Covariational Reasoning and Quantification in a College Algebra Course

Kevin C. Moore
Arizona State University
Kevin.C.Moore@asu.edu

Stacey A. Bowling
Arizona State University
Stacey.Bowling@asu.edu

Introduction

It is well documented that even high performing precalculus and calculus students have weak understandings of the function concept (Carlson, 1998). Studies have revealed that the ability to reason covariationally (e.g., consider formulas and graphs as representing the varying magnitude of two quantities as they change in tandem) is critical for understanding functions and central concepts of calculus (Carlson, 1998; Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; P. W. Thompson, 1994; Zandieh, 2000) and differential equations (Rasmussen, 2001). Drawing from this literature, curriculum and instructional supports that take a covariational approach to teach ideas of variable, rate of change, function, function composition, function inverse, and exponential growth were developed for college algebra. Homework assignments and in class instruction also emphasized meaningful communication (both verbal and written) about functions as representations of covarying quantities. The purpose of this study was to investigate the impact of the curriculum and instructional approach on students' emerging covariational reasoning abilities.

Background

The study discussed here is guided by a central topic in college algebra: students' covariational reasoning abilities. Saldanha and Thompson describe understanding covariation as "holding in mind a sustained image of two quantities' values (magnitudes) simultaneously" (Saldanha & Thompson, 1998). This image of covariation is considered developmental. In other words, one first coordinates two quantities' values (e.g., think of the first quantity, and then the other, think of the first quantity, and then the other, etc.). Then, as a student's image of covariation develops, her/his understanding of covariation begins to involve understanding time as a continuous quantity. Thus, the ability to imagine continuous changing quantities begins to form (e.g., as one quantity changes, an individual has the realization that the other quantity changes simultaneously).

Following this call to the importance of the ability to reason covariationally, Carlson et al. investigated the complexity of students' images of covariation. Namely, the "construction of mental processes involving the rate of change as it continuously changes in a functional relationship" was investigated (2002). The covariational reasoning abilities of high-performing 2nd-semester calculus students were and during this investigation, a theoretical framework was created and refined. Initially, multiple behaviors of students involved in interpreting and representing dynamic function situations were identified (Carlson, 1998). In order to classify the behaviors exhibited, a framework that consists of five mental actions (MA#) and behaviors associated with these actions was developed. This alone was not adequate to describe a student's covariational reasoning *ability*, which can be inferred from the collection of behaviors and mental actions exhibited when responding to a problem. In order to analyze this collection, the covariation framework was extended to describe multiple *levels* (L#) of covariational reasoning resulting in a framework consisting of five distinct developmental levels that are composed of the five mental actions. One's covariational reasoning ability is said to reach a given level (e.g. L4) when it supports the mental actions associated with that level (e.g., MA4) and the mental actions associated with all lower levels (e.g., MA1-

MA3). For instance, a student who is determined to exhibit L3 reasoning (quantitative coordination) is able to reason using MA3 (determining the amount of change of one variable with changes in the other variable) as well as MA1 and MA2. In other words, he or she is also able to coordinate the direction of change with one variable with changes in the other variable. It is noted that the word "levels" entails that higher level reasoning (e.g., L5) implies lower level reasoning (e.g., L3). However, this is not the case with the mental actions (e.g., an MA5 behavior does not imply the lower mental actions.)

The study presented here focused on college algebra students' ability to move among mental actions 1, 2 and 3 and identify the level of covariational reasoning relative to engaged problems. In order to characterize the students' development relative to the class in which they were enrolled, this investigation involved examining the interaction between students' covariational reasoning abilities and topics covered in the course. Also, due to the setting in which the covariational framework was developed (e.g., high performing calculus students), the framework remained open to modification and it was conjectured that due to the population being analyzed, modification would occur.

In the analysis of a student's covariational reasoning abilities, Carlson et al. made an important observation: students often exhibit behaviors that appear to reveal a high level of development (e.g., L5), but when these behaviors are probed, a student may not be able to justify or support the reasoning he appeared to display. This occurrence can be described as a student exhibiting pseudo-analytical behavior (Vinner, 1997). Pseudo-analytical behavior is the situation in which a student does not have the understandings required to meaningfully describe the behavior in which they exhibited. Thus, we define the mental action that produced the behavior as a pseudo-analytical mental action. As mentioned, a student is classified as having a specific covariational reasoning ability level only if he or she is able to perform the mental action relative to that level and all levels below. Thus, if a student reveals MA4, he is only classified as level four if he also exhibits MA1-MA3. If he is not able to exhibit MA1-MA3, the MA4 behavior is a pseudo-analytical mental action.

We also conjectured that this population of students would have difficulty during the *orienting* phase of problem solving, as defined in Carlson and Bloom's multidimensional problem-solving framework (2005). Often, the situations presented required the identification and use of covarying quantities. Thus, it was necessary that a student be able to conceptualize quantities, which is part of the orientation process.

A meaning of *quantity* that provides a useful theoretical perspective is that of Thompson (1989), where a quantity is defined as being a conceived attribute of something (e.g., a perceived situation interpreted from a problem statement) that admits a measurement process. This meaning of a quantity as a conceptual entity provides an additional perspective for analyzing the orientation phase of students' problem solving behavior by looking at students' distinction of quantities and the process of *quantification*. Quantification is defined to be the cognitive process of assigning numerical values to attributes. In order for an individual to quantify an attribute of a conceived situation, one must imagine an explicit or implicit[1] act of measuring the attribute. It is in this process that an attribute becomes truly quantified. That is, to comprehend a quantity, an individual must have a mental image of an object and

[1] An implicit or explicit measurement act implies that it is not actually making the measurement that results in a quantity. Rather, it is *conceived ability* to make the measurement, whether or not it is carried out, that results in a quantity.

attributes of this object that can be measured (e.g., a car in a race with attributes weight, height, speed, distance traveled, etc.), an implicit or explicit cognitive act of measurement that produces the quantity (e.g., measuring distance traveled), and a number, or value, which is the result of that measurement.

Methods

The subjects and setting in which the data was collected

The subjects for the study were college algebra students from a large public university. The classroom from which the students were drawn was part of a design research study where the initial classroom intervention was informed by theory on the processes of covariational reasoning and select literature about mathematical discourse and problem-solving (Carlson & Bloom, 2005; Carlson et al., 2002; Clark, Carlson, & Moore, in preparation). The curriculum was in its first iteration for college algebra students and data was collected to study its effectiveness relative to these constructs. The data was analyzed to produce both insights about the effectiveness of the interventions and useful formative knowledge for their refinement.

Data collection and analysis methods

The section of college algebra from which the students were drawn was video taped each class session and student written work was collected and digitally scanned. Clinical interviews, which are the primary focus of this work, were also conducted with ten students. The interviews occurred at least eleven weeks into the semester and were composed of four mathematical problems with not more than thirty minutes allocated for each problem. The interviews were videotaped and digitized for analysis. The students participating in the interviews were monetarily compensated for interview time.

Our clinical interview approach followed that described by Clement (2000). This clinical interview approach allows the study of knowledge structures and reasoning processes through an open-ended questioning technique. Furthermore, when paired with appropriately designed tasks, using this approach creates a focus on inferring the cognitive actions of the subjects rather than focusing on easily defined outcomes such as patterns of correct and incorrect answers by subjects (Goldin, 2000). By engaging in construction activities, students are more likely to reveal understandings and conceptions they hold, as the student is placed in a situation where mathematizing[2] situations is promoted. Also, a talk-aloud approach was included in our design, to generate insight into the mental processes being performed by the subjects by encouraging students to verbalize their approach to the tasks at hand (Carlson & Bloom, 2005).

In order to reveal as much student thinking as possible, our approach to conducting the interview (e.g., the place of the interviewer in the interactions) followed that of the method described by Goldin (2000). Each task-based interview was conducted such that the exploration into the students' understandings unfolded in the four stages described by Goldin. Although each stage is important in the exploration of student understandings, it is noted that the first stage (free problem solving) was of most interest. By allowing a student free problem solving time, the student was able to act in a manner that was not guided or influenced by interviewer questions.

The covariation framework described by Carlson et al. (2005) provided the foundation for the design of the interviews. Although a theoretical framework had been

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[2] Mathematizing situations refers to the process of identifying and conceiving the quantities of the situation in a way such that they can be structured using mathematical relationships, such as the relationship between distance, time, and speed.

chosen, it was possible that the study would result in the generation of a modified framework. Thus, open coding (Strauss & Corbin, 1998) was first utilized in an attempt to identify and analyze discrete instances of the student's covariational reasoning abilities without being limited to the framework described by Carlson. The framework developed by Carlson was compared to the analyzed emerging behaviors. The open coding and axial approach was also taken to identify behaviors relative to the orientation process of the students. Again, discrete instances of behaviors believed to be part of the orientation process were identified and the characteristics of each behavior were analyzed and compared.

In the analysis of the data, a conceptual analysis, described by Thompson (2000), was performed using the data collected. The students' actions were examined in an attempt to model and understand the thinking of the subjects. Mathematical thinking is dependent on mental operations and thus the goal was to infer, based on student actions, what mental operations were producing the behavior of the students.

In the classification of the mental actions of students, it was important to take into account the possibility of pseudo-analytical behavior relative to covariational reasoning (Vinner, 1997). These are the cases in which a student appears to show a high level of covariational reasoning (e.g., MA5), but if asked to unpack lower level mental actions, he or she is unable to do so. Thus, the interviews were designed to reveal these pseudo-analytical behaviors. This involved persisting in asking for explanations, designing tasks that ask for multiple levels of reasoning, etc.

Results

In this study, we present the results of one student's engagement with the Box Problem (Appendix A). This student's interactions were chosen because they were reflective of the performance of the group as a whole. Relative to the student's general behavior during the interview, the student did not appear to reflectively orient himself to the problems, which may have caused him difficulty in identifying the quantities of the situation. Rather, he worked the problems in a manner such that he chose what information he needed from the problem statement as he worked the problem and encountered difficulties. Analysis of the interview also revealed insights about the student's covariational reasoning abilities. The student was seen consistently operating on the *Direction Level* (L2) and showed multiple instances of behaviors that were suggestive of MA3, MA4, and MA5 reasoning (e.g., speaking of rate, slope, and attempting to consider changes in output while considering successive equal changes in input). However, when probed about apparent MA5 reasoning, he did not exhibit behaviors supported by MA4 and MA3 reasoning. What follows are data that illustrate the student's problem solving behaviors and covariational reasoning ability.

When engaging in the tasks of the Box Problem, rather than orienting to the problem space and planning his approach, Matt appeared to rush to working the problem and only chose to use minimal and crude heuristics (e.g., quickly drawing a picture). For instance, on Task 2, after reading the problem statement he first referred to volume as "height times length times depth," where he referred to depth as the cutout. After the interviewer asked for clarification, the student noticed his duplicate use of height and depth and corrected volume to be height times length times width. He then proceeded to use the piece of paper to describe the situation. This interaction occurred as follows.

Table 1

Matt:	The length is going to be this (*pointing to the length of the paper*), the 11 inches, the width is going to be 8 inches (*point to the width of the paper*), and the depth, or height, is going to be x
Int:	Ok.
Matt:	(*Pause*) So write a formula that predict (*long pause*). So then, if you had, if you had like, if you say the cutout was 1 inch, so that would mean, that would mean, 7 times 10 times 1. Because you know that if 1 inch has been cut off, or if .5 inches have been cut off, because .5 .5 (*pointing to the two corners of the paper*), this would be .5.

In this interaction, Matt decided to use the piece of paper to illustrate the dimensions that he was defining (1-3). However, he did not decide to label the paper that he used to model the situation, nor did he draw a separate picture of the situation or label each dimension and how it was related to the cutout. Also, when speaking of the width and length of the box, he referred to the original width and length rather than the resulting box's width and length (he also referred to the width incorrectly, using 8 inches rather than 8.5 inches) (1-2). Next, he decided to use a specific cutout of 1 inch to discuss the dimensions of the resulting box rather than a general formula (6-8). It appeared that he did this in order to describe the situation first using a static image, as this may have initially been easier for him. Yet, he first described the situation incorrectly, as revealed by his calculating the length and width using a cutout of .5 inches rather than 1 inch. He realized this mistake and immediately corrected himself by stating that the cutout is .5 inches (8-9). These inconsistencies - referring to the length and width of the box as the length and width of the paper - are possibly due to a lack of quantification. It appeared that the student had not conceived of a measurement process when determining the length and width of the box, and instead used the original length and width of the paper. These mistakes in quantity distinction (e.g., interchanging the length and width of the box with the length and width of the paper) could likely be caused by his inattentiveness to forming a formal image of creating the box. As the interaction above revealed, Matt used a crude representation of the situation by quickly describing, but not recording, the dimensions using the piece of paper given. Also, when attempting to describe a static situation, he incorrectly described this static relationship between the cutout and resulting dimensions.

Immediately following this interaction, he was asked to describe the situation again. At this time, he still only used the paper given to him to describe the situation, but made moves to describe cutting away from both sides and how this influences the dimensions of the box, thus revealing he had formed an image of the corners being cut away and at least a partially developed measurement process. Yet, when he attempted to formulate the volume as a function of the cutout, he first wrote $(8.5 - x)(11 - x)x = V$. He described x as the cutout and height, but at first did not realize his error of using x rather than $2x$, since double the cutout is removed from the length and width of the box. Eventually, when describing the formula, he caught his mistake and defined the correct formula. This again is a small error that could have possibly been caused by his hastiness when working through the problem and an informal mental image of the quantities involved.

In addition to revealing orientation behaviors of the student, the Box Problem also offered insight into the student's covariational reasoning abilities. For instance, the following interaction occurred when responding to Task 1.

Table 2

Int:	Ok, so do you remember like the little tool, the finger tool we used?
Matt:	Ya.
Int:	Could you maybe do it describing that, maybe use it?
Matt:	So, uh, ok, we'll make this finger cutout (*referring to his right finger*). Which is the x-axes, cause that's the input I guess, and you get out the volume, which would be y.
Int:	Mm K.
Matt:	So it would go, if the cutout is going like this (*moving his right finger to his right*), the volume would be going like this (*moving his left finger up*). And then once the cutout starts going like this (*continuing to move his finger to the right after slightly moving it left*), the volume becomes to drop again (*moving is left finger down*).
Int:	Ok, so when you say the cutout's going like this, what do you mean, what's going on?
Matt:	So now, from 0 to like 4 inches, or like 1 inch or 2 inches, as it's increasing in inches, the volume increases until it reaches a certain number (*moving his right finger to the right and his left finger up*), and then the volume comes back down even though the cutout is increasing (*moving his right finger to the right and his left finger down*) (MA2).

The interviewer asked the student to use the "finger tool" (1), a tool that was presented in class in order to track the variation of one quantity or the covariation of two quantities. The tool involves tracking the magnitude of one quantity with the right index finger (e.g., moving your finger right is increasing) and tracking the magnitude of another quantity with the left index finger (e.g., moving your finger up is increasing). Initially, the student described the volume and cutout varying in terms of his movements (e.g., "cutout starts going like this"); thus, not revealing his understanding of the covariation of the two quantities (8-12). When asked to describe what "going like this" referred to, he was able to explain that the volume increases and then "comes back down even though the cutout is increasing" while coordinating his fingers properly (15-19). These actions exhibit MA2, and due to his verbalizing of the coordination of two quantities, he can be classified as exhibiting L2 covariational reasoning.

Although the student exhibited only L2 covariational reasoning behaviors, at times he appeared to exhibit MA3-MA5 behaviors. He described functions using instantaneous rates (MA5) (e.g., decreasing at an increasing rate), but was unable to unpack this efficiently using MA3 or MA4 reasoning. This was revealed on Task 3 of the Box Problem after the student correctly described that the volume decreased as the length of the cutout increased from 1.8 to 1.9 inches. This was followed by a question that prompted him to describe how the volume decreased. This caused the student to segment several successive intervals of input on the graph and consider how the output changed. The student then mentioned steady rate and was asked to explain more.

Table 3

Matt:	And see that if every line is equivalently apart (*referring to changes in output*).
Int:	And so if every line...
Matt:	Is equivalently apart, then the more you go up (*pointing to the x-axis*), it goes down at a steady rate (*making hand motions going down*) (MA4). For every increment that you go up, it goes down an increment (*pointing to the x-axis and then y-axis*) (MA3).
Int:	It goes down an increment. And that would tell us what?
Matt:	That every inch that you cut out, it varies with the volume (MA1).
Int:	Ok, and now, how does it vary with the volume? (*Pause*) So you said it varied steady and that, so what exactly do you mean by steady?
Matt:	That means if you go down from 1.8 to 1.9, this is if it is steady, or if you go from 3.0 to 3.1, it's going to be the same increment of decrease in volume (MA3).

The student revealed that he was able to discuss incremental changes of input and how the output would change if the relationship had a constant, or "steady," rate (MA3, 4-7, 12-14). Thus, relative to constant rate, this suggests that the student was able to engage in L3 covariational reasoning behaviors. This leads to the question of whether or not the student's covariational reasoning abilities can be classified, relative to this problem, as L3 for varying rates. The following interaction sheds light into this question, with the interaction occurring immediately after the above interaction.

Table 4

Int: What if it wasn't a steady change?
Matt: Then it wouldn't go down in an equal increment every time.
Int: Ok, so how might it go down?
Matt: Uh, it might go down at a decreasing, increasing rate (MA5).
Int: And so what would that maybe look like?
Matt: Uh, the line would go down like that (*drawing a decreasing, concave down sketch*), steeper steeper steeper steeper.
Int: Ok, so how's that tell us we have...
Matt: Because it's going down, when it gets steeper, it's going down more over less time. So, that's weird. When it goes down steeper, here it's going down less as steep, so it's going down less as much. Here it's going down steeper, because it's getting more straight, it's going dramatically down (*student comparing a section that isn't as steep to a section of the graph that is steeper*) (MA5).

When asked to describe a situation that didn't have a constant rate of change, the student described that this means the decrease of output wouldn't be the same (2). This revealed possible MA3 behavior relative to changing rates. However, after drawing a decreasing at an increasing rate graph, the student described it using the shape of the graph, or steepness of the graph (7, 10-14). The student first mentioned that it went down "less as much" (11), but did not compare this amount of change to other amounts of change in order to describe what the change was "less as much" relative to. Instead, the student chose to speak of the steepness of the graph and use relative terms such as dramatically down, possibly revealing a pseudo-analytical L5 behavior, as it was not clear if the student was continuing to attend to the covariation of the quantities involved or remained focused on solely the shape of the graph (shape thinking). The student's conception of the situation was further revealed as the interaction continued.

Table 5

Int:	Ok. Is there any way, so here we talked about well, if we go the same, we have the same (*referring to the steady rate conversation and work*), so how can we describe this situation in terms like that?
Matt:	It terms of what, this? I don't know, because I haven't checked if it goes down the entire way equally, the whole time. But if it did then for every increment here, there is an equal increment going down. But if it didn't, then for every increment going across (*pointing to x-axis*), there might be a different increment in decrease (*pointing to y-axis*) (MA3).
Int:	Ok, so lets say that this graph was doing that, then what would happen as...
Matt:	Well, then it depends, if the graph was getting steeper, but I can't tell by the naked eye.
Int:	So lets just say, lets draw a new graph in there that is steeper by your definition.
Matt:	I draw it? So if it was going down like that (*draws a decreasing, concave down graph*), then we know for every increment (*pointing to the x-axis*), it's not going down for a certain increment here (*pointing to the y-axis*). It's going to start increasing faster and faster, so it's going to go down faster (*making motions down the y-axis*). But if the graph was going like (*draws a decreasing, concave down graph, but not as steep as previous graph*) this, then it would be going down even slower. It would still be going down, but slower. But if the graph went down, in an exact, every increment (*pointing to x-axis*) a certain increment (*pointing at y-axis*) which maybe would look more symmetrical, and everything like that, then we would know it was going down at a certain increment (*pointing to x-axis*) for every increment in volume (*pointing to y-axis*) (MA3).

The student was again able to describe constant rate using MA3 behavior (5-6) and when probed further to describe a graph that is decreasing and concave down, the student referred to increments of input (15) and attempted to describe changes in output

(16-21) relative to these increments of input. During this description, the student described "it," which could refer to the output or shape of the graph, as "going to go down faster." Thus, it appeared the student was possibly considering changes of output for equal changes of input, an MA3 behavior and thus a L3 classification. However, the student then revealed that he was likely shape thinking when using the phrase "going to go down faster" (18-21). After drawing another graph that was decreasing and concave down (but not as steep as the previous graph), the student appeared to describe this graph relative to the previous graph, explaining it "would still be going down, but slower." His shape thinking was further revealed after this interaction when the student incorrectly described the second graph created as decreasing at a decreasing rate and then incorrectly explained that the change in output was decreasing for successive changes of input because it wasn't as steep as the first decreasing and concave down graph he drew. Although the student appeared to exhibited MA3-MA5 behaviors, his difficulty to unpack these behaviors and his tendency to focus on the shape of the graph limited his covariational reasoning behaviors relative to this task to L2.

The Box Problem offered insight into both the student's problem solving behaviors and his covariational reasoning abilities. First, the student did not appear to participate in meaningful and descriptive orientation behaviors. Rather, he used crude heuristics, such as drawing a rough sketch, with which he did not explicitly identify each quantity of the situation. This, in turn, may have limited his conception of the varying quantities as processes of measurement and his understandings of the relationships between these quantities; thus causing quantity distinction and identification issues. For instance, when describing the created box, he was able to describe the height as the cutout, but described the length and width as the original paper's length and width. It appeared that rather than constructing a well-developed mental image of the box and its construction, the student instead relied on only briefly referring to the situation. Also, it is noted that the student's description that occurred during Task 2, where the student was asked to determine a formula, differed from his description on Task 1. Of importance was that on Task 1, the student gave a very descriptive explanation of the dynamic situation (Table 2) but then appeared to abandon this image on Task 2. Relative to the student's covariational reasoning ability, Matt was able to reason at L2 of the framework. He was able to describe the direction of change in volume as the length of the cutout increased. He also appeared to reveal pseudo-analytical behaviors where at times he exhibited what may have been MA3-MA5 behaviors, but when he attempted to unpack these ideas and describe the covariation of the quantities, he revealed behaviors that implied he was using the shape of the graph to describe the quantities, rather than the magnitudes of the two quantities. For instance, he described two decreasing, concave down graphs differently. According to the student, the steeper graph was decreasing at an increasing rate and the other graph was decreasing at a decreasing rate because it was going down "slower."

Discussion

It was not unexpected that the students showed difficulty exhibiting higher than L2-L3 covariational reasoning behaviors. This is consistent with observations of high-performing 2nd semester calculus students made by Carlson et al. (2002). Similar to the study of the calculus students, students consistently made attempts to coordinate the amount of change of the output variable while considering changes in the input variable (MA3; Tables 3 & 5). However, when probed further, the students had difficulty speaking of amounts of change when describing the coordination of the two quantities, especially in the cases of varying rates (Tables 4-5).

An unexpected finding was the emergence of the importance of quantification. Our analysis suggests that the students exhibited few behaviors indicative of building a mental image of the values and variables of the situations as quantities (attributes admitting some conceived measurement process). This caused the students to incorrectly identify and describe various quantities in the situation (Table 1). This raises the question of why the student did not participate in reflective orientation behaviors in order to build a meaningful image of the situations and the quantities involved. It is possible that they were accustomed to traditional mathematics courses where procedures are the primary focus, and the students are unintentionally trained to devalue modeling situations in order to mathematize the situation.

Future Research

This research provides useful knowledge about both college algebra students' ability to reason covariationally and students' orientation behaviors. Carlson et al.'s (2002) covariational framework was extended to describe the reasoning of college algebra students and insight was gained relative to the orientation process of students. Specifically, the act of quantification emerged as a critical aspect of the orientation process and a necessary prerequisite to covariational reasoning. Identifying and promoting students' processes of quantification promises to be an important and useful area of research. This research is also relevant to the continued investigation of students' ability to reason covariationally when presented with problems that involve real-world situations. In order to reason covariationally about dynamic real-world situations, it is necessary to identify and quantify the variables of a situation. The results above revealed a needed focus on modeling and quantifying situations. The construct of quantification also should be useful in identifying and promoting the development of students' ability to build meaningful mathematical representations of physical situations.

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Appendix A – The Box Problem

Starting with an 8.5" x 11" sheet of paper, a box is formed by cutting equal-sized squares from each corner of the paper and folding the sides up.

Task 1: Describe to me how the length of the side of the cutout and the volume of the box covary.

Task 2: Write a formula that predicts the volume of the box from the length of the side of the cutout.

Task 3: Given a graph, describe how you would use this graph to describe how the volume changes as the length of the side of the cutout varies from 1.8 inches to 1.9 inches.