(Self) Assessment: Aiding Awareness of Achievement

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Abstract

In this paper I present a tactic used, which could be described as being a combination of measurement tool and miniexperiment, intended to entice students in a sophomore linear algebra class to construct their own examples of algorithmic-skill-level problems. The tactic is a questionnaire which asks the reader to identify their comfort level via a scale intended to simultaneously pinpoint various achievement levels of the student in the cognitive and affective domains and to communicate to the student desirable and advantageous abilities for success in the course. The questionnaire is part of, and tailored to, an advanced organizer distributed each class meeting.

1 Introduction

What I write about here is a product of my first attempt at teaching a 27-student sophomore-level linear algebra class at a research university. Class met three days each week for 50 minutes each meeting day. My teaching style was probably best described as direct instruction; I plan a lesson with examples speaking to the relevant content and concepts, definitions, illustrations, motivation for topics, but would also try to engage my class using questions whose answer requires more than memory-level thinking and would call on students by name uniformly. For example, I may utter to a student named Dean: "Dean, give me an example of a 3×3 non-invertible matrix in which no row or column is simply the multiple of another row or column." In class, my students' performance was acceptable, as was their performance on textbook-based homework assignments, but after assessing the first exam I realized that, in spite of the richness of the lectures and their clear ability to perform well on homework, something was missing. The textbook adopted (which shall remain nameless) arguably contained problems primarily targeting algorithmic skill and only one paradigm for matrix multiplication was stressed – the row-dot-column one.

I tried to focus on concepts in the lectures and discussed things like the four fundamental subspaces of a matrix à la Gilbert Strang, and found it best to think in terms of the column space of a matrix and hence used the paradigm of linear combinations of columns for matrix multiplication. I mention these things not because I want to argue a certain pedagogical perspective on linear algebra, but to make the reader aware of the situation I was facing: I was compelled to use a perspective different from the adopted text, but did not want to take the time to construct new and more appropriate homework sets or tailor my teaching to meet the text's structure. Instead, I tried something which I hoped would shift some responsibility to the student, while giving them a perspective on study habits that would help them in their future endeavors with Mathematics. I asked myself these questions:

- How can I instill in students a habit of regular meaningful reflection which leads to better, more optimized, preparation, and more confidence and willingness to try?
- Can students be encouraged to create their own examples and problems to solve; can they be guided to learn to practice without their textbook?
- If a student becomes comfortable with creating examples, what effect will this have on their achievement?
- How can I give a structure to the course so that the discussions during the lecture are not lost, thereby making the lectures and time I spend preparing for them, more worthwhile?

2 Tactic

My concern was also a little more specific than indicated by the generic questions I stated above. After assessing the first exam, whose histogram of scores is in Figure 1, it was clear that achievement was not at its best.

Inspired by [4] and [5] and also by a colleague, Dr. Chris Call, in the College of Natural Resources at Utah State University, I decided to use a self-assessment, which I will describe below. First I'll list some characteristics of the course at the time I began implementing the self-assessment. Each lecture was accompanied by an advanced organizer distributed at the

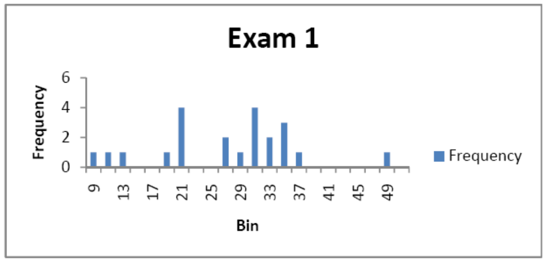

Figure 1: Histogram of scores for the first exam.

beginning of class with the four categories: *Content*, *Key Ideas and Relationships*, *Recall*, and *Agenda Items*. The first three categories contain details relevant to the current unit and what will be developed or used again from an earlier unit. The section *Agenda Items* contains an outline of the activities or examples that will be carried out during the lecture. The self-assessment was then attached to the advanced organizer, and for each example or topic in the agenda items, a corresponding prompt asking a student to construct an example, say of a certain type of matrix, and carry out some task, say finding its nullspace. Each prompt on the self-assessment asked the student to circle a number (ranging from 0 to 4) identifying the most appropriate description of their state of mind regarding the topic associated to the prompt. I tried to refine the scale and define the quantifiers in such a way that both cognitive and affective traits could be identified and ultimately to clarify what it means to be prepared for an exam.

I give an example. Here is an agenda item from an advanced organizer:

3. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a transformation defined by $L(\vec{v}) = (v_1 + v_2, 0)^T$.
   (a) Verify that L is a linear transformation.
   (b) Construct a matrix which represents L with respect to the standard basis.
   (c) Verify that $\mathcal{B} = [(1,1)^T, (-1,1)^T]$ is a basis for $\mathbb{R}^2$.
   (d) Construct a matrix representing L with respect to $\mathcal{B}$.

The corresponding prompt on the self-assessment was:

3. Student can produce a transformation from $\mathbb{R}^2$ into $\mathbb{R}^2$ as in, but different from, agenda item 3.
   (a) Student can verify whether L is a linear transformation.
   (b) Student can construct a matrix for the transformation constructed.
   (c) Student can find a basis $\mathcal{B}$ for $\mathbb{R}^2$ different from the standard one and different from the one in agenda item 3c.

(d) Student can construct a matrix representing L with respect to $\mathcal{B}$.

The student is asked to respond to each prompt by circling a numeric indicator ranging from 0 through 4 defined as follows:

- 0 := Student believes they have no understanding whatsoever of the concept involved or any idea how to complete the task.
- 1 := Student has no confidence in correctness of their response, but can make attempt.
- 2 := Student can make an attempt at completing the task and has some understanding of the concepts involved.
- 3 := Student can, with confidence in the correctness of their response, complete the task and believes they have a good understanding of the concepts involved.
- 4 := Student can direct a classmate in the completion of the task, while explaining to them the reasons and concepts involved.

I note that my class was instructed on the proper interpretation of the indicators and a discussion was had which concluded in the unanimous acceptance of the scale and the notion that a self-ranking of $x+1$ is "better" than a ranking of x, for $x \in \{0, 1, 2, 3\}$. These indicators were chosen – by me with no input from my class – and worded as they are for several reasons. These are they:

- The rank of 4 – the "best" ranking a student can give themselves – represents a sense of understanding or achievement on the comprehension and communication level, a learning level not typically addressed in textbooks. I note that a bi-weekly study session was organized through which students could test this self-assessment.
- The terms "confidence", "believes", and the phrase "can make an attempt" are intended to entice the student to reflect on the affective domain of learning Mathematics.
- The progression of the rankings and transference of responsibility to the student is intended to reflect characteristics of *mathematical maturity*.*

* Although the scope of this paper is not the analysis of what is meant by *mathematical maturity*, I list a few characteristics I decided were reasonable ones to foster in this type and level of course:

- A sense of confidence in the face of new concepts, but given sufficient indoctrination into notation and definitions.
- An ability to recognize what you know and don't know – an ability and tendency to self-assess.
- An ability to teach yourself.

Incidentally, I interviewed several colleagues about what they thought mathematical maturity is and the three I listed, more or less, were part of every list I received. However, one colleague simply said, in response to the question "How would you define *mathematical maturity*?", "We know it when we see it."

Each self-assessment was assigned as a high-priority homework assignment to be turned in the next class meeting. Although I did not give points for doing the self-assessment, nearly every student turned one in nearly every class meeting. I note that I did not change my teaching style or schedule based on the responses in the self-assessments and my intentions to never do that were made clear. I therefore felt that I was doing all I could in order to communicate the responsibility for learning linear algebra was theirs and to provide a helpful framework for developing good study habits and a way to assess their achievement level.

3 Results

At the end of the semester, I had approximately 20 self-assessments from each of 15 lectures, so about 300 documents with data corresponding to confidence levels related to concocting examples relevant to procedural and conceptual linear algebra topics. Many prompts were repeated from questionnaire to questionnaire as many as four times so that any progressions could be tracked if I desired. What I will show here is simply the two histograms from the sequel exams and the one from the first for reference – they are shown in Figure 2. I note that I conducted an analysis of the exams' reliability coefficient, using Kuder-Richardson type analysis with adjustments made for criterion-referenced measurements and found the reliability coefficient of each test to fall within the interval $[0.85, 0.89]$ (cf. [3]).

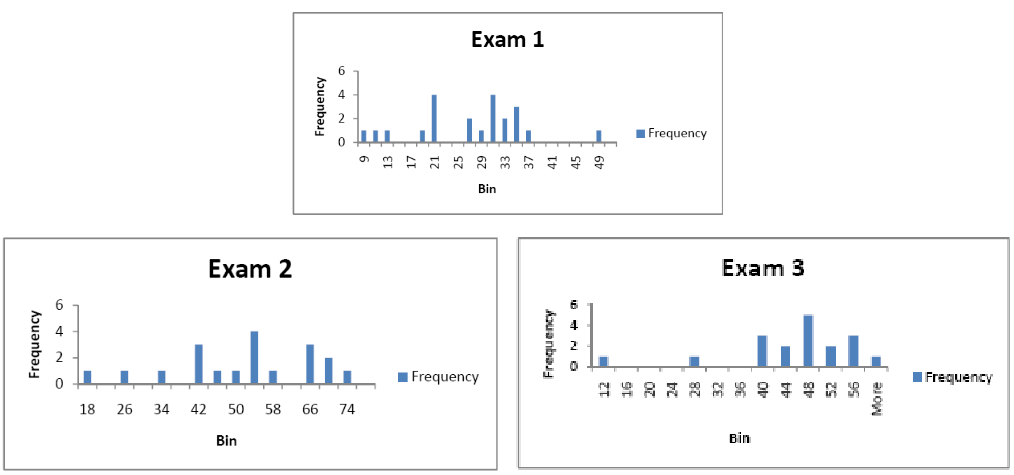

Figure 2:

The self-assessment data was also examined in the following way. Let E_i be the exam score of student i, and S_i be the total self-assessment points student i gave themselves on the relevant self-assessments. The ordered pairs $\{(S_i, T_i)\}$ were plotted and a strong positive correlation was observed.

4 Conclusion

The trend in the exam scores could of course be due to many factors, but I have begun using self-assessments like the one described here in other classes. Some refinement of the scale can be made, or perhaps it should be coarser, and the wording used in the definitions of the scores are things I am trying to hone. I am satisfied with the development of the self-assessment I describe here in very few ways, but find it to be a pleasing component of my teaching preparation which takes very little additional time. Indeed, I have simply handed back the self-assessments before exams in lieu of any review, thereby sparing class time.

Soon before exam 3, I asked, on a self-assessment, "Have you done anything to adjust your study habits in order to achieve a higher score on our third exam?" All students that said they had, referenced the self-assessments and gave details about how they were used to focus their studies. Furthermore, all students that said they had adjusted their study habits achieved a higher score on the third exam.

In regards to further research, other than fine-tuning the self-assessment and its implementation in a linear algebra course, and testing for its efficacy, I find the prospect of communicating and fostering characteristics of mathematical maturity to be compelling. I think certain Mathematics classes are suitable for developing certain characteristics while other classes are not, and yet other classes simply require some characteristics of mathematical maturity to be present in order to expect a high level of achievement. The first question may therefore be "What are characteristics of mathematical maturity whose development I can foster as a teacher and how?" Next, "How can I develop those characteristics in my Math-X class?", or perhaps "Which characteristics should I focus on in my Math-X class?"

References

[1] Alibert, D., Towards new customs in the classroom, *For the Learning of Mathematics*, **8**, no. 2 (1988) 31 – 43.

[2] Angelo, T.A. and Cross, K. P. *Classroom Assessment Techniques*, 2nd ed., Jossey-Bass, San Francisco, 1993, 148 – 153.

[3] Brennan, R. L., and M. T. Kane, An index of dependability for mastery tests, *Journal of Educational Measurement* 14 (1977) 277-289.

[4] Bressoud, D. M., The One-Minute Paper, *Assessment Practices in Undergraduate Mathematics*, MAA notes # 49, 87 – 88.

[5] Koker, J., If You Want to Know What Students Understand, Ask Them! *Assessment Practices in Undergraduate Mathematics*, MAA notes # 49, 91 – 93.

[6] Miller, R. B. and Behrens, J. T. and Greene, B. A., Goals and perceived ability: Impact on student valuing, Self-Regulation, and Persistence, *Contemporary Educational Psychology* (1993) 2-14.

Appendix

Here is an example of the advanced organizer and its corresponding self-assessment for the class meeting of 10/31/2008.

MATH 2270 – Lecture Twenty Eight, 10/31/08
Fall 2008
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Content: Change of basis. Matrix representation of linear transformation with respect to different bases.

Key Ideas:

- The matrix for a linear transformation $L : V \to W$ is determined by its action on a basis for V. We investigate the circumstance of using different bases for V; specifically, what effect does changing the basis for V have on L's representation as a matrix.

Recall:

- *Change of basis.* Suppose $\mathcal{S}$ is the standard basis for $\mathbb{R}^n$ and $[\vec{v}]_{\mathcal{S}} = (c_1, c_2, \dots, c_n)^T$ is the coordinate vector for $\vec{v} \in \mathbb{R}^n$ with respect to $\mathcal{S}$. Suppose also that we wish to express $\vec{v}$ in terms of some different and better-for-some-reason-or-other basis for $\mathbb{R}^n$, call it $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n\}$. Then we are looking for the constants $\alpha_1, \alpha_2, \dots, \alpha_n$ that satisfy the relationship $\vec{v} = (c_1, c_2, \dots, c_n)^T = \alpha_1 \vec{b}_1 + \alpha_2 \vec{b}_2 + \cdots + \alpha_n \vec{b}_n$. In other words,
$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \alpha_1 \vec{b}_1 + \alpha_2 \vec{b}_2 + \cdots + \alpha_n \vec{b}_n = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_n \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$
The coordinate vector for $\vec{v}$ with respect to the better-for-some-reason-or-other basis $\mathcal{B}$ (in other terms $[\vec{v}]_{\mathcal{B}}$)is $(\alpha_1, \alpha_2, \dots, \alpha_n)^T$.

- Let V be a vector space, and α a scalar. If L is a linear transformation from V to W, where W is another vector space (then $L(\alpha \vec{v}) = \alpha L(\vec{v})$ and $L(\vec{u} + \vec{v}) = L(\vec{u}) + L(\vec{v})$), then a matrix for L with respect to some basis, say $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n\}$ is $\begin{bmatrix} L(\vec{b}_1) & L(\vec{b}_2) & \cdots & L(\vec{b}_n) \end{bmatrix}$

Agenda items:

1. Suppose $L : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $L(\vec{v}) = (v_1 + v_2, 0)^T$. Suppose $\mathcal{B} = [\vec{b}_1 = (1,1)^T, \vec{b}_2 = (-1,1)^T]$ is a better-for-some-reason-or-other basis for $\mathbb{R}^2$. Determine the matrix representation for L with respect to $\mathcal{B}$.

0. Check that L is linear.
1. Determine a matrix for L w.r.t. the standard basis, let's call it A.
2. Determine what $L(\vec{b}_1)$ and $L(\vec{b}_2)$ are. What information does this give?
3. Construct transition matrix U from $\mathcal{B}$ to $\mathcal{S}$ (the standard basis).
4. Compute the matrix for L w.r.t. $\mathcal{B}$ via the relationship $U\vec{c} = \vec{x}$, where $\vec{c}$ is coordinatized w.r.t. $\mathcal{B}$ and $\vec{x}$ is coordinatized w.r.t. $\mathcal{S}$. Call this matrix B.

(a) Note the relationship $B = U^{-1}AU$.

2. $\mathcal{P}_2 = \text{span}(1, x, x^2)$. Suppose $L : \mathcal{P}_3 \to \mathcal{P}_3$ is defined by $L(p(x)) = xp'(x) + p''(x)$.

   (a) Find a matrix A for L w.r.t. $\mathcal{S}_P = [1, x, x^2]$.

   (b) Verify that $\mathcal{B} = [1, x, 1 + x^2]$ is a basis for $\mathcal{P}_2$.

   (c) Find a matrix B for L w.r.t. $\mathcal{B}$.

   (d) Consider $p_S(x) = [p(x)]_{\mathcal{S}_P}$ and $L^n(p_S(x))$ versus $L^n(p_B(x))$, where $p_B(x) = [p(x)]_{\mathcal{B}}$.

Self-Assessment for Lecture Twenty Eight

Student:

Key to indicators:

- 0 := Student believes they have no understanding whatsoever of the concept involved or any idea how to complete the task.
- 1 := Student has no confidence in correctness of their response, but can make attempt.
- 2 := Student can make an attempt at completing the task and has some understanding of the concepts involved.
- 3 := Student can, with confidence in the correctness of their response, complete the task and believes they have a good understanding of the concepts involved.
- 4 := Student can direct a classmate in the completion of the task, while explaining to them the reasons and concepts involved.

Self-Assessment prompts. Please consider each prompt, attempt to complete the task, and then assess the process of completion via circling the appropriate indicator.

1.	(a)	Construct a linear transformation L on $\mathbb{R}^2$.	0	1	2	3	4
	(b)	Build a matrix A representing L from 1a.	0	1	2	3	4
	(c)	Construct a basis $\mathcal{B}$ for $\mathbb{R}^2$ different from the standard one and build a matrix B for L with respect to $\mathcal{B}$	0	1	2	3	4
2.	(a)	Pick a basis for $\mathcal{P}_2$ (my notation) different from the standard one	0	1	2	3	4
	(b)	Build a matrix for the differential operator with respect to the basis $[1, x, x^2]$	0	1	2	3	4
	(c)	Build a matrix for the differential operator with respect to the basis $[x, 1, x^2]$	0	1	2	3	4
	(d)	Build a matrix for the differential operator with respect to the new basis you chose in 2a	0	1	2	3	4
3.		Let $n > 3$. Construct the inverse of an invertible $n \times n$ matrix A.	0	1	2	3	4
4.		Suppose A is a matrix thought of as a mapping from one vector space to another. Without mentioning the determinant, explain why A has an inverse only if it is an $n \times n$ matrix? Please put explanation in the *Comments* section below.	0	1	2	3	4

Comments: