Implications of Undergraduates' Conceptions of Function

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The concept of function has played a crucial role in the development of modern mathematics and has been useful in the development of other disciplines such as physics and economics. Since 1923 there has been a push by mathematics educators to make the concept of function a central focus in secondary education because of its pivotal role in mathematics and other fields (NCMR, 1923). Researches and educators have focused on the reformation of teaching the concept of function, yet students still tend to struggle with understanding the concept of function and its applications (Carlson, 1998; Monk, 1992; Cooney & Wilson, 1996; Vinner & Dreyfus, 1989).

Some students' struggles have been attributed to the dual nature of function, structurally as an object and operationally as a process (Sajka, 2003; Sfard, 1991). Kleiner (1989) shows that throughout the history of the function concept, mathematicians encountered similar struggles as students do today. The concept of function went through several revisions from its first formal definition by Johann Bernoulli; his definition of function focused on the general composition of a variable and constants (Kleiner, 1989). The purpose for each revision of the definition was to accommodate new problems posed by mathematicians and physicists such as Fourier's concerns about heat flow in material bodies (Ponte, 1992). Many mathematicians used the Dirichlet-Bourbaki definition which states a function is a relation between two sets where every variable element of one set is related to a unique variable element of the other (Kleiner, 1989, Ponte 1992). Of the changes that were made to the definition of function, one of the main concerns was which analytic expressions were to be included and excluded (i.e. discontinuous and piecewise-defined relations) (Kleiner, 1989).

Students appear to echo these same concerns of early mathematicians on what is and isn't a function. When Vinner and Dreyfus asked students to identify functions they found that some students would reject relations as being functions because they were discontinuous, while others would accept the same relations as functions because they were discontinuous (1989). The same contradicting explanations occurred for piecewise mappings on the survey. The students that Vinner and Dreyfus surveyed also wrote a range of definitions similar to the variety of past definitions of function (1989). When asked what a function is in their opinion only 27% of the students gave the correct Dirichlet-Bourbaki definition.

In a study done by Even, participants also displayed general confusion of what the definition of function is and what is required for a relation to be a function (1993). Approximately 51% of the participants gave a "modern" definition of function that referred to the arbitrary nature of function; roughly 35% gave an "old" definition of function that was characterized by requiring some regularity. While these studies show that a fair number of students and teachers posses a somewhat modern conception of function, this group is small.

The following study attempts to extend past research by exploring university students' conceptions of function by focusing on: (a) their ability to define function, (b) their ability to describe real world situations which can be modeled using a function, and (c) their ability to identify functions in a mathematical context. Additionally correlations are sought between a student's ability to define function and (1) their ability to describe a real world situation which could be modeled using a function, and (2) their ability to identify functions in a mathematical context.

Method

Participants

Participants who volunteered were given fifteen to twenty minutes of class time to complete the survey. There were 289 undergraduates covering all classes and a variety of majors. At the time of the survey, the participants were enrolled in one or more of the following courses: Calculus II, Calculus III, Calculus IV, Orientation to the Mathematics Major, Linear Algebra I, Methods of Proof, Advanced Mathematics for Teaching, and Senior Seminar. Methods of Proof, Advanced Mathematics for Teaching, and Senior Seminar consisted of mathematics majors only.

Survey

The survey is included below in Figure 1.

Major: ___________________________

Class standing at Cal Poly (Circle One): 1st year Sophomore Junior Senior

Circle classes you have taken: 141 142 143 241 248 306 406 412 413 414 481 482

1. Rate your agreement with the following statements on the scale below: 1-strongly disagree and 5-strongly agree.

   a. Definitions are fundamental to the study of mathematics.

      1 2 3 4 5
      SD SA

   b. The concept of function is vital to the study of multiple branches of mathematics.

      1 2 3 4 5
      SD SA

2. Write a precise mathematical definition of function.

3. Describe a real world situation that can be modeled using a function and state the independent and dependent variables.

   Independent Variable: ________________ Dependent Variable: _________________

4. Is each of these a function? If so, name as many mathematical properties as you can that the function possesses. If not, explain why.

   a. $f:\{-3,-2,1,2,3\}\to\{1,2,3,4,5,...\}$ given by $f=\{(1,2),(2,1),(3,5),(-3,5),(-2,1),(2,2)\}$.

      Function: Yes No

   b. $f:R\to R,$ given by $f(x)=\begin{cases} x+2 \text{ if } x\le 0 \\ x \text{ if } x>0 \end{cases}$

      Function: Yes No

Figure 1. Survey on Conceptions of Functions

The goal of the survey was to explore participants' beliefs about the role of definitions in mathematics, their abilities to define function, their abilities to provide real-world examples that can be modeled by a function and their abilities to recognize functions. Results from questions two through four of the survey will be the focus of the remainder of this paper.

Coding and Organization

When the participants were asked to "Write a precise mathematical definition of function" their definitions were coded using three different rubrics: point, letter, and category. Points were given on a strict basis if the participants' definitions included the four ideas the researchers though vital to the definition of function. Participants were also given the letter codes A, B, C, and D that correspond with each point. Using the letter code allowed the researchers to determine which parts of the definition the participants knew. At most, a participant could earn four points, one point for each of the ideas in Figure 2.

Letter	***Idea***
A	The idea of *mapping*.
B	Stating that a function maps from one *set to another set*.
C	The idea that *every element in the domain* is sent to an image in the co-domain.
D	The idea that the image point is *unique.*

Figure 2. Definition Coding

In the third question on the survey, participants were asked to give a real world example that can be modeled using a function as well as state the independent and dependent variables. The participants' suggested real world example were simply coded correct, incorrect, or no response. For a response to be coded correct participants had to have an example in which all four criteria from the definition question were met. The independent and dependent variable responses were coded either correct, incorrect, switched, or no response. A response was coded incorrect when one or both variables were incorrect and switched when the two variables were correct, but in the incorrect order.

The example of a mapping between sets and the piecewise function were both coded as correct, incorrect, and no response. If a participant answered the mapping question correctly by signifying that it is not a function, then their explanation was coded as correct, incorrect, and no response. If a participant signified that the example in piecewise notation is a function and listed of the following properties: one-to-one, not one-to-one, onto, not onto, continuous, and discontinuous; the properties were noted.

To organize the data and report results the participants were divided into three groups: Pre, Current, and Post. Participants were placed in a group based on whether or not they had taken or were taking the Methods of Proofs course. The Pre participants group had not taken the Methods of Proofs course, the Current group was enrolled in the Methods of Proofs course, and the Post group indicated they had previously taken the course.

Results

Pre Group

The Pre group consisted of 226 participants and the following results highlight their general conceptions of function.

Ability to define function. When asked to give a precise mathematical definition of function, the Pre group scored on average 1.08 on the 4-point scale. Table 1 illustrates the distribution of the coding of the Pre groups definitions and includes the breakdown by letter code. For example, 26% of the group received 1 point for their definition and of those 82% received the point for A the idea of a mapping while the other 18% received the point for D the idea that an image is unique. As can be seen 42% of this group received 0 points for their definition while only 1% received 4 points. Also, very few participants in the Pre group included B, that every element in the domain is mapped to an image in the co-domain in their definition.

Table 1
Coding of Definitions for the Pre Group

Points	Letters	Percent (number) of participants
0		42% (94)
1		26% (60)
	A	82% (49)
	D	18% (11)
2		16% (36)
	AB	19% (7)
	AC	6% (2)
	AD	67% (24)
	CD	8% (3)
3		15% (33)
	ABD	3% (1)
	ACD	97% (32)
4		1% (3)

Ability to give a real world example and recognize functions. Results for the Pre group are included in Table 2 below. Participants in the Pre group described a real world situation that could be modeled using a function on 74% of responses. Most examples given were related to either growth and decay (i.e. population, bacteria, interest, etc.) or physical situations (i.e. velocity, distance, measurement, etc.). Of the participants who correctly described a real world situation, 63% also correctly identified the independent and dependent variables of the associated function.

Fifty nine percent of the participants in the Pre group determined that the mapping $f:\{-3,-2,1,2,3\}\rightarrow\{1,2,3,4,5,...\}$ given by $f=\{(1,2),(2,1),(3,5),(-2,1),(2,2)\}$ is not a function. Of those participants who correctly identified that the mapping is not a function, only 38% gave a correct explanation. When asked if the mapping $f:R\rightarrow R$ given by $f(x)=\begin{cases} x+2 \ if \ x\le 0 \\ x \ if \ x>0 \end{cases}$ is a function, 80% of the participants responded with the correct answer.

Table 2
Results from Pre Group for Giving a Real-World Example and Function Recognition

	Real world example		Recognizing functions		
	Participants' example	Identifying variables	Mapping between sets	Explanation for correct response	Piecewise function
Correct	74% (167)	63% (106)	59% (133)	38% (51)	80% (181)
Incorrect	17% (39)	25% (42)	28% (63)	38% (50)	16% (35)
Switched	NA	8% (13)	NA	NA	NA
No Response	9% (20)	4% (6)	13% (30)	24% (32)	4% (10)

Note. Data is shown as percent (number) of participants

Current

The Current group consisted of 27 participants and the following results highlight their general conceptions of function.

Ability to define function. When asked to give a precise definition of function the Current group scored on average 2.67 out of 4 points. Table 3 illustrates the distribution of coding of the Current groups definitions and includes the breakdown by letter code.

Table 3
Coding of Definitions for Current Group

Points	Letters	Percent (number) of participants
0		15% (4)
1		7% (2)
	A	50%(1)
	D	50%(1)
2		15% (4)
	AB	50%(2)
	AD	50%(2)
3		22% (6)
	ABD	83%(5)
	ACD	17%(1)
4		41% (11)

Forty-one percent of the Current group received 4 points for their definition and the most common correct definition was "f is a function from A to B iff (i) $Dom(f) = A$ (ii) $[(x, y) \in f \wedge (x, z) \in f] \Rightarrow y = z$." The most common missing information for this group was C, that every element of the domain is mapped to a unique image, as only 1 participant who did not write a correct definition included this piece of the definition.

Ability to give a real world example and recognize functions. Sixty-seven percent of the Current group wrote a real world example that could be modeled by a function. Of the 18 participants who gave a correct real world example, 15 participants also stated the independent and dependent variables correctly.

The majority, 93%, of the Current participants correctly stated that the example of mapping between sets is not a function. Of the participants who correctly identified this mapping, 64% gave a correct explanation as to why the mapping is not a function. Also 100% of the Current group correctly recognized the piecewise function. Results from both questions are presented in Table 4.

Table 4
Results from Current Group for Giving a Real-World Example and Function Recognition

	Real world example		Recognizing functions		
	Participants, example	Identifying variables	Mapping between sets	Explanation for correct response	Piecewise function
Correct	67% (18)	83% (15)	93% (25)	64% (16)	100% (27)
Incorrect	33% (9)	0% (0)	7% (2)	28% (7)	0% (0)
Switched	NA	17% (3)	NA	NA	NA
No Response	0% (0)	0% (0)	0% (0)	8% (2)	0% (0)

Note. Data is shown as percent (number) of participants

Post Group

The Post group consisted of 36 participants and the following results highlight their conceptions of function.

Ability to define function. The Post group averaged 2.28 out of 4 points on their definitions of function. Table 5 illustrates the distribution of coding of the Current groups definitions and includes the breakdown by letter code.

Only three of the Post group participants received 0 points for their definition and the rest of the definition scores were evenly distributed among 1, 2, 3 and 4 points. The information in

Table 5 also suggests that almost all of the Post group included A, that a function is a mapping in their definition, but only 8 of the 36 were able to write a precise definition of function.

Table 5
Coding of Definitions for Post Group

Points	Letters	Percent (number) of participants
0		9% (3)
1		22% (8)
	A	75%(6)
	D	25%(2)
2		25% (9)
	AB	56%(5)
	AD	33%(3)
	CD	11%(1)
3		22% (8)
	ABC	25%(2)
	ABD	50%(4)
	ACD	25%(2)
4		22% (8)

Ability to give a real world example and recognize functions. Of the 36 participants in the Post group, 24 were able to provide a real world example that could be modeled by a function. Of those participants, 15 were able to correctly identify the dependent and independent variables. Eighty-one percent of the Post participants were able to recognize that the mapping between sets is not a function and 55% of them were able to give a correct explanation as to why. Also 89% of the Post group was able to recognize that the piecewise function is in fact a function. Results from both questions are presented in Table 6.

Table 6
Results from Post Group for Giving a Real-World Example and Function Recognition

	Real world example		Recognizing functions		
	Participants' example	Identifying variables	Mapping between sets	Explanation for correct response	Piecewise function
Correct	67% (24)	63% (15)	81% (29)	55% (16)	89% (32)
Incorrect	30% (11)	29% (7)	19% (7)	31% (9)	8% (3)
Switched	NA	4% (1)	NA	NA	NA
No Response	3% (1)	4% (1)	0% (0)	14% (4)	3% (1)

Note. Data is shown as percent (number) of participants

Summary of Definition, Real World Situation and Function Recognition Results

With regard to their definitions the Current group provided the most precise definitions as highlight by their average from the points coding. Also there was a significant increase in average from the Pre to current group and a decline in average from the Current group to Post group. With the Pre group the idea that was most often left from their definitions is that a function maps between sets while the Current and Post groups most often omitted that every element of the domain is mapped to an element in the co-domain.

All three groups of participants were able to give a correct real world example more than 67% of the time. Of those participants who were able to give a correct example, the Pre and Post groups demonstrated similar abilities to correctly identify the independent and dependant variables while the Current group exceeded the other two at correctly identifying the independent and dependent variables.

The Current group had a slightly higher percentage of participants than the Post group who were able to correctly identify that the mapping between sets is not a function. The Pre group did not do as well identifying that the mapping is not a function and when they were able to they tended to not be able to give an explanation for why. All three groups had at least 80% of the participants who correctly identify the piecewise function.

Definition of Function as a Non-Predictor for Writing a Real World Example

A graph representing the number of points versus the percentage of participants able to give a correct example can be seen in Figure 3. For both the Current and Post groups, there is no observable evidence that a participant who received more points for their definition had a greater likelihood of giving a correct example of a real world situation that could be modeled using a function. For example 100% of both the Current and Post participants who scored zero points on the definition of function correctly described a real world example that could be modeled by a function, however 64% of the Current and 50% of the Post participants who received 4 points were able to give a correct example.

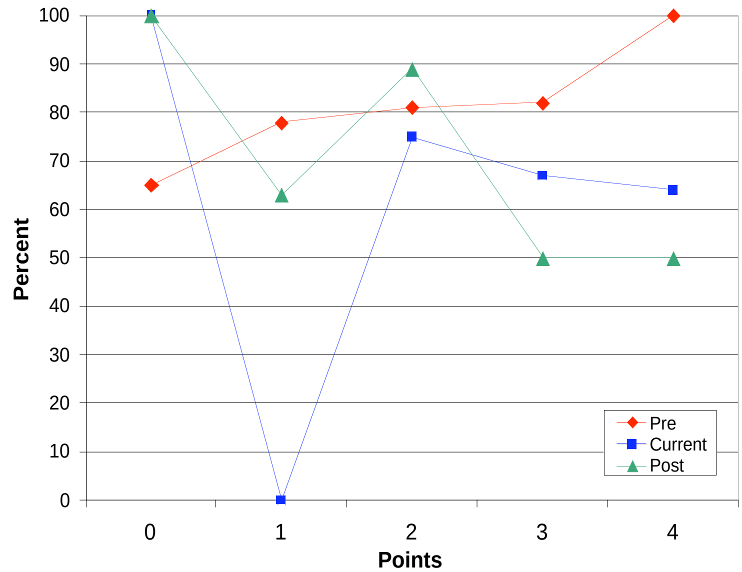

Figure 3. Points for Definition Versus Real World Examples

The only possible correlation between ability to define function and suggesting a real world example is seen in the Pre group where as the number of points earned increases, the percentage of participants who are able to give a real world example increases. This would mean that if a participant could define function then one may be able to reasonably assume they can give a correct example. However, there is only a slight increase in percentages as the points increase. Furthermore, the sample size for the Pre participants with four points is very small in comparison to the other points. Thus, there is not enough evidence to conclude that a participant's ability to define function predicts their ability to give a correct real world example.

The following examples highlight ability to define function as a non-predictor of ability to give a real world example. One participant stated that a function is "an expression whose value depends on a parameter." This definition received zero points, however they gave the following correct example of a real life situation that can be modeled by a function, "distance of a car from a starting line as a function of time" and correctly identified the variables. Another participant gave the following definition "a relation between a domain and a range where each element of the domain maps to only one element of the range." This definition received four points, however they suggested the following incorrect real world example, "where population growth is dependant upon the size of the population."

The data in Figure 4 suggests that in both the Pre and Post groups, the percentage of participants that correctly identified the variables in their given situation increased as the number of points they received increased for participants who received 0 through 2 points.

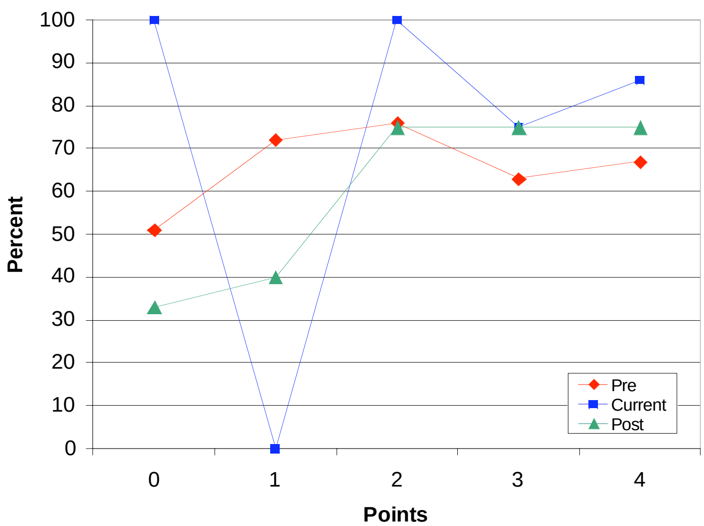

Figure 4. Points for Definition Versus Ability to Identify Variables

In the Current group there is no evidence of an increase in the ability to provide the correct variables as the number of points received for their definition increases. Thus there does not appear to be an overall correlation between ability to define function and ability to correctly identify the independent and dependent variables in a real world situation that can be modeled by a function.

Definition of Function as a Non-Predictor for Recognizing a Function

As with suggesting real world examples, the data does not suggest a relationship between participants' abilities to define function and their abilities to recognize a function. All of the Current group participants recognized that $f:\{-3,-2,1,2,3\}\to\{1,2,3,4,5,\ldots\}$ given by $f=\{(1,2),(2,1),(3,5),(-3,5),(-2,1),(2,2)\}$ is not a function. Considering data from the Pre and Post groups in Figure 5, there does not seem to be a correlation between participants' abilities to define function and their abilities to recognize a mapping between sets that is not a function.

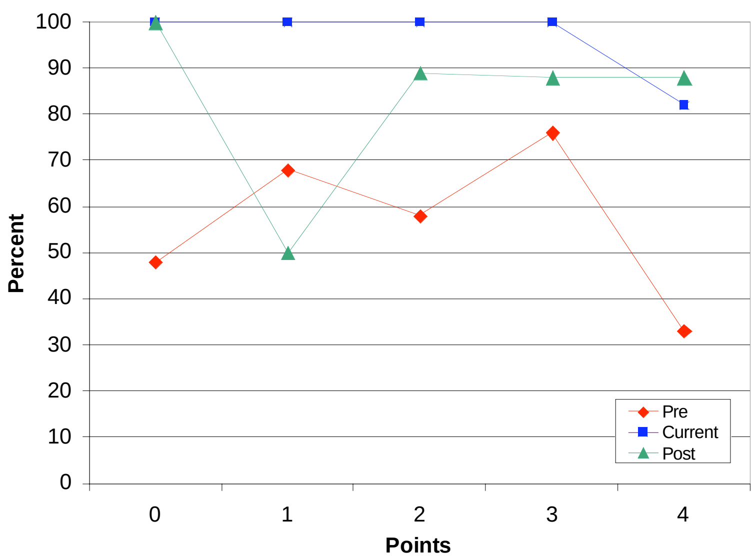

Figure 5. Points for Definition Versus Recognizing the Mapping Between Sets is not a Function

Everyone in the Current group recognized that $f(x)=\begin{cases}x+2 & if\ x\le 0\\ x & if\ x>0\end{cases}$ is a function. In general, the Post participants actually had a decline in percentages. Of the Pre participants, the lowest percentage able to recognize the function was 67% of those who received four points. The data in Figure 5 suggests that there does not seem to be a relationship between participants' abilities to define function and their abilities to recognize a piecewise function.

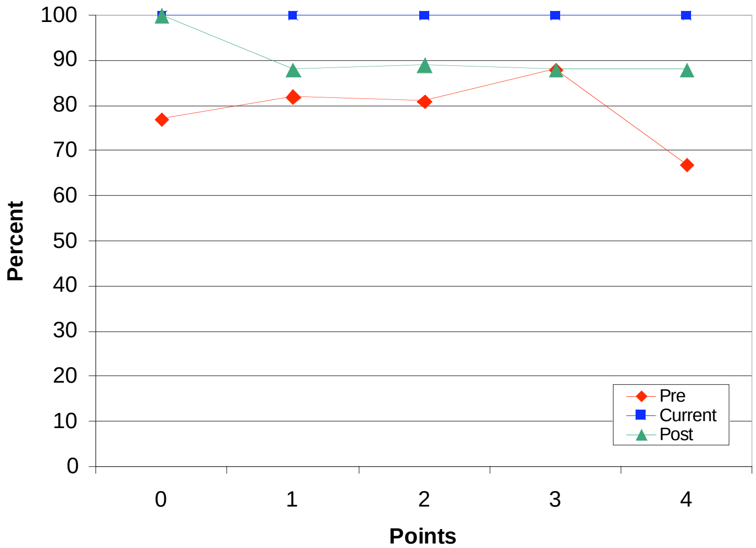

Figure 6. Points for Definition Versus Ability to Recognize a Piecewise Function

Definition of Function as a Predictor for Giving a Correct Explanation

The trend that as the number of points a participant received for their definition increases, so does the their ability to give a correct explanation of why the mapping between sets is not a function is seen in Figure 7.

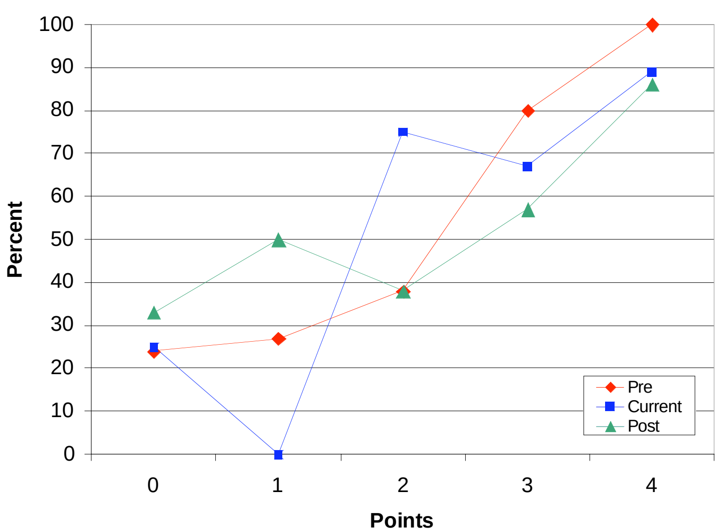

Figure 7. Points for definition versus correct explanation for the mapping between sets

Only 24-33% of the participants who received zero points were able to give a correct explanation where 86-100% of the participants who received four points were able to give a correct explanation. This data supports the notion that we could reasonably predict whether a participant could give a correct explanation for why the mapping between sets is not a function based on their ability to define function.

Discussion

Overall Weak Conception of Function

On average participants struggled to write a precise definition of function. Although the Current group had the highest average of 2.67, this implies that these participants were on average only able to produce definitions of function that included just over half of the four key ideas used for coding. Compared to the Current group the Post group had a lower average of 2.28 and the Pre group had an average score of 1.08, the lowest of all the groups. These findings are similar to those of Even (1993) and Vinner & Dreyfus (1989) who suggest a generally weak modern conception of function held by students and teachers.

Pre Participants' Basic Understanding of Function

Results suggest that the Pre group has a general weak conception of functions. When asked to define function, this group on average only stated one of the four key ideas coded for very few of the participants in the group included in their definition that a function maps from a set to a set.

Possible contradictory is the fact that Pre group had the highest percentage of participants who could give a correct example of a real world situation. However, this group also had the lowest percentage of participants who were able to correctly identify the independent and dependent variables in their given example. Hence, it is evident that while they were able to recall examples that they have seen in their coursework, but they do not seem to fully comprehend or understand these situations.

Calculus is often taught from the perspective that students have a deep understanding of functions. These Pre participants were either in their second, third, or fourth calculus course and their conceptions of function could hinder their abilities to be successful in calculus. Therefore it may be necessary for a formal introduction to functions to happen before or at the beginning of the calculus sequence. Furthermore, since many students' majors do not require a Methods of Proofs course they may never receive a formal introduction to functions and this could affect their performance in upper division classes. Further research may be needed to explore if the minimal conception of functions that students like the Pre group in this study may have are related to performance in advanced coursework or even in the workplace.

Loss of Retention after the Methods of Proofs Course

The results suggest that for these participants there was generally a loss of retention from the Current group to the Post group. First, the Current group has a slightly higher average number of points received for the definition of function than the Post group. Although the Current and Post groups were equally able to give an example of a real life situation that can be modeled by a function, the Post group of participants were significantly less able to correctly identify the independent and dependent variables.

There is also a decrease in the percentage of Post participants who are able to recognize that the mapping between sets is not a function. Furthermore, participants in the Post group were less likely to give a correct explanation as to why the mapping is not a function. Fewer Post participants recognized that the piecewise mapping is a function and the same percentage of Current and Post participants were able to list the properties of the piecewise function that were of interest.

Generally these results suggest that even though they are taking classes that rely heavily on their understanding of function the Post group was less likely to provide a precise definition of function, recognize a mapping between sets that was not a function and identify the independent and dependent variables in real world examples. This lack of retention may be surprising because of the large role that function plays in many upper division mathematics courses. Since this research did include the same participants a different stages of their undergraduate careers the results imply the necessity to explore retention within the mathematics major, including what concepts are not retained and why.

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