Exploring Epistemological Obstacles to the Development of Mathematics Induction

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This paper examines findings from two teaching experiments. The first involved undergraduate mathematics and science majors. The second is ongoing and involves advanced 6th grade students. The purpose of the paper is to explore similarities and differences in the students' approaches to mathematical induction appropriate tasks and then to use the multi-age comparison to explore a potential epistemological obstacle to mathematical induction.

Introduction

Research on undergraduates' understandings of proof by mathematical induction has shown that undergraduates experience difficulties with this method of proof (Robert & Schwarzenberger, 1991, Dubinsky, 1989; Movshovitz-Hadar, 1993a, 199b). Harel and Sowder (1998) and Brown (2003) have questioned the extent to which these difficulties are due to traditional instructional approaches that tend to hastily introduce the definition and that do not facilitate the development of mathematical induction as a means to solve a class of problems. In an effort to distinguish between those difficulties that are primarily didactical in nature and those that are primarily epistemological, this paper examines findings from two teaching experiments. The first involved undergraduate mathematics and science majors. The second is ongoing and involves advanced 6th grade students. The purpose of the paper is to explore similarities in the students' approaches to PMI-appropriate tasks[1] and then to use the multi-age comparison to investigate a potential epistemological obstacle to mathematical induction.

Theoretical Perspective

This work is informed by the *Theory of Didactical Situations* (Brousseau, 1997), which views students' errors and, more generally, their difficulties as being of particular

[1] The phrase "PMI-appropriate tasks" is used to denote tasks for which mathematical induction is a viable proof technique.

importance. This position on students' errors and difficulties stems in part from what Balacheff (1990) refers to as the constructivist hypothesis, which is the hypothesis that students' mathematics, even their errors, arises from students adapting their ways of knowing to a *milieu*[2]. It also stems from an idea central to the Theory of Didactical Situations, namely, the idea of an *obstacle*, as described by Brousseau:

> … errors and failures do not have the simplified role that we would like them to play. Errors are not only the effect of ignorance, of uncertainty, of chance … but the effect of a previous piece of knowledge which was interesting and successful, but which now is revealed as false or simply unadapted. Errors of this type are not erratic and unexpected, they constitute obstacles
>
> (Brousseau, 1997, p. 82)

An obstacle is a way of knowing that functions productively in some settings, while supporting the manifestation of errors in others. This productivity or success is said to entrench these ways of knowing, and therefore, make them resistant to change -- hence, the name "obstacle."

Within the theory, obstacles can take three forms: *ontogenic*, *didactical*, and *epistemological*. *Ontogenic* obstacles are developmental obstacles, that is, they are obstacles related to the stages of mental development of the child. *Didactical* obstacles are those that arise as a result of instructional choices and therefore, are avoidable through the development of alternative instructional approaches (what Brousseau refers to as didactical engineering). *Epistemological* obstacles, in contrast, are those that arise regardless of the instructional approach, for their origin is the concept itself; in other words, "to overcome the obstacle is part of the construction of the meaning" of the

[2] *Milieu* in this context refers not only to the environment but also the expectations of how one is to function in that environment, what other might refer to as socio-cultural norms and practices.

concept (Balacheff, 1990). For the purpose of this paper, we will be concerned with obstacles of the latter form, namely, epistemological obstacles.

The phrase *epistemological obstacle* often is mistaken for something "bad" -- a snag, hindrance, or stumbling block -- as opposed to how it is intended within the theory. Namely, it is intended to denote a way of knowing that functions productively in some settings and is essential to the development of the concept. Epistemological obstacles can be construed as faulty ways of thinking but such a perspective ignores their importance, their developmental necessity, and their productivity in specific settings.

For example, one can argue that students' production of solutions to PMI-appropriate tasks that do not include a base case are a result of ways of knowing mathematical induction that arise from didactical choices and therefore, are indicative of a didactical obstacle. The claim stems from the results of teaching experiments (Brown, 2003) in which these particular ways of knowing were not observed and in which students quickly identified the error in false proofs, whose flaw related to the omission of a base case. In other words, the irreproducibility of the results of Dubinsky (1989), who reported on students' omission of and difficulty with the base case, suggests that this way of knowing is a didactical, as opposed to epistemological, obstacle. On the other hand, conceiving of a limit as something that is never reached may be an epistemological obstacle (Sierpinska, 1987).[3] There are limits that you cannot "reach," for example, $\lim_{x \to \infty} e^{-x}$, and limits that you can, for example $\lim_{x \to 2} 3x - 1$. Making sense of the latter example, when one's way of knowing the concept of limit are rooted in the idea that a limit is something that is never reached, therefore, requires shifts in ones ways of

[3] I would like to thank Anna Sierpinska for the discussions at the RUME 2008 conference about epistemological obstacles to the notion of limit that are the basis for this portion of the paper.

knowing limits that were productive in some settings, potentially necessary, and are now producing errors. To progress, as argued by Sierpinska, “the student will have to rise above his convictions, to analyse from outside the means he had used to solve problems in order to formulate the hypotheses he had admitted tacitly so far, and become aware of the possibility of rival hypotheses” (Sierpinska, 1987, p. 374).

One can gather evidence of epistemological obstacles through historical analyses (Brousseau, 1997; Sierpinska, 1987). The reason for this being, modern notions of the concept in question must either have arose from the resolution and evolution of the epistemological obstacle, that is, the particular ways of knowing, or have arisen from some historical trajectory which avoided these ways of knowing. Thus, either the obstacle is unavoidable and it was encountered at some point in the history of the concept or the obstacle is avoidable.[4] The position taken in this paper is, in addition to historical analyses, researchers can further elaborate the nature of potential epistemological obstacles through the analysis of teaching experiments that modify instructional settings and approaches. This position aligns with Sierpinska’s, who has explored epistemological obstacles related to limits by modifying instructional settings (Sierpinska, 1987). This is not to say that by modifying instructional approaches (or settings) we do not create anew the same didactical obstacles but rather that by altering instructional approaches (or settings), we can attempt to reduce the likelihood of perpetuating these obstacles, while increasing the likelihood of their identification.

[4] One can argue that this is a false dichotomy, for an epistemological obstacle can be exaggerated by one’s didactical choices. Harel and Sowder have made a similar point (Harel & Sowder, 2005).

Background

The work described in this paper is ongoing. It began with a series of teaching experiments that employed alternative instructional approaches to mathematical induction. These teaching experiments (a total of 5) involved small cohorts of undergraduate mathematics and science majors between the years of 1999 and 2001. These teaching experiments were a reaction to two aspects of then current research on mathematical induction. First, research in this area tended to focus on the identification of student difficulties (see for example, Dubinsky, 1986,1989; Movshovitz-Hadar, 1993a, 1993b; Reid, 1992). Much of this research, however, was conducted post-instruction and therefore, students' difficulties were described apart from the particular curricular and pedagogical choices involved. Consequently, the students' understandings, or in this case, misunderstandings were portrayed as "a *phenomenon of the student*; that is, as being independent of the students' interactions with teachers, other students, curricula, classroom discussions, classroom materials and of the problems the student has solved (or failed to solve)" (Brown, 2003, p. 1). Second, research in this area had yet to produce a model of students' development of mathematical induction rooted in students' evolving conceptions of what is means to solve PMI-appropriate tasks.[5]

Since these experiments, models of the evolution of students' understanding of mathematical induction have been proposed (Harel, 2002; Brown, 2003; Harel & Brown, in press). In Brown (2003), in addition to identifying a model of the evolution of students' understanding of mathematical induction, potential epistemological obstacles to mathematical induction were explicitly identified. Building on this work, I report in this

[5] Dubinsky and Lewin (1986) had put forth a genetic decomposition of mathematical induction. This is, however, a decomposing of mathematical induction in terms of the relevant mathematical concepts not a developmental model of students' conceptions of what entails solving PMI-appropriate tasks.

paper on an ongoing teaching experiment involving mathematically advanced 6th grade students – students whose mathematical backgrounds differ significantly[6] from typical undergraduate students. The aim of this project is to analyze the students' responses to PMI-appropriate tasks, contrast them to the undergraduates' responses, and to use this analysis to further explore potential epistemological obstacles to mathematical induction.

Operationalizing the Notion of an Epistemological Obstacle

Epistemological obstacles are ways of knowing that function productively in some settings, while fostering errors in others (Brousseau, 1997; Balacheff, 1990). What is the form of such ways of knowing? Sierpinska argued that there is a "property of duality of epistemological obstacles" (1987, p. 5), that is, that epistemological obstacles can manifest themselves in terms of coupled ways of knowing, which are incompatible. Furthermore, Sierpinska demonstrated how obstacles can be characterized in terms of coexisting conceptual understandings and perspectives of mathematical knowledge. From this viewpoint, it is important when exploring epistemological obstacles to identify potentially incompatible ways of knowing, and they ways in which students' conceptual understandings relate to particular perspectives of mathematical knowledge.

The idea that ways of knowing function within a system of coexisting conceptual understandings and perspectives of mathematical knowledge aligns well with Harel's (1998) *Dual Assertion*; the idea that not only do students' *ways of thinking* affect the meanings students attribute to mathematical concepts (*ways of understanding*), but also that students' ways of understanding affect their ways of thinking. Thus Sierpinska's approach can be viewed as supporting the exploration of epistemological obstacles through the consideration of students' ways of thinking and ways of understanding. It is

[6] Consider, for example, a simple measure such as years of schooling in mathematics.

this approach that was taken in Brown (2003), and in the work described in this paper, when describing epistemological obstacles to mathematical induction.

Data Collection

Data collection for the teaching experiments with undergraduate mathematics and science students (UGS) occurred during a series of five teaching experiments conducted at two, large, urban state universities. All experiments were either audio or videotaped and occurred in the context of either a seminar offered within a mathematics department or one of four teaching experiments in the form of 6-week experimental courses for undergraduate students who were concurrently enrolled in the second semester of calculus. Between 3-8 students were enrolled in the seminar and courses. All participants were asked to complete a *Curricular History Questionnaire* and to respond to a *Pre-Experiment Assessment*. These documents were aimed at identifying students who had previously received instruction on mathematical induction through explicit questions and through tasks that would prompt use of mathematical induction, as indicated by prior pilot work with undergraduates. Enrollment in the experimental courses was limited to undergraduates whose responses to the *Curricular History Questionnaire* and to the *Pre-Experiment Assessment* indicated no prior exposure to mathematical induction. Data collected for these experiments includes audio or videotapes of all sessions, transcripts of each session, students' written work, and instructor field notes.

Data collection for the teaching experiment with advanced 6th grade students occurred in the context of a supplemental math course at a public elementary school, during the students' normal school day. This course was offered to 6th grade students who completed the 6th grade curriculum as 5th graders. This work is ongoing. Students meet with the instructor (the author) twice a week for 1-hour. Data collected for this experiment includes students' written work, field notes, and instructor notes.

Instructional Considerations

The teaching experiments reported in this paper were conducted with an alternative instructional treatment – a didactical engineering of mathematical induction – that was deeply rooted in ideas described in Harel and Sowder's (1998) discussion of mathematical induction task sequencing.[7] In addition to the use of an alternative sequencing of tasks, didactical situations were designed to address the development of two intellectual needs: the development of an intellectual need for non-empirical reasoning; and, the development of an intellectual need for hypothetico-deductive reasoning in the context of mathematical proof (Brown, 2003). In addition to selecting and designing tasks to support the development of specific didactical situations, steps were taken to foster a local didactical contract that placed the verification and justification of solutions in the hands of the students. A full discussion of the development of this local didactical contract is beyond the scope of this paper, so it will not be discussed here. However, it is important to note that the theoretical perspective taken in this work does not place the development of the need for justification solely in the realm of social considerations – but rather views it as a result of the creation of a milieu in which students can engage in mathematics *that warrants such practices*.

Findings

The purpose of this paper, and this comparison, is to explore the extent to which a potential epistemological obstacle identified with undergraduate students (UGS) arose or

[7] For more information about the alternative task sequencing, see also Harel (2002), Brown (2003) and Harel and Brown (in press).

was observed with the 6th grade students (6GS). To fully describe the potential epistemological obstacle it is important to first note at which stage in students' development of mathematical induction the obstacle arises. To do this, I will first briefly describe the model. Brown's (2003) model of the evolution of students' understanding of mathematical induction is a model of students' evolving conceptions of what constitutes a general solution to a PMI-appropriate task; in other words, it is a model of how mathematical induction might arise as a means to solve a class of problem. This model describes students' conceptions as progressing through three stages: the *pre-transformational stage*, the *restrictive transformation* stage, and *transformational* stage.

Key to the transition from one stage to the next is the overcoming of obstacles related to students' conceptions at that stage. During the restrictive transformational stage, students' approaches to PMI-appropriate tasks do not entail reasoning from empirically verified patterns (as is the case in the pre-transformational stage) but rather entails focusing on relations between consecutive cases. In other words, during the restrictive transformational stage students begin to attend to structural relations between consecutive cases. Their ways of understanding these relations, however, foster difficulties with particular PMI-appropriate tasks and are indicative of a potential epistemological obstacle to mathematical induction – the *infinite processes* obstacle. Thus, the transition from the restrictive transformational stage to the transformational stage is marked by shifts in the students' ways of knowing infinite processes. To illustrate how the infinite processes obstacle manifests itself after the students have advanced beyond purely empirical approaches, I will describe the UGS and 6GS responses to the L-tiling task.

The L-Tiling Task

In each of the UGS teaching experiments and in the 6GS teaching experiment, students worked in groups to solve what I will refer to as the L-tiling task (see Figure 1.)

L-Tiling Task*: For any positive integer n, can a 2^nx2^n grid with one square removed be tiled with L-tiles (shown below)?*

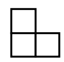

Figure 1. The L-Tiling Task

In the series of UGS teaching experiments, it was found that, after an initial exploratory phase, UGS tend to approach the question of how to tile an infinite collect of grids first by using a divisibility argument and then a decomposition approach. The *divisibility argument* UGS propose begins with students arguing that an L-tile is composed of 3 square tiles. The students then argue that the area of the 2^nx2^n grid with one square removed, is (2^nx2^n)-1. Thus, one simply needs to verify that (2^nx2^n)-1 is divisible by 3.

> Paula: Aren't they all divisible by three though? … (pause) … like four squared and you take a, like four squared is sixteen. You take away one, so you'd have a total of fifteen.
> […]
> Paula: Just divide by three and it gives you an even number like for all of them so it would always work.
> Susan: That's true.

Through algebraic work, the students verified that 3 divides (2^nx2^n)-1. After which the students were asked to consider the geometry of the grid and whether or not a "different" grid of 15 squares could be L-tiled (see Figure 2). These explorations led the students to reject the divisibility argument as a general solution to the L-tiling task.

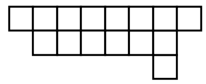

Figure 2. 15-Square Grid

The *decomposition approach* proposed by UGS begins with the students using examples to show how one can partition a grid into four quadrants, each of which either has no square removed, a corner square removed, or a non-corner square removed. In the case with a non-corner square removed, the students continue to partition that quadrant of the grid into four smaller quadrants until the missing square is in the corner of one of the smaller quadrants (see Figure 3).

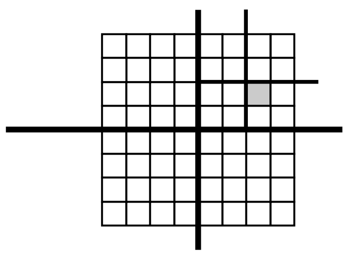

Figure 3. Example of Partition of $2^3 \times 2^3$ Grids

After which, the students claimed that the grid with the missing square can be tiled with a previously L-tiled grid with a corner removed and that the remaining quadrants can be "built up" by arranging three grids with corners removed in such a way that the removed squares form an L-tile (see Figure 4).

It is interesting that in several of the teaching experiments the decomposition approach to the L-tiling task was the only acceptable solution for many of the students; for even after another student suggested an alternative solution that entailed selecting the appropriate $2^{n-1} \times 2^{n-1}$ grid to tile the quadrant with the missing tile rather than partitioning that quadrant into smaller and smaller grids, as is done with the decomposition approach, the students maintained that the decomposition approach was the only "convincing" solution.

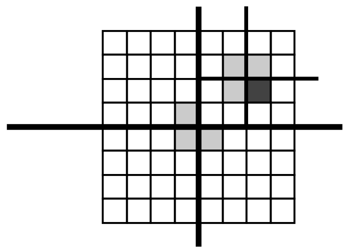

Figure 4. L-tiling of quadrants with no missing tile (n = 3).

Surprisingly, the 6GS problem solving trajectory with the L-tiling task, and often other PMI-appropriate tasks, mirrored the UGS students' trajectory. In the case of the L-tiling task, after having explored a series of grids (e.g., n = 1,2,3) and L-tiling various collections or classes of grids (e.g., the class of all $2^2 \times 2^2$ grids) the 6GS recognized that as n increases, the number of grids being considered increases and that the question concerned an infinite collection of grids, whose dimensions are of the form $2^n \times 2^n$. Recognizing that one cannot produce infinitely many L-tiled grid, groups of students then suggested the divisibility argument, as described above. Also like the UGS, once the divisibility argument was rejected the students developed the decomposition approach. For example, Tina used the $2^4 \times 2^4$ grid as a generic example and argued:

> Tina: you have a 16×16 grid. Obviously, one-fourth of the grid is an 8×8 grid. That's the one with the real square missing.

Tina then continued by explaining that the 8×8 grids consist of four 4×4 grids, the 4×4 grids consists of 2×2 grids. She then argued:

> Tina: Obviously, we can L-tile this (points to an L-tiled 2×2 grid) case. And by knowing I can L-tile this case, I know I can L-tile any two to the *n* (2^n) case.

Furthermore, when the 6GS engaged in a whole class discussion of their solutions to the L-tiling task, three of the four student groups (with each group having 3-4 students) presented the decomposition approach. Interestingly, only one student, Joel, argued that it would be easier to "just use whatever grid you need from before," in other words, to select the appropriate grid from the class of $2^{n-1}\times2^{n-1}$ grids to tile the quadrant with the missing square. Joel, however, remained in the minority, with the majority of 6GS preferring the decomposition approach solution, as was the case with the UGS.

Discussion of UGS and 6GS Student Responses

It is surprising that, having recognized the need for a general solution, both the UGS and 6GS restricted their explorations to the numeric aspects of the L-tiling task, as opposed to jointly considering both the geometric and the numeric. With both the UGS and 6GS teaching experiments, however, the divisibility approach arose after the students indicated an awareness of two issue related to the L-tiling task; namely, that the question concerned an infinite set of objects and that one cannot verify each element (in this case, grid) of the set. Thus, it appears that the students neglected the geometric aspects of the task and reduced the task to an algebraic question in order to address issues of generality. Moreover, I would also argue that the divisibility argument allowed them to address these concerns, while also enabling the students to avoid hypothesizing the existence of a collection or class of L-tiled grids.

Having rejected the divisibility argument, the students are left having to address the question of whether or not one can L-tile an infinite collection of grids, without a means to reduce the question to a single statement or diagram. Student initiated discussions about how to create "larger" grids from smaller grids, however, quickly enabled the students to recognize a structural relation between consecutive cases when restricting one's considerations to the grid with a corner square removed. Unlike other cases, one can "build" an infinite sequence of grids with a corner square removed, simply by iterating the process illustrated in Figure 5.

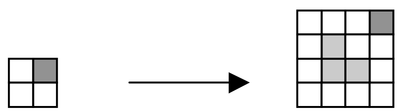

Figure 5. Constructing Grids with Corners Removed.

This leads to the following question, what distinguishes the solution for an arbitrary grid from the solution for the grid with a corner square removed? What is it that supports students reducing the problem to this particular case? The position taken in this paper is that the decomposition approach provides the means, from the students' perspective, to "construct" any grid from known objects. In contrast, a general solution that does not rely on such a decomposition necessarily entails partitioning an arbitrary $2^n \times 2^n$ grid into four $2^{n-1} \times 2^{n-1}$ grids, tiling three of the four quadrants using the $2^{n-1} \times 2^{n-1}$ grid with a corner tile removed, and then tiling the remaining quadrant by assuming the existence of the class of L-tiled $2^{n-1} \times 2^{n-1}$ grids with a square removed, without having "constructed" this class of objects. In other words, one must engage in hypothetico-deductive thinking about a collection of objects of which only a subset has been "constructed." The decomposition approach, therefore, allows students to avoid such reasoning in favor of constructive approaches. Thus, the decomposition approach is indicative of students' use of the *constructive proof scheme*, where "students doubts are removed by the actual construction of objects – as opposed to mere justification of the existence of objects" (Harel & Sowder, 1998, p. 272). It is also, I would argue, indicative of the infinite processes obstacle.

The Infinite Processes Obstacle

The infinite processes obstacle consists of a *way of thinking about iterative processes* and a *way of understanding implications*. The way of thinking about iterative processes manifests itself when one recognizes an imperfective, iterative process and then uses the process to attribute the property in question to the entire set. As such, this way of thinking about iterative processes relates to Lakoff and Nunez's (2000) description of the basic metaphor for infinity. The way of understanding implications is that implications are causal relationships; a way of understanding implications that, as noted by Harel (1999), has historical precedents within the context of mathematics.

Having recognized the need for a general solution both the UGS and the 6GS turned to an approach that would enable them to avoid assuming the existence of an infinite class of objects, with the exception of the instance in which they recognized an imperfective, iterative process for generating L-tilings for grids with corner squares removed. Thus, it appears that the recognition of such a process enabled the students to attribute the property in question to the infinite collection of grids with corner squares removed. Building upon this basis of belief, the remaining grids are then reduced to elements of this collection. In other words, it is the existence of such a process that enables the students to verify the entire collection of grids. But why not simply assume that the previous class of objects exists? Why rely on "constructed" objects? One cannot assume the existence of a class of objects, for instance, the class of all $2^3 \times 2^3$ grids, use these grids to construct the class of all $2^4 \times 2^4$ grids and then *know* that the class of all $2^4 \times 2^4$ grids actually exists, if implications are viewed as causal relationships. In other words, if $a \rightarrow b$ or "*a* causes *b*" then *a* must exist for *b* to exist; that is, I must be able to construct *a*, as opposed to assume the existence of *a*, in order to know *b*.

But why consider such ways of thinking and ways of understanding an obstacle? How is it that they function productively? Where do they lead students to errors? I would argue that the students' decomposition approach is an example of the infinite processes obstacle functioning productively, for it led the students to produce a viable solution –one reminiscent of the method of infinite descent. I would also argue, and as was the case with the UGS, that it inhibits the production of solutions that rely on the students' ability to engage in hypothetico-deductive reasoning. For example, when asked to solve the Two-Color problem (Figure 6), many undergraduates view each class of maps as consisting of infinitely many possibilities due to potential variations in the points of intersection.[8]

The Two Color Problem: Consider any "map" formed by drawing *n* straight lines in a plane to represent boundaries. Is it possible to color the countries using two colors, if no two adjoining countries (those with a line segment as a common border) have the same color?

Figure 6. The Two-Color Problem

Thus, to generate a general solution to the Two-Color Problem, the student must recognize the need to assume the existence of a collection of objects that have not been

[8] Clearly, a mathematician would recognize that, for example, for the case $n = 2$ there are only two possibilities. However, this is often not the perspective of undergraduates and the issue here is not mathematical but psychological.

constructed either directly or through an imperfective, iterative process. In other words, the student must overcome the infinite processes obstacle. This claim stems from data collected in the UGS teaching experiments where the Two-Color Problem was used to create an intellectual need for including an inductive hypothesis and was, for the students, the final step towards formulating the method of proof.

Discussion

The similarities in the UGS and 6GS approaches to the L-tiling task, and other tasks not discussed here, support the claim that the development of mathematical induction as a means to solve a class of problems necessarily entails shift in students ways of knowing infinite processes – in particular, their ways of thinking about iterative processes and ways of understanding implications. As such, this work aligns well with prior work on students' difficulties with mathematical induction (Dubinsky, 1989; Movshovitz-Hadar, 1993b) that has indicated students' struggle to understand, describe, and use the inductive step, in particular, the inductive hypothesis.

The findings also indicate two areas that need further elaboration. First, and as suggested in Brown (2003), it appears that PMI-appropriate tasks that involve class-to-class sequences may pose unique epistemological issues for students not encountered with case-to-case PMI-tasks. A PMI-appropriate task that involves class-to-class sequences is a task that concerns relations between consecutive classes of objects, as opposed to specific cases. For instance, the L-tiling task concerns relations between the class of $2^{n-1} \times 2^{n-1}$ grids (which consists of $2^{2(n-1)}$ grids) and the class of $2^n \times 2^n$ grids (which consists of 2^{2n} grids) whereas, tasks such as the Towers of Hanoi concern the relationship between the case of n-1 disks and the case of n disks. Thus, this finding indicates either an alternative categorization of PMI-appropriate tasks or a need to further elaborate or extend Harel's (2002) dichotomy of PMI-appropriate tasks, which focuses on the distinction between explicit and implicit recursion and non-recursion tasks. Second, the comparison of UGS and 6GS provides further evidence indicating a potential epistemological obstacle to mathematical induction, which, in turn, suggests that instructional approaches to mathematical induction should aim to support shifts in students' ways of knowing infinite processes. As such, this work has implications for the development of and necessity for alternative curricular approaches to mathematical induction. Finally, it is noteworthy that the results with the 6GS mirrored those of the UGS. Certainly, one can argue that the 6GS provide an example of what Harel and Sowder (2005) refer to as "advanced mathematical thinking at any age."

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