Beyond Static Imagery: How Mathematicians Think About Concepts Dynamically

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Abstract: Researchers have emphasized the important role of visualization, and visual thinking, in mathematics, both for mathematicians and for learners, especially in the context of problem solving (see Presmeg, 1992). In this paper, we examine the role that motion and time—which engage similarly sensory modes of thinking—play in mathematicians' conceptions of mathematical ideas. In order to expand the traditional focus on (and distinction between) visual and analytic thinking (see Zazkis, Dubinsky, and Dautermann, 1996), we employ gesture studies, which have arisen from the more recent theories of embodied cognition. Expanding on Núñez's (2006) work, we show how mathematicians' gestures express dynamic modes of thinking that have been hitherto underrepresented.

Introduction

At least since Pólya, scholars have been interested in the aspects of problem solving that do not depend strictly on logical or analytic modes of thinking. For example, a large number of researchers have focused on the role of visualisation in mathematical thinking. More recently, the theories of embodied cognition have drawn attention to the essentially metaphorical ways that mathematicians and learners alike come to understand mathematical concepts. This kind of research points to the very important role that extracognitive processes play in the development of mathematical knowledge. While visual imagery has long been an explicit component of mathematics (through the use of diagrams and figures), it has often been represented and communicated in static terms, making dynamic imagery much less common in mathematical writing. However, recent research suggests that humans tend to conceive of objects as being in motion (even when they're not), and use these dynamic conceptualisations in thinking and problem solving. Our goal in this paper was to investigate the extent to which professional mathematicians could be seen as employing such dynamic forms of thinking, even about purportedly static objects. We were motivated in part by the emergence of dynamic representation technologies of the past 20 years and their affordances for mathematics learning.

In this paper, we first present a brief overview of research on the different modes of thinking, including the visual and analytic, and connect this research to emerging theories and methodologies from embodied cognition. We then present the analysis of mathematicians' verbal and non-verbal expressions in describing two concepts: quadratic functions and eigenvectors. Finally, we present a discussion about our findings, and offer suggestions regarding the use of gestures in teaching mathematics.

Background

In mathematics education literature on students' mathematical thinking, researchers have proposed distinct modes of thinking. Krutetskii (1976) distinguishes verbal/logical thinking from visual/pictorial thinking. The former is an indicator of level of mathematical abilities whereas the latter indicates a type of mathematical giftedness. Krutetskii's work led mathematics educators to inquire further about the visual/pictorial mode of thinking, and to emphasise its importance in mathematical thinking (see Bishop, 1989, Eisenberg and Dreyfus, 1986 and 1991, Presmeg, 1992; Zimmermann and Cunningham, 1991). In her study of mathematicians' ways of coming to know mathematics, Burton (2004) interviews seventy mathematicians working in different fields of mathematics. She identifies three primary modes of thinking: visual/pictorial, analytic/symbolic and conceptual.

While many researchers have distinguished the visual from the analytic, as two modes of thinking (see Clement, 1982), Zazkis et al. (1996) focus on the relationships between the visual and the analytic modes of thinking. They argue that these two modes of thinking are not dichotomous, and propose the Visualization/Analysis (VA) model to describe students' ways of thinking in problem solving. In defining the visual category, Zazkis et al. draw attention to the sometimes dynamic nature of visual imagery, something that Burton (2004) also does, in describing her visual/pictorial category as "often dynamic." According to Zazkis et al., perceiving dynamic processes and objects creates more complex mental images than perceiving static objects. Further, the very act of perceiving static objects involves dynamic actions, as the eye moves across the visual field to build the static objet (Piaget, 1969). Although both Burton and Zazkis et al. recognise the presence of dynamic visual imagery, they do not offer many examples of such imagery, nor do their models of mathematical thinking accord it a primordial role.

More recent research, drawing on theories of embodied cognition (see Lakoff and Núñez, 2000), suggests that dynamic thinking (and not just image-based dynamic thinking) plays an important role in conceptual development. For example, Núñez (2006) argues that mathematical ideas and concepts are ultimately embodied in the nature of human bodies, language and cognition. He has shown that static objects can be unconsciously conceived in dynamic terms through a fundamental embodied cognitive mechanism called 'fictive motion;' he illustrates this mechanism using the concepts of limits, curves and continuity.

In addition to studying mathematicians' linguistic expressions, Núñez broadens the methodological scope by including analyses of mathematicians' metaphors and gestures, which are key to revealing more dynamic thinking processes. As such, Núñez's approach differs from that taken by the researchers cited above, who focus mainly on linguistic expression, and who minimize the role of dynamic thinking in their models of mathematical thinking. We note that the inattention to (and sometimes ignorance of) the role of time and motion in mathematical thinking has strong historic roots: not only does mathematics tend to detemporalise mathematical processes (Balacheff, 1988; Pimm, 2006), but several mathematicians have expressed discomfort at the idea of moving objects (see Frege, 1970).

The almost exclusive focus on linguistic expression may be partly responsible for the lack of attention to dynamic modes of thinking. In turning to gestures for insight into the nature of mathematical thinking, Núñez suggests that gestures have been "a forgotten dimension of thought and language" (p. 174). Recent research, however, has shown that speech and gesture are two facets of the same cognitive linguistic reality. In particular, research claims that gestures provide complementary content to speech content (Kendon, 2000) and that gestures are co-produced with abstract metaphorical thinking (McNeill, 1992). This research supports our methodological approach in this paper, which is to analyse both speech and gesture in describing mathematical thinking. In particular, given the motion aspect of gesturing, we hypothesise that analysing gestures will provide more insight into the dynamical thinking process of mathematicians.

Research Context and Participants

In our larger study, we extend Núñez's work to explore concepts other than limits and continuity. This paper focuses on concepts relating to functions, matrices and eigenvectors. While Núñez studied mathematicians as they gave lectures, we chose to adopt the approach of Burton (2004), who used interviews to examine the nature of mathematical thinking. We designed our interviews using a set of questions aimed at eliciting mathematicians' concept imagery around a variety of mathematical concepts, spanning K-12 and undergraduate mathematics. In this paper, we present data from interview with four mathematicians whose research interests were in both pure and applied mathematics, and who were all members of a medium-sized mathematics department in Canada. Each interview lasted between 1 and 1.5 hours. Interviews were videotaped and transcribed. We reviewed the video clips and selected to analyse mathematicians' speech, gestures, analytic and visual thinking about quadratic function and eigenvector.

Analysis of Study

We refer to Núñez's framework, conceptual metaphor and fictive motion to analyse mathematicians' linguistic and non-linguistic expressions. We also use McNeill's gesture classification and transcription to analyse the movements of the mathematicians' hand and arm as they described mathematical concepts. Verbal and gestural excerpts from interviews follow.

Analysis of Speech and Gestures: Quadratic Function

In our first analysis, we illustrate the way in which linguistic expression by itself can include evidence of dynamic thinking. In response to our prompt about quadratic function, LG first says:

“Well I guess I see, I picture a parabola, right, a parabola which is, um, or a conic section if it’s a quadratic function of two variables.” In addition to the visual image of a graph of a parabola, he also talks about variables, which point to more analytic/symbolic thinking. It seems that his thinking moves flexibly between visual and analytic thinking which supports the VA model.

LG then continues to say: “I don’t think I picture just one. […] I know that there’s only one parabola up to scaling. If you took any two parabolas, you can always rotate it, put them side by side, zoom in on one and it will look just like the other.” LG not only visualizes the graph of a parabola, but also thinks about the graph in motion, as he translates it, rotates it, and zooms in on it. He conceives a static entity (the parabola, the equation of the parabola) in dynamic terms, as illustrated by the verbs *translate*, *rotate*, *zoom*. In other words, his concept of quadratic function doesn’t include just the graph, or the equation, but the parabola in motion: in the language of Sfard (2008), he uses the dynamic aspect of the parabola as a “saming” technique, to make all the parabolas, whatever their shape, size, orientation, be one single object; as he later says “there’s only one quadratic function really.”

In our next example, we show how the linguistic expression and the non-linguistic expression can illustrate different aspects of mathematical thinking. Once again, in response to our prompt of quadratic function, NN begins by referring to a real object: “Something like a goblet, yeah so both a parabola and a goblet.” She uses the goblet metaphorically to describe the shape of a parabola. While both ‘parabola’ and ‘goblet’ evoke visual images, instead of dynamic ones, her speech coincidences with a set of gestures. In Figure 1 below, her right hand is cupped under, with fingers pointed upward, as if holding the goblet. In MacNeill’s scheme, this is an *iconic gesture*, which presents an image of concrete entity (goblet). Then, she uses her index finger to trace out a parabola starting from left to right and then returning from right to left (see Figure 2). This gesture could be classified as a *metaphoric gesture*, which ‘points’ to an abstract object (a parabola—insofar as a parabola counts as an abstract object). Note that in this gesture, the finger is moving, as if tracing a curve, or drawing a parabola; it is not a static gesture, as the one used to accompany the word "goblet."

	
Figure 1. shows NN's right hand which depicts a parabola	Figure 2. shows NN's index figure while tracing out a parabolic curve

In our third case, instead of producing the gesture along with the speech, the mathematician replaces speech by gesture. Again, in response to our prompt, JJ says: "initially I thought of algebraically, then I thought of [index figure depicts a concave down parabola] one of these [index figure depicts a concave up parabola] one of these." His gesture resembles that of NN, but differs also in several ways: he draws two different parabola, one concave and one convex, and also, draws them right in front of his body, at chest level. In contrast, NN goes back and forth along one parabola, and draws her in a region above, and to the right or her head. For both NN and JJ, the dynamic gestures are metaphorical, referring as they do to abstract objects. However, whereas NN evoked the metaphor of the goblet and the visual imagery of the parabola, JJ speaks first about the algebraic interpretation of the quadratic function, signalling an initial analytic—and very static—conception.

Our fourth and final case PT, combines various aspects of the first three, but in slightly different ways. His thinking is analytic/symbolic, while he says "this would be a function that is ay ex squared plus bee ex plus cee**,** and then, you could represent that by a parabola." But, he uses a set of gestures (see Figure 3) to actually write out the symbols ax^2+bx+c. He then draws out a very big parabola (see Figure 4), in his upper left spatial field, with his index finger, and says "going like that.” His gesture points to an abstract object. He then says “of course, in that you can include a line, you can imagine a line in there, […] though a line is technically a quadratic function.” In his accompanying gesture, his whole hand moves from left to right, fingers extended, as if cutting out a plane (see Figure 5). That he sees the parabola becoming a line (as the parabola flattens out), it also appears that he sees the parabola moving continuously from a curved line to a straight one—whereas LG saw the parabola move continuously across transformations.

		
Figure 3. PT gestures the quadratic equation.	Figure 4. PT draws a parabola.	Figure 5. PT’s gestures line as parabola.

Analysis of Speech and Gestures: Eigenvectors

In response to our prompt about eigenvectors, LG first says: “I guess eigenvector might be a resonance so if you are in a big tunnel and you start singing and hit the right note it starts to go really loud resonating your ears.” He uses resonance metaphorically to describe abstract objects, eigenvectors. He evokes a visual image and conceives it in dynamic terms, as he uses the verb *to go*. LG’s linguistic expression alone reveals the presence of dynamic thinking. Unlike the examples above, LG’s dynamic thinking is not necessarily image-based; rather, the dynamism is in the echo, which starts to “go really loud.”

In our next example, we analyse NN’s linguistic and non-linguistic expressions to illustrate dynamic aspects of her mathematical thinking. In response to our prompt about eigenvectors, she says: “stresses, so if I am thinking about a plate being pulled out so it’s gonna move along principles.” She uses ‘stresses’ as a metaphor that refers to eigenvectors. She evokes a visual image of a plate and uses motion, as illustrates by the verb *pull out*, to describe her concept image of eigenvectors. Her speech coincidences with a set of gestures: Figure 6 shows how she embodies a dynamic imagine of eigenvector in the context of a real world example, “a plate being pulled out.” She clenches her hands and moves her arms back and forth, as if holding a horizontal steering wheel, to accompany her verbal expression. This is another *metaphoric gesture*.

Figure 6. Shows NN’s hand and arm movements in describing eigenvector.

Our third case, JJ, first says “one idea is, you have the idea of matrix as a linear transformation and, um, so you take a vector and you map it to something else.” This seems to describe a visual image of mapping on vector to another, though his description “matrix as a linear transformation” also indicates a more analytic conception of eigenvector. He then continues to say that “you set the matrix up by some inputs, they’re gonna come inside and then obviously you say what is the important direction when the two line up of course. So, that is one idea that I use to say that there is something special about that direction.” Here, he uses his hands to demonstrate a vector and its transformations: with his index fingers (on both hands) he rotates one finger toward the other (see Figures 7 and 8). His hand movements, which coincidence with the verbal description quoted above, show how he conceives the process of transformation dynamically, as something coming together in the same “direction.”

Figure 7. Shows JJ’s use of hands to depict a vector and its linear transformation.

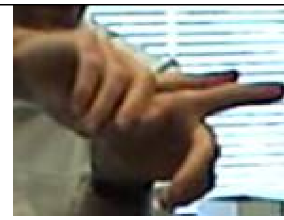

Figure 8. Shows how JJ’s line up a vector and its transformation to illustrate the concept of eigenvector.

Our fourth and final case PT, in response to our prompt, says “I think of a matrix, I think of applying the matrix to the vector, and then what you get out is another vector that’s in the same direction but either stretched or shrunk.” Again, his linguistic expression reveals the presence of fictive motion in his conception of an eigenvector, which he describes as something you “get out,” that is “stretched or shrunk.” His speech coincidences with arm and hand movements that are similar to NN’s gesture: starting with his hands and arms extended (as in Figure 9), he brings them toward each other as he says “same direction” and moves them away again when he says “stretched or shrunk.” Unlike NN, who is referring to plates and stresses, PT seems to be thinking about the vectors themselves, and also using metaphorical gestures in describing them.

Figure 9. Shows PT’s arm movements in illustrating result of transformation, stretched or shrunk in eigenvector.

Discussion and Reflections

The results of our analysis indicate that: first, the mathematicians use gestures and metaphors to express their thinking about concepts. Second, their linguistic and non-linguistic expressions comprise a dynamic component. However, while sometimes this dynamic component is visual in nature, other times it is no. This would suggest that some forms of dynamic thinking are non-visual, and more time-based. In fact, in Thurston's (1994) categorisation of the different "facilities of mind," he includes both a "vision, spatial, kinaesthetic (motion) sense" category and a "Process and time" category, where the latter refers to a facility for thinking about processes or sequences of actions.

As Núñez (2006) points out, the dynamic component of gestures and metaphors promote understanding mathematical concepts (Núñez, 2006). Following Zazkis et al.'s (1996) work, which draws attention to the important interaction between the visual and the analytic, we hypothesise that dynamic thinking is potentially a bridge between visual and analytic thinking: further research on this hypothesis seems warranted.

Given the strong presence of motion-based thinking we found in our mathematicians, we believe that dynamic thinking might be an important component of learners' conceptual development. We propose that increasing their exposure to dynamically represented concepts—as offered in dynamic geometry software environments—scaffold students' dynamic imagery of concepts and mathematical objects. We are currently engaged in investigating this hypothesis in the context of linear algebra.

On a final note, turning now to the teaching and learning of mathematics: we suggest that the instructional use of gestures warrants further study. Cook, Mitchell and Goldin-Meadow (2008) reported that requiring students to gesture while learning a new concept helped to retain the knowledge they had gained during instruction. Another recent study on teachers' gestures shows that the teacher uses gestures to scaffold students' comprehension of the algebraic relations (see Alibali & Nathan, 2007). It seems reasonable to assume that not all gestures will work in this way; however, drawing on the gestures that teachers may use to scaffold students' understanding and mathematicians use to think about concepts may well provide guidance to educators looking to identify productive gestures for instruction.

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