A Local Instructional Theory for the Guided Reinvention of the Quotient Group Concept.

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Abstract

In this paper, we briefly describe a local instructional theory that has resulted from a series of design experiments focused on the quotient group concept. This local instructional theory will consist of 1) a generalized instructional sequence intended to support the guided reinvention of the quotient group concept and 2) a theoretical and empirical rationale for this generalized instructional sequence. We will describe the design experiments that informed the development of the instructional sequence and, in order to illustrate and motivate the local instructional theory, we will describe key aspects of the reinvention process in terms of the participating students' mathematical activity.

Introduction

This research reported here is part of a larger project that has as its goal the creation of an innovative research-based inquiry-oriented curriculum for abstract algebra. Here we will focus on the research and design work that supported the development of the quotient group unit of the curriculum. Our primary purpose will be to describe the resulting *local instructional theory* (Gravemeijer, 1998) for supporting the guided reinvention of the quotient group concept.

It has been observed that students struggle with the quotient group concept (Dubinsky, Dautermann, Leron, and Zazkis, 1994). Burn (1996) noted that the quotient group can be seen in the elementary example of even/odd parity which suggests that in some sense the concept may be accessible to students at least in an informal sense. Of course it has also been noted that "seeing the general in the particular' is one of the most mysterious and difficult learning tasks students have to perform." (Mason & Pimm, 1984). The goal of our research was to develop instruction that would support students in reinventing the quotient group concept by building on their intuitive notions of parity.

Theoretical Perspective

This research was influenced by the instructional design theory of Realistic Mathematics Education (RME) in three ways. First, the primary goal of the project is to develop an approach to the teaching of the quotient group concept that is consistent with the RME idea of *guided reinvention*. Gravemeijer and Doorman (1999) explain that the idea of guided reinvention "is to allow learners to come to regard the knowledge that they acquire as their own private knowledge, knowledge for which they themselves are responsible" (p. 116). Second, our analysis of the participating students' mathematical activity and our design efforts were guided by Gravemeijer's (1999) description of the ingredients of a local instructional theory. In particular, we worked to:

1) Identify student ideas and strategies that anticipated the quotient group concept.
2) Learn how these ideas and strategies could be evoked.
3) Learn how these ideas and strategies could be leveraged to support the reinvention of the quotient group concept.

Additionally, our research and design work was influenced by two RME related theoretical ideas. The first is the *emergent models* design heuristic from RME. According to Gravemeijer (1999) emergent models are used in the RME approach to promote the evolution of formal knowledge from students' informal knowledge. The idea is that the model initially emerges as a *model-of* students' activity in experientially real problem situations and the model later evolves into a *model-for* more formal activity. The second specific RME related theoretical idea that guided our work is Lakatos' (1976) framework for mathematical discovery, the process of *proofs and refutations*. This framework was developed to explain historical processes of mathematical discovery and was adapted by Larsen & Zandieh (2007) to explain processes of guided reinvention in undergraduate mathematics education. Briefly, the process of proofs and refutations is one in which conjectures are revised (and concepts developed) as proofs are analyzed in light of counterexamples. Larsen & Zandieh argued that this framework could also serve as a suite of instructional design heuristics. These two design heuristics were used to develop initial conjectures as to how the quotient group concept might be reinvented.

We conjectured that the quotient group concept could emerge as a model-of the students' informal mathematical activity as they searched for parity in the group D_8 (the symmetries of a square). We expected that the students could then further mathematize this activity by first considering whether the even/odd partition they found formed a group in some sense, and then by generalizing to more complex groups of partitions (e.g. partitions into four subsets consisting of pairs of elements). Then as the students analyzed various partitions we conjectured that, in light of examples that did and did not work, the students might develop conjectures as to what conditions are needed to make a partition function as a group. Then through the process of proofs and refutations, we imagined that the concept of normality (among others) might be reinvented as a condition needed to prove a given partition formed a group.

Relationship to the Research Literature

This research fits within a growing body of research that is exploring the utility of RME for supporting the learning of undergraduate mathematics. For example, there are ongoing RME-guided instructional design projects at the undergraduate level in the areas of geometry (Zandieh and Rasmussen, 2007) and differential equations (Rasmussen and King, 2000). Like these projects, the present project aims to contribute specific knowledge about the learning and teaching of particular mathematical topic (quotient groups), and contribute to the ongoing development of RME. For example, our findings suggest that the process of proofs and refutations might represent a productive approach by which to support the transition from model-of to model-for. In this case, we will see that a process akin to that described by Lakatos (1976) supported students in fleshing out the details of their emerging quotient group concept.

Research Method

The research reported here consists of a series of design experiments. Analyses of data collected during each experiment informed the ongoing development of the emerging local instructional theory and the design of specific instructional tasks. Here we focus on the first two phases of the research and design cycle:

- The first experiment was conducted with a pair of undergraduate students. We met with these students for ten 60-90 minute sessions.
- The second experiment consisted of two implementations of the emerging instructional sequence in a group theory course, one taught by a research mathematician.

The resulting data corpus is being subjected to multiple cycles of iterative analysis. One of the purposes of this analysis is to further refine and articulate the local instructional theory for the guided reinvention of the quotient group concept.

Results

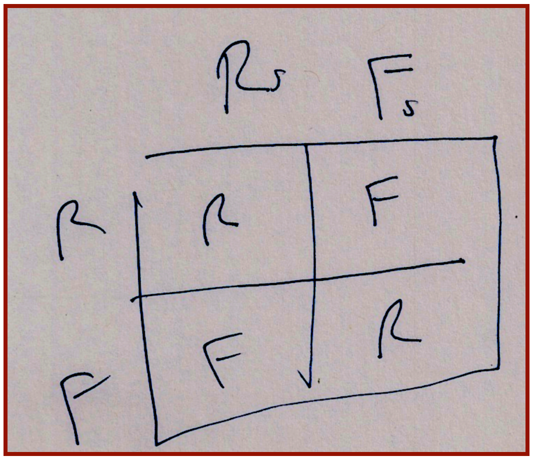
In this section we describe five key steps in the reinvention process and illustrate these with examples from the design experiment data. These five steps constitute the framework for our local instructional theory and represent a generalized instructional sequence by which the concept of quotient group might be reinvented by building on students' understanding of parity.

Step 1: Identifying Evens and Odds in a Finite Group

In each of the design experiments, the first step in the reinvention process was the identification of something like even/odd parity in the group of symmetries of a square (a group that was very familiar to the participating students). The students (Rick and Sara) who participated in the first design experiment had developed their own set of symbols for the symmetries of a square, using additive notation: $\{0, R, 2R, 3R, F, F+R, F+2R, F+3R\}$. Here R represents a 90-degree clockwise rotation and F represents a flip across a vertical axis.

> *Rick:* Evens are those moves that added to themselves give the moves in that same group that we called even. Odds are moves when we add them to other [moves] in the odd group we get evens.
>
> *Researcher: So which ones are the odds and which ones are the evens?*
>
> *Rick:* The evens are 0, $2R$, F, and $F+2R$. The odds are R, $3R$, $F+R$, and $F+3R$.

Note that Rick describes "evenness" in terms of the operation and not in terms of the form of individual elements. This is important because it highlights the group structure of the resulting partition, setting the stage for generalizing to more complex groups of subsets. The realization that the even/odd partitions in fact form groups whose elements are subsets is a key. In the following excerpt we see Sara and Rick coming to terms with this idea. (During this interaction the students have two operation tables in front of them, one was the full table for D_8 and the other was a 2x2 table representing the partition of D_8 into flips and rotations. See Figure 1.)



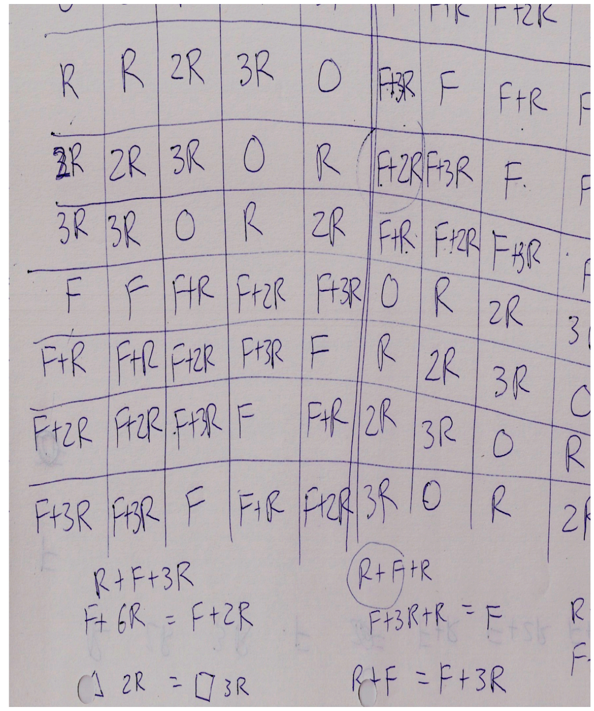


Figure 1. A 2x2 flip/rotation table and the full 8-element operation table for D_8.

Researcher: ...[Points to the full D_8 table] How many elements does this group have in it?

Rick: Eight.

Researcher: How many does this one have [points to the 2x2 table]?

Rick: I don't think this is a group. I don't think.

Sara: It does have eight.

Researcher: What if I said it had two?

Rick: Then it wouldn't be a group.

Sara: Well if you want to make meta-groups.

Researcher: Ok, let's make meta-groups.

Rick: No wait, maybe it would be a group. It would be a group for me.

Sara: Yeah.

Researcher: What would the elements of the group be?

Rick: Rotations and flips. My identity would be rotations.

Step 2: Conjecturing and Proving that One of the Subsets Must be a Subgroup

We found in the design experiments that the students very quickly conjectured that one of the subsets must be a subgroup in order for a partitioning to form a group (or partition into evens/odds). In the following excerpt from the second design experiment (in the group theory course) a student explains that there can be no more than three ways to partition D_8 into evens and odds because D_8 has only three subgroups of order four.

Erin: There's not going to be any more because you have to have one be a subgroup…

Teacher: So the claim is that one of the two sets actually has to be a subgroup… Why do you think that one of them needs to be a subgroup?

Erin: If it's a subgroup, then it with itself is just going to make that subgroup back…

Teacher: The evens with itself is the evens. Ok, so you claim that based on that reason that some element with itself has to equal itself. You're making the case that you need to have a subgroup to be one of your sets

Erin: Yeah, so that subgroup will act as the identity.

The idea that this subgroup must act as an identity subset is crucial throughout the reinvention process. Below, we will see that this idea can be used to build the idea of coset formation and to formulate a necessary condition for a partition (into cosets) to form a group (normality).

Step 3: Generalizing to More Complex Groups of Subsets

The third step in the reinvention process is to generalize by shifting the focus from parity to the group structure. In our design experiments, the students were asked to find other ways to subdivide the group of symmetries of a square to form groups with more than two elements.

This step is non-trivial. We found that the students struggled to move beyond the notion of parity. Those students who had not attended to the fact that the even/odd partitions they had found earlier were actually groups (whose two elements were actually subsets) needed to shift their way of thinking as they engaged with this new task. In each of our design experiments, the students identified the one partition into four subsets that actually forms a group. (Note that the students "multiplied" subsets by taking each element of one subset and multiplying it by each element of the second subset, and this "worked" if the result was one of the subsets in the partition.) In addition to this one partition that works, the students typically try out a number of other partitions that do not work, setting the stage for identifying necessary conditions.

Step 4: Determining How to Partition the Rest of the Group

After it has been established that one of the subsets must be a subgroup, the next step is to determine how to partition the rest of the group. It turns out that there is only one way to do this (by forming cosets). In the Figure 2 we see Rick's algorithm for partitioning the group.

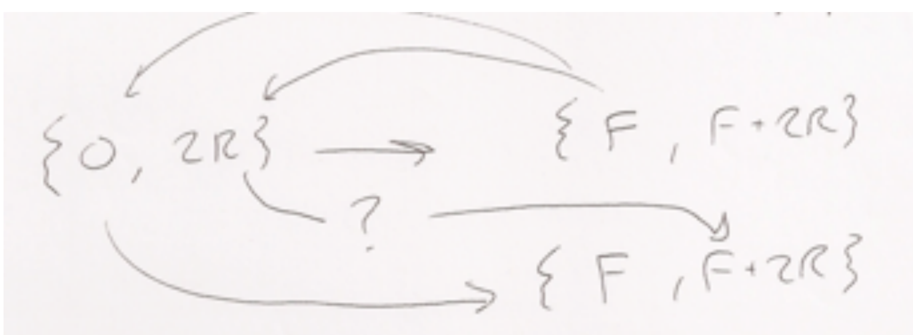


Figure 2. Rick's algorithm for partitioning a group.

In this example, Rick started with the subgroup $\{0, 2R\}$ of D_8. He then chose the element F from outside the subgroup. He stated that the result of combining the subset $\{F, \_\}$ with the identity subset $\{0, 2R\}$ must be the subset $\{F, \_\}$. Then adding F to F and F to $2R$, he obtained the subset $\{F, F + 2R\}$ and then deduced that F must be paired with $F + 2R$. Notice that this algorithm can easily be used to formulate the definition of *coset*.

Step 5: Finding a Necessary Condition for a Partition of Cosets to Form a Group

After it has been determined that the only way to successfully partition a group to form a group of subsets is by forming cosets, the final step of the reinvention process is to determine what additional condition is needed to ensure that such a partition forms a group. Again this can be done by focusing on the need for the subgroup to act as the identity element. In the following excerpt from the group theory course, two students formulate such a condition and explain why it is needed in a specific case.

> *Chuck*: I think that I got a pretty good explanation. The left coset and the right coset don't match.
>
> *Luc*: Yeah
>
> *Chuck*: Just use the one element and you get that. I used R and ... I did RI and RFR and I got these, and then I did IR FRR and I got these two. So you can't have F be a part of this subset and FR^2 be a part of … it doesn't work…You started generating a ton of elements
>
> *Luc*: and it doesn't work it on the right. It creates the yellow-purple group. Instead of just…

Here the students are considering the subgroup $H = \{I, FR\}$. They then consider the subset that contains the element R. Multiplying R through in both directions, they generate four different elements. However, since H must work like the identity element these products should produce only elements from the two-element coset that contains R. Thus they conclude that the subgroup H did not behave like an identity element because the right cosets did not equal the left cosets. Once this condition gets established (and used to define normality) the formal details of the quotient group can be worked out. For example, one can show that the set of cosets is closed under the set multiplication operation by proving that two cosets of a normal subgroup aH and bH will multiply to produce the coset abH. This result can motivate the idea of multiplying cosets by representatives (which can be shown to be well-defined when the subgroup is normal), which in turn can be used to simplify the rest of the formalization of the theory, including the proof that normality is a sufficient condition.

Conclusions

Through a sequence of design experiments we have developed a local instructional theory for supporting the guided reinvention of the quotient group concept. In this paper we have briefly described and illustrated this instructional theory in the form of five key steps in the reinvention process.

References

Burn, Bob. (1996). What are the fundamental concepts of group theory? *Educational Studies in Mathematics*, *31*, 371-378.

Dubinsky, E., Dautermann, J., Leron, U., & Zazkis, R. (1994). On learning the fundamental concepts of group theory. *Educational Studies in Mathematics*, *27*, 267-305.

Dubinsky, E., Dautermann, J., Leron, U., & Zazkis, R. (1997). A reaction to Burn's "What are the fundamental concepts of group theory?" *Educational Studies in Mathematics*, *34*, 249-353.

Gravemeijer, K. (1998). Developmental research as a research method. In A. Sierpinska & J. Kilpatrick (Eds.), *Mathematics Education as a Research Domain: A Search for Identity* (pp. 277-296). Dordrecht, The Netherlands: Kluwer.

Gravemeijer, K. (1999). How emergent models may foster the constitution of formal mathematics. *Mathematical Thinking and Learning, 1*(2), 155-177.

Gravemeijer, K., & Doorman, M. (1999). Context problems in realistic mathematics education: A calculus course as an example. *Educational Studies in Mathematics, 39,* 111-129.

Lakatos, I. (1976). Proofs and refutations. Cambridge: Cambridge University Press.

Larsen, S. & Zandieh, M., (2007), Proofs and refutations in the undergraduate mathematics classroom. *Educational Studies in Mathematics*. 67 (3) 205-216.

Mason, J. & Pimm, D. (1984). Generic Examples: Seeing the General in the Particular. *Educational Studies in Mathematics* 15(3), 277-290.

Rasmussen, C., & King, K. (2000). Locating starting points in differential equations: A realistic mathematics approach. *International Journal of Mathematical Education in Science and Technology, 31,* 161-172.

Zandieh, M., & Rasmussen, C. (2007). A case study of defining: Creating a new mathematical reality. *Mathematical Thinking and Learning.* (Manuscript submitted for publication).