Learning Proof by Mathematical Induction

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Abstract: This qualitative study of six pre-service teachers' perceptions and performance around proof by mathematical induction indicates strengths and challenges for collegiate teaching and learning. We report on constant comparative analysis of student mathematical work and on two focus group interviews of three students each.

Background

The National Council of Teachers of Mathematics' (2000) *Reasoning and Proof Standard* calls on school teachers to help students create and validate (determine the truth of) logical arguments. As part of their preparation, pre-service secondary teachers take courses such as discrete mathematics in college where they learn to make, test, and prove conjectures about mathematical patterns and relationships. In particular, they work with *proof by mathematical induction* (PMI). Though there are several studies on how learners might experience, understand, and use proof (Hanna, 2000; Selden & Selden, 2003; Tall, 1991; Weber, 2001, 2004), there are few on how ideas of proof by mathematical induction are taught or how they are perceived by learners who are prospective secondary teachers (Brown, 2008; Harel, 2002). One approach to examining the development of mathematical understanding is to consider the developmental, instructional, and learner aspects involved (e.g., Brousseau's (1997) ontogenic, didactical, and epistemological obstacles or, at a smaller grain size, Gray and Tall's (1994) proceptual synthesizing of language/symbol evocation, skill with process, and richly connected concept image). In this report, we have focused on the learner. In a separate report, we have focused on

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the instructor (Tsay, Hauk, Yestness, Davis, & Grassl, 2009). As a consequence, the results reported here relate first to learners' perceptions and address the question: *What is the nature of pre-service secondary teachers' perceptions and performance in learning proof by mathematical induction in introductory discrete mathematics courses?*

The theoretical foundations for the design and analysis of the study were constructivist and informed by Brousseau's theory of didactical situations. We also relied on the work of Harel (2002) regarding the role of learner perception of intellectual necessity in coming to understand proof by mathematical induction. Harel compared two forms of secondary school instruction for learning about inductive proof: traditional and necessity-based (see Table 1).

Table 1. Aspects of Traditional and Necessity-based PMI Instruction – Based on Harel (2002).

Traditional Instructional Approach	Necessity-based Approach
T.1. Teacher presents examples of how a formula with a single, positive, integral variable (like the sum of the first *n* integers) is generalized from observations and an observed pattern. **T.2.** Teacher talks about why examples are not enough to prove a proposition *P*(*n*) is true for all positive integers *n*. **T.3.** Teacher demonstrates the principle of mathematical induction as a proof technique involving two steps: Step 1: Show that *P*(1) is true. Step 2: Show *P*(*n*) implies *P*(*n*+1) for all *n*. **T.4.** Students practice applying the steps to mostly algebraic examples (e.g., formulaic and symbolic recursive relationships).	**N.1.** Students work with implicit recursion problems to develop pattern generalization skill; **N.2.** Students work with explicitly recursive relationships using quasi-induction as a method of testing conjectures. **N.3.** Teacher presents math induction as an abstraction of quasi-induction that meets students' felt need for a rigorous method of proof. **N.4.** Students make, test, and prove conjectures about a variety of mathematical statements using the language and procedures of mathematical induction.

Though both of the mathematicians who taught the discrete mathematics courses we observed used traditional and necessity-based ideas, the balance of their use differed across the two instructors. In this sense, the study was informed by variety of didactical situations (and of didactical obstacles – see the end of the Results section). Nonetheless, the main result is about learning and the kind of conceptual restructuring that may be needed for any learner of proof by mathematical induction in any didactical situation. That is, though the language/symbol set and procedure for proof by mathematical induction can be taken up and used by students in many successful ways, a reorganization of thinking about mathematics, particularly about what constitutes “problem-solving” and about the nature of “proof” appears to be necessary in coming to a conceptually rich and connected concept image for proof by mathematical induction.

Methods

Setting

The undergraduates in our study were enrolled in two sections of discrete mathematics at the same 12,000-student doctoral-extensive university in the United States. Most students in the two classes (65%) were planning on becoming secondary school mathematics teachers and some (about 25%) were planning to be primary school teachers with a specialty in mathematics. About half of the students in both classes had graduated from high schools within a 200-mile radius of the university and most had not encountered proof by mathematical induction before in a high school or college mathematics course. Like the U.S. secondary teaching population, the students were mostly from middle socio-economic status, majority culture, backgrounds. One instructor, Dr. Isley taught largely in Harel’s (2002) traditional style and the other, Dr. Vale, often used a necessity-based approach.

The instructors were both mathematicians. Dr. Isley, with a PhD in combinatorial algebra, had taught college mathematics for more than 20 years and was the author of the text used in the class. He had taught discrete mathematics more than 20 times, and generally used lecture with occasional in-class activities. During the three weeks of PMI focus, Dr. Isley lectured 60% of the time and the class spent 40% of the time attending to in-class lecture presentations of inductive proofs (on overhead transparencies) by student teams. Before students presented, they met with Dr. Isley in his office, where he helped them validate their work. Dr. Vale, with a PhD in logic and model theory, had 10 years college teaching experience though this was his first time teaching proof by mathematical induction and the first time he had taught discrete mathematics. Dr. Vale used Isley's textbook, and developed additional activities for class, using a mix of traditional and necessity-based activities in class. During the observed lessons on PMI, Dr. Vale lectured about 35% of the time with the balance of about 65% of class time spent on students working individually and in groups to make, test, and prove conjectures about recursive and closed-form expressions. A notable distinction between the experiences of students in Dr. Isley's class and those in Dr. Vale's class was that students in Dr. Vale's class validated each others' inductive proofs during in-class group work and regularly had proof-validation tasks where they analyzed potential proofs provided by Vale.

Each focus group had students with course grades of A, B, and C respectively (in what follows, student pseudonyms begin with letters corresponding to course grades). Alan, Brooke, and Chuck were the focus group students in Dr. Isley's class. Anna, Beth, and Celia were the focus group students from Dr. Vale's class. At the time of the interviews, all were planning to be school teachers: Alan, Chuck, Anna, and Celia were in a bachelor's degree program in mathematics for secondary teachers while Brooke and Beth were in a degree program to prepare elementary mathematics specialists.

Data Collection and Analysis

We relied on information from four data gathering activities. First, we observed (in person or from a video recording) seven 50-minute class meetings on PMI for each instructor and completed related PMI textbook reading and activities (data set A). Second, at the end of the semester, we conducted 90- to 120-minute video-recorded interviews with focus groups of three students each – one group from each of the two discrete math classes (data set B). The third form of data (set C) was a 90- to 120-minute video-recorded interview with each instructor about mathematics, about proof by mathematical induction in particular, and about the teaching and learning of both. Finally, the fourth set of data (set D) was student work on two PMI-related common final exam items (*n*=49), one requiring students to generate a proof, one asking students to validate a purported proof. For the work reported here, we spent the greatest analytic effort on the student-generated forms of data (set B – focus group interviews and set D – student final exam work). Figure 1 summarizes our iterative process for qualitative open coding for themes within data sets and axial coding for categories and sub-categories across data sets.

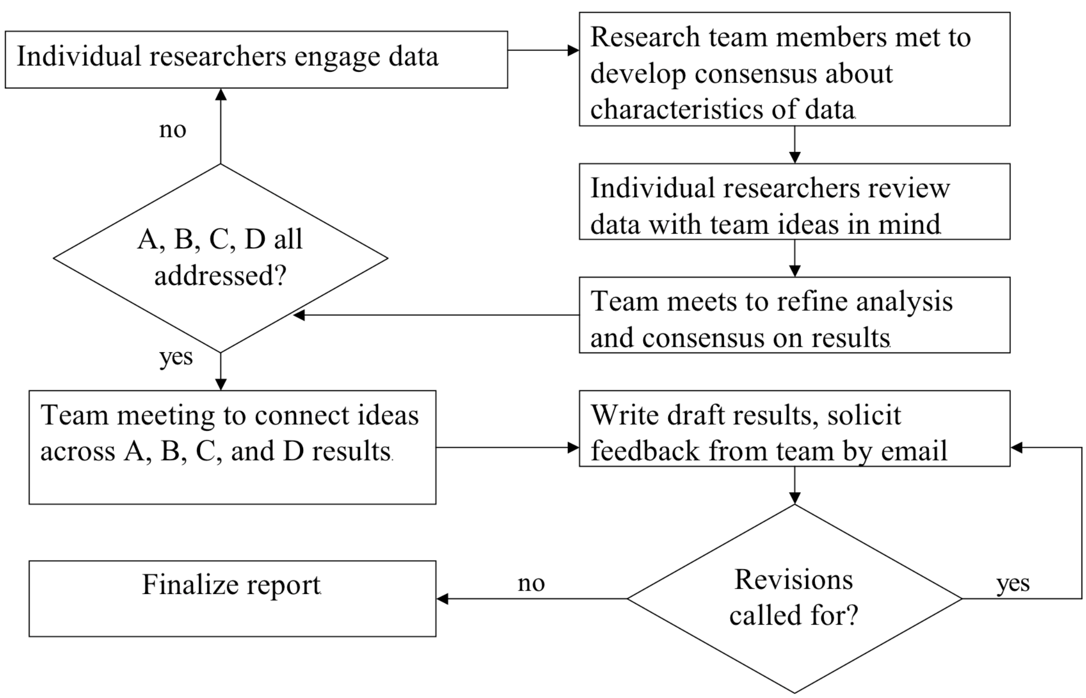


Figure 1. Flow chart of data analysis process.

For example, we started with open coding of focus group interviews to generate potential categories. Based on these categories, we used axial coding of student solutions to final exam items to refine categories and identify potential sub-categories. Then we revisited and reviewed the video of focus group interviews, using the hypothesized categories and sub-categories as a basis for inductive analysis.

Results

Students talked explicitly about three categories of challenges in learning proof by mathematical induction. A fourth potential category emerged from analysis of student work on final exams. All challenges were common to students from both classes. Students noted they had begun to develop a "different kind of thinking" from what they had done to date in "solving problems" in mathematics. They also commented on the difficulty of connecting what seemed to them a jumble of procedures: from the geometry of *n*-gons to rules for exponents. A third kind of cognitive restructuring for students, particularly those in Dr. Vale's class, occurred when faced with proof-validation tasks. Finally, the problematic nature of coming to understand an infinite iterative process was a category that was not discussed explicitly by students, but that emerged through different symbolic/linguistic mechanisms in student written work on the final exam.

Challenge 1: Problem-Solving

All six students spoke about "doing regular problems" and the "hard problem-solving" needed for generating a proof. In particular, they reported "solving the problem" of "going from $P(k)$ to $P(k+1)$" and that it required flexible use of previous learning.

> *Alan:* There's a lot of manipulation you have to do.
>
> *Beth*: The hardest part that I find is trying to actually prove for $k+1$… that math.
>
> *Brooke*: You need a lot of background algebra knowledge and different mathematics knowledge … like 3^{n+1} is 3^n times 3.

Also, some students relied on the language of problem solving to talk about proving:

> *Chuck*: When I was watching the presentations [by other students] I was trying to make notes about how people solved them - the key points.

During both focus groups, students mentioned a desire for "real-world problems." Dr. Vale's students spent 10 minutes generating possible applications, mostly about recursive or iterative growth models. Dr. Isley's students said they did not recall any applications, except that PMI had to do with computers being right every time, and did not generate examples on their own.

Challenge 2: Connecting

All of the focus group students spoke about the idea of synthesizing or "pulling together" declarative and procedural knowledge from multiple sub-domains in mathematics for PMI. This extended beyond problem-solving and ranged from using strategies from geometry, algebra, and calculus, to seeing "how things are related, deeper math," including specialized knowledge about things like the Fibonacci numbers. Students reported on the restructuring of their understanding of mathematics and of their perceptions about their own role in making mathematics. They perceived a challenge in developing autonomy for setting up the relationship between $P(k)$ and $P(k+1)$ in preparation for using the inductive hypothesis.

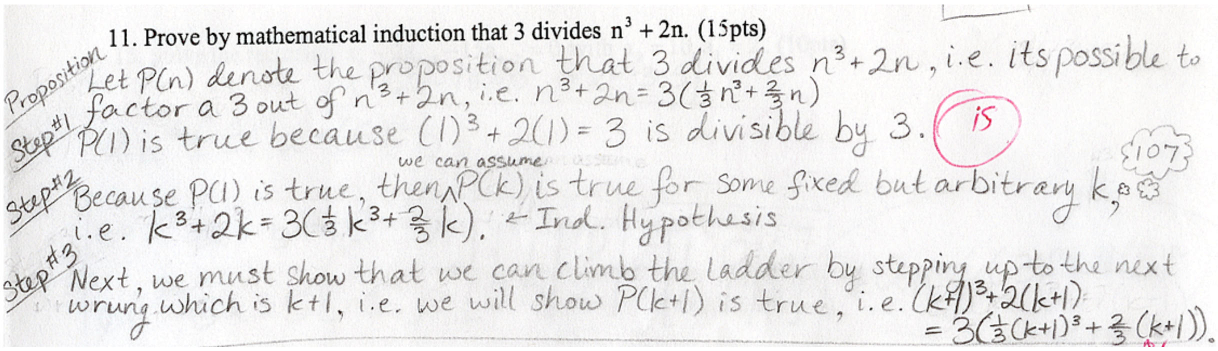
> *Anna*: I really like how it was set out: you're always going to have a basis, you're always going to have the induction step…*but* it would have been really helpful for me if there had been more, smaller steps. …And sometimes I would have the right thing, but I didn't know it was right.
>
> *Celia*: I didn't know exactly how [I] was supposed to pull every thing together [to use the inductive hypothesis].
>
> *Brooke:* I kind of stumbled… If I don't see all the steps, I get caught up on it and then I have to figure it out on my own. … I had taken good notes, I knew all the steps, but it's hard to know how to connect, how to come up with the things that *will* connect later on.

Challenge 3: Validating

In both classes, students had experiences with reflecting on and validating proofs by induction written by other people. However, these opportunities differed. All six focus group students commented on the challenge of "working backwards" in validating a proof and all preferred proof-generation to proof-validation.

> *Anna*: I thought it was easier to create my own [proof] as opposed to correcting one that was already done that was wrong.

Though all of the students were planning on becoming teachers and acknowledged that grading was part of being a teacher, none reported seeing a connection between the idea of validating a purported proof and the work of grading that they would do as teachers. This may have been, in part, a result of the didactical situations in the two classes around validation. In Dr. Isley's class, students met with the instructor in his office and he led them through the validation and revision of their proofs. Dr. Isley had exhorted students "to have it right first" before presenting. Dr. Isley noted in his interview that he expected students to validate each other's proofs during the presentations but that he expected no significant errors in the proofs presented. So, though his students had engaged in validation of their own work, neither they nor Dr. Isley had expected to engage in substantive validation during their classmates' presentations. Students who experienced Dr. Isley's largely traditional approach felt a procedural competence in asserting the framework for proof by mathematical induction (for an example of the detail with which students presented the procedure in writing a proof, see Figure 2). Just under half of the students in both courses wrote a complete and correct proof by mathematical induction on the final. Also, about half (not necessarily the same students) in both classes completely and correctly validated a proof by mathematical induction on the final exam.


11. Prove by mathematical induction that 3 divides $n^3 + 2n$. (15pts)

Proposition: Let $P(n)$ denote the proposition that 3 divides n^3+2n, i.e. its possible to factor a 3 out of n^3+2n, i.e. $n^3+2n = 3(\frac{1}{3}n^3+\frac{2}{3}n)$

Step#1: $P(1)$ is true because $(1)^3+2(1) = 3$ is divisible by 3. is

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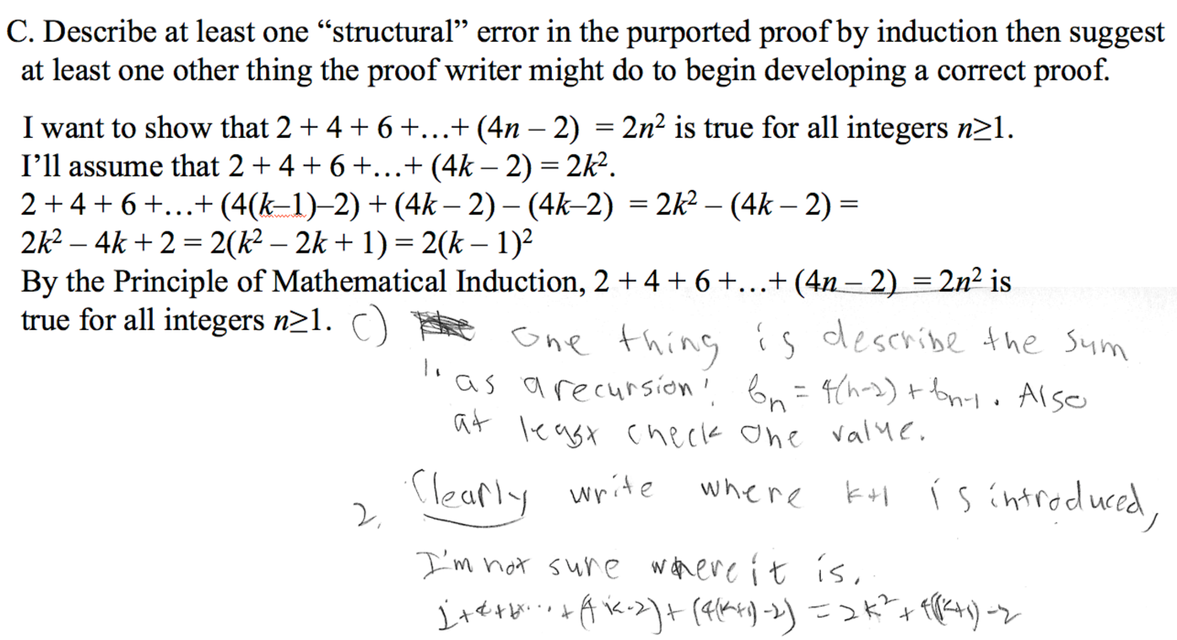
Step#2: Because $P(1)$ is true, then we can assume $P(k)$ is true for some fixed but arbitrary k, i.e. $k^3+2k = 3(\frac{1}{3}k^3+\frac{2}{3}k)$. ← Ind. Hypothesis

Step#3: Next, we must show that we can climb the ladder by stepping up to the next wrung which is $k+1$, i.e. we will show $P(k+1)$ is true, i.e. $(k+1)^3+2(k+1) = 3(\frac{1}{3}(k+1)^3+\frac{2}{3}(k+1))$.

Figure 2. Student proof by mathematical induction, from Dr. Isley's final.

In Dr. Vale's class, students had in-class tasks where they were meant to validate each other's proofs, but students reported not always being sure that the group had created a valid proof unless it was reviewed by Dr. Vale. Additionally, Dr. Vale regularly provided faulty proofs with structural and syntactic/symbolic errors for students to validate (see Figure 3). Dr. Vale noted in his interview that he would likely use student proofs with "authentic errors" the next time he taught PMI.


C. Describe at least one "structural" error in the purported proof by induction then suggest at least one other thing the proof writer might do to begin developing a correct proof.

I want to show that $2 + 4 + 6 + \ldots + (4n - 2) = 2n^2$ is true for all integers $n \geq 1$.
I'll assume that $2 + 4 + 6 + \ldots + (4k - 2) = 2k^2$.
$2 + 4 + 6 + \ldots + (4(k-1)-2) + (4k - 2) - (4k-2) = 2k^2 - (4k - 2) =$
$2k^2 - 4k + 2 = 2(k^2 - 2k + 1) = 2(k - 1)^2$
By the Principle of Mathematical Induction, $2 + 4 + 6 + \ldots + (4n - 2) = 2n^2$ is true for all integers $n \geq 1$.

C) 1. One thing is describe the sum as a recursion! $b_n = 4(n-2) + b_{n-1}$. Also at least check one value.

2. Clearly write where k+1 is introduced, I'm not sure where it is.
$2+4+6+\cdots+(4k-2)+(4(k+1)-2) = 2k^2+4(k+1)-2$

Figure 3. Student validation, from Dr. Vale's final.

Challenge 4: Infinite Iteration

A fourth potential category of epistemological obstacle emerged from what we had, at first, seen as an issue of didactical obstacle. Coding of student work on the common final exam items led us to identify two additional challenges, each associated with a particular instructor. In Dr. Isley's case, the use of non-standard terminology in-class appeared to be associated with idiosyncratic language/symbol use in communicating proofs. For example, in Figure 2 the student asserted "Next, we must show that we can climb the ladder by stepping up to the next wrung." While Dr. Isley and his students saw the ladder analogy as useful in learning about PMI, such a statement on a proof in another context (e.g., in another instructor's class) might not have had much meaning. In Dr. Vale's class, students worked a great deal with recursive representations of relationships (e.g., defining a_k in terms of a_{k-1} – note that in Figure 3 the student suggests defining the relationship recursively). The symbolic foundation of working with recursion appeared to be associated with errors in some students' proofs, such as several students writing or validating an inductive step based on showing $P(k)$ implies $P(k-1)$ rather than $P(k)$ implies $P(k+1)$. In both cases, we suggest that the underlying issues for learners were the iteration and infinity encapsulated by the inductive step in a proof by induction.

Discussion

Through examination of student perceptions we identified three categories of potential epistemological obstacles in learning about proof by mathematical induction and through examination of their performance on final exam items, came up with a fourth category. These categories included aspects of problem-solving, connecting, and reasoning (in particular, validation), and infinite iteration. While the first three are key areas of the NCTM (2000) standards for K-12 mathematics, the fourth involves the concepts of infinite process and infinity, which are usually associated with advanced mathematics. Nonetheless, recent work by Brown (2008) suggests that the ideas of infinity, particularly of infinite iteration, are accessible to 12-year-olds. As has been noted in other work, the iterative and infinite nature of PMI, particularly in understanding the inductive process itself, is both a teaching challenge and a learning challenge (Brown, 2008; Harel, 2002). Future work will need to include explorations with learners and tasks that provoke attention to, and conversation about, each of the four proposed categories. One path to such tasks will be to talk with advanced undergraduates, graduate students, and professors in mathematics to develop, pilot, and refine infinite iteration concept-eliciting tasks.

From re-analysis of student work on common final exam PMI items, we identified further support for the three categories that arose from the focus groups and we have begun developing sub-categories. Several cycles of refinement have led to planning for future work that will include inductive coding of student final exam work and focus group interview data in terms of a coding triple based on the proceptual development categories:

> Gray & Tall (1994, p. 121) describe a *procept* essentially as the amalgam of three things, a *pro*cess (such as addition of three and four), a con*cept* produced by that process (the sum) and a symbol that evokes either concept or process (e.g. 3+4). Following Davis (1983), they distinguish between a process which may be carried out by a variety of different algorithms and a procedure which is a "specific algorithm for implementing a process" (p. 117). A procedure is therefore cognitively more primitive than a process (DeMarois & Tall, 1996, p. 2).

The PMI procept triple would involve a judgment, based on the nature of student perceptions communicated during existing (and possibly additional) focus group interviews, about (a) the evocations associated with symbols/language of PMI, (b) the process of generating the "steps" in

PMI – with particular attention to the idea of infinite iteration, and (c) validating (as a component of concept development). From such coding, we hope to identify instructor and learner approaches to proof by mathematical induction that facilitate the engagement and resolution of the kinds of epistemological obstacles we have proposed here.

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