifted the focus of the class to a discussion solely about the direction of change of the height and volume of water in the bottle. This then led to the class determining that the height and volume of water in the bottle are both increasing, which resulted in the class concluding that the height of the water in the bottle would be increasing at a constant rate and this would be represented by a straight line sloping upward. However, the class appeared to remain unable to discuss what increasing at a constant rate meant in terms of the quantities of the situation.

Mentioned previously was that it was difficult to determine, using classroom data, the reasons why students were unable to speak with meaning relative to rates of change in the context of the bottle problem. In order to investigate students' understandings and possible misconceptions, task-based interviews were analyzed in an attempt to gain insight into these difficulties. These interviews revealed that students did not appear to build models of the situations that the questioning was situated in. That is, it appeared that students did not build a developed image of the quantities they were asked to reason about. Here, the term quantity refers to an attribute of an object or situation that admits a measurement process. Without a developed image of the quantities one is asked to reason about, it can be expected that reasoning about these quantities will be very difficult.

Discussion and Conclusions

In order to establish a classroom where ideas are shared and discussed, it is important for the teacher to be aware of what his/her students are saying and how that can be used to allow them opportunities to create deeper understandings of the mathematics. By encouraging students to speak with meaning, the teacher is helping to establish the kind of mathematical discourse necessary to create these opportunities. It is within this setting that the teacher must actively make decisions about when to follow a student shift in the discourse and when it is necessary to initiate a shift.

The teacher contributes to the establishment of speaking with meaning by modeling it when giving an argument, encouraging students to give sufficient mathematical arguments to back up their claims and by highlighting instances when students *speak with meaning*. This highlighting can be achieved in one of two ways. First, the teacher can verbally reward a student for speaking with meaning; second, the teacher may choose to use a student's sufficient response as a point of negotiation (example 1). In this case, the teacher decided to follow this student initiated shift in order to allow students the opportunity to reorganize their thinking about a concept. In both cases the student responses are actions, which, when vindicated by the teacher, become objects that the students can reflect upon. By encouraging the students to speaking with meaning, reflective discourse occurred creating the opportunity for collective reflection.

It is also our contention that students' abilities to speak with meaning can help a teacher infer the progress of the classroom. As noted in our first two examples, when the students were able to speak meaningfully about the concepts, reflective discourse and collective reflection have the opportunity to occur. When students are unable to speak meaningfully about a mathematical concept, as in the third example, further analysis is needed to uncover their misunderstandings and contributing factors to this inability. In this latter case we want do not claim that collective discourse and collective reflection cannot take place, but rather that it is much more difficult to determine what objects the students may be reflecting upon.

We find speaking with meaning to be a very powerful term. It has the dual nature of being a sociomathematical norm regarding what counts as appropriate mathematical discourse in the classroom and it can be used by teachers to operationalize what is meant by appropriate mathematical behavior. In this paper we hope to have highlighted how teachers can create learning opportunities in the classroom by encouraging this type of discourse. Further we want to illustrate how students are engaged in reflective discourse when they are contributing to the sociomathematical norm of speaking with meaning. In this paper we are not claiming that when speaking with meaning occurs, learning does as well, but rather learning opportunities are created. It is up to the individual students to take these opportunities to reorganize their understandings of the concept. The only way to determine if they have is through individual psychological analysis.

The goals of this paper are twofold: theoretical and practical. We look to add to the theoretical literature regarding speaking with meaning. In this case we demonstrate how contributing to speaking with meaning can influence the mathematical discourse in the classroom and how it can create learning opportunities for students. These connections help demonstrate the value of the construct and will allow it to be used as a lens in examining classroom discourse.

With regards to practical applications, as we noted earlier, in-service teachers were able to find speaking with meaning very valuable in directing the mathematical discourse of their classrooms. Thus speaking with meaning has implications for pre-service and in-service teacher development. We hope this research is the beginning of a training sequence that will help teachers establish appropriate mathematical discourse in their classrooms. In order to do so, teachers need to be aware of what it means to speak with meaning, how to encourage it to emerge as normative, how to initiate shifts in the discourse by utilizing meaningful student contributions, and how to initiate shifts when student contributions are not meaningful.

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