The Great Gorilla Jump: A Riemann Sum Investigation

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The Great Gorilla Jump (see appendix) is an activity that was designed by the authors to introduce students to the topic of the Riemann sum within the familiar context of position, velocity, and acceleration. Prior to the implementation of the gorilla activity, the students had worked with several examples of finding the velocity of an object given the position. The gorilla activity was intended to have the students reverse their thinking by being required to find the position when given the velocity. This paper will illustrate the development and rationale for each piece of the Great Gorilla Jump. We will point to a few key findings from data analysis on students' understanding of Riemann integrals, but the focus of this paper will be on activity development and not data analysis.

First, it is important to note that the intent was *not* to have students learn to use anti-differentiation. Instead the goal was to have students discover the underlying structure of the Riemann sum and by the end of the activity, the Riemann integral. In fact, due to the nature of the activity, it was not possible for the students to use andtiderivatives. They were given a discrete number of points (velocity of the gorilla at half-second intervals) instead of an algebraic formula representing the velocity.

Background

Currently there is very little mathematics education research on the topic of student reasoning about Riemann sums and definite integrals. Exceptions are the work of Thompson (1994), Orton (1983), and Sealey (2008). Sealey's work (2008) focused on developing a conceptual understanding of the underlying structure of the Riemann integral prior to students being introduced to the Fundamental Theorem of Calculus, and Thompson's work (1994) focused on developing a conceptual understanding of Riemann sums in a way that makes the Fundamental Theorem of Calculus intuitively obvious to students.

Orton also studied student understanding of definite integrals, but via means of documenting students' ability to evaluate Riemann sums and definite integrals. The article focused on the calculations involved in finding Riemann sums and finding and interpreting area under a curve for the definite integral. It concluded that one does not need to understand the structure of the Riemann sum and definite integral in order to evaluate it. Many of Orton's students could perform routine procedures for finding the area under a curve, but the students rarely could explain their procedures and some even admitted that they did not understand why they were performing the procedures (Artigue, 1991).

We believe that an understanding of the structure of the Riemann integral is important for students to understand for many reasons. It will help students be able to set up a definite integral when the context is less familiar to the student (e.g. work, energy, pressure, etc.). In addition, there are many functions that do not have an andtiderivative that can be expressed in terms of elementary functions. Thus, the Fundamental Theorem of Calculus could not be applied, and numerical approximation methods would be needed. Finally, we believe that an understanding of the structure of the Riemann integral will enable students to understand common numerical methods, such as Simpson's rule or the trapezoid rule since these are based on the structure of the Riemann sum.

Theoretical Perspective

Piaget's structuralism (1970; 1975) was used as the theoretical perspective for the design of the study as well as the data analysis. Structuralism is a form of constructivism that is based on the premise that students “construct” understanding. In terms of mathematics, students need the opportunity to engage in activities that reflect the structure of the concepts to be learned, in this case the structure of a definite integral, consisting of the limit of a sum of products.

It is expected that this lesson be introduced to students before any formal instruction on integrals and give students an intuitive example from which to develop the formal notation and understandings of Riemann integrals. Our hypothetical learning trajectory (Simon, 1995; Simon & Tzur, 2004) for the Great Gorilla Jump activity was based on Oehrtman’s (2002, 2004, 2008) approximation framework. This framework asks students five questions involving the determination of an appropriate approximation, determination of an error for this approximation, determination of a bound the error, and making the error as small as possible. The students who were observed engaging in this activity had been using Oehrtman’s approximation framework throughout the semester. It was expected that answering the questions in this activity would come somewhat naturally to them, building on existing knowledge both within the approximation framework as well as with the concepts of distance and velocity.

Hypothetical Learning Trajectory of the Activity

The Activity: A gorilla (wearing a parachute) jumped off of the top of a building. Using a CBR, you were able to record the velocity of the gorilla with respect to time once each half-second. Note that he touched the ground just after 5 seconds. The data for the experiment was shown in a table. The students were then asked the following five questions:

1. *Approximate how far the gorilla fell during each half-second and fill in the table below.* (Table omitted for space purposes.) In the original version of the activity, the students were expected to graph the data. It was decided that the activity be modified so that the data was presented in a table with only 11 data points instead of a continuous graph for two reasons. First,

the finite number of data points simplifies the problem. To approximate the velocity on each interval, the students only had two choices—the velocity at the beginning of the interval or the velocity at the end of the interval. The second reason for presenting the data in this manner was based on students who may have already seen Riemann sums in a previous calculus class. During the initial activities, we did not want the students to reason in terms of area under a curve. Students in the pilot study had difficulty when working with area under a curve because they did not have an understanding of the relationship between area under a curve and the quantities used (force, distance, and energy in the pilot data). Based on that study, it was determined that students needed a stronger understanding of the structure of the Riemann sum *before* being introduced to the connection to area under a curve (Sealey, 2006, 2008; Sealey & Oehrtman, 2005).

It was hypothesized that the students would struggle a little with question 1 (approximating the distance traveled during each half-second time interval-product layer), because the velocity during each time interval was not constant. It was hoped that the students would decide to use the velocity at one of the endpoints of the interval, applied to the entire interval to obtain an approximation after a brief class discussion of this idea.

2. *Approximate the total distance the gorilla fell from the time he jumped off of the building until the time he landed on the ground.* Based on the pilot data, we did not believe that students would have difficulty with question number 2. Students never seemed to have difficulty understanding that the entire quantity is the sum of the parts.

3. *Is your approximation an overestimate, an underestimate, or is it the exact value? How can you tell? Explain your answer clearly.* 4. *Is there a way to bound the error? If so, find a bound and explain your reasoning. If not, explain why it is not possible.* 5. *If you were able to*

find a bound for the error, how small can you make the bound? Explain your reasoning clearly. Questions 3, 4, and 5 were intended to get the students to start thinking about ways to find better approximations, with the ultimate goal of developing the limit structure of the Riemann integral. Problems with these questions were expected to be minimal as the students had been answering similar questions all semester.

Methods / Subjects

A class of second semester calculus students was observed engaging in the Great Gorilla Jump activity. This lesson was videotaped and transcribed for analysis. The analysis was performed using Strauss and Corbin's (1990) coding techniques and a coding scheme based on a Riemann integral framework in Sealey's dissertation (2008). This lesson was one of a series of three activities that made up a classroom teaching experiment (Simon & Tzur, 2004). Thus, hypothetical learning trajectories were created for each activity and were modified based on the results of the experiment.

Before the implementation of the three activities, the class was led in a brief discussion about distance, velocity, and time in two contexts: one in which the velocity was constant over the entire interval and one in which the velocity changed during the interval. This discussion was focused on making sure the students understood that the formula *distance = velocity * time* only applies to situations in which the rate was constant over the time interval in question. After finishing this discussion, the students divided themselves into groups, and the gorilla activity was passed out.

Results

One particularly interesting aspect of the students' development was seen in their inappropriate use of a product structure. When asked how he determined the distance the gorilla traveled in each half-second, one student stated that his answer corresponded to “change in *distance* times change in time”, when in reality, it should have been the product of *velocity* and *change in time*. A short time later, he changed his wordage to match his calculations, which corresponded to the product of *change in velocity* and *change in time*. While this is closer to the correct method, he should not have been using the *change* in velocity in his calculation.

It is also interesting to note that one student suspected that the gorilla problem was related to antiderivatives. However, since the structure of the antiderivative was not a well-developed structure, he was unable to describe the connection, and thus, was unable to use his previous knowledge to help him solve the gorilla problem. If fact, since the students were not given an algebraic formula that described the gorilla’s velocity, it would have been impossible for him to have applied antiderivatives to the problem.

A third notable result from this activity was the students’ relative ease with the concept of limit within the problem. Quite easily, the students were able to determine if their approximation was an overestimate or an underestimate. They also readily computed a bound on their error and were able to determine that a more accurate approximation could be found if the time interval between data points was decreased. Thus, the students had little difficulty with the concept of limit, but significant difficulty determining the appropriate product structure. This result was quite unexpected, since the limit concept is one that is typically viewed as quite difficult for introductory calculus students.

In summary, we want to emphasize that this paper is intended to introduce the reader to the development of the Great Gorilla Jump Activity. Analysis of student understanding of the Riemann integral is beyond the scope of this paper, but it was important to point to a few of the highlights in the analysis. The students’ difficulty with the product layer illustrates the need for using an activity like the Great Gorilla Jump because it is based on concepts familiar to the students (position and velocity). This activity provides a powerful starting point for students' development of the complicated structure of the Riemann integral.

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Appendix

Activity 1: The Gorilla Problem

A gorilla (wearing a parachute) jumped off of the top of a building. We were able to record the velocity of the gorilla with respect to time twice each second. The data is shown below. Note that he touched the ground just after 5 seconds.

Time (in seconds)	**Velocity (in feet per second)**
0	0
0.5	5
1.0	7
1.5	8
2.0	11
2.5	11.5
3.0	12
3.5	13
4.0	15.5
4.5	18
5.0	19

1. Approximate how far the gorilla fell during each half second interval and fill in the table below.

Time interval (in seconds)	**Approximate distance traveled**
0 – 0.5	
0.5 – 1.0	
1.0 – 1.5	
1.5 – 2.0	
2.0 – 2.5	
2.5 – 3.0	
3.0 – 3.5	
3.5 – 4.0	
4.0 – 4.5	
4.5 – 5.0	

2. Approximate the total distance the gorilla fell from the time he jumped off of the building until the time he landed on the ground.

3. Is your approximation an overestimate, an underestimate, or is it the exact value? How can you tell? Explain your answer clearly.

4. Is there a way to bound the error? If so, find a bound and explain your reasoning. If not, explain why it is not possible.

5. If you were able to find a bound for the error, how small can you make the bound? Explain your reasoning clearly.