Exploration of the role of mathematical discourse in constructing mathematical object

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Semiotic analysis on mathematical discourse contributes to describing prospective teachers' construction and communication of mathematical objects. The purpose of this study is to build up a language for use in describing, deliberating, and assessing mathematical objects in focus. The data for the study came from observations of two mathematics classes for prospective teachers. A preliminary theoretical analysis using Gray and Tall's Procept (1994) framework indicates that there are significant discrepancies on participants' perceptions of signifier-signified-and-referent in mathematical discourse. Classroom implications and future directions for this study are explored.

BACKGROUND

Recently, the mathematics education society at large has paid more and more attention to *conceptual understanding*. The book entitled, "Adding it up: helping children learn mathematics" published by the National Research Council (NRC), puts conceptual understanding as one of the key strands of mathematical proficiency. In this book, conceptual understanding is defined as "comprehension of mathematical concepts, operation, and relations" (Kilpatrick, Swafford, and Findell, 2001). It seems the majority of people in the mathematics education society have been realizing the fragility of knowledge accompanied only by procedural knowledge and skills. Nonetheless, what counts as conceptual understanding and how conceptual understanding evolves may remain debatable, as Devlin (2007) criticized NRC's description of conceptual understanding, which barely elaborates the question of what it might be, but not really answers the issue of how to acquire the understanding, knowledge and comprehension of the mathematical concepts.

Influenced by Piaget's idea of how "actions and operations become thematized objects of thought" (1972), several theories have been proposed to describe such *objects of thought*, e.g., Dubinsky (1991), Sfard (1991), and Gray & Tall (1991), in which transformations performing on physical/mental objects is brought up to our attention. While Dubinsky and Sfard emphasize on its developmental stages, e.g., action/process vs. object/schema and operational vs. structural, Gray and Tall adopt the idea of *procept* to describe an amalgam of three components: a mathematical *concept* is produced by a *process*, and the usage of *symbol* is flexible enough to present its corresponding concept and process as well. In a sense, if a symbol is used as a signifier to refer a signified, i.e., procept, a successful learner should be able to *see* a process acting on an input to produce an output as concept. Moreover, later on, the learners can perform actions/transformations on the signified they already perceived. A comparison between NRC's conceptual understanding and Gray & Tall's procept, the idea of procept might encompass what NRC refers as concepts, operation, and relation, and, more important, could serve as a measurable construct for an occurrence of learning through an analysis on discourse to measure to what extent the learner actually see such signified.

Another criticism from Devlin to conceptual understanding is his questioning the idea of "students who should understand [concept] before they do [mathematics]," which is a common belief of the advocates of conceptual understanding. He proposed *functional understanding*, students' minimum and sufficient understanding of a mathematical idea, to ensure making progress and allowing further refinement and/or correction of their understanding. Similar idea was also proposed by Sfard (2000). She suggests that students can simply implement mathematical discourse in exchanging meaning before they have full understanding of the concept, in which the learners may gradually symbolize *mathematical reality* into being. In this study, I will refer Gray & Tall's procept and/or Sfard's mathematical reality as mathematical objects for indicating what people talk and write about when they do mathematics.

Granted Sfard's claim that the learners approach their understanding of mathematical objects through mathematical discourse, it surely is important for mathematics researchers to build up a language for describing, deliberating, and assessing mathematical objects in focus. That is, it is very possible that far before a mutual agreement has been clarified and settled on the referents or the signified in focus among the participants, the introduced signifiers have been adopted for operation and/or manipulation. How the perception of a sign system evolves in recognizer's mind would be a good indicator to what extent of a mathematical object is constructing.

THEORECTICAL PERSPECTIVE

Procept Theory (Gray and Tall, 1994) is adopted as a theoretical framework for this study for there is a non-linear progressive and recursive relationship between signifier and signified in constructing and communicating a mathematical object. This study investigates symbol-signifier being both a process and a concept in referring to the signified in focus. Semiotic analysis on mathematical discourse demonstrates discursive participants' adoptions in language, visual images and mathematical symbols to convey their perception of mathematical objects in focus in a given context, i.e., a system of signs, consisting of signifier and signified, along with the given context as referent.

SETTING, DATA COLLECTION and METHODOLOGY

The participants were preservice elementary teachers enrolled in two sections of a first-semester mathematics course, belonging to a two-course sequence. Basic course topics included quantitative reasoning, meanings of whole numbers/place value with operations, meanings of integer with operations, and meanings of rational number with operations. Two sections were taught by instructors Mary and Jose, both pseudonyms, respectively. Mary had more than twenty years teaching experience in this subject matter, while Jose was a Graduate Teaching Assistant with two years middle school and one and one half years college teaching experience. One Supplemental Instructor, Robert, was in Mary's class to assist her students after class by creating more comprehensible notes and utilize them as a basis of instruction/enhanced tutoring. Jose had been Mary's Supplemental Instructor before he taught this session.

As manipulatives and cooperative groups become more widely used in elementary and middle mathematics classes, these aids and situations are not just learning tools for assisting our pre-service teachers learning mathematical concepts but also themselves are targeted learning objectives of the preservice teachers. In Mary's and Jose's classes, they both adopted hands-on lab activities, which were originally created by Mary. This presentation draws upon transcribed video clips from Mary's and Jose's lab activities of numeral systems. The main objective of the lab is to develop a better understanding of base-ten numeral system by exploring bases other than ten.

The hands-on lab activity adopted by Mary and Jose for developing a sense of numeral system contains three sub-activities: (A) counting by grouping; (B) counting in different bases; and (C) converting from one base to another. The manipulatives for implementing this lab is multilink cubes set, which consist of 1cm x 1cm x 1cm cubes with feature of easy to stack and count exercises. There are three tasks in activity (B), which request group of three preservice teachers to execute (1) counting multilinks by grouping, say group of five and then group of five of fives; (2) simulating counting multilinks by using abacus/counter; and (3) recording the counting procedure demonstrating in (1) and (2) by using base-five numeral system. In terms of APOS theory, activity (A) is for learner to *experience* an action performing on multilinks; activity (B) is for leaner to *perceive* the actions as a totality to *compress* them into an object; and activity (C) is for learners to be able to perform actions or transformations on the objects constructed in activity (B).

The data for the study came from observations of classroom events, written artifacts, and individual interviews. Discourses between instructors and students were video-taped and transcribed verbatim for analyses. The transcriptions of discourses are analyzed using the constant comparative method (Lincoln & Guba, 1985) with Gray & Tall's procept theory as a framework. The preliminary result indicates there are significant discrepancies on participants' perceptions of signifier-signified-and-referent in mathematical discourse.

From the perspectives sketched above, this study poses two leading questions: (1) How can we tell if a student constructs a math object through mathematical discourse? (2) What does the role of mathematical discourse play in constructing math objects?

DATA and RESULT

As described above, Mary is an experienced instructor. During the interview, her supplemental instructor Robert and instructor Jose complimented her fluency of dictation, which indicates her competence to provide her students with clearer direct instruction and to lead her students to have better performance in what she intended. Figure 1 below, from Robert's notebook, shows Mary's demonstration of activity (A) counting group of five and then group of five of fives. Her dictation leaded students to record the remainders of all steps of actions as the answer for the number of cubes in base-five. In this stage, the learners may not know the reason why together the remainders of actions is the answer, although they had experienced the actions.

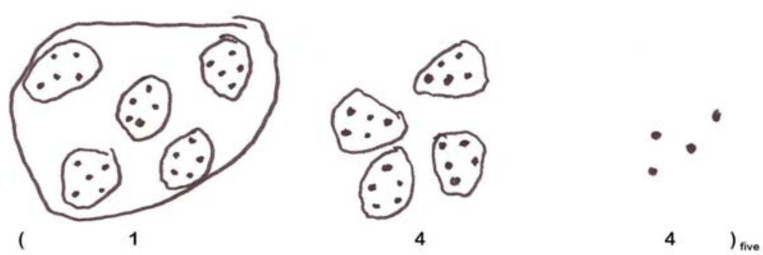

Figure 1. Counting by grouping

According to task description of activity (B), learners in this stage were supposed to reflect on the performed actions and to compress them into an object-like of thought. Figure 2 shows example of Mary's demonstration of counting by base-five numeral system. As usual, Mary made a demonstration with a fluent manner. Nonetheless, she never introduced or mentioned the terms or names, formal or informal, regarding place value. Mary put some particular emphasis on the expressions, such as “no numeral name as 5 in base-five system” and “five groups of fives *and so on.*” In an informal conversation after session, she seemed to believe that students should perceive the announced rule of “and so on” and connect the recursive pattern to the concept of place value. She also considered that the introduction of any new terms or names related place value burdens the learners’ cognitive demand, for students already struggled in grasping the meaning behind the lab activities.

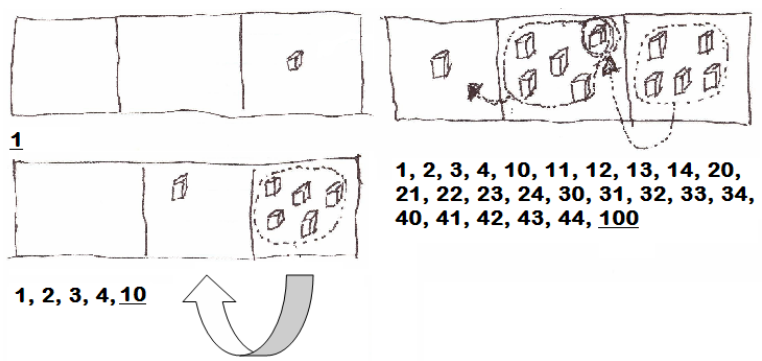

Figure 2. Mary’s demonstration of counting and recording by base-five numeral system: the sketch here only partial of the demonstration to emphasizes the concept of grouping, i.e., place value.

Activity (C) is to convert the expressions of numbers from one base to another, such as converting the number expressed by $(123)_5$ to the expression of base-ten. The demonstration of Mary was mainly algorithmic, e.g., $(123)_5 = 1(5^2) + 2\,(5^1) + 3(1) = 33$.

From the observation of Mary’s lab practice, besides it going smoothly, it is not easy to tell what student actually gained from the lesson. Indeed, in interviews with students from Mary’s class, it was found that students did not *see* the connection between the algorithm, e.g., $(123)_5 = 1(5^2) + 2(5^1) + 3(1) = 33$, and the multilink cubes, e.g., group of five and group of five of fives. At least, the students did not have the language to describe the connection, if they ever made any.

Jose's lab implementation did not go as smooth as Mary did, which is not a surprise, since he was less experienced in classroom delivering than Mary was. Jose basically went through the same procedures like Mary did, but with a vaguer direct instruction and combining activities (A) and (B) together as one shoot. In a twenty-five minutes small group activity, Jose had to go back and forth to the front of the class to provide more instruction of what to go next. Generally, most of the students had no clue why the digits ended at 4 in base-five and could not do the grouping beyond the second place value, which is exactly what happened to Mary's students who were in the interviews mentioned above. With Jose's assistance, his students seemed to be able to following the instruction toward the end of this period.

Although Jose had always been in a hurry during this summer session worrying about not covering enough course material, out of my surprise on the next day, Jose picked up the lab activities (B) again, by saying:

> I don't want to take the whole time for the lab but I do want you to get the practice. So follow the instructions on the lab. Don't just say here is a group of five, here a group of five, no do what it is telling you. Count one, two, three, four, one zero, one one, one two, one three, one four, two zero. Actually go through the motion and there is a reason for that …

This time, he insisted (1) all three members in a triplet group have to go through all three tasks simultaneously: counting multilinks by grouping, simulating counting by using counters, and recording the counting procedure; and (2) communicate with a set of informal terms, e.g., units, longs (group of five), and flat (group of five of fives). During the interview, Jose expressed his frustration of two things. First, Jose was reluctant to provide his students with direction hoping his students can figure out the underlying structure of the lab activity, which his students failed to. Second, Jose felt awkward in communication with no names or terms to refer to what he was talking about. Although not all the students follow his instruction of going through the motion of linking the multilinks, we could observe, from the following excerpts, some changes since then.

> Day 1:
> Student: For five of these [five groups fives] it will be one of these [one five-fives] ….
> Jose: For these fives groups of fives…you group the fives into fives… and then you group those fives into what… fives right.
>
> Day 3:
> Jose: In our base five system we have ones units then we have not tens like in our base ten, but….
> Student: longs
> Jose: Ok, we have fives or longs. That is why we did the lab. So we can picture what is going on. Fives are longs. Five units linked together. And then we have flats, which are what?
> Student: Five longs, or 25 units.
> Jose: Twenty-fives units, right. And, then after that we have our blocks, which is what?
> Students: 125's
> Jose: And so on and so forth

In Day 3, a teaching episode from Jose's demonstration of converting $(12344)_5$ into base-ten numeral system ($4 + 4 \times 5^1 + 3 \times 5^2 + 2 \times 5^3 + 1 \times 5^4$) shows how students were constructing the object of place value in the fifth place of numeral display:

> Jose: … We have 4 ones 4 longs 3 flats and 2 cubes. What does this look like 625, in base five?
> Cindy: six flats
> Gina: no, four of the cubes

Students already had experience about ones, longs, flats, and cubes, which they had performed actions on multilinks. For them, the new unit, consisting of 625, was new to them. Their intuitive guess of 625 as six groups of 25 or as four groups of 25 (the rule of no more than four) indicates they did have some sort of understanding in this regard.

> Jose: (Does some arithmetic: 625 / 125 = 5)… How many cubes do we have?
> Gina: Its five cubes.

Jose: What it is, we have five cubes right, if we put five cubes together what does that look like?
Student: (Inaudible)
Jose: Is it going to make a bigger cube?
Student: Yes.
Gina: Long
Jose: What it is going to do is [to] make a long. If we put five cubes together, we get a long. Right. So it is a long made of cubes (*showing the gestures of putting cubes together and creating the long made of cubes*). What's after that?
Student: Another long.
Jose: What happens when we put five 625's together we get an F… (*starts with an F*).
Student: Flat
Jose: We get a five by five made of cubes right. Of those one-zero-zero-zero-flats, first we get a long. And with five of these we get a flat. (*Start drawing a picture*). Each of those squares, five-by-five (cubes), one-zero-zero-zero, three zeros, makes the flat.
Gina: Like we were doing yesterday staking them.
(G*estures before the response: she pinched her fingers slightly and she moved her hand forward five times as if she was constructing a long. She then responds to Juan by saying "like we were doing yesterday staking them." She said this while simultaneously using both hands as if she were handling flats and stacking them upon on another higher and higher as if she was creating a cube.*)

What Gina did was to perform a mental action on a mental object (a 5x5x5 cube) to construct new mathematical objects, e.g., long made of cubes (625 units), flat made of cubes, and so on.

The result stated above shows somewhat convincing information that Jose's students seemed to be able to construct mathematical objects which we did not observe in Mary's class. This behooves us to wonder what actually happened in Jose's class, but not in Mary's class, so that such object construction occurred.

TENTATIVE CONCLUSIONS and FUTHER RESEARCHES

Cottrill et al. (1996) mentioned there are at least two ways of constructing objects-from processes and from schemas. In this study, we find it quit difficult to tell whether students constructed a mathematical object from compressing a series of process into an object or just simply performed action following the instructor's dictation. However, it is much easier for us to tell whether the learners construct a mathematical object through an action performing on object constructed. The example mentioned above shows students' (*potential*) construction of long made of cubes and flat made of cubes through mathematical discourse. This observation may lead to the answer to our first research question, "How to tell if a student constructs a mathematical object through mathematical discourse." For further researches, a protocol involving the construction of new mathematical object from schema, if any, could serve as a research tool and/or a teaching tool to evaluate or enhance such construction.

Nonetheless, to have tentative hypotheses to our second question, "the role of mathematical discourse plays in constructing math objects," is still not so clear. Our tentative hypothesis is the naming activity, e.g., calling the products as long, flat, etc., and using the terms for exchanging and negotiating their ideas. Our further research design may plan to involve students in the process of creating their signifiers for communication and negotiation, and to observe this involvement interacting with mathematical objects construction.

REFERENCES

Cottrill, J., Dubinsky, E., Nichols, D., Schwingendorf, K., Thomas, K. & Vidakovic, D. (1996). 'Understanding the limit concept: beginning with a coordinated process schema', *Journal of Mathematical Behavior*, 15. 167-192.

Gray, E. M., & Tall D. O. (1991). Duality, Ambiguity and Flexibility in Successful Mathematical Thinking, *Proceedings of PME XIII*, Assisi, Vol. II 72-79.

Gray, E. & Tall, D. (1994). Duality, ambiguity, and flexibility: A "proceptual" view of simple arithmetic", *Journal for Research in Mathematics Education 25*(2), 116-140.

Devlin, K. (2007). What is conceptual understanding? Retrieved March 10, 2007, from http://www.maa.org/devlin/devlin_09_07.html

Dubinsky, E. (1991). Reflective Abstraction in Advanced Mathematical Thinking. In D. Tall (Ed.) *Advanced Mathematical Thinking*, Kluwer: Dordrecht, 95-123.

Lincoln, Y. S., & Guba, E. G. (1985). *Naturalistic inquiry.* Beverly Hills, CA: Sage.

Piaget, J. (1972). *The Principles of Genetic Epistemology*. London: Routledge & Kegan Paul.

Sfard, A. (1991). On the Dual Nature of Mathematical Conceptions: Reflections on processes and objects as different sides of the same coin, *Educational Studies in Mathematics*, 22, 1-36.

Sfard, A. (2000). Symbolizing mathematical reality into being: How mathematical discourse and mathematical objects create each other. In P. Cobb, K. E. Yackel, & K. McClain (Eds), *Symbolizing and communicating: perspectives on Mathematical Discourse, Tools, and Instructional Design.* (pp. 37-98). Mahwah, NJ: Erlbaum.