Implications of historical development for mathematics education: The case of limit

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Introduction

Experienced teachers, in addition to knowing the content they teach, have been shown to draw upon pedagogical content knowledge—knowledge of how to teach particular subject matter. One important component of pedagogical content knowledge (Shulman, 1986; Grossman, Wilson, & Shulman, 1989) is knowledge of student thinking which includes knowledge of strategies and difficulties students have about the content. Researchers have demonstrated that elementary school teachers with richer knowledge of typical student difficulties and strategies create more productive learning opportunities for their students . Until recently, this research was primarily limited to topics at K-12 levels, but the expanding body of research on how students think about and learn undergraduate mathematics makes it possible to examine these issues at the college level. This work focuses on calculus teachers' knowledge of student thinking about limit and, using historical development as a lens, explores the nature of the difficulties associated with the concept. More specifically, the study addresses the following questions: (a) What do the teachers of calculus know of their students' thinking about limit?, and (b) How can the historical development of the concept help us make sense of college teachers' knowledge of student thinking about limit?

Theoretical framework

At the undergraduate level, understanding of limit is considered to play a key role in the understanding of fundamental calculus concepts such as continuity, derivative and integral. The limit concept, on the other hand, presents major difficulties for students (Bezuidenhout, 2001; Cottrill et al., 1996; Tall & Vinner, 1981; White & Mitchelmore, 1996; Williams, 1991). These studies highlight that the formal understanding of the concept is unlikely to occur unless students first have the intuitive understanding of the concept. However, they also argue that students' intuitive understanding of the concept relies heavily on infinitesimals and dynamic motion, which might hinder the understanding of the formal aspect of limit. In that respect, some of the representational tools (e.g., verbal, visual and symbolic) used by students while thinking about limit lead to additional difficulties .

Limit was also a difficult concept for mathematicians to develop. The historical development of the concept indicates that mathematicians' initial understanding of the concept was based on the (implicit or explicit) use of infinitesimals and the consideration of motion as a continuous process which can be found in the calculus of Newton, Leibniz and Cauchy (Boyer, 1970; Kleiner, 1991, 2001; Schubring, 2005). In the process, mathematicians encountered difficulties such as "whether a variable can go beyond the limit and whether a variable can definitely reach the limit" (Schubring, 2005, p. 293), which are also mentioned in the literature on student thinking about the concept (Williams, 1991). It is only towards the late 18^{th} century, through the arithmetization of calculus, that the intuitive notions of space, natural continuity and motion were eliminated from Cauchy's definition of limit, which led to the formal ε-δ definition of Weierstrass. Therefore, the formal and the informal aspects of the limit concept require different conceptions of continuity and are based on different metaphors (Lakoff & Núñez, 2000). In that sense, the formal definition of limit is relatively new in the progress of thinking about limit and it was mostly through the intuitive dynamic view that mathematicians made sense of the concept.

Research design and methods

Participants for this project consisted of nine mathematics doctoral students from a large, public university in the Midwestern U.S. The participants had at least two years of college-level teaching experience and all taught calculus before. Each doctoral student was interviewed individually. Interviews were designed around three tasks modeled after research on student thinking about limit . The first task was about finding the limit of piecewise functions at a critical point. The doctoral students were first asked to solve this task and then anticipate possible student strategies and difficulties related to the task. For the second task, the doctoral students were asked to create some quiz questions about the convergence of sequences. The goal of the task was to understand whether the participants knew the common student difficulties related to finding limits at infinity. For the last task, the doctoral students were asked to write an exam question to test their students' skills in finding limits at particular points on the graph of a given function. For this, the doctoral students created their own graphs and were asked which aspect of the limit concept they were testing and why. The interviews were audio-recorded and transcribed. Analysis was done using an inductive, cross-task approach for identifying themes . Some coding categories were adapted from research on student thinking about the topics and tasks; others were iteratively developed. Comparisons were made between participants' knowledge of student thinking and relevant findings from research on student thinking about limit. The analysis was carried in conjunction with a similar study about graduate students' knowledge of student thinking about derivative (Speer et al., 2005). The foci of the analysis were on cataloging the richness of the knowledge of particular strategies and difficulties each graduate student mentioned and on comparing this knowledge with findings from research on student thinking about limits. In addition to this, the tasks used in the study were analyzed using the historical development of limit as a lens.

Findings

The analyses indicated that rather than giving a holistic and consistent picture of what the doctoral students knew of their students' thinking, the tasks evoked fragmented pieces of that knowledge. More specifically, each task elicited different parts of the knowledge doctoral students had of their students' thinking. For example, although the first task was considered a typical limit task by the participants, they did not mention any of the possible student difficulties about limit highlighted in the literature except for one. Instead, they focused on the possible student difficulties arising from the underlying concepts of limit (e.g., difficulties related to piecewise functions and algebraic computations). The purpose of the second task was to prompt the doctoral students about possible student difficulties about limit arising from boundedness and monotonicity. This task, however, hardly served this purpose and instead evoked some other underlying concepts about limits (e.g., convergence of series, different representations and patterns of sequences, infinity). Hence, the student difficulties mentioned by the literature about these aspects of the limit concept were only implicitly mentioned by the doctoral students. On the other hand, the doctoral students did mention the most common student difficulties indicated by the literature in the third task. In that respect, the third task was the most productive task in terms of eliciting doctoral students' knowledge of student thinking about limit. It should be noted, however, that none of the graduate students mentioned the student difficulties associated with the dynamic view of limit for this task. Overall, this study gave less information compared to graduate students' knowledge of student thinking about derivative. The graduate students primarily focused on the underlying concepts of limit rather than limit itself, especially for the first two tasks. The different responses the graduate students gave for each task indicated a differential treatment of tasks.

Discussion and implications for further research

The differential approach given to tasks by the graduate students might be due to the different nature of the tasks. For example, the first task put the graduate students in a learner mode (they solved the task) and then the teacher mode (they talked about how their students would solve the task) whereas the second and the third tasks put them only in a teacher mode (they wrote an exam question). It is also possible that the graduate students' responses to the tasks reflect the progress of thinking about the concept over history. For example, the piecewise function in the first task was given in algebraic form. In that respect, the function did not have a unique rule and there was no explicit indication of dynamic motion which might be the reason why the graduate students focused more on the function and algebra related difficulties than limit related ones. Similarly, the historical development of the limit concept indicates that limit was more closely associated with the notions of instantaneous rate of change and area than sequences. This might be why none of the graduate students talked about limit at infinity but instead focused on the patterns of their sequences and infinity for this task. Unlike the first two tasks, the third task explicitly highlighted the dynamic view and motion through its emphasis on the graphs of functions. It should be noted that this task was the most productive task and could keep the graduate students' focus on the concept of limit rather than its underlying concepts. Moreover, it elicited from the graduate students the most common student difficulties indicated by the literature. It might be because this task was most faithful to the intuitive and commonly perceived aspects of limit indicated by the historical development of the concept. Then why the calculus teachers did not mention any student difficulties related to the dynamic view for this task? In order to answer this question, it seems necessary to include more tasks about the dynamic view. Given the cognitive difficulties associated with the concept throughout its historical development, it might also be necessary to assess the graduate students' knowledge of limit before exploring their knowledge of student thinking about the concept. Finally, it might be necessary to include some tasks related to the formal definition of limit in order to understand whether the graduate students recognize the distinct metaphors than govern the informal and the formal aspects of the concept.

The development of tasks that measure calculus teachers' knowledge of student thinking about limit plays a key role in the diagnosis stage of what they know as well as how we can improve what they know. The tasks which keep the graduate students' focus primarily on ideas of limit can give us a relatively complete picture of their knowledge in this area and give them opportunities to display what they know of their students' thinking about the concept. The historical development of the concept seems to be a useful source for the development of such productive tasks.

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