

The Role of Quantitative Reasoning in Solving Applied Precalculus Problems

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Introduction

Mathematics educators have emphasized the necessity of students having the ability to reason conceptually relative to functions and conceptualize patterns of change (Kaput, 1994; Monk, 1992; NCTM, 1989, 2000; Sfard, 1992; Thorpe, 1989; Vinner & Dreyfus, 1989). Recently, the National Council of Teachers of Mathematics (NCTM, 2000) *Standards* called for students to be able to analyze patterns of change in various contexts, such as interpreting the statement “the rate of inflation is decreasing.” The NCTM *Standards* also suggested that students develop deeper understanding of mathematically representing the ways in which quantities change (NCTM, 2000). Although it is not clear if mathematics curricula is adequately responding to these calls, it is well documented that high performing precalculus and calculus students have weak understandings of the function concept and difficulty modeling function relationships involving the rate of change of one variable as it continuously varies with another variable (Carlson, 1998; Monk & Nemirovsky, 1994; Oehrtman, Carlson, & Thompson, 2008; Thompson, 1994). This ability to reason covariationally has been shown to be critical in the understanding of functions, central concepts of calculus (Carlson, 1998; Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Thompson, 1994), and differential equations (Rasmussen, 2001; Zandieh, 2000).

In response to the continued difficulty of students constructing and modeling function relationships, we conducted an investigation of undergraduate precalculus students' developing quantitative and covariational reasoning abilities. Specifically, this investigation focused on a detailed examination of possible factors that contribute to or hinder the ability of a student to reason about quantities in a dynamic situation. It was the intention of this investigation to gain insights about the quantities that students do construct and the influence of these conceptions on reasoning about functions and covariational situations. This involved an attempt to identify various mental actions in the process of identifying quantities and the relation of these mental actions to a student's ability to conceive of functional relationships and reason covariationally with regards to these relationships. When referring to "identifying various mental actions in the process of identifying quantities," this should not be interpreted as stating that all students do identify quantities. In fact, we conjectured that the opposite is often the case and students do not necessarily identify measurable quantities of a situation. This investigation intended to identify what conceptions of quantities students possibly develop, how the conceptions of quantities and their relationships influence reasoning about functions and dynamic relationships, and what difficulties exist in the process of identifying quantities and their relationships.¹

Background

Although multiple studies have investigated students' conceptions of functions and their development and ability to reason covariationally (Carlson, 1998; Carlson, et al., 2002; Harel & Dubinsky, 1992a), the notion of a quantity is consistently mentioned but not described in each study. For instance, Dubinsky and Harel describe a *process* conception of function involving "a

¹ The term "difficulties" is meant to range from not building a developed image of the situation to attitudes and beliefs (e.g., math is not about identifying quantities).

dynamic transformation of *quantities...*” (1992b, p. 85). Covariational reasoning has been defined as “cognitive activities involved in coordinating two varying *quantities* while attending to the ways in which they change in relation to each other” (Carlson, et al., 2002, p. 354). Thus, these studies revealed student difficulties in attending to how two quantities change in tandem; however, they did not address how students come to conceptualize quantities when solving applied problems. If students are to conceive of function relationships between quantities and covariation is composed of mental actions “on” quantities, then the cognitive processes of conceptualizing quantities and their relationships have significant implications relative to student reasoning.

Quantitative reasoning (Smith III & Thompson, 2008; Thompson, 1989) offers a theory that emphasizes a learner conceiving of quantities in a manner that enables reasoning that we, as educators, would call mathematical. Quantitative reasoning refers to the mental actions of an individual conceiving of a situation (e.g., a runner in a race), constructing quantities of his or her conceived situation (e.g., time elapsed or distance ran), and both developing and reasoning about relationships between these constructed quantities (e.g., a relationship between distance ran, time elapsed, and average speed). This mental structure that is created is then a foundation from which the student can reflect upon to develop mathematical understandings and reasoning.

A *quantity*, as described by quantitative reasoning, is a conceived attribute of something that admits a measurement process, where this “something” could be an image of a situation interpreted from a problem statement or a mathematical object (e.g., a graph), admits a measurement process (Smith III & Thompson, 2008). By this definition of quantity, a quantity is a cognitive object. That is, a quantity is conceived of (not about) and this conception consists of an identified object of a situation and an attribute of the object that has some explicit or implicit

measurement process.² This definition of quantity implies that one has *cognitively* constructed an object or objectified a phenomenon that has attributes that can be measured. This cognitive distinction of a quantity stems from the stance that an individual's experience is completely his or her own. For instance, an individual could conceive of a quantity such that the conception includes or does not include the ability to imagine the quantity varying.

Thompson (1989) further defines *quantification* to be the process by which one assigns numerical values to attributes. In order for an individual to comprehend a quantity, the individual must have a mental image of an object and attributes of this object that can be measured (e.g., a car in a race with various attributes), an implicit or explicit act of measurement that produces the quantity (e.g., measuring distance traveled), and a value, which is the result of that measurement. It is in this last characteristic that the process of quantification occurs, the process of mapping a result of measurement to the characteristic. The result of identifying a number as a measurement of an attribute is said to be a *value*, opposed to be a number.

It is necessary to briefly describe what the term "result" (of measurement) refers to when describing the role of mapping a result to a characteristic. If taken literally, the term result implies an actual number (e.g., "3.3"). However, result is used here in a more general sense. Result can be used to reference both determined and undetermined magnitudes. In addition to referring to an explicit magnitude such as "3.3," result could refer to an imagined magnitude. For instance, in the process of measuring a quantity, it is possible that one may not have the

² An implicit or explicit measurement act implies that it is not actually making the measurement that results in a quantity. Rather, it is *conceived ability* to make the measurement, whether or not it is carried out, that results in a quantity.

necessary physical tools to measure the quantity but mentally they are able to imagine the measurement being made and resulting in “3.3” of something.

Additional definitions of use are a *quantitative operation* and a *quantitative relationship*.

A quantitative operation can be defined as the conception of two quantities being taken to produce a new quantity. A quantitative relationship is the image of three quantities, two of which determine the third by a quantitative operation.

With these definitions in place, Thompson (1989) has defined *quantitative reasoning* as the analysis of a situation into a *quantitative structure*, that is, a network of quantities and quantitative relationships. According to Thompson (1989), *Quantity-based arithmetic* consists of quantitative reasoning, determination of appropriate operations (inferred according to relationships among quantities) to calculate quantities' values, and the propagation of calculations. *Quantity-based algebra* is described as the same as quantity-based arithmetic, except representations of situations are under-constrained in terms of quantities' values (i.e., there is not enough numerical information to propagate calculations), some value or values are represented symbolically, and formulas are propagated instead of values being propagated.

In general, quantitative reasoning stresses the importance of students conceiving of situations and measurable quantities of a situation. Although each individual in a classroom develops understandings and images that are completely unique to that individual, this is not to say that students cannot develop understandings and images that are consistent with the instructional goals. Thus, instruction must account for the initial development of situations and quantities that the students are to reason about such that the images held by the students enable reasoning consistent with the instructional goals (e.g., covariational reasoning and modeling).

Enabling algebraic reasoning and covariational reasoning also has the potential to influence reasoning with regards to the conception and use of formulas and functions. Algebraic reasoning, and specifically formalizing algebraic reasoning, is reliant on the need of formulas, as formulas are often intended to generalize reasoning and relationships. In other words, formulas are used to fit and describe a situation in a generalized manner. Allowing students the opportunity to immerse in a situation in which they can conceptualize quantities and their relationships enables formulas and functions to emerge in a way such that they describe a conceived situation. Note that a process conception of function can somewhat “come free” through this perspective. With an image of a situation developed that includes how two quantities covary, and a function or formula that is a reflection of this image, a student has the foundations from which they can understand that the function involves “a dynamic transformation of quantities according to some repeatable means that, given the same original quantity, will always produce the same transformed quantity” (Harel & Dubinsky, 1992b, p. 85). Furthermore, with a properly developed image of a situation, the foundations are placed for a “self-evaluating” view of function that supports the ability to coordinate entire intervals of inputs and outputs. As a result from a mental model that includes covariation of the two quantities already being in place, the formula or function can be conceived in a manner such that it formalizes the covariation of the two quantities.

Methods

Subjects and Setting

The subjects for this study were six undergraduate college algebra and precalculus students from a large public university in the southwest United States. All subject participation was

voluntary and each subject was given monetary compensation for participating. All six subjects were 18-25 years of age and full-time students.

The classrooms from which the students were drawn was part of a design research study where the initial classroom intervention was informed by theory on the processes of covariational reasoning and select literature about mathematical discourse and problem-solving (Carlson & Bloom, 2005; Carlson, et al., 2002; Clark, Moore, & Carlson, accepted). The instructor of each class was the first author of this report. The classroom instruction consisted of direct instruction, whole class discussion, and collaborative activity. Each module's design was based on a conceptual analysis of the cognitive activities conjectured to be necessary to develop understandings of the module's topic. Specific topics of focus were developing conceptual understandings of rate of change, proportionality, functions, linear functions, exponential functions, logarithmic functions, rational functions, and trigonometry (at the precalculus level). It is noted that enrollment in the courses was voluntary and, upon enrollment, the class sections were not distinguishable from other sections at the university.

Data Collection and Analysis Methods

The researchers conducted clinical interviews with multiple students that consisted of tasks intended to gain insights to the subjects' quantitative and covariational reasoning abilities. The clinical interviews conducted followed the methodology described by Clement (2000) and Goldin's (2000) principles of structured, task-based interviews (e.g., four stages of free problem-solving, suggesting heuristics minimally, guiding the use of heuristics, and exploratory, metacognitive questions). Each subject was prompted by the interviewer to read each problem aloud and asked to explain his or her thinking at various times when solving each task. This interview approach allowed a method of gaining insights to the knowledge structures and

reasoning processes of the subjects through an open-ended questioning technique. This was very important in the investigation of a subject's quantifying processes, as these processes can often have the tendency to be strictly cognitive actions, which are not typically revealed in a classroom setting or collected student work. In order to promote behaviors that can be used to infer the cognitive actions of the subjects, a talk-aloud approach was included in our design of the interviews. This approach generates insight into the mental processes being performed by the subjects through the promotion of verbalizing their approach to the tasks at hand (Carlson & Bloom, 2005).

The data collected was analyzed following an open coding approach (Strauss & Corbin, 1998) with an attempt to identify emerging behaviors relative to quantifying situations. These discrete instances believed to reveal insight into a student's conception of quantities were then analyzed in an attempt to determine the mental actions that contributed to the emerging behaviors. The mathematical constructions and interactions that occurred between the subject and interviewer were examined in an attempt to model and understand the thinking of the subject. Mathematical thinking is dependent on mental operations and thus the goal was to infer, based on actions, what mental operations contribute to the behavior of the student. This data analysis approach has been described as performing a conceptual analysis of the data in an effort to accurately portray the thinking that lead to written or verbal product (Thompson, 2000).

Results

Analysis of the students' problem solving behaviors and inferred thinking on two tasks are provided, followed by a summary of overall findings. The difficulties and non-routine nature of the subjects' problem solving processes offered insight into the subjects' abilities to

conceptualize and reason about quantities that were related and covaried in a dynamic relationship.

The Box Problem (Q1)

Starting with an 11" x 13" sheet of paper, a box is formed by cutting equal-sized squares from each corner of the paper and folding the sides up.

Write a formula that predicts the volume of the box from the length of the side of the cutout.

When first engaging in Q1, all subjects participated in some use of heuristics. This ranged from drawing diagrams to using the actual paper that the task was printed on to discuss the situation. As an example, one subject (Travis) initially oriented himself to the problem by drawing a picture of a piece of paper (Figure 1) with which he denoted the square cutouts on the piece of the paper. As he drew and labeled his diagram, he discussed the diagram (Table 1).

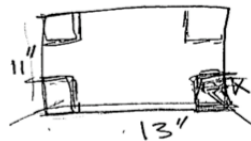


Figure 1 – Travis' Drawing of the Box

Table 1.

1	Travis:	Uh, I'm drawing the corners that are going to be cutout so they are equal
2		sides. So, I will label one of the sides of the cutouts as x , and, uh, it's 11 by
3		13, so I'll label that (<i>labeling length and width of the paper</i>). Um (<i>pause</i>) a
4		formula that predicts the volume of the box. The volume of the box is length
5		times width times height (<i>writing</i>). Um (<i>pause</i>) so we would take the length,
6		which is 13, times 11, which is the width, and then the height is x , whatever
7		the side of the length of the cutout is. And. uh, that will be the volume.

In this excerpt, Travis initially created a diagram to orient himself to the problem situation. As Travis labeled his diagram, he denoted the cutout and also the original length of the width of the 11” by 13” paper (lines 2-3). After stating what appeared to be his identified goal of the problem, “a formula that predicts the volume of the box” (line 3), Travis identified the general formula for the volume of a box (line 4) and then attempted to relate this to his conceived box (lines 5-7). However, he used the original length and width of the paper, a static measurement, as the length and width of the box. Although he appeared to identify the relationship between the height of the box and the cutout (lines 5-7), his response suggested that his conception of the length and width of the box did not include the proper relationship with the length of the cutout. Rather, his conception of length and width of the box was the fixed measurement of the original length and width of the paper. This misconception is further illustrated as the interview continued (Table 2).

Table 2.

1	Int.:	Ok, so that's the volume of what then?
2	Travis:	The box from the sheet that you originally had taken the cutout the corner of
3		and folded up.
4	Int.:	Ok, ok, so what's the 13, the 11, and the x represent again then?
5	Travis:	The 13 is the length of the paper, 11 inches is the width of the paper, a x is
6		height of the cutout from a corner.
7	Int.:	Ok, so that's the height, so why does that end up being the height of the box?
8		Why is the cutout...
9	Travis:	Um, because when you take this away from the paper and you fold this up, say
10		we took that out. This length would be the same as this length (<i>referring to the</i>

- 11 *cutout and height*), the height. When you would fold it up it would be the same
- 12 as the length of the cutout.
- 13 Int.: Ok.
- 14 Travis: So if you fold it up, then it goes up, which would be the height.
- 15 Int.: Ok. So what are the, what are the various quantities of this situation that we are
- 16 working with?
- 17 Travis: Um, quantity of, number of inches for the length, number of inches for the
- 18 width, number of inches for the height (*tracing the full dimensions of the*
- 19 *paper*).
- 20 Int.: Ok. So what do the 13 and 11 represent then again?
- 21 Travis: Number of inches of length of the side of the box, 11 is the number of inches of
- 22 the width of the side of the box, and x is the number of inches in the height of
- 23 the box.

During this interaction, Travis discussed the height of the box as a varying quantity that was equal to the length of the side of the cutout (lines 2-3 & 9-12), an image that appeared to include an image of the construction of the box relative to this dimension. His apparent understanding was revealed when he explained, “you take [the cutout] away from the paper and you fold this up... the height. When you would fold it up it would be the same as the length of the cutout.” However, Travis did not appear to conceive of the proper measurements of the length and width of the box and their relationship to the original sheet of paper and the length of the cutout, as revealed by him referring to both the length and width of the box and the length and width of the paper as 13 inches by 11 inches (lines 5-6 & 21-23). Instead, it appears that Travis had an image of a box with a static base but a height that could vary. Thus, it appears that

the dimensions of length and width were objects in this situation, with both the length and width of the paper and box being the same objects.

When working Q1 (with an 8.5"x11" paper), and similar to Travis' actions, Matt first attempted to describe how the paper related to the constructed box. However, rather than drawing a representation of the situation, Matt chose to use the paper that the task was presented on to discuss the constructed box (Table 3).

Table 3.

1	Matt: The length is going to be this (<i>pointing to the length of the paper</i>), the 11 inches,
2	the width is going to be 8 inches (<i>pointing to the width of the paper</i>), and the
3	depth, or height, is going to be x
4	Int: Ok.
5	Matt: (<i>Pause</i>) So write a formula that predict (<i>long pause</i>). So then, if you had, if you
6	had like, if you say the cutout was 1 inch, so that would mean, that would mean, 7
7	times 10 times 1. Because you know that if 1 inch has been cut off, or if .5 inches
8	have been cut off, because .5 .5 (<i>pointing to the two corners of the paper</i>), this
9	would be .5.
10	Int: Ok, so now why would that be .5 then?
11	Matt: Because you're cutting off, if the width, if the width is 8, and, if we know that
12	before the cutout, oh my...the width is 8.5.

When using the 8.5" by 11" paper to describe the width and length of the box, he referred to the same measurements as the width and length of the paper (he also referred to the width incorrectly, using 8 inches rather than 8.5 inches) (lines 1-2). Thus, it appears that he had not conceived of the proper measurement processes of the length and width of the box relative to the cutout. Next, Matt decided to use a specific cutout of 1 inch to discuss the dimensions of the

resulting box rather than a general formula (lines 6-8). When first describing the resulting box, it appears that he had reconceptualized the measurement processes of the length of the box and width of the box relative to the cutout (lines 6-7). However, he first described the situation incorrectly, as revealed by his calculating the length and width using a cutout of .5 inches rather than 1 inch. Again, he refined his image of the situation and his conceived relationship between the measurement of the length of the box and the length of the cutout, and then he corrected himself by stating that the cutout is .5 inches (lines 8-9). Finally, he caught his mistake of using 8 inches as the original width, rather than 8.5 inches (lines 11-12).

In general, it appears that Matt continually refined his image of the situation during this interaction, including the various measurement processes of multiple quantities in relation to other quantities. It appears that Matt did not initially conceive of a measurement process for the length and width of the box. Instead, he saw the length and width of the paper as fixed values that were useful for computing the volume of the box. His inability to conceptualize these two quantities as distinct from the original sheet of paper indicates that he did not view the length and width of the box as assuming various measurements. As the interaction above revealed, Matt used a crude representation of the situation by quickly describing, but not recording, the dimensions of his conceived box using the 8.5" by 11" sheet of paper. However, as he worked through the task and explained his reasoning, he refined his image of the box to include the correct quantitative relationship of the quantities of a specific box and the length of a cutout.

As revealed in both of the interactions with Matt and Travis, a situation that may be interpreted as a simple context by a teacher offered much conceptual difficulty to both subjects. The finding that the subjects did not construct a correct image of quantities and their relationships was common across all six subjects. For instance, one subject referred to the length

of the paper, width of the paper, length of the cutout, length of the paper, and width of the paper as “the length.” Although this could easily be interpreted as a poor choice of wording on the subject’s part, the subject’s difficulties in solving the problem revealed that she likely had not imagined quantities and their variation; nor had she developed an image of the relationships among quantities in the problem context. Thus, the student appeared to view each dimension as an object. This was followed by her interchanging various dimensions as she continued working the task, implying she had not cognitively distinguished the various “lengths.”

As each subject continued to work each task and receive probing from the interviewer, each subject found the need to refine his or her image of the situation. Once this refinement was completed, each subject then had minimal difficulty determining the correct volume formula and describing the formula by referencing the quantities they conceived and relationships between these quantities.

As an example of a subject refining his image of the situation, consider the actions of Travis immediately after the interaction in Table 2. Travis was first asked about the relationship between the sides of the box and the sides of the paper. Immediately, Travis stated that he “did it wrong” and noted that “cutting out a piece” would “reduce the length and the width,” implying that he had altered his image of the situation. Then, unprompted, he described how this understanding would result in changes to his characterization of the dimensions of the box and hence the formula for the volume of the box (Table 4).

Table 4.

1	Travis:	So, the actual, the actual length would be 13 minus 2 x , because there's two
2		cutouts on the side, so you have to multiply the x by 2. So that would be
3		length, width would be 11 inches minus 2 x , and the height would be x again.

-
- 4 *(pause)*
- 5 Int.: Ok, so say again real quick, why the $13 - 2x$, and $11 - 2x$.
- 6 Travis: Um *(pause)* because on one side of the paper, when you would take the
- 7 cutouts, you would take two cutouts and one cutout is x , is the length of one
- 8 side of the cutout *(pointing to the cutout in the diagram)*. So, since you're
- 9 taking two away, you'd have to subtract it by $2x$'s, by two cutouts. And the
- 10 same thing with the width, two times also.
- 11 Int.: Ok, so what would the resulting volume be then, of the box?
- 12 Travis: Um, it would be the length *(pause)* times the width times the height.
- 13 Int.: Ok, so what does x again represent here?
- 14 Travis: That's, uh, the length, or the number of inches in the side of a cutout of the
- 15 paper.
-

As a result of his new, and accurate (from our observing perspective), characterization of the length and width of the base of the box as varying with the length of the side of the cutout, Travis was able to correctly construct the algebraic expressions for the length and width of the box (lines 1-3 & 6-10). Furthermore, it appears that he had conceived of these algebraic expressions as representations of measurements of his conceived situation. When describing why $13 - 2x$ and $11 - 2x$ would represent the length and width of the box, respectively, he described the terms of these expressions in relation to his new image of measurements of the physical situation. For instance, he described that subtracting by $2x$'s resulted from taking two cutouts away, where x represented the length of one side of the cutout (lines 7-9). Also, Travis understood x as representing *the number of inches* of the length of the side of a cutout (lines 14-15). Thus, it appears that Travis had quantified the length of the side of a cutout and also had

developed a correct understanding of the relationships between the dimensions of the box and this measurement. Travis apparently developed this understanding after his realization that the cutout length would reduce the length and width of the box base. He also appeared to be able to covary the cutout length with the box's base length, base width, height and volume. Thus, Travis' constructed quantitative structure included the proper relationships between the various quantities of the situation and how they covaried.

Overall, Q1 reveals the importance of a problem solver having the correct mental image of a situation when attempting to reason about quantities and their relationships (e.g., developing quantitative relationships/structures), including the development of meaningful mathematical formulas that characterize those relationships. In the case of both subjects presented above, the creation of an expression to represent the volume was driven by their images of the situation. Hence, with incorrect images of the situation, incorrect relationships were constructed and incorrect algebraic representations for the volume were produced. It is also important to note that the subjects had little difficulty creating and explaining their incorrect formulas, and thus the mistakes appeared to be solely lying within their constructed image of the situation and resulting relationships (e.g., lacking an appropriately quantified length and width of the box). When their mental image of the situation was reconstructed, the subjects easily altered their volume expressions and were able to describe the formulas produced in terms of the physical relationships and measurements. That is, it was not the case that the subjects lacked the reasoning abilities to create the proper formulas; rather, the formulas they created, correct or incorrect from our point of view, were directly a result of their constructed image of the situation.

The Ferris Wheel Problem (Q2)

Consider a Ferris wheel with a radius of 36 feet that takes 1.2 minutes to complete a full rotation. April boards the Ferris wheel and begins a continuous ride on the Ferris wheel. If the platform to board the Ferris wheel is 8 feet off of the ground, sketch a graph that relates the total distance traveled by April and her vertical distance from the ground.

Q2 provided additional insights into the subjects' quantitative reasoning abilities.

Prompts for them to produce and explain their graphical representation of the covariation of conceived quantities provided additional insights into the subjects' abilities to reason dynamically about quantities and their values. We also note that only three of the subjects were given this task during their interview. When first attempting Q2, none of the three subjects drew a diagram of the situation to discuss their reasoning. Each individual correctly identified the minimum vertical distance from the ground as 8 feet because of the platform. However, Charles explained that 72 feet was the maximum vertical distance (the diameter of the Ferris wheel), while Brad explained that 36 and 42 feet was the maximum vertical distance (the radius of the Ferris wheel and an incorrect addition of the radius and 8 feet), and Travis explained that 64 feet was the maximum vertical distance (the difference of the diameter of the Ferris wheel and the 8 feet). As the subjects continued to work the problem each eventually identified the correct maximum value for the vertical distance, while also referencing the quantities of the initial vertical distance and the diameter of the Ferris wheel. Thus, it appears that although all three subjects identified the initial height stated in the problem statement, they did not conceive of this height in relation to the location of the Ferris wheel and the individual's height *from the ground*.

In producing a graph, all three subjects drew graphs that correctly characterized April's increasing and decreasing distance from the ground as the distance that she traveled around the circumference of the Ferris wheel increased. However, the concavity of all three subjects' graphs

of the covariation of April's distance from the ground and her distance traveled was incorrect. When asked to verbally explain the meaning of his or her graph, each subject explained that as the total distance increased, April's vertical distance from the ground increased and then decreased. Even when prompted they did not explain how these two quantities' values changed in tandem beyond explaining April's distance from the ground increasing or decreasing as her distance traveled increased. Thus, it appears that at this point in their solutions, their graphs were based solely on a directional covariation of the two quantities. Important to note though was that each subject did refer to the distance traveled as increasing, inferring that they had identified this attribute of the situation. This is important as the graph could be produced in an iconic manner by transferring April riding "up" and then "down" the Ferris wheel without explicitly reasoning about her distance traveled.

After each subject was asked to explain why he drew a "curved" graph, they all responded with a focus on the amounts of change of the total distance and vertical height. As an example Charles first drew a graph that was concave down for the first three-quarters of a revolution after April boarded at the bottom of a Ferris wheel. His subsequent explanation described that his graph was concave down because, "the change in output is decreasing, as, for every equal amount of input (*denoting changes of input and output on his graph*)." Charles' explanation was correct relative to the concavity of *his produced graph*. Yet, the graph itself was incorrect relative to the situation over the first quarter of a revolution (e.g., concave down opposed to concave up). Thus, it appears that his graph was driving his description of the situation. In response, the interviewer asked Charles to explain the shape of his graph using the Ferris wheel (Table 5).

Table 5.

1 Charles: So, as the total distance is increasing (*tracing the arc-length*), we
2 notice that height is increasing but for every successive change of
3 total distance (*making marks at equal changes of arc-length*), lets say
4 right here it's eight, well if I drew a bigger one, I'd be able to show it
5 more precise.

6 Int: Here, go ahead and, uh, I've got some extra pieces of paper. Go ahead
7 and if you want to draw it on there somewhere (*handing him a sheet*
8 *of paper*) a little bigger.

9 Charles: (*Drawing a larger circle and drawing a vertical-horizontal crosshair*
10 *in the middle of the circle*) So, as, I guess we can assume this is
11 ninety degrees (*referring to the compass*) we can make an angle
12 (*attempts to use protractor on compass*)...

13 Int: So what are you trying to do right now?

14 Charles: Well I see, this thing moves (*referring to the compass*), I'm trying to
15 show that, um, I'm trying to make, well I could use the protractor,
16 I'm just trying to change, show successive change in input.

17 Int: Could you use the Wikki Stix to do that?

18 Charles: Well, actually, yes I can.

19 Int: So, you're trying to show successive changes in what?

20 Charles: In input, which would be the total distance (*marks a distance on a*
21 *Wikki Stix mumbling to himself, then marking off successive arc-*
22 *lengths on the circle*). Ok, so, for every change in total distance, right
23 here (*referring to arc-length*), he, well, hmmm.

- 24 Int: What makes you go hmmm?
- 25 Charles: Because I was thinking the last time I did this represented right
- 26 (*referring to the top-half of the circle*), hmm.
- 27 Int: So what's making you go hmm now?
- 28 Charles: Because it seems as total distance increases (*referring to the arc-*
- 29 *length*), the actual change in the height is increasing instead of
- 30 decreasing.

In this excerpt, Charles reiterated the directional covariation of the total distance traveled and the height of the individual (lines 1-3). Next, in an attempt to explain the curvature of his graph, Charles described that he wished to use successive changes of input, which he identified as the distance traveled (line 20), and that he could do this by segmenting the arc-length of the Ferris wheel (lines 9-12 & 14-16). Then, after marking off equal intervals of distance traveled on his diagram of the Ferris wheel, Charles questioned his original graph (lines 20-23). This resulted in Charles describing that the changes in height should be increasing rather than decreasing (lines 28-30).

In this case, Charles' refined image of the situation was the basis for his explanation relative to corresponding amounts of change of the quantities and how these would be reflected in the graph. On the other hand, Charles' first description of the situation relative to corresponding amounts of change of the quantities was driven by the graph, as opposed to his image of the situation. That is, he produced a graph based on a directional image of the situation and *then* interpreted the graph relative to amounts of change. In both cases, his description of *his* graph was correct. However, in the first instance, his graph appears to have shaped his description of the covariation of the two quantities; in the second instance, his constructed

contextual image of the relationship between the two quantities was the basis for his reasoning when he constructed his graph and explained the covariation of the quantities characterized by this graph. Thus, similar to the observations of the subjects when responding to Q1, the “incorrectness” of Charles’ first graph was not a result of him having weak reasoning abilities. Rather, in this case, it was the focus of Charles attention that influenced his description of the relationship between the quantities.

These behaviors by Charles highlight the role of quantitative reasoning and covariational reasoning in representing the relationship between two quantities. In order to represent the (correct) relationship between two covarying quantities, a student must first construct an image of the situation, including attributes of the situation that can be quantified such that these conceptions enable the student to correctly constructing and representing the relationship between quantities. Although Charles correctly identified the directional relationship of distance traveled and distance from the ground, his image of the situation did not initially include a focus on the covariation of the amounts of change of the distance traveled and the amounts of change of the distance from the ground. Charles’ actions also revealed the difficulty of covarying two quantities’ values. Charles’ descriptions remained attentive to the quantities he was relating, but he was not able to produce a correct graph (from an observer’s perspective) until he focused on specific intervals of arc-length relative to the contextual situation. Thus, similar to the findings from Q1, in this case it was not an inability to engage in the reasoning needed to successfully complete the task. Rather, his products were tied to his incorrect image of the situation, which he eventually refined over the course of completing the task.

Summary

The images that the subjects constructed of each of the task situations appeared to influence the subjects' subsequent actions and products. As one example, when responding to Q1, multiple subjects did not initially conceive of the length and width of the box as differing from the length and width of the paper; nor did they develop an image of the quantitative relationships between the length and width of the box, the length of the cutout, and the length and width of the paper that included an image of their covariation as the length of the side of the cutout took on different values. In the absence of this image, it is unreasonable to expect the subject to construct a correct and meaningful algebraic or graphical model of the situation.

Although the subjects' initial images of the situation were not consistent with the intended situation, each subject was able to continually refine her or his image to produce correct solutions and provide meaningful descriptions of the quantities and their relationships. This finding would not be revealed in a typical testing or homework setting, and emphasizes the necessity of curriculum and instruction to support students' in learning to identify and conceptualize relevant quantities and construct relationships between these quantities before formalizing these relationships through the use of formulas.

Discussion and Implications

This research provides information about the modeling behaviors of students and should therefore be useful to curriculum designers and instructors. Even though it is common practice to ask undergraduate mathematics students to work problems set in contextual situations, the curriculum and instruction rarely attends to the development of students' ability to conceptualize the relevant quantities and their relationships. This research suggests that supporting students in learning to create their own quantitative structure of a problem context could be extremely

valuable to their success in creating formulas and providing meaningful descriptions of their solutions and graphs.

In order to promote students developing cognitive images of situations such that these images promote mathematical reasoning, gaining further insight into students' images of quantities and relationships between quantities will be an important focus of our future research. As discussed previously, an understanding of a situation that includes a coherent image of quantities and their relationships can possibly promote algebraic (mathematical) reasoning. Thus, it is important to not only investigate the mental actions that take place as students conceive of problem situations and problem goals, but also investigate how these conceptions can influence reasoning as students attempt to "formalize" their conceived relationships. Students may benefit if curriculum designers and teachers become more aware of these mental processes and their importance. Mathematical models should be a reflection of the situation, or a way of formalizing the relationships of the situation, where these relationships are developed cognitively by the solver and then formalized. If algebraic models are presented without properly developed cognitive models of the contextual situation under investigation, the algebraic model will likely fail to be meaningful to the students; and if we expect students to find mathematics as a way to reason about the experiential world, then it is necessary that mathematical representations be meaningful relative to the experiential world, or this expectation is highly unrealistic.

References

Carlson, M. (1998). A cross-sectional investigation of the development of the function concept.

In E. Dubinsky, A. H. Shoenfeld & J. J. Kaput (Eds.), *Research in collegiate mathematics education, III. Issues in Mathematics Education* (Vol. 7, pp. 115-162).

- Carlson, M., & Bloom, I. (2005). The cyclic nature of problem solving: An emergent multidimensional problem-solving framework. *Educational Studies in Mathematics*, 58(1), 45-75.
- Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, 33(5), 352-378.
- Clark, P. G., Moore, K. C., & Carlson, M. P. (accepted). Documenting the emergence of "speaking with meaning" as a sociomathematical norm in professional learning community discourse. *The Journal of Mathematical Behavior*.
- Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of Research Design in Mathematics and Science Education* (pp. 547-589). Mahwah, NJ: Lawrence Erlbaum Associates, Inc.
- Goldin, G. A. (2000). A scientific perspective on structured, task-based interviews in mathematics education research. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of Research Design in Mathematics and Science Education* (pp. 517-545). Mahwah, NJ: Lawrence Erlbaum Associates, Inc.
- Harel, G., & Dubinsky, E. (1992a). The concept of function: Aspects of epistemology and pedagogy, *MAA Notes* (Vol. 25). Washington, D.C.: Mathematical Association of America.
- Harel, G., & Dubinsky, E. (1992b). The nature of the process conception of function. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (pp. 85-106). Washington, D.C.: Mathematical Association of America.

- Kaput, J. J. (1994). Democratizing access to calculus: New routes to old roots. In A. H. Schoenfeld (Ed.), *Mathematics and cognitive science* (pp. 77-156). Washington, DC: Mathematical Association of America.
- Monk, S. (1992). *A study of calculus students' constructions of functional situations: The case of the shadow problem*. Paper presented at the Annual Meeting of the American Educational Research Association.
- Monk, S., & Nemirovsky, R. (1994). The case of Dan: Student construction of a functional situation through visual attributes. *CBMS Issues in Mathematics Education*, 4, 139-168.
- NCTM (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: NCTM.
- NCTM (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.
- Oehrtman, M., Carlson, M., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' function understanding. In M. P. Carlson & C. Rasmussen (Eds.), *Making the Connection: Research and Teaching in Undergraduate Mathematics Education* (pp. 27-42). Washington, D.C.: Mathematical Association of America.
- Rasmussen, C. L. (2001). New directions in differential equations. A framework for interpreting students' understandings and difficulties. *Journal of Mathematical Behavior*, 20, 55-87.
- Sfard, A. (1992). Operational origins of mathematical objects and the quandary of reification-The case of function. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy*, *MAA Notes*, Vol. 25 (pp. 59-84). Washington, DC: Mathematical Association of America.

- Smith III, J., & Thompson, P. W. (2008). Quantitative reasoning and the development of algebraic reasoning. In J. J. Kaput, D. W. Carraher & M. L. Blanton (Eds.), *Algebra in the Early Grades* (pp. 95-132). New York, NY: Lawrence Erlbaum Associates.
- Strauss, A. L., & Corbin, J. M. (1998). *Basics of qualitative research: Techniques and procedures for developing grounded theory* (2nd ed.). Thousand Oaks: Sage Publications.
- Thompson, P. W. (1989). *A cognitive model of quantity-based algebraic reasoning*. Paper presented at the Annual Meeting of the American Educational Research Association.
- Thompson, P. W. (1994). Students, functions, and the undergraduate curriculum. In E. Dubinsky, A. H. Schoenfeld & J. J. Kaput (Eds.), *Research in Collegiate Mathematics Education: Issues in Mathematics Education* (Vol. 4, pp. 21-44). Providence, RI: American Mathematical Society.
- Thompson, P. W. (2000). Radical constructivism: Reflections and directions. In L. P. Steffe & P. W. Thompson (Eds.), *Radical constructivism in action: Building on the pioneering work of Ernst von Glasersfeld* (pp. 412-448). London: Falmer Press.
- Thorpe, J. A. (1989). Algebra: What should we teach and how should we teach it? In S. Wagner & C. Kieran (Eds.), *Research issues in the learning and teaching of algebra* (pp. 11-24). Reston, VA: National Council of Teachers of Mathematics.
- Vinner, S., & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education*, 20, 356-366.
- Zandieh, M. (2000). A theoretical framework for analyzing student understanding of the concept of derivative. In E. Dubinski, A. H. Schoenfeld & J. J. Kaput (Eds.), *Reserach in collegiate mathematics education, IV* (Vol. 8, pp. 103-127). Providence, RI: American Mathematical Society.