An examination of the knowledge needed by a mathematician to teach an inquiry-oriented course in differential equations

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In this paper we report on an investigation into elements of pedagogical content knowledge (PCK) and mathematical knowledge for teaching (MKT) needed by a mathematician in his first use of an inquiry-oriented curriculum for an undergraduate course in differential equations. Our research is driven by two primary objectives. First, little is known about the teaching practices of college mathematics instructors, and even less about those attempting to change their instructional practices. Since the PCK and MKT developed by teachers may be particular to the practices and curricula with which the teachers are most experienced, research of this sort is needed to learn about the challenges faced by instructors adopting new teaching practices. Second, studies of K-12 teachers have highlighted the importance of PCK and MKT in understanding teachers' practices, but these studies may be prone to limitations due to teachers' limited content knowledge. Since the content knowledge of mathematicians is typically strong, studies of their teaching practices may offer clearer access to the role of other types of knowledge in teachers' practices.

Expanding on our earlier analysis of the challenges faced by a mathematician while teaching an inquiry-oriented differential equations course for the first time (Wagner, Speer & Rossa, 2007), we focus here on the roles of PCK and MKT in supporting teachers as they lead large-group discussions. In particular, we examine how a mathematician whose primary opportunities to develop PCK and MKT have taken place in more traditional, lecture-oriented classrooms, may be under-prepared to motivate and direct mathematically productive discussions in a student-centered class. We argue that PCK is necessary for an instructor to anticipate students' thinking and students' conceptual difficulties so as to address them as they spontaneously arise, and that types of MKT may be needed to understand students' developing ideas and direct them in mathematically useful ways. Limitations in these types of knowledge may curtail the instructor's ability to direct class discussions in mathematically profitable ways.

Motivation for Research

The foci of our research in this area are centered on examining the following questions:

- What do teachers find challenging about using inquiry-oriented approaches to instruction?
- Why are certain aspects of this kind of teaching difficult for some teachers?
- Which specific cognitive resources (kinds of knowledge, beliefs, etc.) are needed to teach in these ways?

Progress in answering these questions will leave researchers better-equipped to answer one other, extremely important, question: How can we help teachers develop the resources and practices they need to teach in these ways?

The research base that has emerged from studies of teachers' efforts to change their teaching practices has helped the education community identify some key factors that shape teachers' instructional practices as well as their success in effectively implementing elements of instructional reform such as collaborative group work and inquiry-oriented activities. Much is known about teachers, their practices, and the challenges they face in implementing reform. We contend, however, that the research on roles of various kinds of knowledge in teaching could benefit from investigations conducted with teachers whose command of mathematical content is extremely strong. In studies of mathematicians, the possibility that their challenges in implementing instructional reform stem from weak content knowledge is small and, as a result, such studies have the potential to provide a clearer picture of the roles of PCK and MKT.

Our investigation of these questions began with a study of a mathematician, Prof. Gage, who was teaching an undergraduate differential equations course using a curriculum informed by research on student learning. This “inquiry-oriented” curriculum emphasized small group problem solving and whole-class discussions. Although the mathematician had taught differential equations in the past, this was his first experience with a reform-based curriculum. In the initial study (Wagner et al., 2007), we analyzed what he found most challenging, why these aspects of teaching were challenging for him, and which cognitive resources he used or needed to inform his instructional decisions. Below we briefly describe the findings from that initial study. We then present our current investigation in which we narrow our focus and analyze specific challenges he experienced in orchestrating whole-class discussions.

Findings From Initial Study and Focus of Current Study

In addition to articulating various challenges throughout the semester, Prof. Gage described the four he felt were most significant in a written reflection he produced at the end of the semester. Prof. Gage perceived the most challenging aspects of teaching this new course to be identifying what students were learning at particular times, determining how much students were learning, deciding how the ideas should be organized and distributed across the semester’s classes, and orchestrating classroom discussions. Further analysis suggested that major factors contributing to these challenges included PCK that he had not needed when teaching other versions of this course and elements of MKT that would have enabled him to understand and follow students’ mathematical ideas as they surfaced during classroom discussions.

For the current study, we zoom in on the challenges Prof. Gage faced in orchestrating discussions. We examine specific instances of such challenges as they occurred in class and the factors that made this aspect of teaching the course so difficult for him. We focus on whole-discussions because of their central role in reform-oriented teaching, and because problems with orchestrating discussions figured prominently in the post-class interviews with Prof. Gage.

Research on Teachers' Knowledge and Practices

We take a cognitive approach to our analysis—an approach shared by other researchers who examine teachers' knowledge and the roles knowledge plays in shaping teaching practices (e.g., Borko & Putnam, 1996; Schoenfeld, 2000; Sherin, 2002). In such an approach, knowledge is seen as one of several factors influencing teachers' goals and the approaches teachers take to accomplish those goals as they plan for, reflect on, and enact instruction. While it is undoubtedly the case that teachers need knowledge of mathematics content, researchers have found it challenging to establish relationships between measures of teachers' content knowledge and student achievement (Ball, Lubienski, & Mewborn, 2001; Wilson, Floden, & Ferrini-Mundy, 2002). These and other findings about resources teachers use have directed researchers' attention to other kinds of knowledge. Of particular note are the influences researchers have found of *pedagogical content knowledge* and *mathematical knowledge for teaching*.

PCK is the label used to describe what teachers know about (among other things) which mathematical topics typically cause students difficulty, how different mathematical ideas tie together and are organized in curricula, and how particular examples or explanations can be useful in teaching particular mathematical concepts. Since the identification of this type of knowledge (Shulman, 1986), researchers have found that PCK plays important roles in teachers' practices and the learning opportunities such practices create for students. For example, researchers have shown that teachers' knowledge of the different strategies that their students would use to approach problems is positively correlated with student achievement (Fennema et al., 1996). In addition, findings indicate that teachers who participated in programs designed to enrich their PCK of students' thinking tended to modify their practices to include listening and attending more closely to students' mathematical reasoning and to adopt other practices associated with education reform (Fennema, Franke, Carpenter, & Carey, 1993).

In addition to having PCK at one's disposal, teachers engage in a type of mathematical work to follow and understand the ideas and solution strategies that students generate. Researchers have described and examined the knowledge needed to do this work and the connection of this mathematical knowledge for teaching to student achievement (Ball & Bass, 2000; Hill, Rowan, & Ball, 2005; Hill, Schilling, & Ball, 2004). MKT is a resource that teachers draw upon to accomplish a variety of teaching-related tasks such as following students' mathematical thinking, evaluating the validity of student-generated strategies, and making sense of a range of student-generated solution paths.

Although many factors come together to shape teachers' instructional practices, we focus our analysis on PCK and MKT for two reasons. First, when describing challenges he faced, Prof. Gage often explicitly mentioned his unfamiliarity with elements of PCK and expressed frustrations stemming from limitations that we will argue are related to his MKT. In addition, most studies of PCK and MKT have involved K-12 teachers and, as described earlier, those teachers' knowledge of mathematics content can be weak. Focusing on the PCK- and MKT-related issues that limit a mathematician's instructional practices may be a productive direction for research when the objective is to disentangle and refine what is known about various types of knowledge that shape teaching practices.

Setting, Research Design, and Methods

The goal of this analysis was to understand the reasons for some of the challenges Prof. Gage faced while orchestrating large-group discussions. To accomplish this goal, the research design included data on Prof. Gage's in-class instructional practices and interviews tied to those specific practices. Below we describe the research setting and how the data were collected, the approach we took to selecting data for the particular analyses in this paper, and the methods used to conduct the analyses that revealed the factors that shaped his teaching practices.

The Setting

Data collection occurred at a private university in the Midwest. The university enrolls approximately 7,000 students, of whom approximately 4,000 are undergraduates. Prof. Gage was teaching an undergraduate course in Differential Equations using the Inquiry-Oriented Differential Equations (IO-DE) materials developed by Rasmussen (2002). Prof. Gage had 17 years of university teaching experience at the time and had taught DEs in the past using a traditional text and more traditional instructional methods. Of the 19 students in the class, most were pursuing majors or minors in mathematics and/or biology, chemistry or physics.

The IO-DE curriculum consists of student and instructor materials developed from research on student learning and include several series of problems, activities, and accompanying Java applets designed to guide students through discovery of the core concepts of a dynamical systems approach to DEs. Students are expected to work collaboratively on problems and to participate in whole-class discussion through which they acquire graphical, numerical and analytic techniques for analyzing, interpreting and solving DEs. Problems and activities were designed to challenge and encourage ways of thinking about the mathematics and to lead students to discover important ideas. While the top-level content areas of a traditional course in DEs are readily visible in the course materials, the particular "path" through which students encounter them differs from most traditional texts.

Data Collection Methods

During the semester, almost all meetings of Prof. Gage's class were videotaped. After almost all of these classes, interviews were conducted with Prof. Gage by one of the authors. During these interviews, Prof. Gage discussed his perceptions of how the class went as well as the challenges he faced. He typically and spontaneously focused his comments on the aspects of the class with which he was least satisfied and which were sources of the most frustration. At times, Prof. Gage requested feedback and/or advice, and sometimes the discussions included collaborative planning for the next class. In contrast with other kinds of interviews (e.g., clinical interviews) for which the objective is to refrain from influencing the interviewee, these discussions likely shaped some of Prof. Gage's decisions about his teaching. The discussions during which Prof. Gage sought assistance, rather than being problematic for analysis, became rich sources of data on the kinds of knowledge he lacked.

After the course was over, Prof. Gage wrote a short essay about what he considered to be the biggest challenges he faced in teaching the course. In addition, he provided a written description of his prior experiences of planning and teaching differential equations.

Data Selection Methods

The methods used for gathering data generated a tremendous quantity of video and audio recordings (approximately 30 hours of classroom video and 18 hours of recorded interviews). In the prior study, we used Prof. Gage's post-semester essay about his challenges as a means of narrowing the focus of the analysis. For the current study, we concentrate specifically on the challenges he faced while orchestrating large-group discussions.

Since all class days included large-group discussions, we needed a strategy for selecting the days we would subject to analysis. We chose to focus our analysis on classes that were especially challenging (from the instructor's and/or the researchers' perspectives). In particular, we analyzed classroom data if (a) during interviews Prof. Gage expressed high levels of frustration and/or described the class as especially challenging, or (b) the classroom video data suggested that the large-group discussions were problematic in some way (e.g., the main mathematical point did not surface clearly, the discussion occupied significantly more class time than was recommended in the curriculum, etc.). Additional details about the characteristics of these discussions are provided in the subsequent sections on data analysis and findings. We used both the classroom video data and the interview data from these chosen classes.

Data Analysis Methods

For each class selected for analysis, we transcribed the corresponding interview and the portions of large-group discussions that were mentioned during that interview. Findings from the first study (described above) suggested that some of the challenges Prof. Gage faced stemmed in part from his lack of familiarity with different ways students would approach or think about problems in the curriculum, as well as from difficulties he had following the mathematical ideas students contributed during the discussions. We hypothesized that these issues related to his PCK and MKT would also be evident in the specific classroom discussions chosen for analysis. To examine this hypothesis, we first developed a set of general, top-level codes for identifying interview segments related to Prof. Gage's PCK or MKT. With this set of codes, we took a grounded theory approach (Strauss & Corbin, 1990) to the analysis of the interview transcripts. Each author independently coded the same transcript, generating additional codes and sub-codes as seemed necessary. We compared the two codings and, through competitive argumentation (VanLehn & Brown, 1982), developed a revised set of codes as well as criteria for assigning those codes. This coding scheme then became the basis for analysis of other transcript data.

We approached the analysis of the classroom video data in a similar manner, coding transcripts for recurring themes (such as excessively long discussions on a single topic, or insightful student contributions that appeared to go unnoticed) and we looked for evidence that the sources of Prof. Gage's difficulties were knowledge-related. Finally, we looked for patterns that emerged across both coding processes. In particular, we focused on themes in the interviews that were consistently associated with specific challenges faced by Prof. Gage during the large-group discussions. We took the fact that Prof. Gage repeatedly mentioned particular issues as evidence that those issues were major influences on his decisions and practices.

In the end, multiple sets of classroom and interview data corresponded to each of our claims. In this paper, we illustrate our claims with examples that we believe best illustrate our findings with the least amount of background information. That is, we selected classroom episodes for which it would be relatively simple to describe the mathematical ideas at stake and the discussion that preceded those episodes.

Episode Analysis

Leading large-group discussions in an inquiry-oriented classroom setting requires a teacher to possess and enact a constellation of skills to guide students in a mathematically productive direction. "Inquiry" is a central activity not only for the students, but also for the instructor. Teachers inquire into students' thinking to make sense of their ideas, to recognize the potential mathematical utility of those ideas, and to highlight both correct and incorrect mathematical understandings that may lead toward productive arguments or problem solutions. These efforts to make sense of students' ideas, often presented in nascent or imprecise forms unlikely to meet more formal mathematical standards, require careful attention to students' contributions as well as the ability to recognize good mathematical thinking even when hidden beneath informal or incorrect language, representations, or reasoning. Additionally, teachers utilize their PCK to anticipate students' ideas or conceptual difficulties, to address them as they arise, and to refocus or redirect students' thinking when useful mathematical ideas do not arise spontaneously. All this must occur as teachers simultaneously orchestrate discussions, balance the relevant objectives of the activities at hand, and work within the time constraints under which individual activities and the entire course must progress.

In this episode, we analyze an exchange in Prof. Gage's classroom that reveals how certain elements of PCK and MKT are needed for an instructor to direct an otherwise "stalled" class discussion productively. Two aspects of this classroom excerpt are examined here. First, Prof. Gage found himself poorly prepared to move the class discussion forward when students were unable to clarify for themselves a point of mathematical confusion that he had expected them to handle easily. Second, even though Prof. Gage had listened to a student's contribution well enough to summarize it for the class, he was nevertheless unable to recognize the value of the contribution for illuminating the very point that he had attempted to make just minutes earlier—a point that Prof. Gage himself later identified as passing unacknowledged by the class.

The students had been asked to propose a DE capable of modeling simple population growth under ideal circumstances. The activity was designed to lead them to recognize that $dP/dt = kP$ was a reasonable model, with P representing the population size at time t, and k being a constant of proportionality. After discussing a variety of possible models, the class had narrowed their choices to two: $dP/dt = P$ and $dP/dt = e^t$. Students commonly suggest the latter model, in part because they have prior experience with exponential population models and also because of the conceptual challenge in identifying the *difference* between the two models.

Because $P(t) = e^t$ is a solution to both DEs, understanding how the two models differ and why the first (but not the second) is a reasonable model for population growth can be difficult.

The following excerpt from the class transcript begins as Prof. Gage made a suggestion in an effort to help students recognize the difference between the two models:

G: If you take e^t, then if you differentiate it, you get it back. Some-. So are they the same or not? [*14 seconds silence*] How about, uh, you know, something so, like 2 times e^t? [*8 seconds silence*] $P(t)$ is 2 times e^t?

S: I really don't understand … what they mean by "the same."

G: OK. Can somebody may, uh, phrase what may be meant by 'these are the same'? Robert?

R: I think what was said is, OK, let's say you say $P(t)$ is where P is equal to e^t. Then if you take the derivative you'll get dP/dt is equal to e^t. But I think that's, I, while I can't deny the truth of that because you can just, by going back to the original equation, you can just substitute between P and e^t, you can derive the equation that says P is equal to e^t. I think that's a specific circumstance, that, you know, where that happens to work.

By suggesting that the students consider $P(t) = 2e^t$, Prof. Gage hoped that they would notice that $P(t) = 2e^t$ solved only the first of the two DEs, thereby helping them to begin to distinguish between the two models. His suggestion, however, was met with lengthy silence. This lengthy silence was unusual for Prof. Gage's class, suggesting not only that students did not understand the point of his suggestion, but also that Prof. Gage himself did not know how to respond to their silence. Prof. Gage corroborated this interpretation in the post-class interview. The interviewer suggested that it is useful to have mathematical "bait" at hand to spur discussion when it stalls:

G: Well, I mean, I asked, for example, well, how about 2 times *e* to the *t*? It satisfies one of them, but not the other one. Showing that they're not the same. Nobody reacted to it. You know … OK, it doesn't make any sense to them. They didn't know what to do with that. Um, and I didn't have a whole collection of, you know if that doesn't work I'm gonna try this, and if that doesn't work, then I-, because I don't even know if I'm going to get into that situation. That's why the planning is so hard. […] To put out-, to, to, to be ready to have all this "bait" in case I need it.

As noted above, the students' difficulties in making sense of the subtle difference between the two differential equations is common. In fact, the activities provided in the curriculum materials were designed precisely to encourage students to confront this mathematical nuance that is typically presented as unproblematic in traditional texts. Prof. Gage, however, did not anticipate the extent of students' possible difficulties with this point, and when his suggestion to consider $P(t) = 2e^t$ failed to elicited the response for which he had hoped, he was at a loss:

G: I say, OK, yeah, this should work. … But then it doesn't. […] I didn't plan it out carefully enough. Because I was convinced it wouldn't be such a big deal. I thought this was gonna, this would go pretty quickly, once we review the time independence and all this kind of stuff, I was convinced it would go pretty quickly.

In this instance, Prof. Gage's lack of certain elements of PCK hindered him in moving the class forward. He was unable to anticipate students' difficulties with the conceptual challenge underlying this activity, despite its inclusion in the curriculum to raise just such a challenge. As a result, he was unable to prepare himself to address these difficulties when they arose. His one suggestion was met with silence from the class and his inability to understand the nature of the conceptual problems the students faced left him stymied as to what to do next.

Prof. Gage noted that he "was convinced it would go pretty quickly," suggesting that he was not merely unengaged with the planning process but had in fact made efforts to anticipate problematic points during his preparation for class. At other points in the interview, Prof. Gage described the significant amount of time he spent thinking about other mathematical points that he expected to arise. Those points, however, never came up. It is not surprising that he lacked extensive knowledge of these potential difficulties for students, since in most DE curricula simple population models are presented primarily for students to solve, not derive, and so the challenge to create such a model never arises.

The remainder of the classroom exchange presented above highlights the role that MKT can play as an instructor attempts to make sense of students' thinking. The silence met by Prof. Gage's suggestion was finally broken when a student asked for clarification of the question at hand. Robert (R), then observed that even though one could "just substitute between P and e^t," he believed that to be "a special circumstance … where that happens to work."

Robert's contribution was directly related to the point that Prof. Gage was trying to make by asking students to consider $P(t) = 2e^t$. This solution solves the first differential equation, but not the second. At the same time, it also "happens to work" that $P(t) = 2e^t$ solves both $dP/dt = P$ and $dP/dt = 2e^t$, thereby paralleling the situation under consideration. Because of this connection to the very issue with which students were struggling, Robert's suggestion could have been used in a number of ways to direct the conversation in the productive direction that Prof. Gage had hoped. One might, for example, inquire further into what precisely Robert meant by "that happens to work," or one might repeat the suggestion of $P(t) = 2e^t$ and ask Robert (or the class) how it related to the point he was trying to make. Prof. Gage, in fact, began such an inquiry:

G: Uh, sorry, where what happens to work?[1]

R: That P is eq-, P is the same as e^t.

S: Can you come up with one where it doesn't work?

G: Yeah, can you come up with something where it wouldn't?

R: Well, let's say your, let's say your $P(t)$ was $P + t$.

[*transcript omitted*]

G: OK, so Robert is saying that, you know, this kind of feels like some particular instance where something is happening, but we can't right now come up with something, ah, that kind of supports, supports that. Melanie?

Prof. Gage initially inquired into Robert's reasoning, but during the conversation that followed, Robert made the unhelpful suggestion, "say your $P(t)$ was $P + t$," resulting in, as one student put it, "a recursive definition for P." After clarifying the trouble with Robert's suggestion (transcript

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[1] An instructor who recognized the potential usefulness of Robert's suggestion would probably ask him to clarify it in order to highlight his reasoning for the class and propel discussion forward. Prof. Gage's apologetic tone, however, suggests that he asked the question because he did not fully understand Robert's point.

omitted here), Prof. Gage summarized what Robert had said and called on another student, thereby putting closure on the discussion of Robert's contribution and encouraging the conversation to move in a new direction.

Prof. Gage's choice to end the discussion of Robert's contribution and to permit the discussion to move off in a new direction suggests that he did not recognize the significance of Robert's idea as it emerged. Prof. Gage's summary included only the most surface-level details of the contribution ("this kind of feels like some particular instance where something is happening"), suggesting that even though he was attending to Robert's words, he failed to recognize or understand the significance of Robert's observation that the *something* that "happened to work" was closely related to the point Prof. Gage had attempted to make by suggesting $P(t) = 2e^t$ just minutes earlier. The opportunity to reconsider $P(t) = 2e^t$ in light of Robert's observations was lost, and $P(t) = 2e^t$ was not mentioned again until more than 20 minutes later– after the debate between the two models had already been resolved.

What contributed to Prof. Gage's inability to recognize the potential in Robert's observation? First, we eliminate some obvious possibilities. It is highly unlikely that Prof. Gage was simply distracted or inattentive to Robert. He began to respond by asking Robert to clarify his meaning; he reiterated a pointed question posed by another student; and he ended the discussion with a summary of Robert's words. His summary, however, was essentially empty of any mathematical content, suggesting that he may not have followed or understood the ideas that Robert offered. It is not the case, however, that Prof. Gage lacked the mathematical content knowledge needed to understand Robert's comment. Robert's point was directly related to the same point that Prof. Gage attempted to make just a few minutes earlier. It is also not the case that Prof. Gage lacked the discussion management or pedagogical skills needed to elicit students' ideas. He asked several questions in an attempt to clarify Roberts' point, and he engaged in significant mathematical discussions with students at other times in the course.

Having eliminated these other potential knowledge-related sources, we contend that there is evidence that Prof. Gage's failure to recognize the potential in Robert's contribution was rooted in an absence of some MKT relevant to the mathematical discussion at hand. To follow students' mathematical reasoning in real time requires the ability to recognize mathematical ideas in and/or infer those ideas from students' vaguely expressed or partially formed ideas. The mathematical work one does to recognize and infer the relevant mathematics in such circumstances may require more than the more formal, precise, and standard mathematical knowledge typical of mathematicians. Post-class interviews revealed that Prof. Gage found following students' reasoning quite difficult at times– a situation that was exacerbated by the variety of instructional tasks for which he was responsible at any time:

G: I need to do too many things. I need to try to follow the train of thought carefully, and I need to try to figure out when is it a good place to do something. I have to look for certain clues that I am sensing are good ones to do something, and I – which is kind of a detached observation kind of thing. And the other is to be part of the thought process, to really follow it along, to try to direct it a little. There's a conflict there. I can't do both.

On the one hand, orchestrating discussions required Prof. Gage to make decisions about when and how to direct the conversation, which he perceived as "kind of a detached observation kind of thing." On the other hand, following students' thinking carefully demanded that he "be part of the thought process." In combination, these roles of being simultaneously "detached" and "a part of the thought process" left him feeling conflicted, as well as overwhelmed by his "need to do too many things."[2]

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[2] In addition to the roles of *knowledge* examined here, it is also evident that Prof. Gage held certain *beliefs* about mathematics, teaching, learning, etc., that influenced his instructional practices. Although we do not present findings from our analysis of his beliefs in this paper, Prof. Gage had beliefs about his role in large-group discussions that shaped decisions he made about when and how to direct the

Perhaps most telling in regard to this particular classroom episode is Prof. Gage's struggle with having to do "too many things." At the beginning of this section, we highlighted the need for a teacher to possess and enact a constellation of skills to direct discussions fruitfully in inquiry-oriented classrooms. At the time he taught this course, Prof. Gage had 17 years of teaching experience, including experience teaching DEs. In a more traditional classroom setting, Prof. Gage, by his own accounts, did not experience the kinds of difficulties that he identified in this inquiry-oriented class. That is, the knowledge and skills that he had accumulated over the years enabled him to teach comfortably in a more traditional lecture classroom but were insufficient to support his instructional practice in this new classroom context. We contend that the sense of being overwhelmed by having to do "too many things" was very real to Prof. Gage precisely because he lacked—or was in the process of learning—certain knowledge and skills associated with teaching in these new ways. In this particular case, we suggest that his inability to recognize the potential usefulness of Robert's contribution can be attributed to his lack of some MKT that would have enabled him to follow Robert's reasoning while simultaneously attending to other, on-going needs (e.g., orchestrating the discussion in general). Ultimately, despite the strength of his mathematical content knowledge and his clear attention to Robert's contribution, Prof. Gage was unable to detect the mathematical potential in Robert's ideas.

What specific knowledge might have helped Prof. Gage under the circumstances? It is difficult, if not impossible, to identify such knowledge by its absence. As we noted above, even though Prof. Gage likely possessed all the formal mathematical knowledge of the material under discussion, "unpacking" such knowledge into underlying conceptual challenges related to students' informal and intuitive ideas was needed. Making sense of Robert's contribution

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conversation. Those decisions were substantially influenced by the PCK and MKT he had available and so we chose to focus this analysis only on knowledge-related issues. This should not be taken to suggest that there are not other factors influencing the nature of the instruction Prof. Gage provided.

required a series of inferences. Prof. Gage needed to infer meaning behind Robert's imprecise language: "that's a specific circumstance … where that happens to work." (Prof. Gage attempted to question Robert on just that point.) Then, even without necessarily being clear of what Robert himself perceived, Prof. Gage would have needed to recognize that $P(t) = 2e^t$ also "happens to work," because, in Robert's words, "just, by going back to the original equation [$dP/dt = P$], you can just substitute between P and e^t [$2e^t$], you can derive the equation that says P is equal to e^t [$2e^t$]." Finally, he would then have needed to judge how such a connection, once made with Robert and the rest of the class, might be used further to assist students in understanding the difference between the two differential equations at hand, as well as the reason that one rather than the other better modeled the population situation in question. All of this required that Prof. Gage be able to take a student's unfamiliar and perhaps surprising way of perceiving the mathematical situation, understand the student's perspective, and simultaneously "translate" it into a more mathematically normative and useful idea. In short, a significant amount of "mathematical work" would be required of him on-the-spot and while feeling overwhelmed by his classroom roles. We contend that Prof. Gage's difficulties were therefore rooted in an absence of just such MKT that would have allowed him to understand more quickly Robert's idea and its value for the discussion. Instead, Prof. Gage needed to engage in more complex cognitive efforts to follow Robert's reasoning—which, in this case, were unsuccessful in revealing the important and pedagogically useful mathematical ideas beneath Robert's words.

Conclusions and implications

Our analyses suggest that, despite many years of teaching and while possessing strong content knowledge, mathematicians may still face challenges similar to those that some K-12 teachers face when enacting aspects of reform-oriented instruction. This lends credence to the claim that there are types of knowledge important for teaching that are tied to specific content and ways of teaching that content, and that such knowledge is not derived exclusively from one's knowledge of the content.

Although the inquiry-oriented approach to instruction was new to Prof. Gage, the general pedagogical skills and knowledge with which he began the course were sufficient to support the development of practices for helping students participate in discussions and contribute ideas. In a similar vein, although the mathematical ideas were organized differently than in other DE courses he had taught, his mathematical content knowledge was certainly sufficient to enable him to recognize and understand the content of the IO-DE curriculum. The difficulties arose when he needed to rely on PCK and/or MKT to provide direction and structure to the discussions. Although he likely possessed PCK that he had developed and used while teaching other differential equations courses, that PCK was inadequate for anticipating what students would think and do with the IO-DE materials and insufficient to help him out of the various difficult situations in which he found himself when students did not offer productive ideas.

In the midst of large-group discussions, he was also sometimes unable to leverage his considerable mathematical content knowledge to follow and interpret students' ideas. The ideas students generate are particular to the problems they attempt to solve, and since those problems were new to him and of a different sort than those in his other DE courses, he had no prior opportunities to observe the ideas students might come up with. As a result, following and interpreting students' ideas required considerable effort and attention. Such effort, however, was not always possible for Prof. Gage as he simultaneously attended to his other instructional responsibilities. If he had the MKT necessary to recognize and quickly understand students' ideas, he might have found the overall challenge of orchestrating discussions less daunting.

We assume that through additional experience with these materials and the instructional practices that accompany them, Prof. Gage would likely develop PCK and MKT specific to the teaching of this course. A continued research goal, however, is to develop ways to support and perhaps expedite this learning for teachers such as Prof. Gage and others who wish to develop and expand their teaching practices.

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