How to Act?
A Question about Encapsulating Infinity

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The focus of this paper is to outline briefly the key ideas of my presentation at the 12th SIGMAA on RUME conference. This paper addresses aspects of an on-going study of mine which attends to learners' conceptions of infinity as manifested in their engagement with variations of a well-known paradox: The Ping-Pong Ball Conundrum. The APOS Theory (Dubinsky & McDonald, 2001) postulates a framework for interpreting learners' understanding of two distinct ideas of mathematical infinity: potential and actual infinity. According to Fischbein (2001), *potential infinity* can be thought of as a process which at every moment in time is finite, but which goes on forever. A*ctual infinity* is described as a completed entity that envelops what was previously potential. Through the mechanisms of *internalisation* and *encapsulation*, Dubinsky et al. (2005a,b) propose that learners construct meaning for the concept of infinity as a *process* (potential infinity) and infinity as an *object* (actual infinity). This paper takes a closer look at the specific features connected to infinity the *object*.

Theoretical Background

In the terminology of the APOS Theory (Dubinsky & McDonald, 2001), an understanding of a mathematical entity begins with an action conception, which is recognised by the need for an explicit expression to manipulate or evaluate. Eventually, an action may be interiorised as a mental process. Once an action has been interiorised, the individual can imagine performing an action without having to directly execute each and every step. If the individual realises the process as a completed totality, then encapsulation of that process to an object is said to have occurred. Encapsulation of a process is a sophisticated step which requires appreciating the mathematical entity as a completed object that can be acted upon. Also, a mathematical concept may be composed of more than one entity, involving several actions, processes, and objects, which must be coordinated into a mental *schema*.

Dubinsky et al. (2005a,b) suggested that interiorising infinity to a process corresponds to the idea of potential infinity, i.e. infinity is imagined as performing an endless action, though without imagining carrying out each step. Encapsulating this endless process to a completed object corresponds to a conception of actual infinity. As in the general case, encapsulation of infinity is considered to have occurred once the learner is able to think of infinite quantities "as objects to which actions and processes (e.g., arithmetic operations, comparison of sets) could be applied" (Dubinsky et al., 2005a, p.346). Dubinsky et al. (2005a) also observed that "in the case of an infinite process, the object that results from encapsulation transcends the process, in the sense that it is not associated with nor is it produced by any step of the process" (p.354). Brown, McDonald, and Weller (in press) introduced this possibility, and termed the encapsulated object of infinity a *transcendent object*.

Two questions arise: (1) *How* does a learner act on infinity (ie how are arithmetic operations applied)? and (2) What can the 'how' tell us about an individual's understanding of infinity?

Setting and Methodology

Data for this study were collected from two participants: Jan and Dion. Jan was mathematics major in a southeastern state university in the USA. She was in her final year of the program and was very interested in the concept of infinity both from a mathematical and philosophical point of view. Jan had prior experience with Cantor's Theory of Transfinite Numbers through formal instruction during her undergraduate studies. In particular, she was familiar with comparing sets via one-to-one correspondences. Dion was an instructor in mathematics education at a university in eastern Canada. He held a master's degree in mathematics education and a bachelor's degree in mathematics. Dion taught prospective secondary school teachers in mathematics and didactics, the curriculum for which included aspects of Cantor's theory, such as establishing a one-to-one correspondence between the sets of natural and even numbers.

Data were collected from an interview with the participants, who were asked to engage with the two variants of the Ping-Pong Ball Conundrum which are discussed below. The interviews began by presenting participants with the first paradox, PP. Following their responses and a discussion of the normative resolution to PP, participants were asked to address the variant, PV. A discussion of the normative resolution to PV, akin to the explanation below, ensued. After this discussion, participants were encouraged to reflect on the two thought experiments and their outcomes.

The Ping-Pong Ball Conundrum

> You have an infinite set of numbered ping-pong balls and a very large barrel; you are about to embark on an experiment that will last exactly 1 minute. Your task is to place the first 10 balls into the barrel and then remove number 1 in 30 seconds. In half of the remaining time, you place balls 11 – 20 into the barrel and remove number 2. Continue ad infinitum. After 60 seconds, at the end of the experiment, how many ping-pong balls are in the barrel?

The normative resolution to the Ping-Pong Ball Conundrum (PP) involves coordinating three infinite sets: the in-going ping-pong balls, the out-going ping-pong balls, and the intervals of time. In order to make sense of the resolution to this paradox, a normative understanding of actual infinity is necessary. Although there are more in-going ping-pong balls than out-going ping-pong balls at each time interval, at the end of the experiment the barrel will be empty. An important aspect in the resolution of this paradox is the one-to-one correspondence between any two of the three infinite sets in question (see Mamolo & Zazkis, 2008, for a more detailed discussion). Given these equivalences, at the end of the experiment, the same amount of ping-pong balls went into the barrel as came out. Moreover, since the balls were removed in order, there is a specific time for which each of the in-going balls was removed. Thus at the end of the 60 seconds, the barrel is empty.

A variation of the Ping-Pong Ball Conundrum can easily be imagined. Consider the following:

> Rather than removing the balls in order, at the first time interval remove ball 1; at the second time interval, remove ball 11; at the third time interval, remove ball 21; and so on… At the end of the experiment, how many balls remain in the barrel?

This Ping-Pong Ball Variation (PV) begins in much the same was as the original Ping-Pong Ball Conundrum (PP). In one minute, an experiment involving inserting and removing infinitely many ping-pong balls from a barrel is carried out. However, the distinction lies in the fact that the Ping-Pong Ball Variation calls for the removal of balls numbered 1 at time one, ball number 11 at time two, ball number 21 at time three, and so on. Thus, despite the one-to-one correspondences between all of the infinite sets in question, at the end of the 60 seconds there will remain infinitely many balls in the barrel. In this experiment there will never be a time interval wherein balls 2 to 10, 12 to 20, 22 to 30, and so on, are removed. The seemingly minor distinction between removing balls consecutively, as in PP, versus removing them in a different ordering, as in PV, has a profound impact on the resolution of the paradoxes: while in one instance subtracting infinitely many balls from infinitely many balls yielded zero, in the other it yielded infinitely many. Taken together, the two paradoxes illustrate the indeterminacy of transfinite subtraction.

Results

Both Dion and Jan were easily able to resolve PP by establishing the appropriate one-to-one correspondences. Similarly, they also recognised a one-to-one correspondence between in- and out-going balls in PV. Dion concluded that the resolutions to PP and PV should be the same, arguing that in PV "after you go [remove] 1, 11, 21, 31, … 91, etc, you go back to 2." He was reluctant to accept the solution of a non-empty barrel, stating "if ball number 2 is there, so is 2 to 10, etc… so, infinite balls there? I have trouble with that." He went on to observe that while "on one hand $\infty - \infty = 0$, on the other it's ∞." Eventually, Dion conceding that he was "convinced" of the normative solution to PV. Dion's revelation that "on one hand $\infty - \infty = 0$, on the other it's ∞" suggests that accommodating actual infinity goes beyond the ability to act on an object, and includes an understanding of *how* to act on that object.

In contrast to Dion's struggle, Jan was able to resolve PV, recognising that "transfinite cardinal arithmetic doesn't work exactly like finite cardinal arithmetic". Jan connected her understanding of correspondences between infinite sets to explain the indeterminacy of transfinite subtraction. She remarked:

> "Even though there is a bijection between the set of balls put into the barrel and the set of balls removed, there are still an infinite number of balls left in the barrel after the minute is up! … we can easily create an infinite sequence of balls that are not removed".

Jan also reflected on the relationship between the two paradoxes and noted, "we seem to have done the exact same thing (physically) in both cases, but due to some arbitrary numbering system that we have imposed upon the set of balls removed, we have changed the remaining number from zero to infinity!"

Discussion

How does a learner act on infinity? Dubinsky et al. (2005a) suggest two ways an individual may act upon the object of infinity – apply arithmetic operations and compare sets. Focusing on the former, this study identifies two different ways learners 'acted'. Dion, who revealed an understanding of infinite set comparison in his resolution of PP, suggested that 'anything' subtracted by itself should be zero, and had "trouble" with the idea that the barrel would not be empty. When Dion was faced with a non-routine problem about transfinite subtraction, he 'acted' by generalizing his intuition of subtracting real numbers, and struggled with the indeterminacy of transfinite subtraction. In contrast, Jan's ability to deduce consequences of a set being equinumerous with one of its proper subsets contributed to her understanding of the indeterminacy of transfinite subtraction, and allowed her to 'act' in a way that was consistent with normative standards.

What can the 'how' tell us about an individual's understanding of infinity? Dion's difficulty acting on actual infinity in the normative way suggests that acknowledging the distinction between *how* actions, such as arithmetic operations, behave differently when applied to transfinite versus finite entities is an integral part of accommodating the idea of actual infinity. Further, it suggests that how actions are applied may be relevant to the encapsulation of an object. This study opens the door to further investigation into how learners act on infinity and what, if anything, can be inferred about an individual's conceptualisation based on *how* that individual applies actions and *which* actions are applied.

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