College Students' Understanding of Rational Exponents: A Teaching Experiment

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The study examines first through a pre-interview the understanding five college students (at a Midwestern university) have of rational and negative exponents (including the zero exponent), and the students' justifications for their notions of exponents. A teaching intervention (experiment) is then conducted with the same students to test the strength and limitations of a conjecture[1] *for teaching rational and negative exponents, using an approach based on the concept of relative rate of growth or change (Confrey, 1994), the concept of factor of multiplication and a procedure to calculate decimal exponents. Results suggest that the definition of exponents students learn in school provide the primary lens for conceptualizing rational and negative exponents. The laws of exponents pl ay no foundational role in this process. Post-interviews with the same content as the pre-interview, held after the teaching experiment, showed improved responses on most questions.*

Research questions

The aim of the study was to answer the following questions:

1. *What are the students' concepts of rational and negative exponents before the teaching intervention?*

2. *What is the role of the laws of exponents in the process of developing rational and negative exponents as emerging from the first part of the study, and during and after the teaching experiment?*

3. *What is the impact of the teaching experiment on the knowledge of the participating students?*

Brief review of literature

Studies of students' conceptions of exponential functions are relatively recent (Rizzuti, 1991; Confrey, 1994; Confrey and Smith, 1995). What has not been investigated so far is the actual development of the students' notions of exponents, in particular the rational and negative exponents. Studies on students' modes of understanding and use of definitions (Edwards, 1997; Edwards & Wards, 2004) and logical statements (Selden & Selden, 1995; Selden & Selden 2005)

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[1] A conjecture refers here to an inference based on incomplete evidence, which serves as a guide within a framework and can be modified or adapted during or after the teaching intervention (Confrey & Lachance, 2000)

suggest that mathematical definitions and logical statements are usually not the points of departure for students in trying to understand concepts with complex consequences. Concept images, concept statements and concept definitions (Tall & Vinner, 1981) play a very important role in the development of students' notions of mathematical concepts.

Theoretical framework

Learning mathematical concepts develops in phases from the stage of actions on physical or mathematical objects to mental processes that are internalized and encapsulated into new mathematical objects and transformed into schemas that are again acted upon through new actions (Dubinsky, 1994). *Understanding* mathematical concepts also require a refined framework that includes the development and interaction of internal (mental) representations and external physical and symbolic representations involved in systems of learning and problem solving (Goldin & Kaput, 1996), where external representations are tools for thinking (Goldin & Herscovics, 1991). In this study I try to integrate the framework of Goldin and colleagues (1996, 1991) on internal and external representations with the theory of actions on mathematical objects that evolve into schemas for the learner and eventually into new mathematical objects (Dubinsky, 1994). The purpose is to create a framework for understanding the growth of rational exponent concepts in students, and a framework for the teaching experiment to transform aspects of the learning process for rational exponents.

Description of the study

Semi-structured interviews were held with five undergraduate students and two graduate students to gain insight into their ideas of positive integer exponents, rational exponents, the exponents zero and one, negative exponents and the laws of exponents in general. The inclusion of the graduate students in the first phase was to get more insight in differences and similarities in the responses of the novice students compared with the responses of the (mathematically) more mature graduate students. The graduate students were not involved in the rest of the investigation. Written and verbal answers on the connection between positive integer exponents and rational exponents were documented. Special attention was given to the meaning the students proposed for the rational, decimal, negative and zero exponents and the justifications for their values.

Next, over a period of four weeks, a conjecture based transformative teaching experiment (Confrey, J. & Lachance, A., 2000) was conducted. The teaching experiment was conducted by the researcher in two separate groups. Three interviews were held with each student during the teaching experiment to determine his or her understanding of the content of the teaching material. After the teaching experiment each student was interviewed with the same set of questions as in the initial, pre-teaching experiment interview, to document the changes in students' notions on exponents. All sessions were video and audio taped and transcribed verbatim. The transcripts and video tapes were analyzed, coded and ordered into qualitative tables for comparison, which were later developed into constructs.

Results of the investigations

Research question 1: *What are the students' concepts of rational and negative exponents before the teaching intervention?*

The analysis of the first set of interviews with the five novice students suggest that all of the novice students were familiar with and had a robust idea of positive integer exponents. I call this concept of positive integer exponents the Common Definition of Exponents (CDE). Only one student was familiar with the term Laws of Exponents. Four of the five students knew why the zero exponents carried the value 1 for the general case, but they did not know why. They all indicated that the zero exponents either had a "weird nature" or suggested that it meant that there are zero factors and the value should be "nothing or zero", or that the value was just a different "rule of the game". One student tried to give an explanation but did not clarify the steps. Negative exponents were described as reciprocals of factors but nobody knew why that was the case: "the teachers said so"; "nobody told me why"; "the concept is 'weird' because it is logically inconsistent"; "negative exponents indicate the absence of the base number".

All the students related rational exponents to radicals, but the definition was perceived as an "impossible one" because "you can't have 1/3 times a base number; that is impossible!" One student said that rational exponents are radicals, but $5^{1/3}$ is not 1/3 times 5. ("I do not know why this definition is like this; the teacher told us"). Decimal exponents were mostly converted to fractions to understand what they are. The decimal digits had no other meaning then indicating how big the numbers were. Fractions in exponents were "equal" as soon as the fractions could be reduced through common factors. This was seen as a natural property of exponents independent of the definition through radicals and a direct result of the properties of fractions used as exponents.

None of the students had a proposal to unify all the definitions of exponents into an all encompassing concept or notion for exponents. The students kept repeating the Common Definition of Exponents (CDE) as their basic notion of exponents.

> Research question 2: *What is the role of the laws of exponents in the process of developing rational and negative exponents as emerging from the first part of the study, and during and after the teaching experiment?*

The data suggest that the Laws of Exponents are not a central concept for the students and they do not use them to base their notions of rational, zero or negative exponents on. The CDE seems to be the basis for their constructions of the concept of exponents but they are unable to give meaning to the rational, zero, or negative exponents reasoning from their first notions. The bridge from positive integer exponents to rational, decimal or negative exponents seemed to be closed to them.

The teaching experiment results suggest that the notions of relative rate of growth (change) and the corresponding factors of multiplication[2] are difficult initial steps for all the five novice students. The concepts were introduced in the context of population growth with a fixed rate of growth over equal periods of time. When the students were asked to find the rate of growth over multiple periods the first impulse of all the students was to try an additive approach of adding the rates of growth over the given periods. Only gradually did the importance of the factor of multiplication become clear to the students.

Rational exponents were introduced as factors of multiplications involving non-integer periods of time in the population model. Radicals were treated as new base numbers that could produce the initial base number through repeated multiplication. For example the factor of multiplication 5 (or base number 5) could be the result of two periods with factor of multiplication 2.23606 (= $\sqrt{5}$). The number 2.236068 was called "half a factor of 5" because two factors were needed to achieve what factor 5 did in one multiplication. This notion was difficult, but by using the terminology consistently it became more accepted during the teaching experiment. A special procedure for calculating decimal exponents was discussed and connected to the laws of exponents and the decimal roots (i.e. numbers of the form $\sqrt[10]{5} \approx 1.174617$; $\sqrt[100]{5} \approx 1.016225$, etc) of the base. The place value system of the decimal exponents was given a central place in these discussions and each student developed her or his own method to find the decimal

[2] For example: if a quantity grows r% per unit (of time), then the factor of multiplication is $(1+ \frac{r}{100})$

digits of exponents from equations like "Solve for X: $5^X = 10$". The procedure was practiced by the students and made into their own system of notation and organization. Lists of decimal roots of the base were produced by two students and systematically used to find all the digits through division and multiplications. Other students carried out their calculations in a less organized way. One student used columns to organize his decimal calculations.

The zero exponents were explained in relation to a situation with zero growth. The zero growth was then illustrated in both a graph (using shorter and shorter intervals) and in tables to correspond to a factor of multiplication of one (1). This teaching episode was not obvious to the students, but the emergence of the multiplication factor one (1) through higher and higher decimal roots seemed to make an impression with the students. The place value system for decimal exponents also offered a way to use zero exponents as actual place holders for factors that do not appear in a numeral and where the explanation "absence of factors" actually makes sense.

Negative exponents were treated as tools to undo a multiplication and the negative sign was linked to reverse movements on an (exponential) graph. The fact that numbers with negative signs could indicate reverse movements or inverses in various contexts seemed to make it more acceptable to students that exponents linked to reverse operations and movements naturally carry a sign to denote such movements.

Research question 3: *What is the impact of the teaching experiment on the knowledge of the participating students*?

The responses to the post-interview suggest that all the students made some progress in understanding rational and negative exponents. Not one student mentioned inconsistencies or lack of logic in their responses to the questions. Two of the five students produced an actual, generalized notion of exponents as measures of how much multiplication of a certain base is used. Three students did not propose any generalized notion but they were able to talk about the process of finding a decimal exponent and they were able to explain every digit in the decimal form of an exponent and explain negative exponents using the movement metaphor.

Final notes

The conjecture needs further polishing in the area of rates of growth and factors of multiplication. Students approached the multiplicative context with additive or linear notions. The procedure for calculating decimal exponents seemed to help students make sense of what rational and decimal exponents stand for. The laws of exponents are re-established when working with rational and decimal exponents and may help some students to encapsulate the whole process of mentally constructing the notion of rational exponents.

References

Confrey, J. & Lachance, A. (2000). Transformative teaching experiment through conjecture driven research design. In A.E. Kelly& R.A. Lesh (Eds.) *Handbook of research design in mathematics and science education* (pp. 231-265). Mahwah: NJ. Lawrence Erlbaum Associates, Publishers.

Confrey, J. (1994). Splitting, similarity and rate of change: A new approach to multiplication and exponential functions. In G.Harel & J. Confrey, (Eds.) *The development of multiplicative reasoning in the learning of mathematics* (pp. 291-330). Albany, NY: State University of New York.

Confrey, J. & Smith. E. (1994). Exponential functions, rates of change, and the multiplicative unit. *Educational Studies in Mathematics, 26*, 135-164.

Dubinsky, E. (1994. A theory and practice of learning college mathematics In A. H. Schoenfeld (Ed.), *Mathematical thinking and problem solving* (pp. 221-243). Hillsdale, NJ: Lawrence Erlbaum Associates, Publishers.

Edwards, B. (1997). An undergraduate student's understanding and use of mathematical definitions in real analysis. In J.A. Dosey, J.O. Swafford, M. Parmantie & A.E. Dossey (Eds.). *Proceedings of the19th Annual Meeting of the North American Chapter of the International* *group for the Psychology of Mathematics Education (Vol. 1)*, (pp. 17-22.) Columbus, OH: The Eric Clearinghouse for Science, Mathematics and Environmental Education.

Edwards, B. & Ward, M. B. (2004). Surprises from mathematics education research. Students' (mis)useof mathematical definitions. *American Mathematical Monthly, 111*, 411-424.

Goldin, G.A. & Herscovics, N. (1991). Toward a conceptual-representational analysis of the exponential function. In F.Furinghetti (Ed.), *Proceedings of the Fifteenth International Conference for thePsychology of Mathematics Education (Vol. 2*), (pp. 64-71). Genoa, Italy: Department of Mathematics.

Selden, J. & Selden, A. (1995). Unpacking the logic of mathematical statements. *Educational Studies in Mathematics, 29*, 123-151.

Selden, A. & Selden, J. (2005). Perspectives on advanced mathematical thinking. *Mathematical Thinking and Learning, 7*, 1-13.

Tall, D. & Vinner, S. (1981). Concept image and concept definitions in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*, 151-169.