e-Proofs: Student Experience of Online Resources to Aid Understanding of Mathematical Proofs

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Abstract

This paper outlines a theoretical basis for the design of *e-Proofs*, a specialist use of educational technology for the support of proof comprehension in undergraduate mathematics. I begin by framing the problem of teaching for proof comprehension, giving research background and an argument about what lectures do not (and cannot) do to address this. I then discuss the limitations of two possible alternative solutions before describing the e-Proofs and the way in which they do address the identified problem. I give an overview of the use of first versions of e-Proofs in an Analysis course, including data on student usage and feedback. Finally, I suggest some features that the current e-Proofs do not have but that would be pedagogically desirable, and review the limitations of e-Proofs with regard to providing an overall educational experience.

Framing the problem

Research background

As lecturers we want our students to understand the proofs in their undergraduate mathematics courses. It is not clear that our tests always reflect this desire, however. Conradie and Frith (2000) made a compelling argument that ordinary "state and prove"-type examination questions do not test for proof comprehension, and gave illustrations of an alternative approach in which students are given a proof and asked various types of question about its content and structure. Similar question types were also suggested by Smith, Wood, Coupland and Stephenson (1996), as part of range of tasks that would test different types of knowledge and skill. Others have investigated the types of reasoning currently required in textbook and examination problems. In Sweden, Bergqvist (2007) and Lithner (2003) found that a large proportion of textbook and examination tasks do not require any creative reasoning on the part of the student and can instead be solved by imitative reasoning.

There is not a great deal of research about the process and skills associated with proof comprehension, although Yang and Lin explicitly studied this in the context of geometry with grade 9 and 10 students (Lin & Yang, 2007; Yang & Lin, 2008). They distinguished four levels of proof comprehension, summarised as:

- Comprehension of surface (no analysis);
- Comprehension of recognised elements (recognition of premises, conclusions, properties applied);
- Comprehension of chaining elements (logical chaining of premises, properties, conclusion);
- Comprehension of encapsulation (interiorising as a whole, ability to apply).

The learning resources described here aim to draw students' attention to the possibility of understanding proofs at the higher of these levels by highlighting logical relationships between premises, properties used and conclusions, and by breaking down the whole proof into distinct chunks or subproofs.

Of course, much of the research and theoretical argument around proof construction is relevant for identifying the skills that might contribute to success in proof comprehension. For instance, to fully understand a proof, a student would have to at least recognise when a deductive argument is being given, although this might not be the type they would use themselves (eg. Harel & Sowder, 1998). They would need, in particular, to be aware of relevant definitions and recognise where these are used (cf. Vinner, 1991), and to have sufficient grasp of logical language to establish whether the framework of a purported proof could establish what is claimed (Selden & Selden, 2003). At the more detailed level, they would also need to be inclined and able to infer warrants (Alcock & Weber, 2005), identifying how each line follows by deduction from the premises of the theorem, or the lines above, or known theorems.

Research typically indicates that undergraduate students cannot be assumed to possess these skills, and one branch of research that was widely reported at the RUME conference responds to this by designing instruction that involves students in the process of proving. This instruction typically involves students in working collaboratively through carefully sequenced tasks designed to develop their conceptual understanding and lead them to construct their own arguments and refine these into proofs (see the papers in these proceedings by Larsen, Bartlo, Johnson & Rutherford and by Rasmussen, Zandieh & Wawro). This, in theory, means that the students "own" their proofs and have built them with comprehension from the beginning. However, implementation of such curricula demands small classes, and in the UK at least it is common to have over 100 students in a lecture. Resource constraints mean that this is not likely to change, and the development work reported here takes a different approach to providing support for proof comprehension in a standard lecture environment.

The problem of lecturing

There are, of course, ways of promoting student reasoning and interactions in a lecture environment, and most students also have access to problems classes in which they are expected to actively engage in problem solving. However, many proofs are still presented in lectures, with the expectation that students will gain comprehension from this experience.

Of course, lecturers do not simply write proofs on boards, or otherwise present them. They also give substantial extra explanation. They might provide a motivating overview of the argument, state rationales for certain approaches, explain warrants for each line and explain the overall structure of a proof once this is written down. However, in terms of supporting proof comprehension, there are a number of problems with this model. First, students' attention may not be directed precisely enough, especially if the explanation involves looking at two lines in different places in the proof. Second, even if their attention is in the right place(s), they may not be able to grasp logical relationships quickly enough to understand them in real time, especially if this involves recalling an earlier theorem or results from several earlier lines. Third, the explanation is ephemeral and is typically no longer available when the student comes to re-read their lecture notes. Hence the student must do much of the work of proof comprehension during independent study time and with minimal instruction on how to approach this.

Possible solutions

Once this problem is recognized, one possible solution might be to simply record the lecture. This would allow a student to see and hear explanations again, so it addresses their ephemeral nature. However, it leaves us with the problems of directing attention precisely and of seeing relationships in real time. Even with wellplaced sectioning of a video so that desired sections can be located and watched again, there are likely to be slips and hesitations in the spoken explanation, quick shifts back and forth and vagueness in indication of what is supposed to be the focus. In addition, both visuals and audio are unlikely to be optimally clear, and there may well be extraneous distractions in either. One could, of course, attempt a professional-quality recording in a studio rather than a real lecture, but this is extremely resource heavy and impractical for most lecturers, especially as it remains likely that there will be mistakes that cannot be easily edited out.

Another approach would be to provide additional written information to accompany the proof. This is done, in a two-column format, by the professor studied by Weber (2004). My concern about this approach is that long proofs can already be intimidating, and such annotations could easily more than double the length. For weaker students, extra writing may be interpreted as simply "more to learn", and not distinguished from the chain of arguments in the proof itself. Indeed, this might actually serve to obscure the structure of the proof. It is not that annotations and further explanation are not useful, but I argue that adding these as additional text may not be an optimal delivery method, and that a technological solution can do better.

e-Proofs

Initial design

e-Proofs are designed to address all of the difficulties outlined above by making the structure and reasoning used in a proof more explicit without cluttering its presentation. Each consists of a sequence of screens such as that shown in Figure 1 below. Each screen shows the theorem and the whole proof, but much of this is "greyed out" to focus attention on particular lines. To highlight relationships, each screen also has boxes and arrows indicating links among the lines. Each screen is also accompanied by an audio file that the student can listen to as many times as they like by clicking an icon.

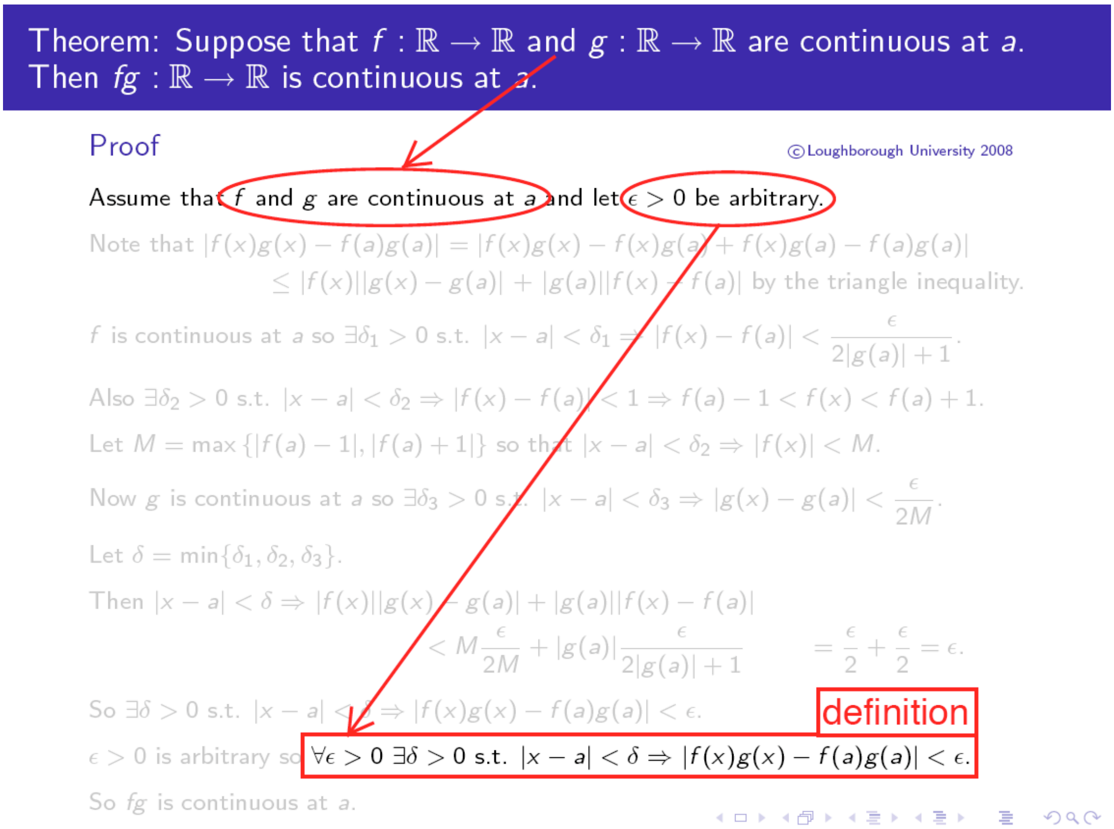

Figure 1: A screen shot from an e-Proof for the product rule for continuous functions. The accompanying audio for this screen says, "In the first line, we state our assumption that *f* and *g* are continuous at *a*, which corresponds to the premise of our theorem. We also let epsilon greater than zero be arbitrary, because we want to show that *fg* satisfies the definition of continuity at *a*, which we will achieve by the end of the proof. Doing so involves showing that something is true for all epsilon greater than zero, so choosing an arbitrary epsilon means that all our reasoning from now on will apply to any appropriate value."

The screen in Figure 1 comes from what I have termed the *line-by-line* version of this e-Proof. I also constructed *chunk* versions, the aim of which is to focus attention on the global structure of the proof by breaking it into relatively self-contained sections or subproofs. Figure 2 shows a screen from the chunk version of the same e-Proof.

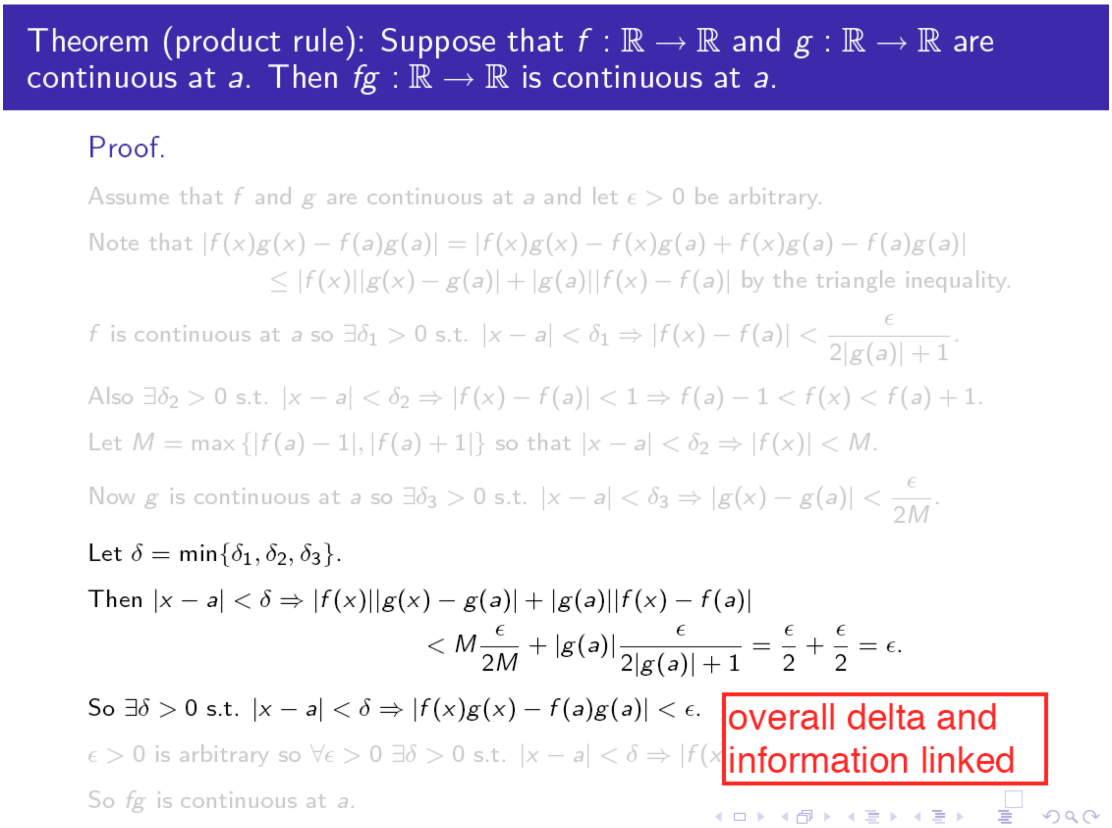

Figure 2: A screen shot from a *chunk* version of an e-Proof for the product rule for continuous functions. The accompanying audio says, “In the third chunk, we set up an overall delta value, and put together the information from the second chunk to show that if the modulus of *x* minus *a* is less than this delta, then our original modulus expression is less than epsilon.”

Improvements in a new version

The e-Proof screens shown above are from versions made for a course in real analysis and used in Autumn 2008. They were constructed by using Beamer to convert a LaTeX file into a pdf presentation, which was then annotated and separated into screens. The audio was recorded using Audacity. This content was then uploaded to the university’s virtual learning environment (VLE), making use of one of its standard lesson structures. This was a somewhat clumsy process involving uploading screens and audio separately, and was restricted by the content and structure of the rest of the VLE’s standard layout.

Figure 3 shows a screen shot from a prototype improved version made using Flash. This has two particular improvements. First, it allows the annotations to be better synchronized with the audio content, so that as the audio proceeds, the arrows and boxes appear and disappear exactly when they are needed. Second, it has an improved navigation structure in which the student can more easily find their way to a particular line that they wish to hear explained. With the support of JISC[1] Learning and Teaching Innovation Grant, work is now underway to develop an open-source web-based tool called ExPOUND (Explaining Proofs: Offering Understanding through Notated Demonstrations) that will allow academics to easily construct e-Proofs of this type.

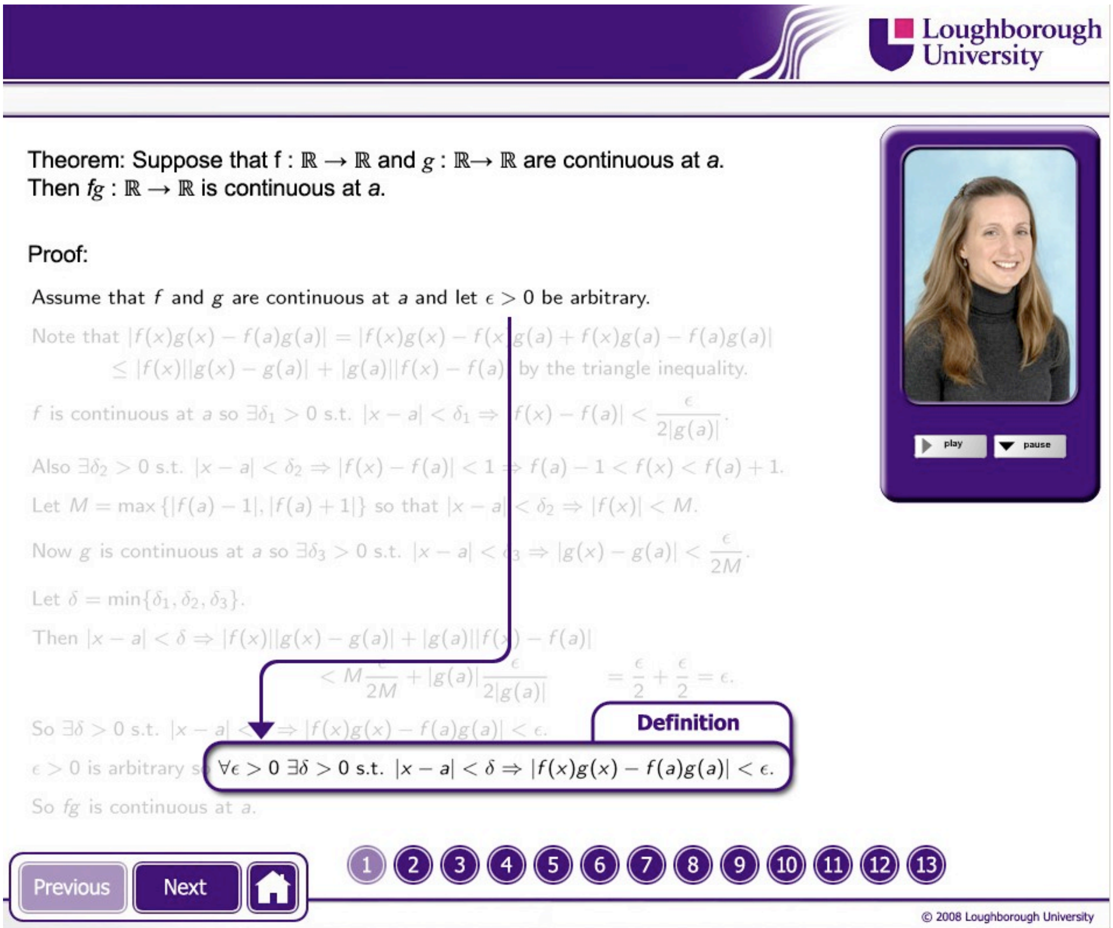

Figure 3: A screen shot from an improved Flash version of an e-Proof.

[1] Joint Information Systems Committee, see www.jisc.ac.uk.

Addressing the problems of lecturing

The design of the e-Proofs, particularly in the improved version, means that:

- Attention is directed precisely;
- Logical relationships are explicitly highlighted by coordinating the appearance of annotations with audio commentary;
- Audio can be replayed as many times as the student wishes;
- Visuals are clear and without extraneous distractors;
- Small sections of audio can be scripted and recorded separately for clarity;
- Annotations appear one at a time and do not permanently add content, so the integrity of the proof is preserved without clutter or "more to learn";
- Navigation to a specific point of difficulty is straightforward.

Overall, the coordination of the static underlying proof and the dynamic annotations and audio mean that reasoning one needs to do to understand a proof is made explicit in a way that could not be achieved in a lecture or a book.

Usage and feedback

Usage data

The VLE collects usage data for all the posted documents and other types of activity, so it is possible to ascertain how much the e-Proofs were actually used by the students. The eight available e-Proofs were for the product rule for continuous functions, the intermediate value theorem, the extreme value theorem, Rolle's theorem, the generalized mean value theorem, Taylor's theorem, the equivalence of Riemann's condition with the definition of integrability, and additivity of the Riemann integral. Table 1 shows the usage data, broken down week-by-week, for weeks 2-11 of the Autumn term, the Christmas vacation, and the two revision weeks before the examination (the examination was on day five of the second of these weeks). Each count represents a student accessing either a line-by-line or a chunk version of that e-Proof. There is some unevenness in this data, since some students only looked at one screen of the e-Proof then moved on whereas others viewed the whole thing. However, most accesses involved viewing at least half of the available screens.

	Term										Vacation				Revision		
Week	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*1*	*2*	*3*	*4*	*1*	*2*	Total
prod	73	43	11	1	2	4		2	1	7	16	14	15	10	36	51	286
IVT		8	11	1	3	3	2	2	1	3	13	3	9	15	39	65	178
EVT			14		1	1	1			1	7	3	14	11	27	54	134
Rol					4	4				1	4	1	4	11	19	56	104
GMV					3	8	2	2			1	3	5	6	10	38	78
Tay						26	7					1	3	6	11	62	116
Rie								6			3	3		5	12	50	79
Add									3	2	3	1	4	2	6	30	51
Tot	73	51	36	2	13	46	12	12	5	14	47	29	54	66	160	406	1026

Table 1: Usage data for the eight e-Proofs.

The bottom right total of 1026 means that on average students accessed seven e-Proofs (though of course there is massive variability in how much individual students used any of the resources available). For comparison, Table 2 below shows the total number of downloads of the solutions to the weekly (not for credit) problems, indicating a similar level of use.

Class	1	2	3	4	5	6	7	8	9	10
Downloads	256	227	137	166	142	95	117	72	89	68

Table 2: Number downloading problem class solutions.

Of course, a relatively high level of use does not mean that students necessarily attained a good level of proof comprehension, and the heavy usage in the examination week may indicate panic rather than a genuine attempt to understand. But evidently these resources were used, particularly during revision time when access to the lecturer was no longer routine.

Feedback

Feedback on the e-Proofs was solicited from the students who attended two revision lectures in the first revision week. A total of 74 students (of 144 registered for the course) filled in a questionnaire and the results are summarized in Table 2.

	Average
e-Proofs helped me understand particular proofs.	4.3
e-Proofs helped me remember particular proofs.	3.8
e-Proofs helped me see why lines of proofs are valid.	4.3
e-Proofs helped me see the overall structure of proofs.	4.3
e-Proofs are easier understand than ordinary written proofs.	4.1
e-Proofs helped me learn how to ask myself why each line is valid in ordinary written proofs.	3.9
e-Proofs helped me learn how to look for the overall structure in ordinary written proofs.	3.9
e-Proofs helped me to be more confident that I can succeed in proof-based mathematics.	3.6
This course would have been harder without the e-Proofs.	4.1
I would like to have e-Proofs for other courses.	4.3

Table 3: Feedback on the e-Proofs. Scores are averages from a Likert scale on which 1=strongly disagree, 5=strongly agree.

These responses are, of course, very subjective. Students may not attribute the same meaning to "understand" that we would, and any resource that is perceived as having been provided with good intentions is likely to be viewed positively. Further, the responses are given by the students who came to revision lectures. These students are likely to be those who are more organized and doing better overall, and therefore to have a more positive view than their peers of all the learning resources available. However, together with the usage data, they do indicate that the e-Proofs were experienced as a positive addition to the course.

Discussion

What e-Proofs don't do yet

In addition to improved synchronization between audio and annotations, there are many features that e-Proofs could beneficially incorporate. They could, for instance, have boxes showing one or more types of additional information such as definitions and theorems that are used in the proof, static or dynamic diagrams, or indications of where similar structures or algebraic "tricks" can be found in other proofs from the course. e-Proof-like structures could be made for different types of course elements, for instance for the introduction and explanation of definitions. Some of these possibilities will be investigated in the ExPOUND project; from June 2009, progress can be followed on the project website at http://expound.lboro.ac.uk.

What e-Proofs don't do

I have argued here that in theory, e-Proofs can focus attention on the process of understanding a proof by making explicit the search both for warrants for line-by-line validity and for an overview of structure. However, it is important to recognize the limits of such a resource in terms of what it can contribute to the overall learning process. Essentially, an e-Proof allows the teacher to articulate their own understanding of a proof. I have argued that it allows them to do this better than might be achieved in a lecture or a book, because it can direct attention explicitly to the structure of the proof and the relationships among the lines. This might therefore be a better explanation than is more usually available, but it is still just an explanation. There is some interactivity built in, but this is only in the weak sense that the student controls the pace and sequence of the content and can replay parts at will (cf. Laurillard's (2002) discussion on interactive media). A lecturer can attempt to address likely points of difficulty, but the content does not change in response to the student's current understanding.

Research questions: What e-Proofs might do

The work discussed here has been done on the basis of research on proof comprehension, but thus far this has been a development project rather than a research project. However, the availability of any new type of resource raises research questions, and here these might include:

- Does study of an e-Proof improve proof comprehension?
- Does study of a number of e-Proofs lead to the development of general proof comprehension skills that can be transferred to standard written proofs?
- How do students use e-Proofs, both at the detailed interactive level and at the level of incorporating their study into overall study time?
- What design elements make an e-Proof more or less effective in promoting proof comprehension?
- What are the effects upon learning if students are asked to work collaboratively to construct their own e-Proofs?

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