An Investigation of One Instructor's Mathematical Knowledge for Teaching: Developing a Preliminary Framework

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Introduction

One of the more intriguing questions in current mathematics education research involves what mathematical knowledge is necessary and/or helpful when teaching mathematics. Shulman (1986), Ball, Hill, and their colleagues (1994, 2004, 2008), and Ferrini-Mundy, Senk, & Schmidt (2004) are just a few researchers who have attempted to address this question about knowledge for teaching. Certainly most people would agree that there is both skill and art involved in the process of teaching any subject at any level. Additionally, intuition tells us that if we know the subject (mathematics) better, then we will be better educators. Unfortunately, until twenty years ago, there was little research in this area; fortunately, many mathematics educators are studying this issue today. Following the development of constructs for mathematical knowledge of teaching, we need to study how instructors use this mathematical knowledge during the work of teaching (Hill, Schilling &Ball, 2007). Once we move towards looking for some answers to those two questions, we can address one more important question: how can we as mathematics educators provide ways for current and future teachers to learn the mathematics and pedagogy that will lead to improved student performance?

Hill, Ball and Schilling (2008) recently presented a developing framework to study the many types of knowledge that teachers use. In particular, they suggested that there are different kinds of subject matter knowledge: *common content knowledge, specialized content knowledge,* and *knowledge at the mathematical horizon;* pedagogical content knowledge may be divided into *curriculum knowledge, knowledge of content and students, and knowledge of content and teaching*. Their work is particularly associated with developing understanding and assessing elementary teachers' knowledge for teaching. One question that arises is whether the constructs for mathematical knowledge for teaching at the elementary school level are appropriate at the high school or university level. Therefore, we conducted a limited investigation of knowledge for teaching by looking at one particular mathematician as a case study (Glaser & Strauss, 1967). Future research might extend this to a larger study of more mathematicians at the university level.

Undergraduate mathematics education researchers have begun exploring this area and we build on their work. Wagner, Speer, and Rossa (2007) investigated the teaching issues that a mathematician struggled with as he implemented inquiry-oriented teaching for the first time. They found that in their case study, the professor struggled with the pedagogical issues but these authors did not address the mathematical knowledge issues in that particular situation. Researchers at Portland State University began addressing the work of teaching mathematics, as they looked at collaboration between a mathematician and a math educator (Bartlo, Larsen, Lockwood, 2008). Their work shows that the mathematical knowledge that a professor brings to an abstract algebra classroom is broad but that there are teaching moments where making the connections and understanding student thinking provides growth opportunities for the professor. Rasmussen and Marrongelle (2006) also posit that pedagogical content tools which require special mathematical knowledge can extensively further the mathematical discourse and learning in a university level inquiry oriented differential equations class.

For this report, we investigated a mathematician as he implemented an inquiry-oriented differential equations (IO-DE) curriculum. Our research question proposes to investigate the validity of extending Hill and Ball's (2008) framework into the college classroom classroom. We ask: What mathematical knowledge for teaching does one mathematics professor teaching in an IO-DE course draw upon to further the mathematical agenda of the class?

Setting, data collection, and analysis

Data collection was conducted in Spring 2008 in a college level Differential Equations class in the southeastern United States (enrollment of 25) using a classroom teaching experiment methodology (Cobb, 2000). Most students in the class were mathematics, science, or engineering majors, had finished Calculus III, and about one third of the students had taken at least one prior course with this particular mathematics professor. The professor had been using inquiry-oriented strategies in his other courses (e.g., Abstract Algebra, Mathematical Reasoning) for several years, but had only taught Differential Equations once about 7 years prior and was implementing the specific inquiry-oriented differential equations materials (Rasmussen, 2003) for the first time that semester. Prior to each teaching session, the professor met with the researchers to discuss the material to be taught and make a planned trajectory. They also met immediately after class for debriefing sessions to reflect on the lesson and discuss any issues or questions that arose that may affect the content and teaching strategies used for the next class.

The class was designed to be inquiry-oriented with each class session involving cycles of learning: whole class discussion, followed by small group discussion, followed by whole class discussion. The learning environment of the classroom established by the professor required students to discuss the mathematics they were learning, express their own ideas, and make sense of, and agree or disagree with others' ideas.

The data used for analysis for this paper was drawn from the videotaped class episodes, field notes from a non-participant observer, and video/audio taped debriefing sessions held immediately after class. To begin our analysis, we reviewed videotapes and field notes of most class sessions throughout the semester. We then chose 6 class sessions that came from different points in the semester with different content foci to use in our analysis. For each class session we created descriptive timelines (chunked by 5-10 minutes) that provided an overview of the content, student work, and teacher moves during that class. We reviewed the descriptive timelines from each of the six class sessions, as well as the field notes, to identify episodes that appeared to contain several lengthy sections where the professor was facilitating mathematics discussions.

We began to code the instructor's words by using the mathematical activity before and after the words to help determine the mathematical knowledge involved. Coding started as top down, in that we were coding using the three types of knowledge, *common content knowledge, specialized content knowledge,* and *knowledge at the mathematical horizon* described earlier as subject matter knowledge (Hill & Ball, 2008). We used the teacher's words or actions, confirming the code by examining the student's conversation before and after the teacher's words or inscriptions in the episode identified. Because the mathematician was working as part of the research team, we believe that we can be assured of some validity in the coding. We soon agreed that the coding would need to be broader than just coding a sentence or two. To really understand the knowledge for teaching, whole episode of several minutes may need to be considered.

As the coding activity continued, we found that we were unable to cleanly identify using these three definitions for what mathematics the professor, Dr. Lee, used. At this point we began to consider revising the classifications or working with other possibilities. Our process toward developing a framework is described below.

Results

"The Egg is Broken!" This statement began to emerge from our analysis as we were unable to see how these categories of subject matter knowledge, as described by Hill and Ball, fit with the knowledge we identified from the episodes were considering. For example, "common content knowledge" in elementary school is what users of mathematics in work and other situations are expected to know. What is "common content knowledge" in differential equations? This was a conundrum. What is "specialized knowledge" in differential equations when it is taught using inquiry-oriented, conceptually focused materials as opposed to differential equations taught in a traditional lecture style? Is "mathematics on the horizon" only what is coming, or is it both forward, backward and lateral? As these issues continued to arise, we decided to make major changes in the coding to allow for what we were seeing in the data.

The first area we worked on refining was common content knowledge (Hill & Ball, 2004). Because knowledge of differential equations is not common to many and inquiry-oriented differential equations (IODE) is even less commonly known, there did not seem to be a way to code for "common knowledge" in differential equations. Possibilities included considering the calculus and earlier mathematics that students would bring into the course, the differential equations knowledge that any college mathematics instructor might have, or the differential equations knowledge that anyone using DEs in their work would know. Each of these ideas was considered and we tried to code our selected episodes. However, none of these iterations allowed us to really view the instructor's knowledge in a way we found helpful.

After several iterations, we agreed upon a definition of common content knowledge as knowledge any person that develops a deep understanding of the content being taught will have after completing this specific course (e.g., the items listed on a syllabus or the major ideas that the professor intends to teach). We agreed upon this as it allows common content knowledge to be situated in a particular college level (or secondary level) class and allows for the issue of the huge differences in mathematical content in classrooms at these levels. We contend that it is best to study this assuming that the common content is specific for a particular area. (See Figure 1).

We next addressed the issues of coding specialized content knowledge for teaching. Several times we were unable to code what our ideas were for specialized content knowledge. In many cases, we found that the knowledge we thought the professor was drawing on was most likely similar to what other mathematicians used. The interesting and "pedagogical" piece of the coding was *when* and *how* he used the mathematical knowledge, not *what* he used. This we decided was primarily, then, a pedagogical content issue and we address that in another paper (Lee, Lee, Keene, Holstein, Eley, and Early, 2009).

As we continued struggling with the coding, we turned to the issue of *knowledge of mathematics at the horizon*. In elementary mathematics, the horizon is considered to be further mathematics that teachers might know the students will learn, or need to know in more advanced mathematical learning circumstances. Here, many of these students were taking no more mathematics and possibly not using advanced mathematics in any more situations. The everyday notion of horizon led us to consider horizon as both “in front of” and “behind” the current mathematics. This coding plan seemed reasonable until we were unable to find anything “in front of” in our episodes. We still felt that there is knowledge of mathematics to come that a mathematics professor will use, but we did not see this in the chosen episodes. For example, the professor claimed that he talked about Linear Algebra during the classes, but we did not see evidence of this. We did see the professor talk about mathematics from other courses many times and connect it to the current instruction. We also decided to use an everyday term and call it *knowledge of the big picture;* eventually this evolved one more time into *mathematical depth knowledge* as a subcategory of “specialized mathematical knowledge”.

As we worked with the idea of coding for *knowledge of the big picture,* the final piece of the characterization of special knowledge emerged. The professor whose classroom we were coding often made it clear that his purpose did not involve just teaching content, but that the intent was to bring the students into the mathematical community by using his knowledge of the work of mathematicians. We grew to believe that this was an exceptionally important component of this particular professor’s knowledge that he was using in the classroom, and his use of it made it a specialized knowledge used in teaching. His words for this were “helping students to develop and use their mathematical gut”. Thus, we somewhat tentatively agreed on the use of three subcategories of the specialized knowledge for teaching. (See Figure 1)

Category	Definition	Example
Common Content Knowledge	Knowledge any person that develops a deep understanding of the content being taught will have after completing this specific course (items listed in a syllabus, major ideas that the professor intends to teach)	Euler’s method to create numerical approximate solutions to an ODE
Specialized Mathematical Knowledge for Teaching	Special mathematical or other content knowledge that the professor brings in to improve the students’ understanding of the current mathematics; includes mathematical depth knowledge (knowledge from other previous mathematics courses that connects to the content being learned and is intended to just “reinforce” or “remind” students)	+/- charts in calculus as it can be connected to flow lines
	Mathematical knowledge that the professor uses because he knows about future mathematics	?
	Knowledge of the habits of mind of mathematicians the professor introduces to assist students in developing their mathematical “gut”	Recognizing when there is a question in the mathematics that needs to be investigated

Figure 1. Kinds of mathematical knowledge for teaching

Discussion and Illustrations of the Preliminary Categories

1. **Common Content Knowledge.** There are of course many examples of common content knowledge. According to our working definition, this would be the mathematics that is included in the course materials and thus what the students will be expected to learn. There is an unspoken implication here that the content knowledge a teacher uses is probably more robust and deeper than what the students may come to know, but it is still the primary mathematics in the course.

By looking at the materials, this knowledge can be identified. Euler's method is a method that the class reinvents to use as a way to find solutions to differential equations, particularly those that are not solvable with analytic techniques. Euler's method is based on the idea that a differential equation can provide the slope of a tangent line at a given point to the solution that passes through that point. By using that slope to create a short line segment, the solution is approximated. After a short length of time, a new slope and line segment are calculated. The professor understood this procedure and its underlying conceptual basis. As he facilitated the reinvention of Euler's method, he drew upon his deep knowledge, and the class moved ahead in their mathematical study.

2. **Specialized Mathematical Knowledge for Teaching: Depth Knowledge.** Depth knowledge can be defined as content from courses other than the current course, probably already taken by the students, which the professor introduces to connect to the current material. This introduction of the mathematics serves two important purposes: enrich and clarify the current material and promote deeper understanding of the other mathematics. One of the important representations that students are expected to know after this IODE course is the flow line. The flow line is a one dimensional vertical representation of all the solutions to an autonomous ordinary differential equation (ODE). The materials and teacher notes for the course indicate this as part of the students' toolkit of representations that they will use to describe and reason with as they study the structure of the solution space for the ODE. Early in the course, the class discussed several different solutions to an ODE and the flow line was used to think about and understand the relationship between the different solutions. The professor used his knowledge of a calculus representation to connect and enhance the students understanding of the flow line.

3. **Specialized Mathematical Knowledge for Teaching: Knowledge of Mathematicians Habits of Mind.** The professor was very interested in our coding the instances where he was trying to indoctrinate the students into what it is like to study mathematics, or be a mathematician. His knowledge of being a mathematician is, of course, first hand. We believe that his explicit ideas of how one is a mathematician are a specialized knowledge. Many mathematicians may not have identified the skill and art that is required to successfully function as a mathematician and in order to provide opportunities for students to become mathematicians, this explicit knowledge needs to be used by the professor.

As an example, the students were studying a differential equation $\frac{dh}{dt} = -h^{1/3}$. This specific ODE has the quality that it is possible for more than one solution to intersect at specific points on the *t* axis. Mathematicians find this a problem, and actually use a theorem, the uniqueness theorem for solutions to differential equations, to know when solutions do not intersect. However, this is likely not something students know, and additionally, they probably do not think of it as something that mathematicians would avoid (more than one solution at a specific point). When the students were talking together, the professor asked them “What do you find disturbing about this situation?” As the conversation continued and the students finished working in groups to discuss this, one student said, “It is not disturbing, but it is certainly inconvenient!” We see here how the professor used his knowledge of how mathematicians think to bring to the forefront of the conversation the mathematical “gut,” and students were integrated into the mathematical community in one small way.

4. **Specialized Mathematical Knowledge for Teaching: Knowledge of Future Mathematics.** Although we did not code any of the episodes for knowledge of future mathematics, we hypothesize that in other classes of differential equations as well as other university level courses, this is an identifiable specialized knowledge for teaching. We connect this to Shulman's (1986) vertical curriculum knowledge, in that this knowledge is drawn in to specifically prepare students for mathematics (and possibly other content courses such as physics) that they will encounter at a later time. For example, if students in this class were to take a dynamical systems course (which is not offered at this university) there are certain concepts and methods that are used in that course that would be foreshadowed in the IODE course. Since the professor knows this future knowledge, she or he could bring that to bear in the classroom, tying it in a way that makes the current class more meaningful and so that when the students are presented with it later, they are able to use their past knowledge in effective ways.

Conclusion

In this paper, we offer kinds of mathematical knowledge for teaching. These results are tentative, and we are not convinced that these characterizations are appropriate or helpful in our study of university level mathematics professors. However, we suggest these characterizations of knowledge to the mathematics and mathematics education community to begin a dialogue.. We suggest that mathematical knowledge for teaching does not look the same at more advanced mathematical levels as in elementary school, where current work has been reported . We found it helpful to return to Shulman's descriptions of knowledge that teachers draw upon to do the work of teaching (1986). For example, he introduced the ideas of lateral and vertical curriculum knowledge to be two type of teachers' mathematical knowledge for teaching. This is parallel and supports our results that there is mathematical depth knowledge or lateral curriculum knowledge as well as knowledge about future and past mathematics or vertical curriculum knowledge. This curriculum knowledge may lead us to consider our specialized categories in different ways.

Mathematicians are becoming more interested in the scholarship of teaching and are beginning to look to mathematics educators to provide some answers. Our goal is not only to encourage mathematics educators to consider this problem, but to bring it to the mathematicians' community and continue discussions with them so that mathematics teaching at the university might be improved if we know more about the knowledge for teaching at the tertiary level.

Finally, we need to consider that the construct of mathematical knowledge that a professor uses may need some revisioning. Instead of the *knowledge* a professor uses, the interesting *professor's work of teaching* at the university level was a more interesting and profitable arena to study. The professor involved in this research had a passio for bringing the students into the community of practicing mathematicians which seems to be important from his view and for the students' futures. Therefore, this idea of bringing mathematics students new habits of mind may be a focus of future work.

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