Taiwanese Undergraduates' Proof Performance in the Domain of Continuous Functions

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Abstract

Recently, a growing number of studies in the United States have provided evidence that students have difficulty with proofs in advanced mathematics courses (Moore, 1990, 1994; Weber, 2001). Few research studies, however, have focused on undergraduates' abilities to produce proofs and counterexamples in the domain of continuous functions and, more specifically, on Taiwanese undergraduates' abilities. In this study, we examine Taiwanese undergraduates' performance constructing proofs and generating counterexamples in the context of continuous functions. While this study is not designed as a comparative study, our analysis provides results that may be compared with existing empirical studies. Such comparisons can provide insight into performance differences among undergraduate mathematics students—insights that may be particularly meaningful given that Taiwanese school mathematics students score consistently high on international achievement tests. More importantly, our study has broader implications for instructors who would like to improve undergraduates' proof performance in advanced mathematics courses.

Introduction

Many consider proof to be central to the discipline of mathematics and the practice of mathematicians. Accordingly, undergraduates in advanced mathematics courses are expected to master the skills required to both construct proofs and generate counterexamples. Existing empirical studies in the United States, however, have shown that many undergraduate students have difficulty with proof (Harel & Sowder, 1998), particularly with constructing formal proofs in advanced mathematics courses, such as introductory group theory and other introduction courses to higher mathematics (Moore, 1990, 1994) and abstract algebra (Weber, 2001). Yet few studies have focused specifically on undergraduates' proof performance in the domain of continuous functions, and even fewer on Taiwanese undergraduates' proof performance.

In this study, we examine thirty-six Taiwanese undergraduate mathematics majors' performance constructing proofs and generating counterexamples in the domain of continuous functions—an important area within the undergraduate mathematics curriculum. This study was guided by the following two research questions: (1) How well do Taiwanese undergraduate mathematics majors[1] construct proofs and generate counterexamples in the domain of continuous functions? (2) What errors appear in the proofs students construct or the counterexamples they generate? The data consist of students' responses to a written assessment in which they either constructed proofs for statements they considered to be true or generated counterexamples for statements they considered to be false.

While this study is not designed as a comparative study, it does provide results that can be considered in relation to existing empirical studies in the United States. Given the successful performance of Taiwanese elementary and secondary school students on international

[1] Further references to the Taiwanese undergraduate mathematics majors will be abbreviated to the Taiwanese undergraduates for the sake of simplicity.

mathematical achievement tests, such comparisons may provide insight not only into performance differences among undergraduate mathematics students, but more importantly into the design of curriculum and instruction that may lead to improvements in undergraduate students' proof performance in advanced mathematics courses.

Theoretical Perspective

In this section, we discuss the relationship between concept definitions and either proofs or counterexamples. Vinner and colleagues (Tall & Vinner,1981; Vinner, 1983; Vinner & Dreyfus, 1989) define concept definition as a *formal definition that can be written or spoken in mathematical language*. According to Chin and Tall (2000), students often develop an understanding of concepts informally before learning their formal definitions. Moreover, informal concept definitions are essentially informal descriptions of syntactic knowledge with students' own language and often with partial correct understanding of the formal definition.

From Tall's (1989) point of view, a mathematical proof requires that "*clearly formulated definitions and statements*" or "*agreed procedures*" are used to "*deduce the truth of one statement from another*" (p. 5). Moore (1990, 1994), however, notes that students often lack a relevant understanding of concept definitions or they often do not know how to apply the concept definitions in the specific domain of writing proofs. Zaslavsky and Peled (1996) found that mathematics teachers, who received an undergraduate degree in mathematics, and student teachers, who have completed several advanced mathematical courses, also have difficulties generating counterexamples due to their lack of conceptual understanding. Taken together, proofs and counterexamples involve checking true and false statements. Without adequate understanding of the content of informal and formal definitions, it is likely difficult for students to determine the truth or falsity of a statement as well as to produce a proof if the statement is deemed true or a counterexample if deemed false. In this study, we take the view of concept definitions regarded as informal and formal definitions to explore Taiwanese undergraduates' proof performance in the domain of continuous functions.

Methods

Participants

Participants in this study were selected by convenience sampling as participants were contacted by colleagues of the researchers, and were selected on the basis of their willingness to participate in the study. Participants in this study included thirty-six Taiwanese undergraduates enrolled in Advanced Calculus I in Fall 2007 at a national university in Taiwan. Continuous functions were addressed in a previous calculus course, thus all of the students participating in this study had some relevant domain knowledge. Students in Advanced Calculus I are usually college-aged and disproportionately male. This study does not target participants based on age, gender, or other characteristics, but the course enrollment gender distribution provided more men than women as participants.

Instrument

The instrument was written in English because English is used in advanced calculus courses at the university in Taiwan. The instrument, which was comprised of five mathematical statements constructed by the researchers, was designed to provide a measure of students' concepts of continuous functions as listed in Table 1. The mathematical statements were designed to (a) reflect understanding of continuous functions, (b) be representative of basic types of proofs and counterexamples in continuous functions, and (c) be completed by each participant in approximately 30 minutes. The instrument was finalized after pilot testing with Taiwanese undergraduates and graduate students with a major in mathematics.

Table 1

Five Propositions Used in This Study

Proposition number	Mathematical statement	True or false
1.	Let *f* and *g* be functions defined on a set of numbers *S*, and let $a \in S$. If *f* is continuous at *a* and *g* is discontinuous at *a*, then *fg* is discontinuous at *a*.	False
2.	Let *f* be a function defined on a set of numbers *S*, and let $\lvert f \rvert$ be the function whose value at x is $\lvert f(x) \rvert$. If *f* is continuous at $a \in S$, then $\lvert f(x) \rvert$ is continuous at $a \in S$.	True
3.	Let f^2 be a function defined on a set of numbers *S*, and let $a \in S$. If f^2 is continuous at *a*, then *f* is continuous at *a*.	False
4.	Let f be a continuous function from [0 , 1] onto [0 , 1]. Then there exists a point $x_0 \in [0, 1]$ such that $f(x_0) = x_0$.	True
5.	Let *D* = $[0, 1] \cup (2, 3]$ and define $f : D \to R$ by $f(x) = \begin{cases} x & if\, 0 \le x \le 1 \\ x-1 & if\, 2 < x \le 3 \end{cases}$, then $f : D \to R$ is continuous.	True

Note. Proposition 4 is from the master's entrance examination of the department of mathematics of National Tsing Hua University in Taiwan (2001). Proposition 5 is from *Advanced calculus: A course in mathematical analysis* by P. M. Fitzpatrick (1996, p.56).

Data Collection

The primary source of data was students' written responses to the above instrument. The instrument was administered to the students in their advanced calculus classes after completing all course instruction in continuous functions. Students were asked to construct proofs for statements they believed to be true and to generate counterexamples for statements they believed to be false. Students were asked not to include their names in order to maintain anonymity.

Data Analysis

Data were gathered on concept definitions through students' proofs, correctness of proofs and counterexamples, and errors manifested in the students' answers. In this section, we discuss the coding of each item in the following manner.

Coding the concept definitions of proof

Table 2 provides a brief description of the codes used to distinguish types of concept definitions in continuous functions.

Table 2

Descriptions of Codes for Concept Definitions of Proofs

Code	Description
No response	Left blank
No basis of definitions	No relevant syntactic knowledge presented
Informal definitions	An informal description of syntactic knowledge with students' own language and with partial correct understanding
Formal definitions	An essentially correct formal definition in full detail using mathematical language
Incorrect formal definitions	A partially correct description using formal definitions in mathematical language

Coding the correctness of proofs and counterexamples

Table 3 and Table 4 provide a brief description of the codes that were used when coding the correctness of proofs.

Table 3

Descriptions of Codes for Correctness of Proof

Code	Description
No response	Left blank
Counterexample	Giving an incorrect counterexample instead of a proof
No basis for constructing a proof	No relevant syntactic knowledge presented like a guess
Not proof but relevant information presented	Only narrating relevant syntactic knowledge presented
Result achieved with some reasoning omitted	Almost presenting a complete proof but making minor errors
Completeness	A complete proof

Table 4

Descriptions of Codes for Correctness of Counterexample

Code	Description
No response	Left blank
Proof	Giving an incorrect proof instead of a counterexample
No basis for generating a counterexample	No relevant syntactic knowledge presented like a guess
Not counterexample but relevant information presented	Only narrating relevant syntactic knowledge presented
Result achieved with some reasoning omitted	Almost presenting a complete counterexample but making minor errors
Completeness	A complete counterexample

The manifested errors in students' proofs and counterexamples

The manifested errors in the students' attempts to construct proofs and to generate counterexamples were investigated by analyzing student's written work.

Coding reliability

To check the reliability of the coding, the investigators and an independent coder worked separately, and ten students' responses were selected randomly to be coded by an independent coder. Agreement between the investigators and the independent coder was 84% for coding the concept definitions of proof, 81% for coding the correctness of proof, and 86% for coding the correctness of counterexample.

Results

We first focus on the quantitative data of Taiwanese undergraduates' responses to a written instrument in continuous functions, and then on what errors manifested in students' written work of proofs and counterexamples.

Quantitative Data of Taiwanese Undergraduates' Responses

Table 5 displays the types of concept definitions used by students when writing proofs for true statements in continuous functions. As Table 5 shows, none of the students used formal definitions in continuous functions to construct proofs.

Table 5

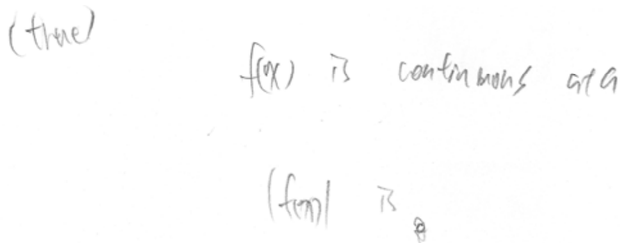
Types of Students' Concept Definitions in Writing Proofs

Proposition number	No response		No basis of definitions		Informal definitions		Formal definitions		Incorrect formal definitions	
	n	%	n	%	n	%	n	%	n	%
2.	19	53	5	14	8	22	0	0	4	11
4.	13	36	7	19	14	39	0	0	2	6
5.	4	11	24	67	8	22	0	0	0	0

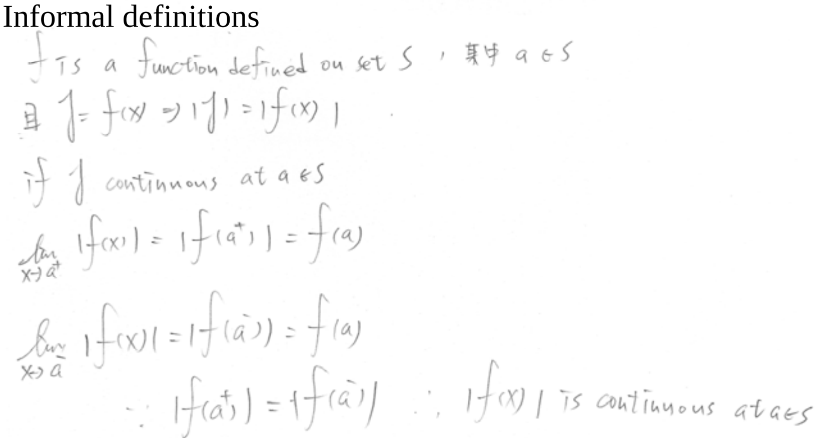
The following are examples of the codes (a) no basis of definitions, (b) informal definitions, and (c) incorrect formal definitions when students constructed the proof for the Proposition 2 by using concept definitions.

Proposition 2: Let *f* be a function defined on a set of numbers *S*, and let $|f|$ be the function whose value at x is $|f(x)|$. If *f* is continuous at $a \in S$, then $|f(x)|$ is continuous at $a \in S$.

No basis of definitions



Informal definitions



Incorrect formal definitions

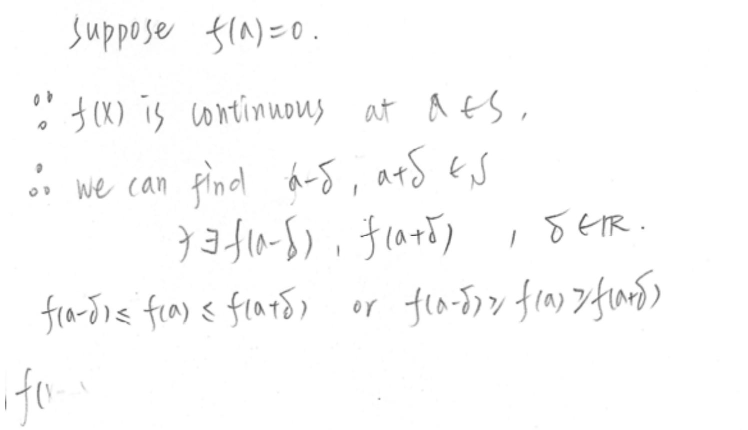


Table 6 shows students' performance in constructing proofs for true statements in continuous functions. As Table 6 shows, none of the students provided a complete proof for each true mathematical statement.

Table 6

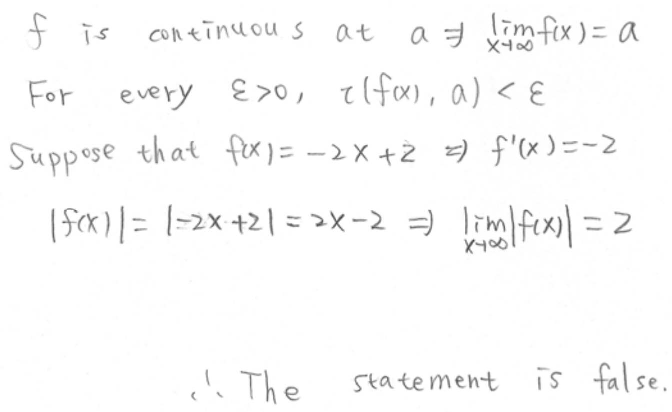
Students' Performance Constructing Proofs

Proposition number	No response		Counter-example		No basis for constructing a proof		Not proof but relevant information presented		Result achieved with some reasoning omitted		Complete-ness	
	n	%	n	%	n	%	n	%	n	%	n	%
2.	18	50	4	11	3	8	11	31	0	0	0	0
4.	13	36	4	11	11	31	8	22	0	0	0	0
5.	7	19	1	3	20	56	8	22	0	0	0	0

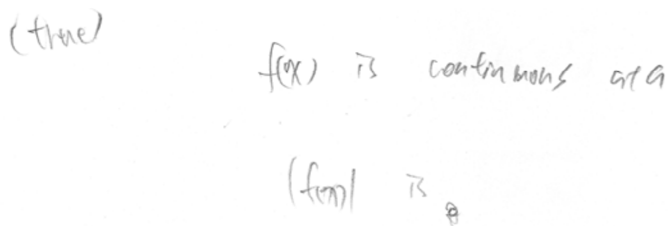
The following examples are for (a) a counterexample, (b) no basis for constructing a proof, and (c) not a proof but relevant information presented when students constructed a proof for the Proposition 2.

Proposition 2: Let f be a function defined on a set of numbers S, and let $|f|$ be the function whose value at x is $|f(x)|$. If f is continuous at $a \in S$, then $|f(x)|$ is continuous at $a \in S$.

Counterexample



No basis for constructing a proof



Not proof but relevant information presented

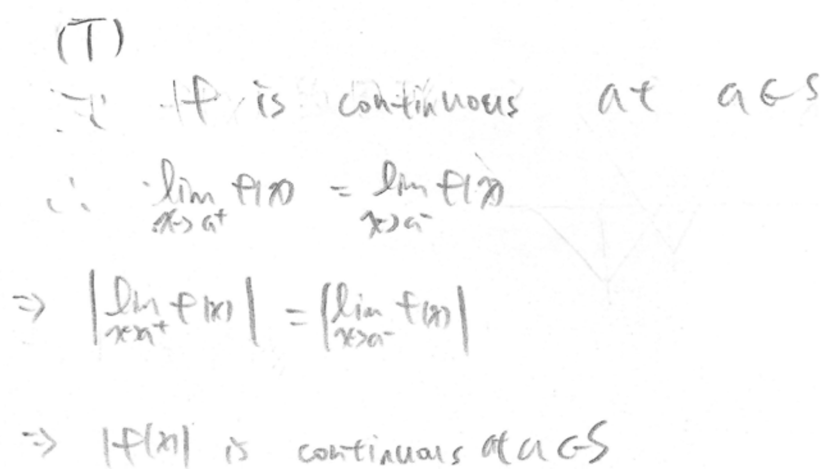


Table 7 shows students' performance generating counterexamples for false statements in continuous functions. As Table 7 indicates, nine and seven students generated complete counterexamples for the Propositions 1 and 3, respectively.

Table7

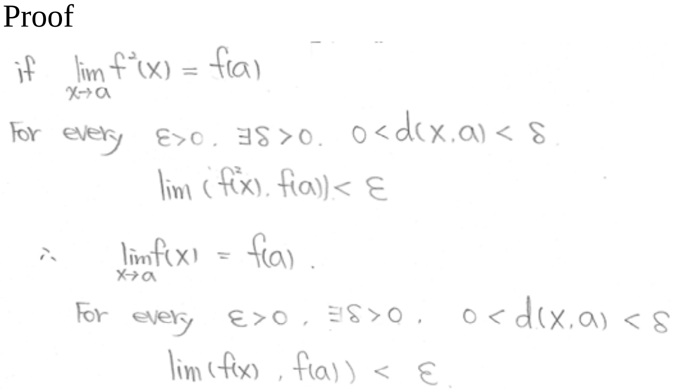
Students' Performance Generating Counterexamples

Proposition number	No response		Proof		No basis for generating a counter-example		Not counterexample but relevant information presented		Result achieved with some reasoning omitted		Complete-ness	
	n	%	n	%	n	%	n	%	n	%	n	%
1.	10	28	14	39	0	0	0	0	3	8	9	25
3.	16	44	6	17	5	14	0	0	2	6	7	19

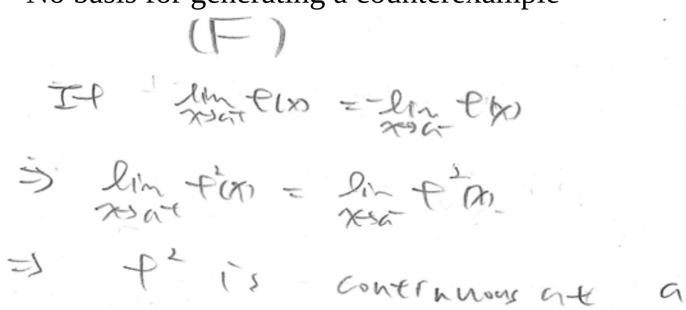
The following examples are for a) a proof, b) no basis for generating a counterexample, c) not a counterexample but relevant information presented, d) result achieved with some reasoning omitted, and e) completeness when students generated a counterexample for the Proposition 3.

Proposition 3: Let f^2 be a function defined on a set of numbers *S*, and let $a \in S$. If f^2 is continuous at *a*, then *f* is continuous at *a*.

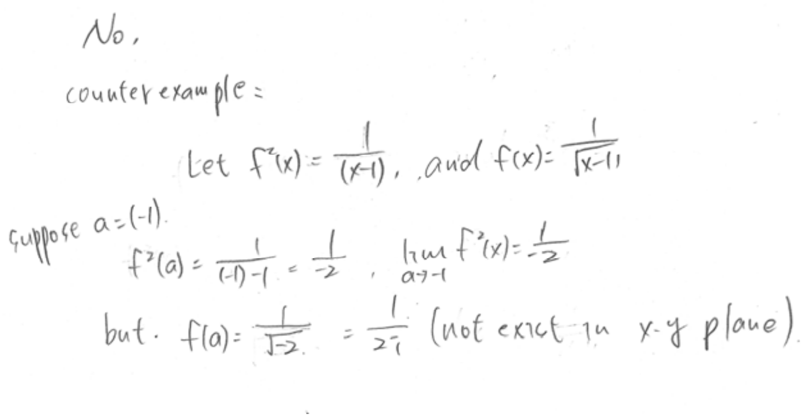
Proof



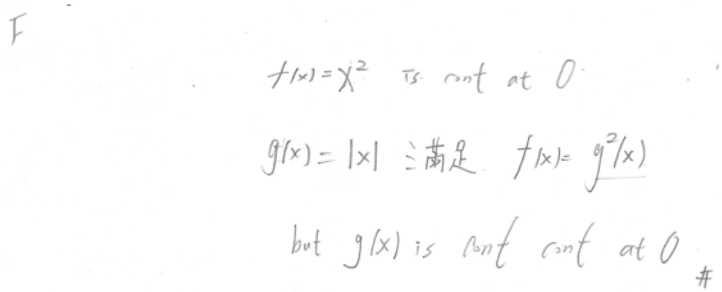
No basis for generating a counterexample



Result achieved with some reasoning omitted



Completeness



Errors Manifested in Students' Written Work

Errors manifested in students' written work were analyzed in detail for each proof and counterexample. The analysis of the data revealed a number of errors, and the main sources are discussed below.

1. In general, many students were incapable of applying formal mathematical definitions to construct formal proofs for Proposition 2, 4 and 5, because they only presented incorrect or partially incorrect understanding of the informal mathematical definitions. In light of this, they had difficulty determining true or false mathematical statements or producing complete proofs for true mathematical and counterexamples for false mathematical statements.

2. With respect to constructing formal proofs, some students only narrated relevant syntactic

knowledge presented with their own language and with partial correct understanding of concept definitions in continuous functions.

3. Perhaps most surprising, none of the students were able to construct complete formal proofs in this study although they provided relevant mathematical knowledge with a partial understanding of concept definitions in continuous functions. Therefore, participants were unable to apply formal mathematical definitions to write formal proofs with mathematical language.

4. With respect to generating counterexamples, several students (39% for Proposition 1 and 17% for Proposition 3) believed the false mathematical statements to be true, and attempted to provide a proof.

Summary

Even though all participants in this study had some relevant domain knowledge in continuous functions because the topic was addressed in a previous calculus course, the data indicate that most participants were not able to determine whether the false mathematical statements were true or not due to their partial correct understanding of concept definitions. Additionally, several of them tried to produce a counterexample for a correct proposition, and many of them were not able to produce complete correct proofs for the true mathematical statements because they seemed to lack the understanding of the mathematical language needed when writing formal proofs. Finally, the percentage of students able to provide counterexamples was much higher than the percentage of students able to produce proofs, although the percentages are not very large in either case. In other words, generating a correct counterexample seemed slightly easier for students than constructing a correct proof.

Although this study confirms findings from other studies in the United States (Moore, 1990, 1994; Weber, 2001), the findings are somewhat unexpected given that the successful performance of Taiwanese elementary and secondary school students on international mathematics achievement tests might lead one to expect the proof performance of such students to be better. Given the mathematical backgrounds of the participants, it is surprising that they still had a considerable difficulty generating proofs and counterexamples. Such difficulty was unanticipated in part because of so little existing literature on undergraduate Taiwanese students and proofs. Overall, the majority of participants did not show an understanding of concept definitions in continuous functions.

Conclusions

In this study, participants did not seem to understand the relevant concept definitions needed to determine whether given mathematical statements were true or false, and did not seem to know how to use their concept definitions to construct proofs when asked to write proofs. These findings are consistent with Moore's work on undergraduates and their construction of proofs (1990, 1994). Participants also used informal language instead of mathematical language with partial understanding of concept definitions to construct proofs (Rin, 1983), so they had difficulty producing proofs. Although this study only asked participants to construct proofs for statements that they believed to be true and to generate counterexamples for statements they believed to be false, and only for five mathematical statements, these results provide suggestive evidence regarding Taiwanese undergraduates' understanding of concept definitions in continuous functions are revealed in their written work.

Additionally, the comparisons between this study and existing empirical studies in the United States (Moore, 1990, 1994; Rin, 1983) show that Taiwanese undergraduates have similar difficulties to American undergraduates when writing proofs. Even though the successful performance of Taiwanese elementary and secondary school students on international mathematical achievement tests is well known, such comparisons allowed by this research suggest that Taiwanese undergraduates encounter similar challenges as their American counterparts in constructing proofs.

In mathematics, true propositions are supported by proofs while false propositions are refuted by counterexamples. In order to determine if propositions are true or false, students should have a deep understanding of concept definitions in the specific domain because mathematical propositions involve formal mathematical definitions. In this view, concept definitions play an important role in mathematics courses for learning to produce proofs and counterexamples. Although formal definitions are taught in undergraduate mathematics courses, students are not generating these concepts, which could be explained by multiple factors: (a) these concepts seem to be too abstract for students, (b) students need more time to finish the mathematical statements, or (c) the mathematical statements do not make any sense to students and so they cannot generate mathematical language on their own even with a complete understanding of mathematical definitions.

In order to develop undergraduates' understanding of concept definitions, mathematics instructors and professors may consider a stronger emphasis on formal definitions when teaching new sections. To assist undergraduates in writing proofs and generating counterexamples, it is possible that mathematics instructors and professors should pay more attention to students' written work to help them write increasingly formal proofs and complete counterexamples. Since this requires changing the pedagogies of mathematics instructors and professors with regard to concept definitions, proofs, and counterexamples, it will not be easy. But in undergraduate mathematics courses, such an enhancement may be a necessary condition for enabling students' understanding of concept definitions and performance in writing proofs and counterexamples.

Currently, few research studies have specifically focused on undergraduates' abilities to produce proofs and counterexamples, especially in Taiwan. In order to gain more insight into the relationship between undergraduates' concept definitions and either proofs or concept definitions and counterexamples, further research in Taiwan as well as in the United States needs to further explore undergraduates' performance producing proofs and counterexamples by designing more mathematical statements and conducting intensive interviews with students to understand their perspective. We hope this paper highlights the need to call more attention to empowering undergraduates in their learning to write complete proofs and counterexamples with a deep understanding of concept definitions in mathematics courses in Taiwan as well as in the United States.

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