Learning advanced mathematical concepts by reading text

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Abstract. In this paper, we analyze the way two groups of students—successful mathematics majors and less successful mathematics majors—attempted to learn a new mathematical concept. We presented both groups of students with a textbook-style presentation of the concept and then asked them to complete homework-type exercises. The successful mathematics students were more likely to reformulate the concept definition in their own words, connect the new concept to prior mathematical concepts that they had studied, try to understand what a theorem is asserting before reading its proof, and try to understand the statements in their exercises before attempting to prove them.

1. Introduction

As undergraduate majors enter their advanced proof-oriented courses, there is a dramatic shift in how they are expected to reason about mathematical concepts (Moore, 1994). Advanced mathematical concepts are defined in precise, unambiguous terms, usually using formal logical notation. Properties of advanced mathematical concepts must be justified by deducing them from the concept's definitions (Tall, 1989). A large body of literature demonstrates that undergraduates have serious difficulties in understanding and reasoning about mathematical concepts; many students do not understand the role that mathematical definitions play in advanced mathematics (e.g., Vinner, 1991), have difficulty in understanding and reasoning about the logical notation in a concepts' definition (e.g., Dubinsky et al., 1994), cannot write proofs about them (e.g., Moore, 1994; Weber, 2001), and believe that concepts possess properties that they do not possess (e.g., Tall & Vinner, 1981; Vinner, 1991).

The purpose of this paper is to address the following research question: How do successful mathematics majors develop an understanding of mathematical concepts by reading traditional textbooks? The answers to this question can be used to address the broader issues of how students can successfully develop an understanding of advanced mathematics and what types of instructional activities can foster their development.

2. Related literature

2. 1. Presentation of advanced mathematical concepts in traditional settings

When professors introduce new concepts in advanced mathematics courses, they typically do so using a "definition-theorem-proof" format (Weber, 2004). That is, after introducing a concept and its definition, the professor then states theorems about the concept and provides rigorous proofs to justify these theorems. Davis and Hersh (1981) characterize this practice by averring that "a typical lecture in advanced mathematics… consists of definition, theorem, proof, definition, theorem, proof in solemn and unrelieved concatenation" (p. 151). Raman's (2004) analysis of textbooks used in advanced mathematical courses reveals that textbooks introduce concepts in a similar manner. Many mathematics educators and mathematicians are critical of this practice (see Weber, 2004, p. 116-117 for a brief review of this literature), in part because understanding a mathematical concept involves more than knowing its definition and drawing logical deductions from it.

2. 2. Learning advanced mathematical concepts

The issue of how students can learn to understand advanced mathematical concepts is of obvious importance to collegiate mathematics educators and there has been considerable research on this topic. Most of this research has been conducted using design research (Cobb et al., 2003) or a similar methodology. Within this paradigm, researchers engage in an iterative cycle in which they first postulate how students might come to understand a concept and then design activities to foster this understanding. They then implement this instruction with a group of students. An analysis of this implementation is used to refine the theoretical assumptions of how students can learn and the instructional activities. In Asiala et al.. (1996), leading researchers in collegiate mathematics education advocate for this type of research. Research within this framework has produced effective instruction for advanced mathematical courses (e.g., Dubinsky et al., 1994; Asiala et al., 1996; Larsen, 2004).

2. 3. How students learn advanced mathematical concepts in traditional settings

Research on how students learn advanced mathematical concepts under traditional instruction is sparse. Dahlberg and Housman (1997) observe that generating examples of advanced mathematical concepts creates learning opportunities for students, but example generation is not a process that most students engage in during traditional instruction (e.g., Moore, 1994). Several other studies observe how students learn advanced mathematical concepts over the course of a semester (e.g., Pinto & Tall, 1999; Alcock & Simpson, 2004). However these studies focus on stages that students progress through as they come to understand a concept but do not focus on *how* students develop this understanding.

Traditional instruction of advanced mathematics is widely criticized and is generally regarded as unsuccessful. Nonetheless, some students are able to learn in traditional classrooms. Understanding how these students learn can be useful for two reasons. First, this may suggest possible learning trajectories that might be useful for less successful students. Second, while the design-based research described in the previous section has produced effective instruction, this instruction has not enjoyed widespread application, in part because its implementation requires mathematics professors to radically alter their approach to teaching. Understanding the strategies that successful students use in traditional settings may lead to instructional recommendations that do not require professors to completely overhaul their instruction.

2. 4. Learning by reading text

In advanced mathematics courses, students are expected to learn in large part by studying definitions (Dahlberg & Housman, 1997) and reading proofs (Selden & Selden, 1995). There is a large body of work on reading comprehension that delineates the cognitive processes and describes the cognitive strategies used by readers to comprehend text. Palinscar and Brown (1984) describe metacognitive strategies that are used by good readers. When reading, good readers will engage in strategies such as summarizing the paragraph that they just read, relating what they read to their prior knowledge, and predicting the direction in which they think the argument is heading. These strategies serve two purposes. The first is to *foster comprehension* by allowing the reader to construct an interpretation of the text that is meaningful to them. The second is to *monitor comprehension*; if the reader is unable to execute a metacognitive strategy (e.g., if they are unable to summarize a paragraph or an argument takes a turn that they did not expect), this serves as a cue to the reader that comprehension is not proceeding smoothly and remedial action, such as re-reading the text or seeking clarification, is necessary.

Chi and her colleagues argue that generating self-explanations is necessary for students to understand scientific texts (Chi et al., 1989, 1994). Because scientific texts would be impossibly long if every detail of the argument was spelled out, even quality expositions require the reader to fill in substantial details. Successful students will routinely generate explanations for how new sentences in the text they read connect to previous sentences and their own informal understandings. Chi et al. (1994) conjecture that generating self-explanations facilitates learning because this generation requires readers to constructively interpret the text that they are reading, connect what they are reading to their prior knowledge, and at times compel them to resolve inconsistencies between what is being asserted in the text and what they believe to be true.

While there is little literature on how students learn mathematics by reading text, we believe that good students may use analogous strategies. Because understanding a concept involves more than being able to state its definition (Vinner, 1991), students must engage in activities that allow them to build personally meaningful images of the concept that are aligned with the concept definition. Because proofs would be intractable if every logical detail was spelled out (e.g., Davis & Hersh, 1981), details of how new assertions follow from previous ones are often left to the reader sometimes requiring the reader to invoke his or her prior knowledge (Weber & Alcock, 2005). A central purpose of this study is to understand the strategies that successful students use to learn advanced mathematical concepts from traditional text.

3. Methods

3. 1. Materials

This study examines how students came to understand the concept of *function preservation*, which has the following definition: Let f be a real–valued function and A be a subset of the real numbers. f is defined to be **preserved on** A if $f(A)$ is a subset of A.

The first author generated this concept, believing it would be appropriate for this study for the following reasons. First, this definition is based on basic concepts from set theory that were covered in the students' transition-to-proof course. Thus, the participants, who had all successfully completed a transition-to-proof course, should possess the background knowledge to comprehend the definition. Second, the concept of function preservation is not directly analogous to other concepts in advanced mathematics so all participants would be learning this concept for the first time. Third, despite its relative simplicity, this concept was rich enough that it could be represented in a variety of ways (symbolically, by prototypical examples, graphically, diagrammatically) and interesting theorems could be proven about it.

The first author created a written presentation of this concept that we believe is representative of how textbooks in undergraduate mathematics courses introduce new concepts. In pilot studies, mathematicians and undergraduate students agreed that this prose was representative of a textbook presentation. The written presentation consisted of four parts, each on a separate sheet of paper: (a) the definition of the concept, (b) two examples and one nonexample of the concept, (c) three theorems and proofs about the concept, and (d) nine exercises about the concept. This presentation can be found in the Appendix of this paper.

3. 2. Participants

Two groups of students participated in this study:

- Eight undergraduate mathematics majors who completed a course in real analysis and earned greater than a 3.4 GPA in their proof-oriented mathematics courses. These students are referred to as the *successful mathematics majors*, or SMMs.
- Eight undergraduate mathematics majors who completed a course in real analysis and earned lower than a 3.0 GPA in their proof-oriented mathematics courses. Their analysis teachers indicated that these students put forth adequate effort in their real analysis course but still struggled. These students are referred to as the *less successful mathematics majors*, or LSMMs.

3. 3. Procedure

Participants each met individually with the first author for a task-based interview that was audio-taped. Participants were asked to "think aloud" while they attempted to understand a new mathematical concept. At the start of the interview, they were given the definition of function preservation that they were allowed to read until they felt they understood it. They were then given a sheet of paper with examples of the concept. When they indicated that they understood the prose on this sheet, they were given a sheet of paper with theorems and proofs. Next, participants were asked to complete the nine exercises in any way they saw fit. (Participants' written work was collected as data.) Finally, participants were asked a series of open-ended questions about their mathematical study habits in general (e.g., "Describe for me some of the things that you do when you encounter a new mathematical concept?") and about interesting behavior they exhibited during the interview. Participants were given as long as they liked to complete each stage of the interview; most interviews lasted about one hour.

3. 4. Analysis

All audiotapes were transcribed and the data were analyzed using Chi's (1997) method for coding verbal data. In a first pass through the data, an inductive coding scheme was generated to describe the strategies that participants used to understand the function preservation concept. After this coding scheme was operationalized, two coders independently coded all the data. This coding allows us to count occurrences of strategy usage and present our qualitative findings in a quantitative form.

4. Results

4. 1. Reading the definition

The strategies used by both groups of participants are presented in Table 1.

TABLE 1. Reading the definition

	SMMs	LSSMs
Reformulate definition	8	5
Relate definition to another concept	5	0
Draw a diagram	3	0
Recall related definitions	1	2

Reformulating the definition—When reading the definition of the concept, the majority of the participants attempted to provide an alternative formulation of the concept. Three examples of reformulation are provided below:

> SMM6: Okay, I guess that means that if the operation that *f* performs on the elements of A is closed over A. I think I understand that.
>
> SMM5: Preserved. *f* maps a subset on itself. Okay. I understand that.
>
> LSMM2: So, does this just mean that the output of everything that is in A is also in A? Yeah, okay, the output of f is in the set A. Okay.

In most instances of reformulation, the participants expressed the ideas of the definition in informal mathematical language, but this was not always the case. For example, SMM4 provided a logically equivalent definition to the one provided, noting that f was preserved on A if f restricted to A was a function from A to A. All of the SMMs reformulated the concept, but three of the LSMMs did not.

Relating function preservation to other mathematical concepts. When reading the definition of function preservation, five SMMs compared function preservation to other concepts that they had learned, such as linear transformations and scalar multiplication in vector spaces and closed group operations. No LSMM attempted to compare function preservation with another mathematical concept.

Other observations. Three other observations are worth noting. First, no participant generated an example of a function preserved over a particular set when they read the concept definition. This differs from Dahlberg and Housman's (1997) finding that some strong students will spontaneously generate examples of new concepts but is consistent with Moore's (1994) claim that undergraduates are frequently unwilling or unable to generate examples of concepts without prompting. Second, there were three instances in which the SMMs considered a diagram of what it would mean for a function to be preserved on a set. No LSMM considered a diagram. Third, the participants spent relatively little time studying the definition of the concept. Thirteen of the 16 participants spent under two minutes studying the concept; the other three participants (one SMM, two LSMMs) spent longer primarily because they had difficulty recalling what it meant to evaluate a function over a set.

Summary. Overall, the SMMs processed the definition of function preservation more extensively than the LSMMs. The SMMs were more likely to reformulate the definition, connect function preservation to other mathematical concepts, and draw or imagine a diagram illustrating the concept. Dahlberg and Housman (1997) argue that reformulating concept definitions allows students to build an image of the concept definition, something they believe is necessary for students to use the definition in problem solving and proof writing. Drawing analogies between function preservation and other concepts enables students to build links between what they are learning and what they already know, a constructive process that Chi et al. (1994) argued is essential for understanding scientific text. As Hiebert and Lefevre (1986) argued that conceptual understanding consists of the links one has between mathematical concepts, comparing new concepts to old ones may be directly beneficial to students.

4. 2. Reading examples, theorems, and proofs

The strategies that both groups of students used to read the examples, theorems, and proofs are presented in Table 2.

TABLE 2. Reading examples and proofs

	SMMs	*LSMMs*
Reading the claim to be proven		
Reformulate the claim	5(8)	2(2)
Recall relevant definitions	3(3)	1(1)
Evaluate/prove statement prior to reading the proof	6(11)	3(5)
Re-read the statement	5(9)	4(5)
Ask for clarification	4(6)	0(0)
Reading the proof of the claim		
Rephrase part of the argument	5(9)	6(12)
Justify a step within a proof	4(7)	7(17)
Anticipate arguments while reading the proof	3(4)	0(0)
Re-read the proof	3(3)	0(0)

The table lists number of distinct participants, with the number of instances in parentheses.

4. 2. 1. Reading the statement prior to reading the proof

Reformulating the statement. There were eight instances in which the SMMs reformulated the statement that was being proved prior to reading its proof. In the excerpt below, SMM5 reformulated Theorem 2. Like many participants, SMM5 had initial difficulty understanding what Theorem 2 was asserting, but resolved this difficulty by reformulating the statement in a way that made sense to him.

> SMM5: [reading the theorem] r is real and there is a set that's not real and r is in that set. OK, that much I don't get […] Oh, OK, it's not the whole number line, all the real numbers. So r is in this set and f is closed on this set, a non-trivial subset of the real numbers containing the number.

Evaluating the veracity of the statement. SMMs were twice as likely as LSMMs to consider whether the statement was true prior to reading its proof. In some cases, the participants attempted to make this judgment by constructing a justification or proof in support of the statement; in others, they provided an intuitive argument or no argument at all (e.g., saying, "that seems true" or "that makes sense"). Two instances of participants evaluating a statement's veracity are provided below:

> SMM8: [Reading Theorem 1] Then f is preserved on A union B. OK. Makes sense. If x is in either A or B, then $f(x)$ should be in either. Let's see… proof.
>
> LSMM5: [For Example 1] [Student writes: $-1 \leq x \leq 1$ // $1 \leq x^2 \leq 1$// $x^2 \leq 1$ // x^2 is in [0, 1]]

Clearly LSMM5's reasoning for why $f(x)=x^2$ is preserved on [0, 1] was invalid, but as this study focuses on strategies students employ, rather than their success in employing them, this was still coded as evaluating the statement.

Re-reading the statement. SMMs were more likely to re-read the statement. In general, they did so when they were unable to comprehend what they just read, as is illustrated in the two excerpts on Theorem 2 below:

> SMM6: "Theorem two, let r be a real number. Then there is a set C sub r which is not in the real numbers, so it's a complex number, right? Such that r is a complex number. What? Wait, wait, wait. Okay. r is a real number."
>
> SMM2: "There is a non-trivial subset of the real numbers containing that number that f is preserved on. Okay, what does that mean? Let R be a real number, C_r not equal to R such that R is an element of C_r."

In the first excerpt SMM6 is rephrasing Theorem 2 in his own words as he is reading it. Upon reading $C_r \neq \mathbf{R}$, he erroneously infers that C_r must consist of complex numbers. From there, he infers r must be complex after reading r is an element of C_r. To SMM6, this contradicts the earlier claim that r is real. Upon recognizing this contradiction, he returns to the beginning of the statement to read it again. Similarly, when SMM2 is unable to understand an informal summary of Theorem 2, he re-reads the statement from the beginning.

Request for clarification. There were six instances when SMMs asked the interviewer for clarification on some aspect of the statement to be proven. Generally, this consisted of attempts to understand notations or conventions in the proof. No LSMM did this. As it seems reasonable to expect that SMMs would be at least as familiar with mathematical notation and convention as LSMMs, this implies LSMMs were more likely to accept a notation they did not understand without asking for assistance.

Summary. Selden and Selden (1995) observe that students often have difficulty understanding what mathematical propositions are actually asserting and consequently will not find proofs of these propositions to be convincing. In general, the SMMs spent more time trying to understand what was being asserted than the LSMMs. The SMMs' strategies for developing this understanding where analogous to Palinscar and Brown's (1984) comprehension-fostering and comprehension-monitoring metacognitive strategies. The SMMs tried to foster comprehension by understanding what the proposition was asserting (by describing it in their own words) and seeing if it made sense (by evaluating the statement's veracity). When they were unable to do so, they would take remedial actions such as re-reading the text or asking for clarification.

4. 2. 2. Reading the proofs

Summary. Perhaps surprisingly, the LSMMs were more likely to rephrase aspects of the proofs in their own words and justify particular steps within a proof. There are (at least) two hypotheses that can account for this observation. First, perhaps more of the statements were obvious to the SMMs than to the LSMMs, so the SMMs felt less need to justify these statements. (In a study conducted with mathematicians reading proofs, they offered no justification for the large majority of assertions within a proof that they read (Weber, 2008)). Second, it is possible that focusing on the line-to-line logic within a proof is less important for mathematical learning than the first author has previously asserted elsewhere (e.g., Weber & Alcock, 2005; Weber et al., 2008). It may be the case that stronger students focus on the larger ideas in the proof and spend less time verifying the mundane details within a proof. More research is needed to determine which hypothesis (if either) is correct.

4. 3. Completing the exercise

Not surprisingly, the SMMs collectively outperformed the LSMMs on the exercises, getting a greater percentage correct (82.3% vs. 53.4%), despite trying more of the advanced exercises. The strategies each group used to complete the exercises are presented in Table 3.

TABLE 3. Completing the exercises

	SMMs	*LSMMs*
Reading the problem statement		
Reformulate the statement	6(15)	1(1)
Evaluate truth or falsity of statement prior to proof writing	8(26)	5(6)
Re-read the exercise	4(8)	3(5)
Recall definitions	7(14)	7(12)
Referring to text while completing the exercise		
Refer to text at all	5(10)	8(33)
Use text as a template for a proof	0(0)	7(14)
Use text to find useful results	3(3)	3(3)
Use text to see how new argument should be written	2(2)	3(4)
Strategies used while writing proofs		
Give justification or informal argument prior to writing proof	7(12)	3(5)
Consider truth or falsity of statement after proof has begun	6(12)	2(4)
Construct a diagram	6(8)	0(0)
Consider an example	3(3)	3(4)
Evaluate proof for correctness	2(6)	2(3)

The table lists the number of distinct participants, with the number of instances in parentheses.

4. 3. 1. Prior to attempting to write a proof

Reformulating the statement. SMMs frequently would reformulate a statement prior to writing a proof. Two examples of this are provided below:

> SMM3: [For number 6] f of Z is preserved on Z and g of Z is preserved on Z and we want that f plus g is preserved on Z. So...wait a minute...f and g are real-valued functions. So, maybe we're saying that if you take an integer and you apply function f or g to that integer, then you get an integer. So then, this is basically just saying that the sum of two integers is an integer.
>
> SMM5: [on number 8] We've got a subset [referring to E]. So by subset you might mean that…it could probably mean D as well….So we're looking for..there exists a set D and f is preserved on E. And E contains the real numbers. E could be a lot of different things. We don't know if *f* is preserved on D…question… we could show the epsilon-delta crap and proofs of 311 [advanced calculus].

These two episodes illustrate the benefits of reformulation. In the first episode, SMM3's reformulation of exercise 6 essentially is an informal version of the proof he then produced. In the second episode, SMM5's reformulation of exercise 8 enabled her to recognize it as a "for all-there exists" statement, which suggested an appropriate proof framework (Selden & Selden, 1995).

Evaluating the veracity of the statement. There were 26 instances in which the SMMs evaluated the veracity of the statement prior to constructing a proof, as compared with only 6 from the LSMMs. That the SMMs made these evaluations so often was not surprising—there were five exercises in which students were asked to "prove or disprove" so determining whether a statement was true or false was necessary to decide how to proceed. In contrast, LSMMs frequently assumed the "prove or disprove" statements were true and attempted to prove them.

Summary. Selden and Selden (1995) observe that students frequently have difficulty determining what is being asserted in a mathematical statement. Consequently, these students often have trouble proving statements, since they cannot determine what to assume and what to conclude. The SMMs' reformulation of the statements and determination of whether to produce a proof or counterexample allowed them to understand what they needed to prove, which helped them in their proof construction. Schoenfeld (1985) observed that good problem solvers spent more time understanding their task than weak problem solvers, who tended to dive right into calculations. This finding is also present in our data; SMMs spent more time determining what they needed to prove, whereas LSMMs spent less time processing the statement to be proven.

4. 3. 2. Constructing the proofs

Referring to text. When doing the exercises, participants were permitted to refer to the written text they had previously read. Both SMMs and LSMMs did so, although LSMMs did so three times as often and earlier in their proof attempts. The SMMs and LSMMs used the text in different ways. The SMMs used the text primarily to look up definitions or see how a proof they were writing should be structured. The LSMMs would often use the text to write *template-based proofs* (cf., Weber, 2005), locating a proof of a theorem similar to the statement they were proving, and producing a nearly identical proof by changing variables within the proof. For instance, LSMM5 wrote the following proof for Exercise 2:

Let $f(x) = x/2$. Show that f is not preserved on Z.
Counterexample. $A = [0,2]$
Let $x \in A$, picking $x = 1$
$f(1) = 1/2$ □ $[0,2]$.
So $f([0,2])$ □ $[0,2]$.

By itself, this argument looks bizarre. However LSSM5's work makes sense given that she based her proof directly on the proof given for Example 3. (Both Exercise 2 and Example 3 required proofs showing a particular function was not preserved over a given set). For Exercise 4, which asks students to prove or disprove that if a function *f* was preserved on A and B, then it was preserved on A ∩ B, LSMM6 wrote a proof nearly identical to that of Theorem 1, simply replacing every union with an intersection. Hence, her proof had invalid inferences, such as "since *f(x)* is a member of A, *f(x)* is a member of A ∩ B". In her post-interview, I asked LSSM6 about this statement:

LSSM6: [surprised] Oh… Wow! That's not true at all.

K: Would you be able to say it […] Would you be able to see a way to…?

LSSM6: Yeah, yeah. Because we know *x* is in A and *x* is in B, which means that *f* of *x* is in A and f of x is in B since f is preserved on A and B. So then *f* of *x* is in the intersection of A and B.

This excerpts illustrates two important points. First, LSMMs relying on template-based proofs would write assertions that, at least with reflection, they knew were not true. (Similarly, it seems unlikely that LSMM5 believed that 1/2 was not in the interval [0, 2] as she wrote in the invalid proof presented above). In other words, the LSMMs sometimes suspended sense-making in their template-based proofs. Second, in some cases, the LSMMs could have produced a valid proof had they chosen to rely on their own mathematical resources, instead of relying on templates.

Other observations. The SMMs were more likely than the LSMMs to give an intuitive justification for why an assertion was true prior to writing a formal proof (SMM3's reformulation of Exercise 6 presented earlier in the text is an example of this). SMMs also were more likely to reconsider the veracity of the statement they were proving during their proof attempt. Although the SMMs sometimes produced diagrams (usually Venn diagrams) and the LSMMs did not, these diagrams were generally not helpful and sometimes misleading. For instance, several SMMs used diagrams to convince themselves that Exercise 4 was false, when in fact it was true.

Summary. Elsewhere we argue that what an individual has the opportunity to learn by constructing a proof depends on what that individual focuses on during the construction (Weber, 2005). LSMMs often relied on template-based proofs, which is consistent with the finding that less successful students are likely to abandon sense-making when using worked examples to solve new problems (e.g., VanLehn, Jones, & Chi, 1991). While this may provide the LSMMs practice with learning procedures that can prove a class of mathematical statements, producing template-based proofs will likely not provide LSMMs with the opportunity to refine their understanding of the concepts they are studying or obtain conviction or understanding in the assertions they are proving. The SMMs sometimes produced an intuitive argument for why assertions should be true before providing formal justifications, while the LSMMs rarely did this. Basing proofs on informal ideas (cf., Weber & Alcock, 2004) can provide students with the opportunity to evaluate, refine, and expand their understanding of the concepts being studied.

5. Discussion

The four central findings from this study were:

- Although both SMMs and LSMMs studied the definition of function preservation quickly, the SMMs did more to build an understanding of the definition. They were more likely to reformulate the definition, connect the definition to other concepts that they had learned, and express the concept diagrammatically.
- When reading theorems and their proofs, the SMMs used comprehension-fostering strategies to understand what was being asserted took remedial actions such as re-reading the statement or asking for clarification if they did not comprehend what they had read.
- When completing the exercises, the SMMs spent more time understanding what was being asserted in the exercises prior to attempting to construct proofs or disproofs. In particular, they were more likely to reformulate a statement in their own words and judge whether the statement in question was true or false. In contrast, LSMMs were more likely to dive directly in their proof attempts.
- The LSMMs sometimes proved statements by writing template-based proofs, a process during which they suspended sense-making and focused on manipulation.

What is striking is not that the LSMMs were unable to rephrase the definitions, theorems, and exercises that they read or connect what they learned to their prior knowledge, it is that they *usually did not try to do so*. Hence, their failure to build understanding of advanced mathematical concepts appears due, at least in part, to an unproductive epistemology of what advanced mathematics is about, what it means to understand it, and how it can be learned. A potential direction toward improving advanced mathematics courses would be to engage more students in the sense-making strategies that SMMs used. How this can be done effectively is a topic in need of research.

Two notes of caution should be expressed. First, it might be tempting to think that perhaps the instructor should do the strategies for the students—e.g., the instructor could reformulate definitions and theorems when he or she presents them. While we wouldn't discourage this practice, we question whether it would be effective. Based on a meta-analysis of 19 studies, Webb (1989) found that when students *provided* elaborate explanations for a phenomena, substantial learning tended to occur; however, receiving others' elaborate explanations was only weakly correlated with student achievement. The benefits from the SMMs' learning strategies may be due, in part, to the fact that the SMMs actively constructed ways of understanding that were personally meaningful to them (cf., Chi et al., 1994). An instructor's explanations would not have this benefit.

Second, it is unclear if simply asking LSMMs to engage in the learning strategies that SMMs used would, by itself, improve their understanding of advanced mathematical concepts, since the LSMMs might not have the abilities or understandings to implement these strategies effectively. Although Chi et al. (1994) found that simply asking students to provide self-explanations improved their comprehension of scientific text, Palinscar and Brown's (1984) students required intensive training before they were able to successfully implement the comprehension-fostering and comprehension-monitoring strategies that these researchers endorsed.

Learning advanced mathematical contexts involves drawing connections and making inferences that go beyond what is explicitly stated by a professor or written in text. Successful mathematics majors employ strategies for building this understanding that less successful students do not use. One way to improve traditional instruction in advanced mathematics is to create activities or environments that would engage *all* students in using these strategies. Designing these activities or environments may not be trivial, but would be a productive direction for future research.

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APPENDIX

Definition**:** Let f be a real-valued function. Let C $\subseteq \mathbb{R}$. Then f is **preserved on** C if and only if f(C) $\subseteq$ C.

Example: Suppose $f(x) = x^2$.

Claim 1: $f(x)$ is preserved on [0, 1].

Proof. Let $x \in [0, 1]$. Since $x \geq 0$, $x^2 \geq 0$. Since $|x| \leq 1$, $x^2 \leq 1$. Therefore $0 \leq x^2 \leq 1$. So if $x \in [0, 1]$, then $f(x) \in [0, 1]$. Hence, $f([0, 1]) \subseteq [0, 1]$.

Claim 2: $f(x)$ is preserved on **Z**. (Note: **Z** denotes the set of integers).

Proof: Let x be an integer. Then x^2 is an integer, because an integer times an integer is an integer. Therefore, if $x \in$ **Z**, $f(x) \in$ **Z**. Hence, f(**Z**) $\subseteq$ **Z**.

Claim 3: $f(x)$ is not preserved on [0, 2].

Counter-example: $1.5 \in [0, 2]$. $f(1.5) = (1.5)^2 = 2.25 \notin [0, 2]$. Therefore, f([0, 2]) $\not\subseteq$ [0, 2].

Proposition 1. For any real-valued function f, f is preserved under $\mathbb{R}$ and $\varnothing$.

Proof. Let $x \in \mathbb{R}$. Since f is a real-valued function, $f(x) \in \mathbb{R}$. Thus, $f(\mathbb{R}) \subseteq \mathbb{R}$, and f is preserved on $\mathbb{R}$. Since $f(\varnothing) = \varnothing \subseteq \varnothing$, f is preserved on $\varnothing$.

Theorem 1. Let f be a real-valued function and A and B be subsets of $\mathbb{R}$. If f is preserved on A and f is preserved on B, then f is preserved on A $\cup$ B.

Proof. Let $x \in$ A $\cup$ B. Then $x \in$ A or $x \in$ B. Suppose $x \in$ A. If $x \in$ A, then $f(x) \in$ A because f is preserved on A. Since $f(x) \in$ A, then $f(x) \in$ A $\cup$ B. Similarly, if $x \in$ B, then $f(x) \in$ B, and $f(x) \in$ A $\cup$ B. Therefore, if $x \in$ A $\cup$ B, $f(x) \in$ A $\cup$ B. This is sufficient to show that f is preserved on A $\cup$ B.

Theorem 2. Let r be a real number. Then there is a set $C_r \neq \mathbb{R}$, such that $r \in C_r$ and f is closed on C_r. In other words, for every real number, there is a non-trivial subset of the real numbers containing that number that f is preserved on.

Proof. Let $r \in \mathbb{R}$. Define r_i for $i \in \mathbb{N}$ as follows. (Note, $\mathbb{N}$ denotes the set of natural numbers (1, 2, 3,…)). Let $r_1 = f(r)$. Let $r_2 = f(r_1)$. Let $r_3 = f(r_2)$. In general, let $r_{i+1}=f(r_i)$. Let C_r be the set containing r and r_i for all $i \in \mathbb{N}$. Clearly $r \in C_r$.To show that f is preserved under C_r, choose an $x \in C_r$. Either $x = r$ or $x = r_i$ for some $i \in \mathbb{N}$. If $x = r$, then $f(x) = r_1$ so $f(x) \in C_r$. If $x = r_i$ for some $i \in \mathbb{N}$, then $f(x) = r_{i+1} \in C_r$. Thus, $f(C_r) \subseteq C_r$ so f is preserved on C_r.

Exercises

(Note, $\mathbb{R}$ denotes the set of real numbers. **Z** denotes the set of integers. Note, $\mathbb{N}$ denotes the set of natural numbers (1, 2, 3,…)).

1. Let $f(x) = 3x$. Show that f is preserved on **Z** and $\mathbb{N}$. Find a set C that f is not preserved on.
2. Let $f(x) = x/2$. Show that f is not preserved on **Z**.

3. Let $f(x) = x/2$. Let A be the set {1, 1/2, 1/3, 1/4, …, 1/n, …}. Show that f is preserved on A.

For exercises 4-7, assume f and g are real valued functions and A and B are subsets of .

Prove or disprove the following statements:

4. If f is preserved on A and f is preserved on B, then f is preserved on A □ B.
5. If f is preserved on A and g is preserved on A, then $f(g(x))$ is preserved on A.
6. If f is preserved on **Z** and g is preserved on **Z**, then $f + g$ is preserved on **Z**.
7. If f is preserved on A and g is preserved on A, then $f + g$ is preserved on A.
8. Let D be a finite subset of the real numbers and let f be a real-valued function. Show that there exists a set E such that D □ E and f is preserved on E.
9. Let f be a bijective function that is preserved on a set A. Prove or disprove that f^{1} is preserved on A.