Looking at calculus students' understanding from the inside-out:
The relationship between the chain rule and function composition

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> "The chain rule is one of the hardest ideas to convey to students in Calculus I. It is difficult to motivate, so that most students do not really see where it comes from; it is difficult to express in symbols even after it is developed; and it is awkward to put it into words, so that many students can't remember it and so can't apply it correctly" (Gordon, 2005).

The quotation above highlights that while the chain rule is straightforward to write out, simply $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$, or in Leibniz notation $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$, teaching and learning the chain rule is more complex. The complexity of the chain rule deserves exploration because students struggle to understand it and because of its importance in the calculus curriculum. Mathematicians interviewed by Webster (1978) described the chain rule as "essential," "crucial," "fundamental," and "about the most useful tool there is" (p. 2). Additionally the chain rule and its applications occupy approximately half of the differentiation chapter in a typical calculus textbook (e.g., Thomas' Calculus 11th ed.).

Despite the importance of the chain rule in the differential calculus curriculum and its difficulty for students, the chain rule has been scarcely studied in mathematics educational research (Clark, Cordero, Cottrill, Czarnocha, DeVries, Tolias, Vidakovich, 1997; Gordon, 2005; Uygur & Ozdas, 2007; Webster, 1978). These difficulties include both Leibniz and function notation (Gordon 2005; Tall, 1992; Webster 1978), the inability to apply the chain rule to

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This project was completed under the *FAST (F*uture *A*cademic *S*cholars in *T*eaching) Fellowship Program sponsored by the Graduate School at Michigan State University and the Center for the Integration of Research Teaching and Learning. Opinions expressed here are those of the author and are not necessarily those of the sponsors.

unfamiliar functions or in novel problems (Capistran, 2005; Webster, 1978), and difficulties associated with composing and decomposing functions (Clark et al., 1997; Cottrill, 1999; Hassani, 1998).

The connection between calculus students' difficulties with the chain rule and the concept of function is consistent with the well-recorded importance of function within the larger realm of calculus (Swanson, 2006; Carlson, 1998; Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Selden & Selden, 1992). Research on function has indicated that students view functions to be represented only by algebraic formulas and that a graph without a formula either has no meaning or is only a minor addendum to a formula (Ferrini-Mundy & Graham, 1991; Monk, 1994; Vinner & Dreyfus, 1989).

Previous research emphasizes the importance function composition in understanding the chain rule, but the ability to say more than that has been elusive (Clark et al., 1997; Cottrill, 1999; Hassani, 1998). This exploratory study was designed to focus on how do students understand the concept of function composition as seen through chain rule problems. Specifically, how do students understand function composition with functions with which they are familiar, somewhat familiar, and not familiar?

Examples of familiar functions are polynomials and trigonometric functions. They are functions that the students have had many experiences with both before taking calculus as well as in examples and homework problems in their calculus class, including the chain rule. Exponential and logarithmic functions are examples of somewhat familiar functions. These are functions that the students have experienced before calculus, but have had no direct experience with them yet in calculus. The participants in the study use the 11th edition of Thomas' Calculus and it is not until Calculus II that these types of functions are introduced in the calculus curriculum. An example of a function that students are not familiar with is the following[1].

Task 1. Suppose $roses(x)$, $violets(x)$, $red(x)$, and $blue(x)$ are all functions that have the following derivatives:

$$\frac{d}{dx}\left(roses(x)\right) = red(x) \qquad \frac{d}{dx}\left(violets(x)\right) = blue(x)$$

$$\frac{d}{dx}\left(red(x)\right) = red(x) \qquad \frac{d}{dx}\left(blue(x)\right) = \frac{3x+1}{x^2}$$

Find $f'(x)$ where $f(x) = roses(violets(x))$.

They have not encountered such a function either before or in calculus, thus students are not familiar with it in any context.

Methods

This study involved two rounds of data collection. In the spring 2007 semester 14 freshman Calculus I students were audio-recorded while working in small groups (3-4 students) for six consecutive class periods. Two weeks later, six of these students were audio-recorded during interviews conducted in pairs. These six students were chosen because the quality of the classroom recording was better for them than others. In the fall 2007 semester another 4 interviews were conducted with volunteers who were also freshman Calculus I students.

In both the classroom and interview situations the students were asked to solve both routine and non-routine chain rule problems. The tasks either had the function fully composed or as a composition of two or three ‘smaller’ function (see Tasks 2 and 3 in the next section). The non-routine tasks involved functions similar to problem 1 listed above as well as logarithms, exponentials, and inverse trigonometric functions. The recordings were transcribed and analyzed for patterns across function type.

Findings

[1] This problem was modified from Webster (1978).

Overall the students were successful on problems with familiar function. However students were not as successful on problems involving unfamiliar and somewhat familiar functions. On these two problems students consistently substituted function multiplication in place of function composition. From the in class portion of the study Jane[2] wrote the following answer to Task 1 above and explained it to another student in the following way. (italics added)

Written answer: $f'(x) = red(violets(x)) \cdot blue(x)$

Jane: "Well I kind of treated this like if you have cosine or something else like an inside function - outside function, so you take the derivative of the outside *times* the inside then take the derivative of that inside function. So that is why I got red *times* violet times blue."

Typically one would expect the word *of* to be used instead of *times* in a situation like this involving a composite function. In the interview that followed Task 2 shown below was given.

Task 2. Use the fact that $\frac{d}{dx}(\ln x) = \frac{1}{x}$ to answer the following questions.
Let $h(x) = (f \circ g)(x)$ where $f(x) = \ln(x)$ and $g(x) = -x^3$. Find $h'(x)$.

In response to Task 2 Jane answered as follows.

Written work: $h'(x) = \frac{1}{x}(-x^3)(-3x^2)$

Jane: I said that since the derivative of natural log is one over x *times* the inside function of negative x cubed and then I took the derivative of… then I took the derivative of the inside function and got negative three x squared.

When considering Jane response to Task 1 with her response to Task 2 we see that when she used the word *times*, it was not accidental. She meant that the part of the answer outside of the parentheses and the part on the inside were to be multiplied together and not composed.

2 All names are pseudonyms.

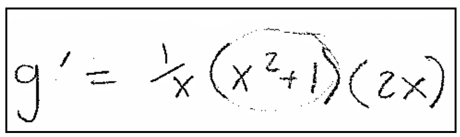
Half of the students interviewed (3 in spring 2007 and 2 in fall 2007) interchanged function composition with function multiplication. Only one student, Tina, from the second round of interviews, was able to reconceptualize the problem and arrive at the correct answer. Her first answer was similar to Jane's on the following task. However, she was not satisfied with her answer and she rethought about her answer to Task 3.

Task 3. Use the fact that $\frac{d}{dx}(\ln x) = \frac{1}{x}$ to answer the following questions.

Find $g'(x)$, where $g(x) = \ln(x^2 + 1)$.

Tina responded in the following manner.

Written work:

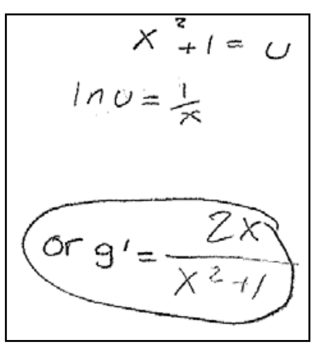


Tina: It says, like, the derivative of the logarithm x is one over x, but I'm confused as to why. This just says log of x squared plus one, so do I put the… *[circles x^2+1]* like it's the whole, I don't know.

She then started the problem over again.

Tina: If I, if it's supposed to look like this one, *[POINTS to derivative of log x = 1/x]* it would look just like log x, then this would have to be x squared plus one so then it would be 2x over x squared plus one.

Written work:



By using *u* as a substitution Tina decomposed the function performed the derivative of natural log and then recomposed. This decomposing and composing helped her to overcome her initial tendency to treat this as function multiplication.

The replacement of function multiplication and function composition was similar yet different to the findings of Meel (1999). The students in Meel's study substituted $f\big(g(x)\big)$ as $f(x)g(x)$ or $\big(f(x)g(x)\big)\cdot x$. In contrast, if the students in this study had substituted in that fashion, it would have been appropriate for them to use the product rule when computing the derivative and that did not happen with any of these students. Further studies need to be conducted to further understand the subtleties happened with function composition.

Conclusions and Implications

These results were similar to the results of Webster (1978) in that students who are successful solvers of routine chain rule problems are not necessarily successful solvers of non-routine problems. It was also shown that even though students wrote down the correct answer, they were not always thinking the right answer and that there may still be subtle misunderstandings hidden in the notation. Specifically from this study, function multiplication and function composition both used parenthesis in mathematical notation.

Additionally, Carlson (1998) reported that "full [function] concept development appears to evolve over a period of years" (p. 143). This implies that students do not have a completed concept of function when they learn the chain rule in first semester calculus. Studies of the kind reported here will further enhance the mathematics education community's knowledge of students' understanding of function surrounding the chain rule, an essential and intermediate calculus topic.

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