Mathematicians, Mathematics Educators and High School Mathematics Teachers Interpretations and Judgments Regarding Calculus Students' Problem Solving Methods

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Mathematicians and mathematics educators have a long history of working together toward the education of our nation's mathematics teachers and the development of mathematics curricula (Bass, 2005). Though mathematics and mathematics education are often viewed as two separate fields, Bass reminds us that the field of mathematics education is in essence a field of applied mathematics. As such, the fields are wedded to each other in that they both are responsible for the education of school mathematics teachers. Mura (1995) pointed out the importance of casting light on the continuity of the influences that prospective teachers are exposed to during their training, in both mathematics and mathematics education courses. Since both mathematicians and mathematics educators work closely with prospective teachers to develop the knowledge needed to teach mathematics (Ball & Bass, 2003), it makes sense to compare what they attend to when drawing upon that specialized knowledge.

This paper will report the findings of an exploratory study in which a group of mathematicians, mathematics educators, and high school teachers were asked to analyze calculus students' problem solving methods, a practice that draws upon ones *mathematical knowledge for teaching* (Ball & Bass, 2003). Specifically it aims to address the following research question: What do mathematicians, mathematics educators, and high school mathematics teachers attend to when making interpretations and judgments about calculus students' problem solving methods?

Conceptual Framework

Teacher knowledge has long been a focus of educational research (Ball & Bass, 2003). Among the types of knowledge that have been shown to be important for teachers to develop are: content knowledge, pedagogical content knowledge and technological pedagogical content knowledge (eg. Ma, 1999; Shulman, 1986; Mishra & Koheler, 2006). Recently Ball and her colleagues at the University of Michigan have brought to our attention the very important and specialized knowledge that mathematics teachers must develop, *mathematical knowledge for teaching* (MKT) (Ball & Bass, 2003). The notion of MKT developed from analyzing the practice of teaching with a mathematical eye, i.e. identifying the mathematical work of teaching (Bass, 2005).

According to Ball and Bass knowing mathematics for teaching requires that teachers be able to "unpack" ideas that are typically compressed into highly abstract and usable forms and to recognize connections both within and across mathematical domains. For example, the mathematical work of teaching includes, but is not limited to: coming up with accurate mathematical explanations, presenting appropriate definitions, making connections between representations, responding productively to student's mathematical questions and curiosities, making mathematical judgments about instructional materials, and interpreting and making mathematical judgments about students questions, solutions, and insights (Ball & Bass, 2003, p. 6-7). Bass (2005) identified four categories of MKT, all of which are drawn on by teachers when they "unpack", recognize connections, and make decisions about what to do next. The four categories of MKT are: 1) common mathematical knowledge, 2) specialized mathematical knowledge, 3) knowledge of mathematics and students, and 4) knowledge of mathematics and teaching. These skills are especially needed when teachers are required to interpret and make judgments about students' problem solving methods.

Methods

Thirty-seven members of the mathematics and mathematics education community were interviewed. The mathematicians were all self identified as research mathematicians and were from three different universities in the southeastern United States. Their experience as research mathematicians ranged from 1 year to 40 years, with an average of 19 years in the field. The mathematics educators were from two U.S. universities (one in the midwest and one on the southeast) and all identified themselves as both researchers and mathematics teacher educators. Their experience in the field ranged from 1 year to 38 years, with an average of 10 years in the field. The high school mathematics teachers were from four schools all within the same large urban school district in the southeastern United States. Their experience in the field ranged from 1 year to 28 years, with an average of 9 years of experience. In addition, three of the high school mathematics teachers were concurrently enrolled in a mathematics education master's degree program. None of the others had graduate school experience.

Each participant was shown three video clips, each of a different student solving a particular calculus problem. Each of the students in the videos used some combination of paper, pencil and graphing calculator as tools to solve the problem. The videos included both the students' paper and pencil work and their graphing calculator work shown side-by-side in real-time. Participants were asked to comment on the students' problem solving methods. Each interview was approximately one hour in length. The interviews were audio recorded, transcribed verbatim and entered into NVivo as cases.

Grounded theory methods (Strauss & Corbin, 1990) were used to analyze the data with respect to MKT. Specifically, the data was first read through and reduced to utterances regarding what Ball and Bass (2003) refer to as interpretations and judgments made about students problem solving methods. The utterances were coded using an open coding process, and then the open codes were chunked into conceptual categories. This process resulted in nine categories that represented what the participants attended to when making interpretations and judgments about the problem solving methods used by the students on this task. The categories included: ability to grade, appropriateness of methods for testing, appropriateness of technology use, communication of solution, connections between representations, evidence of student understanding, speed, perceived affective experiences, and traditional procedures. It is important to note that the categories were not discrete, as sometimes an utterance captured more than one idea. For example, the statement "She didn't communicate what she did so I couldn't grade that" was coded as both communication of solution and ability to grade. Once coding was complete, the data was put back together and reread to test the codes for clarity and reliability. Finally, interpretations were made regarding the conceptual categories and their relationship to MKT within and between groups.

Results

It was found that when asked to interpret and make judgments on the students' problem solving methods the mathematicians and mathematics educators focused on the similar aspects of student work: appropriateness of technology use, connections between representations, and communication of solutions. In contrast, the high school teachers focused on the ability to grade, speed, appropriateness of methods for testing, and traditional procedures. All three groups attended to evidence of student understanding, but in strikingly different ways. The following sections detail the attention of each of the three groups (mathematicians, mathematics educators, and high school teachers) in their analysis of the students' work.

The Mathematicians

Though the mathematicians that participated in this study by and large did not appreciate the graphing calculator as an appropriate mathematical tool, the focus of their analysis of the students problem solving methods were focused on their perceptions of the students' mathematical reasoning. Specifically, beyond the expected attention toward the appropriateness of the technology use, they consistently attended to evidence of student understanding, connections between representations, and communication of solutions.

Evidence of student understanding. In their interpretations of the students problem solving methods the mathematicians consistently attempted to identify what the students did understand. More than half of their utterances with respect to the students work were focused on evidence of student understanding. Furthermore, any claims about student understanding were always supported with evidence. For example, when commenting on a student's use of the graphing calculator one mathematician noted that the student must have understood what the graph created on the calculator represented within the context of the problem and backed it up with the following evidence,

> "..she just graphed the parabola and knew that the top should be somewhere up here as opposed to moving down, that's what it looked like to me anyway she had just a really good idea of what the graph was supposed to look like so she could adjust."

It was never the case that a mathematician claimed a student did or did not understand something without providing evidence to support the claim.

Connections between representations. Given the nature of the work that they were presented (students using graphing calculators as a part of their problem solving methods) it was not surprising that attention was paid to the representations created using the technology and those created on paper. The mathematicians attended to these representations through a mathematical eye consistently looking for evidence that the students' were making those connections. For example, as one mathematician reflected on a student's work in which a graphical representation of the derivative of a function was used to answer a problem significant effort was put towards understanding how the student connected the representation to the problem as is evidenced by the following:

> "What's on paper, taking the theoretical derivative and knowing what it is. It's hard, I think of this as a function and also as a graph…So now I'm torn. You gave me good examples there. And also she seemed to know what she was going for. She knew that she wanted the graph, she wanted to see that numerically."

This kind of analysis of student created graphing calculator representations was typical of the mathematicians as they analyzed the students work.

Communication of solutions. The mathematicians all made mention of the ways in which the students chose to communicate their solutions by either pointing out the completeness of their mathematical arguments or pointing to holes in them. Typically, if the communication of a solution was found to be incomplete these judgments were followed by suggestions for ways that the student could improve or hypothetical ways in which they would structure future instruction to address the insufficiencies. For example, one mathematician pointed out that if a student were to use a graphing method to determine a maximum of a function he would ask,

> "How do you know there are no other solutions outside your window? How do you guarantee that your solution is the correct one? There is a way. Lets' think about it and then the calculus begins. And that's motivation. I'd give them a problem and say let's do it two different ways and see which is more convincing."

It was this type of careful consideration of the students communication of their solution methods and even hypothetical pedagogical moves designed to improve their mathematical communication that was typical of the mathematicians.

The Mathematics Educators

The mathematics educators attended the appropriateness of technology use, as expected due to the context of the study. Unlike the mathematicians, the mathematics educators made no comments regarding the appropriateness of the graphing calculator itself. Beyond the expected focus on the appropriateness of technology use the mathematics educators were very similar to the mathematicians in that they attended to evidence of student understanding, connections between representations, and communication of solutions. However, they also attended to the affect they perceived the students were experiencing.

Evidence of student understanding. More than half of the interpretations of the students work made by the mathematics educators were focused on evidence of what the students understood. Whenever claims were made about student understanding, they were always backed up with evidence from the student's work. For example, one mathematics educator stated the following when judging a students work on problem that required finding the maximum rate of change of a function,

> "She seems to understand the concepts of finding the maximum rate of change. So she seems to understand the rate of change is that first derivative. And so she knows to look for the – she knew the formula and evidently the procedures for finding the maximum rate of change. She was using the first and second derivative test to do that. So I would certainly feel comfortable that this student understands the concept."

This type of evidence seeking to support claims of student understanding was typical of the mathematics educators in this study.

Connections between representations. Again, similar to the mathematicians the mathematics educators attended to the connections that they perceived the students were or were not making between representations created on their graphing calculators and those on their paper. For example, in response to one student's creation of a series of graphs on the graphing calculator and the students written work one mathematics educator remarked, "So it's interesting he did all this work with the second derivative by hand but he never looked at the graph with the second derivative at all." It was careful consideration of what representations were created and how they were used like this that exemplified the mathematics educators' focus on connections between representations.

Communication of solutions. Like the mathematicians, the mathematics educators attended to the ways in which the students communicated both their solution methods and final answers. However, in their consideration of the many ways that the students could and should have communicated their work, 7 of the 10 mathematics educators pointed out the need for a way for students to communicate the work that they do using technology tools like the graphing calculator. Upon reflection on a student who solved a problem using the graphing calculator and only wrote down the final answer one mathematics educator noted the following:

> "In this world however, I believe that it is important for her to have a record just like you had a record [of the graphing calculator work on the video screen] so that now the procedure isn't I press the black box and this is what I came out with. The procedure would be what you showed me when it scrolled. And so now her procedure is entered and generated by the computer that she can show someone.

> It's just that she didn't write it down by hand. And for me that would be more than fine. So I would be perfectly happy with this. But see here she writes this down, she turns in a piece of paper – this would drive old people crazy. What should be there though is the display that we saw that was generated [on the video of the graphing calculator screen] and every time it was generated it would have a screen shot of it. Then I'd think old people would be halfway decent with that because they could see what she was really doing."

Whether it was through paper and pencil or video of graphing calculator produced screens the mathematics educators all focused on the ways in which the students communicated their mathematical work.

Perceived affective experiences. All 10 of the mathematics educators commented at least once on what they perceived a student might be feeling at some point in their problem solving. They commented on feelings such as confusion, comfort, stress, frustration, and even happiness. For example, in response to a student who was examining a table of values a mathematics educator said, "He would feel much more comfortable if there was a specific integer or number in there." The mathematics educators all seemed drawn to the affective side of the student's mathematical work.

The High School Mathematics Teachers

The focus of the high school teachers' interpretations and judgments of the students' work was quite different from the mathematicians and mathematics educators. While they also attended to student understanding, they did so in a much different manner. In addition, as they analyzed the student's work they attended to more pedagogical issues such as: the ability to grade, appropriateness of the problem solving methods for testing, speed, and traditional procedures.

Evidence of student understanding. Instead of attending to what students did understand, the high school mathematics teachers tended to point out what they believed the students did not understand. Furthermore, they often did not provide evidence for such claims even when prompted to do so. For example, claims such as “he doesn’t know what he’s doing” and “I don’t know what she’s doing because she didn’t show any work at all on paper” were quite common. The only exceptions to this type of judgment came from the three high school teachers who were concurrently enrolled in a mathematics education master’s program. These three teachers (referred to from here on out as master’s teachers) made interpretations and judgments about student understanding that were much more consistent with the mathematicians and mathematics educators.

Ability to grade. All of the high school teachers commented at least once on how they would go about grading a student’s work. These comments ranged from why a particular solution would be easy to grade to whether or not a solution would be worthy of partial credit. For example, in response to a student who wrote down in detail every step of her work a teacher commented, “she would be great to grade!” Or, in response to a student who did not show any work, just an answer a teacher said, “I wouldn’t accept that as an answer. That’s the correct answer, but I would take off points. She is obviously not showing me what she should show.” Grading seemed to be forefront on the teachers mind as they evaluated the students’ work.

Appropriateness of methods for testing. Like the mathematicians and mathematics educators the high school teachers did attend to whether or not the ways in which students used the graphing calculator were what they would deem appropriate. However, the guiding lens through which the high school teachers appeared to be making this judgment had to do with whether or not the methods, technology included or not, were appropriate for testing situations. For example, one high school teacher pointed out that graphing calculator methods are "perfect for multiple choice, but not on a free response question." Another commented on a student's use of the table on the graphing calculator saying, "That is a good testing strategy, but not a problem solving strategy." All 14 of the high school teachers commented on whether or not the methods they were observing were appropriate for testing situations at least once.

Speed. Given that the high school teachers attended to the appropriateness of methods for testing, it is not surprising that they also commented on the speed of those same procedures. Speed was not only recognized, but also appears to be valued by the high school teachers. For example, comments like "she did a great job, she got the solution fast" and "she was able to come up with the solution the fastest way, faster than I could have done it, so I think it's impressive," were common among the teachers. One teacher even commented on a student solution using the graphing calculator saying, "She's not fast enough using the calculator."

Traditional procedures. As mentioned above, the mathematicians and mathematics educators focused on the ways that the students communicated their solutions. When they analyzed the students' mathematical arguments they were often referring to the students traditional paper and pencil procedures. The high school teachers also attended to those traditional procedures, but for most of them their reasons for doing so were not focused on the communication of mathematical ideas, but instead simply the expectation of seeing the traditional procedures. For those teachers, the traditional procedures were the only acceptable solution methods and they wanted to see them. Typical comments included, "I would like to see more analytical work shown", "He didn't show any of the work I want to see", and "I mean, I don't have anything to see what he did."

The only exceptions to the expectation of seeing traditional procedures among the high school teachers were the master's teachers. None of the three master's teachers alluded to expecting to see traditional paper and pencil procedures, instead their comments were more in line with the mathematicians and mathematics educators in that they wanted the students to be able to communicate their solutions and solution methods. For example, one of the master's teachers compared two students' solution methods. One of the students had taken a more traditional procedure approach to an extension of a familiar maximum problem while the other had solved it using the graphing calculator. He commented, "[the first girl] was taking more of a traditional calculus textbook approach. Take the first derivative; take the derivative of the rate of change, etc. [the other girl] kind of streamlined that and went through and just found the maximum value rate of change. I think both methods are valuable, and even similar, they have just justified their solutions in very different ways."

Discussion

The results of this study are encouraging in the respect that there were such similarities between the mathematicians and mathematics educators' use of MKT in their analysis of the students work. However, there was a big difference between these two groups and the high school teachers. The mathematicians and mathematics educators made interpretations by "unpacking" the students' mathematical work, making judgments about the mathematics that the students must understand in order to use the methods that they did, and making connections between the representations the students constructed on the graphing calculator and their written inscriptions. All of which requires one to draw their MKT (Ball & Bass, 2003). The high school teachers on the other hand primarily focused on pedagogical issues. The only mathematical work in their interpretations and judgments (with the exception of those who had taken mathematics education graduate courses) came from the connections they made between the students graphing calculator methods and the traditional procedures they wanted to see.

It is not surprising that the high school teachers were more focused on assessment than the mathematicians and mathematics educators given the current focus on assessment in schools. Nevertheless, those teachers without mathematics education graduate experience focused solely on these issues, with little to no effort towards looking at the student work with a mathematical eye. This difference begs the question how and when mathematical knowledge for teaching is best taught. The fact that the high school teachers who have taken mathematics education graduate courses constructed far more of their interpretations and judgments through a mathematical lens might suggest that either analyzing student problem solving methods are not explicitly taught in undergraduate mathematics education like it often is in graduate courses, or possibly that as undergraduates students have not had occasion to make sense of these experiences.

Afterword: Reflections from a High School Teacher Turned Graduate Student

In a very fortunate turn of events, one of the high school teachers that participated in this study recently began a doctoral program in mathematics education. Approximately half way through the first semester of her doctoral program Ellen (a pseudonym) approached the researcher and asked if she could "redo" her interview. When asked why she felt compelled to do the interview again she said that she was embarrassed by what she had previously said about the students' work. The focus of her second interview was in fact very much aligned with the mathematics educators rather than the high school teachers.

The difference between Ellen's first and second interviews suggested that her experiences as a graduate student in mathematics education had impacted the way in which she interpreted and made judgments about student work. When asked to reflect on why her graduate school experience might have had such a big impact she identified the graduate student learning community as being instrumental in her growth. Specifically, she noted that being around colleagues and professors discussing student learning was something that she had never experienced.

> "I sure know a lot of math, but that didn't always help me in the classroom, it didn't help me address student misconceptions or figure out how to pose tasks or how to question in the right way to figure out what to do next with my students. I think the more you're around teachers, I mean I was kind of confined to my classroom and I had my beliefs and how I was taught to do math, what my high school teachers did I had never really talked to other people about it."

Ellen also shared that it was within this community of learners that some of her preconceived notions about what mathematics is and how it should be taught were challenged for the first time. One of her beliefs that certainly impacted her analysis of these students was that about the use of calculators to do math, she explained:

> "Well, one…I thought that you should be able to do math without a calculator. I still…it's kind of one of those hairy areas, but I've noticed that in some of the interviews you did that some of the kids had a really concrete conception for the math they were doing even if they used their calculators and they used it as a short cut."

Though Ellen is still obviously struggling with what she believes the role of graphing calculators should be in the teaching and learning of mathematics, in her second interview she did not immediately dismiss students graphing calculator work but instead studied it carefully trying to determine what the students had to understand mathematically in order to take the actions and draw the conclusions that they did.

Though Ellen's story does not help in identifying when a mathematical eye towards teaching is best developed (in an undergraduate or graduate program), it certainly does suggest an environment in which it can be developed. For Ellen this environment was one in which she was part of a safe community of colleagues where she was challenged to think deeply about student understanding and to critically examine her beliefs about the teaching and learning of mathematics.

Conclusion

Mathematicians and mathematics educators are responsible for the education of our nation's mathematics teachers and as such it is important that there is some consistency in the influences that these groups have on prospective teachers during their training (Bass, 2005; Mura, 1995). The results of this study suggest that mathematicians and mathematics educators are providing similar influences in the ways that they interpret and make judgments about student work, or their practice of MKT. This is particularly good news given that this is the kind of mathematical work for teaching that is done on a daily basis in the classes in which prospective teachers are students. However, the high school mathematics teachers in this study were quite different from the mathematicians and mathematics educators in that they focused mainly on pedagogical issues rather than mathematical ones in their analysis of the students' work. The only exceptions were those teachers who have had some mathematics education graduate courses.

Though the participants in this study represent a very small sample of each community (i.e. mathematicians, mathematics educators, and high school teachers) the consistency within each group does indicate that there is a possibility that the data is representative. As such it provides motivation for continued research in this area. It is important that we continue to build an understanding of how MKT high school mathematics is best developed and the impact that both mathematicians and mathematics educators have on its cultivation.

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