When Students Prove Statements of the Form $(p \rightarrow q) \Rightarrow (r \rightarrow s)$

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Abstract

We explore ways that university students handle proving statements that have the overall structure of a conditional implies a conditional, i.e., $(p \rightarrow q) \Rightarrow (r \rightarrow s)$. We structure our analysis using the theory of conceptual blending. This allows us to see how students combine ideas from the statements of and pictures representing two conditionals as they work to show that one implies the other. Students recruited a proving frame from their experience, which was insufficient for the complexities of the statement. This led them to start with the totality of $(p \rightarrow q)$ in ways that were problematic.

Introduction

The purpose of this paper is to illustrate the power of the theory of conceptual blending to clarify issues that students have in proving statements that have the overall structure of a conditional implies a conditional, i.e., $(p \rightarrow q) \Rightarrow (r \rightarrow s)$. This logical structure occurs often in statements to be proven at the university level. For example, since the definition of A is a subset of B ($A \subseteq B$) is a conditional statement ($x \in A$ implies $x \in B$), then a simple set theory statement such as "If $A \subseteq B$, then $A \cup B \subseteq B$," has this logical form. Another instance of this logical structure occurs when proving the induction step in a proof by induction, i.e., if a conditional statement is true for *k* terms, the same conditional statement is true for *k*+1 terms. Since the definition of a function f to be increasing (if $x_1 < x_2$, then $f(x_1) < f(x_2)$) and the definition of a function f to be one-to-one (if $f(x_1) = f(x_2)$, then $x_1 = x_2$) are conditional statements, a calculus statement such as "if a function f is increasing, then f is one-to-one" has also this logical structure. In addition, this logical structure can be found in real analysis contexts such as proving "every convergent sequence is Cauchy," i.e., if for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $|a_n - L| < \varepsilon$, then for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n, m > N$, $|a_n - a_m| < \varepsilon$.

We will examine the situation of students working to prove one direction of the equivalence of two forms of the parallel postulate of Euclidean geometry. The research literature indicates that undergraduate students struggle with proof writing (e.g., Weber, 2001), understanding the logical structure of the mathematical statements (Selden & Selden, 1995; Dubinsky & Yiparaki, 2000), and completing induction proofs (Brown, 2003; Harel, 2001). This paper adds to that literature by describing students' proving using a logical structure that is common in mathematical problems at this level, but which has not been directly addressed in previous work.

Methods and Setting

The data for this study was collected as part of a semester long teaching experiment (Cobb, 2000) in an upper division geometry course. Data consisted of videotape recordings of each class session, and copies of students' written work. During small group discussion there were two cameras, each focused on a different small group. Group One consisted of students we call Nate, Stacey, Andrea and Paul. Group Two consisted of students we call Valerie, Emily and Alice. The curriculum consisted of a series of activities in which students would need to define, conjecture, and prove results in geometry on the plane and the sphere (Henderson, 2001). This study focuses on one day late in the semester in which students were asked to prove either Euclid's Fifth Postulate (EFP) implies Playfair's Parallel Postulate (PPP) or PPP imples EFP.

Henderson (2001) states EFP as, "If a straight line crossing two straight lines makes the interior

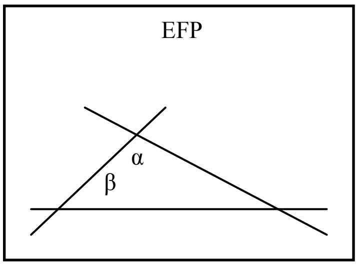

Figure 1: Euclid's Fifth Postulate

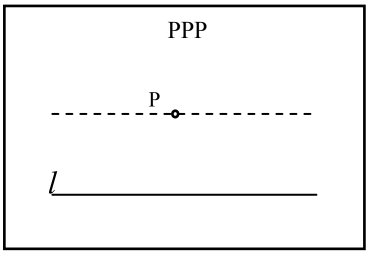

Figure 2: Playfair's Parallel Postulate

angles on the same side less than two right angles, the two straight lines, if extended indefinitely, meet on that side on which are the angles less than the two right angles," and PPP as, "For every line and every point not on the line there is a unique line through the point that does not intersect the original line." The instructor told the students that the two postulates are equivalent and gave them the option to "*use*" EFP in order to prove PPP or vice versa. In her introduction of the task the instructor drew the two figures shown while explaining each of the postulates. The teacher's initial drawing showed only the part of the statement that was given. For example for the PPP picture initially she just drew the bottom line, *l*, and a point, P, not on that line. However, when she explained the conclusion of each statement she completed the picture and these completed pictures were left on the board for students to reference. In addition, as the teacher explained the statements she wrote the terms "unique" and "exist" to the right of the PPP picture and the phrases "$\alpha + \beta < 180$" and "$\alpha + \beta < \pi$" underneath the EFP picture. A second visual reference available to students was found in the textbook. The two pictures in the book for these two statements were also completed pictures similar to what the teacher had drawn.

Theoretical Framework

Fauconnier and Turner's (2002) theory of Conceptual Blending posits the existence of a subconscious process that entails the blending of diverse scenarios or mental spaces (*inputs*) to form a new stable conceptual model for use in reasoning. A mental space consists of an array of elements and their relationships to one another. It is activated as a single unit. In conceptual blending, two (or more) such mental spaces are activated and crucial elements of each are integrated and *mapped* to a third space to form a blended space. As part of completing the blend, a conceptual frame may be recruited to help organize the information in the blend (Coulson & Oakley, 2001). Once the blend is *complete* it can be manipulated to make inferences or answer questions. This manipulation is referred to as *running the blend*. The blended concept is treated as a simulation that can be run imaginatively according to principles and properties that the input spaces bring to the blend. In the discussion that follows we will see how the theory of conceptual blending can be applied to students proving a statement of the form a conditional implies a conditional. Specifically we illustrate students mapping from input spaces to the blend, completing the blend, running the blend and applying a conceptual frame to a blended space.

Results and Analysis

In this section we illustrate how the theory of conceptual blending may be applied to illuminate aspects of student reasoning as they work to construct a proof. At the beginning of this analysis we describe how students put together ideas from the statements of each of the postulates and the pictures drawn on the board by the instructor or printed in the textbook. This initial process is described in the section, "Mapping to the blend of EFP and PPP". Once the students have seen how to fit some of the notions from the two postulates together (and even before that process has been fully carried out) students engage in a process of trying to bring in a proving frame appropriate for structuring their work on this task. We describe this in the section "Completing the Blend". The third section, "Running the Blend", discusses how the students manipulated the combined ideas of EFP and PPP in the blended space to realize an important set of relationships that they recognized as a key idea for the proof. In addition, the students work to structure a proof by using their key idea and the simple proving frame. In this process, they project back to the input spaces, referring to the statement of EFP in ways that seem to imply that they have blended the premise and the conclusion of EFP.

Mapping to the Blend of EFP and PPP

As the students in Group One began to work on the proof, they looked at the pictures and statements of the postulates as given in their textbook. Because the two postulates were described on two sequential pages in the book, EFP on the front side of the page and PPP on the back of that same page, on the video we see the students flipping back and forth between the two pages as they began to think about which direction might be easier to prove and how to prove it. The students seemed to be using the two figures from the book, almost identical to the ones drawn on the board (see Figures 1 and 2), as input spaces to form a blend that related the two postulates. Stacey's comments in the following exchange indicated how this blending occurred:

Stacey: *Because no matter what we can put any P out there at our point of intersection [reaches forward with her arms extended to make a point with her finger tips].*

Nate: *That's a good point.*

Stacey: *And then we know that, that it is unique [indicates a horizontal line with her pencil], that it is not going to come back and intersect somehow [points to something on Andrea's paper].*

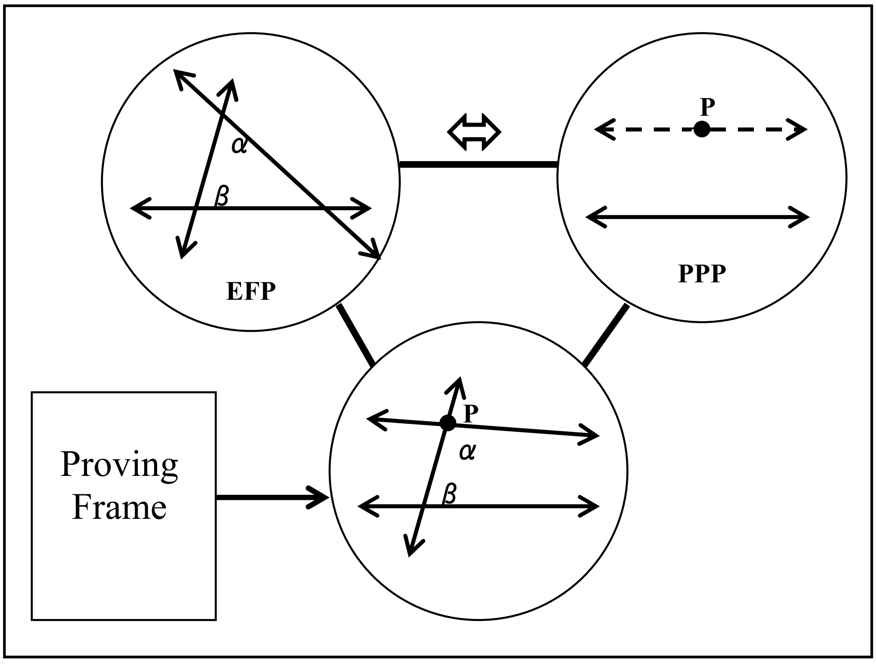

Figure 3: Blending of EFP and PPP

The two figures in the text seemed to function as inputs for a blended space as shown in Figure 3. The bottom line of each of the input spaces was mapped into the bottom line of the blended space. The transversal from the EFP input space was mapped into the blended space. The top line from each space was mapped into a line in the blended space that the students thought of as being mobile in that it could rotate with a pivot at its intersection with the transversal. Note that the dottedness of the top line in the PPP input space was not transferred to the blended space. The point P from the PPP space was mapped into the blended space onto the same place as the intersection of rotation.

Completing the Blend

Once an initial mapping occurs from the input spaces to the blended space, the next step in the blending process is completing the blend. Completing the blend consists of adding connecting ideas that are needed to make sense of the blend. This may include structuring the blend with a conceptual frame.

Initially, most of the students recruited what we call the Simple Proving Frame (SPF). In the Simple Proving Frame, there is a given statement (premise), then a series of implications, then the conclusion (see Figure 4). There is nothing inherently wrong with this proving frame or trying to apply it to a conditional implies a conditional. However, unless a student has particular theorems to work with that allow a direct proof from $(p \rightarrow q)$ to $(r \rightarrow s)$, then the Simple Proving Frame may be inadequate.

Generic Simple Proving Frame	Case 1	Case 2
Given [ ] … Series of Implications … Then [ ]	For the case of $(p \rightarrow q)$ Given p … Series of Implications … Then q	For the case of $(p \rightarrow q) \rightarrow (r \rightarrow s)$ Given $(p \rightarrow q)$ … Series of Implications … Then $(r \rightarrow s)$

Figure 4: The simple proving frame (SPF)

As students tried to apply the simple proving frame to EFP implies PPP, they put EFP in the place of what is given and PPP in the place of the conclusion. For example, as the students in Group One were flipping back and forth between pages, they made statements that indicated that they were framing their initial thoughts about the proof in terms of starting with one postulate and trying to "get from one to the other." Another instance of recruiting the Simple Proving Frame occurred in Group Two, especially for Alice and Emily. Alice referred to the expression "EFP → PPP" that she had written in her notebook, in her use of the simple proving frame below.

Alice: So Okay, so. So when it says, prove EFP implies PPP or PPP implies EFP that means whichever, like if we're doing from this one [points to the letters EFP] to this one [points to the letters PPP], we get to say this [EFP] is true so we have to prove this [PPP].

…

Valerie: Okay, so let's think. So we had a line and we got a point.

Emily: How do you get from one to the other? ...

Valerie: I don't know.

Emily: 'Cause basically if you use one to prove the other, you have to get from one…

Alice and Emily: to the other.

Valerie seemed to frame the problem differently than the Simple Proving Frame. In the place of what is given, she started with a line and a point which is the premise of PPP. If she had continued this line of reasoning, it could lead to structuring the problem by what we call the "Conditional Implies Conditional Frame" (CICF). When using the CICF to prove statements of the form $(p \to q) \Rightarrow (r \to s)$, one starts with *r* and uses a series of implications including $p \to q$, to reach the conclusion, *s* (see Figure 5). As students try to apply the CICF to

EFP implies PPP, they put "there is a line and a point" as what is given and "there is a unique line…" in the place of the conclusion. EFP is then used as one of the implications to get the conclusion of PPP. The CICF seems to become more adequate when student work with a direct proof from $(p \rightarrow q)$ to $(r \rightarrow s)$. The students in both groups eventually used the CICF to prove this statement; however, in this short paper we are illustrating only the use of blending in the opening of their discussions of this problem.

Generic Conditional Implies Conditional Frame	For the case of $(p \rightarrow q) \rightarrow (r \rightarrow s)$
Given ☐ … Use ☐ … Thus ☐	Given r … Then p Since p and $(p \rightarrow q)$ Then q … Thus s

Figure 5: The conditional implies conditional frame

Running the Blend of EFP and PPP

In the section "Mapping to the Blend," Stacy in Group One suggested a blend by putting the point P at the intersection of two lines in the EFP setup (see Figure 3 above). The blended picture then consisted of a bottom line and two other lines with the intersection point P. Immediately after Stacey's suggestion, Paul proposed an idea for thinking about the relationship between the two postulates in the blended space. In particular, Paul described three cases where the sum of the interior angles α and β is less than, greater than, or equal to π (see Figure 6).

Paul: Well, if you assume the first one [EFP] would there be three cases that $\alpha + \beta < \pi$, $\alpha + \beta = \pi$, *or* $\alpha + \beta > \pi$ *and then the uniqueness part of it would be proved by the* $\alpha + \beta = \pi$ *and in that case they wouldn't meet. Well, that wouldn't be unique either because* $\alpha + \beta > \pi$ *wouldn't meet either.*

Andrea: Well, if $\alpha + \beta > \pi$, *then they would intersect on the other side.*

Nate: Then you can look on the other side.

Paul: *The other side, yeah.*

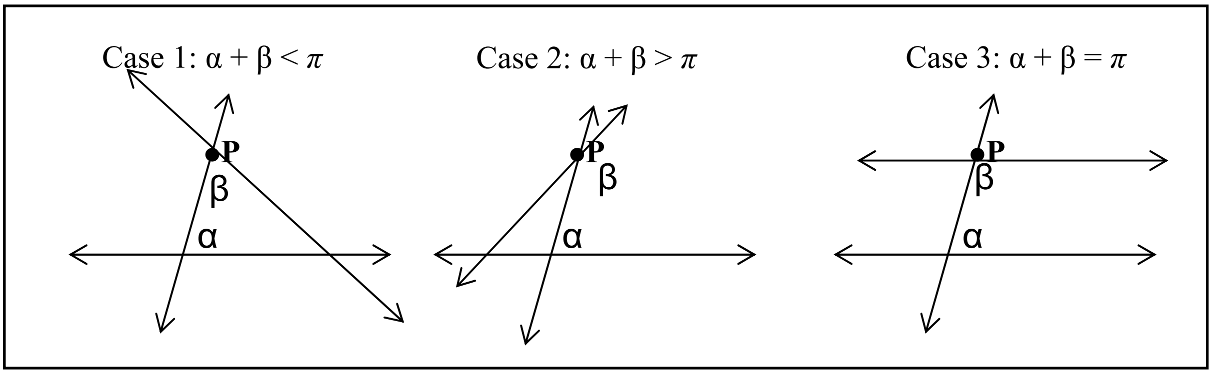

Figure 6: Paul's three cases

We would say Paul's statement comes from running the blend. Running the blend consists of imagining all the different possible locations of that line and the relationship between those positions and the sum of the angles α and β. In this way, the students' running of the blend led them to the key idea (Raman, 2003) of having these three cases for the proof.

As described in the "Completing the Blend" section (see Figure 4), the students have been structuring their proof using the Simple Proving Frame (SPF). Immediately following the transcript above, Stacey contributed to the discussion by bringing the statement of EFP (the given part of the SPF) to bear on the problem. However, she did so in a way that does not retain the implication structure of EFP.

Stacey: If we are assuming the whole thing though, we are assuming they are less than π.

Paul: *Okay, well I'm confused.*

Nate: Well, we are saying that if they equal π, then we don't know anything. Cause we know something if they are greater than π.

Stacey: Yeah.

Nate: Cause we can just flip it over.

Stacey: Yeah, but do we even care? Cause it's all assumed. This whole thing is. We are assuming $\alpha + \beta < \pi$. We don't care if it's equal to π.

Paul: Yeah, that's true.

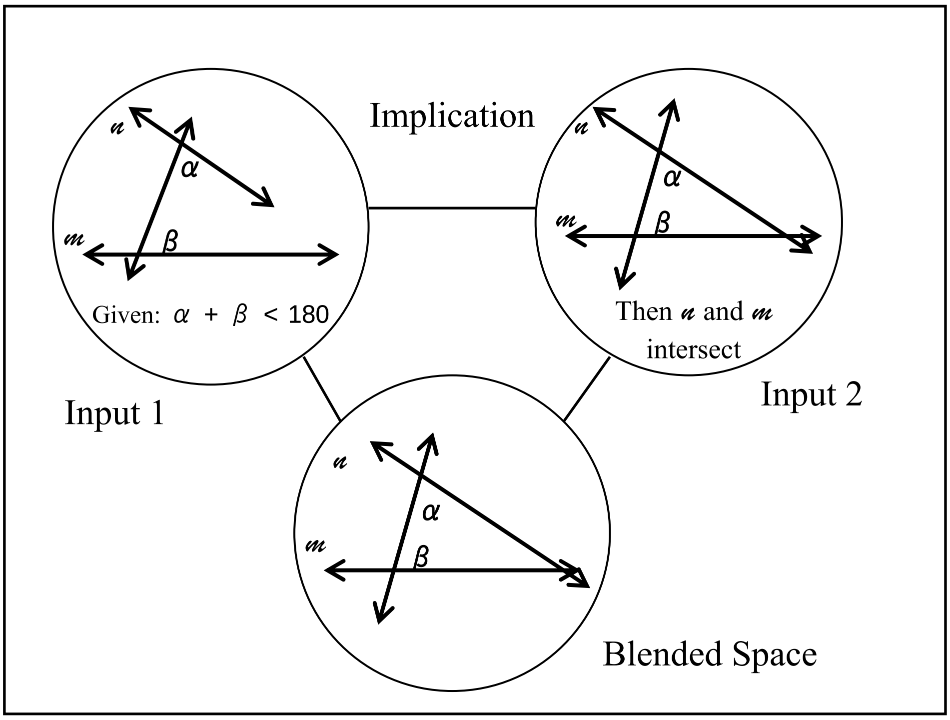

Figure 7: Blending of the premise and conclusion of EFP

Paul's agreement with Stacey seems to indicate that the group in this moment was considering EFP as being given, since it was the premise of the SPF that they were using. In other words, they assumed all ("the whole thing") of EFP was given, *both* its premise and conclusion. We interpret the students to have blended the premise and conclusion of EFP (see Figure 7) prior to mapping EFP into the blend of EFP and PPP. The blending of the premise and conclusion of EFP may have been influenced by the sketch for EFP given by the text and drawn on the board by the instructor. In both cases the figure is left for viewing with both the premise and conclusion information combined in the same figure and without the implication structure being clearly indicated.

In Group Two the students also seem to bring EFP into the blended space and the structure of the SPF in a way that does not maintain the integrity of the implication structure. During a twenty minute period these students made at least five statements indicating that they could use the inverse or converse of EFP. For instance, Emily talked about the converse of EFP: "So we know that $\alpha + \beta$ is less than 180 by this proof if they intersect." Alice on the other hand talked about the inverse of EFP: "…the actual statement says if the [sum of the interior angles] is less than 180, it will intersect on that side. If it's greater [than 180], it's not going to intersect on that side." Another instance was found when Emily applied the inverse of EFP while she was trying to structure her proof: "α' + β' = 180. So on either side you cannot make it intersect."

During this twenty minute discussion, Group Two seems to be running the blend by considering different scenarios for α, β and the intersection of the line. Towards the end of the twenty minutes, the group, led by Valerie, suggested three cases identical to those discussed in Group One and shown in Figure 6. As the students continued to work on the proof, each group eventually began to structure their proofs using the Conditional Implies Conditional Frame (CICF). That discussion and the discussion of how the students argued for uniqueness is the subject of another paper.

Discussion

In this paper we describe students' initial proving ideas in working on a conditional implies conditional proof. Although students later in the class period came to see ways to structure their proof using the more complex CICF, initially students structured their proof using the SPF. In this paper we use the language of conceptual blending to describe both the students' use of a proving frame and the ways in which students seem to combine ideas from the two postulates in their initial reasoning about the proof.

Students blended EFP and PPP in ways that were powerful for creating a key idea for the proof. Running the blend imaginatively allowed them to see relationships between the components of the two postulates that could be useful for creating the desired proof. However, students initially recruited a proving frame to structure the blend that hampered their efforts. This occurred in two ways. First, the students did not have the necessary theorems to complete a proof of this conditional implies conditional statement using a Simple Proving Frame. So, their heavy reliance on this proof frame in the initial discussions slowed their efforts. The use of a CICF could have, and eventually did, lead to a workable proof for these students. Second, the use of the SPF in the blending of EFP and PPP compounded problems from the compression of ideas that occurs in blending. In particular, we see that when the students treated EFP as “all” being given (as in the SPF of Figure 4), they treated EFP as a collection of parts (sum of angles, intersecting line or not) without maintaining the appropriate implication structure between these parts. In this way they blended the premise and the conclusion of EFP in a way that was problematic.

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