An Example of a Non-traditional Pedagogy in an Abstract Algebra Class: Was it Reform Teaching

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To start class, Dr. V. says, "The price of admission today was the ring property sheet. Look over the problems – check with a neighbor to see if you can convince a friend – then you can see if you can convince an enemy," referring to the well-known description that a real proof can convince a friend, an acquaintance, and an enemy.

"Who's the enemy?" one student asks to much laughter in the classroom.

The truth is that for many of these students, this whole Abstract Algebra class is the price of admission to their desired major. The future engineers seem to look at pure mathematics with some skepticism. The pre-service teachers may not *a priori* believe that learning Abstract Algebra can help them be better teachers of the secondary curriculum. For most of them, passing is the ultimate goal, maybe earning a certain grade, but few students chose to take abstract algebra because it sounds like "fun." Moreover, it is naïve to assume that grades will be sufficient to motivate students to then make appropriate day-to-day decisions about their work relative to the class; nearly all teachers have been confronted by students who fail to hand in homework, who skip classes and even exams, yet still claim that passing is their goal. Thus, it is reasonable to suggest that we need a better understanding of the fine-grained mechanisms by which students make day-to-day decisions about their work. Further, we believe that the unique abstract algebra class that we studied can provide insight into students' decision-making processes.

1.1 The place of abstract algebra in the undergraduate curriculum

Abstract algebra is viewed as an important course in the undergraduate preparation of a mathematics major because algebra courses have objectives that include mastering the course content of *groups, rings* and *fields*, as well as mastering proof writing, and seeing connections between previously used mathematical systems such as the integers and polynomials (Cuoco, Goldenberg, & Mark, 1996; Edwards & Brenton, 1999; Findell, 2000). Yet, the literature suggests that students have difficulty meeting the goals their instructors may hold (Dreyfus, 1999; Dubinsky, et. al, 1994; Hart, 1986; Hazzan, 1994; Hazzan, 1999; Hazzan & Leron, 1996; Leron & Dubinsky, 1995; Leron, Hazzan, & Zazkis, 1995; Weber, 2001). Although we know that students are often unsuccessful in abstract algebra, we have little understanding of what helps students succeed in the course. It is in this spirit that mathematics educators have suggested that more research on the relationship between teaching and learning is important, and especially the beliefs, goals, and motivations that enable students to succeed in mathematics (Middleton, et al., 2004).

1.2 Understanding student actions-middle grain size

In order to make sense of student actions on, essentially, the day-to-day decision level we wish to introduce the concept of *raison d'être*. By this term, we mean to indicate the constellation of aligned beliefs, goals and motivations that the students construct in response to the teacher's intended learning goals and *sense of purpose*. We believe that this collection of goals, beliefs and emotions motivates students to act at the day-to-day level.

1.2.4 The social perspective

"(S)tudents' learning is seen as participation in the local, emerging mathematical practices" (Stephen, et al., 2003, p. 67). That is, there are both social and individual aspects of the learning process. While individual students construct their own knowledge, it is always done in relation to their mathematical community of practice; communal development of mathematics happens as a result of the collection of individual learning that comes from classroom activities. Given that learning occurs in context, an analysis of classroom practice serves to both document the existing social situation and to provide insight into past practice (Stephen, et al., 2003).

Significant aspects of a class that should be investigated include: social norms, socio-mathematical norms, and classroom mathematical practices. Social norms the "rules" of classroom behavior, both implicit and explicit (e.g., stated in a syllabus) that are not specific to the mathematics being learned, but rather dictate the environment in which learning will occur. Socio-mathematical norms refer to classroom practices that are discipline-specific, such as the criteria for a complete and correct proof. Classroom mathematical practices can be understood as the taken-as-shared practices related to specific ideas such as the level of detail required in writing a proof (Cobb, 1999; Cobb and Yackel, 1996).

1.2.2 Student Motivation

Research has begun to explore the relationship between students' goals and beliefs and the practices of their classrooms. There is a growing body of evidence that students' beliefs and learning goals influence their participation in classroom activities (Ames, 1992; Cobb et al., 2001; Jansen, 2006, 2008, in press; Lo, Wheatley, & Smith, 1994; Middleton & Spanias, 1999; Stephan, Cobb, & Gravemeijer,).

For example, students who believe that participation has a positive utility value for learning and perceive that the teacher supports active participation in group discussions are more likely to ask questions and attempt explanations, whereas students who want to avoid embarrassment are more likely to simply provide answers (Jansen, 2006). Jansen (2006, 2008) has indicated that prior classroom experience may negatively impact students' willingness to participate in classroom discussions, and potentially other activities; however, in the case of an advanced mathematics class the students have generally been highly successful earlier in their mathematics careers, and we can posit that negative feelings vis-à-vis mathematics classes are not a significant factor.

Lastly, there is a body of research that suggests that classroom activities should be directed at specific learning goals, and that students' motivation is directly related to their interest in meeting these learning goals (Bereiter & Scardamalia, 1989). Yet, even the studies described above state that relatively little is known about student decision-making and that more data are needed to understand what motivates undergraduates to participate in mathematics classes (Jansen, in press).

1.2.3 Student Learning Goals

Research suggests that there is a strong link between students' motivation and the type of goals that they set for themselves (Ryan & Deci, 2000). We know that the goals that the teacher brings and the teacher's vision of class is also important. Research suggests that making the learning goals explicit to students may maximize the potential learning from classroom activities, but perhaps more important is the teacher's ability to convince the students of the importance and validity of those learning goals. This may encourage students to construct their own mastery goals for the course in the place of the usual goal of "achieve a passing grade" (AAAS, nd; Ames, 1992; Bereiter & Scardamalia, 1989; McNair, 2000).

1.2.4 Collective Cognitive Responsibility

A feeling of belonging to a community has major impacts on persistence in both programs of study and in individual courses (Carlson, 1999; Grevholm, Persson & Wall, 2005;

Herzig, 2002; Soto-Johnson, Yestness, & Dalton, 2008). McMillan and Chavis (1986) stated that a sense of community requires that an individual (i) feels a sense of belonging to the community, (ii) feels influential within the group, (iii) feels nurtured by the group, and (iv) feels an emotional connection to the group. Ryan and Deci (2000), among others, have noted the importance of *relatedness* (students feeling that they are respected and cared for) as a means of motivation. Yet, these researchers are not capturing the idea that members of a community feel a sense of responsibility to other members of that community. In an academic community, a sense of responsibility to the group is an important component to consider. For that reason, we will consider the construct of Collective Cognitive Responsibility (CCR) (Scardamalia, 2002).

CCR can be thought of as having the "condition in which responsibility for the success of a group effort is distributed across all the members," and this effort includes community knowledge-building or knowledge-creating (Scardamalia, 2002, p. 2). That is, this expands the concept of sense of community to one in which students are supported, feel a sense of importance and autonomy, and feel responsible for contributing to the growth of communal knowledge.

We propose that in the class we will describe below, Dr. V.'s strength is in offering students multiple reasons for taking the course and helping them to both understand his goals for the course and constructing their own goals and motivations for coming to class. To accomplish this, he included activities that engendered a learning-oriented CCR. That is, Dr. V helped the students to construct a *raison d'être* in which all interests were aligned with mathematical learning and supported good academic habits at the day-to-day level.

3.1 Context and methodology

Eastern State University (ESU) is a Doctoral I university in the Carnegie Classification system.

3.1.1 The Instructor

The instructor of the introductory abstract algebra course in this study was a male professor who had previously taught introductory abstract algebra a number of times. This was the second time that he had taught the course in this manner, using materials he created. His research field was mathematics education and he had a joint appointment in mathematics and education. Two graduate students, both pursing doctorates in mathematics education, assisted with the course instruction.

3.1.2 The Abstract Algebra Course

The Eastern State University online course catalogue gives a short description of an introductory abstract algebra course that was to cover *group theory* and *ring theory*. It was a single-semester course and not recommended for students intending to pursue graduate studies in pure mathematics.

3.2 Data sources

The primary data source was observations of class meetings that included field notes, video recordings and subsequent transcriptions of the recordings. During the observations and transcriptions, I was especially interested in attempted proofs and the discussions of the attempted proofs. Over the course of the semester, I had the opportunity to observe 15 class meetings. I also had a number of conversations with the lead instructor and saw presentations he gave about the course.

Semi-structured follow-up interviews were conducted with seven of the students that were audio recorded and subsequently transcribed six months after the course ended.

3.3 Data analysis

My analysis began with my field notes. My field notes included two separate columns. The first column was meant to capture the flow of the class and generally consisted of *conceptual descriptions* that gave an overview of pedagogical moves and student reactions during the class meetings. All memos were written in an informal style, the aim being to bring to light interesting ideas rather than to produce a finished product (Glasser, 1992).

After taking field notes, I transcribed the video data and then began to construct a narrative based upon both the transcripts and my earlier memos. I constructed this narrative in a way that I believe aligns with Powell, et. al's (2003) suggestions for analysis of video data. Powell, et. al suggested that the first steps in video analysis include carefully viewing the video data, describing the data and identifying critical events.

This case was more complicated than simple video because I was also present in the classes while making the video recordings; because of that, the method of analysis was slightly different. This is not to claim that another researcher would write the same memos, create the same codes or even ask the same questions of the data, but the process allowed the data to drive the code creation.

4. Data and analysis

As argued above, the instructor's *sense of purpose* has substantial impact on the shape and direction of a mathematics course both in terms of day-to-day activities and at the course level. When queried after the course was completed, the instructor described his own set of beliefs about the class. He notes that he wanted students "to confront problems much more often than results," amd "to develop understanding [of content] in a learning trajectory that move from analysis of examples to formulation of theory." He adds, "we wanted to … encourage students to reflect on the nature of mathematics as a process." Moreover, Dr. V regularly communicated these learning goals to the students, and made explicit statements to the students about why learning *group theory* and *ring theory* were important to their mathematical development and the mathematical knowledge of teachers.

In many ways, Dr. V's goals would fit within the purview of any introductory abstract algebra course. Thus, the more interesting questions center on what did the class looked and sounded like, and what the instructor did in order to try to help students internalize and realize those learning goals and build a CCR. Ultimately, I sought to understand what *raison d'être* the students constructed.

Item 1:

Dr. V walks into the room and welcomes the students back to class. He introduces the topics for the day and then starts passing out a handout with the title, "Some thoughts and questions about algebraic proof." Dr. V then says, "One of my agendas in the course is to develop some broad agenda about mathematics, proof and reasoning in mathematics. What I've done is put together some thoughts and questions. I'd like you to take 10 or 15 minutes to talk to some people near you… Spend some time looking at these questions." The questions included:

- What is your reaction to the suggestion that completeness and correctness of proof can only be judged in relation to the intended audience you are trying to convince?
- How should the presentation of a proof to a live audience (with a mixture of written and oral elaboration) differ from a presentation that is being communicated in writing alone?
- The remarks above dealt only with constructing and presenting a proof, not with discovering a provable proposition (or with constructing counterexamples to claims of others). What strategies do you find effective in deciding whether a conjecture is

true or false?

As soon as Dr. V is done speaking, the students read the questions, and then, without further direction, move their desks into smaller groups to discuss the questions posed. In a few minutes, Dr. V called the students back to order by saying, “What are your responses?” The discussion primarily focuses on the ideas of “complete” and “correct.” There is particular interest on how the intended audience of the proof relates to the completeness and correctness of the proof. One student claimed, “Well, my definition of completeness would be, If someone sees a step and says “Why is this true?” and can’t see it immediately, then it’s not complete.” Another student then makes the observation, “I think correctness is objective, but completeness is subjective. If a proposition is true, it has a correct proof.”

Later in that same class, Dr. V turns the students to questions about ring theory. They had, in the previous class, read the definition of rings and determined if proposed structures are rings. To start this class, Dr. V directs their attention to the question that states:

> Complete the following sentences in ways that give significant true statements about all rings (R, +, •) and develop proofs of your conjectures.
>
> For all a, b, and c in R:
>
> a. a(0) = (0)a = ______________
>
> b. a(–b) = (–a)b = ___________
>
> c. if a + b = a + c then __________
>
> d. if x + a = a, then _________
>
> e. (–a)(–b) = _________

Dr. V asked the students to determine what should fill in each of the blanks, and once they had done that, asked them to write proofs of their propositions. In particular, he set them working in groups to produce a that in any ring, $a(0)= 0$. After working in small groups for approximately 15 minutes, three students offered to put their proofs on the chalkboard and explain them to the class.

The proof farthest to the right (and the first completed) was written up by Dan. It had the look of a standard textbook proof:

$$\begin{aligned} 0 &= 0 + 0 \\ \boldsymbol{a}(0) &= \boldsymbol{a}(0 + 0) \\ \boldsymbol{a}(0) &= \boldsymbol{a}(0) + \boldsymbol{a}(0) \\ 0 &= \boldsymbol{a}(0) \end{aligned}$$

As was the custom in the class, Dan was asked to return to his proof and explain and defend it. While Dan was speaking at the front board Dr. V was standing against one of the side-walls of the classroom. The other students were all silent, and seemed to be watching Dan's presentation. When Dan finished, the section of the class where he had been sitting clapped, but no one in the class said anything. Dr. V prompted a brief conversation, and then to close the lesson, he had the following brief discussion with Dan:

> Dr. V: Why was 0 = 0 + 0 the crucial thing? [Then, directed at the student who presented the proof] What would lead you to think about this?
>
> Dan: It has zero and looks like the distributive property.
>
> Dr. V: The distributive property is going turn out to be real useful in rings because it's the bridge between the two operations.

The second proof was offered by a student, Chris. Chris often had original insights, but had impatience with details. His proofs often appeared unpolished.

$$\begin{aligned} \boldsymbol{a}(0) &= \boldsymbol{a}(-\boldsymbol{a} + \boldsymbol{a}) \\ &= \boldsymbol{a}(-\boldsymbol{a}) + \boldsymbol{a}^2 \\ &= -\boldsymbol{a}^2 + \boldsymbol{a}^2 \\ &= 0 \end{aligned}$$

Before he could begin his explanation, another student in the class (designated S, below) challenged his assumption that $a \otimes -a = -a^2$. Chris immediately offered a possible resolution of this concern.

$$
\begin{aligned}
(0)\boldsymbol{a} &= (1 + (-1))\boldsymbol{a} \\
&= (1)\boldsymbol{a} + (-1)\boldsymbol{a} \\
&= \boldsymbol{a} + (-1)\boldsymbol{a} \\
\textbf{\textit{So}}\ \ (-1)\boldsymbol{a} &= -\boldsymbol{a}
\end{aligned}
$$

S: That line, number three, aren't you assuming what you're proving?

Chris: Which?

S: On the left.

Chris: No, we know this, I'm just moving things around.

S: But you're making use of what you show as the last line in your proof.

Chris: Yeah, but I'm just re-writing, all of these lines are the same thing. I was just trying to get this to look like something useful.

A discussion of the validity of each line of the proof ensued, and Chris suggested additional possible revisions. Finally, they arrived at an insurmountable error:

S: How can we do a times minus a equals minus a squared, and even when you elaborate you have [reads aloud] $(a)(-1)(a) = (a)(a)(-1)$ and that would make it commutative.

Chris: Multiplication is commutative.

Dr. V: Well, we don't know that rings have commutative multiplication, in fact, they don't all, nor do they all have unit elements… Well, the consensus is that it seems he's assuming more than is in his theorem.

The third proof used the idea that a•0 meant *a* zeroes added together, but the student who wrote it was unsure himself about its validity. The third student quickly declared that he had realized it was only valid in specific rings and retracted his work without class discussion.

In the class described above, there are a number of discussions and interactions where it is possible to see Dr. V's goals for the class, but the most notable may be in the beginning of the class meeting; it is notable becuse Dr. V explicitly shares his learning goals with the students. In the discussion that follows, the students explicitly demonstrated their understanding of the ideas of completeness and correctness of a proof.

When the class turned to the algebra content for the day, it was possible to observe, in action, the types of thinking that the students had been discussing. They were clearly reading the proposed proofs to determine whether or not they were complete and correct. That is, their work after the discussion mirrored the types of activities that they engaged in. This is an example of the students acting in ways that showcased their developing understanding of the socio-mathematical norms of class and of the discipline. In this way the students could understand their mathematical community as more expansive than just those members of the class but also including the wider-community of mathematicians.

After the first proof that was presented, Dr. V. had to coax the students into a discussion about the proof. He asked them to state the most important idea, and to indicate the problematic line. This discussion and prompting reinforced the social and socio-mathematical norms of the class. For example, it reinforced the social norm that students would always present and defend their work. Dr. V's prompting also reinforced the social norms of the audience for a presentation, in that they were expected to carefully read the proposed proof and ask questions. This prompting for questions was then unnecessary in the discussion of Chris' proposed proof. Moreover, the students quickly located these problematic elements of Chris' proposed proofs and asked him to defend them. The socio-mathematical norm for presentation was more specific; students were expected to state the key idea in the creation of their proofs. The socio-mathematical norms were also specific about the types of questions that students were to ask; in particular, they were to question aspects of the proof that included both the structure and the warrants for individual lines (Toulmin, 1958; Weber & Alcock, 2005).

The key point here is not that two of the students stumbled over a basic proof about rings, but that the entire session was driven by student input and reactions. Many students in the class had developed the mathematical self-confidence and risk taking that encouraged them to try ideas in front of others (and the others had learned how to respond). They all had the sense that they were in a mathematical community whose norms included collective responsibility for advancing content, proof, and proof-verification and the ability to discuss each of these in an abstract way. This discussion of proof included being able to describe the structure of proofs and the key ideas.

Item 2:

The problems 68b and e from above have been left unsolved by the class as they have worked further into the material on ring theory. As students progress through other problems on the board, Dr. V reminds the students that their mathematical work would be easier if they could use 68b, but class convention was such that they could not use it in further work until it had been proved. Finally, one group of students comes up with a proof to 68b, and a student announces this during a break in discussion. Julianne announces that her group has succeeded, but she is reluctant to take credit for her proof. "Well, I just did it by looking at his proof," she says, indicating Rob's work, "otherwise I wouldn't have gotten it." She starts talking through the proof, explaining, "We could just say – instead of 1 + -1 = 0, we could plug a + -a in for 0, and then, um, use the distributive property…" Dr. V interrupts her and asks her to go to the board, where she writes down the proof.

There is a brief discussion during which Julianne recapitulates the key idea and, again, credits Rob's work on a different problem. Dr. V looks at the students, "So, now we've got 68b for sure. Nobody took e, but probably now you're willing to take e."

In fact, two students immediately volunteer to "take e," although each is willing to let the other write the proof on the board. Dr. V invites both students to write their proofs on the board, saying, "Turn your backs to each other, and let's see what we get." Brandon and Chris wrote the following two proofs:

Chris' Proof	Brandon's Proof
(-a)(-b) Define a = -c (-(-c))(-b) c(-b) -cb -[cb] -[(-a)b] -[-ab] -[-(ab)] ab	(a + -a) = 0 (a + -a)(-b) = 0·(-b) (a+-a)(-b) = 0 a(-b) + (-a)(-b) = 0 ←-ab + (-a)(-b) = 0 (-a)(-b) = ab

Dr. V noticed the arrow that Brandon had drawn in his work, "Does this arrow mean you aren't finished?" Brandon responded, "No, that's where I used 68b." Brandon then explains his proof by using the work that has come before, "The key is to go from here to here, which is to use the thing that Michelle proved, which is basically bringing the negative out and keeping the inside." After some discussion, Dr. V suggests that Brandon has given "enough of an idea for everybody" and moves on to Chris' proof.

Chris stands by his proof . Before anyone can ask a question, he says:

> I defined a variable, c, such that a is the additive inverse of c. Then I replaced that a here and I used the property of the additive inverse of the additive inverse is the original, so I got that c times the additive inverse of b is the additive inverse of cb together. So, I used that as a single variable and then re-replaced the additive inverse of a in here and then defined that as a single variable and then used that same property that the additive inverse of an additive inverse is the original.

Looking around the room, Dr. V saw that the class was silent. "OK, take a couple minutes to convince yourself that this is ok. Ask questions such as you have." Even though students think about this and talk briefly with each other, they are unwilling to pose questions to Chris. Dr. V then says, "There are some questions about this one, but you're being quiet." That statement prompts the students to begin an animated discussion during which approximately half of the class participates. They discuss the form of the proof, the validity and warrant of each line of the proof, and link the proof back to previous work in *group theory*. At the close of the conversation, a member of the class observed, "One comment I like about Chris' method is that he saw that at its core it's really 68b, and he just had to convert 68e into 68b and that a equals opposite c is what does it."

In the case of the proofs described above, there are a number of significant actions on the part of the students. Each of the student presenters described the key idea of their proof and gave credit when they drew upon a previous problem, and did so unprompted. When looking at the resulting class dynamics, the conversation following Chris' proof was quite animated with a large number of participants asking questions in an attempt to understand the work and ensure that it was valid. Moreover, one student complimented the structure of Chris' work for the explicit connection it drew to previous work, again demonstrating how thy have internalized socio-mathematical norms.

There is evidence in these classroom incidents that suggests that students had internalized Dr. V's learning goals both for content and mathematics generally, and that they had develop a strong sense of community, including a CCR. Evidence from follow-up interviews corroborates these findings.

4.1 Interview data:

The interviews were originally intended to explore how much mathematics content students could recall six months after the end of class, with a brief set of introductory questions that prompted them to discuss their experiences in the class. A few significant themes emerged. The first was that six of seven students who were interviewed stated that the most important aspect of class was, “Learning as opposed to grades.” Moreover, their comments indicated that they were usually grade-driven in other classes; as one stated, “[Abstract algebra was] probably the only math class where, well, I don’t know how everyone else did, but I was doing well in the class and I felt like I didn’t have to worry about my grade and I could just enjoy the class.” Thus, we have some evidence that the students have taken Dr. V’s communicated learning goals and constructed their own aligned mastery (e.g., content-based) goals for learning as opposed to purely achievement (e.g., grade-driven) goals.

The other major theme that emerged from the interview data was that students recognized their responsibility for the development of the content of the course. As one stated:

> It was very student directed. Dr. [V] would try and keep a low profile, he was basically just there to ask some questions and clear things up. He wouldn’t lecture and he would give us the notes and expect us to discover the math. We were taking a more active role in the class.

Other students reiterated similar senses of responsibility for the material, “If you figure something out, great, come and share it with the class… people went up to the board to share proofs or what they found...”

Lastly, in terms of how work was assigned and why students chose to do it, we again sense an aspect of CCR: “There were some times I was motivated to do a problem I wouldn’t normally be able to do… for the most part I would just do something to have something to present." Thus, we see students remarking on how the students in the class share responsibility for advancing the knowledge of the community, and, at the individual level, that a student felt responsibility to the class in order to have something to present. Thus, from the interviews, we have additional evidence from the interviews that the students have recognized the pedagogical moves that Dr. V has made. They seem to know, and have accepted, that Dr. V is encouraging, and eventually demanding, that the students take primary responsibility for their learning and be responsible for the learning of their classmates.

5. Results

We observed a class in which the instructor had a strong *sense of purpose* and regularly communicated his learning goals to his students. As a result, it contributed to the construction of aligned performance and mastery goals on the part of the students. Dr. V established social norms and socio-mathematical norms through which he communicated to the students that their learning was the most important aspect of the class. He made it so that students were able to feel successful, focus on learning and not worry about their grade in the class.

Dr. V created a classroom community where the student presenters were regularly expected to explain their work, what lead them to their proof, and the key idea of their proof. Similarly, the students who were the audience were expected to ask questions about the individual lines of the proofs so that the presenter could share his or her rationale, and so they could ensure that they understood the thinking behind it. This explicit statement of learning goals and the socio-mathematical norms and classroom practices of justification, explanation, and questioning combined to indicate to the students that Dr. V placed significant value on student learning, not content coverage or some other measure.

Moreover, because Dr. V engaged the students in direct conversation about both content and mathematical thinking, (e.g., the completeness and correctness of proofs in the above example), the students were more able to construct their own mastery learning goals for the course instead of just performance goals. As such, the aggregate of all of these practices allowed the students to recognize Dr. V's interest as indicated in their comments during the interviews.

The socio-mathematical norms and classroom practices were also such that they promoted the students to construct a collective cognitive responsibility – CCR – for the development of the course material. Although the work was not structured exactly as Scardamalia (2002) suggest it could be for giving students all responsibility because Dr. V did create the text and structure the students' work, the fact that the students were not allowed to draw on a result until it was proven potentially mitigated against that fact. In the data above when explaining proofs students explicitly stated the "key idea" in proofs, referenced helpful or inspirational work by their peers and "took" responsibility for problems. Moreover, in their interviews they acknowledged this responsibility. Through these actions, the students were demonstrating responsibility for growing the knowledge of the class, and, for helping each other learn and make sense of both the content and the methods of proof. Similarly, when students were presenting a proof, Dr. V attempted to reinforce that the students were responsible for the work by standing either on the side or the back of the room. That is, he was communicating through his location that the 'work' of the class, proving and proof-verification, was theirs to do.

When students were in the audience, they were expected to ask questions in order to ensure that they understood, but also to reinforce the notion that they had responsibility for proof-validation. That is, they had to agree that a proof was complete and correct before the result could be used in future work. In order to do this, they were expected to carefully look at each assertion in a proof and determine whether or not it was valid and warranted; moreover, they were expected to ensure that the structure of the proof was appropriate. In this way, the class also had the responsibility for ensuring that their knowledge was correct. In short, the students had responsibility to grow the collected knowledge, ensure it was correct, and to help each other learn. Because of this, it is reasonable to suggest that the class developed a strong form of CCR.

Thus, what we have seen is a classroom in which students believed their learning was important. They were able to state the class learning goals and articulate their learning, and, as shown in their classroom actions and in interview data, seemed to develop strong internal motivation to succeed. Furthermore, due to their important role in the proof-production and proof-validation stages, and, as noted in the interview data, they felt a sense of competence, and likely autonomy. Lastly, as noted above, the students seem to have developed a strong sense of community. In total, this suggests that the students were much more likely to be motivated to succeed, and, have a strong *raison d'être* that promoted their academic success on the level of day-to-day decision making. That is, a goal of passing a class is often insufficient when it comes to helping students make day-to-day choices, but due to their strong *raison d'être,* these students were much more likely to make appropriate choices. This was also reflected in the fact that of the 22 students who started the course, all of them completed it and 21 of them earned passing grades. Although there is not much literature about persistence and completion of abstract algebra courses, anecdotal data suggests that this level of success makes this class unusual (Edwards & Brenton, 1999). That is, the students were more likely to complete their work to a high level and earn good marks on exams.

It seems that it was Dr. V's insistence on the importance of proof-validation as an important part of the knowledge-creation process (coupled with the frequent presentation of incorrect proofs) that made the most difference in terms of creation of a CCR. It is important to ask why proof validation can serve as an activity that can engender the creation of such a learning community. I posit the following rationale: first, as a proof was not accepted without a positive assent from the class, it made active participation more important for the building of knowledge, which may have helped each student in class feel a sense of influence and autonomy as the literature suggests is important (Ryan & Deci, 2000). Second, Dr. V insisted on the importance of checking mathematical logic as a means of improving the students' knowledge of mathematics, thus, working through the logic required in proof validation was directly contributing to a learning goal. Lastly, there were incorrect proofs put onto the board with great frequency. As such, Dr. V could remind the pre-service teachers in the group that they would regularly be reading student work looking for errors. The idea that students were doing work to prepare for their future jobs as teachers may have also motivated the students to practice proof-validation.

Lastly, it seems reasonable to claim that the theoretical lens of *raison d'être*, which combined aspects of motivation, *sense of community*, and CCR, was helpful in making sense of the students' actions on a day-to-day basis within the class in terms of their interest and willingness to participate, especially in whole-class discussions. Moreover, the analysis of the classroom data was supported by the follow-up interviews suggesting that it is reasonable to infer student *raison d'être* from their in-class activities. This represents an advance from previous research, especially at the undergraduate level, as previous work was principally done via post-hoc survey, interview, or analysis of assessments (Jansen, 2006, 2008; Soto-Johnson, Yestness, & Dalton, 2008).

6. Implications and directions for future research

This class may have been a very fortuitous combination of personalities and pedagogical techniques in multiple regards vis-à-vis student motivation. First, there is no evidence that Dr. V's actions were either necessary or sufficient for engendering the kind of *raison d'être* that we argue the students constructed. We have hypothesized some actions that impacted student motivation, sense of community and CCR, but more study is absolutely essential to better understand the linkages between enacted and learned curriculum, with special attention paid to the role of the student. This type of investigation into students' beliefs, goals and motivations should be pursued in advanced undergraduate classes, with the intent of determining pedagogical moves that increase the odds that students will be successful.

There is very little literature devoted to describing the practical aspect of why students choose whether or not to engage At the advanced undergraduate level, there seems to be some presumption of a basic level of interest in the material as well as a sense that the pedagogical contract specifies a certain amount of student effort. It is worth further investigating faculty beliefs relevant to student motivation. That is, attempting to understand why faculty believe that students should be interested in the material. Similarly, we should look more closely at how well (or poorly) an interest in 'getting good grades' translates into academically-oriented decision making at the day-to-day level.

These questions should be asked in both traditional and non-traditional classroom settings. For example, in Inquiry-Oriented classes especially, we, as instructors, often assume a 'good-faith' effort on the part of our students to engage with the material, yet should students choose not to engage, such classes will fail miserably. Even though the immediate classroom need of a 'good faith effort' on the part of the students are significantly lessened in a traditional classroom, there is still an underlying belief that significant time and energy should be invested by students in preparation for class.

Thus, we should certainly seek to understand what undergraduate instructors offer their students in terms of *sense of purpose*, sense of community, and help in creating other beliefs and motivations that will lead to the type of decision which are academically useful.
This type of understanding can help researchers and teachers create interventions that could motivate all students, especially those who are struggling, to adopt practices that are aligned with academic success, both on the day-to-day and more global levels. In particular, we can explore whether there are particular interventions that seem to significantly impact students' motivations, sense of community and beliefs. We can then determine how to usse these interventions to make students more likely to participate, do their homework in a timely and complete manner, and develop mastery goals as opposed to performance goals. This research program could have significant impact on undergraduate education and retention of math and math education majors.

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