A Local Instruction Theory for the Development of Number Sense

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Externally-developed local instruction theories are indispensable for reform mathematics education. (Gravemeijer, 2004, p. 108)

Gravemeijer (2004) elaborates on the construct of a *local instruction theory* developed in the context of design research as a means to offer teachers a framework of reference for designing and engaging students in a set of exemplary instructional activities. Gravemeijer (2004) discusses the core elements on an LIT with an example of learning goals, instructional activities, and, in particular, the role of tools and imagery.

We, too, see a local instruction theory as indispensable to the design of instruction. The paucity of examples in the literature suggested to us that this construct needed further elaboration and illustration. In this paper, we offer an empirically-grounded theory on how a set of activities can be used in support of the development of number sense. We first reiterate the differences between local instruction theory and hypothetical learning trajectories (Gravemeijer, 1999; 2004; Simon, 1995). Within the context of our design research in a class for pre-service elementary teachers, we provide an example of a local instructional theory and illustrate the relationship between the local instruction theory and the resulting hypothetical learning trajectories.

OUR SETTING

Within a content class for pre-service elementary teachers with a focus on Number & Operations, we had the aim of developing students' number sense. According to Reys & Yang (1998),

> Number sense refers to a person's general understanding of number and operations. It also includes the ability and inclination to use this understanding in flexible ways to make mathematics judgments and to develop useful strategies for handling numbers and operations. (p. 225)

The development of number sense is a widely accepted goal of mathematics instruction (c.f., NCTM, 2000; Reys, Reys, McIntosh, Emanuelsson, Johansson, & Chang, 1999). In order for teachers to facilitate the development of good number sense, they must exhibit number sense themselves. If we value instructional approaches in which students are asked to reason and communicate their reasoning and teachers are expected to capitalize on these opportunities, teachers need the ability to interpret reasoning and respond appropriately (Carpenter, Fennema, & Franke, 1996; National Research Council, 2001; Sowder, 1992).

In this report, we describe aspects of our teaching experiment with an aim of fostering students' development of number sense with regard to mental computation and computational estimation and articulate the under-girding theory. Mental computation is used in computational estimation and both computational estimation and mental computation are associated with the structure of number and rely on number sense (Sowder, 1992). Number sense can be described as broadly as good intuition about numbers and their relationships (Howden, 1989) but in the classroom teaching experiment we describe here, our instructional focus was on students' sense making, mental computation and computational estimation, and generally a framework of number relations available for flexible mental computation (Gravemeijer, 2004; Stephan, Bowers, Cobb, & Gravemeijer, 2000).

Our planning for the semester-long Number and Operation course included the development of several *hypothetical learning trajectories* (HLTs). An HLT consists of learning goals for students, planned instructional activities, and a conjectured learning process in which the teacher anticipates the collective mathematical development of the classroom community and how students' understanding might evolve as they participate in the learning activities of the classroom community (Cobb, 2000; Cobb & Bowers, 1999; Simon 1995). Hypothetical learning trajectories have been described and articulated for a number of teaching experiments in diverse areas, such as linear measurement, equivalence of fractions, and statistics (c.f. Gravemeijer, 2004; Jones, et al, 2001; Simon, 1995; Simon & Tzur, 2004; Stephan, Bowers, Cobb, & Gravemeijer, 2003).

Our goal of supporting students' development of number sense encompassed more than what was represented in hypothetical learning trajectories such as those described in the aforementioned literature. In contrast, our aim of developing number sense would span a semester and needed to encompass a number of particular mathematical concepts, each of which involved the creation of a unit specific instructional sequence (e.g., place value, meaning for operations, properties). Based on a literature review, course content goals, and our design heuristics, we developed a set of goals and a philosophy that would under-gird the design of instructional sequences with regard to the development of number sense. A conjectured local instruction theory provided a framework for the integration of support for mental computation and computational estimation into the unit-specific instructional sequences or HLTs (Gravemeijer, 1999; 2004; Simon, 1995).

Local instruction theory (LIT) refers to "…the description of, and rationale for, the envisioned learning route as it relates to a set of instructional activities for a specific topic. " (Gravemeijer, 2004, p. 107). In Gravemeijer's (1999) view, (1) the hypothetical learning trajectory deals with a small number of instructional activities and the local instruction theory encompasses a whole sequence, and (2) the hypothetical learning trajectories are envisioned within the setting of a particular classroom, whereas the local instructional theory comprises a framework, which informs the development of hypothetical learning trajectories for particular classrooms. Thus, the distinction between LIT and HLT is two-fold. One distinction is the duration of the learning process and the other is the 'situatedness' in a particular classroom.

In this paper, we discuss our local instruction theory for the development of number sense as manifested in mental computation and computational estimation in terms of goals and rationale and its relationship to ensuing hypothetical learning trajectories. In previous talks and papers, we have focused on how past research has contributed to the development of the LIT and we presented our evidence for students' developing number sense (Whitacre, 2007; Whitacre & Nickerson, submitted). Within this paper, we summarize our LIT and illustrate it with examples from our classroom teaching experiment. We conclude by arguing that the resulting LIT can be useful to teachers designing instruction that facilitates the development of number sense.

THEORETICAL PERSPECTIVE

We take a perspective on learning in mathematics classrooms as a constructive process as individual students participate in and contribute to the norms and practices of their classroom community (Cobb & Yackel, 1996; Lave & Wenger, 1991). As such, we are interested in understanding individual students' thinking, as well as, classroom participation structure, social norms, socio-mathematical norms and practices (Cobb & Yackel, 1996). In adopting such a perspective, we seek to understand a learner's ability to play a role, including his or her ability to anticipate, sense what is feasible within a context, and improvise or adapt accordingly (Hanks, 1991). As we conceptualized our instructional design, we found it useful to frame the goals and envisioned learning route in terms of Greeno's (1991) environment metaphor.

As mathematics educators, we conduct classroom-based research and design instruction in the form of *design research* or *developmental research* (Cobb & Bowers, 1999; Gravemeijer, 1994; 2004). Our instructional design for this inquiry-oriented class was guided, in part, by central tenets of Realistic Mathematics Education (RME) instructional design heuristics: (1) sequences must be experientially real, (2) students should be guided to reinvent significant mathematics for themselves wherein (3) students and the teacher develop a model-of informal activity which becomes a model-for mathematical reasoning (Gravemeijer, 1999, 2004; Richards, 1991; Stephan, Bowers, Cobb, & Gravemeijer, 2003). With regards to design research, the LIT informs the development of HLTs. Likewise, our experiences with each HLT and a retrospective analysis inform the broader goals, envisioned learning route, and instructional activities of the local instruction theory. We began our planning by first reviewing how researchers described the characteristics of individuals with good number sense and the researchers' recommendations for pedagogy in support of number sense.

REVIEW OF LITERATURE ON TEACHING FOR NUMBER SENSE

The research on number sense, summarized elsewhere, suggests that people who have good number sense tend to exhibit the following characteristics when performing mental computation: sense-making approach, planning and control, flexibility, and an appropriate sense of reasonableness (Carraher, Carraher, & Schliemann, 1987; Markovits & Sowder, 1994; Reys, Rybolt, Bestgen, & Wyatt, 1982; Schoenfeld, 1989; Sowder, 1992). But the research also suggests that having an awareness of the characteristics of and the variety of strategies employed by individuals with good number sense does not imply that one can teach to the symptoms of good number sense. Importantly, Sowder (1992) notes that "[t]here is consensus on the fact that number sense should permeate the curriculum beginning in the early grades, rather than being relegated to 'special lessons' designed to 'teach number sense'" (p.386).

McIntosh (1998) made a number of pedagogical suggestions. One of the first is that specific mental computation and computational estimation strategies should not be taught even while giving mental computation a greater priority in teaching. Explicit, direct instruction in the use of productive strategies may, in fact, be counterproductive (Greeno, 1991; Schoenfeld, 1992; McIntosh, 1998). This is due to the fact that flexibility, planning, and control might best be fostered when students can develop in sense-making and the habit of making choices. As Schoenfeld (1992) points out, when strategies are directly taught, "they are no longer heuristics in Polya's sense; they are mere algorithms."

McIntosh recommended that, given time, students will invent novel strategies. His first recommendation was that we not restrict mental-arithmetic sessions to developing the ability to do mental math in short bursts of speed. Second, he suggests that after students have been asked to mentally compute, they get an opportunity to share and discuss, with an emphasis on there being many valid ways to solve the same problem. Third, he suggested that teachers take advantage of student's spontaneous interest in each other's strategies by encouraging students to try out those solution strategies shared by peers. Finally, he encouraged an experience of doing mental math in class that is non-threatening and pleasurable.

Greeno's (1991) environment metaphor helps us to think about direct instruction in contrast with the intended effect of the pedagogy that McIntosh advocates. Whereas, direct instruction is analogous to giving explicit directions to a newcomer who could use them to reach a desired destination from a specified starting point, having explicit directions to every conceivable place is impractical. Ultimately, people need to establish their own lay of the land that involves establishing their own landmarks ad getting lost and finding their own way back to the personal landmarks. Our study used McIntosh's suggestions for pedagogy for the development of number sense as a starting point. We sought to foster students' development of number sense with regard to mental math and computational estimation.

LOCAL INSTRUCTION THEORY FOR NUMBER SENSE

A local instruction theory describes goals, instructional activities or plans of action based on underlying assumptions about teaching and learning. We will begin by delineating our goals and our envisioned learning route with a rationale. We will share some instructional activities not with the intention of offering a portable instructional sequence but offering a theory of how these instructional activities could work to develop number sense. Fundamental to this is a particular classroom culture.

Realizing a problem-centered, inquiry-based learning classroom for students' high-order mathematical reasoning requires particular social norms, such as the need to explain strategies and an accompanying expectation that one attempts to make sense of explanations given by others. In addition to explanations, students provide reasons why in order to help others make sense. Researchers further describe the importance of an inquiry-based classroom culture where students discuss whether a strategy is reasonable, identify its weaknesses and then further strengthen arguments by considering others' perspectives (c.f., Bowers & Nickerson, 2001; Kazemi & Stipek, 2001; Wood, Williams, & McNeal, 2006). Indeed, central to our goals was the need to build intellectual autonomy where students have a means of judging the efficacy of strategies (Yackel & Cobb, 1996). Furthermore, the classroom culture should be one in which the teacher needs to attend to the qualitatively distinct ways in which individual students participate and to view the students' distinct ways of solving problems as resources on which the teacher and the student can capitalize (Cobb & Bowers, 1999).

Goals and Envisioned Learning Route

Our goals for our students with regards to developing number sense are applicable to any mathematics course where students have opportunities to flexibly engage with numbers. The goals can be articulated with three major foci. First, students exhibiting number sense can capitalize on opportunities to use number sensible strategies for problem-solving situations both inside and outside the classroom. Second, students exhibiting number sense draw on deep, connected knowledge of number and operations to develop a repertoire of number sensible strategies. Third, students exhibiting number sense reason with models to build on this understanding and flexibly use new number-sense based strategies. With respect to the environment of number and operations, our goals would be that students would come to act in this environment as one who can recognize opportunities for solution strategies based in number sense, who has many ways to think about number and operations, who can flexibly draw on a repertoire of computational and estimation strategies and make sense of unorthodox strategies.

In order to support our first goal of students' capitalizing on opportunities for number sensible strategies for problem solving, students would be invited to use quantitative reasoning and mental computation and computational estimation throughout the course, regardless of the particular content of the curriculum. Based on our experience with other populations of pre-service elementary school teachers, we conjectured that the students initially exhibit an over-reliance on standard algorithms and estimation strategies, often without sense-making. Students would be taught to conduct a quantitative analysis of problems embedded in context. They would be asked to do mental computation and computational estimation.

Other researchers have suggested that when invited to mentally compute, they would approach mental computation tasks with limited options, many using the mental analogue of a standard algorithm, hereafter referred to as MASA (Hope & Sherrill, 1987; Markovits & Sowder, 1994; Reys, Reys, Nohda, & Emori, 1995). Likewise, students often approach computational estimation problems limited with standard rounding algorithms before computation without consideration of magnitude or the use of other benchmark numbers. We saw the instructor as a role model for identifying opportunities for number-sensible problem-solving strategies as a natural and practical aspect of mathematical activity (Lunenberg, Korthagen, & Swennen, 2007). We expected that as students participated in a classroom with a collective orientation toward making sense of number, with growing knowledge of the domain of number and number operations and properties, students would come to recognize opportunities for number sense-based strategies and realize the benefit of such a disposition.

Our second goal was that students develop a repertoire of number-sense based strategies. Students would be invited to perform calculations mentally in context-embedded problems, which typically results in some degree of strategy invention (Macintosh, 1998). The classroom culture needed to be one in which students shared their mental calculations and made sense of others' shared strategies. The collective math activity would include reflective discourse on the shared strategies. Initially, the strategies might be indistinguishable for students except in terms of surface characteristics. In order to make sense of strategies, it was necessary that students understood important mathematical key concepts, such as place value. After reflecting upon many examples, the instructor and students could then negotiate socio-mathematical norms of mathematical difference and relative ease and efficacy. We conjectured that students would then begin to see these shared strategies as examples of strategies with essential characteristics. In coming to understand difference, they should also develop a repertoire of number sense based strategies. The students could make sense of others' strategies. We expected this would contribute to their development of habits of planning and control (Schoenfeld, 1992).

Our third goal was that as students would listen to others' number sense-based strategies and come to understand their origins, they would flexibly use new mental calculative strategies that build on this understanding. Through reflective discourse on shared strategies, instructors and students would negotiate symbolizing models-of strategies. These models-of students' reasoning through transformational records could come to be used as models-for reasoning and a means to empower students to create their own new strategies.

In sum, our three goals related to developing students' sense-making, planning and control, and flexibility–characteristics of students exhibiting number sense. Specifically, our goals were to enable students to capitalize on opportunities for number sensible strategies in and out of school settings drawing on a repertoire of nonstandard sense-based strategies, at least in part because of their ability to reason with models.

THE LOCAL INSTRUCTIONAL THEORY

Following are some excerpts from the teaching experiment around number sense, chosen to illustrate how support for the development of number sense was integrated into the content course for pre-service teachers (see Whitacre & Nickerson, submitted; Whitacre, 2007).

Goal 1: Identifying Opportunities for Number-sensible Strategies

From an instructional design perspective, we wanted the students to engage in personally meaningful activity, consistent with one of RME's basic principles that instructional sequences must be experientially real. It was essential when planning for instruction that the instructor identify opportunities within the curriculum to invite and model the use of number-sensible strategies both for computation and estimation. *Experientially real* sequences encompassed quantitative reasoning of narrative problems, as well as opportunities for mental computation and computational estimation that were not grounded in a narrative context. Equally important was that mental computation and computational estimation not be treated in isolation. We wanted students to experience authentic opportunities to use mental computation and computational estimation productively, in concert with developing a disposition toward mathematics as a sense-making endeavor, as well as confidence in working with numbers.

The importance of identifying opportunities within the curriculum to integrate quantitative reasoning and mental computation and computational estimation was critical in two ways. First, using mental computation based in number sense in experientially-real contexts supports recognition of its commensurability with other forms of participation (Cobb & Bowers, 1999; Thompson, 1992). Second, at the same time we needed to identify opportunities wherein the teaching of the mathematics of the course supported the mathematical understandings needed for mental computation and computational estimation grounded in number sense. For example, deep understanding of place value is needed for place-value based collection strategies for addition and subtraction to 100 (Gravemeijer, 1999). In other words, the content provided *foundations for* and *occasions for* mental math and estimation activity grounded in sense making.

We believed that the context-based problems provided opportunities for analyzing the structure of a problem with regards to its quantities. Quantitative analysis has a focus on the relationship among quantities, thus opening solution possibilities beyond the calculation needed to solve a problem with a standard algorithm and promoting sense making. Although both context-embedded problems and problems devoid of context can constitute experientially real situations for students, we started with problems posed as ‘story problems.’ The story problems seemed a natural entry point as opportunities for nonstandard problem-solving.

They also presented opportunities for mental computation in the form of calculations required to obtain solutions for these problems. For example, students were presented with a ‘catch-up’ problem in which one competitor starts running a race 600 meters behind the other but can run 25 meters per minute faster than the other competitor, and were asked to find out who would win the race (Sowder, Sowder, & Nickerson, in press). We conjectured that solving a problem such as this would not lead one to use the standard algorithm. Furthermore, once the students have suggested a solution path, it presents opportunities for mental computation. In the course of our teaching experiment, the instructor routinely stopped in the course of solving a problem such as this to ask students to mentally perform calculations, such as how long it would take the second competitor to catch up. The class then discussed a few solutions before continuing to solve the story problem at hand. The instructor guided the development of a social norm that one needs to explain and make sense of explanations by others.

Because problems in context tend to elicit oral computation procedures, as opposed to school-learned procedures (Carraher, Carraher, & Schliemann, 1987), the class began to use computations other than the mental analogue of the standard algorithm (MASA). Later computation estimation strategies were shared. The instructor and class began negotiating the norm that nonstandard solutions were acceptable and making sense of the mathematics of a strategy was always a priority. In guiding opportunities for quantitative reasoning and inviting mental computation strategies, the instructor, as an experienced resident of the environment, directed newcomers (Greeno, 1991).

In sum, our first goal was to have students capitalize on opportunities to solve problems using number sensible strategies. We began by identifying opportunities for mental computation and computational estimation within the curriculum. We engaged students in quantitative reasoning in context problems to support a shift from over-reliance on standard algorithms and strategies. The class established a social norm of sharing strategies using strategies to sense-based approaches to problem solving. The instructor modeled opportunities for mental computation and computational estimation based in understanding of number and its properties.

Goal 2: Students Develop a Repertoire of Number-sense based Strategies

As one aspect of guiding students to reinvent significant mathematics for themselves, the collective math activity included reflective discourse on shared strategies and methods of computation. As students began to perform a mental computation and computational estimation using the number sensible strategy, they shared their solutions and strategies. Because our classes often consist of students educated in a few different countries, the strategies students shared vary naturally. Students shared their own learned strategies—some learned meaningfully and others by rote. In the United States, subtraction is usually taught by regrouping, but in other countries children are taught an “equal additions” or other method. After discussing what different strategies afford and then returning to the problem at hand, the instructor’s believed that from the students’ perspective the answer we sought could have been done by pencil-and-paper and that the mental math was motivated entirely by the instructor.

In order to support: (1) a sense that the mental math activity had not been a digression, and (2) a means of connecting the distinct discussions of solution methods that were separated by class meetings, the instructor began to refer to strategies by the name of the nominating student. The instructor would allude to a method that a student had shared recently by asking (with regard to the calculation at hand) "How might Christine do this calculation?" This "naming" gave the classroom community a means of productively discussing mathematical difference. Such discussions involved increasing reflection on and reference to previous activity.

An understanding of place value, operations, and properties provided foundations for the practice of naming. As other students adopted what came to be seen as similar strategies, students collectively negotiated descriptive names that somehow captured the essence of the strategy. An understanding of operations and properties helped students to be able to talk about the collection of strategies with regard to the operations. The discussion shifted from naming strategies tied to specific examples to being able to look across many examples to discuss what properties they took advantage of. For example, one class section chose the name "Break up to Make up" to refer to partial products multiplication (15 x 24 is treated as 15 x 20 + 15 x 4). Students' taken-as-shared understanding of salient aspects of a strategy became an object, enabling discussion about the details of a single example and facilitating discussion of the structure of number systems. The practice of naming constituted a vertical shift in mathematizing, which facilitated reflective discourse on strategies.

In sum, our second goal was to have students develop a repertoire of number sensible strategies. We note the importance of anticipating the nonstandard strategies that students might use. As students shared and grew in understanding of the number and operations, they negotiated differences among and relative efficacy of shared strategies. The practice of naming contributed to students' development of planning and control.

Goal 3: Students Develop the Ability to Reason with Models

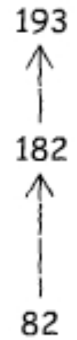

Figure 1. Initial notation.

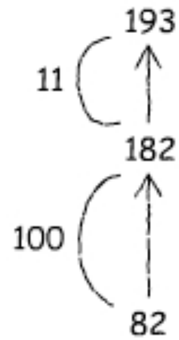

Figure 2. Elaborated notation.

Our third goal was that students would flexibly use (mental) models for reasoning. From an instructional design perspective, we wanted instructors and students to develop a model of informal activity that could become models for mathematical reasoning. It was essential in planning that the instructor anticipate productive, powerful models for reasoning. The instructor must anticipate and capitalize on models of student thinking that can be linked to models for reasoning. Following is one example from the classroom teaching experiment that illustrates the process of symbolization.

As strategies were shared, the class agreed on symbolizations of these strategies that they agreed made sense. For example, when the class was invited to solve a problem that involved comparing heights of two pairs of siblings, students determined they needed to find the difference between 193 cm and 82 cm. Ashley described her method of adding on: “82 plus 100 is 182, then plus 11 more is 193. So, it’s 111.” The instructor then offered to notate her method on the chalkboard with guidance from students to notate two “jumps” upward, as in Figure 1.

When asked how one would know the answer from what had been drawn, students suggested "writing numbers to the side," and the instructor made the additions depicted in Figure 2 to notate the measures of each jump. This became "Ashley's method" as it was applied to find solutions for other difference problems. These early symbolizations assume the role of a *transformational record*. Thus, while they begin as records of student thinking, the symbolizations are "used by students in achieving subsequent mathematical goals" (Rasmussen & Marrongelle, 2006, p. 394). In subsequent problems, the instructor turned this record on its side and the empty number line replaced the initial, informal notation as a shared conventional means of reasoning about addition and subtraction strategies.

The rectangular array model was also used as a powerful model for reasoning about multiplication computation and computational estimation. By way of one example, a student introduced the model as her way of reasoning about 24 x 15 as 40 x 9, an easier computation, by recognizing the five 3's in 15 and the eight 3's in 24.

Discussion of the LIT in terms of the three goals

Our sequence can be summarized briefly in terms of the following general instructional activities:

- The instructor anticipates opportunities for mental computation and computational estimation within a particular content area.
- The instructor models identifying opportunities for occasions for the practical authentic activity of utilizing number sense-based strategies to solve problems.
- The instructor anticipates the strategies and the mathematics implicit in the strategies. He or she must ensure there is support for such strategies both in the problem choice and in the understanding of number structure, operations, and properties.

- Students are expected to use quantitative reasoning in problem solving.
- The instructor anticipates productive models for reasoning. He or she must anticipate so that the instructor can productively capitalize on models of student thinking and links to models for reasoning.
- Students from the beginning of the course are expected to perform calculations mentally and to share the methods for their mental computations. The instructor guides the negotiation of social norms that students discuss shared methods with an aim toward making sense of the mathematics.
- The class collectively negotiates how the strategies can be symbolized; these symbolizations are guided in the sense of transformational records. The class may begin by referring to strategies with reference to specific examples.
- The class negotiates differences and relative efficacy of strategies. Students name the strategies in meaningful ways thus making them objects of reflection. This gradual shift necessitates the maintenance of a cumulative list of shared strategies.
- Keeping a list enables discussions to turn toward whether or not a shared strategy is different from those already seen. Thus, criteria need to be negotiated for aspects of a strategy that are essential, as opposed to incidental.
- A robust repertoire of strategies emerges from the organization of various examples, and taken-as-shared definitions are broadened as more examples are seen.

In sum, the goals in Greeno’s (1991) metaphor include students acting in an environment in which they can capitalize on opportunities for number-sensible strategies, develop a repertoire of number sensible strategies, and an ability to reason with (mental) models. These relate directly to characteristics of people exhibiting number sense: sense-making, flexibility, planning and control. In planning instruction, an instructor must prepare learning in experientially real settings, prepare a curriculum with opportunities for nonstandard strategies, and anticipate robust models for reasoning. The development of particular norms and socio-mathematical norms is crucial in the development of a practice of employing number sensible strategies.

DISCUSSION

Our LIT for the development of number sense included an articulation of three goals, the envisioned learning route, and instructional activities with a rationale. Just as one day's planning and interpretations of students' mathematical activity inform subsequent days' planning and interpretations within a classroom teaching experiment, analogies can be drawn to the development of theory that informs local instruction theory. The cyclic activity can be more broadly considered as a sequence of teaching experiments in which previously conducted research analysis informed our conjectured local instruction theory. Thus, our learning instruction theory is empirically tested and can inform future research into the development of number sense.

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