Workbooks for Independent Study of Group Theory Proofs

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Abstract

This paper reports on a small project investigating students' experience of a workbook designed to promote in-depth study of proofs. The workbook was one of two used as part of the assessment for an abstract algebra course, and was designed by the course lecturer. In this paper, we first review related research literature. We then provide detail on the setting in which the research took place and the intentions of the workbook designers. We discuss the research method, and present the results in three sections: observations of participants' progress in their first hour working on the workbook, themes arising from interviews with participants about this experience, and general feedback from the whole class. We conclude with a brief discussion of issues arising from this research for the design of worksheets for independent study in general.

Introduction

Undergraduate mathematics involves a lot of independent study. As lecturers, we try to present material clearly, but naturalistic studies of mathematical activity have generally taken place in a classroom setting, and we usually have little to no idea about our students' independent study. Those of us who teach proof-based courses do, however, have reason to believe that our students often do not succeed in productively studying the proofs we present. In response to this problem, two mathematics lecturers designed workbooks to replace two standard assessments in a course in abstract algebra. These workbooks were designed to promote careful study of the proofs from two sections of the course; the workbook discussed here covered material associated with Lagrange's Theorem.

Related research

Learning in abstract algebra

Various studies have pointed to the challenges facing students when they begin to learn abstract algebra during undergraduate study. They must learn to work with new types of object which, as in much earlier mathematics, may be introduced first as processes (such as finding cosets in particular groups) but which must be encapsulated very quickly if later material is to make sense (Dubinsky, Dautermann, Leron & Zazkis, 1994). If students are unable to do this, the effect may be that they effectively reduce the level of abstraction, working with these objects as though with objects of simpler types (Hazzan, 1999). Also, as in all subjects at this level, understanding of logical language is necessary for success. Hazzan and Leron (1996) provide an example of the types of error that can occur when students misinterpret the standard conditional statement of Lagrange's Theorem, using it to conclude that Z_3 is a subgroup of Z_6 because 3 divides 6. Further, Weber (2001) found that undergraduates were often unable to construct simple group theory proofs, not because they did not know the requisite theorems and definitions, but because they lacked the strategic knowledge necessary to decide which to apply.

More generally, understanding a proof requires one to unpack the logic of mathematical statements and decide whether the framework of the proof is such that it could possibly prove the theorem (Selden & Selden, 2003), and checking the validity of a proof may involve a complex combination of deductive processes such as constructing subproofs, and more informal reasoning based on checking against examples (Weber, 2008; Inglis, Mejia-Ramos & Simpson, 2007).

Beliefs and study habits

Students' interpretation of the mathematics in courses such as abstract algebra will also be influenced by their more general beliefs about what it means to learn and understand. It is well recognised that students' beliefs about knowledge may be such that they expect everything that is needed to be provided by an authority in a form that they are expected simply to reproduce (eg. Perry, 1988). In mathematics specifically, students often come to proof-based courses with beliefs that do not stand them in good stead for study at this level. For example, they may believe that "[t]here is only one correct way to solve any mathematics problem—usually the rule the teacher has most recently demonstrated to the class" (Schoenfeld, 1992). Their study habits may also be such that they are unlikely to spontaneously engage with proofs at an appropriate level. For example, Crawford, Gordon, Nicholas & Prosser (1994) found that over 70% of students entering a mathematics degree in Australia claimed to approach their study by doing examples for reproduction.

Instructional constraints and choices

Such beliefs do not come from nowhere, and it has been argued that standard transmission-oriented teaching of mathematics may mitigate against the development of more mature epistemological stances and study approaches (Boaler & Greeno, 2000). In response to such concerns, mathematics educators have conducted action research based around designing curricula for inquiry-oriented small group and classroom settings, in a variety of subjects including group theory (Zandieh, Larsen & Nunley, 2008).

However, such curricula are unlikely to be in widespread use in the foreseeable future, partly because many lecturers around the world are constrained by large lecture sizes, and partly because their use involves developing skills for handling unfamiliar types of classroom interactions. Where instructors do have more leeway is in the work they set for students to complete in independent study. In the course under consideration, this work had typically involved calculation or application-style problems. With the introduction of the workbooks, it now involves questions designed to promote proof comprehension, using questions not dissimilar in style to those considered by Conradie and Frith (2000).

The project

Setting

The workbooks were used as part of the assessment for a course entitled Groups, Rings and Fields. This is an abstract algebra course taken by students in the second year of a three-year undergraduate mathematics degree in a UK university (students in the UK specialise earlier, so the whole of the content of this degree is mathematics). There were 25 students registered for this course, which had three 1-hour lectures per week, taught in a traditional lecture style for one 12-week term (the students take a total of eight such courses per year, with all the summative examinations in a third summer term). The course also had a weekly problems class. 20% of the total credit came from four assessments, two of which were workbooks and two of which were standard problem sheets.

Intentions and workbook design

Each workbook was designed to be a self-contained set-piece, recapitulating considerable material from the lecture notes in a format that is more compressed and in an order that allows the main ideas to be given greater prominence. The appendix shows a sample page to indicate the layout and content of the Lagrange's Theorem workbook. The workbook as a whole is structured into the following sections:

1. An application of Lagrange's Theorem (see the appendix);
2. Coset calculations in various groups;
3. Proving that cosets have the same size;
4. Proving that cosets partition the group;
5. Proof of Lagrange's Theorem;
6. A case study of the subgroups of the isometries of the square.

The course lecturer was the principal designer of the workbooks, and he described his intentions thus:

> "The thing I was definitely setting out to do was to say look, in your notes there's a proof which…you can sit down for an hour and you can work hard to understand it. It looks easy because it's four lines long. But, you know, actually you are going to have to spend ten minutes thinking about each line. And here is an example of how you take one of those, and you cut it into lines. So that's, if you like, how to read a proof, if you're not experienced in seeing – in knowing that each line is actually a substantial thing and is there for a reason."

Research method

The research proceeded through several stages. First, the two lecturers involved in the project (the second two authors) were interviewed (by the first author) to clarify their aims for the workbooks and use this clarification to design a feedback form. Second, observations were conducted of student volunteers working for their first hour on the workbook. Students were asked to participate as individuals if they normally worked alone or in pairs if they normally worked together, and two individuals and two pairs were observed. The observations took place in an office and were recorded by two webcams, one filming the students' faces and one filming their work. This made remote observation possible from a neighbouring room, and the first author watched the students work and then returned to the office and interviewed them for approximately 30 minutes about this experience. She asked specific questions about their difficulties after asking the following open-ended questions:

- Please tell me about your general experience of working on that workbook.
- Do you have any comments about specific parts of it?
- Was it different from ordinary study you would do for this module? If so, how?
- Have you learned or understood anything that you didn't know or understand before?

Results

Observations

Table 1 below shows the approximate number of minutes that the participants spent on the workbook tasks. The label "(skip)" indicates that the participants moved on from this task before completing it, usually because they became stuck, but in Penny's case because she judged that some tasks would be straightforward using her notes and she wanted to work on the more challenging aspects first. It is worth noting that there was considerable variation in how much progress they made during this time; some of this appeared related to their familiarity with their lecture notes, a point that we return to in the next section.

	Kelly	Liam & Mick	Nathan & Oscar	Penny
Intro/reading	6.5	10	7.5	2
1.2 Lagrange app	11	20	40.5	11.5
2.2 golden rule	8	7.5	6	3
2.2 computing cosets	4.5	3.5 (skip)		19
2.3 representatives	1.5	6		2.5
3.1 cosets same size	3	1.5 (skip)		1.5 (skip)
4.1 computing cosets	4	8		2
4.1 partitions	2			2
4.2 union of cosets	1.5			5 (skip)
4.2 cosets disjoint	8			
5 Lagrange proof				
6 Isom(sq)				
6.3 Isom(sq) lattice				8

Table 1: Minutes spent by interview participants on workbook tasks.

Interviews

Three learning-related themes emerged from the interview data:

1. The structure provided by the workbooks helped the students to work independently on abstract material including proofs in algebra.
2. The workbooks promoted careful study of the lecture notes (in some cases, study that would not otherwise have taken place).
3. The tasks in the workbooks helped them to analyse proofs in ways they had not done before.

We present illustrative quotations from the interviews for each of these themes.

In relation to theme 1, the interview participants all remarked positively about aspects of the structure of the workbooks. Kelly and Liam, for instance, both noted that they provide more explicit information to help the student locate the relevant material. For Liam this was particularly important as they felt that they often wasted time searching for the information they needed.

K: I quite like these workbooks because I kind of…you get a little bit of information first and not just the question, so it's putting you in the right sort of…starting to think about it. Whereas sometimes you just pick up a question and go, I have no idea what this relates to.

L: …in all the other modules you get like, it'll just have like question 1, question 2, question 3, question 4, and it'll literally say "prove this", and it's just like, well where the hell do I get this from? But that tells you what chapter, where you get it from, so you can work through it and you learn a lot more doing that. Because otherwise it's just like having a stab in the dark.

L: I've spent like a good four hours on a piece of coursework, not being able to find it. And it's like, that's not really very productive.

In relation to theme 2, the interview participants clearly had different habits regarding regular study of their lecture notes. Both Liam and Nathan, for instance, indicated that they did not normally study their notes between lectures.

L: I'll organise them, like when I get my sheet out I'll organise them. But I won't like, read through it, recapping or whatever.

N: I haven't really had time to be honest. Because we started this just after Christmas, and I've got so many assessments going on at the moment, I don't really get time to read through my notes very often.

Penny did study her notes regularly, but she also indicated that the workbooks promoted more careful study.

P: Because like I'll be skipping bits, and I won't actually be reading all the – I won't go through the whole notes. Whereas this really does make you go through all of them. Instead of flick, flick, flick.

In relation to theme 3, some of the interview participants indicated that they were aware of the need to study proofs, since this type of mathematics was different from their earlier experience. They felt prompted to give more detail on the reasoning than they might if not pressed by a specific question.

K: …if it had just said, show that they're equal, you'd have just gone, well it's obvious. Whereas, because they've kind of…prodded you in the right direction, it's like, well you know that's the answer, so you have to show why it's the answer.

N: …the actual questions you get, like exam-style, they are to recite proofs and…it's – I don't think it's…well it's just that I'm less experienced at doing that. I mean last year the modules, basically, were mostly like doing exercises. Learning a method and doing it, sort of thing.

O: sometimes…you can get intimidated by a big proof, I think.

Feedback

The feedback forms first asked for practical information about the amount of time students had spent working on the workbook. This was of concern to the designers because the workbook appears physically large compared with ordinary coursework assignments and they did not want the workload to be unreasonable. In fact, this appeared to be unproblematic. Table 1 shows the responses of the 19 students who completed this part of the feedback to the questions:

H: How many hours did you spend on this workbook?

D: Over how many days were these hours spread?

O: For how many of the hours did you work with other students?

N: For how many of the hours did you actively refer to your lecture notes?

Q	Responses																			Mean
H	7	7	6.5	6	6	6	5.5	5.5	5	5	5	5	4	4	4	3.5	3	3	3	4.95
D	4	4	4	3	2	2	3	2	4	4	2	2	5	3	3	2	2	2	2	2.89
O	4	2.5	0	1	5	2	3	0	1.5	1	3	0	0	1	0	1.5	2	0.5	0.25	1.49
N	3	5	4	6	1.5	5.5	4	2	1	1.5	4	4	3	4	2	2.5	1	2	2.25	3.07

Table 2: Responses to feedback questions about work time.

We can see that on average these students claimed to spend about five hours on the workbook, over about three days, with about two hours spent working with others and three spent actively referring to their notes. However, it would be misleading to think that this was the profile for a notional average student, as there was considerable variation, in particular with regard to the latter two response categories.

The feedback then asked for students' opinions on their learning experience, which are summarized in Table 3.

After working on this workbook…	Mean
I have a better understanding of the material.	4.2
I am more likely to remember the material correctly.	4.1
I have a better understanding of the relationships between the mathematical ideas.	3.9
I have a better understanding of the overall structure of this part of the course.	4.2
I have a better understanding of the proofs.	4.2
I am more likely to remember the proofs.	3.7
I have taken proofs apart line-by-line in a way that I had not before.	3.8
I will take other proofs apart line-by-line like this in future.	3.8
I am more confident that I can read and understand proofs in general.	3.8
I am more confident that I can reproduce and construct proofs in general.	3.5

Table 3: Responses to feedback form questions about experience of the workbooks. Mean is calculated from a Likert scale: 1=strongly disagree, 5=strongly agree.

These numbers indicate a generally positive experience, though it should be noted that the questions are phrased positively and that the students' confidence becomes less pronounced on skills not directly related to the proofs studied in the workbook.

Discussion

The participants in this study talked about their difficulties with proof in a way that is consistent with the literature, in that they recognised that engaging with proofs required a different type of reasoning from that used in their earlier mathematics, and that this was not easy for them. In some cases, their difficulties were exacerbated by poor study habits such as not reading their lecture notes as the course progressed. This meant that there was considerable variation in how much of the workbook the participants worked through in the observation hour, as well as variation among the whole cohort in terms of the amount of time spent working with others, with direct reference to notes etc. Nonetheless, all the participants were able to engage with the workbook and all reported that this was a positive learning experience: in some cases, considerably more positive than working on other types of problem sheet.

Overall, this research gave us cause to consider the following general questions about teaching this type of material at this level. First, it seems worth considering how much we want to modify our problem setting in order to compensate for poor study habits. The workbook discussed here contained much more material than a usual problem sheet, and much more recapitulation of lecture material in particular. Some lecturers may decide that this is inappropriate because the students should be familiar with lecture material through regular study in any case. Others may decide that, since their students are unaccustomed to the careful study required to understand a proof, this sort of support is appropriate. Second, this workbook represents one attempt to address the issue of providing support for proof comprehension in the context of independent work for a standard lecture course. It seems worth also considering whether there are other ways in which we might design instruction to directly support students in their proof reading.

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References

Boaler, J. & Greeno, J. (2000). Identity, agency and knowing in mathematics worlds. In J. Boaler (Ed.) *Multiple perspectives on mathematics teaching and learning*. Westport, CT: Ablex, pp. 171-200.

Crawford, K., Gordon, S., Nicholas, J. & Prosser, M. (1994). Conceptions of mathematics and how it is learned: The perspectives of students entering university. *Learning and Instruction, 4,* 331-345.

Conradie, J. & Frith, J. (2000). Comprehension tests in mathematics. *Educational Studies in Mathematics*, *42*, 225-235.

Dubinsky, E., Dautermann, J., Leron, U. & Zazkis, R. (1994). On learning fundamental concepts of group theory. *Educational Studies in Mathematics, 27*, 267-305.

Hazzan, O. (1999). Reducing abstraction level when learning abstract algebra concepts. *Educational Studies in Mathematics*, *40*, 71-90.

Hazzan, O. & Leron, U. (1996). Students' use and misuse of mathematical theorems: The case of Lagrange's theorem. *For the Learning of Mathematics*, *16(1)*, 23-26.

Inglis, M., Mejia-Ramos, J. P., and Simpson, A. (2007). Modelling mathematical argumentation: The importance of qualification. *Educational Studies in Mathematics*, *66*, 3-21.

Perry, W.G. (1988). Different worlds in the same classroom. In P. Ramsden (Ed.) *Improving learning: New perspectives* (pp.145-161). London: Kogan Page.

Schoenfeld, A. (1992). Learning to think mathematically: Problem-solving, metacognition and sense making in mathematics. In D.A. Grouws (ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp.334-370) Macmillan: New York.

Selden, J. & Selden, A., (2003). Validation of proofs considered as texts: can undergraduates tell whether an argument proves a theorem? *Journal for Research in Mathematics Education*, *34*, 4-36.

Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics*, *48*, 101-119.

Weber, K. (2008). How mathematicians determine if an argument is a valid proof. *Journal for Research in Mathematics Education, 39,* 431-459.

Zandieh, M., Larsen, S. & Nunley, D. (2008). Proving starting from informal notions of symmetry and transformations. In M. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics* (pp. 129-138). Washington, DC: Mathematical Association of America

Appendix

A sample page from the beginning of the Lagrange's Theorem workbook.

Cosets and Lagrange's theorem

1 Lagrange's theorem

Lagrange's theorem is about finite groups and their subgroups. It is very important in group theory, and not just because it has a name.

Theorem 1 (Lagrange's theorem) *Let G be a finite group and $H \subset G$ a subgroup of G. Then $|H|$ divides $|G|$.*

We will prove this theorem later in the workbook. But first we begin to see what the theorem means.

1.1 Understanding the statement

Remember that $|G|$ denotes the number of elements of the group G; it makes perfect sense because G is finite. Similarly, since $H \subset G$, certainly H is also finite and again writing $|H|$ makes sense.

Now Lagrange's theorem says that whatever groups $H \subset G$ we have, $|H|$ divides $|G|$. That's an amazing thing, because it's not easy for one number to divide another. For example, if we had a group G_1 with $|G_1| = 77$, then any subgroup of G_1 could only have size 1, 7, 11 or 77. So if you were working out the elements of a subgroup H_1 of G_1 and you could see 12 different elements of H_1 already, then in fact you would be finished: you would know that $|H_1| = 77$, and so the subgroup would *have to be* the whole of G.

That example is a bit artificial. Nevertheless, seeing how a theorem is used in practice helps you to understand it, so we look next at a true application of Lagrange's theorem.

1.2 A favourite application of Lagrange's theorem

The same counting argument as above (but easier) proves your favourite first corollary of Lagrange's theorem. Remember that 2 is the smallest prime—1 is not a prime.

Corollary 2 *If G is a finite group with $|G|$ prime, then G is cyclic.*

Proof Step 1: Show that $|G| \geq 2$ and conclude that there is some element $g \in G$ which is not equal to the identity 1_G.