Students' Notions of Convergence in an Advanced Calculus Course

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Abstract

The research literature indicates that the limit concept is typically a difficult concept for students to grasp. However, there is little evidence to indicate how students deal with the mathematical definitions of a limit, specifically the definitions related to sequence convergent. The purpose of this paper is to examine students' conceptions of convergent sequences and Cauchy sequences. We examine junior level mathematics students in an advanced calculus course as they prove "if a sequence is convergent, then it is a Cauchy sequence."

Introduction

The concept of limits has been regarded as one of the most fundamental concepts in mathematics. However, educational studies indicate that students have experienced difficulties in understanding the concept of limit (Tall, 1992; Tall & Vinner, 1981; Williams, 1991, Edwards, 1997, Knapp, 2006). Burn (2005), Davis and Vinner (1986), Szydlik (2000), and Williams (1991) reported that it was not easy for students to transition from an informal to a formal understanding of a limit. In particular, when students informally learn the concept of limit, they determine the convergence of a sequence and explore the properties of limits not by the rigorous definition but by images of the limit (Roh, in press). Images formed in this manner therefore influence the process of transition from an informal understanding of the concept of limit to understanding its formal definition (Przenioslo, 2004). Mamona-Downs (2001) pointed out the lack of understanding of the mathematical meaning and logical structure of the words that constitute the $\varepsilon - N$ definition as another source of influence on the transition from informal to formal understanding. Dubinsky (1991) drew attention to the "reflective abstraction" of quantifiers in the definition as a cognitive process that plays a crucial role in understanding the definition. Many researchers contended that students should understand the role of "any" ε (Davis and Vinner, 1986; Roh 2005), the relationship between ε and N (Knapp & Oehrtman, 2005; Roh, 2004), and the idea of approximation (Oehrtman, 2002) in order to properly conceptualize the definition of a limit. However, there is little research to indicate how students perceive the notions of convergent sequences and Cauchy sequences. The purpose of this paper is to describe student understanding regarding the concepts of convergent sequences and Cauchy sequences.

The Studies

The data for this paper is part of two larger studies each of which followed a set of juniors and seniors, majoring in mathematical sciences or mathematics education, enrolled in real analysis courses in 2004 and 2007, respectively. The students had already completed calculus and a proof writing course. In the first study, students were enrolled in one of four sections of advanced calculus/beginning real analysis from various instructors. Students were co-enrolled in a semester long workshop, led by the first author of this paper, exploring their notions of proof, definition usage and development of tools and strategies for proving. In each hour-long weekly session, students would tackle common problems in advanced calculus and work in groups to produce a proof. Group A consisted of students called Doug, Molly, Ben and Jane and Group B consisted of students called Dustin, John, Mark and Lynn. Each group was videotaped as were whole class discussions. Students also were encouraged to post further discussion thoughts on an online class discussion board.

The second study was conducted as part of a larger study from a semester long teaching experiment in an introductory real analysis course taught by the second author of this paper. Data consisted of videotape recordings of all class sessions, office hour sessions, task-based interviews, and copies of students' worksheets, quizzes, and exams. During class sessions, students worked in small groups with proper guidance from their instructor. The main feature of group activities was to construct definitions of mathematical terms, to apply the definitions to prove properties about the term, and to evaluate if arguments given by the instructor were valid as mathematical proofs. In this paper, students' discussion from two groups will be presented. The students in Group C will be called Megan, Sophie, Stacy, and Matt; and those in Group D will be called Sean, Steve, Stan, and Susie. Each day, the instructor assigned one student from each group rotationally as a group facilitator of the day. On that day, Megan and Sean were facilitators of their groups.

Prior to the class day on which we will be reporting students in both studies had encountered bounded sequences and convergent sequences. Students were familiar with discussing concepts and working on proof writing tasks in small groups. On the day in question students in both studies were asked to prove if a sequence is convergent, then it is a Cauchy sequence. We will report on the resulting discussions in each study. The purpose of our report is not to generalize knowledge given from a sample of few students to a population. Rather, we are forming a conceptual model of students' notion of convergence, and their processes of proving convergence, that can be used by other researchers to investigate students' learning in similar settings.

Results

This paper investigates the misstatements as well as the correct statements and usage made by students about the relationship between Cauchy sequences and convergent sequences.

We will examine five short episodes which took place during the class sessions. In the first episode we illustrate the difficulties Group A faced in understanding the notational definition of a Cauchy sequence. In Episode 2 we show Group C connecting the two notions of convergence by using physical devices ($\varepsilon-$strips) provided in previous classes. In the third and fourth episodes we follow Group D. First we examine the obstacles students faced in setting up the proof of the inequality $|a_m - a_n| < \varepsilon$, then in the next episode we show how an important shift in understanding the role of the variable *N* in the proof, allowed the group to overcome the obstacle. Finally, in Episode 5, we illustrate how Group B resolved an obstacle similar to that in Episodes 3 and 4 by focusing on the role of ε, rather than *N*, in structuring their proof.

Episode 1: Understanding of the definition of Cauchy sequences

This episode is reported from the first study conducted by the first author (Knapp, 2006). The class day in question took place in the middle of the semester. In the previous class session they had proved that a bounded sequence has a convergent sub-sequence. Since students were in different advanced calculus courses, some students had already seen the definition of a Cauchy sequence while others had not. The class handout asked them to prove the statement, but did not give a definition for Cauchy sequences. So the students turned to their notes or textbook for a definition. In the first few minutes of the discussion, the students in Group A clarified their definition of a Cauchy sequence for Ben, the classmate who did not yet know the definition.

Doug: Alright. Definition of a Cauchy sequence is basically a sequence, oh okay. Basically you have 2 sequences, $\{a_n\}$, $\{a_m\}$. Uh... Let...

Molly: Well they're both different elements of the same sequence.

Doug: Yeah. Let epsilon be greater than 0, such that....

Molly: There exists an $n = N$ based on your epsilon such that *a* to the... $|a_m - a_n| < \varepsilon$.

Ben: So it’s the idea that sub sequence is convergent?

Molly: For all $n, m > N_\varepsilon$, right?

Jane: Well it’s just that as the sequence elements get closer and closer together. So now you’ve got some big N where...

Ben: I’m picturing a ___[inaudible] in my head where like...

Jane: As long as you’re beyond... yeah.

Ben: ...I have an $e^x \sin x$ and then a sequence that’s getting closer and closer and closer to 0. Or, closer and closer to...yeah that’s right.

This initial discussion centered around Doug, Molly and Jane unpacking the symbolic definition of a Cauchy sequence for Ben. Notice Doug and Molly made several misstatements as they were describing their understanding of the definition of Cauchy sequences for Ben. First Doug was confused that $\{a_n\}$ and $\{a_m\}$ represent two different sequences, then Molly began with $n = N_\varepsilon$. These misstatements seemed to be founded in their unpacking of the symbolic definition. Both students had encountered the notational definition of a Cauchy sequence, but were not able to sufficiently describe the meaning in their own words.

Ben on the other hand was looking for examples to help him understand the concept. Notice his first comment was to think about a Cauchy sequence meaning a sub-sequence is converging rather than the whole sequence. The group did not correct Ben’s conception of Cauchy sequences. Jane’s comment, however, focused the group on the idea that Cauchy sequences get “closer and closer together.” This was the working definition of a Cauchy sequence which the group adopted. This working definition did not help the group to understand the difference in meaning between Cauchy sequences and convergent sequences. Ben’s example $e^x \sin x$ was a function the limit of which approaches 0 as x approaches 0. While this example does in fact "converge" it also did not provide insight into the aspects of the definition of a Cauchy sequence.

Three of the four groups we examined began their discussion by unpacking the logical statement of the definition. Like Group A they looked for connections between the new definition and concepts they had already studied. Students' notions of convergent sequences played a role in their understanding of the definition of Cauchy sequences.

Episode 2: Connecting two notions of convergence

The following episodes from Episode 2 to Episode 4 are from the second study conducted by the second author of this paper. In the previous class sessions, students were engaged in a hands-on activity, called the $\varepsilon-$strip activity, which was specially developed for teaching convergence of sequences (Roh, in press). Each $\varepsilon-$strip was introduced to students as a strip with infinite length and constant width. Since the $\varepsilon-$strips were made of translucent paper, such as patty paper, students were able to observe the graph of the sequence through the $\varepsilon-$strip. In addition, a red line was drawn in the center of each $\varepsilon-$strip to mark an anticipated limit value on the graph. In this case, half of the width of an $\varepsilon-$strip represents an error bound ε in the $\varepsilon-N$ definition. In a particular case when a sequence $\{a_n\}_{n=1}^{\infty}$ is convergent to L, if the center line of an $\varepsilon-$strip is aligned on $y=L$, the terms, or a_n's, that are inside the $\varepsilon-$strip are those satisfying the strict inequality $|a_n - L| < \varepsilon$. Figure 1 illustrates an $\varepsilon-$strip centered on the $x-$axis and lying on top of the graph of the convergent sequence $\{1/n\}_{n=1}^{\infty}$.

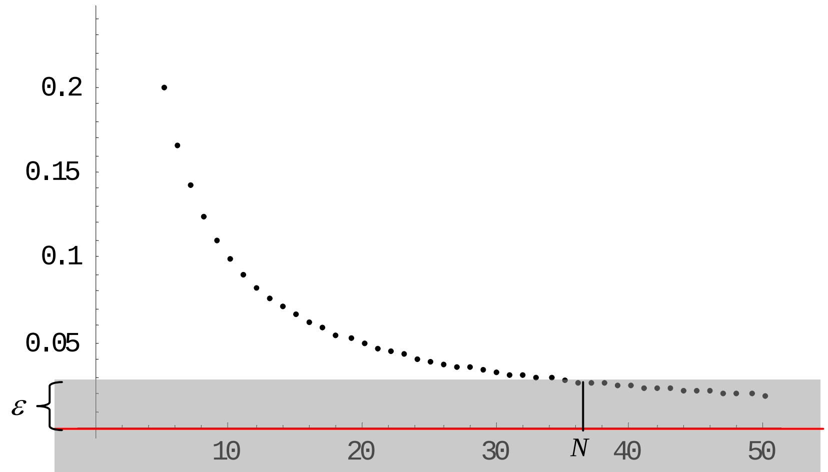

Figure 1 A graph of a sequence $\{1/n\}_{n=1}^{\infty}$ with an $\varepsilon-$strip

Episode 2 illustrates students' ways of using $\varepsilon-$strips to connect the notions of convergence to the definition of a Cauchy sequence. First, students compared differences in symbols between the definition of convergence and the definition of Cauchy sequences. They then realized that the *m*-th term a_m of a sequence $\{a_n\}_{n=1}^{\infty}$ was used in the definition of a Cauchy sequence whereas the limit *L* of the sequence was used in the definition of convergence. Megan, as the facilitator of Group C in the class session, initiated the discussion by asking group members why they could use a_m in the definition of a Cauchy sequence instead of *L* regarding the notion of convergence. Megan's question focused her group members' attention on exploring the meaning of the expression $|a_n - a_m|$ which is less than any given value of ε.

Megan: Why are we picking two…

Sophie: *n*?

Megan: ..Yeah. Why don't we have -- I guess "Why are we using *m* [a_m] instead of *L*, now? Why?" would be the question.

Matt: Uh, so you have -- you have another…

Sophie: Why is that, instead of an *L*, a_n instead?

Matt: Yeah.

Sophie: Like what does that mean for us?

Megan: Yeah.

Matt: It [$|a_n - a_m|$]'s a finite difference, that's what that [$|a_n - a_m| < \varepsilon$] is saying.

Megan: Why do we want to find the difference? And why does it mean it's convergent?

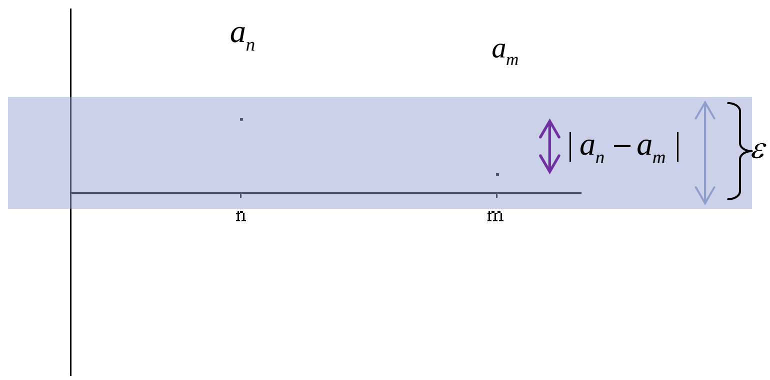

Figure 2 Megan's interpretation of Cauchy sequences

As shown in the transcript above, Matt described the inequality $|a_n - a_m| < \varepsilon$ as a finite difference between two terms a_n and a_m. After Matt's comments on the inequality $|a_n - a_m| < \varepsilon$, Megan re-interpreted Matt's understanding of the inequality $|a_n - a_m| < \varepsilon$ by using an $\varepsilon -$ strip along with her gesture on her hands. To be more precise, Megan gestured with both hands. She took her forefinger and thumb on her left hand and placed them a short distance apart (as if she was pinching them together), similarly she pinched together her forefinger and thumb on her right hand, the two fingers on her left hand were closer to each other than those on her right hand. This can be interpreted that Megan represented the distance between two terms a_n and a_m of a sequence by using her left hand fingers, and the width of an $\varepsilon -$ strip by using her right hand fingers (See Figure 2).

Megan then described that two terms a_n and a_m would be contained in the $\varepsilon-$strip. Her way of using $\varepsilon-$strips in describing the meaning of the inequality $|a_n - a_m| < \varepsilon$ became a turning point for other students in Group C to explore the meaning of Cauchy sequences via $\varepsilon-$strips. Students in Group C eventually rephrased the meaning of a Cauchy sequence as "for any $\varepsilon-$strip, any two points in the sequence are contained within the $\varepsilon-$strip." The following excerpt reveals how the group used $\varepsilon-$strips in their interpretation of a Cauchy sequence.

Matt: Um, let's see. You're looking at –

Megan: You have two points, and if the $\varepsilon-$strip -- if for any $\varepsilon-$strip, we can have it be smaller -- no wait, what's smaller? The difference [$|a_n - a_m|$] is smaller. So all it's saying is the difference between two points is contained within the epsilon? in the $\varepsilon-$strip?

Matt: The difference is -- can be smaller for any…

Sophie: ...can be smaller for the epsilon…

Matt: …all *m, n* greater than *N*. So…

Sophie: So yeah, you can take two points in the sequence, is what it's saying.

Matt: Between any two points in the sequence, there's a finite difference that's less than epsilon.

Megan: We can find an epsilon -- every $\varepsilon-$strip will have those both points contained…

Matt: It's possible to do that, yeah.

In fact, the $\varepsilon-$strip in Figure 2, which Megan illustrated in her motion, is different from the $\varepsilon-$strip illustrated in Figure 1 for the limit of a sequence. In particular, in Figure 2, there is no center line and the width of the strip is not 2ε but ε. Nonetheless, it is worth noting that

$\varepsilon -$strips played a role in connecting two notions of convergence, the limit of a sequence and a Cauchy sequence. By imagining the $\varepsilon -$strip, students could perceive that the task is to not just calculate the difference between two terms of the sequence but to compare such a quantity with an error bound ε.

Episode 3: 2ε is not less than ε.

Episode 3 describes the first obstacle that the students in Group D encountered when they examined how convergence of a sequence would imply the sequence to be a Cauchy sequence. Similar to Group C in Episode 2, Group D also used $\varepsilon -$strips in making sense of the definition of a Cauchy sequence. Students in Group D then set up their proof by assuming that a sequence is convergent. Accordingly, they imagined for any $\varepsilon -$strip, every point but a finite number of points, *N*, of the sequence is contained in the $\varepsilon -$strip. Sean, the facilitator of Group D in the class session, drew a graph of a sequence which is oscillating but is convergent to 0. He marked *N* on the $x -$axis and then drew two lines each of which was parallel to and had the same distance from the $x -$axis, thereafter every term whose index was greater than *N* was contained in the $\varepsilon -$strip. He then called such a distance ε. In fact, the region bounded by these lines is the interior of an $\varepsilon -$strip. Sean hence called such a region an $\varepsilon -$strip and chose two values *m* and *n* each of which was greater than *N* and marked them on the $x -$axis. He then led his group to compare the value of ε with the distance between a_n and the $x -$axis, and with the distance between a_m and the $x -$axis. Since *m* and *n* were greater than *N*, students in Group D could perceive that both a_n and a_m were within the $\varepsilon -$strip, hence their distance from the $x -$axis was less than ε.

Sean: So you have like, here's the big *N* that satisfies it. Let's say this is going like this, and so your $\varepsilon -$strip is such that everything in here is contained past that

N. And you choose two values n and m regardless of where, like -- let's say this one [a_n]'s here and this one [a_m]'s here -- this distance [$|a_n - L|$] here has to be less than this distance [ε] here. Right?

Stan: Because that's positive epsilon. Has to be less than or it wouldn't -- yeah okay.

Sean then realized that based on their reasoning, they could argue the distance between a_n and a_m , $|a_n - a_m|$, is less than 2ε. However, Sean could not conclude "$|a_n - a_m| < \varepsilon$" because he knew that ε was a positive real number and hence 2ε is greater than ε. Sean indeed thought what they had to do was not to prove $|a_n - a_m| < 2\varepsilon$ but to prove $|a_n - a_m| < \varepsilon$. Figure 3 illustrates how Sean and his group members set up their proof in terms of an $\varepsilon -$ strip placed on the graph of an oscillating convergent sequence.

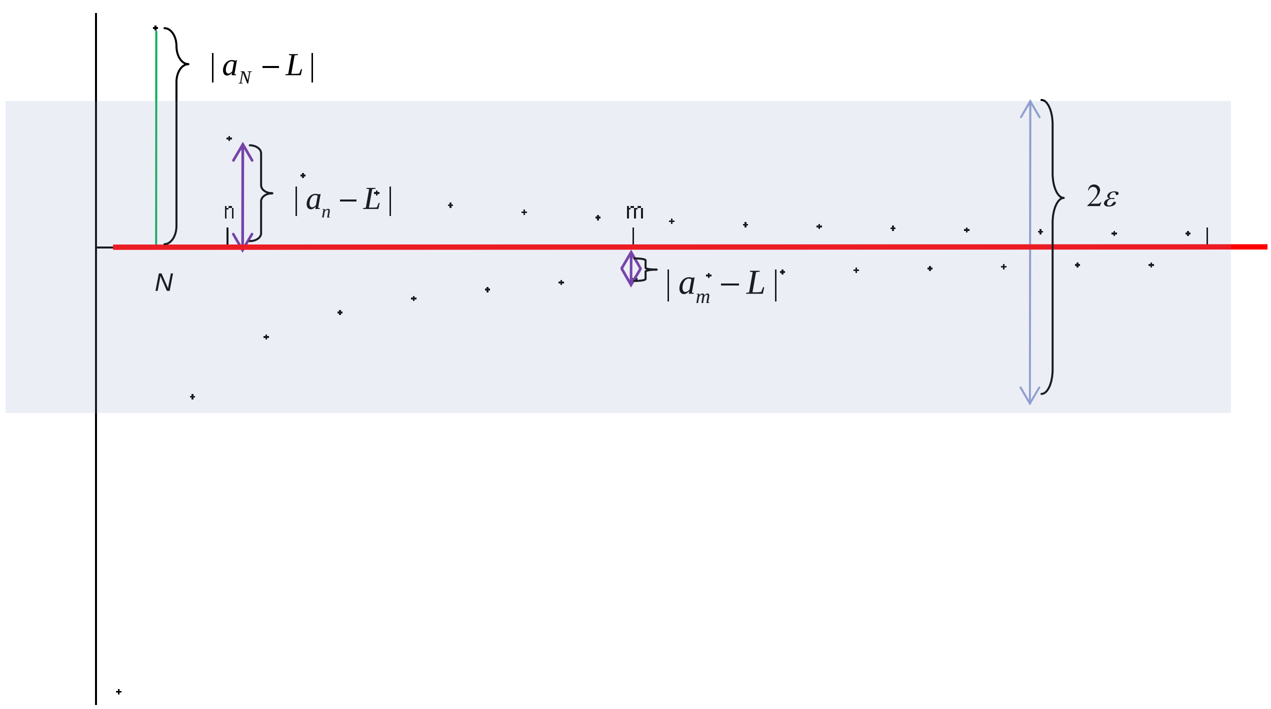

Figure 3 Sean's illustration: $|a_n - a_m| < 2\varepsilon$ but 2ε is not less than ε.

Sean: I mean it would be for sure that this distance here would be less than two times epsilon, because two times epsilon is this whole thing. But how can you be sure that this distance is going to be less than a single epsilon? I mean that's just in the special case, I mean other cases like, [drawing a graph of a monotone decreasing sequence] where it's just that and there's your epsilon strip and your two values; obviously this distance here...[is less than ε]

As seen in the above transcript, Sean also checked whether $|a_n - a_m| < \varepsilon$ if he examined it with a different sequence. He could imagine a sequence which is monotone decreasing and convergent to 0. In this case, he could perceive that $|a_n - a_m| < \varepsilon$. However, he believed that finding an example of a sequence satisfying the inequality $|a_n - a_m| < \varepsilon$ does not mean that such an inequality is always true for any arbitrary sequence.

It is worth noting that, without any guidance from the instructor, students in Group D, by themselves, initiated their proving of the given theorem by assuming a sequence to converge and then comparing the value of ε with distances between term values and the limit *L* of the sequence. In fact, the reasoning behind their proof seems to be very similar to a standard proof of the theorem “every convergent sequence is a Cauchy sequence.” Nevertheless, these students were not sure how they could complete their proof by their way of reasoning. In fact, these students considered their choice of $\varepsilon-$strip to be arbitrary which reveals that they understood the role of ‘any’ ε in the theorem. However, when they chose an $\varepsilon-$strip, they seemed to consider it as a generic element for an error bound ε, thereafter the value ε was fixed for the rest of their reasoning process. Roh (2005) describes that this kind of response seemed to occur because students failed to realize that the arbitrariness of the error bound ε infers the successive change of the value of ε decreasing to 0. Episode 4 illustrates how Sean came to recognize such an error in his reasoning process.

Episode 4: Students' aha moment − Oh wait! There are two different *N*'s

The instructor moved to the whole class discussion by talking to the whole class and collecting students' ideas of proving the given theorem. Sean, on the other hand, was off task and continued his thinking to figure out how to fill the gap between what his group had so far in Episode 3 and what to prove at the end. The aha moment came to him when he realized that the value of *N* in the definition of a Cauchy sequence does not have to be the same as the value of *N* in the definition of the limit of a sequence although the same symbol *N* was used in each definition. He explained to another student Steve in his group that by choosing a bigger value for *N* than the value which appeared in the definition of the limit of a sequence, the difference between two terms a_n and a_m became less than ε. The following excerpt reveals his aha moment in detail.

Sean: [regarding paper] Let me check this out.

[Instructor talking to class]

Sean: [regarding paper] Oh wait…

[Instructor talking to class]

Sean: Yeah, this *N* [in the definition of Cauchy sequence] isn't doing the same thing as it was doing before [in the definition of the limit of a sequence].

Steve: Yeah, exactly.

Sean: So this [*N* in the definition of Cauchy sequence] could be farther down here and this [pointing to a point on the $x-$axis] could be an *N* where it's right here.

Steve: As long as…

Sean: It's farther down.

Steve: …small *m* and small *n* are greater than… yeah.

Sean: Yeah.

It is worth noting that students are not in the status of just comparing symbolic representations any longer. They realized that although the symbol *N* is used in both definitions, it does not mean their values are the same. In fact, these students seem to start seeing the symbol *N* as a dummy variable. Such a shift in understanding the role of the variable *N* seems to resolve these students' obstacles in showing $|a_n - a_m| < \varepsilon$.

On the other, it is noted that Sean suggested choosing a different value not for ε but for *N*. It seems to result from his understanding of the relationship between ε and *N* that is reversed in order from that in the definition of convergence. As seen in Episode 3, Sean picked an index for *N*, then determined the value of ε so that every term after the *N*th term of the sequence is contained in the $\varepsilon-$strip. Roh (2004, 2005) described students' tendency to understand the relationship between ε and *N* in such a reversing way from the formal definition. Courant and Robbins (1996) asserted that such a reversal of the thinking is a source of the psychological obstacles to understanding the formal definition of the limit of a sequence. Kidron and Zehavi (2002) and Knapp and Oehrtman (2005) also reported similar examples of such phenomena in the case of the definition of the limit of a function at a point.

Episode 5: Which epsilon?

Group B did not begin their session by examining the definition of Cauchy sequences. All four students had encountered this term in their advanced calculus course prior to the workshop. So in the first few minutes of the discussion they began writing a proof. Some of the students had a professor who had used $\tilde{\varepsilon}$ in a proof. As they began to write they recalled this instance. Dustin suggested "We can let ε be $\tilde{\varepsilon}/2$. Like over here." This entry sparked the following discussion about the necessity of two epsilons.

John: Why did you do that?

Mark: Yeah, I don't...

Lynn: Remember when that professor...

John: This is the same.

Mark: That's not necessary really.

Dustin: Well see, here's the thing. What we're looking at is like um...

Lynn: Cause when you're using eps...

Dustin: You know what we need. We're gonna need this to be $\varepsilon/2$. So that when we add them together we'll get an ε.

Mark: But this is the same ε as that though.

John: Yeah. That's what I'm like...

Mark: It's just divided by 2. It's the same one.

Lynn: Right. It's to clarify. That like... remember when we had that one professor come in and sub? He used that to clarify. So people could understand why you're dividing it by 2.

John: I'd feel better...

Mark: Yeah, I'd actually... I'm with him, but either way.

John: I'd feel better if this was $\varepsilon/2+\varepsilon/2$ isn't that $\tilde{\varepsilon}/2+\tilde{\varepsilon}/2$?

Mark: Yeah, whatever.

The introduction of $\tilde{\varepsilon}$ caused the students in Group B to wrestle with the relationship between the two epsilons. They had encountered the standard proving method of using $\varepsilon/2$, but were not clear why a separate epsilon was needed. Similar to the situation in Episode 4, the students needed to see the different values inherent in the same generic variable. This discussion surrounding the necessity of epsilon tilde consumed the majority of the workshop time the group spent on this proof. Led by Lynn the group began to recognize the relationship between ε and *N*. As we saw with Group D, Lynn incorrectly determined that each ε was dependent on *N*. Thus they had a problem with the possibility of two *N*'s which would then require two ε's. They did correct the dependence relationship fairly quickly in the conversation; however, they continued to struggle with the reason for including $\tilde{\varepsilon}$.

Mark: I don't see how introducing an $\tilde{\varepsilon}$ would solve any of this.

Dustin: Here we know that for any ε whatsoever we can find that N that is gonna make this true.

Mark: Yeah.

Dustin: For any ε. So if we say that uh, that let...

Mark: If we show that this is less than any ε ...

Dustin: No, any ε includes... any ε would include an ε that's like, that is that $\varepsilon/2$. Right.?

Mark: Yeah but that $\varepsilon/2$ + another $\varepsilon/2$ is just a regular epsilon like that one.

Dustin recognized the arbitrary nature of ε. He also saw the difference between the two epsilons. Lynn also made comments which indicated she understood the different epsilons. Both Lynn and Dustin were not satisfied with the proof the group constructed because of the confusion the two epsilons created. Their dissatisfaction led the group to look for an alternate structure to the proof which would allow them to avoid the use of the $\tilde{\varepsilon}$.

Unlike the other groups, Group B did not at any time invoke an instantiation of a sequence in their proving process. While the examples and ε-strip graphs help the other three groups examine their notions of Cauchy sequences, Group B relied solely on their previous experience with the definition to aid their understanding. Their previous experience with the use of $\tilde{\varepsilon}$ in a proof produced conversation about the nature of both variables *N* and $\tilde{\varepsilon}$ in the proof.

Discussion

The finding of this study supports the previous research in students' understanding of definitions (Edwards, 1997; Knapp, 2006) including the understanding of the meaning of the logical structure in definitions. First of all, we see students' notions of convergent sequences played a role in their understanding of the definition of Cauchy sequences, which is in line with the findings of Przenioslo (2004) and others. In particular, in Episode 4, we see students from Group D picked a value of *N*, then determined the value of ε. This indicates that the students did not conceive the independent role of ε to *N* in the definition of a Cauchy sequence, which supports the work of others in the context of the limit of a function (Kidron & Zehavi, 2002; Knapp & Oehrtman, 2005) and the limit of a sequence (Roh, 2004). In addition, we found that the main difficulty that students encountered was related to the use of the same symbol ε and N in both definitions. We saw that students, especially those from Group B and from Group D were not able to resolve their main difficulty in proving the theorem until they started seeing the variables ε and *N* in both definitions as dummy variables. This suggests that students' understanding of the role of variables in the definition of a Cauchy sequence is pertinent to their ability to use the definition in their proof of the theorem.

Eventually, the students were able to appropriately discuss the meaning of the definition of Cauchy sequences. In fact, we saw the students from Groups A and B correctly justify several statements they made based on the definition of Cauchy sequences and define Cauchy sequences in their own words in their discussion posts. Group A overcame their initial misstatements and was able to use the ideas of Cauchy sequences in their proving. Group B used the construct of an $\tilde{\varepsilon}$ as modeled by a substitute instructor to distinguish between the two epsilons in the proof. Their subsequent discussion led them to better understand the dependent relationship between ε and *N*. Group C effectively used the $\varepsilon -$ strip introduced in previous classes to connect the two notions of convergence. Likewise, without any guidance from the instructor, students in Group D initiated their proving of the given theorem by assuming a sequence to converge and then comparing the value of ε with distances between term values and the limit *L* of the sequence.

The subtle understanding of the role of each variable in the proof of this theorem seemed to be an important notion for students in proving statements involving Cauchy sequences and convergence. Particularly, being able to perceive ε as an arbitrary variable which approaches zero, and recognizing that for each definition there exists a separate *N* based on ε are each skills necessary to complete the proof that every Cauchy sequence converges. A possible direction for further research would be to determine if these notions of variables are valuable to understand further concepts in Advanced Calculus.

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