Investigating Student Approaches to Counting Problems: An Exploration Using the Notion of Actor-Oriented Transfer

Elise Lockwood
Portland State University

Motivation and Theoretical Perspective

According to authors of combinatorial texts, counting can be a nuanced and difficult process. Indeed, Martin (2001) entitled the first section of his book "Counting Is Hard," and Tucker (2002) says that "clever insight" (p. 169), and not merely a predictable approach to solving problems, is a necessary aspect of successful counting. Research on the teaching and learning of combinatorics confirm this fact (Batanero, Navarro-Pelayo, and Godino, 1997; Eisenberg & Zaslavsky, 2004; English, 1991), indicating that students struggle with solving various types of counting problems.

The combinatorics education literature suggests that a potentially important factor in counting is an ability to make connections among counting problems – be it through detecting common structures (English, 1991), or through identifying models of underlying problem types (Batanero et al., 1997). English (2005) reports that "a common finding in many of the studies on combinatorics is that students have difficulty in identifying related problem structures. As a consequence, students' ability to transfer their learning to new combinatorial situations is limited" (p. 135). While English is not specific about how she interprets the word *transfer*, this quote supports a hypothesis that transfer, in some form, could be a useful lens through which to consider how students approach and solve counting problems.

Lobato (2003) introduces what may be a particularly appropriate notion of transfer, called *actor-oriented transfer*, which differs from a more traditional transfer perspective. She describes shifting from "an *observer's* (expert's) viewpoint to an *actor's* (learner's) viewpoint by seeking to understand the processes by which individuals generate their own similarities between problems" (Lobato, 2003, p. 18). She elaborates further upon this view (Lobato & Siebert, 2002) and defines traditional transfer as "the application of knowledge learned in one situation to another situation" (p. 89). Actor-oriented transfer, on the other hand, is "the personal construction of relations of similarity between activities, or how 'actors' see situations as similar" (p. 89). According to Lobato and Siebert (2002), the evidence one looks for when studying these respective views of transfer also differs. In the traditional transfer perspective, evidence of transfer is found through subjects' performance on "paired tasks that are similar from the researcher's point of view" (p. 89). In actor-oriented transfer, however, evidence of transfer is revealed "by scrutinizing a given activity for any indication of influence from previous activities and by examining how people construe situations as similar" (p. 89).

The purpose of this preliminary study is to make a case for actor-oriented transfer as a useful lens by which to study student counting. As Lobato and Siebert (2002) mention, the method for uncovering instances of actor-oriented transfer consists of scrutinizing students as they work and seeing what connections *students themselves* make to prior situations. This paper puts forth three examples that emerged from such a methodology; these examples highlight the potential utility that actor-oriented transfer could offer as a theoretical perspective.

Study

Five pairs of students were videotaped during interviews that ranged from sixty to ninety minutes in length. One pair was from a class for pre-service elementary school teachers, one was from a group theory class, and three were from an advanced combinatorics class – the group theory students and two of the advanced pairs are discussed here. Motivation for pairs research stems from studies (e.g. Larsen, 2004; Swinyard, 2008) which indicate the benefits of such a methodology.

During the interview, students were given a list of twenty counting problems to complete; the respective pairs completed between eight and eighteen of them. The purpose of the interviews was not to guide the students through a particular instructional sequence, nor was it to ensure that students would solve all of the problems completely. Instead, the goal was to provide an opportunity for the researcher to observe the students work on a variety of problems, particularly in order to observe any connections they were making among problems. The students were thus instructed to work freely. In addition, because the researcher sought to have students make connections between problems, it was preferable to have them do more problems, even if incompletely, than to completely solve only one or two problems in the allotted time. Thus, at times the researcher intervened in order to move the students on to the next problem.

The list of problems was compiled by the researcher[1]; the goal was to have a wide variety of problems, ranging in difficulty and problem type. The problems were chosen as representative of common types of problems as categorized by combinatorial texts (e.g. Martin, 2001; Tucker, 2002) and some of the combinatorics education literature (e.g. Batanero, et al., 1997). These included both basic and complex examples of arrangement and selection problems, both with and without repetition. The researcher viewed and analyzed the videotape data and transcribed relevant episodes. Keeping Lobato & Siebert's (2002) definition of actor-oriented transfer in mind, analysis consisted of a) scrutinizing episodes for evidence of ways in which students made connections to previous ideas, and b) examining how students construe situations as similar.

[1] The list of problems was compiled from a variety of sources including Tucker (2002) and Martin (2001). The three problems in this study all come from Tucker (2002).

Relevant Counting Problems

In order for the reader to appreciate the student responses, three of the twenty problems must be discussed in detail. Each of these problems, along with their solutions[2], will be elaborated upon here.

The "Over Three Million" Problem. **How many numbers greater than 3,000,000 can be formed by arrangements of 1, 2, 2, 3, 6, 6, 6?** A solution to this problem involves a case breakdown, determined by whether a 3 or a 6 is the first digit. If 3 is the first digit, the problem becomes a matter of arranging the numbers 1, 2, 2, 6, 6, and 6, which can be done in $\binom{6}{3}\cdot\binom{3}{2}\cdot 1$ ways. This solution can be seen by considering six positions in which to place the numbers; the solution is found by first choosing three positions in which to put the 6's, then choosing two of the remaining three positions to put the 2's, and then choosing one more position in which to place the 1. In a similar way, the solution to the case in which a 6 is in the first position is $\binom{6}{2}\cdot\binom{4}{2}\cdot 2$. In textbooks, this type of problem (after the case breakdown) may be referred to as an "arrangement with limited repetition problem" – it consists of arranging some number of objects where some of the objects are indistinguishable from one another.

The "Test Question" Problem. **A student must answer 5 out of 10 questions on a test, including at least 2 of the first 5 questions. How many subsets of 5 questions can be answered?** One solution to this problem uses a case breakdown, based on whether exactly two, three, four, or five of the first five questions are answered first. Such a solution yields

---

[2] There are multiple ways of expressing the solutions to each problem, many of which elicits a different counting technique. However, the solutions elaborated upon in this section are most relevant to what the students actually did and to the discussion at hand.

$\binom{5}{2}\binom{5}{3}+\binom{5}{3}\binom{5}{2}+\binom{5}{4}\binom{5}{1}+\binom{5}{5}\binom{5}{0}$. A common incorrect answer to this problem is $\binom{5}{2}\binom{8}{3}$.

The reasoning behinds this solution tends to be something like this: "As long as I choose two of the first five problems, then I have satisfied the 'at least two' requirement. Thus, I can pick three of any of the remaining eight problems and I'll still have satisfied the requirement." The problem with this strategy is that some of the solutions can be counted more than once. For example, in utilizing this strategy, consider choosing problems 1 and 2 as the first step (the "five choose two"). Then, in the next step choose problems 3, 7 and 8. Thus, the subset of 5 questions to be answered is {1, 2, 3, 7, 8}. However, this subset could be found in a different way using this strategy, namely, by first choosing problems 1 and 3, and then choosing problems 2, 7, 8. Thus, the solution $\binom{5}{2}\binom{8}{3}$ actually counts some solutions more than once.

The "Piles of Cards" Problem. **How many ways can a deck of 52 cards be broken up into a collection of four piles of 13 cards?**

The solution to this problem is $\dfrac{\binom{52}{13}\binom{39}{13}\binom{26}{13}\binom{13}{13}}{4!}$. The numerator is found by first choosing 13 of the 52 cards to be in the first pile, then choosing 13 of the remaining cards to be in the second pile, then choosing 13 more and 13 more. In employing this solution, we now have 4 distinguishable piles of 13 cards – the first, second, third, and fourth piles. However, since the question asks for collections of piles of cards (collections are unordered), we must divide by a factor of 4!

Results and Discussion

Videotape analysis of these interviews consisted of looking for instances in which students themselves made connections to previous situations and/or construed situations as similar. In conducting this analysis, three examples came to light. Each of these instances provide insight into what connections students make between previous situations, and these connections often differ from those that experts may see. Thus, these examples indicate ways in which actor-oriented transfer may be a useful construct through which to analyze students' treatment of counting problems.

Example 1

The first example follows two students, Chris and Keith, as they solve the Over Three Million problem. These two students were in the advanced combinatorics class; one of them had previously taken discrete mathematics, but neither had extensive experience studying counting problems. When they solved the Over Three Million problem, they made several statements that indicated ways in which they related the problem to previous experiences. In the excerpt below, they had already considered the case breakdown, and they were trying to figure out the number of ways to re-arrange the numbers 1, 2, 2, 3, 6, 6.

K: And in fact this is identical to a word with, uh, indistinguishable letters.
C: Right.
K: Yeah. So how do we solve that? …

This excerpt suggests that Keith sees the problem as identical to others in which there are indistinguishable objects. Thus, he appears to recognize something about the problem structure that is independent of the context; for him, it does not matter if the problem involves numbers or letters, there is an underlying commonality between the two. While he was not pushed on this commonality during the interview, subsequent interviews might have provided opportunities to push Keith on why he saw the current problem as "identical" to another context.

As the students continued to think about the problem, the exchange below emerged. By way of explanation, some weeks before the interview they had been assigned a problem in their combinatorics class which said "How many words, of any length, can be formed using the letters from the word Estate?" This is what they refer to when they say "Estate" in the excerpt below.

K: In fact, um, this is identical to, um,
C: Estate?
K: Estate, yeah, except we're taking out the cases where it has fewer than, uh, than six letters.
C: Right…Um, well see it's not even estate, because in Estate we were counting words as being, like, there was a one-letter word.
K: Yes, yes, that's what I mean. It's like Estate only with the 6-letter words. Yeah. Except that no, no it's not exactly the same because there are two pairs here and there are…
C: There are two pairs in Estate.
K: Yeah, yeah – okay, you're right.

The above excerpt allows us to glean from the students what things they attended to when comparing the Over Three Million problem to the Estate problem. For example, while both students recognized that they were similar problems, both students were attuned to the fact that the two problems were not exactly the same. In particular, there are two noteworthy differences that they mentioned. First, they both highlighted the fact that Estate required all of the possible words of any length, whereas the Over Three Million problem only involved words of length six. This was important enough for both of them to bring up individually. It seems, then, that they realized that they needed to adjust their strategy for solving the Estate problem if they were to solve this problem; they could not just copy an identical solution they had used in the Estate problem. Second, the last three utterances of the excerpt above highlight another differentiation they recognized. The word ESTATE does have exactly two pairs of repeated letters (E's and A's), and the set of numbers {1, 2, 2, 3, 6, 6} also do have exactly two pairs of repeats (2's and 6's). However, initially Keith miscounts and does not think that the two have the same sets of repeated objects. Though Chris corrects him and they end this discussion by saying that they are indeed the same, it is nonetheless noteworthy that Keith makes this distinction. He was focused enough on the specifics of how many letters were involved, and it stands to reason that if the two contexts did have different numbers of repeated objects, that would have been a relevant distinction for him. This discussion highlights the similarities and differences that the students focused on, as opposed to accentuating the comparison that an observer might emphasize.

Maher and her colleagues (e.g. Maher & Martino, 1996) have shown evidence that students can utilize a prototypical problem type when solving counting problems. For instance, students in her study often referred to "the tower problem" as a prototypical example involving the multiplication principle. In the case of Chris and Keith, though, their reference to the Estate problem indicates that they did not have a prototypical problem type in mind. If they did, they would likely be less focused on details like the exact numbers of repeated objects. Instead, it appears that they evaluated the Estate problem as a specific example in their arsenal, and they used that example to consider the Over Three Million problem; it is the closest thing they had at hand to compare with their current situation.

Example 2

While solving the Over Three Million problem, another instance of student-constructed similarity between problems occurred. Before tackling the Over Three Million problem, Chris and Keith had solved the Test Questions problem. They had, in fact, solved this correctly, breaking the solution down appropriately into cases as discussed above. During the interview, the researcher told them about the alternative incorrect solution $\binom{5}{2}\binom{8}{3}$ and asked them to try and come up with a solution that might be counted twice by the incorrect solution and counted only once by the correct solution. Chris and Keith successfully answered this; they were able to explain why the incorrect solution is actually too large.

Chronologically during the interview, this insightful discussion of the Test Question problem came before the Over Three Million problem. During the Over Three Million problem, Chris had successfully come up with a solution for the case when the first number is a 6 – he had written $\binom{6}{2}\binom{4}{2}\cdot 2$ on the board, after which the following exchange took place.

C: So does that double count anything like the… But I'm, I'm worried. So that, that uh, example which she gave us before, the problem we were doing with…the wrong counting problem (looks at paper with problems on them).
K: Yeah, is it double counting?
C: Right. Okay so it's not double counting, and here's why.
[Chris goes on to explain successfully why this situation does not double count].

We see here that the students (Chris especially) constructed a clear connection between the problem they were working on and the Test Question problem. Although they were never asked directly about why they initiated this connection, we can make some guesses as to why they acted as they did. It seems most likely that they were influenced by the visual appearance of the two binomial coefficients being multiplied together. That is, in the Test Questions problem, over-counting occurred in the solution $\binom{5}{2}\binom{8}{3}$, when sets of items were chosen successively. Chris had written on the board $\binom{6}{2}\binom{4}{2}\cdot 2$ as a partial solution to the Over Three Million problem, and so the sight of the successive binomial coefficients might have triggered something for him. Another factor could have been problem proximity – because they had previously solved the Test Questions problem, they were primed to be very wary of over-counting. Thus, Chris' attention might have been attuned to over-counting, resulting in a desire on his part to check for it in every possible situation. It is impossible to know exactly what triggered this reference back to this particular previous problem, but either of these seems like a reasonable possibility. In future studies, further questions about such an action could be pursued in subsequent interviews.

It should be noted that generally in textbooks (and, arguably, from an observer's perspective), The Test Question problem does not have any clear relationship to the Over Three Million problem. They are not of similar problem structure, they do not have similar components, and they would generally not be categorized together. Here, however, the students made a clear connection between the Over Three Million and their prior work on the Test Questions problem. Thus, this is an important example of actor-oriented transfer – namely, it is an instance in which the students constructed similarities among prior situations.

There are potential implications for such a finding. The over-counting example that occurs in the Test Questions problem is a common pitfall that students can face, and it may be easy to assume that once students have seen it they will know exactly when to recognize it in subsequent work. We see in this example, though, that even after success with that particular over-counting issue, it was not immediately clear to these students when and how the issue could arise. Thus, instructors should not to assume that appropriate avoidance of such a pitfall will necessarily occur. Also, it should be mentioned that although Chris and Keith introduced a connection when, in fact, the two situations were not the same, their work should be applauded. They provide here an example of the kind of questioning and pondering that makes for successful counters. Students should be encouraged to assess, as they did, each new counting situation to determine similarities and differences to prior activities. Thus, this example also highlights a spirit that instructors should seek to foster in their counting students – one of curiosity that will cause them to think critically and to assess for themselves how problems relate to one another.

Example 3

A final example is mentioned here to further emphasize the potential value of actor-oriented transfer as a theoretical construct for studying counting – this time by highlighting a pitfall of *non*-actor-oriented transfer. That is, we see here an instance in which the researcher clearly is not attuned to how the students are approaching a problem and fails to attend to the students' mathematical activity. This example is not an instance of actor-oriented transfer, but rather it points to situations that can arise when a researcher is overly focused on her own way of thinking and is not open to the students' activity.

To set up this example, we recall the Piles of Cards problem. Prior to the interview discussed here, the researcher had conducted an interview with another set of advanced combinatorics students, Brad and Ben. These were two extremely capable students who had taken a recent counting-heavy course. In the interview they displayed their abilities and answered all but two of the problems quickly and correctly. Interestingly, however, the two problems that they missed involved the same mistake – they employed a counting procedure that resulted in solutions that were ordered when they had intended to have their solutions be unordered (thus leading to answers that were too big than the correct solution). In the Piles of Cards problem, one of the two that they missed, they obtained an answer of $\binom{52}{13}\binom{39}{13}\binom{26}{13}\binom{13}{13}$, neglecting to divide by the 4!. Prodding in their interview revealed that they believed this solution was correct; they felt it gave the proper number of piles of cards (even though this has over-counted the solutions by a factor of 4!). Because they had been so successful in most of the interview, and because they had made this mistake twice, the researcher was particularly interested in analyzing their response to this question. In her analysis, she was able to explain to herself (and to justify mathematically) the error that they had made: the students had employed a counting procedure in which they inadvertently ordered their solutions when they did not intend to.

Directly after having interviewed Brad and Ben and analyzing their video, the two group theory students, Alex and Mark, were interviewed. On the whole they were less successful than Brad and Ben had been, but they answered the Piles of Cards problem correctly. A partial explanation of their solution is in the excerpt below.

M: Because if, if the order mattered, like, if it, if you had players and the order mattered like who got which group, I think that would be the answer right there [points to just the numerator]. But I think if it doesn't matter, if you're just looking at 4 groups kind of like a set, like it doesn't really matter the order they come, then I feel like we're counting some of them more than once.
A: Yeah.
M: I can't figure out the ones we're going to count two times or three times.
A: Well so there's not too many options as far as that goes, uh, is there?
M: Oh, so there's, there'll be 4! different ways that we could arrange those groups. So may if we divide by 4!
A: That would make sense.
M: That makes sense to me.

They thus arrived at the correct solution of $\dfrac{\binom{52}{13}\binom{39}{13}\binom{26}{13}\binom{13}{13}}{4!}$. At this point in the interview, the researcher made the statements below, seeking confirmation from the students. We see, however, that the researcher's questions reveal an assumption she made about how these students solved the problem. Because of her analysis of Brad and Ben's work, she was convinced that this must have been how Alex and Mark solved the problem. However, in Alex's response, we see that he was not viewing his solution as the researcher assumed he was. She assumed that they recognized that they were introducing order in their counting procedure, and he saw the issue more as being one of having duplicate solutions.

R: Okay, good, and your guys' technique of choosing 13 cards then choosing 13 more then choosing 13 more, do you feel that that ordered those piles – well, I think that you do think that – why do you…that solution without the denominator there, why do you feel that ordered those piles? That solution of 52 choose 13 times 39 choose 13, why is there, like, an implicit ordering in those piles? Or why do you feel that in order to –

A: Well it just gives you duplicate answers. And the duplicate answers shouldn't be counted unless where the cards end up matters.

Both of these ways of viewing the problem are mathematically correct, but we have here evidence of a researcher making an assumption about how the students were thinking. There is information that can be gleaned from this exchange. First of all, since the researcher was clearly focused on her own interpretation of the problem, she missed what the students were actually doing. Thus, there was a squandered opportunity, perhaps, to learn more about how these students had approached the problem and how they had constructed similarities between previous activities. Second, this excerpt reveals a potential pedagogical implication about ordering versus over-counting. The researcher had ordering on her mind, but the students were thinking of having "duplicate solutions," or solutions that would be over-counted. These are mathematically equivalent ideas, resulting in the same kinds of solutions, but it could be that pedagogically one is more natural for students than another. Further interviews need to be conducted before any serious conclusions can be drawn, but there is some indication here that over-counting, rather than ordering, may be a useful construct for students as they learn about and solve counting problems.

Conclusion

Given the nature of counting problems, being able to determine similarity among problems, problem types, situations and techniques is a vital aspect of being a successful counter. The examples provided in this paper lend credence to the idea that actor-oriented transfer, which proposes a student-focused perspective through which to analyze data, could in fact be a useful lens through which to study the teaching and learning of counting problems at the undergraduate level.

References

Batanero, C., Godino, J., & Navarro-Pelayo, V. (1997). Combinatorial reasoning and its assessment. In I. Gal & J. B. Garfield (Eds.), *The Assessment Challenge in Statistics Education* (pp. 239-252): IOS Press.

Eizenberg, M. M., & Zaslavsky, O. (2004). Students' verification strategies for combinatorial problems. *Mathematical Thinking and Learning, 6*(1), 15-36.

English, L. D. (1991). Young children's combinatorics strategies. *Educational Studies in Mathematics, 22*, 451-474.

English, L. D. (2005). Combinatorics and the development of children's combinatorial reasoning. In G. A. Jones (Ed.), *Exploring probability in school: Challenges for teaching and learning* (Vol. 40, pp. 121-141): Kluwer Academic Publishers.

Larsen, S. (2004). Supporting the guided reinvention of the concepts of group and isomorphism: A developmental research project. Unpublished Doctoral dissertation. Arizona State University.

Lobato, J. (2003). How design experiments can inform a rethinking of transfer and vice versa. *Educational Researcher, 32*(1), 17-20.

Lobato, J., & Siebert, D. (2002). Quantitative reasoning in a reconceived view of transfer. *Journal of Mathematical Behavior, 21,* 87-116.

Maher, C. A., & Martino, A. (1996). Young children invent methods of proof: The gang of four. In P. Nesher, L. Steffe, P. Cobb, B. Greer & J. Goldin (Eds.), *Theories of mathematical learning* (pp. 431-447). Manwah, NJ: Lawrence Erlbaum Associates.

Martin, G. E. (2001). *The Art of Enumerative Combinatorics*. New York: Springer.

Swinyard, C. A. (2008). Students' reasoning about the concept of limit in the context of reinventing the formal definition. Unpublished Doctoral dissertation. Portland State University.

Tucker, A. (2002). *Applied Combinatorics* (4th ed.). New York: John Wiley & Sons.