An Investigation Into Precalculus Students' Conceptions of Angle Measure

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Introduction

Trigonometry and trigonometric functions have been important parts of the high school and undergraduate mathematics and science curriculum for the past century. Trigonometry and trigonometric functions also offer one of the earlier mathematical experiences for students that combine geometric, algebraic, and graphical reasoning with functions that cannot be computed through algebraic computations. Though trigonometry has been a part of the mathematics and science curriculum for over a century and can offer rich student mathematical experiences, it is frequently the case that students and teachers have difficulty reasoning about topics reliant on trigonometric function understandings (Brown, 2005; Fi, 2003; Thompson, Carlson, & Silverman, 2007; Weber, 2005). To further complicate things, few studies have inquired into the reasoning abilities and understandings that are foundational in developing trigonometric understandings.

If students are to develop trigonometric understandings, it is implied that they also develop understandings of the "things" that trigonometric functions are about. Although trigonometric functions are often introduced in multiple contexts (e.g., triangle trigonometry and unit circle trigonometry) for somewhat different purposes (e.g., determining the side of a triangle or finding a coordinate), there are common foundations to the use of trigonometric functions in each context. As an example, angle measure presents a commonality between both contexts. As Thompson (2008) has recently argued, the meanings and foundations that are common between

Acknowledgement: Research reported in this paper was supported by National Science Foundation Grants No. EHR-0412537. Any conclusions or recommendations stated here are those of the authors and do not necessarily reflect official positions of NSF.

the various trigonometry contexts, such as angle measure, should be built upon in order to promote coherent student understandings across the trigonometry contexts.

What follows in this report is a description of an approach to trigonometry intended to promote coherent student trigonometric understandings by developing foundational understandings of angle measure. In addition to the suggestions by Thompson (2008) relative to angle measure, the described approach to angle measure and trigonometry is informed by theories of quantitative reasoning and covariational reasoning (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Smith III & Thompson, 2008). This report also describes findings from an investigation into student conceptions of angle measure in the setting of an instructional sequence whose design was based on the described approach to angle measure. This investigation consisted of a teaching experiment that included multiple individual interviews and three teaching sessions with subjects from the undergraduate precalculus level. Through the use of a teaching experiment and individual interviews, insights were gained into students' developing images of angle measure.

Background

The difficulties that students encounter in developing coherent trigonometric understandings is likely multifaceted. First, as Thompson (2008) suggests, trigonometry may be a difficult topic for students due to an incoherence of previous understandings and a lack of developed reasoning abilities that are foundational to building understandings of trigonometry. If students are to develop strong understandings of trigonometry and trigonometric functions, then they *must* also construct deep understandings of the mathematical objects (e.g., angle measure and ratios of lengths) trigonometry and trigonometric functions are about. Also, the understandings students construct must support the construction of coherent and connected understandings of trigonometric concepts.

Trigonometry may also be a difficult topic for students because trigonometric functions require that students develop a process view of function (Harel & Dubinsky, 1992; Oehrtman, Carlson, & Thompson, 2008). In addition to the necessity of reasoning about a context, trigonometric functions are often one of a student's initial experiences with a function that cannot be computationally evaluated. Hence, reasoning about trigonometric functions relies on an individual constructing an image of a situation and two quantities composing the situation in a manner that the individual can imagine the relationship between the values of the two quantities without actually performing computations.

Quantitative reasoning (Smith III & Thompson, 2008; Thompson, 1989) offers a theory that calls attention to the need for developing foundational reasoning abilities and student images regardless of the mathematical topic of interest. Quantitative reasoning describes the mental actions of an individual conceiving of a situation (e.g., motion on a circle), constructing quantities of the conceived situation (e.g., angle measure and a vertical position), and both developing and reasoning about relationships between these constructed quantities.

A *quantity* is defined as being a conceived attribute of something (e.g., this "something" could be an image of a conceived situation) that admits a measurement process (Thompson, 1989). For instance, relative to the focus of this research, an angle is an object (the "something") that has a measurable attribute of openness. Note that a conceived quantity can and will differ from individual to individual. For instance, an individual may conceive of an angle's openness such that they do or do not imagine an arc-length subtended by the angle.

In order for an individual to comprehend a quantity, the individual must have a mental image of an object and attributes of this object that can be measured (e.g., an object traveling on a circular path with attributes of velocity and arc-length traveled), an implicit or explicit act of measurement that produces the quantity (e.g., measuring the arc-length traveled or the ratio of change of distance to the change of time), and a value, which may vary, that is the result of that measurement (e.g., two radians or 3π radians per minute). Also, note that a quantity may be the result of a conceived relationship between two other quantities (e.g., velocity relative to distance and time traveled).

In general, quantitative reasoning stresses the importance of students conceiving of situations and measurable quantities of a situation. Although each individual in a classroom develops understandings and images that are completely unique to that individual, this is not to say that students cannot develop understandings and images that are consistent with the instructional goals. Thus, instruction must account for the initial development of situations and quantities that the students are to reason about such that the images held by the students enable reasoning consistent with the instructional goals. For instance, trigonometric functions are reliant on angle measure as an input. Hence, if students are to reason about trigonometric functions as describing the covariation of two quantities, the students must first obtain an image of an angle and angle measure that includes the ability for the angle measure to vary.

An Approach to Angle Measure and Trigonometry

Above, angle measure was described as a possible common foundation between the various trigonometry contexts. I imagine many would argue, and argue legitimately, that current approaches to trigonometry are already reliant on angle measure. However, this reliance does not necessarily imply that trigonometry is developed on foundations of angle measure, or that angle measure itself is fully developed. I do not mean to imply that angle measures are not often found in mathematics curriculum, but instead that the measure of the angle is often used for a different purpose than as a measurement or developing trigonometric functions. For instance, in triangle trigonometry, an angle is typically used in a manner such that it exists only somewhere within a triangle of interest. In this case, the measure of the angle is used to reference a static object opposed to explicitly identifying a measurement of an attribute of the object that could vary.

In addition to the use of angle measure as referencing an object in triangle trigonometry, the US mathematics curriculum's approach to angle measure is inconsistent and lacks an explicit focus on the *mathematical process* of angle measure. Upon the introduction of angles, angle measure is often described as an amount of a rotation or a measurement of an arc-length. At this time, the degree is often imposed as the common unit of angle measurement and students are told that a protractor is used to measure the angle. However, the *process* by which an angle's measure in degrees is determined is rarely addressed beyond the *use* of a protractor that has already been created by someone other than the student. The ability to *use* a protractor is significantly different than understanding the *process* by which a protractor is created in relation to measuring an angle. Thus, although a student may understand that an angle measure references an arc-length or an amount of rotation, this understanding of the student may not include the mathematical structure behind angle measurement. This lack of understanding can create multiple conceptual hurdles for the student. For instance, in the case that a student solely understands "angle measure measures arc-length," if circles of different radii are used to measure the angle, the linear magnitude of the arc-length varies, possibly resulting in a perturbation in the student's image of angle measure.

Although the approach to angle measure in degrees is frequently left vague in terms of the process by which the measurement is based, angle measurement in radians is often explicitly described as a measurement of an intersected or subtended arc-length in a number of radii. This description of angle measure is significantly different than the typical descriptions of angle measure in degrees first presented to students. For instance, the “magnitude” of the unit of measurement is explicitly defined in the case of radian measurement. A possible result of this is that students spend many years constructing conceptions of angle measure given in degrees, but are then asked to develop understandings of angle measure in radians that contradict or are not related to their current conceptions of degree angle measures.

This treatment of angle measure may explain the research findings of teachers and students lacking meaningful understandings of the radian as a unit of angle measure and that students and teachers appear much more comfortable with degree angle measurements, although the studies do not discuss what the subjects understood relative to degree measurements. As an example of a common research finding, Fi (2003, 2006) observed that teachers could often easily convert between radian and degree angle measures, but they were unable to give a well-defined description of radian measure beyond this conversion. Also, multiple studies (Akkoc, 2008; Fi, 2003, 2006; Tall & Vinner, 1981; Topçu, Kertil, Akkoç, Kamil, & Osman, 2006) reported that teachers did not view π as a real number when discussed in a trigonometry context. Rather, they were observed graphing π radians as equal to 180, where other teachers described π as the unit for radian measure (e.g., a radian is so many multiples of π).

This finding that individuals hold multiple conceptions of the value of π and limited understandings of the radian unit of measurement may be due to an impoverished conception of what attribute angle measure refers to and the process angle measure is based on. Rather than having an understanding of the number as a value that represents a measurement of an attribute of an angle, an individual's understanding of angle measure may only include an image of an object or a position on a circle. A conception of angle measure as a position or an object does not necessarily entail a measurement of anything. In such a case, an individual's ability to reason about trigonometric functions may become reliant on their ability to memorize positions on a circle or specific triangles. For instance, Weber (2005) observed that students could not approximate $\sin(\theta)$ for various values of θ. Instead, students often stated that they were not given enough information to accomplish this task and that they needed an appropriately labeled triangle.

In response to these shortcomings in student reasoning and the apparent incoherence present in the US mathematics curriculum on angle measure, an instructional sequence was designed based on a conceptual analysis (Thompson, 2008) of the lesson concepts to be covered. This conceptual analysis attempted to 1) describe ways of knowing that are immediately *and* developmentally beneficial for learning and 2) analyze ways of understanding a body of ideas based on describing the coherence between their meanings. This conceptual analysis is briefly discussed to set a foundation for the data analysis presented. I also acknowledge that much of this conceptual analysis is based on the groundwork laid by Thompson (2008). The body of ideas presented revolves around the following threads:

- The process of measuring of the openness of an angle is based on determining the subtended arc-length's fraction of the corresponding circle's circumference.
- The unit of a radian as the ratio of a linear measurement of arc-length to the length of a radius, giving a number of radius lengths
- The construction of the unit circle using the radius as a unit of measurement.

- Sine and cosine functions with an input quantity of angle measure, measured in radians, and an output quantity of a ratio of lengths.

Developing students' images of angle measurement as a measurement based on an arc-length's fraction of circumference can possibly avoid a vagueness of angle measurement while also preparing students for the multiple contexts of trigonometry. From this perspective, angle measurements given in degrees and radians can both be presented as based on a fraction of a circle's circumference. In terms of degrees this implies the focus of one degree referring to an the arc-length of a circle that is $1/360^{th}$ of the circumference of a circle; in terms of radians, this implies the focus of one radian referring to an arc-length of a circle that is $1/2\pi^{th}$ of the circumference of a circle. Regardless of the unit of angle measurement, there are a fixed total number of units that rotate any circle centered at the vertex of the angle and measuring the openness of an angle involves determining the fraction of the total circumference of a circle and how many of the total units correspond to this fraction.

With angle measure based on the quantitative relationship of an arc-length's fraction of circumference, an angle's measurement varies when the fraction of the circumference subtended by the angle varies. Rather than an angle and its measure of openness being one static object in a right triangle or a position on a circle, the openness of an angle is a measurable attribute (e.g., a quantity) of the angle that can vary.

An important facet to this approach to angle measure is that it is not reliant on the radius of the circle that is chosen, as long as the circle is centered at the vertex of the angle. Although a linear measurement of the magnitude of the unit of measurement may vary when considering circles of different radius, the fraction of the circumference subtended by the angle, regardless of the circle, remains constant.

In addition to an arc-length's fraction of a circle's circumference, an angle measurement made in radians is the result of the quantitative operation of determining the ratio between a length (e.g., arc-length) and the length of a radius. Thus, if an angle has a measurement of π/4 radians, the length of the arc subtended by the angle is π/4$^{\text{ths}}$ the radius used to create the arc. Note the focus here remains on an arc-length subtended by the angle as a measurable quantity. The focus of the angle measurement is not on a location on a circle, which may often be inferred by a student from the labeling of the unit circle. Also, it is by convention radian measurements are provided in multiples of π, but it is important to remain aware that these conventional radian measurements are measurements of an angle's openness (e.g., values) opposed to locations on a circle.

The use of a radius as a unit of measurement is also important to *construction* of the unit circle. Textbooks often claim the unit circle to be a circle with a radius of $r = 1$ with no further explanation. As Weber (2005) suggests, we must promote that students understand the process by which the unit circle is produced if we wish the students to attach the meanings intended to the unit circle. With the length of one radius as a unit of measurement, all circles can be considered as the unit circle as all have a radius of length one radius. As a consequence, rather than trigonometric functions being related to only a circle of $r = 1$, where 1 is not presented as a multiplicative comparison of lengths, trigonometric functions are connected to any circle through the use of the length of a radius of the circle as a unit of measurement.

At this time, sine and cosine can be defined as *functions* that have an input of angle measure, in radians, and an output that is a multiplicative comparison of lengths. This definition, along with the ideas of angle measurement presented above, allows the development of cosine and sine coherently in each context (Figures 1-3). Relative to the unit circle, the output of cosine is the abscissa of the terminus of the arc subtended by the angle and the output of sine is the ordinate of the terminus of the arc subtended by the angle, with both measured as a fraction of one radius (a ratio of lengths). With regards to right triangle trigonometry, sine and cosine have an input of angle measure, measured in radians, and output of a fraction of the hypotenuse (where the hypotenuse of a right triangle can be used to form a radius). The output of sine and cosine is a value (formed by a ratio) regardless of the context and as the radius of the circle increases (the hypotenuse of the right triangle), the outputs of cosine and sine remain constant due to triangle similarity. Furthermore, if the radius (length of the hypotenuse) is held constant and the angle measure varies, the output values of sine and cosine vary accordingly.

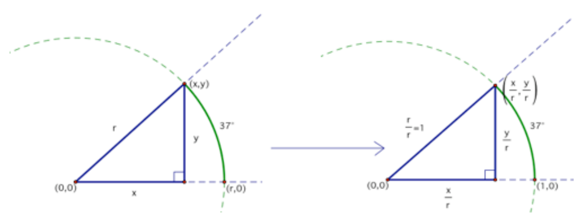

Figure 1 - An arc-length image of angle measure and its connections to unit circle trigonometry

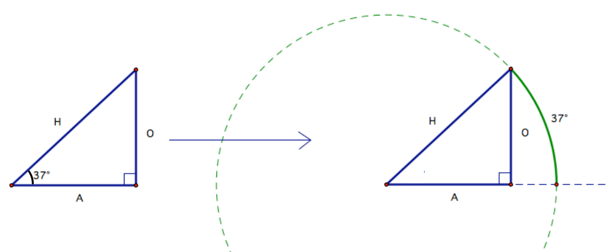

Figure 2 - An arc-length image of angle measure and its connections to triangle trigonometry

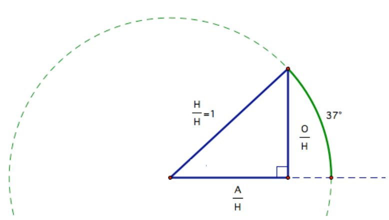

Figure 3 - An arc-length image of angle measure and its connections to triangle trigonometry and unit circle trigonometry

Methods of the Study

The purpose of this study was to gain insights to student understandings of angle measure, the radian as a unit of measurement, and the construction of the unit circle in light of the described approach to trigonometry presented above.

Subjects and Setting

The study was conducted with three students from an undergraduate precalculus course at a large public university in the southwest United States in which the researcher (myself) was the instructor. The subjects were chosen on a volunteer basis and monetarily compensated. The precalculus classroom from which the subjects were drawn was part of a design research study where the initial classroom intervention was informed by theory on the processes of covariational reasoning and select literature about mathematical discourse and problem-solving (Carlson & Bloom, 2005; Carlson, et al., 2002; Clark, Moore, & Carlson, accepted).

All three subjects were full-time students at the time of the study. Travis was a Caucasian male in his mid-twenties, an architecture student, and received a high “C” as his final grade. Brad was a Caucasian male in his late teens, a computer systems engineer major, and received a “C” as his final grade. Charles was a Caucasian male in his late teens, a psychology major, and received a low “A” as his final grade.

Data Collection Methods

The design of the study included a 75 minute pre-interview with each subject, a three session teaching experiment with each session lasting approximately 65 minutes, a 90 minute exploratory teaching interview with each subject after the first teaching session, and a 90 minute post-interview with each subject.

The pre- and post-interviews followed the design of a clinical interview (Clement, 2000) and Goldin's (2000) principles of structured, task-based interviews (e.g., four stages of free problem-solving, suggesting heuristics minimally, guiding the use of heuristics, and exploratory, metacognitive questions). The interviews were conducted with each subject before the teaching experiment and immediately after the teaching experiment.

The pre-interviews were composed of tasks intended to gain insight into the problem solving and quantitative reasoning abilities of the students. The pre-interviews also included multiple problems intended to gain insights into the subjects' (pre)conceptions of angle measure. The post-interviews followed the same clinical interview approach as the pre-interviews. However, the design of the post-interviews was solely focused on the instructional topics of the teaching experiment. Thus, the interviews were intended to gain insights into the resulting understandings and conceptions the students developed during the teaching experiment.

A teaching experiment (Steffe & Thompson, 2000) was conducted with the three subjects and consisted of three 65 minute teaching sessions within a span of eight days. Each teaching session included all three subjects, the instructor (myself), and an observer. Immediately after each session, I debriefed with the observer to discuss various observations during the teaching sessions and possible refinements to make to the future teaching sessions and interviews.

Exploratory teaching interviews were conducted with each student between the first and second teaching sessions. The focus of these interviews was on angle measure (in degrees). The purpose of the exploratory teaching interviews was to gain additional insights into the developing conceptions of the subjects. The interviews followed Goldin's (2000) principles of structured, task-based interviews with one significant addition: the interviews also involved instruction and the posing of additional questions based on the actions of the subjects. This allowed additional insight to the possible limitations in the subjects' current ways of thinking and the necessary constructions of the students to overcome these limitations.

The interviews and teaching sessions were analyzed following an open and axial coding approach (Strauss & Corbin, 1998). The coding was intended to identify emerging student behaviors and patterns or connections between these behaviors. Instances believed to reveal insights into a student reasoning and understanding were analyzed in an attempt to determine the mental actions that contributed to the emerging behaviors and possible patterns in the students' mental actions.

Results

This section provides an overview of findings into the investigation of three students' conceptions of angle measure. First, their initial conceptions of angle measure are characterized. This is followed by data and a discussion that illustrates how these three students refined their conceptions of angle measure over the course of the teaching experiment. The section concludes with a discussion of the common conceptions and conceptual obstacles that these students encountered as they engaged with interview tasks.

Pre-interviews

In general, each subject had difficulty explaining the meaning of performed calculations and why they chose to make various calculations during the pre-interviews. Charles and Travis often justified their calculations by checking units and Brad often described calculations he could make by using the result of his calculations. Also, Charles self-admittedly often attempted to recall formulas from memory. However, this recollection did not appear to be quantitative in nature. As an example, he recalled a formula that was correct symbolically ($s = r\theta$), but he was unable to describe why this formula related the quantities of interest and used an incorrect measurement unit for one of the values (a degree measurement for θ).

Relative to angle measure, all three subjects appear to have held a loose coordination of arc-length and angle measure. Both Travis and Brad reasoned about angle measurement by using a specific circle to determine a linear arc-length measurement that would correspond to one degree. This way of reasoning allowed both to solve a problem that asked for an angle measurement given an arc-length measurement and the radius of a circle. However, this way of reasoning about one degree of angle measure as referring to a linear length, rather than a proportion, offered obstacles that the students did not overcome when the students were not given a specific circle to work with.

Contrary to Brad and Travis, Charles did not discuss angle measure in terms of arc-length other than when he attempted to recall a formula that involved arc-length. Charles described angle measure as an amount a ray was elevated from another ray and that the measurement of this elevation was found using a formula or trigonometric functions. The description given by Charles was the only mention of trigonometric functions by any subject during the pre-interview. He subsequently described trigonometric functions as providing an angle measure based on a coordinate, but he described that he was unsure and could not remember how this was done.

Exploratory Teaching Interviews

The exploratory teaching interviews occurred after the first teaching session, which consisted of an introduction of angle measure in terms of a number of degrees and other contrived units (e.g., a unit such that 16 rotated a circle). During the interviews, both Travis and Brad initially gave explanations of angle measure that consisted of arc-length, circumference, and area of a circle without explaining a correct or incorrect relationship between the measures of each quantity. Contrary to this, when Charles was asked to describe angle measure, the following discussion occurred (Table 1).

Table 1.

1	Charles:	The way we measure an angle is, what an angle is, like this one
2		(*drawing an angle*). That angle corresponds to an arc-length (*drawing*
3		*an arc-length*). So, if it's one degree, depending on how big the
4		radius of the circle is corresponds to a certain arc-length.
5	Int:	Ok. So, uh, could you say a little more about corresponds to a certain
6		arc-length? Talk a little more on that.
7	Charles:	So, a standard unit-circle, which has a radius of one-inch (*drawing a*
8		*circle and labeling the radius of one-inch*). Um, I know it's not a
9		perfect circle but, lets say this is (*drawing a ninety degree angle*), I
10		don't know, lets just make it ninety degrees. This ninety degrees has a
11		set value on the circumference of the whole circle. So if it's ninety
12		degrees with a radius of 1 inch, this 90 degrees corresponds to
13		(*pause*). It's two pi so, it'd be one-half pi (*writing* $(1/2)\pi$). That's how
14		much the arc-length would be.
15	Int:	So how'd you come up, so…
16	Charles:	The reason I know the arc-length would be one-half was because I
17		know the, well the formula for circumference is two-pi-r (*writing*
18		$C = 2\pi r$). So if, so in this circle (*referring to his drawn circle by*
19		*tracing the circumference*) with the radius of 1, the circumference is
20		two pi. And, we know there is 360 degrees in a circle, (*writing* 360
21		*and drawing a circle*) so 90 degrees is a fourth of it. So, a fourth of
22		two pi would equal one-half-pi (*writing* $2\pi/4 = (1/2)\pi$).

During this interaction, Charles immediately identified angle measure as relating to arc-length and that this was dependent on the radius used to make the arc-length (lines 1-4). He continued by giving an example of how arc-length and circumference related to angle measure by using what he claimed was the unit-circle (which hadn't been introduced to this point). Charles continued to describe that the proportion of the angle measure to the 360 degrees should be the same proportion of the arc-length to the circumference of the circle (lines 16-22). Thus, it appears that Charles had constructed an image of angle measure that included reasoning about an arc-length's fraction of a circle's circumference.

As the interviews continued, each subject continually refined their image of angle measure and its relationship to arc-length and circumference. One factor that possibly contributed to this refinement was the need to explicitly describe an attribute of a circle and how it related to angle measure as a result of the presented tasks. Hence, the subjects altered and refined their descriptions of angle measure as they encountered situations with which they needed to find various measurements by using other given measurements, which often involved constructing a circle. Furthermore, after the subjects successfully completed and described a task, the interviewer (myself) provided the students a calculation in open form and prompted them to interpret the calculation and the result of the calculation. This often resulted in the subjects comparing how they could use the various calculations presented to solve the problem. Although the subjects did not naturally reveal certain reasoning abilities when solving each task, the interviewer was able to present certain scenarios and calculations to determine if the subjects could reason in various ways consistent with the instructional goals.

As an example, on a task that each subject added two angle measures of different units by converting one of the measurements, the interviewer focused each subject on explaining how the expression of $\frac{22.3}{360}+\frac{3.1}{16}$ (e.g., the addition of each angle's subtended fraction of a circle's circumference) could be used to solve the task. Brad's response focused on each calculation of the expression (Table 2).

Table 2.

Brad:	Ok, so basically you're saying that if they just took this specific equation and added these two fractions together…
Int:	And they got a number out of it, and you can find the number they got out of it too. Kind of talk about, if you have any insights to what they were doing, where they're going.
Brad:	(*Calculating value of expressions*) Ok, so then this fraction plus this fraction (*pointing to the two fractions*) gives you, um, 0.26.

Int: Ok, so we have that equation right, so could you think about what the student did? Like I said, you might want to try breaking it down, what does this mean, what does that mean (*pointing to the expression*).

Brad: Ok, well, this would be your (*pointing to* 22.3/360) degree measurement and this is gonna, dividing this is gonna give you, um, essentially this is going to be your percentage of, uh, this will be your percentage of degrees for the whole 360 degrees. And then you're going to add that percentage to your percentage of 3.1 quips, your total quips. So essentially what he's saying is that, um, this is going to equal (*pointing to the expression*) 26 percent of your total, total degree measure. Now, or your total circumference. So, essentially you could say 26% of 360 (*calculating on calculator*), probably about 92 (*referring to previously obtained answer*). Or 93.6.

After calculating the result of the expression (line 7) and being asked to explain the thinking of a student that used this expression (lines 8-10), Brad described each term of the expression as a percentage (lines 11-16) of a total number of units. Brad continued by describing the entire expression and the result of calculating the expression as a percentage of total degree measure *and* circumference (lines 16-20). Immediately after this interaction, Brad continued by describing how he could use this percentage to find a specific length of arc-length or angle measurement of the combined angles. Thus, although Brad initially solved the task correctly by converting one unit of angle measurement to the other, after the interviewer posed a calculation to Brad, he revealed the ability to reason about angle measure relative to a percentage of total units or circumference.

The posing of the above calculation resulted in each subject describing the meaning of each value and operation of the expression (e.g., the result of 22.3/360 describes the percentage the degree measure is of 360 degrees) and relating these operations to arc-length and circumference. I note that the subjects' description of an arc-length's *percentage* of a circle's circumference was spontaneous and not promoted previous to the interviews. This percentage description of angle measure appears to be a powerful way of reasoning for the subjects, as they were frequently observed describing and using this constructed understanding to complete various interview tasks.

Another important finding from the exploratory interviews was the importance of the subjects' abilities to distinguish between linear measurements and angular measurements of arc-length and circumference. As the subjects refined their image of angle measure such that this image included an explicit distinction of the two measurements, they were seen generalizing their statements about an arc-length's percentage of a circle's circumference to include any unit of measurement and any circle used to measure the angle. For instance, on the task described above, Brad described that the total percentage of arc-length to circumference wasn't reliant on the chosen unit and could be used to determine any unit of arc-length measurement and for any circle. Also, the ability to distinguish between linear and angular measurements enabled all three subjects to describe how a varying radius of a Ferris wheel would influence the angular and linear speeds of an individual. Consider Table 3 for the description given by Brad.

Table 3.

Int.:	So if we maintain the same angular speed, what happens as the radius increases?
Brad:	You'll still have to travel, well, if you're looking at angular speed and your radius is increasing then your radial speed is going to stay the same, you'll still have to keep that to make it. But, as your looking at it, your linear speed, in order to keep that exact same radial speed, you'd have to increase your linear speed.
Int.:	Ok, so you'll have to increase, and if we shrunk the radius?
Brad:	You'd have to decrease.
Int.:	Decrease. Ok, but lets say we kept the same linear speed…as we increase our radius what happens to our…
Brad:	You'd have to decrease your radial speed.
Int.:	Decrease your radial speed. Ok, good, and if we decrease the radius?
Brad:	You'd have to increase your radial speed.

The ability of Brad to identify how angular (or radial) speed is related to tangential (or linear) speed may have been enabled by his image of the relationships of arc-length, circumference, and angle measure. This image included the ability to consider multiple lengths of a radius while distinguishing linear measurements from angular measurements.

Post-interviews

The post-interviews were conducted after the last teaching session. The second teaching session focused on the use of a radian as a unit of measurement, while the third teaching session focused on circular motion (e.g., graphing the relationship of arc-length traveled and vertical distance above the center of the circle) and the construction of the unit circle through the use of the length of a radius as a unit of measurement.

All three subjects' initial descriptions of angle measure focused on defining angle measure as an arc-length's percentage of an entire circumference. All three students described that a changing radius would not influence the percentage of circumference an angle cuts off. In addition, they indicated that the ratio of the linear measurements of arc-length and circumference wouldn't change when using various circles to measure an angle, while the linear measurement of the subtended arc-length and the circumference would vary. Thus, it appears that the subjects developed an image of angle measure that included distinguishing angle measure as a percentage of a circle's circumference and linear measurements as referring to the magnitude of the measured arc-length. That is, the subjects had identified and clearly distinguished each measurement and the quantities referenced by each measurement.

Relative to the radian as a unit of measurement, each subject first used and described the relationship between arc-length and circumference to determine an angle measurement in radians. In response to this, I posed the question of the meaning of a radian measurement (e.g., "What does 0.51 radians mean?"). All three students initially explained radian measurement by describing it in terms of a percentage of a circle's circumference. For instance, Travis responded by explaining that it was based on a percentage of "all the radians, 6.28" and that the measurement was a percentage of one radius (e.g., 0.51 radians was 51% of a radius), eventually explaining that the measurement can be found by dividing the arc-length by the length of a radius to find "how many radius were in the arc-length." Both Brad and Charles also explained that a radian angle measurement could be given by the ratio of arc-length to the length of a radius, with Brad describing that this operation would result in how many radii go into the arc-length.

Although each subject described a relationship between arc-length and a radius when discussing the radian as a unit of measurement, the subjects were most frequently observed reasoning about radian measure and arc-length as a percentage of a circle's circumference. As an example of the subjects' propensity to reason about arc-length as a percentage of a circle's circumference, on a task that presented a labeled image and asked the students to produce an algebraic relationship between the length of a radius, *r*, and angle measure, θ, and a subtended arc-length, *s*, each subject utilized ratios that defined a percentage of the total circumference in order to determine an algebraic relationship (e.g., $\frac{s}{2\pi r} = \frac{\theta}{2\pi}$). Then, after being prompted to simplify the expression with θ measured in radians and obtaining $s = \theta r$ or $\theta = \frac{s}{r}$, each subject discussed how an arc-length as a percentage of the length of a radius related to each formula. Thus, it appears that the subjects were able to construct a relationship between angle measure in radians and the length of a radius, but an image of angle measure as a percentage of a circle's circumference dominated their actions.

The subjects' responses to a problem that asked them to graphically represent the relationship between an individual's vertical distance from the ground and total distance traveled on a Ferris wheel ride also offered insights to their trigonometric reasoning abilities. First, all three subjects initially drew incorrect graphs relative to concavity but correct in terms of directional change. Each subject initially justified his graph by explaining that as the total distance increased, the vertical distance from the ground increased and then decreased. When asked to explain the curvature of the graph, each subject then refined his explanation to include amounts of change of total distance and vertical height based on the graph produced, opposed to based on a diagram or currently constructed image of the situation.

As an example, after Charles first drew a graph that was concave down for the first three-quarters of a revolution, he was asked to explain his graph. He first explained his graph in terms of the vertical height increasing and then decreasing. He then explained that his graph was concave down because the "change of vertical distance" was increasing and the "change in the change of vertical distance" was decreasing. When asked to elaborate and after referring to the graph, he noted that he meant to describe, "the output's increasing, but the change, ok, I see what I'm saying, the change in output is decreasing, as, for every equal amount of input (denoting changes of input and output on his graph)."

Although Charles' explanation of the concavity of *his produced graph* was correct, the graph itself was incorrect over the first quarter of a revolution (e.g., being concave down opposed to concave up). In response, I asked Charles to explain the shape of the graph using the Ferris wheel (Table 4).

Table 4.

Charles:	So, as the total distance is increasing (*tracing the arc-length*), we notice that height is increasing but for every successive change of total distance (*making marks at equal changes of arc-length*), lets say right here it's eight, well if I drew a bigger one, I'd be able to show it more precise.
Int:	Here, go ahead and, uh, I've got some extra pieces of paper. Go ahead and if you want to draw it on there somewhere (*handing him a sheet of paper*) a little bigger.

Charles:	(*Drawing a larger circle and drawing a vertical-horizontal crosshair in the middle of the circle*) So, as, I guess we can assume this is ninety degrees (*referring to the compass*) we can make an angle (*attempts to use protractor on compass*)…
Int:	So what are you trying to do right now?
Charles:	Well I see, this thing moves (*referring to the compass*), I'm trying to show that, um, I'm trying to make, well I could use the protractor, I'm just trying to change, show successive change in input.
Int:	Could you use the Wikki Stix to do that?
Charles:	Well, actually, yes I can.
Int:	So, you're trying to show successive changes in what?
Charles:	In input, which would be the total distance (*marks a distance on a Wikki Stix mumbling to himself, then marking off successive arc-lengths on the circle*). Ok, so, for every change in total distance, right here (*referring to arc-length*), he, well, hmmm.
Int:	What makes you go hmmm?
Charles:	Because I was thinking the last time I did this represented right (*referring to the top-half of the circle*), hmm.
Int:	So what's making you go hmm now?
Charles:	Because it seems as total distance increases (*referring to the arc-length*), the actual change in the height is increasing instead of decreasing.

Charles first explained that as the total distance increased, the height of the individual also increased (lines 1-3). Next, Charles attempted to identify successive changes of arc-length by relating arc-length to angle measure (lines 9-12 & 14-16). Thus, it appears that Charles had related the path of the individual to angle measure. Charles continued by using a Wikki Stix to mark equal changes of arc-length, which resulted in him questioning his original graph (e.g., lines 20-23), and eventually concluding that changes in height should be increasing rather than decreasing (lines 28-30). Through the use of a diagram, Charles was able to describe the correct relationship between total distance and vertical height using amounts of change and correctly construct the graph representing this relationship. On the other hand, the graph first drawn by Charles was possibly created without reasoning about amounts of changes of height and amounts of changes of total distance. Rather, Charles drew a curved graph, possibly because of an iconic transfer of the shape of the Ferris wheel or an assumption that there was a varying rate of change, and *then* interpreted the graph relative to amounts of change. I note that his description of *his* graph was correct in both instances. However, in the first instance, his graph appears to have molded his description of the two quantities; in the second instance, his reconstructed contextual image of the relationship between the two quantities drove the construction of the graph. Thus, the incorrectness of his first graph was not a result of a lack of reasoning ability. Rather, it was the focus of his attention that influenced his description of the relationship between the quantities.

This insight highlights the role of quantitative reasoning and covariational reasoning in representing the relationship between two quantities. In order to represent the (correct) relationship between covarying quantities, a student must first construct an image of the situation and attributes that can be quantified. These attributes can then be quantified and become conceptual objects the students can reason with in a manner that results in meaningful descriptions and representations. Although Charles' descriptions remained attentive to the quantities he was relating (e.g., arc-length and vertical distance), he was not able to produce a correct graph until he focused on specific measurements of arc-length.

Summary of Findings

Initially, the subjects did not identify the specific role that arc-length and circumference play in conceptualizing angle measure. However, after instruction and encountering tasks that necessitated reasoning about angle measure in terms of arc-length and circumference, the students revealed multiple behaviors indicative of them reasoning about a relationship between angle measure, arc-length, and circumference. Specifically, the subjects' spontaneously conceived of angle measure in relation to an arc-length's *percentage* of a circle's circumference. This conception allowed the subjects to make many meaningful constructions, including unit conversion formulas based on the quantitative relationships defined by the formulas. I conjecture that the use of a percentage may be powerful to the subjects as their images of a percentage may reference a portion of a whole (e.g., 24% of a total 100%), where a fraction or decimal (e.g., 0.24 or 6/25) may not be as readily conceived as describing a relationship between two quantities.

Another observation made was the importance of the subjects' abilities to explicitly distinguish between a linear measurement and an angular measurement of arc-length or circumference. Making this distinction is important, as understanding angle measure includes understanding that the linear magnitude of the unit of angle measurement will change depending on the circle chosen to make the measurement. The ability to make this distinction enabled the subjects to discuss how a circle of any radius could be used to determine an angle's measure. In addition, the separate identification of a linear measurement and angular measurement appears to have enabled the subjects discussing circular motion using multiple units of measurement while describing relationships between the multiple units of measurement. For instance, given a constant angular speed, the subjects' were able to describe how the linear speed would vary for varying values of the radius.

Relative to the radian as a unit of measurement, the subjects were most frequently observed reasoning about radian measure and arc-length as a percentage of a circle's circumference, opposed to as a percentage of the radius. This may have been a result of the instructional emphasis on angle measure as a fraction of a circle's circumference. In addition, it may have been an easier construction for students to relate arc-length to circumference given that the circumference is an arc-length. On the other hand, relating the length of a radius to the measurement of arc-length requires an image of using the magnitude of the radius as a unit of arc-length measurement. Furthermore, this image may be even more difficult when attempting to reason about a radius of varying length.

Conclusions

Over the course of the investigation, all subjects continued to refine their image of angle measure in relation to the quantities of arc-length and angle measure. When working the interview tasks, the subjects did not always spontaneously reveal the various reasoning abilities articulated as instructional goals of the teaching sessions. However, through probing and the posing on additional questions, each subject was able to reason about angle measure in a variety of ways, most of which were consistent with the intentions of the lesson. Thus, the subjects' actions were highly reliant on the situation or task that they conceived. Furthermore, it appears that the subjects' images of angle measure being based on the measurable attributes of arc-length and circumference enabled the subjects to coherently reason about angle measure in a variety of ways depending on the situation they constructed.

The reliance of the subjects' reasoning on the task they were working on highlights the necessity that the design of classroom (and interview) tasks reflect the mathematical reasoning that an instructor wishes to develop. If an instructor wishes to promote reasoning about a mathematical topic in a specific manner (e.g., angle measure as an arc-length's fraction of a circumference), tasks should be designed that necessitates the desired reasoning patterns. The subjects of this study were able to solve the tasks in a variety of ways. However, the subjects' methods of solving the problems could *not* be used to claim the subjects could not reason about the task in a different manner. Hence, through the exploratory interview design, the interviewer was able to pose additional tasks to test and promote the development of the subjects, where these tasks were based on the understandings under investigation.

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