Function, Visualization, and Mathematical Thinking among College Students with Attention Deficit Hyperactivity Disorder

April D. Judd

University of Northern Colorado

april.judd@unco.edu

For many people, AD[H]D is not a disorder but a trait, a way of being in the world. When it impairs their lives, then it becomes a disorder. But once they learn to manage its disorderly aspects, they can take full advantage of the many talents and gifts embedded in this sparkling kind of mind. (Halloway & Ratey, 2005, p. 4)

According to the American Psychiatric Association (2000), between 3% and 7% all of school age children have ADHD. Since the enactment of the American Disabilities Act of 1990, elementary and secondary schools have become more efficient at identifying and addressing the needs of students with ADHD (Gordon & Keiser, 1998). In turn, increasing numbers of young adults with ADHD are enrolling in universities across the nation (Javorsky & Gussin, 1994; Wolf, 2001). It is estimated that between 4% and 6% of all college students in the U.S. have some form of ADHD (DuPaul et al., 2001; Heiligenstein, Conyers, Berns, & Smith, 1998; Weyandt, Linterman, & Rice, 1995), and it is expected that these numbers will continue to rise (Wolf, 2001).

It is now known that symptoms of ADHD *do* impact the academic success of students at the college level. Symptoms of inattention, specifically, have been linked with lower GPAs among the ADHD college student population. In addition, college ADHD students report low self-esteem and challenges adjusting to the academic setting.

Although only a few published studies (Weyandt et al., 1995; Murphy & Barkley, 1996; Stage & Milne, 1996; DuPaul et al., 2001) have reported on the prevalence of ADHD symptoms among college students, these studies suggest that 4% to 7% of college students will experience at least six of the DSM-IV criteria for ADHD. Additionally, up to 11% of all college students with ADHD reported symptoms that are 1.5 standard deviations greater than the sample mean within these studies. That is, the frequency with which these symptoms occurred was 1.5 standard deviations higher for college students with ADHD than the frequency symptoms occurred for all college students in these studies. In particular, in a comparison of university students documented as ADHD and a non-ADHD control group, Heiligenstein, Guenther, Levy, Savino, and Fulwiler (1999) found significant differences between the groups on several academic measures. For example, students in the ADHD group had lower grade point averages (GPAs) and were more likely to be put on academic probation. In another study of students attending a large Catholic university, ADHD students reported poor academic, social, and emotional adjustment to college life (Shaw-Zirt, Popali-Lehane, Chaplin, & Bergman, 2005). They also reported low self-esteem, a result shared by Dooling-Litfin and Rosen (1997).

In addition, current literature suggests that symptoms experienced by *college students* with ADHD may differ from the general population of ADHD adults. For example, Frazier, Youngstrom, Glutting, and Watkins (2007) conjecture that "college students with ADHD are likely to have (a) higher ability levels, (b) greater academic success during primary and secondary school, and (c) better compensatory skills than individuals with ADHD from the

general population" (p. 54). This does not change the fact that college students with ADHD are still at risk for academic failure. It highlights the extent to which ADHD symptoms can impact an individual's educational success. Specifically, college students with ADHD are likely to have higher ability levels than the average ADHD adult, but they also have lower GPAs than *non-ADHD* college students and experience greater academic failure. For example, Schwanz, Palm, and Brallier (2007) studied 316 college students (51% women and 49% men) from a 4-year liberal arts university in the southeast United States. Data from this study indicated that symptoms of inattention and hyperactivity accounted for 9% of the prediction of GPA scores for college students in the study.

What is not known is how to take full advantage of the many talents and gifts characteristic of learning among ADHD students, as suggested by Halloway and Ratey (2005). The educational community has a great deal to learn about how ADHD symptoms may be reflected in the way college students learn and think about mathematics. The significance of this study lies in its contribution to what educators know about the cognitive impacts of ADHD in learning mathematics at the collegiate level. Currently, there is little information to guide college mathematics faculty in adjusting instructional practices to promote the mathematical success of ADHD learners. Additionally, the project contributes to the field of research in collegiate mathematics education through the development of additional theory about mathematical learning in general. The driving questions for this study were as follows.

1. Are the difficulties college students with ADHD encounter while learning the concept of function different from those outlined in published literature for all college students learning mathematics? 2. If so, how does college ADHD student learning about the concept of function differ from that outlined in the literature about college students in general?

To address these research questions, I used a purposeful case–sample of three university students, over the age of 18, enrolled in mathematics during the Fall 2004 term and based the design on a grounded theory perspective (Strauss & Corbin, 1998). The three main participants in this study (two female, one male) were undergraduate students over the age of 18 enrolled at one of two western universities and registered with the campus disability services office (DSO) as ADHD. All three participants were from a white, middle class background, and all three felt they had definite learning problems when it came to studying mathematics. Study participants who felt comfortable "thinking out loud" were selected to provide the richest data.

Ally was a returning junior at University B, a western Tier I Research university. She was 21 at the time of the study and enrolled in Pre-calculus. Beth was in her first semester at University A, a western Carnegie Doctoral I university, and classified as a sophomore. She was 21–years–old and enrolled in Calculus I. Chris was a returning junior at University A and was enrolled in Topics in Calculus. He was 24 at the time of data collection. In addition to these participants, I also interviewed two mathematics instructors. Dr. Beaverton held a Ph. D. in computer science and taught Beth's Calculus I course. Ms. Calloway held a M.A. in Mathematics and taught Chris' Topics in Calculus course.

I organized, analyzed, and reported the data through information-rich case studies, following standard case study methodology (Yin, 2003). My analysis was inductive in nature and followed the constant comparison methods of a grounded theoretical approach (Miles & Huberman, 1994; Strauss & Corbin, 1998). For interpreting and theory building from my analysis, I relied on three foundations established for describing human learning: Bandura's (1986) social cognitive theory and Barkley's (1994) unified theory of ADHD, and Pirie and Kieren's (1994) theory of mathematical learning.

Results

Although analysis of the data suggest possible answers for both research questions, this report will focus on possible differences in the way college ADHD students learn about the concept of function and information outlined in the literature about how college students in general learn about the concept of function. During the course of tutoring sessions with Ally, Beth, and Chris, I noticed three recurring themes: student concept image of function; a preference for using a graphical/pictorial approach to solve problems; and difficulties with symbolic, analytic, and verbal representations. The following is a description of the second and third themes for each participant.

The problem depicted in Figure 1 gives an illustration of Chris' engagement with graphical representations versus symbolic representations. I read part (a) of the problem aloud, following the words with my pen, and I pointed to Figure 2.12 in the text to make sure Chris was looking at the correct graph. Chris immediately answered the question, almost before I could finish, saying that "the rate of change between zero and three is far greater than it is between three and five. It's increasing so much more than – ." Chris did not finish his sentence, but he did point to the segment between x = 3 and x = 5 to indicate what he would have said. Notice that Chris used the graph to determine and explain his answer. This indicated to me that Chris understood how to graphically interpret average rate of change for a function. As an instructor of mathematics, , I knew Chris would be expected to justify his answer algebraically as well as graphically. So I told Chris I wanted to show him how to solve the problem arithmetically. Chris replied, "the hard way," laughed and sat back in his seat. I tried to assure Chris by telling him that it was good that he could solve the problem graphically. Chris replied, "I think it's the actual technical method where I get screwed up first." This pattern of answering a problem graphically but feeling uncomfortable (and physically withdrawing) when asked to use algebraic reasoning continued throughout all our tutoring sessions and was common to all ADHD participants in the study.

- 5. Figure 2.12 shows the cost, y = f(x), of manufacturing x kilograms of a chemical.
 - (a) Is the average rate of change of the cost greater between x = 0 and x = 3, or between x = 3 and x = 5? Explain your answer graphically.
 - (b) Is the instantaneous rate of change of the cost of producing x kilograms greater at x = 1 or at x = 4? Explain your answer graphically.
 - (c) What are the units of these rates of change?



Figure 1: Problem 5 as shown on page 99 of Hughes–Hallett et al., 2003.

Like Chris, Beth also gave privilege to approaches with a focus on using graphs and pictures. Beth seemed to struggle with symbolic, analytic, and verbal representations. For example, in reviewing an exam from Beth's Calculus I class, I noticed that Beth completed tasks involving graphical and tabular representations, including recognizing trends in the slope of a function (i.e., distinguish between exponential and linear growth) given several values in a table. But Beth did not complete exam problems where she was asked to find the formula for a function, solve for x in an equation (e.g., $2x - 1 = e^{2lnx}$), or identify features of a function defined by an equation.

In a member check, I asked Beth to look at a graphical situation that involved matching several algebraic equations to the appropriate graph. Beth consistently identified and coordinated the salient features of the algebraic and graphical representations, but she did not always explain how to use these features in *connecting* the two representations. For example, Beth immediately identified the first graph as equation c (5 = y). I asked Beth how she figured this out so quickly. Beth replied, "Well I saw it's a straight line. So [pause] y has to be equal to that [pointing to equation (c)]. And I saw that y = 5. So there's no x or anything in it. It's saying that y is 5 [inaudible - all the time?]."

I gave Beth a few seconds to think. Then I asked, "Now what are you thinking?"

Beth: I'm trying to remember like [pause] 'cause I know if something is [pause] if there is a minus sign or [pause] uhm. Yeah, if there is a minus or a positive it depends on which way the line slants down. If it slants to the left or if it slants to the right. [10 second pause] For some reason I think this one's f. For some reason my brain is thinking that this [pointing to graph (iii)] looks simple 'cause there's a line that goes through the origin and this looks simple too [pointing to the equation f]. [inaudible] without anything else added to it. That's why my brain is putting 9. Match the graphs in Figure 1.20 with the following equations. (Note that the x and y scales may be unequal.)



Figure 2: Problem 9 as shown on page 11 of Hughes–Hallett et al., 2003.

those two together.

There was an 8 second pause while Beth continued to think about the problem. Then she matched the rest of the graphs to the remaining equations.

Beth: I think this one's e [pointing to graph (ii)]. This one's a [pointing to graph (v)] because [inaudible] it's x - 5 and that's x + 6 and they look very similar. This one's on that side of the line and this one's on this side of the line [pointing from equations to graphs]. [5 second pause] Uhm [pause] this one's d. That one's b; bhas a negative 3x plus 4 and d has a negative 4x minus 5 [inaudible].

We see Beth's comfort using graphical representations to solve a problems in her first exam and in her discussion of identifying graphs with their equations. As mentioned above, in Beth's first exam, we also see that she did not always complete problems involving a primarily symbolic solution method.

Like Beth and Chris, Ally was challenged by tasks that used primarily symbolic, analytic, and verbal representations and showed a preference for graphical or pictorial approaches to problem solving. In fact, Ally often persisted in working with a mathematical task graphically long after I tried to present a more symbolic representation. For example, Ally began our third tutoring session by asking me if we could go over a problem from her homework assignment concerning transformations of functions. The exercise in the textbook asked students to match several functions (e.g., $y = 2 + \sqrt{x}$, $y = -2\sqrt{x}$, $\sqrt{2+x}$, and $y = -\sqrt{-x}$) written in symbolic form to their respective graphs.

As Ally talked about the exercise, she moved to the edge of her seat and leaned forward with her right forearm resting on the table in front of her. She pointed to the exercise with her pencil as she read the directions.

Ally: I thought this one was a little strange. I did make a note to go over that with you. Number 78 says match each function to its graph. The negative square root of a negative x? I just don't get it. Okay, the negative means it should be shaped like that; the open part facing up right there and then a negative x. I don't know. It kinda seems like it should make it open maybe here [pointing to graph (c)].

Notice that Ally's first engagement with this problem was visual. She started by thinking about how the signs within the functions representation could be used to determine how a canonical form (in this case, \sqrt{x}) can generate the given form $-\sqrt{-x}$.

I suggested that Ally consider "what kind of values [she] could plug into x in order for the

square root of a negative number to work out." I wanted Ally to consider the effect of the "inner" negative coefficient on the domain of the function $f(x) = \sqrt{x}$. Ally continued to look at her text while I talked. She agreed to consider my suggestion, but she was very hesitant, pausing frequently, and repeatedly saying she did not understand. It took a lot of prompting to persuade Ally to work through the problem using an empirical approach. As Ally began to work through the task, she moved closer to the table and wrote in her notebook. Ally was very uncomfortable using negative values as input for the function $-\sqrt{-x}$. At first, she said she did not "know how to square root negative numbers." I encouraged her to try using -1 as an input first. Ally said, "Okay," and plugged -1 into the expression. But Ally still was not comfortable with her calculation.

I wanted Ally to see the overall pattern, so I encouraged her to just keep going. Ally considered the input -2 and said, "Negative two would just end up negative square root of two. On and on." So Ally recognized the pattern quickly. However, when I asked her about zero as an input value for the function, Ally looked at me and said, "That doesn't make sense at all. Because there's not a square root or negative for zero. I asked her if she remembered how we defined the square root of a number. Ally interrupted me before I could finish the question and said, "Oh! Okay. I think. Okay. [3 second pause] I don't know what it would be though." I asked Ally again, "Do you remember how we defined the square root of a number? She said, "No, I don't remember what we said exactly, but I think I get it, though. It's like the same thing multiplied together that will equal zero. [2 second pause] I just don't know what that would be."

I asked Ally what number, when squared, would equal zero. She answered zero and observed that zero does work, but she did not seem satisfied. Her exact words were, "Okay, I guess it does work." Then she went back to considering the problem graphically. She said, "I thought that this negative sign [pointing to the first negative sign] just meant that it's flipped, that it's being reflected. But it will also change the values too, right?"

Notice that, despite my prompting to use an empirical approach, Ally went right back to considering a graphical approach to the problem. However, Ally also asked about the effect of the negative signs on the function values. This seemed to be evidence that Ally was trying to understand, or assimilate, the information from the empirical approach that I suggested. That is, it seemed that Ally was trying to reconcile the cognitive conflict she encountered between her learned empirical associations with the canonical form, \sqrt{x} , and the more abstract visualization methods she wanted to use.

I confirmed that the input value would change from negative to positive within the square root, but tried to point out that the entire function still gives a negative output. We went through this explanation twice. Ally understood that the function related negative inputs to negative outputs. But she did not connect this with the graphical representation of the function. I wanted Ally to see the pattern between inputs and outputs for the function. That is, I wanted Ally to notice that the negative sign under the square root (her original concern) "flipped" the domain of the function from $[0, \infty)$ to $(-\infty, 0]$ and the negative sign in front of the square root (i.e., a coefficient of -1) "flipped" the range of the function from $[0, \infty)$ to $(-\infty, 0]$. But my questions did not lead Ally in the direction I hoped.

Conclusions and Implications

The main goal of this study was to develop theory about mathematical learning around the function concept among students with ADHD at the college level. This report focuses in possible answers for Research Question 2. I also discuss directions for future research and implications for teaching.

The full analysis of data suggests that there are similarities and differences between the difficulties ADHD college students encounter and those outlined in published literature for all college students learning mathematics. In particular, both sets of data (from this study and that reported by the literature) suggest that non–ADHD *and* ADHD students find making connections among graphical, tabular, symbolic, and verbal representations of functions challenging.

The literature on student understanding of function reports several challenges, including what is and is not a function, the idea of correspondence, linearity, representations of functions, interpretations of graphs, the concept of covariation, and notation (Leinhardt, Zaslavksy, & Stein, 1990). In particular, past research has indicated that students learning about function tend to: gravitate to a linear understanding of all functions and define a function as a relation that produces a linear pattern when graphed; have a tendency to interpret graphs as an iconic representation or picture of a situation, rather than interpreting the graph as a covariational relationship; and have difficulty making connections between representations of functions (e.g., moving from the graph of a function to its equation). I did not find evidence of the first two tendencies in tutoring sessions with Ally, Beth, and Chris. In contrast, all three ADHD participants fluently discussed the covariational relationship represented by linear and nonlinear graphs. Given a graph of a function y = f(x), Chris identified and compared average rates of change over two intervals using a purely graphical interpretation of the situation. Beth discussed several covariational relationships represented by graphs while matching several graphs to their symbolic equations.

The difference between the ADHD participants in this study and non–ADHD college learners lies in the *types* of representations each group works with successfully. The literature reports that college students, in general, experience several difficulties around graphical representations of functions. In particular, they have difficulty moving from graphs to equations and prefer moving from equations to graphs (Leinhardt et al., 1990). In contrast, Ally, Beth, and Chris did not encounter difficulties around graphical representations. They experienced difficulty with symbolic representations. In other words, Ally, Beth, and Chris moved from graphs to equations successfully and encountered difficulties moving from equations to graphs (e.g., Ally's work on translations of functions). This seemed to be the key difference between the difficulties of the ADHD college student participants in this study and those reported for college students without ADHD. Both groups have difficulty making connections between representations of functions, but the difficulties they encounter are categorically different.

Directions for Future Work

The results of this study indicate that college students with ADHD may approach mathematical learning differently than the general university population. The results also suggest that college students with ADHD have different learning strengths than generally assumed of college students in the mathematics classroom. This has implications for both future research and instruction.

The small sample inherent in this purposeful case study design calls for additional research to determine the replicability and transferability of these results. Is there truly a difference in the connections ADHD students make between representations of functions and the connections non–ADHD students make? In addition, the narrow scope resulting from focusing on student learning around the concept of function indicates that further investigations of how ADHD learners approach mathematics should be conducted at several levels of college mathematics (e.g, intermediate algebra, college algebra, and upper level undergraduate mathematics). Is visualization a general learning strength among ADHD learners in mathematics? How is this addressed within the instruction of concepts of mathematics?

We already know from the literature that college students learning the concept of function prefer equation-to-graph connections/translations, and that there is a struggle to coordinate graphical representations with symbolic representations (Leinhardt et al., 1990). In addition, the literature reports difficulty interpreting information represented in a graphical form. Over the last 40 years, the curriculum has traditionally started from a symbolic approach (Chappell, 2006; Star & Smith, 2006). Reform calculus is an attempt to develop a more robust concept image of function through an early emphasis on graphical and tabular representations, providing students with more practice around graph-to-equation translations and interpretations. The purpose of providing students with more opportunities to develop skills in graph-to-equation translations and interpretations is to facilitate a greater flexibility in moving among *all* types of representations of function.

However, the results of this study suggest that these might not be ideal instructional

strategies for the ADHD learner. The traditional strategy asks an ADHD learn to begin with a representation they have difficulty understanding in order to work with a representation they prefer. Although the reform curriculum begins from a position of strength for the ADHD learner (i.e., graph-to-equation representations), the curriculum assumes that students are familiar with equation-to-graph representations and may not emphasize developing skills in this direction. Thus, instead of starting from a position of strength and extending this strength in some way, reform calculus begins from a position of strength but does not use this strength to develop skills in translating from equations to graphs, a possible area of difficulty for the three ADHD learners in this study.

Why is knowing this a good thing? The results of this study bring awareness to the different strengths of college students learning the concept of function. College students with ADHD may not be the only group that prefers working with graph to equation connections. Awareness of different strengths than those outlined in the literature is essential for the development of a complete concept image of function for all students. This study illustrates the importance of remaining sensitive to the different learning strengths students bring to the classroom, both as researchers and instructors. For example, if I had only focused on the difficulties Ally, Beth, and Chris encountered during their tutoring sessions, I would not have learned about their strength with graphical and tabular representations. I might have only reported that Ally, Beth, and Chris experienced difficulty with equation-to-graph translations, while the literature reports that students, in general, prefer equation-to graphtranslations. This does not bring forth a complete picture of the situation. In fact, we see that although Ally, Beth, and Chris have difficulty with an area in which other students might not experience difficulty, Ally, Beth, and Chris also have a strength that is not shared by all students learning the concept of function. Hence, the situation is not about what Ally, Beth, and Chris *can not* do. It as about the *way* Ally, Beth, and Chris learn mathematics. From a research stance, focusing on learning patterns (e.g., preferences, strengths, and difficulties) allows for a more complete picture than using the deficit model perspective. As an educator, focusing on learning patterns allows greater flexibility and creativity in instruction and assessment. This shift in perspective also provides a more equitable learning environment for *all* students.

References

- American Psychiatric Association. (2000). Diagnostic and statistical manual of mental disorders (4, revised ed.). Washington, DC: Author.
- Bandura, A. (1986). Social foundations of thought and action: A social cognitive theory. Englewood Cliffs, NJ: Prentice-Hall.
- Barkley, R. A. (1994). Impaired delayed responding: The unified theory of attention-deficit hyperactivity disorder. In D. K. Routh (Ed.), *Disruptive behavior disorders in childhood* (pp. 11–59). New York: Plenum.
- Chappell, K. K. (2006). Effects of concept-based instruction on Calculus students' acquisition of conceptual understanding and procedural skill. In F. Hitt, G. Harel, & A. Selden (Eds.), *Research in collegiate mathematics education vi* (pp. 27–60). Providence, R.I.: American Mathematical Association.

Dooling-Litfin, J. K., & Rosen, L. A. (1997). Self-esteem in college students with a child-

hood history of attention deficit hyperactivity disorder. Journal of College Student Psychotherapy, 11, 69–82.

- DuPaul, G. J., Schaughency, E. A., Weyandt, L. L., Tripp, G., Kiesner, J., Ota, K., & al. et. (2001). Self–report of ADHD symptoms in university students: Cross–gender and cross–national prevalence. *Journal of Learning Disabilities*, 34(4), 370–379.
- Frazier, T. W., Youngstrom, E. A., Glutting, J. J., & Watkins, M. W. (2007). ADHD and achievement: Meta–analysis of the child, adolescent, and adult literatures and a concomitant study with college students. *Journal of Learning Disabilities*, 40(1), 49–65.
- Gordon, M., & Keiser, S. (1998). Accommodations in higher education under the Americans with Disabilities Act (ADA). New York: Gordon Systems Publications.
- Halloway, E. M., & Ratey, J. J. (Eds.). (2005). Delivered from distraction: Getting the most out of life with attention deficit disorder. New York: Ballantine.
- Heiligenstein, E., Conyers, L. M., Berns, A. R., & Smith, M. A. (1998). Preliminary normative data on DSM–IV attention deficit hyperactivity disorder in college students. *Journal of American College Health*, 46(3), 185–188.
- Heiligenstein, E., Guenther, G., Levy, A., Savino, F., & Fulwiler, J. (1999). Psychological and academic functioning in college students with attention deficit hyperactivity disorder. *Journal of American College Health*, 47(3), 181–185.

Javorsky, J., & Gussin, B. (1994). College students with attention deficit hyperactivity disor-

der: An overview and description of services. *Journal of College Student Development*, 35, 170–177.

- Leinhardt, G., Zaslavksy, O., & Stein, M. K. (1990). Functions, graphs, and graphing: Tasks, learning, and teaching. *Review of Educational Research*, 60(1), 1–64.
- Miles, M., & Huberman, M. (1994). Qualitative data analysis: An expanded sourcebook (2 ed.). Thousand Oaks, CA: Sage Publications.
- Murphy, K. R., & Barkley, R. A. (1996). Prevalence of DSM-IV symptoms of ADHD in adult licensed drivers: Implications for clinical diagnosis. *Journal of Attention Disorders*, 1, 147–161.
- Pirie, S., & Kieren, T. (1994). Growth in mathematical understanding: How can we characterize it and how can we represent it? *Educational Studies in Mathematics*, 26(2/3), 165–190.
- Schwanz, K. A., Palm, L. J., & Brallier, S. A. (2007). Attention problems and hyperactivity as predictors of college grade point average. *Journal of Attention Disorders*, 11(3), 368–373.
- Shaw-Zirt, B., Popali-Lehane, L., Chaplin, W., & Bergman, A. (2005). Adjustment, social skills, and self-esteem in college students with symptoms of ADHD. Journal of Attention Disorders, 8, 109–120.
- Stage, F., & Milne, N. (1996). Invisible scholars: Students with learning disabilities. The Journal of Higher Education, 67, 426–445.

- Star, J. R., & Smith, J. P. (2006). An image of Calculus reform: Students' experiences of Harvard Calculus. In F. Hitt, G. Harel, & A. Selden (Eds.), *Research in collegiate mathematics education vi* (pp. 1–26). Providence, R.I.: American Mathematical Association.
- Strauss, A., & Corbin, J. (1998). Basics of qualitative research (2 ed.). Thousand Oaks, CA: Sage.
- Weyandt, L. L., Linterman, I., & Rice, J. A. (1995). Reported prevalence of attention deficits in a sample of college students. Journal of Psychopathology and Behavior Assessment, 17, 293–304.
- Wolf, L. (2001). College students with ADHD and other hidden disabilities. Outcomes and interventions. Annals of the New York Academy of Sciences, 931, 385–395.
- Yin, R. (2003). Case study research: Design and methods (3 ed.). Thousand Oaks, CA: Sage.