The Influence of Symbols on Students' Problem Solving Goals and Activities

Introduction

Symbols are the components of the mathematics language that make it possible for a person to communicate, manipulate, and reflect upon abstract mathematical concepts. However, the symbolic language is often a cause of great confusion for students (Rubenstein & Thompson, 2001). The expert mathematician or math teacher is able to work with and to "see" the mathematics through its symbolic representations, whereas students often struggle in this endeavor; they may need to be told what to see and how to reason with mathematical symbols (Bakker, Doorman, & Drijvers, 2003; Kinzel, 1999; Stacey & MacGregor, 1999).

The typical approach for helping students who are failing is to provide them with more practice problems. However, if instructors are not aware of students' networks of understanding, more practice may only reinforce misunderstanding (DeMarois, 1996; Dubinsky, 1991). It is possible that some students are actually developing different, incorrect techniques for problem solving due to their personal interpretations of the symbols involved (Gray & Tall, 1994). For example, one student in the current study was given the task: Solve for x given the equation $x^3 + 2x - 4 = 8$. Her first thought on the problem was that she needed to plug 8 in for x, but, she claimed, "I'm used to seeing it set up like f(x), so it's a little different, but I'll try it anyway." She immediately saw the symbol 8 as being equivalent to f(8), and plugged 8 in for x by hand and then checked it on a graphing calculator and was satisfied with her answer. The equal sign and the 8 were symbols that she ignored, misunderstood, or had given her own meaning in order to complete the task.

The purpose of this study is to conduct a case study of college pre-calculus students' with a focus on their mathematical thinking about symbols in order to identify what students are "seeing" in the symbolic structure of a problem. Using a combination of two conceptual frameworks and analysis of students' discussions of their work on carefully chosen tasks, the researcher focuses on clearly articulating, to the extent possible, how or why students solve a problem the way they do. The research question to be addressed is: In what ways do the symbols in a problem influence college pre-calculus students' goals, activities, and organization of results when solving mathematical problems?

Literature Review

Researchers have found that students have preconceived ideas from personal experiences about what math symbols are supposed to represent, and often base their interpretations on these experiences, falsely assuming that all symbol use is related (Stacey & MacGregor, 1997). Teachers and researchers of mathematics education have observed that many difficulties in mathematics can be attributed to students' problems with manipulating and understanding algebraic symbols (Driscoll, 1999; Gray & Tall, 1994; Kinzel, 1997; Pimm, 1995; Stacey & Macgregor, 1999). One reason for this difficulty that is identified in the research comes from the way in which individuals apply personal meaning to symbols. According to Kinzel (1999), mathematical notations can only be thought of as potential representations that do not become representations until someone constructs an interpretation for them. One person's interpretation may differ from another's. Students' own interpretations are based in the prior experiences that they bring to the classroom. As Stacy and MacGregor (1999) point out, students already have their own ideas about the uses of letters and symbols in their world, and their prior experiences often hinder understanding of mathematical language and notation. Kirshner and Awtry (2004) give evidence that students working with algebraic expressions often respond spontaneously to familiarity with visual notational patterns when making decisions instead of relying on mathematical rules. Students often do not reason about an overall goal or the concepts involved in a problem, but instead look for an implied procedure inherent in the symbols.

In the context of this study, the term *symbol* refers specifically to mathematical symbols. These include letters, numbers, equal signs, plus and minus signs, parentheses, square root signs, etc. (Arcavi, 1994). *Symbolic or algebraic representations* involve such symbols and manipulations of symbols. Arcavi (1994) and others have identified the underlying understanding of algebraic symbols and their uses as *symbol sense*, which Arcavi explains as "a quick or accurate appreciation, understanding, or instinct regarding symbols" (p. 31) that is involved at all stages of mathematical problem solving. Working fluently with symbols in mathematics requires developing strong symbol sense. Arcavi does not attempt to formally define symbol sense, claiming that to do so is difficult because it interacts with other senses such as number sense or function sense, but instead provides an extensive list of examples of what it might mean to have symbol sense. Arcavi suggests that many students fail to see algebra and it's symbols as a tool for understanding, communicating, and making connections, and he sees development of symbol sense as a necessary component of sense-making in general in mathematics. It is a tool that allows students to read into the meaning of a problem and to check the reasonableness of symbolic expressions.

Framework

In order to look in depth at students' problem solving activities and understanding of the symbolic language of mathematics, two conceptual frameworks are used as a lens for analysis of the data. The first is Simon, Tzur, Heinz, and Kinzel's (2004) Reflection on Activity-Effect Relationship framework (AER). This framework, which is an expansion of Piaget's (2001) notion of reflective abstraction, can be used in several ways: a) to help identify the goals and activities used in problem solving, b) to explain the process by which changes in conceptual structures take place, and c) to inform teaching to guide the design of situations to foster specific conceptual changes. For this study, only the first objective will be used. It is the researcher's intention to explore the cycle of goal setting, activity choices, and effects that students progress through on specific problems and to examine how symbols might directly influence this process.

Clarification of some of the terminology in this framework may be necessary for further understanding the use of the framework in analysis: The use of the word *conceptions,* which is equivalent to use of the terms schema or conceptual entities, refers to a learner's way of knowing (Tzur & Simon, 2004). Conceptions are explicitly goal-directed and are constructed when a person perceives or recognizes a certain situation, performs an action or activity associated with that situation, and comes up with a desired or expected result. The term *activity* refers to the mental process, or set of mental processes, that generate mental or observable actions; *goal* refers to a desired state, established by the learner, to which the learner refers for focusing attention and evaluating progress in an activity; and *effect* refers to the experiences that learners consider as outcomes of their activity. It is important to keep in mind that the learner's goals may not be the same as a teacher's goals for the same task, and that effects are structured and constrained by the goals and prior conceptions that the learner brings to the situation (Simon et al, 2004; Tzur & Simon, 2004).

The reflection on AER framework is useful for explaining how learners develop new mathematical conceptions beyond those already available to them. It begins with a goal-directed mental activity, where the learner continually monitors the effects and results of the activity. The learner creates mental records of the relationships between each execution of the activity and the effect produced. By reflecting on these records and looking for patterns between the activities and their effects, the learner abstracts a new activity-effect relationship, which is the basis for a more advanced conception (Simon et al., 2004; Tzur & Simon, 2004).

The second conceptual framework used in this study involves a subset of Arcavi's (1994) symbol sense constructed by Pierce and Stacey (2001, 2004) called Algebraic Insight. It is a lens that can be used to identify specific instances of symbol sense, particularly at the solution stage of problem solving. The authors divide algebraic insight into two parts: a) *algebraic expectation* is the insight needed for working within a symbolic expression, and b) *linking representations* is the insight needed to make connections between symbolic and graphic forms or symbolic and numeric forms. Table 1 further illustrates the characteristics of both the AER and Algebraic Insight frameworks. Incorporating the instances of algebraic insight is important because the stages in AER are often mental processes that are impossible to see directly. The researcher

Table 1. Reflection on AER and Algebraic Insight Frameworks

Progress in AER	**Instances of Algebraic Insight**
Setting a Goal	Identify Objects
Selecting an Activity	Identify Forms
Carrying Out the Activity	Identify Key Features
Reflecting on Records	Identify Dominant Term
Identifying Patterns	Know Properties of Operations
Abstracting an AER	Know Meaning of Symbols
	Know Order of Operations
	Linking Form
	Linking Key Features

cannot know what the students are thinking or understanding, but can try to recognize instances of symbol sense from their actions and shared thoughts and use this information to further understand their goals, activity choices, and reflections.

Methods

To address the research question for this study, a qualitative case study was conducted using six college students who were enrolled in summer sections of a pre-calculus course. Each student participant represents one case. In this paper, only analysis of the case study for Jill, a female sophomore student, will be described.

Data collection

The participants took part in a series of different interviews with the researcher over the course of five weeks. In an initial interview, the researcher met individually with participants and asked questions to gage past experiences and attitudes toward mathematics and observed the students working on four algebraic tasks. Students were encouraged to talk out loud to explain their thinking on the tasks, and an interview protocol was followed to ask students about their problem solving goals, expectations, strategies, and reflections.

Three additional meetings involved students working in pairs on their course homework, which was completed using an online homework tool called Webassign (North Carolina State University, 1997). Homework questions came from students' textbooks, but only answers were submitted online. Students received immediate feedback for their answers, and had ten attempts to correctly answer each problem. The homework sessions took place in a research classroom equipped with 18 cameras in the ceiling. The researcher recorded and observed students' work from a video console in an adjoining room, and tried to have minimal interference in the sessions. This was in following with research that has identified that students may work differently if they are being interviewed then they would normally work on their own (Berry, Graham, & Smith, 2005).

The final two interviews were individual meetings following the completion of two course tests. Students took tests in the classroom, and within two days, met with the researcher to discuss their work. Several problems from the homework that were similar to problems on the tests were discussed in detail. The goal was to have students identify and clarify differences in their approaches to the problems during the homework and test situations. To help students recall their approaches on the homework, the researcher showed and discussed video clips of the homework sessions, asking the participants to explain their goals and activities on the tests and to compare these to their work on the homework.

In each interview, a graphing calculator was available for use if needed. Each interview was videotaped, and any work done on the graphing calculator was digitally recorded on a computer using Windows Movie-Maker software and a TI-Presenter. Written transcripts of the interview and calculator videos were analyzed by the researcher.

Analysis

To analyze the data for Jill, transcripts of her work on problems in all three interview settings were coded by looking for instances of goal setting and activity choices on the problems. To help connect Jill's goals and activities with her symbol sense, additional coding was done to identify instances of algebraic insight, or to identify places where she lacked the algebraic insight necessary to work on the task correctly. Throughout the six interviews, Jill worked on a total of 29 problems, 21 of which have been coded and evaluated in detail. Several of these will be used as examples in the findings below.

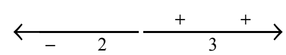
Findings

With this study, the researcher hopes to provide some answer to the question: In what ways do the symbols in a problem influence college students' goals, activities, and organization of results when solving mathematical problems? In this paper, the findings for Jill's case study will be organized around the three components of this research question.

How do symbols influence goals?

Jill's goals as she worked through many of the mathematical tasks were greatly influenced by the symbolic structure of the problem. Symbolic structure refers to any symbol or group of symbols that brings to mind some experience and thus some goal and activity for a step in the problem. For example, Jill worked on the task shown in Table 2 on one of the course tests. Immediately following the test she met with the researcher to discuss her work. She shared that this problem brought to mind steps that she had memorized, and that her initial goal, given an inequality, was to get zero on one side. However, this goal was altered by the rational structure of the problem. She changed her goal to first get rid of the fraction because she said, "it looked funny to her." Her class notes show these steps next to examples of polynomial inequalities, so it is possible that the different symbolic structure had to be adjusted so that she could follow her procedural understanding of the problem. The first row of work in Table 2 shows that she cross-multiplied to change the structure to a polynomial inequality, and her goal then became to find the roots. To accomplish this, she tried different activities; she first expanded on both sides, moved everything to one side and reached the line $x+22>0$, but she did not complete the problem with this method. Instead, she reflected on the effect of her activity and the progress that it was making toward her goal, and for her, this form was not conducive for meeting her goal. Her anticipation, based on her previous experiences and recollection of familiar symbolic structures,

Table 2. Influence of symbolic structure on Jill's goals

Task: Solve for x: $\frac{5(x-2)}{(x-3)} \geq 4$			
STRUCTURE	**GOAL**	**ACTIVITY**	**WORK**
Rational Inequality	Get zero on one side	Get rid of the fraction	$5(x-2) \geq 4(x-3)$
Polynomial Inequality	Find the roots	Expand and collect terms on one side	$5x-10-4x+12 \geq 0$ $x-22 \geq 0$
Linear Inequality	Find the roots from what is inside	Keep parenthesis and collect terms on one side	$5(x-2)-4(x-3) \geq 0$ $x = 2,3$
x- values	Determine the sign in the intervals	Set up a sign chart and plug test points into left-hand expression	

was that she would find the roots by looking at what was in parentheses. Thus, she looked back to the previous step where the form did include parentheses, and adjusted her activity and tried again. This time she found the roots by pulling them out of the parentheses.

How do symbols influence activities?

Another finding related to both the example above and the one in Table 3 is that Jill's activity choices were guided by what she was seeing in the symbols, and these activities often led her to succeed in reaching her personal goals despite a lack of algebraic insight. In the problem above, the additive structure of the problem did not hinder her next goal of determining the sign of the function before and after each solution. The existence of the parentheses was familiar enough for her to consider this activity as making positive progress towards her goal.

During the interview, she was shown a video clip of her work on the homework task: Solve the inequality $(x-1)(x+3)(x+5)<0$, and asked to make comparisons. She said, "The difference with that one was it was already factored out whereas this one [on the test] we had to go through the steps and factor it out and then find the zeros." In other words, she saw the expression in row 3 of Table 2 as a factored form for the problem. She did not correctly identify the form or pay attention to the properties of operations, both of which are important aspects of algebraic insight. Instead she allowed the parentheses symbols to serve as a cue for her actions, and even adjusted the language for rules of mathematics to justify her work, saying things like "a positive minus a positive is always positive" to explain how she found the signs on her sign chart.

A second example of how Jill's activities were guided by her familiarity and experiences with the symbols instead of algebraic insight comes from the example in Table 3. This task was one that Jill had on both the homework and test, and on which she made the same mistakes in both settings. She was given the graph of a quadratic function and asked to find the equation for the graph. Jill had done similar problems in class, and so chose to use an alternate form of a quadratic equation and filled in the given values for the vertex (h,k) and another (x,y) point on the graph. The result of her work was the equation $5(x+2)^2-3$. However, she did not see this as being in the normal form that the teacher might want, so she wanted to get it into the form "y=something x^2 plus something x." Her first activity choice was a familiar mistake of distributing the squared term to inside the parenthesis, but she stopped because the resulting form did not meet her stated goal. Her second attempt was to do the correct expansion, but she then demonstrated a lack of algebraic insight with knowing order of operations by subtracting 3 from 5 in the expression $5(x^2+4x+2)-3$. She sees in the structure two like terms and is combining them without attending to rules of mathematics. She then viewed the video of her work on the

Table 3. Influence of symbols on Jill's activities.

Task: Find the quadratic function whose graph is shown			
STRUCTURE	**GOAL**	**ACTIVITY**	**WORK**
Quadratic Graph	Get quadratic function of the form $y=a(x-h)^2+k$	Substitute values for parameters and variables and solve for a	$(-2,-3)=(h,k)$ $(3,2)=(x,y)$ $2=a(3+2)^2-3$ $2=25a-3, \quad a=5$ $5(x+2)^2-3$
Alternate form of quadratic	Get in the form "y=something x^2 + something x…"	Distribute squared term	$5(x^2+4)-3$
Quadratic with no "x" term	Get in the form "y=something x^2 + something x…"	Do binomial expansion and then add constant terms	$5(x^2-4x-4)-3$ $=2(x^2-4x-4)$ $=2x^2+8x+8$

homework where she had made a similar mistake, and discussed what she was thinking when doing the test activity:

> J – I was just doing basic subtraction, 5-3. Oh, that should have been plus 2.
>
> I – What should have been plus 2?
>
> J – Instead of it being multiplied by 2 I think it…now I would probably put it as plus 2, like I did here [on the homework].

In this last line, she is now seeing something different in what she did on the homework problem, and decides that when she subtracted 5-3 on the test, the result should have been added to the quadratic instead to get $(x^2+4x+4)+2$. Jill is focused on manipulations that she sees as inherent in the symbolic structure of the problem and is not attending to arithmetic rules. It is certain that she knows these rules because as soon as the researcher questions the validity of this manipulation she decides that this is not allowed and that she should have distributed first and then added. Jill knows the order of operations in general, but chooses the same incorrect action on both the homework and test based on what she is seeing in the symbols.

How do symbols influence the organization of results?

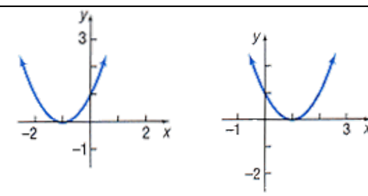
There were instances will Jill can be seen as abstracting relationships and building (mis)conceptions based on how she related to the final answer. There was a need for solutions to look familiar for Jill, which is not a surprising finding, but does demonstrate that she had some expectation for what the end result of a problem was going to look like. When the symbolic result matched her expectation, such as in Table 1 with the presence of parentheses, or in Table 2 with the appropriate quadratic form, Jill did not question her activities. She reflected on her activities only in the sense that she made sure the structures that she created matched her expected form.

In the example in Table 4, Jill and her homework partner abstracted an incorrect conception of transformations of functions. The first two homework tasks asked them to find the graph of $f(x) = x^2 - 1$ and $f(x) = -x^2 - 1$, which they easily accomplished mentally by recalling rules for transformations of functions. This homework accompanied a lesson on quadratic functions, so for tasks 3 and 4 in the table, the teacher's goal would have been for the students to find the vertex or intercepts to identify the graph. Jill's work on the first two tasks, however, caused her to try to apply the same ideas to tasks 3 and 4. When she could not do so, Jill decided to just graph the functions on the calculator, the results of which can be seen in the table. Jill's goal was to identify what effect the $+2x$ or $-2x$ terms had on transforming the graph. The activity that she used to help was to use the graphing calculator, and the effect of this activity led Jill to abstract a relationship between adding or subtracting an x term in the quadratic form and a horizontal shift of the graph. She and her partner both concluded the task assuming that “the x term in the middle would shift left and right.”

Table 4. Influence of symbols on Jill’s abstracted conceptions

Task: Match graphs with the given equations

1. $f(x) = x^2 - 1$
2. $f(x) = -x^2 - 1$
3. $f(x) = x^2 + 2x + 1$
4. $f(x) = x^2 - 2x + 1$

STRUCTURE	**GOAL**	**ACTIVITY**	**WORK**
Quadratic function of form $ax^2 + b$	Identify graph	Use transformation rules	Mental visualization of result. Successfully matches graphs
Quadratic function of form $ax^2 + bx + c$	Apply transformation rules to new form	Use graphing calculator	

Conclusions and Implications

It is evident in Jill’s work that there were elements in the symbolic structure of the problems that directed her toward prior experiences and influenced her goals. The activities that she chose were intended to make progress toward that goal, and the effect of each activity was reflected upon in the sense that she had to identify the form of the problem again and reevaluate the goal to chose the next activity. One implication for teaching that can be drawn from this evidence is that inconsistencies exist between teacher’s goals and students’ goals in problem solving. Teachers need to be attentive to this in order to help guide students’ goal making and activity choices.

Kirshner (1989) suggests that, for some students, visual cues serve as the dominant incentive for syntactic decisions instead of algebra rules. This idea was especially evident with Jill when she confused or ignored rules of algebra to meet her goals. A detailed look into the ways in which students interpret mathematical symbols can hopefully be useful in identifying ways to strengthen students' understanding of symbols and to improve their mathematical capabilities. It may also improve college teachers' awareness of the networks of understandings that students have developed about mathematical symbols and they ways in which they learn to "see" the mathematics.

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